SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853

TECHNICAL REPORT NO. 716

August 1986

Revised August 1988

ROUNDING OFF TO POWERS OF TWO IN CONTINUOUS RELAXATIONS OF CAPACITATED LOT SIZING PROBLEMS

by

Robin Roundy

This research was supported by NSF Grant DMC–8451984, and also by AT&T Information Systems and DuPont Corporation.

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ABSTRACT

In the capacitated version of the Divide and Conquer algorithm for lot sizing in multi–stage production/inventory problems, feasibility is often lost when the reorder intervals are rounded off to powers of two. We propose a new algorithm for rounding off the reorder intervals which always produces a feasible policy. We have shown that the relative increase in cost that occurs when the intervals are rounded off using to this algorithm can not exceed 44% , and that for systems with a single capacity–constrained machine (including the ELSP), the cost increase can not exceed 6%. Computational experience with industrial data sets indicates that the algorithm performs very well.

1. INTRODUCTION.

Consider the problem of scheduling a multi–machine facility which produces a number of different items. One or more items may be consumed in producing a given item. External demand can occur at a constant rate for any or all of the items. Stockouts are not allowed. The setup costs, setup times, production rates, and holding cost rates are known, deterministic, and constant. The objective is to minimize the average setup and holding cost per year over an infinite time horizon.

The following steps have been proposed by researchers for problems of this type. First, approximate production frequencies for the items are computed by suppressing the sequencing aspects of the problem and minimizing EOQ–type cost functions. These production frequencies are expressed as order intervals, the intervals of time between successive production runs of a product. The second step is to round off the approximate order intervals to integer multiples of β for some positive number β. The third step is to use these order intervals to find a feasible schedule.

The most extensively researched problem of this type is the economic lot scheduling problem, hereafter called the ELSP. The ELSP is a special case of this problem in which there is only one machine and only one stage on manufacture. Dobson [5] applied these three steps to the ELSP in sequence. Other authors have implemented some or all of these steps in different ways, often combining them with myopic improvement procedures [1,2,3,4,5,6,7,8,9,11,12,13,14,15,17,21]. The first two of these steps have been applied to multi–item, multi–stage production systems by Maxwell and Muckstadt [18], Roundy [20], and Jackson, Maxwell, and Muckstadt [16], and research on the third step is under way.

In the first step, that of finding approximate order intervals for the products, most authors do not consider setup times or machine capacity explicitly. The choice of order intervals is guided solely by the setup and holding costs. Exceptions to this rule are Fujita [10], Dobson [5], and Jackson, Maxwell, and Muckstadt [16]. When setup times are present, the more successful approach is the one used by Dobson [5] for the ELSP and by Jackson, Maxwell, and Muckstadt [16] for multi–item, multi–stage production systems. In this approach the setup and holding costs are minimized subject to the constraint that the average amount of time spent per year in setting up each machine does not exceed the total yearly amount of time available for setups. This problem is solved by an extension of the Divide and Conquer algorithm [16], which can be viewed as a generalization of Dobson's approach. A precise formulation of this problem is given in Section 2.

The second step is the subject of this paper. We follow other authors in requiring that the order intervals be powers of two times β rather than allowing them to be arbitrary integer multiples of β [5,11,12,13,20]. Powers of two are chosen for two reasons. First, for the ELSP it has been empirically observed that within the class of policies in which all items are produced in equal amounts and at equal intervals of time, power–of-two policies are almost always optimal, and when not optimal they are near–optimal [7,8,13,19]. In addition, the special structure of power–of-two policies makes the third step easier to solve and makes the policies easier to implement on the factory floor [18].

The purpose of this paper is twofold. First, we propose a new way of rounding off the reorder intervals to powers of two times β, $\beta > 0$. Second, we show that the cost penalty incurred in so doing is at most 6.1% for problems with only one capacity–constrained machine (including the ELSP), and at most 44% for multi–item, multi–stage production systems. The latter bound is tight. Computational results with industrial data sets indicate that the average cost penalty is much smaller. Therefore the advantages of power–of–two order intervals can be had at a moderate cost.

We treat β as a variable. In many situations it is important that β be treated as a constant, dictated by some aspect of the system being modeled [18]. For example, β might correspond to length of the time intervals used in planning production. The problem of optimally rounding off the reorder intervals when β is a constant is reminiscent of bin–packing problems. Solution methods for that problem are also likely to be similar to bin–packing heuristics, and to share the same strengths and weaknesses. Jackson, Maxwell, and Muckstadt propose an extremely simple solution to this problem that does not guarantee feasibility. They show that if for each machine, the number of items produced on the machine that have distinct reorder intervals is sufficiently large, then the central limit theorem implies that the solution is approximately feasible with high probability [16].

For the ELSP, many different methods have been used to perform the third step, that of finding a feasible schedule. We will not discuss them in detail. Methods that require the items to be produced in equal amounts and at equal intervals tend to be complex and often do not guarantee feasibility, probably because the problem of determining whether a feasible equal–order-interval schedule exists for a given set of order intervals is NP–complete [15]. The third step is much simpler when the order intervals are allowed to be unequal [3,5,17]. For multi–item, multi–stage production systems only very special cases of the third step have been studied. For multi–stage systems, research on the third step is currently under way at Cornell University.

The remainder of this paper is organized as follows. In Section 2 we review the essential aspects of the way the approximate order intervals are computed, and describe our algorithm for rounding off the order intervals to powers of two. The worst–case relative cost increase incurred in rounding off the order intervals is studied for general systems in Section 3 and for systems with a single bottleneck machine in Section 4. At the end of Section 3 we summarize our computational experiments for multi–item, multi–stage systems using industrial data sets, and in Section 5 we present our conclusions.

2. THE ROUNDOFF ALGORITHM

In this section we review the essential elements of the approach taken by Jackson, Maxwell, and Muckstadt [16] and Dobson [5] for the first step, that of finding approximate order intervals for the items. Then we present our algorithm for the second step, rounding off the order intervals to power–of–two multiples of β.

For multi–item, multi–stage production systems, the mathematical formulation of the problem of finding approximate order intervals for the products takes on the following form:

(P)
$$\min: \sum_n \left[\frac{K(n)}{T(n)} + H(n)T(n) \right] \tag{1}$$

$$\text{such that: } \sum_n \frac{\tau(n,m)}{T(n)} \leq \rho_m \quad \forall \quad 1 \leq m \leq M \tag{2}$$

$$T(\ell) \geq T(n) \quad \forall \quad \ell \rightarrow n \in A \tag{3}$$

where $K(n)$, $H(n)$, $\tau(n,m)$, and $T(n)$ are respectively the setup cost, holding cost coefficient, setup time on machine m, and order interval for item n. M is the number of machines, ρ_m is the fraction of the total operating time that is available for setups on machine m, and $\ell \rightarrow n \in A$ is an arc from item ℓ to item n representing a bill–of–material relationship. For the ELSP, $M = 1$ and $A = \phi$ [5]. It is assumed that $H(n) \geq 0$, $K(n) \geq 0$, and $\tau(n,m) \geq 0$ for all n. Often for each item n, $\tau(n,m) > 0$ for at most one machine m, but this is not necessarily the case. For example, the time of a skilled worker can be modeled in this way.

The solution to (P) is obtained by applying lagrangean relaxation to (2). The algorithm that solves (P) groups the items into <u>clusters</u> C_i, $1 \leq i \leq I$ that share a common reorder interval $T_i = T(n) \;\forall\; n \in C_i$. Let $K_i \equiv \Sigma_{n\in C_i} K(n)$, $H_i \equiv \Sigma_{n\in C_i} H(n)$, and $\tau_{im} \equiv \Sigma_{n\in C_i} \tau(n,m)$. The optimal order intervals $\mathscr{T}^* \equiv (T_i^*, 1 \le i \le I)$ solve

$$(P_1) \qquad \min: \; C(\mathscr{T}) \equiv \sum_i \left[\frac{K_i}{T_i} + H_i T_i \right] \tag{4}$$

$$\text{such that: } \sum_i \frac{\tau_{im}}{T_i} \le \rho_m \quad \forall \quad 1 \le m \le M . \tag{5}$$

The reason that constraint (3) no longer appears is that if (3) is binding for $\ell \to n$ then items ℓ and n are grouped into the same cluster.

It is easily verified that

$$T_i^{*2} = \frac{K_i + \lambda_m \tau_{im}}{H_i} \tag{6}$$

where $\lambda_m \ge 0$ is the dual multiplier for (5) and thus is complimentary slack with that inequality. For the ELSP each cluster contains a single item. $\lambda_1 \equiv \nu^+$ is the positive part of ν where ν solves

$$\rho_1 = \sum_i \frac{\tau_{i1} \sqrt{H_i}}{\sqrt{K_i + \nu \tau_{i1}}} . \tag{7}$$

We now proceed to describe the algorithm for rounding off the the order intervals T_i^* to powers of two times β , $\beta > 0$. For all i let z_i and the integer p_i be defined by

$$T_i^* \equiv z_i 2^{p_i} , \quad 1 \le z_i < 2 . \tag{8}$$

A simple approach to computing $\mathscr{T}$ would be to set $T_i = \beta \times 2^{p_i}$ for all i, and choose β to minimize $C(\mathscr{T})$ subject to (5). Note however that for any $d > 0$, in (8) we could have defined z_i and p_i using $d \leq z_i < 2d$. Computing $\mathscr{T}$ using our simple approach, different values of d would give rise to different $\mathscr{T}$'s. Therefore setting $d = 1$ seems rather arbitrary. The heuristic we propose can be described as applying our simple heuristic to all d, $\frac{1}{2} < d \leq 1$, and selecting the best $\mathscr{T}$ identified in this way.

We assume that the clusters are indexed so that $z_i \leq z_{i+1}$ for all i. For each k, $1 \leq k \leq I$, we consider a solution of the form $\mathscr{T}^k \equiv (T_i^k, 1 \leq i \leq I)$ where

$$T_i^k \equiv \alpha^k 2^{q_i^k}, \tag{9}$$

$$q_i^k \equiv \left\{ \begin{array}{ll} p_i - 1, & i \leq k \\ p_i, & i > k \end{array} \right\}, \tag{10}$$

and α^k is a positive scalar. Note that $p_i - q_i^k = 0$ if $i > k$ and that $p_i - q_i^k = 1$ if $i \leq k$, so (8) implies that

$$z_k \leq z_i 2^{p_i - q_i^k} = \frac{T_i^* \alpha^k}{T_i^k} \leq 2z_k \quad \text{for all } i. \tag{11}$$

We wish to choose α^k so as to minimize $C(\mathscr{T}^k)$ subject to (5). Let

$$K^k \equiv \sum_i K_i 2^{-q_i^k} \text{ and } H^k \equiv \sum_i H_i 2^{q_i^k}. \tag{12}$$

Then the cost of $\mathscr{T}^k$ can be written as $C(\mathscr{T}^k) = K^k/\alpha^k + H^k \alpha^k$. The value of α^k that minimizes $C(\mathscr{T}^k)$ is clearly

$$\beta^k \equiv \sqrt{\frac{K^k}{H^k}} . \tag{13}$$

However (5) implies that α^k is greater than or equal to

$$\gamma^{km} \equiv \frac{1}{\rho_m} \sum_i \tau_{im} 2^{-q_i^k} . \tag{14}$$

Since $C(\mathcal{T}^k)$ is a convex function of α^k, the optimal value of α^k is given by

$$\alpha^k \equiv \max(\beta^k, \max_m \gamma^{km}) . \tag{15}$$

This identifies I policies, one for each value of k. We propose that among these I policies, the policy $\mathcal{T}^{k^*}$ that minimizes $C(\mathcal{T}^k)$ over k is selected. See Figure 1.

<u>Figure 1. The Roundoff Algorithm.</u>

<u>Step 1.</u> Determine z_i and p_i for all i using (8).

<u>Step 2.</u> Re–index the clusters so that $z_i \le z_{i+1}$ for all i.

<u>Step 3.</u> For each k, $1 \le k \le I$, apply (10) and (12) through (15) to compute $C(\mathcal{T}^k)$.

<u>Step 4.</u> Select the vector of order intervals $\mathcal{T}^{k^*}$ with minimal cost.

We now show that $\mathcal{T}^{k^*}$ can be computed in $O(IM + I \log I)$ time. The sort in Step 2 requires $O(I \log I)$ time. Clearly q_i^1, K^1, H^1, α^1, and $C(\mathcal{T}^1)$ can be computed in $O(IM)$ time. Since $q_i^k = q_i^{k+1}$ for all $i \ne k$, it is easy to verify that given q_i^k, K^k, H^k, and γ^{km}, $1 \le m \le M$, we can compute q_i^{k+1}, K^{k+1}, H^{k+1}, $\gamma^{k+1,m}$ and $C(\mathcal{T}^{k+1})$ in $O(M)$ time. Thus we can compute $\mathcal{T}^{k^*}$ in $O(IM + I \log I)$ time.

3. WORST-CASE EFFECTIVENESS FOR MULTI–ITEM, MULTI–STAGE PRODUCTION SYSTEMS

In this section we show that the worst–case increase in cost incurred by applying the Roundoff Algorithm is $1/\log 2 - 1 \cong 44\%$ for multi–item, multi–stage production systems. We also present computational results on a number of industrial data sets. Recall that $\mathcal{T}^*$ solves

$$(\mathrm{P}_1) \qquad \min: \ C(\mathcal{T}) \equiv \sum_i \left[\frac{K_i}{T_i} + H_i T_i \right] \tag{16}$$

$$\text{such that: } \sum_i \frac{\tau_{im}}{T_i} \le \rho_m \quad \forall \ \ 1 \le m \le M \,. \tag{17}$$

The Roundoff Algorithm produces a $\mathcal{T}^{k^*} = (T_i^{k^*}, 1 \le i \le I)$ that approximately minimizes (16) subject to (17) and to

$$T_i = \beta \times 2^{p_i} \,, \ \beta > 0 \,, \ p_i \text{ integer}, \ \ \forall \ \ 1 \le i \le I \,. \tag{18}$$

We want to find the worst–case cost of $\mathcal{T}^{k^*}$ relative to $\mathcal{T}^*$, i.e., to maximize $C(\mathcal{T}^{k^*})/C(\mathcal{T}^*)$ over all data sets. We solve this problem in several steps. In the first of these steps we assume that the value of $\mathcal{T}^*$ that solves (P_1) is known, and we select the data M, τ_{im} and ρ_m for constraint (17). Consider the following problem:

(D_1) Given $\mathcal{T}^*$, select M, $(\tau_{im} : 1 \le i \le I, 1 \le m \le M)$, and $(\rho_m : 1 \le m \le M)$, so as to maximize the relative cost $C(\mathcal{T}^{k^*})/C(\mathcal{T}^*)$ subject to three constraints: $\tau_{im} \ge 0$, $\rho_m > 0$, and $\mathcal{T}^*$ must solve (P_1).

Lemma 1. An optimal solution to (D_1) is

$$M = I\ ,\ \ \tau_{im} = \begin{cases} 1 & \text{if } i = m \\ 0 & \text{if } i \neq m \end{cases}\ ,\ \text{and}\ \ \rho_i = 1/T_i^*\ . \tag{19}$$

Under (19), (17) becomes

$$T_i \geq T_i^* \quad \forall\ 1 \leq i \leq I\ . \tag{20}$$

Proof. Let (P_1^*) be the problem of minimizing (16) subject to (20). Under (19), (P_1) and (P_1^*) are the same problem. Since $\tau_{im} \geq 0$ and $\mathcal{T}^*$ satisfies (17), any $\mathcal{T}$ that satisfies (20) must also satisfy (17). This and the fact that $\mathcal{T}^*$ solves (P_1) imply that $\mathcal{T}^*$ solves (P_1^*). Furthermore α^k is more tightly constrained by (20) than by (17), so $C(\mathcal{T}^k)$ is at least as high for (P_1^*) as it is for (P_1). Since this is true for each value of k, $C(\mathcal{T}^{k^*})$ is at least as high for (P_1^*) as it is for (P_1). □

In the remainder of this section we assume that (19) holds, so (17) can be replaced by (20). We now consider the problem of selecting the setup costs K_i. Assuming as before that the solution $\mathcal{T}^*$ to (P_1) is known and given, consider the following problem:

(D_2) Given $\mathcal{T}^*$, and assuming that (19) holds, select setup costs $(K_i : 1 \leq i \leq I)$ so as to maximize the relative cost $C(\mathcal{T}^{k^*})/C(\mathcal{T}^*)$ subject to two constraints: $K_i \geq 0$ and $\mathcal{T}^*$ solves (P_1).

Lemma 2. The solution to (D_2) is

$$K_i = 0 \ \text{ for all } i\ .$$

<u>Proof.</u> Let $\mathcal{S}^*$ solve (P_1). Let (P_1^*) be the version of (P_1) with $K_i = 0$ for all i. Clearly $\mathcal{S}^*$ solves (P_1^*). Let $\mathcal{S}$ satisfy (18) and (20). Then $[\Sigma_i K_i/T_i]/[\Sigma_i K_i/T_i^*] \le 1 \le C(\mathcal{S})/C(\mathcal{S}^*) = [\Sigma_i (K_i/T_i + H_iT_i)]/[\Sigma_i (K_i/T_i^* + H_iT_i^*)]$, so $C(\mathcal{S})/C(\mathcal{S}^*) \le [\Sigma_i H_iT_i]/[\Sigma_i H_iT_i^*]$. Therefore if $\mathcal{S}$ satisfies (20) then $C(\mathcal{S})/C(\mathcal{S}^*)$ is at least as high for (P_1^*) as it is for (P_1). □

We assume that $K_i \equiv 0$ throughout the remainder of this section. Under (19), (14) becomes

$$\gamma^{km} = T_m^* 2^{-q_m^k} = \begin{Bmatrix} 2z_m , & m \le k \\ z_m , & m > k \end{Bmatrix}$$

by (8) and (10). Since $\beta^k = 0$ we have $\alpha^k = 2z_k$ and $T_i^k = 2z_k \times 2^{q_i^k}$. Let

$$w_i = \frac{H_i z_i 2^{p_i}}{\sum_j H_j z_j 2^{p_j}} .$$

By (8),

$$\frac{C(\mathcal{S}^k)}{C(\mathcal{S}^*)} = \frac{\Sigma_i H_i z_k 2^{q_i^k+1}}{\Sigma_i H_i z_i 2^{p_i}} = \sum_i w_i \frac{z_k}{z_i} 2^{q_i^k+1-p_i} = \sum_{i \le k} w_i \frac{z_k}{z_i} + \sum_{i>k} 2w_i \frac{z_k}{z_i} ,$$

i.e.,

$$\frac{C(\mathcal{S}^k)}{C(\mathcal{S}^*)} = \sum_{i \le k} w_i \frac{z_k}{z_i} + \sum_{i>k} 2w_i \frac{z_k}{z_i} . \qquad (21)$$

The relative cost of the order intervals calculated by the Roundoff Algorithm is $R \equiv \min_k C(\mathcal{S}^k)/C(\mathcal{S}^*)$. Let $z_{I+1} \equiv 2z_1$ and $\omega \equiv \Sigma_{1\le i\le I} (z_{i+1}/z_i - 1)$. Setting $z_0 \equiv \frac{1}{2} z_I$ we have $\omega = \Sigma_{1\le i\le I} (z_i/z_{i-1} - 1)$.

Lemma 3. $R \le 1/\omega$. Furthermore this bound is tight.

Proof. Let $v_k \equiv (z_{k+1} - z_k)/\omega z_k$. Then $v_k \ge 0$ and $\Sigma_{1\le k\le I} v_k = 1$. By (21), $R \equiv \min_k C(\mathcal{S}^k)/C(\mathcal{S}^*) \le \Sigma_k v_k C(\mathcal{S}^k)/C(\mathcal{S}^*) = \Sigma_i \frac{w_i}{z_i} \{\Sigma_{k\ge i} v_k z_k + \Sigma_{k<i} 2v_k z_k\} = \Sigma_i \frac{w_i}{\omega z_i} (z_{I+1} - z_i + 2z_i - 2z_1) = \Sigma_i \frac{w_i}{\omega} = 1/\omega$. Setting $w_i = (z_i/z_{i-1} - 1)/\omega$ in (21), we see that the bound given by Lemma 3 is tight. □

Applying Lemma 3 is equivalent to choosing H_i to maximize R . The final step in determining the worst–case effectiveness of the heuristic is therefore choosing I and z_i to maximize $1/\omega$, subject to the constraints $1 \le z_i \le z_{i+1} < 2$ for all $i < I$, and $z_{I+1} = 2z_1$.

Theorem 4. $R \le 1/\log 2 \cong 1.44$. Furthermore, this bound is tight.

Proof. Setting $y_i = \log (z_{i+1}/z_i)$, the problem of choosing I and z_i to minimize ω becomes

(D$_3$)
$$\min: \sum_i (e^{y_i} - 1)$$
$$\text{such that: } \sum_i y_i = \log 2$$
$$y_i \ge 0 .$$

Since e^y is convex, the solution to (D_3) is $y_i \equiv \frac{1}{I} \log 2$ for all i, i.e., $\frac{z_{i+1}}{z_i} = 2^{1/I}$ for all i. Thus $R \le 1/\{I(2^{1/I}-1)\}$, a non-decreasing function of $I \ge 1$ that tends to $1/(\log 2)$ as I tends to ∞. □

Summarizing, the worst case occurs when (19) holds, $K_i = 0$, $w_i = (z_i/z_{i-1} - 1)/\omega$, and I is large. Of these conditions, (19) stands out as being highly unrealistic and highly restrictive. Clusters are connected components of the bill–of–materials network. Consequently they tend to follow the flow of products through the plant rather than spreading out to include all of the products on a bottleneck machine. One would thus expect bottleneck machines to intersect a number of different clusters.

To test the performance of the algorithm on real–world problems, we have run it on five different industrial data sets, three small ones and two large ones. The results of these runs are summarized in Table 1 below. Data sets one, two, three, four, and five come respectively from companies that manufacture computers, industrial chain, aircraft, pneumatic air tools, and automobiles.

We compare the cost of the policy computed by the Roundoff Algorithm to the lower bound obtained by solving (P), and to the cost of the algorithm proposed by Jackson, Maxwell, and Muckstadt (called the Simple Algorithm) in which the order intervals are rounded off to powers of two without regard to feasibility. For this algorithm we also report the maximum value of the ratio of the left-hand side of (2) to the right-hand side. For a feasible policy this ratio will not exceed one.

The results in Table 1 confirm that the Roundoff Algorithm works much better on industrial data sets than its worst–case performance bound indicates.

Table 1: Computational Results.

Data Set	Nodes	Arcs	Machines	Roundoff Algorithm: Cost Ratio	Simple Algorithm: Cost Ratio	Simple Algorithm: Maximum Feasibility Ratio
1	8	4	2	1.000	0.786	1.35
2	14	14	2	1.066	0.976	1.15
3	27	37	4	1.016	1.167	1.08
4	407	311	31	1.053	1.188	1.14
5	269	776	27	1.117	0.995	1.19

4. THE WORST–CASE EFFECTIVENESS FOR THE ELSP.

In this section we show that if there is only one capacity-constrained machine ($M = 1$ in (P)), then the worst–case increase in cost that can result from applying the Roundoff Algorithm is 6% , i.e., that $C(\mathcal{T}^{k^*})/C(\mathcal{T}^*) \leq \frac{1}{2}(\sqrt{2} + \sqrt{.5})$. We also show that $C(\mathcal{T}^{k^*})/C(\mathcal{T}^*)$ can be as large as $1/[2(\log 2)^2] \cong 1.04$. Since there is only one machine, in this section we drop the machine subscripts. Therefore in (5), (6), and (14) we use ρ for ρ_1 , τ_i for τ_{im} , λ for λ_1 , and γ^k for γ^{km} . There are two cases to consider, $\Sigma_i \tau_i/T_i^* < \rho$ and $\Sigma_i \tau_i/T_i^* = \rho$.

Case 1: $\sum_i \frac{\tau_i}{T_i^*} < \rho$.

Suppose we were to decrease the value of ρ from its current value to $\Sigma_i \tau_i/T_i^*$. The cost $C(\mathcal{T}^*)$ of $\mathcal{T}^*$ would be unaffected because $\mathcal{T}^*$ is still feasible for (P). However the cost $C(\mathcal{T}^k)$ of $\mathcal{T}^k$ would either increase or remain unchanged. Therefore the relative cost for Case 1 is bounded from above by the relative cost for Case 2.

Case 2: $\sum_i \frac{\tau_i}{T_i^*} = \rho$.

Suppose we were to use $\bar{\mathscr{T}}^k \equiv (T_i^k \equiv \gamma^k 2^{q_i^k}, 1 \le k \le I)$ instead of using $\mathscr{T}^k$. This would be tantamount to choosing α^k so as to make (2) tight for all k , even though this may be a suboptimal choice of α^k . Therefore $C(\bar{\mathscr{T}}^k) \ge C(\mathscr{T}^k)$ for all k .

An upper bound on $C(\mathscr{T}^{k^*})/C(\mathscr{T}^*)$ can be found as follows. The minimum of $C(\bar{\mathscr{T}}^k)$ over k is an upper bound on $C(\mathscr{T}^{k^*})$. We use a weighted average of $C(\bar{\mathscr{T}}^k)$ over k as an upper bound on the minimum over k of $C(\bar{\mathscr{T}}^k)$. The weight assigned to k is w_k where

$$\begin{aligned} w_k &\equiv \log_2(z_{k+1}/z_k) , \quad k \ne I ; \\ w_I &\equiv \log_2(2z_1/z_I) . \end{aligned} \tag{22}$$

Let

$$z_i^k = z_i \text{ if } i > k \text{ and } z_i^k = 2z_i \text{ if } i \le k . \tag{23}$$

The following lemma is used in the proof.

Lemma 5. If $i < j$ then $\sum_k w_k \left[\frac{z_j^k}{z_i^k} + \frac{z_i^k}{z_j^k} \right] \le (\sqrt{2} + \sqrt{.5})$.

Proof. By (22) and (23)

$$\sum_k w_k \left[\frac{z_j^k}{z_i^k} + \frac{z_i^k}{z_j^k} \right] = f\left[\frac{z_i}{z_j} \right]$$

where

$$f(x) \equiv \left[x + \frac{1}{x} \right] \log_2 (2x) + \left[2x + \frac{1}{2x} \right] \log_2 \left[\frac{1}{x} \right] .$$

Since $i < j$, (8) implies that $\frac{1}{2} \le \frac{z_i}{z_j} \le 1$. On the interval $\left[\frac{1}{2}, 1\right]$, the function f is concave and attains its maximum value of $\sqrt{2} + \sqrt{.5}$ at $x = \sqrt{.5}$. □

<u>Theorem 6.</u> $C(\mathcal{T}^{k^*}) \le \frac{1}{2}(\sqrt{2} + \sqrt{.5})\, C(\mathcal{T}^*) \cong 1.06\, C(\mathcal{T}^*)$.

<u>Proof.</u> Note that (23), (10), and (8) imply that

$$z_i^k \times 2^{q_i^k} = z_i \times 2^{p_i} = T_i^* .$$

Let $A_i \equiv \tau_i/T_i^*$, $B_i \equiv H_i T_i^*$, and $D_i \equiv K_i/T_i^*$. Then by (12) we have

$$K^k = \sum_i D_i z_i^k \text{ and } H^k = \sum_i \frac{B_i}{z_i^k} .$$

Let $a_i \equiv \frac{A_i}{\Sigma_j A_j}$. Note that $a_i \ge 0$ and $\Sigma_i a_i = 1$. By assumption $\rho = \Sigma_i \tau_i/T_i^* = \Sigma_i A_i$, so (14) implies

$$\gamma^k = \frac{\Sigma_i A_i z_i^k}{\Sigma_i A_i} = \sum_i a_i z_i^k .$$

We can write $C(\bar{\mathcal{T}}^k)$ as

$$C(\bar{\mathcal{T}}^k) = \frac{K^k}{\gamma^k} + H^k \gamma^k = \frac{\Sigma_i D_i z_i^k}{\Sigma_i a_i z_i^k} + \left[\sum_i a_i z_i^k\right] \left[\sum_i \frac{B_i}{z_i^k}\right] . \tag{24}$$

By (6) we have $B_i = \lambda A_i + D_i$, so

$$C(\mathscr{T}^*) = \sum_i (B_i + D_i) = \sum_i 2D_i + \sum_i \lambda A_i \tag{25}$$

and

$$\sum_k w_k\, C(\bar{\mathscr{T}}^k) = \sum_k w_k \left[\frac{\Sigma_i\ D_i z_i^k}{\Sigma_i\ a_i z_i^k} + \left[\sum_i a_i z_i^k \right] \left[\sum_i \frac{D_i}{z_i^k} \right] \right]$$

$$+ \sum_k w_k \lambda \left[\sum_i a_i z_i^k \right] \left[\sum_i \frac{A_i}{z_i^k} \right] . \tag{26}$$

The remainder of the proof is divided into two parts. In Part 1 we show that the first term of (26) is at most $\frac{1}{2}(\sqrt{2} + \sqrt{.5})$ times the first term of (25), and in Part 2 we show that the second term of (26) is at most $\frac{1}{2}(\sqrt{2} + \sqrt{.5})$ times the second term of (25). Since $C(\mathscr{T}^{k^*}) = \min_k C(\mathscr{T}^k) \le \min_k C(\bar{\mathscr{T}}^k) \le \Sigma_k\ w_k C(\bar{\mathscr{T}}^k)$, this will complete the proof.

<u>Part 1.</u>

Let $d_i \equiv \dfrac{D_i}{\Sigma_j\ D_j}$. Then $d_i \ge 0$ and $\Sigma_i\ d_i = 1$. We need to show that $D \le \sqrt{2} + \sqrt{.5}$ where

$$D \equiv \sum_k w_k \left[\frac{\Sigma_i\ d_i z_i^k}{\Sigma_i\ a_i z_i^k} + \left[\sum_i a_i z_i^k \right] \left[\sum_i \frac{d_k}{z_i^k} \right] \right] .$$

By Jensen's inequality and by Lemma 5,

$$D \le \sum_k w_k \left[\left[\sum_i d_i z_i^k \right] \left[\sum_i \frac{a_i}{z_i^k} \right] + \left[\sum_i a_i z_i^k \right] \left[\sum_i \frac{d_i}{z_i^k} \right] \right]$$

$$= \sum_{j<i} \left[a_j d_i + d_j a_i \right] \sum_k w_k \left[\frac{z_j^k}{z_i^k} + \frac{z_i^k}{z_j^k} \right] + \sum_i 2 a_i d_i$$

$$\le \left[\sqrt{2} + \sqrt{.5} \right] \left[\sum_{j<i} \left[a_j d_i + d_j a_i \right] + \sum_i a_i d_i \right]$$

$$\le \left[\sqrt{2} + \sqrt{.5} \right] \left[\sum_i a_i \right] \left[\sum_i d_i \right] = \sqrt{2} + \sqrt{.5} .$$

Part 2.

We need to show that $A \le \frac{1}{2}(\sqrt{2} + \sqrt{.5})$ where

$$A = \sum_k w_k \lambda \left[\sum_i a_i z_i^k \right] \left[\sum_i \frac{a_i}{z_i^k} \right]$$

By Lemma 5,

$$A = \sum_{j<i} \left\{ a_j a_i \sum_k w_k \left[\frac{z_j^k}{z_i^k} + \frac{z_i^k}{z_j^k} \right] \right\} + \sum_i a_i^2$$

$$\le \frac{1}{2} \left[\sqrt{2} + \sqrt{.5} \right] \left[\sum_{j<i} 2 a_j a_k + \sum_i a_i^2 \right] = \frac{1}{2} \left[\sqrt{2} + \sqrt{.5} \right] . \quad \square$$

Theorem 7. $C(\mathcal{S}^{k^*})/C(\mathcal{S}^*)$ can be as high as $\dfrac{1}{2(\ln\ 2)^2} \cong 1.04$.

Proof. Let $K_i \equiv 0$. Then $\beta^k = 0$ for all k , so $\mathcal{S}^k = \bar{\mathcal{S}}^k$ for all k . (6) implies that $B_i = \lambda A_i$ and $T_i^{*2} = \lambda \tau_i / H_i$, so $D_i = 0$, $A_i = \tau_i / T_i^* = \sqrt{\tau_i H_i / \lambda}$, and we can choose τ_i and H_i so that $T_i^* = z_i = 2^{(i-1)/I}$ and $A_i = 1/\sqrt{\lambda}$. Thus $a_i = 1/I$ for all i . Equations (24) and (25) imply that

$$C(\mathscr{T}^{k^*})/C(\mathscr{T}^*) \leq \left[\sum_i a_i z_i^k\right]\left[\sum_i \frac{a_i}{z_i^k}\right] = [2I^2(2^{1/I} + 2^{-1/I} - 2)]^{-1} ,$$

a non-decreasing function of $I \geq 1$ that converges to $\frac{1}{2(\ln\ 2)^2}$ as I tends to ∞. □

5. CONCLUSIONS.

We have proposed an algorithm for rounding off the reorder intervals computed by the capacitated version of the Divide and Conquer algorithm for lot sizing in multi–stage production/inventory problems, for capacity-constrained, multi–item lot sizing problems to powers of two. We have shown that the relative increase in cost that occurs when the intervals are rounded off using to this algorithm can not exceed 44% , and that for systems with a single bottleneck machine (including the ELSP), the cost increase can not exceed 6%. Computational experience with industrial data sets indicates that for multi–machine problems the algorithm performs much better than the worst–case bound indicates.

ACKNOWLEDGEMENTS

The author is indebted to Rafaela del Pilar for obtaining the computational results reported herein. This research was supported by the National Science Foundation under grant number DMC–8451984, and by AT&T Information Systems and DuPont Corporation.

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