Integration Constraints in Parametric Design of Physical Objects*

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ABSTRACT

Design by constraint is a powerful approach to improve *CAD* systems and designer productivity. This paper addresses the topic of incorporating integration constraints concerning object mass and inertia in a *CAD* system. Two classes of generic objects are discussed that contain affine and conformal images of a given solid, usually derived from the initial design solution. Domain-derivatives are partial derivatives needed to solve the existence problem of an object instance satisfying all constraints and, eventually, to find an optimal design solution. In the paper it is shown that these derivatives are closely linked to the topology of the solid. They are symbolically expressable as integrals over domains having a lower geometrical dimension than the original solid.

1. Introduction

In engineering design activities, one of the main tasks is to determine a suitable shape for a set of design components. The actual problem usually consists of finding suitable dimensions and positions for the design components, whose primitive shape is suggested by the professional practice or by previous successful designs. In most problems the actual shape of a component is itself not important, but because it determines the static and dynamic behaviour of the object component, as well as its geometric relationships with other components in a complex assembly.

Traditional engineering design procedures are exceedingly iterative. A rough initial design, having an approximately dimensioned shape for each component, is subject to various inspections, and some dimensions are eventually modified. Then another set of checks is performed that usually lead to a new dimensioning phase, and so on, in a trial and error verification process. Such a cyclic procedure may be very expensive and time wasteful. When one component is changed, adjacent components may also have to be modified to maintain design integrity, leading to a theoretically endless process of modification. Such a process is actually bounded by the maximum time and cost allocated for the activity, and it is usually shortened by good professional designers.

In recent years, Computer Aided Design (*CAD*) techniques have successfully reduced the design effort by allowing explicit declaration of functional and geometric constraints to be satisfied by the design solutions, and various prototipe *CAD* systems that partially automate the dimensioning of 2− and 3−*D* objects have been developed[4,6,3].

Design by constraint is performed by identifying, in a class of parameterized object instances, the one that satisfies all constraints. In the oldest approaches to the parametric design, that Requicha[18] denotes as *primitive instancing*, the object families were used mainly for representational issues, without exploiting the involved problem-solving capabilities. Instead, the most interesting feature of generic or parameterized objects is their capability of maintain functionality while varying in dimension[10].

Most of the work on the automatic dimensioning and positioning of parts in an assembly is related to the pioneering work of Sutherland [20] and Ambler and Popplestone[1]. Automatic dimensioning and positioning where then exploited by[9,16,14]. The Light and Gossard [16] method, called *variational geometry*, introduced the dimensioning of a geometric model through the solution of a simultaneous non-linear system of equations. The only integration constraint allowed in this approach concerned the shape area. In Lee[14] some concepts from Ambler [1] and Gossard[16] are implemented for computing the spatial relationship among the components in an assembly. More recently, design by constraint using Prolog [2,6] has been experimented.

This paper addresses the topic of incorporating in a *CAD* system the integration constraints concerning the required values of either mass and inertial properties of the design. Both in traditional design procedures and in the most of currently used *CAD* systems, the evaluation of inertial properties is performed on the current design solution, by using approximated numeric methods. Only recently, closed formulas that allow for the symbolic evaluation of inertial properties of linear non-convex polyhedra where given by Lien and Kajiya[15] and, independently, by Cattani and Paoluzzi[7], by using the Timmer-Stern method[21,17]. Section 2 is therefore dedicated to recall the mathematical background and symbolic formulas for the inertia of polyhedrons. In section 3 we analize two different object classes corresponding to a given generic object shape. The first class is that of objects obtainable as *affine images* of a generic object representative of the class, usually an initial shape solution. The second class is that of *conformal images* of the generic object. In section 3 characterizations of affine and conformal objects classes are presented and relationships between inertial properties of different object instances for both classes are given. Such relationships may be used to verify the *existence* of an object instance satisfying a given set of integral and geometric constraints by transforming each of them in an equation, usually non-linear in the design parameters, and looking for their simultaneous solution. If the set of real simultaneous solutions of constraints contains more than one element, i.e. more than one feasible object instance, it is possible to look for optimal design. For instance the object mass, or the object inertia, or the object external surface may be minimized. Some examples of optimal parametric design problems and the relative statement of inertial constraints will be presented in sections 5.

We show in this paper that volume integrals concerning inertial constraints can be expressed as nonlinear polynomials in the design parameters. Hence, both in deciding about the existence of feasible design solutions, and in looking for global or local optimum design, it may be necessary to arrange a symbolic expression of their partial derivatives, or to evaluate in suitable points the gradient of the objective function or the jacobian matrix of constraints.

If the design problem concerns the finding of a feasible conformal object instance, the partial derivatives of volume integrals are expressible as low-degree polynomials in the design variables, but with coefficients that are, on the contrary, very complex expressions of the position of vertices of the generic object. Section 4 is therefore dedicated to the study of geometric properties of a class of partial derivatives of volume integrals, that we call *domain-derivatives*, with the aim of avoiding the necessity of invoking a symbolic algebraic manipulation system when looking for a feasible or optimal object instance.

Our main result, in studying domain-derivatives of volume integrals, concerns their property of being computable as integrals over domains having a lower geometrical dimension than the original solid. It is interesting to discover that the domain-derivatives are closely linked to the topology of the solid. In particular, we prove a set of theorems showing that the first, second and third derivatives of a volume integral, with respect to the displacement parameters, (when non-zero) are respectively equal or proportional to:

(a) a surface integral of the integrand function g over a face of the integration domain;

(b) a line integral of g over an edge of the integration domain;

(c) the value of g, evaluated on a vertex of the integration domain.

Such theorems, together with the formulas (3-5) of section 2, give a very simple and computationally efficient way for arranging symbolic expressions for domain-derivatives. Moreover, the theorems are very general, as they hold for every volume integral over a polyhedral domain, and for each kind of integrand function. Their validity is consequently not restricted to the derivatives of inertial properties of a physical object.

2. Mass and inertial properties

Mass and inertial properties of solid objects are defined as volume integrals of some very simple and low-degree polynomial functions, always the integration being done over the portion of the 3-space occupied by the object under consideration. If P is the solid, then the mass $M(P)$, the first moments $M_x(P)$, $M_y(P)$, $M_z(P)$, the second moments $M_{xx}(P)$, $M_{yy}(P)$, $M_{zz}(P)$, and the products of inertia $M_{xy}(P)$, $M_{yz}(P)$, $M_{zx}(P)$, are defined as

$$\iiint_P p(x,y,z)\, dm = \iiint_P p(x,y,z)\, \rho(x,y,z)\, dV \quad , \tag{1}$$

where $\rho(x,y,z)$ is the local density of the solid, and the function $p(x,y,z)$ is respectively equal to: 1 (mass); x, y, z (first moments); x^2, y^2, z^2 (second moments); xy, yz, zx (products of inertia). The coordinates of the centroid $x_G(P)$, $y_G(P)$, $z_G(P)$ are then defined as $M_x(P)/M(P)$, $M_y(P)/M(P)$, $M_z(P)/M(P)$. The moments of inertia with respect to coordinate axes $M_x^2(P)$, $M_y^2(P)$, and $M_z^2(P)$ are in turn defined as the integrals of the square distances from the considered coordinate axis and are constructed, respectively, as $M_{yy}(P)+M_{zz}(P)$, $M_{zz}(P)+M_{xx}(P)$ and $M_{xx}(P)+M_{yy}(P)$.

Very frequently the density ρ of the solid can be considered constant. In this case the solid is said to be ***homogeneous***, and the density term can be expressed outside the integral. Notice that for homogeneous solids the position of the centroid doesn't depend on the density, but only on the geometry of the object.

Analogous expressions are used for defining the mass and inertial properties of lines and surfaces in the 3-space. These are evaluated by means of line or surface integration, using exactly the same integrand functions we have seen for the solid case. In[8] and[7] exact and closed formulas for the evaluation of line, surface and volume integrals of general trivariate polynomials over linear and regular 1- 2- and 3-

polyhedrons in 3-space are given. In particular, in [7] it is shown that any such integral may be computed as a linear combination of the following terms, called *structure products*:

$$\int_{P_1} x^\alpha y^\beta z^\gamma \, dl \quad or \quad \iint_{P_2} x^\alpha y^\beta z^\gamma \, dS \quad or \quad \iiint_{P_3} x^\alpha y^\beta z^\gamma \, dV \ , \tag{2}$$

where P_1, P_2 and P_3 are respectively a polygonal path, a polyhedral surface and a polyhedral solid; α, β and γ are non-negative integers. Some pretty resolution formulas for structure products[7] follow:

$$\int_{P_1} x^\alpha y^\beta z^\gamma \, dl = \tag{3}$$

$$= \sum_{i=0}^{n-1} \|\mathbf{v}_{i+1}-\mathbf{v}_i\| \sum_{h=0}^{\alpha} \binom{\alpha}{h} x_i^{\alpha-h} (x_{i+1}-x_i)^h \sum_{k=0}^{\beta} \binom{\beta}{k} y_i^{\beta-k} (y_{i+1}-y_i)^k \sum_{l=0}^{\gamma} \binom{\gamma}{l} z_i^{\gamma-l} (z_{i+1}-z_i)^l \frac{1}{h+k+l+1}$$

$$\iint_{P_2} x^\alpha y^\beta z^\gamma \, dS = \tag{4}$$

$$= \sum_{f\in P_2} \|\mathbf{n}(f)\| \sum_{i=1}^{n_f-1} \sum_{h=0}^{\alpha} \binom{\alpha}{h} x_0^{\alpha-h} \sum_{k=0}^{\beta} \binom{\beta}{k} y_0^{\beta-k} \sum_{l=0}^{\gamma} \binom{\gamma}{l} z_0^{\gamma-l} \cdot$$

$$\cdot \sum_{p=0}^{h} \binom{h}{p} (x_i-x_0)^{h-p}(x_{i+1}-x_0)^p \sum_{q=0}^{k} \binom{k}{q} (y_i-y_0)^{k-q}(y_{i+1}-y_0)^q \sum_{r=0}^{l} \binom{l}{r} (z_i-z_0)^{l-r}(z_{i+1}-z_0)^r \cdot$$

$$\cdot \frac{1}{h+k+l+1-p-q-r} \sum_{s=0}^{h+k+l+1-p-q-r} \binom{h+k+l+1-p-q-r}{s} \frac{(-1)^s}{s+p+q+r+1}$$

$$\iiint_{P_3} x^\alpha y^\beta z^\gamma \, dV = \tag{5}$$

$$= \frac{1}{\alpha+1} \sum_{f\in\partial P_3} \frac{n_x(f)}{\|\mathbf{n}(f)\|} \sum_{i=1}^{n_f-1} \sum_{h=0}^{\alpha+1} \binom{\alpha+1}{h} x_0^{\alpha-h+1} \sum_{k=0}^{\beta} \binom{\beta}{k} y_0^{\beta-k} \sum_{l=0}^{\gamma} \binom{\gamma}{l} z_0^{\gamma-l} \cdot$$

$$\cdot \sum_{p=0}^{h} \binom{h}{p} (x_i-x_0)^{h-p}(x_{i+1}-x_0)^p \sum_{q=0}^{k} \binom{k}{q} (y_i-y_0)^{k-q}(y_{i+1}-y_0)^q \sum_{r=0}^{l} \binom{l}{r} (z_i-z_0)^{l-r}(z_{i+1}-z_0)^r \cdot$$

$$\cdot \frac{1}{h+k+l+1-p-q-r} \sum_{s=0}^{h+k+l+1-p-q-r} \binom{h+k+l+1-p-q-r}{s} \frac{(-1)^s}{s+p+q+r+1}$$

In the above formulas, $\mathbf{v}_i = [x_i\ y_i\ z_i]$ is the position vector of the vertex v_i; the number of vertices in the polygonal path P_1 is $n+1$; f is the generic face of the polyhedral surface P_2 and of the boundary δP_3 of the polyhedral solid P_3; n_f+1 is the number of vertices around the face f; $\mathbf{n}(f)$ is an external normal vector to f. Furthermore, for the sake of precision, the position vector $\mathbf{v}_i$ should be written in eqs. (4) and (5) as $\mathbf{v}_{if} = [x_{if}\ y_{if}\ z_{if}]$, and read as the $i-th$ vertex in a circular and counterclockwise ordering around f. If multiple connection of faces is allowed, f should correspond to the single boundary loop, oriented counterclockwise if external and clockwise if internal to another loop. Only one level of loop nesting is allowed.

In order to state optimization problems for the design of a physical object, it is useful to express the whole set of inertial properties in a compact matricial form:

$$\mathbf{M}(P) = \begin{bmatrix} M_{xx} & M_{xy} & M_{xz} & M_x \\ M_{yx} & M_{yy} & M_{yz} & M_y \\ M_{zx} & M_{zy} & M_{zz} & M_z \\ M_x & M_y & M_z & M \end{bmatrix} . \tag{6}$$

This matrix contains ordinately the first and second moments, the products of inertia and the mass of the object P. In the following we prefer, for homogeneous objects having constant density ρ, the notation:

$$\mathbf{M}(P) = \rho\ \mathbf{I}(P) \ , \tag{7}$$

where $\mathbf{I}(P) = [i_{hk}]$, with $h,k=1,\ldots,4$. This matrix contains the geometrical features of an (homogeneous) object, and does not depend on the object's density or material.

3. Affine and conformal object classes

In this section we present two different useful object classes corresponding to a given generic object shape. The first is that of objects obtainable as affine images of the generic object representative of the class. The second class is that of conformal images of the generic object. Both classes can be used in parametric design. In the following we show how the inertia of an affine object instance is related to the inertia of the corresponding generic object. Then we will see how to express symbolic expressions of integral properties for conformal images of the generic object.

3.1 Affine transformations of objects

Affine transformations are defined as the composition of a translation and an invertible linear transformation. It is well known that such transformations of 3-space can be represented as linear transformations in the space $\mathbf{R}^4$, by using homogeneous coordinates and, for convenience, a normalized representation of vectors. If vectors are represented as column, then the fourth row of the normalized matrix of an affine transformation is always [0 0 0 1].

We want to discuss now the relationship between the mass and inertial properties of an affinely transformed solid object and that of a similar, but untransformed object. In particular, Theorem 1 below shows that inertial properties are computable as linear functions of the properties of the untransformed object (considered variable), if the transformation is fixed; or they are computable as ratios of polynomials in the coefficients of the affine transformation matrix, if the values of the inertial properties of the untransformed object are fixed and the transformation is unknown.

Let $\mathbf{P}$ be the $4 \times m$ matrix containing, arranged by column, the position vectors of vertices of the polyhedron P, and let $\mathbf{I}(P)$ be the 4×4 matrix (7). Then the following theorem holds:

Theorem 1 Let $\mathbf{Q}$ be the matrix of an affine transformation and P_Q be the transformed object ($\mathbf{P}_Q = \mathbf{Q}\,\mathbf{P}$). Then:

$$\mathbf{I}(P_Q) = |\det \mathbf{Q}|^{-1}\, \mathbf{Q}\, \mathbf{I}(P)\, \mathbf{Q}^T. \tag{8}$$

This theorem is a generalization of those given in Mechanics under the names of *Rotating axes theorem* and *Translating axes theorem*[12,11], both valid only for rigid transformations. Theorem 1 allows not only translation and rotation of the space, but also scaling and stretching. We will see that in problems of affine optimization (see section 5) we are mainly interested in properties of this kind. In particular, Theorem 1 allows the *existence problem* to be solved when looking for affine instances of a generic object satisfying inertial constraints.

3.2 Conformal transformations of objects

The statements in section 3.1 hold for any object, because they operate on the space in which the objects are embedded. At the contrary, this subsection will concern only polyhedral solids; an extention to more general objects is under study.

Let P_0 be a regular linear polyhedron, and F_0 be the set of its faces, with $n=\#F_0$. Then let P be another polyhedron, with the same topology of P_0, and derived from it by allowing translation of planes containing the faces. In this case, if $A_i x + B_i y + C_i z + D_{0i} = 0$ is the cartesian equation of the face $f_{0i} \in F_0$, with unit external normal $\mathbf{n}_i$, then the corresponding face f_i of P has equation:

$$[x \;\; y \;\; z] \cdot \mathbf{n}_i = \frac{D_{0i}}{\sqrt{A_i^2+B_i^2+C_i^2}} + h_i, \tag{9}$$

where h_i is the signed distance between the planes containing f_{0i} and f_i, respectively. In the following we shall call such parameters *face displacements*. If the face displacements maintain the topology of the polyhedron, i.e. if the boundary of P is described by the same number of vertices, edges and faces of P_0 and by the same incidence relations among them, then we can consider the polyhedron P as a function of the original polyhedron P_0 and of the face displacements h_i:

$$P = H(P_0, h_1, \cdots, h_n) \tag{10}$$

It is easy to see that P is the result of a *conformal transformation* of P_0, because all angles, planar in the 2-dimensional case and solid in the 3-dimensional one, are invariant. For a pictorial representation of a two-dimensional example, see fig. 1. An example with two different conformal images of a 3-dimensional polyhedron is shown in fig. 2.

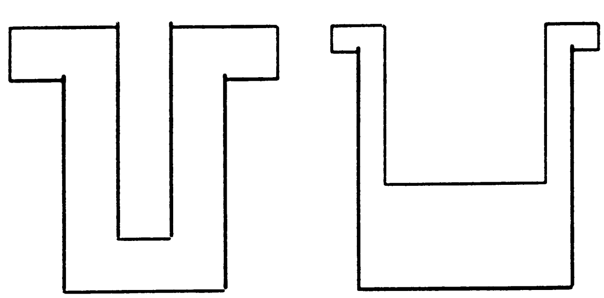

figure 1. conformal instances of a generic 2-D object

Such an object class, containing homeomorphic and conformal images of a given polyhedron P_0 will be called *conformal object class*. A conformal object class is parametrized by the set of face-displacements h_i, $i=1,...,n$. The mapping between object instances and tuples of face-displacements is not, unfortunately, bijective, because while each well-formed object instance corresponds to a different tuple of parameters, the converse is not always true. As a matter of fact, the object corresponding to a given tuple of parameters might self-intersect, and so not maintain the object class topology.

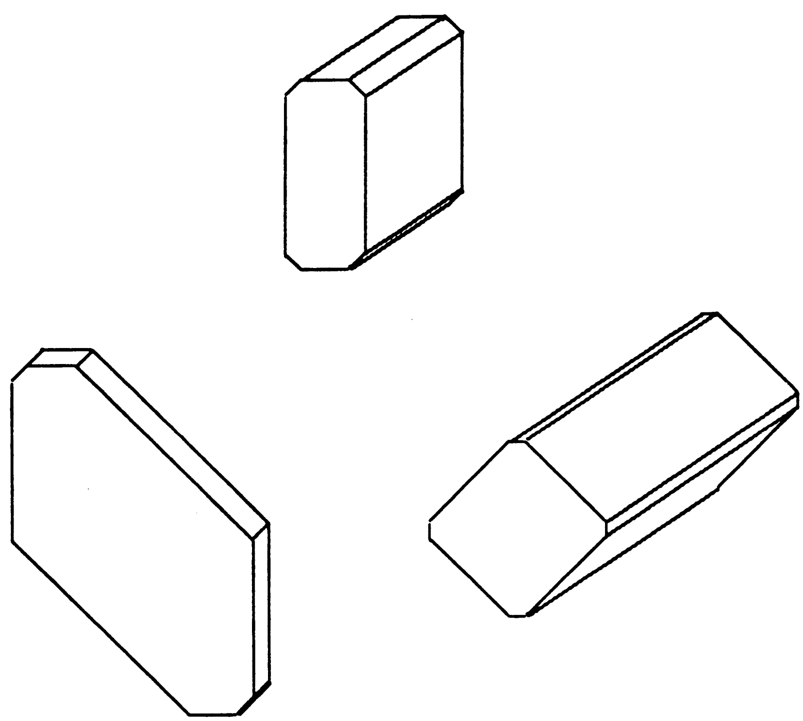

figure 2. Three conformal instances of a generic 3-D object

The necessity of avoiding self-intersections is the main problem arising in the practical use of the conformal transformations. This problem is not dramatic when looking for optimal object instances, because usually an initial solution is known, and the optimization process may proceed choosing a better feasible solution at each step. In this case it is sufficient to verify that self-intersections don't arise at each step, suitably modifying the decision criterion.

In order to obtain an explicit relationship between the known values of the inertial properties of the class representative object P_0 and the unknown values of its generic conformal image $P = H(P_0, h_1, \cdots, h_n)$, it is sufficient to substitute, in the formulas (3-5), the coordinates of the object vertices, whose inertial properties are to be evaluated, with an expression containing explicitly the coordinates of P_0's vertices and the face displacements h_i. Such a substitution is suggested by the following Theorem 2.

Let $a_{ih}x+b_{ih}y+c_{ih}z+d_{ih}=0$ be the cartesian equation of one of the faces incident on the vertex v_i, and

$$\mathbf{A}_i = \begin{bmatrix} a_{ih} & a_{ik} & a_{il} \\ b_{ih} & b_{ik} & b_{il} \\ c_{ih} & c_{ik} & c_{il} \end{bmatrix}$$

be the matrix of normal vectors to three linearly independent planes, containing faces incident on v_i, and $\mathbf{d}_i = [d_{ih}\ d_{ik}\ d_{il}]$. Then we have:

Theorem 2 Position vectors $\mathbf{v}_i = [x_i\ y_i\ z_i]$ of the vertices of a polyhedron under a conformal transformation, and displacements h_i of faces incident on vertices, are linearly related:

$$\mathbf{v}_i = \mathbf{R}_i\ \mathbf{h}_i + \mathbf{s}_i \tag{11}$$

where $\mathbf{h}_i$ is a vector containing the displacements of three non parallel faces incident on the vertex v_i, $\mathbf{R}_i$ is a 3×3 matrix of constants and $\mathbf{s}_i$ is a vector of constants, with $\mathbf{s}_i = -\mathbf{A}_i^{-1}\ \mathbf{d}_i$ and $\mathbf{R}_i = -\mathbf{A}_i^{-1}\ \mathbf{A}_i^T \circ \mathbf{A}_i$.

The operation denoted as $\circ$ is specified as follows:

$$\mathbf{B} \circ \mathbf{C} = [d_{hk}]\ ,\ with\ \ d_{hk} = \begin{cases} (\mathbf{b}_h \cdot \mathbf{c}_k)^{\frac{1}{2}} & (\textit{if } h=k) \\ 0 & (\textit{otherwise}) \end{cases} \tag{12}$$

where $\mathbf{b}_h$ and $\mathbf{c}_k$ are the h-th row and the k-th column of $\mathbf{B}$ and $\mathbf{C}$, respectively.

4. Domain-derivatives of a volume integral

Inertial constraints are symbolically expressed as non-linear equation in the design parameters. So, both in deciding about the existence of feasible design solutions, and in looking for global or local optimum design, it is necessary to arrange a symbolic expression of their partial derivatives, or to evaluate in suitable points the gradient of the objective function or the jacobian matrix of constraints (see appendix 1).

If the design problem concerns the finding of a feasible conformal object instance, such derivatives are expressible as low-degree polynomials in the face displacements, but with coefficients that are, on the contrary, very complex expressions of the position of vertices of the generic object P_0. This section is therefore dedicated to the study of geometric properties of a class of partial derivatives of volume integrals, with the aim of avoiding the necessity of invoking a symbolic algebraic manipulation system when arranging symbolical derivatives of a volume integral.

Consider, first of all, a volume integral over a polyhedral domain P in 3-space:

$$G(g,P) = \iiint_P g(x, y, z)\, dV \tag{13}$$

The integral G exists if g is continous (or sectionally continous) in P. If the integration domain is a conformal image of another domain, i.e. if P depends on an initial volume P_0 and on the orthogonal displacements $h_1, \ldots, h_n$, then the integral [3] becomes a composite function of g, P_0, and h_i:

$$G(g,P) = G(g, H(P_0, h_1, \ldots, h_n)). \tag{14}$$

A partial derivative of a volume integral G with respect to some face-displacement h_i is called a *domain-derivative*. In the following we will study the properties of the domain-derivatives of G, having fixed the integrand function g and the integration domain P_0.

Our main result, in studying domain-derivatives of volume integrals, concerns their property of being computable as integrals over domains having a lower geometrical dimension than the original solid. In particular, the following theorem states that a first order derivative of a volume integral is a surface integral.

Theorem 3 The first domain-derivative of the volume integral of a continous function g with respect to a face displacement h_i, is equal to the surface integral of g over the displaced face f_i:

$$\frac{\partial}{\partial h_i} \iiint_{P_0} g(x, y, z)\, dV = \iint_{f_i} g(x, y, z)\, dS\,. \tag{15}$$

Another result states that a second order domain-derivative is proportional to the line integral along the edge shared by two displaced faces. Their ratio will depend on the relative slopes between the faces and the common edge. The ratio assumes a maximum value when the two faces are orthogonal, and it is zero when the faces are coplanar:

Theorem 4 The second domain-derivative of the volume integral of a continous function g, with respect to two different face displacements h_i and h_j, is either: a) zero, if the faces f_i and f_j are not adjacent, or b) proportional to the line integral of g over the edge $e_{ji} = f_i \bigcap f_j$:

$$\frac{\partial^2}{\partial h_i \partial h_j} \iiint_{P} g(x, y, z)\, dV = \begin{cases} 0 & (\text{if } f_i \bigcap f_j = \emptyset) \\ \mathbf{n}_i \cdot \mathbf{e}_{ji} \times \mathbf{n}_j \int_{e_{ji}} g(x, y, z)\, dl & (\text{if } f_i \bigcap f_j = e_{ji}) \end{cases} \tag{16}$$

The following corollary specifyies the physical meaning of the constant ratio between a second order domain-derivative and the line integral along an edge: it is equal to the sin of the oriented angle between the two considered faces.

Corollary 4.1 A non zero second domain-derivative (with respect to two face displacements) of a volume integral, is equal to (the product of) the angle between the normals to the faces, times the line integral along their common edge

$$\frac{\partial^2 G}{\partial h_i \partial h_j} = \sin \alpha_{ji} \int_{e_{ji}} g(x, y, z)\, dl \quad (\text{if } f_j \bigcap f_i = e_{ji}) \tag{17}$$

Corollary 4.2 The second order domain-derivatives of a volume integral G are independent from the ordering of the derivation:

$$\frac{\partial^2 G}{\partial h_i \partial h_j} = \frac{\partial^2 G}{\partial h_j \partial h_i} \tag{18}$$

The corollary 4.2 states a symmetry property important for the structure of the Hessian matrix of a volume integral. From this result, the continuity of both first and second order domain-derivatives follows trivially.

Corollary 4.3 First and second order domain-derivatives of a volume integral are continous.

And, finally, the following theorem relates the value of a third order domain-derivative to the value of the integrand function evaluated on a vertex of the integration domain, and to the relative slopes among three displaced faces incident on the vertex.

Theorem 5 The third order domain-derivatives of the volume integral of a continous function g, with respect to three different face displacements h_i, h_j and h_k, are either: a) zero, if the faces f_i, f_j and f_k don't intersect in a vertex, or b) proportional to the value of the integrand function g evaluated on the common vertex v_{kji}:

$$\frac{\partial^3}{\partial h_i \partial h_j \partial h_k} \iiint_P g(x, y, z)\, dV = \begin{cases} 0 & (\mathit{if}\ f_i \cap f_j \cap f_k = \varnothing) \\ \dfrac{\sin\alpha_{kj}}{\mathbf{n}_i \cdot \mathbf{n}_k \times \mathbf{n}_j}\, g(x, y, z)\, \Big|_{v_{kji}} & (\mathit{if}\ f_i \cap f_j \cap f_k = v_{kji}) \end{cases} \tag{19}$$

The theorems we have shown in this section are interesting for three different reasons. First of all, they model formally the designer's intuitive understanding about the rate of change of the mass and inertial properties, when the shape of the designed object is subject to local changes. While it is very easy and natural to imagine that the maximum volume increment of the object is obtained by translating the face having maximum surface, for other more complex properties, also involving distances of the object's points from coordinate planes, the answer is not so intuitive. The theorems we have given help to balance, intuitively but correctly, the contributions of the sizes of the modified faces, with those of the relative slopes of the adjacent faces, and those of their average distances from coordinate planes. Secondly, such theorems, together with the formulas (3-5), give a very simple and computationally efficient way for arranging symbolic expressions for domain-derivatives. Last but not least, the theorems in this section are very general, as they hold for every volume integral, and for each kind of integrand function.

5. Design affine optimization

We show in this section how mass and inertial properties may be used to define both the constraints and/or the objective function of a mathematical optimization problem, in order to find a feasible or optimal object instance in physical object design problems. We assume that an initial value for the shape of the object to be designed is given, whereas the elements of an affine transformation matrix, to be applied to the shape to be optimized, are unknown, and will be considered as *design variables*. We call such a problem *affine optimization problem* (*AOP*).

In stating an affine optimization problem, many different situations may arise. In the simplest case, only one homogeneous object (a) has to be considered; in a slightly more complex situations, only one object has yet to be considered, but it is made of different homogeneous parts having different densities (b). In the most complex situations, the design problem may concern the optimization of a complex assembly, either made of homogeneous objects (c), or with some object eventually constituted of parts with different densities (d). In the cases (a) and (b), only one unknown transformation matrix is sufficient, because every part, as the whole space, will be subjected to the same transformation; in the cases (c) and (d) a different transformation matrix for each object in the assembly will be necessary.

In the following, we state the affine optimization problem for the most complex case, taking into account an assembly with more then one homegeneous part in each component object. So, we will indicate an *assembly* as a set of homogeneous parts $\{O_{ij}\}$, where the indexes refer to the $j-th$ homogeneous part of the $i-th$ component object ($i=1,\ldots,N$).

5.1 variables and objective functions

The design variables are the elements of the 3×4 superior submatrices of the matrices $\mathbf{Q}_i$ of unknown affine transformations associated to the N objects of the assembly to be optimized. So, in a *AOP* there are $12N$ scalar variables.

The quality functions to be extremized will usually concern some integral qualities of the design or some weighted combination. Three examples of such kind of objective function follow:

minimum mass

$$\min\ z = \sum_{i=1}^{N}\sum_{j=1}^{N_i} \rho_{ij} \iiint_{O_{ij}} dV \tag{20}$$

$$= \sum_{i=1}^{N} |\det \mathbf{Q}_i|^{-1} \sum_{j=1}^{N_i} \rho_{ij}\, \mathbf{q}_{i4}\, \overline{\mathbf{I}}_{ij}\, \mathbf{q}_{i4}^T = \sum_{i=1}^{N} |\det \mathbf{Q}_i|^{-1} \sum_{j=1}^{N_i} \rho_{ij}\, \overline{\iota}_{ij44}$$

minimum polar moment

$$\min\ z = \sum_{i=1}^{N}\sum_{j=1}^{N_i} \rho_{ij} \iiint_{O_{ij}} (x^2+y^2+z^2)\, dV \tag{21}$$

$$= \sum_{i=1}^{N} |\det \mathbf{Q}_i|^{-1} \sum_{j=1}^{N_i} \rho_{ij} \left(\mathbf{q}_{i1}\, \overline{\mathbf{I}}_{ij}\, \mathbf{q}_{i1}^T + \mathbf{q}_{i2}\, \overline{\mathbf{I}}_{ij}\, \mathbf{q}_{i2}^T + \mathbf{q}_{i3}\, \overline{\mathbf{I}}_{ij}\, \mathbf{q}_{i3}^T \right)$$

$$= \sum_{i=1}^{N} |\det \mathbf{Q}_i|^{-1} \left[\mathbf{q}_{i1} \left(\sum_{j=1}^{N_i} \rho_{ij}\, \overline{\mathbf{I}}_{ij} \right) \mathbf{q}_{i1}^T + \mathbf{q}_{i2} \left(\sum_{j=1}^{N_i} \rho_{ij}\, \overline{\mathbf{I}}_{ij} \right) \mathbf{q}_{i2}^T + \mathbf{q}_{i3} \left(\sum_{j=1}^{N_i} \rho_{ij}\, \overline{\mathbf{I}}_{ij} \right) \mathbf{q}_{i3}^T \right]$$

minimum area

$$\min\ z = \sum_{i=1}^{N}\sum_{j=1}^{N_i} \int\!\!\int_{S \subseteq \partial O_{ij}} dS = \sum_{i=1}^{N}\sum_{j=1}^{N_i} \int\!\!\int_{S \subseteq \partial O_{ij}} x^0 y^0 z^0\, dS = \sum_{i=1}^{N} |\det \mathbf{Q}_i|^{-1} \sum_{j=1}^{N_i} S_{ij} \tag{22}$$

In the above equations, $\mathbf{Q}_i$ is the $i-th$ unknown matrix, $\overline{\mathbf{I}}_{ij}$ is the 4×4 matrix of constants (7) evaluated, by using the formula (5), on the initial design solution; ρ_{ij} is the density of the design component O_{ij}, assuming homogeneity; $\mathbf{q}_{ik}$ is the $k-th$ row of the unknown matrix $\mathbf{Q}_i$.

It is important to note that the initial design solution has two main roles: to define the topology and some geometric invariants of the shape that will not change during the *AOP* solution process. In particular, the ratios between distances of points of the same object will not change: if $\mathbf{v}_a, \mathbf{v}_b, \mathbf{v}_c$ are the position vectors of any three points belonging to the same object, then $(\mathbf{v}_b-\mathbf{v}_a)^2/(\mathbf{v}_c-\mathbf{v}_a)^2$ is constant under any affine transformation. So, the optimal solution is obtained by translating, rotating, scaling and stretching the initial one, but maintaining in any case some similarity with the initial shape.

The integral objective functions we have shown are, at most, rational functions of second degree in the numerator and of third degree in the denominator, in the unknowns q_{ihk}, $1 \le i < N$, $1 \le h < 4$, $1 \le k < 3$. As a matter of fact $|\det \mathbf{Q}_i|$ is an unknown polynomial of degree 3; this follows from the fact that the $4-th$ row of an affine transformation matrix is the unit vector.

5.2 Integration constraints

Each mass and inertial property (6), either of an elementary part O_{ij}, or of a single component object $O_i = \bigcup_j O_{ij}$, or of the whole assembly, may be forced to assume a fixed value or to satisfy an inequality constraint:

$$m_{ijhk}^{\min} \le m_{ijhk} = |\det \mathbf{Q}_i|^{-1}\, \rho_{ij}\, \mathbf{q}_{ih}\, \bar{\mathbf{I}}_{ij}\, \mathbf{q}_{ik}^T \le M_{ijhk}^{\max} \qquad (1 \le i \le N, 1 \le j \le N_i, 1 \le h \le 4, 1 \le k \le 3) \tag{23}$$

or

$$m_{ihk}^{\min} \le m_{ihk} = |\det \mathbf{Q}_i|^{-1}\, \mathbf{q}_{ih} \left(\sum_{j=1}^{N_i} \rho_{ij} \bar{\mathbf{I}}_{ij} \right) \mathbf{q}_{ik}^T \le M_{ihk}^{\max} \qquad (1 \le i \le N,\ 1 \le h \le 4,\ 1 \le k \le 3) \tag{24}$$

or

$$m_{hk}^{\min} \le \sum_{i=1}^{N} m_{ihk} = \sum_{i=1}^{N} |\det \mathbf{Q}_i|^{-1}\, \mathbf{q}_{ih} \left(\sum_{j=1}^{N_i} \rho_{ij} \bar{\mathbf{I}}_{ij} \right) \mathbf{q}_{ik}^T \le M_{hk}^{\max} \qquad (1 \le h \le 4,\ 1 \le k \le 3) \tag{25}$$

where m_{ijhk}, m_{ihk}, m_{hk} represent, respectively, the h,k element of the inertia matrix relative to the assembly part O_{ij}, to the assembly component O_i, or to the assembly as a whole. Correspondingly, $m^{\min}$ and $M^{\max}$ represent minimal and maximal feasible values for the property h,k.

Of course, other constraints involving integration properties could be built up, to guarantee the design solution matches all the design goals. For example, in order to guarantee the *static equilibrium* of the component O_i, the coordinates of its centroid might be confined to remain inside a restricted region:

$$x_{G_i} \le \frac{\sum_{j=1}^{N_i} \rho_{ij}\, i_{ij14}}{\sum_{j=1}^{N_i} \rho_{ij}\, i_{ij44}} = |\det \mathbf{Q}_i|^{-1}\, \mathbf{q}_{i1} \frac{\sum_{j=1}^{N_i} \rho_{ij}\, \bar{\mathbf{I}}_{ij}}{\sum_{j=1}^{N_i} \rho_{ij}\, \bar{i}_{ij44}}\, \mathbf{q}_{i4}^T \le X_{G_i} \tag{26}$$

$$y_{G_i} \le \frac{\sum_{j=1}^{N_i} \rho_{ij}\, i_{ij24}}{\sum_{j=1}^{N_i} \rho_{ij}\, i_{ij44}} = |\det \mathbf{Q}_i|^{-1}\, \mathbf{q}_{i2} \frac{\sum_{j=1}^{N_i} \rho_{ij}\, \bar{\mathbf{I}}_{ij}}{\sum_{j=1}^{N_i} \rho_{ij}\, \bar{i}_{ij44}}\, \mathbf{q}_{i4}^T \le Y_{G_i} \tag{27}$$

The boundary values $x_{G_i}, y_{G_i}, X_{G_i}, Y_{G_i}$ might be expressed, in turn, as functions of the transformed positions of some vertex belonging to other components of the assembly.

5.3 Geometric feature constraints

Together with integral constraints, other kind of geometrical constraints may be necessary to obtain the correct object solution. Specific knowledge about the desired geometric properties of the design solution can be easily transformed in specific constraints in the *AOP*. Such constraints may a) directly involve geometric features of feasible object instances or b) apply to the unknown transformation matrices, and constrain their structure, and, as a consequence, the shape and position of the assembly. The constraints of first type, studied by Ambler and Popplestone[1] and Lee[13,14], usually concern the required spatial relationships between pairs of components of the assembly. They are referred to as *against*, *fit* and *hinge* constraints.

"Against" constraints

Two faces in objects O_i and O_j may be constrained to remain coplanar, by forcing their external normal vectors to be opposite, and forcing one point of the face of O_i to belong to the plane for the face of O_j. Such constraints can be expressed as follows:

$$\mathbf{Q}_i\ (\mathbf{v}_{i1}-\mathbf{v}_{i0})\times(\mathbf{v}_{i2}-\mathbf{v}_{i0}) = -\mathbf{Q}_j\ (\mathbf{v}_{j1}-\mathbf{v}_{j0})\times(\mathbf{v}_{j2}-\mathbf{v}_{j0}) \ ; \tag{28a}$$

$$\det\ [\mathbf{Q}_i\mathbf{v}_{i0}\ \ \mathbf{Q}_j\mathbf{v}_{j0}\ \ \mathbf{Q}_j\mathbf{v}_{j1}\ \ \mathbf{Q}_j\mathbf{v}_{j2}] = 0 \ , \tag{28b}$$

where $\mathbf{v}_{i0}$, $\mathbf{v}_{i1}$, $\mathbf{v}_{i2}$ and $\mathbf{v}_{j0}$, $\mathbf{v}_{j1}$, $\mathbf{v}_{j2}$ are two suitable triples of points from the considered faces. The equation (28a) states that the normals to two given faces must be opposite after the transformation; the equation (28b) constraints one transformed point of O_i to belong to the transformed plane for $\mathbf{v}_{j0}$, $\mathbf{v}_{j1}$, $\mathbf{v}_{j2}$.

"Fit" constraints

If two axes, belonging respectively to objects O_i and O_j, have to remain aligned, as in the case of a solid cylinder that must fit inside a cylindrical hole, the following pair of constraints can be applied, which state the alignment between a pair of points $\mathbf{v}_{i1}$, $\mathbf{v}_{i2}$ in O_i and a pair $\mathbf{v}_{j1}$, $\mathbf{v}_{j2}$ in O_j:

$$\mathbf{Q}_j(\mathbf{v}_{j2}-\mathbf{v}_{j1})\times\mathbf{Q}_i(\mathbf{v}_{i2}-\mathbf{v}_{i1}) = 0; \tag{29a}$$

$$(\mathbf{Q}_j\mathbf{v}_{j1}-\mathbf{Q}_i\mathbf{v}_{i1})\times\mathbf{Q}_i(\mathbf{v}_{i2}-\mathbf{v}_{i1}) = 0; \tag{29b}$$

"Hinge" constraints

Two objects may be forced to share a common vertex, or a common edge, in order to model a *ball and socket* or a *hinge* constraint: such modeling may be carried out by stating the equality of one or two pairs of transformed points:

$$\mathbf{Q}_i\mathbf{v}_{i1} = \mathbf{Q}_j\mathbf{v}_{j1}; \tag{30a}$$

$$\mathbf{Q}_i\mathbf{v}_{i2} = \mathbf{Q}_j\mathbf{v}_{j2}; \tag{30b}$$

In the first case (ball and socket constaint) only one vector equation will be necessary; in the second case (hinge constraint) both equations are needed.

Matrix constraints

Some constraints of the second type, concerning the required structure of matrices $\mathbf{Q}_i$, might be expressed by forcing the considered unknown matrix to be equal to the product, in a given order, of other matrices representing elementary linear transformations. For example, the matrix $\mathbf{Q}_i$ could be constrained to represent a scaling transformation with parameters s_{ix}, s_{iy}, s_{iz}, and fixed point x_{i0}, y_{i0}, z_{i0}. Such constraints can be expressed in the following way:

$$\mathbf{Q}_i = \begin{bmatrix} 1 & 0 & 0 & x_{i0} \\ 0 & 1 & 0 & y_{i0} \\ 0 & 0 & 1 & z_{i0} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{ix} & 0 & 0 & 0 \\ 0 & s_{iy} & 0 & 0 \\ 0 & 0 & s_{iz} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_{i0} \\ 0 & 1 & 0 & -y_{i0} \\ 0 & 0 & 1 & -z_{i0} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or, equivalently:

$$q_{i11} = s_{ix}\ ;\ q_{i22} = s_{iy}\ ;\ q_{i33} = s_{iz}\ ;$$
$$q_{i14} = x_{i0}(1-s_{ix})\ ;\ q_{i24} = y_{i0}(1-s_{iy})\ ;\ q_{i34} = z_{i0}(1-s_{iz})\ ;$$
$$q_{i12} = q_{i13} = q_{i21} = q_{i23} = q_{i31} = q_{i32} = 0\ .$$

In the same way, $\mathbf{Q}_i$ may be forced to represent a general rotation, by imposing:

$$\mathbf{q}_{i1}^2 = \mathbf{q}_{i2}^2 = \mathbf{q}_{i3}^2 = 1;\ \mathbf{q}_{i1}\cdot\mathbf{q}_{i2} = \mathbf{q}_{i1}\cdot\mathbf{q}_{i3} = \mathbf{q}_{i2}\cdot\mathbf{q}_{i3} = 0.$$

Obviously, any other kind of geometrical constraint might be applied, being careful to express each relationship between transformed coordinates. For example, a constraint on the desired distance d between a couple of points $\mathbf{v}_{i1}$, $\mathbf{v}_{i2}$ belonging to the same object i will be written: $\mathbf{Q}_i(\mathbf{v}_{i2} - \mathbf{v}_{i1}) \cdot \mathbf{Q}_i(\mathbf{v}_{i2} - \mathbf{v}_{i1}) = d^2$

APPENDIX 1

Nonlinear optimization

The standard form for mathematical optimization problems with both nonlinear objective function and with nonlinear *inequality* constraints is:

$$\textit{maximize: } z = f(\mathbf{x}), \ \ \textit{subject to: } g_i(\mathbf{x}) \leq 0 \quad (1 \leq i \leq m) \tag{31}$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]$. Minimization problems are reduced to form (31) by changing the objective function $f(\mathbf{x})$ to $-f(\mathbf{x})$. Analogously, constraints of the form $g_i(\mathbf{x}) \geq 0$ are easily reduced to the to the standard form (31). If *slack variables* $x_{n+1}^2, x_{n+2}^2, \ldots, x_{n+m}^2$ are added to the left sides of the constraints, then each inequality may be converted to an equality, reducing the problem (31) to the standard form with *equality* constraints:

$$\textit{maximize: } z = f(\mathbf{x}'), \ \ \textit{subject to: } g'_i(\mathbf{x}') = 0 \quad (1 \leq i \leq m) \tag{32}$$

with $\mathbf{x}' = [x_1, .., x_n, x_{n+1}, .., x_{n+m}]$, and $g'_i(\mathbf{x}') = g_i(\mathbf{x}) + x_{n+i}^2$. The Lagrangian function, L, is defined as a linear combination of the *Lagrange multipliers* λ_i with the objective function $f(\mathbf{x})$, the original constraints $g_i(\mathbf{x})$ and the slack variables:

$$L(\mathbf{x}') = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \left[g_i(\mathbf{x}) + x^2_{n+i} \right]. \tag{33}$$

If the system of the *Kuhn-Tucher conditions*:

$$\begin{aligned} \frac{\partial L}{\partial x_j} &= 0 \qquad (j=1,\ldots,n+m) \\ \frac{\partial L}{\partial \lambda_i} &= 0 \qquad (i=1,\ldots,m) \\ \lambda_i &\geq 0 \qquad (i=1,\ldots,m) \end{aligned} \tag{34}$$

admits solutions, then one of them will be the solution to program (31), iff $f(\mathbf{x})$ and each $g_i(\mathbf{x})$ have continous first partial derivatives.

One of the most frequently used procedures for solving the problem (31) is the *method of feasible directions*[5], applicable only if the region of feasible solutions has an interior, i.e. if any pair of inequalities in (31) isn't derived from the conversion of an equality constraint. To apply this method, it is necessary to evaluate in suitable points the Jacobian matrix of constraints, defined as the $m \times n$ matrix

$$\mathbf{J} = \left[\frac{\partial g_i}{\partial x_j} \right] \qquad (1 \le i \le m,\ 1 \le j \le n), \tag{35}$$

and the gradient of the objective function:

$$\nabla f = \left[\frac{\partial f}{\partial x_j} \right] \qquad (1 \le j \le n). \tag{36}$$

In concluding this synthetic summary ot some main concepts of non-linear optimization, we remark that quite all methods of solution of non-linear problems may require either the knowledge, in closed form, of derivatives of the objective function and/or the constraints, or the numeric evaluation of such derivatives in suitable points. For this reason the main part of this paper is dedicated to the study of domain-derivatives of volume integrals, that we claim can be used to constrain practical problems of parametric design of physical objects.

APPENDIX 2

Proof of theorem 1

The proof is given only for two elements of the matrix $\mathbf{I}(P_Q)$. Analogous derivations could be given for the other matrix elements.

a) At first, we want to show that

$$i_{44}(P_Q) = |\det \mathbf{Q}|^{-1}\, \mathbf{q}_4\, \mathbf{I}(P)\, \mathbf{q}_4^T \tag{37}$$

where $\mathbf{q}_4$ is the fourth row of the matrix $\mathbf{Q}$ of the affine transformation. By definition of mass, we have:

$$i_{44}(P_Q) = \iiint_{P_Q} dV \tag{38}$$

and, by exchanging the integration domain, according to the transformation represented by the matrix $\mathbf{Q}^{-1}$, J being the Jacobian of the linear transformation:

$$i_{44}(P_Q) = \iiint_P |J| \, dV = |\det \mathbf{Q}^{-1}| \iiint_P dV \,. \tag{39}$$

Then, for a property of determinants: $i_{44}(P_Q) = |\det \mathbf{Q}|^{-1} \, i_{44}(P)$. As for an affine matrix the fourth row is always [0 0 0 1], the eq. (37) derives immediately.

b) Secondly, we want to show now that

$$i_{11}(P_Q) = |\det \mathbf{Q}|^{-1} \, \mathbf{q}_1 \, \mathbf{I}(P) \, \mathbf{q}_1^T. \tag{40}$$

Remember that we have, for the position vector, $\mathbf{v}' = [x' \; y' \; z' \; 1]^T = \mathbf{Q}\,\mathbf{v}$; so, by definition of second moment:

$$i_{11}(P_Q) = \iiint_{P_Q} x'^2 \, dV \tag{41}$$

and, by substituting the coordinates and the integration domain after having applied the affine transformation $\mathbf{Q}$:

$$i_{11}(P_Q) = \iiint_P (q_{11}x+q_{12}y+q_{13}z+q_{14})^2 \, |J| \, dV \tag{42}$$

$$= |\det \mathbf{Q}^{-1}| \iiint_P \Big[(q_{11}x+q_{12}y)^2+(q_{13}z+q_{14})^2+2(q_{11}x+q_{12}y)(q_{13}z+q_{14}) \Big] \, dV$$

$$= |\det \mathbf{Q}|^{-1} \, [\, q_{11}^2 i_{11}(P)+q_{12}^2 i_{22}(P)+q_{13}^2 i_{33}(P)+q_{14}^2 i(P)$$
$$+2q_{11}q_{12}i_{12}(P)+2q_{12}q_{13}i_{23}(P)+2q_{11}q_{13}i_{13}(P)$$
$$+2q_{11}q_{14}i_1(P)+2q_{12}q_{14}i_2(P)+2q_{13}q_{14}i_3(P) \,]$$

and, recalling that $\mathbf{I} = \mathbf{I}^T$ and then $i_{hk} = i_{kh}$, we can write:

$$i_{11}(P_Q) = |\det \mathbf{Q}|^{-1} \sum_{i=1}^{4} \sum_{j=1}^{4} q_{1i} \, i_{ij}(P) \, q_{1j} = |\det \mathbf{Q}|^{-1} \, \mathbf{q}_1 \, \mathbf{I}(P) \, \mathbf{q}_1^T \quad \square \tag{43}$$

Proof of theorem 2

The system of the equations of three generic planes parallel to the original faces for the vertex v_i may be written as $\mathbf{A}_i\mathbf{v}_i = -\mathbf{d}^*_i$. Therefore, if the planes are linearly independent, the position vector $\mathbf{v}_i$ of vertex v_i in displaced position is

$$\mathbf{v}_i = -\mathbf{A}_i^{-1}\mathbf{d}^*_i \tag{44}$$

with

$$d^*_{ij} = d_{ij} + (a_{ij}^2+b_{ij}^2+c_{ij}^2)^{\frac{1}{2}}\, h_{ij} \qquad (1 \le j \le 3) \tag{45}$$

where $d_{ij}\ (a_{ij}^2+b_{ij}^2+c_{ij}^2)^{-\frac{1}{2}}$ is the distance from the origin of the original plane for f_{r_j}, and $h_{ij} = h_j$ the face displacement in the conformally transformed object. By arranging face displacements in vector form and by substituting (45) in (44), linear equation (11) is directly obtained. □

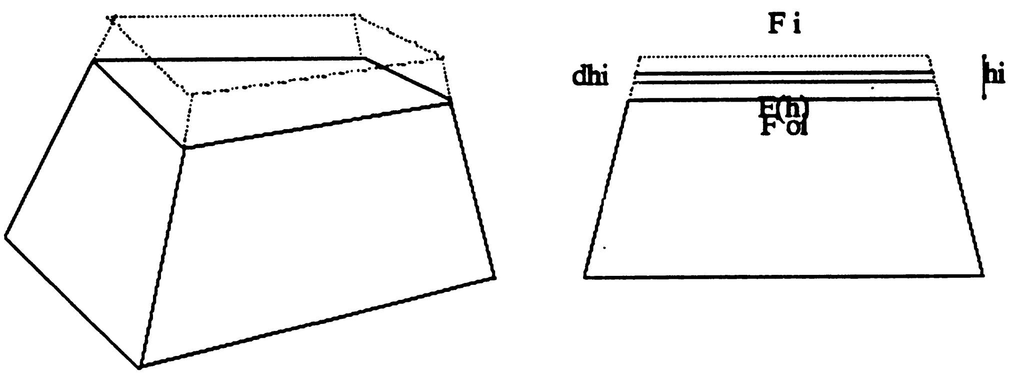

figure 3. The volume ΔP_i, corresponding to a displacement h_i of the face F_i.

Proof of theorem 3

By the definition of partial derivative and by the property of domain-additivity of integrals, being $P = P_0 \bigcup \Delta P_i$, and $P_0 \bigcap \Delta P_i = \varnothing$, we may write:

$$\frac{\partial G}{\partial h_i} = \lim_{h_i \to 0} \frac{G(P) - G(P_0)}{h_i} \tag{46a}$$

$$= \lim_{h_i \to 0} \frac{G(P_0) + G(\Delta P_i) - G(P_0)}{h_i} \tag{46b}$$

$$= \lim_{h_i \to 0} \frac{G(\Delta P_i)}{h_i} \tag{46c}$$

So, recalling the expression (13) for G, we have:

$$\frac{\partial G}{\partial h_i} = \lim_{h_i \to 0} \frac{1}{h_i} \iiint_{\Delta P_i} g(x, y, z)\, dV \tag{47}$$

Now, observe figure 3: the volume increment ΔP_i corresponding to a single displacement h_i has two parallel faces, namely f_{0i} and f_i, and hence the volume integral in (47) may be computed as an iterated integral of the following kind:

$$\frac{\partial G}{\partial h_i} = \lim_{h_i \to 0} \frac{1}{h_i} \int_0^{h_i} \left[\iint_{f(h)} g(x, y, z)\, dS \right] dh \quad . \tag{48}$$

where $f(h)$ is a generic planar surface between f_{0i} and f_i, and dh is a line differential in the direction orthogonal to the surface. For fixed g and P_0, the integral inside brackets in (48) is a function of h, and we may apply the mean-value theorem for integrals, and derive the following expression, the surface integral being evaluated over a suitable planar surface $\overline{f(h)}$ at distance $\bar{h}$ from f_{0i}:

$$\frac{\partial G}{\partial h_i} = \lim_{h_i \to 0} \frac{1}{h_i} \iint_{f(\bar{h})} g(x, y, z)\, dS \left[h \right]_0^{h_i} \quad , \tag{49}$$

Therefore, we have:

$$\frac{\partial G}{\partial h_i} = \lim_{h_i \to 0} \iint_{f(\bar{h})} g(x, y, z)\, dS \tag{50}$$

And, for the continuity of integral in (50) with respect to the variation of the integration domain:

$$\frac{\partial G}{\partial h_i} = \iint_{f_{0i}} g(x, y, z)\, dS \quad \square \tag{51}$$

Proof of theorem 4

The two faces f_i and f_j, corresponding to the displacements h_i and h_j, may be either disjoint or intersect along the edge e_{ji}. In both cases we have, by theorem 3:

$$\frac{\partial^2 G}{\partial h_i \partial h_j} = \frac{\partial}{\partial h_i} \iint_{f_j} g(x, y, z)\, dS \tag{52}$$

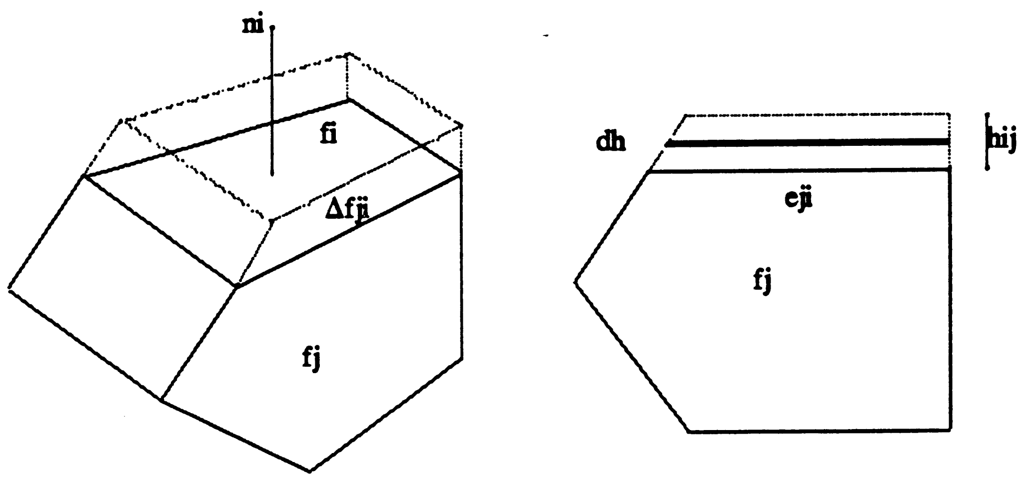

figure 4. Variation ΔF_{ji} of the face F_j for a displacement h_i of the adjacent face f_i

Part a) If faces f_j and f_i are not adjacent, the surface integral in (52) is not affected in any way by the displacement of face f_i, so we have:

$$\frac{\partial}{\partial h_i}\,(\textit{function not dependent by } h_i) = 0 \quad .$$

Part b) If faces f_i and f_j are adjacent, consider figure 4, and observe that, by definition of partial derivative, we have to compute a surface integral over the region Δf_{ji}, adjacent and coplanar to f_j. Notice, from the figure, that $f_{ji} = f_j \bigcup \Delta f_{ji}$, and that $f_j \bigcap \Delta f_{ji} = \emptyset$. Hence, by definition of derivative:

$$\frac{\partial}{\partial h_i} \iint_{f_j} g(x, y, z)\, dS = \lim_{h_i \to 0} \frac{S(f_{ji}) - S(f_j)}{h_i} \quad , \tag{53}$$

where $S(f) = \iint_{f_j} g(x, y, z)\, dS$. Therefore, for domain-additivity:

$$\frac{\partial^2 G}{\partial h_i \partial h_j} = \lim_{h_i \to 0} \frac{S(\Delta f_{ji})}{h_i} \quad , \tag{54a}$$

$$= \lim_{h_i \to 0} \frac{1}{h_i} \iint_{\Delta f_{ji}} g(x, y, z)\, dS \tag{54b}$$

At this point, observe that the face increment Δf_{ji} is a trapezoid (see figure 4) with base e_{ji} and heigth h_{ji}. So, it is possible to solve the integral in (54b) by the iterate method, as a summation of line integrals over lines $e(h)$ parallel to e_{ji}:

$$\frac{\partial^2 G}{\partial h_i \partial h_j} = \lim_{h_i \to 0} \frac{1}{h_i} \int_0^{h_{ji}} \left[\int_{e(h)} g(x, y, z)\, dl \right] dh \ . \tag{55}$$

Then, by applying the mean-value theorem for integrals:

$$\frac{\partial^2 G}{\partial h_i \partial h_j} = \lim_{h_i \to 0} \frac{h_{ji}}{h_i} \int_{e(\bar{h})} g(x, y, z)\, dl \ , \tag{56}$$

where $0 \le \bar{h} \le h_{ji}$. Finally, it is easy to see that h_{ji} is the projection of the vector $h_i \mathbf{n}_i$ in the direction of the vector $\mathbf{e}_{ji} \times \mathbf{n}_j$, being $\mathbf{e}_{ji}$ the unit vector in the direction of the edge e_{ji}, oriented counterclockwise (for the face f_j):

$$h_{ji} = h_i\ \mathbf{n}_i \cdot \mathbf{e}_{ji} \times \mathbf{n}_j \ . \tag{57}$$

So, by substituting (57) in (56), we obtain:

$$\frac{\partial^2 G}{\partial h_i \partial h_j} = \lim_{h_i \to 0}\ \mathbf{n}_i \cdot \mathbf{e}_{ji} \times \mathbf{n}_j \int_{e(\bar{h})} g(x, y, z)\, dl \ , \tag{58}$$

and, for continuity of integrals with respect to changes in integration domain, we have finally:

$$\frac{\partial^2 G}{\partial h_i \partial h_j} = \mathbf{n}_i \cdot \mathbf{e}_{ji} \times \mathbf{n}_j \int_{e_{ji}} g(x, y, z)\, dl \quad \square \tag{59}$$

Proof of corollary 4.1

From theorem 2, we have only to show that:

$$\mathbf{n}_i \cdot \mathbf{e}_{ji} \times \mathbf{n}_j = \sin \alpha_{ji} \ , \tag{60}$$

where α_{ji} is the oriented angle between $\mathbf{n}_j$ and $\mathbf{n}_i$. First of all, notice that $\mathbf{e}_{ji}$ is a unit vector, perpendicular to both unit vectors $\mathbf{n}_i$ and $\mathbf{n}_j$, and therefore it may be computed in the following way:

$$\mathbf{e}_{ji} = \frac{\mathbf{n}_j \times \mathbf{n}_i}{\|\mathbf{n}_j \times \mathbf{n}_i\|} = \frac{\mathbf{n}_j \times \mathbf{n}_i}{\sin \alpha_{ji}} \tag{61}$$

And hence we have, by applying in ($62a$–$62b$) a theorem from vector calculus[19]:

$$\mathbf{n}_i \cdot \mathbf{e}_{ji} \times \mathbf{n}_j = \frac{1}{\sin \alpha_{ji}} \mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_i) \times \mathbf{n}_j \tag{62a}$$

$$= \frac{1}{\sin \alpha_{ji}} \mathbf{n}_i \cdot [(\mathbf{n}_j \cdot \mathbf{n}_j)\mathbf{n}_i - (\mathbf{n}_i \cdot \mathbf{n}_j)\mathbf{n}_j] \tag{62b}$$

$$= \frac{1}{\sin \alpha_{ji}} [\, 1 - (\mathbf{n}_i \cdot \mathbf{n}_j)^2 \,] \tag{62c}$$

$$= \frac{1}{\sin \alpha_{ji}} (\, 1 - \cos^2 \alpha_{ji} \,) \tag{62d}$$

$$= \sin \alpha_{ji} \qquad \square \tag{62e}$$

Proof of corollary 4.2

Case a) If the two faces f_i, f_j are no adjacent, then we have, for the case a) of theorem 2:

$$\frac{\partial^2 G}{\partial h_i \partial h_j} = \frac{\partial^2 G}{\partial h_j \partial h_i} = 0 \;\; ; \tag{63}$$

Case b) otherwise, if $f_i \cap f_j = e_{ji}$:

$$\frac{\partial^2 G}{\partial h_i \partial h_j} = \sin \alpha_{ji} \int_{e_{ji}} g(x, y, z)\, dl \tag{64a}$$

$$= -\sin \alpha_{ji} \cdot \left(-\int_{e_{ji}} g(x, y, z)\, dl \right) \tag{64b}$$

$$= \sin(-\alpha_{ji}) \int_{e_{ij}} g(x, y, z)\, dl \tag{64c}$$

$$= \sin \alpha_{ij} \int_{e_{ij}} g(x, y, z)\, dl \tag{64d}$$

$$= \frac{\partial^2 G}{\partial h_j \partial h_i} \quad \square \tag{64e}$$

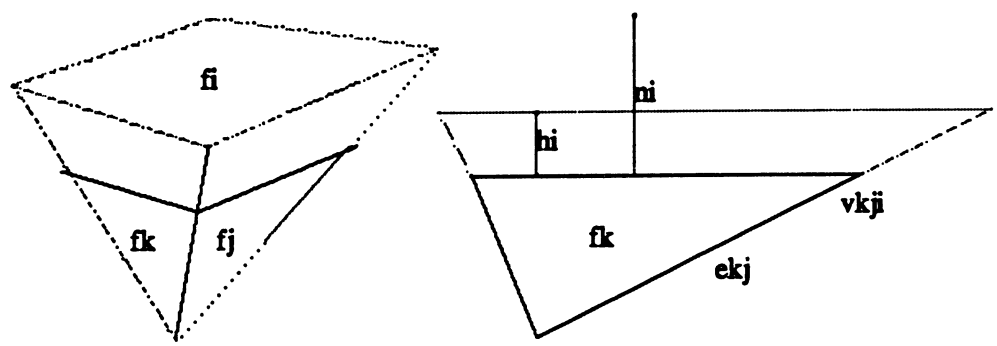

figure 5. Topological and geometrical parameters involved in the displacement of three incident faces f_k, f_j and f_i.

Proof of theorem 5

In both cases we have:

$$\frac{\partial^3 G}{\partial h_i \partial h_j \partial h_k} = \frac{\partial}{\partial h_i} \frac{\partial^2 G}{\partial h_j \partial h_k} \tag{65}$$

Part a) If $f_j \bigcap f_k = \varnothing$ then $\frac{\partial^3 G}{\partial h_i \partial h_j \partial h_k} = 0$ by theorem 4. Otherwise

$$\frac{\partial^3 G}{\partial h_i \partial h_j \partial h_k} = \frac{\partial}{\partial h_i} \int_{e_{kj}} g(x, y, z)\, dl \tag{66a}$$

$$= \frac{\partial}{\partial h_i} \textit{(function not dependent by } h_i) = 0 \quad . \tag{66b}$$

Part b) Suppose, now, that f_k, f_j and f_i intersect in the vertex v_{kji}. In this case we can write, by virtue of theorem 4:

$$\frac{\partial^3 G}{\partial h_i \partial h_j \partial h_k} = \frac{\partial}{\partial h_i} \int_{e_{kj}} g(x, y, z)\, dl = \lim_{h_i \to 0} \frac{L(\, e_{kji}\,) - L(\, e_{kj}\,)}{h_i} \quad , \tag{67}$$

where $L(e) = \int_e g(x, y, z)\, dl$. So, by using a demonstration scheme similar to that of theorems 3 and 4, we have:

$$\frac{\partial^3 G}{\partial h_i \partial h_j \partial h_k} = \lim_{h_i \to 0} \frac{1}{h_i} L(\Delta e_{kji}) \tag{68a}$$

$$= \lim_{h_i \to 0} \frac{1}{h_i} \int_{\Delta e_{kji}} g(x, y, z)\, dl \tag{68b}$$

$$= \lim_{h_i \to 0} \frac{1}{h_i} \int_0^{h_{kji}} g[x(h), y(h), z(h)]\, dh \tag{68c}$$

$$= \lim_{h_i \to 0} \frac{1}{h_i} g(x, y, z)\,\Big|_{v(\bar{h})} \Big[h \Big]_0^{h_{kji}} \tag{68d}$$

Where $0 \leq \bar{h} \leq h_{kji}$ is a suitable curvilinear coordinate value along the edge increment Δe_{kji}. Therefore we have:

$$\frac{\partial^3 G}{\partial h_i \partial h_j \partial h_k} = \lim_{h_i \to 0} \frac{h_{kji}}{h_i} g(x, y, z)\,\Big|_{v(\bar{h})} \tag{69}$$

From figure 6, we can see that h_{kji} is the measure of the variation of the length of the edge e_{kj}, under a face-displacement h_i. So, we have $h_{kji} = h_i \,/\, \mathbf{n}_i \cdot \mathbf{e}_{kj}$, and hence:

$$\frac{\partial^3 G}{\partial h_i \partial h_j \partial h_k} = \lim_{h_i \to 0} \frac{1}{\mathbf{n}_i \cdot \mathbf{e}_{kj}} g(x, y, z)\,\Big|_{v(\bar{h})} = \frac{\sin\alpha_{kj}}{\mathbf{n}_i \cdot \mathbf{n}_k \times \mathbf{n}_j} g(x, y, z)\,\Big|_{v_{kji}} \tag{70}$$

□

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