RANDOM WALKS ON SOME CLASSES OF SOLVABLE GROUPS

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In the first part of this dissertation we determine the behavior of the return probability of simple random walks on the free solvable group $S_{d,r}$ of derived length d on r generators and some other related groups. In the second part, we study the decay of convolution powers of a large family $\mu_{S,a}$ of measures on finitely generated nilpotent groups. Here, $S = (s_1, \dots, s_k)$ is a generating k-tuple of group elements and $a = (\alpha_1, \dots, \alpha_k)$ is a k-tuple of reals in the interval $(0, 2)$. The symmetric probability measure $\mu_{S,a}$ is supported by $S^* = \{s_i^m, 1 \leq i \leq k,\ m \in \mathbb{Z}\}$ and gives probability proportional to

$$(1 + m)^{-\alpha_i - 1}$$

to $s_i^{\pm m}$, $i = 1, \dots, k$, $m \in \mathbb{N}$. We determine the behavior of the probability of return $\mu_{S,a}^{(n)}(e)$ as n tends to infinity. This behavior depends in somewhat subtle ways on interactions between the k-tuple a and the positions of the generators s_i within the lower central series $G_j = [G_{j-1}, G]$, $G_1 = G$. In the third part, we prove tightness properties of some random walks on groups of polynomial volume growth driven by spread-out measures, including the measures $\mu_{S,a}$ studied in the second part.

BIOGRAPHICAL SKETCH

Tianyi Zheng was born in a village in Southeast China. Growing up with all the happy kids in the village, it never occured to her that someday she might study mathematics for her life. After surviving the national higher education entrance exam, she went to Tsinghua University in Beijing. With a lot of luck, she was accepted to the Math Department at Cornell and worked with Laurent Saloff-Coste. With Laurent, she explored random walks on various group structures. She has to admit that bare hands comparison of Dirichlet forms is her main technical tool. To make furthur progress, there are much more to be learned in the future. At this point, she is very close to finish her thesis, and feels a bit embarrassed to write this autobiography.

This thesis is dedicated to my family, who supported me each step of the way.

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This thesis is based on joint work with Laurent Saloff-Coste.

First I would like to express deepest gratitude to my advisor, Laurent Saloff-Coste, for so much energy and time he devoted to teaching and helping me. Having you as my advisor is the best thing that ever happened to my life. I would like to thank Professor Leonard Gross and Robert Strichartz for being on my committee. Many thanks go to my undergraduate advisor Zhi-yin Wen for his encouragement and support through the years. I would like to thank Christophe Pittet, for his wonderful work on random walks on groups with Laurent, without him this dissertation would never form its shape. Also I would like to express gratitude to the Cornell Math Department in general, to all the professors from whom I learned a lot of math, and to the staff members who have always been so friendly and helpful.

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CHAPTER 1

INTRODUCTION

Random walk on groups is a beautiful subject at the intersection of probability, analysis, geometry and algebra. A guiding principle is to investigate the interplay between the behavior of random walks and the algebraic/geometric structure of the underlying group. The subject is rich; a wealth of theories and techniques can be applied to study various aspects of the behavior of random walks.

Random walk on countable groups was first studied systematically by Kesten in his thesis [23]. Given a countable group G and a probability measure μ on G, the random walk driven by μ can be thought of as a way to randomly explore the group by taking independent steps distributed according to μ. More precisely, the random walk driven by μ (started at the identity element e of G) is the G-valued random process $X_n = \xi_1 \dots \xi_n$ where $(\xi_i)_1^\infty$ is a sequence of independent identically distributed G-valued random variables with law μ. In the classical case of simple random walk on G with finite generating set S, μ is uniform on $S \cup S^{-1}$. If $u * v(g) = \sum_h u(h)v(h^{-1}g)$ denotes the convolution of two functions u and v on G then the probability that $X_n = g$ is given by $\mathbf{P}_e^\mu(X_n = g) = \mu^{(n)}(g)$ where $\mu^{(n)}$ is the n-fold convolution of μ. The main focus of this dissertation is to study the problem of describing the behavior of the return probability to the identity, namely, the decay of the quantity

$$\mathbf{P}_e(\xi_1 ... \xi_n = e) = \mu^{*n}(e).$$

Many aspects of the behavior of these random processes are closely related to the algebraic and geometric property of the underlying group G. See [48] for a systematic exposition of the subject. One of Kesten's fundamental results states that, for a random walk driven by a symmetric measure with generating support, the probability of return, $\mathbf{P}_e(\xi_1 \dots \xi_n = e)$, decays exponentially fast if and only the group G is non-amenable. See [22, 23]. In this dissertation we will focus on solvable groups. Solvable groups are always amenable and the decay of $\mu^{*n}(e)$ is always slower than exponential decay.

To further explain the interplay between random walk behavior and algebraic structure of the underlying group, we briefly review the notions of nilpotent, polycyclic, solvable and metabelian groups below. See [32, Chapter 5].

For $h, k \in G$, set $[h, k] = h^{-1}k^{-1}hk$. For two subsets A, B of G, the commutator subgroup $[A, B]$ is the subgroup of G generated by all the elements of the form $[a, b]$, $a \in A$, $b \in B$. The lower central series of G is the non-increasing sequence of subgroups defined by $G_1 = G$, $G_2 = [G, G]$, $G_{i+1} = [G_i, G]$. A group is nilpotent if there is an integer k such that $G_k = \{e\}$.

The derived series is defined by $G^{(1)} = G$, $G^{(2)} = [G, G]$, $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. By contruction, $G^{(i)}/G^{(i+1)}$ is abelian. A group is solvable if there is an integer k such that $G^{(k)} = \{e\}$. A basic result in group theory is that any nilpotent group is solvable.

A group is polycyclic if it admits a finite decreasing sequence of subgroups $H_1 = G \supset H_2 \supset \cdots \supset H_{k-1} \supset H_k = \{e\}$ such that H_{i+1} is normal in H_i and H_i/H_{i+1} is cyclic. Polycyclic groups are always solvable. Finitely generated nilpotent groups are always polycyclic.

A group G is metabelian if its commutator group $[G, G]$ is abelian. Observe that metabelian group are obviously solvable. They can be polycyclic or not, nilpotent or not. See Figure 1.1 for an illustration of inclusion relations of these classes of groups.

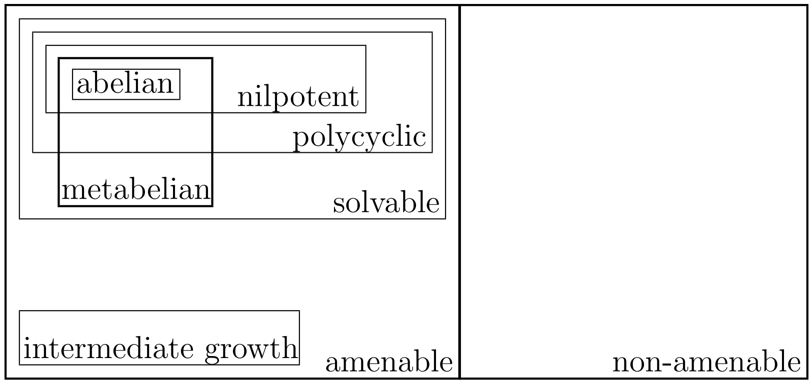

Figure 1.1: The inclusion relations between various classes of finitely generated groups. Figure taken from [29].

Next we review what is known concerning return probability of simple random walks on a group G in relation to the volume growth and the algebraic structure. See the introduction of [29] and the references given there. Here and in the rest of this dissertation, we say that $f \lesssim g$ if there are constants c_1, c_2 such that for all t, $c_1 f(c_2 t) \leq g(t)$; $f \gtrsim g$ if there are constants c_3, c_4 such that for all t, $g(t) \leq c_3 f(c_4 t)$. We say $f \simeq g$ both $f \lesssim g$ and $f \gtrsim g$ hold.

Return probability of random walks driven by symmetric finitely supported probability measures enjoys great stability properties. By [30, Theorem 1.4], if μ_i, $i = 1, 2$, are symmetric (i.e., $\mu_i(g) = \mu_i(g^{-1})$ for all $g \in G$) finitely supported probability measures with generating support, then the functions $n \mapsto \phi_i(n) = \mu_i^{(2n)}(e)$ satisfy $\phi_1 \simeq \phi_2$. By definition, we denote by Φ_G any function that belongs to the $\simeq$-equivalence class of $\phi_1 \simeq \phi_2$. Let $|g|$ be the word-length of G with respect to some fixed finite symmetric generating set. Let $V(n) = |\{g \in G : |g| \leq n\}|$ be the volume growth function. The results below are mostly due to Varopoulos, which relate the behavior of Φ_G to the volume growth and algebraic structure of the underlying group G.

1. If $V(n) \simeq n^d$ then $\Phi_G(n) \simeq n^{-d/2}$ (Varopoulos [46]); See also [20, 45]).
2. If $V(n) \geq c_1 \exp(c_2 n^\alpha)$ for some $c_i, c_2 > 0$ and $0 < \alpha \leq 1$, then $\Phi_G(n) \leq C \exp\left(-cn^{\alpha/(\alpha+2)}\right)$ (Varopoulos [44]); See also [20, 45]).
3. If G contains a polycyclic subgroup of finite index having exponential growth then $\Phi_G(n) \simeq \exp(-n^{1/3})$. (Varopoulos [44], Alexopoulos [1]. See also [20]).

In [29], the authors discussed the return probability of simple random walks in terms of classical versus exotic behavior. We recall the following classical results about structure of discrete linear groups. Let Γ be a discrete subgroup of a Lie group having finitely many connected components (here, discrete refers to the topology induced on the subgroup by the topology of the ambient group). Then either Γ is non amenable or Γ is amenable and then it must contain a polycyclic subgroup of finite index. In particular, in the second alternative, Γ must be finitely generated and its volume growth V must either be of exponential type $V(n) \approx \exp(n)$ or of polynomial type $V(n) \approx n^d$ for some integer d. See [42], [49]. Note that this implies that many solvable groups are excluded from the class of discrete linear groups, e.g., all those containing subgroups that are not finitely generated.

From the results described above, three behaviors of the return-probability function Φ emerge as the only possible behaviors for finitely generated discrete subgroups of Lie groups having finitely many connected components. See [29] and references given there.

1. Polynomial behavior: $\Phi(n) \approx n^{-d/2}$ for some integer d. This happens exactly if G contains a nilpotent subgroup of finite index.

2. $\exp(-n^{1/3})$ behavior: $\Phi(n) \approx \exp(-n^{1/3})$. For discrete subgroups of Lie groups having finitely many connected components, this is the case if and only if G has exponential growth and is amenable. This behavior also appears in some other examples that are not discrete subgroups of Lie groups.

3. Exponential behavior: $\Phi(n) \approx \exp(-n)$. This happens exactly if G is non-amenable.

These three behaviors are referred to as the classical behaviors.

For solvable groups that are discrete subgroups of some Lie group having finitely many components, only the first two behaviors above can arise since solvable groups are always amenable. In this case, the behavior of the return probability function Φ can be characterized in terms of the volume growth. Namely, $\Phi(n) \approx n^{-d/2}$ if and only if $V(n) \approx n^d$ whereas $\Phi(n) \approx \exp(-n^{1/3})$ if and only if $V(n) \approx \exp(n)$, and these are the only possible behaviors. In [29], the authors exhibited solvable groups having a behavior that is different from the polynomial and $\exp(-n^{1/3})$ behaviors above. Among other results, they proved the following theorem.

Theorem 1.0.1. *[29, Theorem 1.1] (1) For any finitely generated metabelian group, there exists $\epsilon \in (0,1)$ such that*

$$\phi(n) \geq \exp(-c_1 n^{1-\epsilon}) \quad \textit{for } n \textit{ large enough.}$$

(2) For each small $\delta > 0$, there exists a finitely presented metabelian group such that

$$\phi(n) \leq \exp(-c_2 n^{1-\delta}) \quad \textit{for } n \textit{ large enough.}$$

(3) There exists a finitely generated solvable group (not metabelian) for which for any $\delta \in (0,1)$ there exists c_δ such that

$$\phi(n) \leq \exp(-c_\delta n^{1-\delta}) \quad \textit{for } n \textit{ large enough.}$$

More concrete examples and other behaviors are given in [29]. All of them are obtained through the same algebraic construction known as a wreath product. In Chapter 2, we describe another construction that gives interesting exotic behaviors. Recall that $G^{(i)}$, the derived series of a group G, is defined inductively by $G^{(1)} = G$, $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$. The main result of Chapter 2 determines decay of Φ_G for G in the family of free solvable groups.

Theorem 1.0.2. *Let $\mathbf{S}_{d,r}$ be the free solvable group of derived length d on r generators, that is, $\mathbf{S}_{d,r} = \mathbf{F}_r/\mathbf{F}_r^{(d)}$ where $\mathbf{F}_r$ is the free group on r generators, $r \geq 2$.*

- *If $d = 2$ (the free metabelian case) then*

$$\Phi_{\mathbf{S}_{2,r}}(n) \simeq \exp\left(-n^{r/(r+2)}(\log n)^{2/(r+2)}\right).$$

- *If $d > 2$ then*

$$\Phi_{\mathbf{S}_{d,r}}(n) \simeq \exp\left(-n\left(\frac{\log_{[d-1]} n}{\log_{[d-2]} n}\right)^{2/r}\right).$$

Note that the free solvable groups are the free objects in the family of finitely generated solvable groups, in the sense that any solvable group of derived length d on r generators can be realized as a quotient of $\mathbf{S}_{d,r}$. In particular, if G is a solvable group of derived length d on r generators, then $\Phi_G \gtrsim \Phi_{\mathbf{S}_{d,r}}$.

Besides simple random walks, there are many types of measures with unbounded support one can consider. The possibility of long range jumps often makes the problem quite interesting. In Chapter 3 we first introduce a natural class of such measures that are supported on powers of generators. Let G be a finitely generated group, and let $S = (s_1, ..., s_k)$ be a generating tuple. We attach a k-tuple of positive reals $a = (\alpha_1, ..., \alpha_k)$, $\alpha_i \in (0, \infty]$ to the generating tuple $S = (s_1, ..., s_k)$, and consider measure $\mu_{S,a}$ supported on powers of generators given by

$$\mu_{S,a}(g) = \frac{1}{k}\sum_{i=1}^{k} c(\alpha_i) \sum_{m\in\mathbb{Z}} \frac{1}{(1+|m|)^{1+\alpha_i}} \mathbf{1}_{\{s_i^m = g\}}.$$

Note that the steps taken along the different one parameter subgroup $\{s_i^n, n \in \mathbb{Z}\}$ are driven by different laws. On $\mathbb{Z}$, the power law $\mu_{\alpha_i}(m) = \frac{c(\alpha_i)}{(1+|m|)^{1+\alpha_i}}$ can be viewed as a discrete version of the symmetric α_i-stable law and $\mu_{\alpha_i}^{*n}(0) \simeq n^{-1/\alpha_i}$. For each generator s_i, let μ_{s_i,α_i} denote the pushforward of the discrete α_i-stable law μ_{α_i} under the homomorphism $\mathbb{Z} \to \langle s_i \rangle$ given by $n \to s_i^n$. Then the measure $\mu_{S,a}$ defined above is the average of such μ_{s_i,α_i} for $s_i \in S$.

From the way we construct the measure $\mu_{S,a}$, different generators support different stable laws, we expect the behavior of $\mu_{S,a}^{*n}(e)$ to change when we change the generating tuple S or the index tuple a. In this sense the decay of $\mu_{S,a}^{*n}(e)$, unlike simple random walk, is not a group invariant. But if we think of choosing S and a as experiments of random walks on G, then taken collectively, the behavior of $\mu_{S,a}^{*n}(e)$ contains a wealth of information about the algebraic and geometric structure of the underlying group G.

Fourier transform is a powerful tool to study random walks on abelian groups. See [41]. These tools in general are not available on noncommutative groups. The study of random walks on noncommutative groups is often based on techniques that are rather different from the classical Fourier transform techniques used in the abelian case. This is certainly the case for our problem here. The most interesting part is to investigate the interaction between the long jumps and the structure of the underlying group. Below is a brief sketch of some of our results.

On nilpotent groups we have a relatively complete understanding of behavior of $\mu_{S,a}^{*n}(e)$. Our main result, in its simplest form, can be stated as follows.

Theorem 1.0.3. *Let G be a nilpotent group equipped with a generating k-tuple $S = (s_i)_1^k$, and $a = (\alpha_i)_1^k \in (0, \infty]^k$. Assume that the subgroup generated by $\{s_i : \alpha_i < 2\}$ is of finite index in G. Then there exists a real $D \geq 0$ depending on (G, S, a) such that*

$$\mu_{S,a}^{*n}(e) \simeq n^{-D}.$$

This statement suggests further questions including the following three:

1. Can we compute D? how does it depends on S, a and G?
2. What happen if the subgroup generated by $\{s_i : \alpha_i < 2\}$ is not of finite index in G?
3. What happens on other groups?

The first question will be answered completely in Chapter 3. The exact value of D depends in an intricate and interesting way on (a) the commutator structure of G, (b) the position of the generators s_i in the commutator structure of G and (c) the values of the parameters α_i. Briefly speaking, we can compute D explicitly from a filtration on G adapted to the index tuple $a = (\alpha_i)_1^k$. In the special case where all α_i are equal to $\alpha \in (0, 2)$, the filtration coincides with the lower central series of G, and $D = d/\alpha$, where d is the degree of polynomial volume growth of G. The filtration adapted to a is built as follows. Regarding S as a formal alphabet, we assign a system $\mathfrak{w}$ of weights to formal commutators of S. Assuming that all $\alpha_i \in (0, 2)$, the generator s_i is given weight $1/\alpha_i$, the commutator $[s_i, s_j]$ is given weight $1/\alpha_i + 1/\alpha_j$, and so on. Let $G_j^{\mathfrak{w}}$ be the subgroup generated by the images of all formal commutators of weights.In this way, we obtain a descending normal series with abelian quotients

$$G = G_1^{\mathfrak{w}} \trianglerighteq G_2^{\mathfrak{w}} \trianglerighteq ... \trianglerighteq G_{j_*}^{\mathfrak{w}} \trianglerighteq G_{j_*+1}^{\mathfrak{w}} = \{e\}.$$

Similar to Malcev coordinates, there is a system of approximate coordinates adapted to this filtration. The number D in the theorem is given by

$$D(S,a) = \sum_{1}^{j_*} \overline{\omega}_j \mathrm{rank}(G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}).$$

The second question is rather subtle and will not be completely elucidated in this dissertation although some partial results are obtained in Chapter 4. In its full generality, the third question is too wide ranging to be discussed here in details. On the free metabelian group $\mathbf{S}_{2,r} = \mathbf{F}_r/[N,N]$, $N = [\mathbf{F}_r, \mathbf{F}_r]$, we have the following result.

Theorem 1.0.4. *Let $\{\mathbf{s}_1, ..., \mathbf{s}_r\}$ denote the free generating set of the group $\mathbf{F}_r$. Let $S = \{s_1, ..., s_r\}$ be the projection of the generating set to $\mathbf{S}_{2,r}$. On the free metabelian group $\mathbf{S}_{2,r}$, for $a = (\alpha_1, \dots, \alpha_r) \in (0,2)^r$, we have*

$$\mu_{S,a}^n(e) \simeq \exp\left(-n^{r/(r+\alpha)}[\log n]^{\alpha/(r+\alpha)}\right)$$

where

$$\frac{1}{\alpha} = \frac{1}{r}\left(\frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_r}\right).$$

In Chapter 4 we prove we prove tightness properties of some random walks on groups of polynomial volume growth driven by spread-out measures, including the measures $\mu_{S,a}$ on nilpotent groups studied in the second part. Some other complementary bounds are also included.

CHAPTER 2

RANDOM WALKS ON FREE SOLVABLE GROUPS

2.1 Introduction

2.1.1 The random walk group invariant Φ_G

Let G be a finitely generated group. Given a probability measure μ on G, the random walk driven by μ (started at the identity element e of G) is the G-valued random process $X_n = \xi_1 \dots \xi_n$ where $(\xi_i)_1^\infty$ is a sequence of independent identically distributed G-valued random variables with law μ. If $u * v(g) = \sum_h u(h)v(h^{-1}g)$ denotes the convolution of two functions u and v on G then the probability that $X_n = g$ is given by $\mathbf{P}_e^\mu(X_n = g) = \mu^{(n)}(g)$ where $\mu^{(n)}$ is the n-fold convolution of μ.

Given a symmetric set of generators S, the word-length $|g|$ of $g \in G$ is the minimal length of a word representing g in the elements of S. The associated volume growth function, $r \mapsto V_{G,S}(r)$, counts the number of elements of G with $|g| \leq r$. The word-length induces a left invariant metric on G which is also the graph metric on the Cayley graph (G, S). A quasi-isometry between two Cayley graghs (G_i, S_i), $i = 1, 2$, say, from G_1 to G_2, is a map $q : G_1 \to G_2$ such that

$$C^{-1} d_2(q(x), q(y)) \leq d_1(x, y) \leq C(1 + d_2(q(x), q(y)))$$

and $\sup_{g, \in G_2} \{d_2(g, q(G_1)) \leq C$ for some finite positive constant C. This induces an equivalence relation on Cayley graphs. In particular, $(G, S_1), (G, S_2)$ are quasi-isometric for any choice of generating sets S_1, S_2. See, e.g., [12] for more details.

Given two monotone functions ϕ, ψ, write $\phi \simeq \psi$ if there are constants $c_i \in (0, \infty)$, $1 \leq i \leq 4$, such that $c_1\psi(c_2 t) \leq \phi(t) \leq c_3\psi(c_4 t)$ (using integer values if ϕ, ψ are defined on $\mathbb{N}$). If S_1, S_2 are two symmetric generating sets for G, then $V_{G,S_1} \simeq V_{G,S_2}$. We use the notation V_G to denote either the $\simeq$-equivalence class of $V_{G,S}$ or any one of its representatives. The volume growth function V_G is one of the simplest quasi-isometry invariant of a group G.

By [30, Theorem 1.4], if μ_i, $i = 1, 2$, are symmetric (i.e., $\mu_i(g) = \mu_i(g^{-1})$ for all $g \in G$) finitely supported probability measures with generating support, then the functions $n \mapsto \phi_i(n) = \mu_i^{(2n)}(e)$ satisfy $\phi_1 \simeq \phi_2$. By definition, we denote by Φ_G any function that belongs to the $\simeq$-equivalence class of $\phi_1 \simeq \phi_2$. In fact, Φ_G is an invariant of quasi-isometry. Further, if μ is a symmetric probability measure with generating support and finite second moment $\sum_G |g|^2\mu(g) < \infty$ then $\mu^{(2n)}(e) \simeq \Phi_G(n)$. See [30].

2.1.2 Free solvable groups

This chapter is concerned with finitely generated solvable groups. Recall that $G^{(i)}$, the derived series of G, is defined inductively by $G^{(0)} = G$, $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$. A group is solvable if $G^{(i)} = \{e\}$ for some i and the smallest such i is the derived length of G. A group G is polycyclic if it admits a normal descending series $G = N_0 \supset N_1 \supset \cdots \supset N_k = \{e\}$ such that each of the quotient N_i/N_{i+1} is cyclic. The lower central series $\gamma_j(G)$, $j \geq 1$, of a group G is obtained by setting $\gamma_1(G) = G$ and $\gamma_{j+1} = [G, \gamma_j(G)]$. A group G is nilpotent of nilpotent class c if $\gamma_c(G) \neq \{e\}$ and $\gamma_{c+1}(G) = \{e\}$. Finitely generated nilpotent groups are polycyclic and polycyclic groups are solvable.

Recall the following well-known facts. If G is a finitely generated solvable group then either G has polynomial volume growth $V_G(n) \simeq n^D$ for some $D = 0, 1, 2, \dots,$ or G has exponential volume growth $V_G(n) \simeq \exp(n)$. See, e.g., [12] and the references therein. If $V_G(n) \simeq n^D$ then G is virtually nilpotent and $\Phi_G(n) \simeq n^{-D/2}$. If G is polycyclic with exponential volume growth then $\Phi_G(n) \simeq \exp(-n^{1/3})$. See [1, 21, 44–46] and the references given there. However, among solvable groups of exponential volume growth, many other behaviors than those described above are known to occur. See, e.g., [14,29,38]. Our main result is the following theorem. Set

$$\log_{[1]} n = \log(1+n) \text{ and } \log_{[i]}(n) = \log(1 + \log_{[i-1]} n).$$

Theorem 2.1.1. *Let $\mathbf{S}_{d,r}$ be the free solvable group of derived length d on r generators, that is, $\mathbf{S}_{d,r} = \mathbf{F}_r/\mathbf{F}_r^{(d)}$ where $\mathbf{F}_r$ is the free group on r generators, $r \geq 2$.*

- *If $d = 2$ (the free metabelian case) then*

$$\Phi_{\mathbf{S}_{2,r}}(n) \simeq \exp\left(-n^{r/(r+2)}(\log n)^{2/(r+2)}\right).$$

- *If $d > 2$ then*

$$\Phi_{\mathbf{S}_{d,r}}(n) \simeq \exp\left(-n\left(\frac{\log_{[d-1]} n}{\log_{[d-2]} n}\right)^{2/r}\right).$$

In the case $d = 2$, this result is known and due to Anna Erschler who computed the Følner function of $\mathbf{S}_{2,r}$ in an unpublished work based on the ideas developed in [14]. We give a different proof. As far as we know, the Følner function of $\mathbf{S}_{d,r}$, $d > 2$ is not known.

Note that if G is r-generated and solvable of length at most d then there exists $c, k \in (0, \infty)$ such that $\Phi_G(n) \geq c\Phi_{\mathbf{S}_{d,r}}(kn)$.

2.1.3 On the groups of the form $\mathbf{F}_r/[N,N]$

The first statement in Theorem 2.1.1 can be generalized as follows. Let N be a normal subgroup of $\mathbf{F}_r$ and consider the tower of r generated groups $\Gamma_d(N)$ defined by $\Gamma_d(N) = \mathbf{F}_r/N^{(d)}$. Given information about $\Gamma_1(N) = \mathbf{F}_r/N$, more precisely, about the pair $(\mathbf{F}_r, N)$, one may hope to determine $\Phi_{\Gamma_d(N)}$ (in Theorem 2.1.1, $N = [\mathbf{F}_r, \mathbf{F}_r]$ and $\Gamma_1(N) = \mathbb{Z}^r$). The following theorem captures some of the results we obtain in this direction when $d = 2$. Further examples are given in Section 2.5.3.

Theorem 2.1.2. *Let $N \trianglelefteq \mathbf{F}_r$, $\Gamma_1(N) = \mathbf{F}_r/N$ and $\Gamma_2(N) = \mathbf{F}_r/[N,N]$ as above.*

- *Assume that $\Gamma_1(N)$ is nilpotent of volume growth of degree $D \geq 2$. Then we have*
$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n^{D/(D+2)}(\log n)^{2/(D+2)}\right).$$
- *Assume that*
  - *either $\Gamma_1(N) = \mathbb{Z}_q \wr \mathbb{Z}$ with presentation $\langle a, t | a^q, [a, t^{-n}at^n], n \in \mathbb{Z}\rangle$,*
  - *or $\Gamma_1(N) = \mathrm{BS}(1,q)$ with presentation $\langle a, b | a^{-1}ba = b^q\rangle$.*

  Then we have
$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-\frac{n}{(\log n)^2}\right).$$

In Section 5, Theorem 2.5.4, the result stated here for $\mathrm{BS}(1,q)$ is extended to any polycyclic group of exponential volume growth equipped with a standard polycyclic presentation.

Obtaining results for $d \geq 3$ is not easy. The only example we treat beyond the case $N = [\mathbf{F}_r, \mathbf{F}_r]$ contained in Theorem 2.1.1, i.e., $\Gamma_d(N) = \mathbf{S}_{d,r}$, is the case when $N = \gamma_c(\mathbf{F}_r)$. See Theorem 2.6.14.

Remark 2.1.3. Fix the presentation $\mathbf{F}_r/N = \Gamma_1(N)$. Let $\boldsymbol{\mu}$ be the probability measure driving the lazy simple random walk $(\xi_n)_0^\infty$ on $\mathbf{F}_r$ so that

$$\mathbf{P}_{\mathbf{e}}^{\boldsymbol{\mu}}(\xi_n = \mathbf{g}) = \boldsymbol{\mu}^{(n)}(\mathbf{g}).$$

Let $X = (X_n)_0^\infty$ and $Y = (Y_n)_0^\infty$ be the projections on $\Gamma_2(N)$ and $\Gamma_1(N)$, respectively so that

$$\Phi_{\Gamma_2(N)}(n) \simeq \mathbf{P}_{\mathbf{e}}^{\boldsymbol{\mu}}(X_n = e) \text{ and } \Phi_{\Gamma_1(N)}(n) \simeq \mathbf{P}_{\mathbf{e}}^{\boldsymbol{\mu}}(Y_n = \bar{e})$$

where e (resp. $\bar{e}$) is the identity element in $\Gamma_2(N)$ (resp. $\Gamma_1(N)$.) By the flow interpretation of the group $\Gamma_2(N)$ developed in [13, 28] and reviewed in Section 2.2.2 below,

$$\mathbf{P}_{\mathbf{e}}^{\boldsymbol{\mu}}(X_n = e) = \mathbf{P}_{\mathbf{e}}^{\boldsymbol{\mu}}(Y \in \mathfrak{B}_n)$$

where $\mathfrak{B}_n$ is the event that, at time n, every oriented edge of the marked Cayley graph $\Gamma_1(N)$ has been traversed an equal number of times in both directions. For instance, if $\Gamma_1(N) = \mathbb{Z}^r$, the estimate $\Phi_{\Gamma_2(N)}(n) \simeq \exp(-n^{r/(2+r)}(\log n)^{2/(2+r)})$ also gives the order of magnitude of the probability that a simple random walk on $\mathbb{Z}^r$ returns to its starting point at time n having crossed each edge an equal number of time in both direction.

2.1.4 Other random walk invariants

Let $|g|$ be the word-length of G with respect to some fixed finite symmetric generating set and $\rho_\alpha(g) = (1 + |g|)^\alpha$. In [5], for any finitely generated group G and real $\alpha \in (0,2)$, the non-increasing function

$$\widetilde{\Phi}_{G,\rho_\alpha} : \mathbb{N} \ni n \to \widetilde{\Phi}_{G,\rho_\alpha}(n) \in (0,\infty)$$

is defined in such a way that it provides the best possible lower bound

$$\exists\, c, k \in (0,\infty),\ \forall\, n,\ \ \mu^{(2n)}(e) \geq c\widetilde{\Phi}_{G,\rho_\alpha}(kn),$$

valid for every symmetric probability measure μ on G satisfying the weak-ρ_α-moment condition

$$W(\rho_\alpha, \mu) = \sup_{s>0}\{s\mu(\{g : \rho_\alpha(g) > s\})\} < \infty.$$

It is well known and easy to see (using Fourier transform techniques) that

$$\widetilde{\Phi}_{\mathbb{Z}^r,\rho_\alpha}(n) \simeq n^{-r/\alpha}.$$

It is proved in [5] that $\widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq n^{-D/\alpha}$ if G has polynomial volume growth of degree D and that $\widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq \exp\left(-n^{-1/(1+\alpha)}\right)$ if G is polycyclic of exponential volume growth. We prove the following result.

Theorem 2.1.4. *For any $\alpha \in (0,2)$,*

$$\widetilde{\Phi}_{\mathbf{S}_{2,r},\rho_\alpha}(n) \simeq \exp\left(-n^{r/(r+\alpha)}(\log n)^{\alpha/(r+\alpha)}\right).$$

The lower bound in this theorem follows from Theorem 2.1.1 and [5]. Indeed, for $d > 2$, Theorem 2.1.1 and [5, Theorem 3.3] also give

$$\widetilde{\Phi}_{\mathbf{S}_{d,r},\rho_\alpha}(n) \geq c\exp\left(-Cn\left(\frac{\log_{[d-1]} n}{\log_{[d-2]} n}\right)^{\alpha/r}\right).$$

The upper bound in Theorem 2.1.4 is obtained by studying random walks driven by measures that are not finitely supported. The fact that the techniques we develop below can be applied successfully in certain cases of this type is worth noting. Proving an upper bound matching the lower bound given above for $\widetilde{\Phi}_{\mathbf{S}_{d,r},\rho_\alpha}$ with $d > 2$ is an open problem.

2.1.5 Wreath products and Magnus embedding

Let H, K be countable groups. Recall that the wreath product $K \wr H$ (with base H) is the semidirect product of the algebraic direct sum $K_H = \sum_{h \in H} K_h$ of H-indexed copies of K by H where H acts on K_H by translation of the indices. More precisely, elements of $K \wr H$ are pair $(f, h) \in K_H \times H$ and

$$(f, h)(f', h') = (f \tau_h f', hh')$$

where $\tau_h f_x = f_{h^{-1}x}$ if $f = (f_x)_{x \in H} \in K_H$ (recall that, by definition, only finitely many f_x are not the identity element e_K in K). In the context of random walk theory, the group H is called the base-group and the group K the lamp-group of $K \wr H$ (an element $(f, h) \in K \wr H$ can understood as a finite lamp configuration f over H together with the position h of the "lamplighter" on the base H). Given probability measures η on K and μ on H, the switch-walk-switch random walk on $K \wr H$ is driven by the measure $\eta * \mu * \eta$ and has the following interpretation. At each step, the lamplighter switches the lamp at its current position using an η-move in K, then the lamplighter makes a μ-move in H according to μ and, finally, the lamplighter switches the lamp at its final position using an η-move in K. Each of these steps are performed independently of each others. See, e.g., [29, 34] for more details. When we write $\eta * \mu * \eta$ in $K \wr H$, we identify η with the probability measure on $K \wr H$ with is equal to η on the copy of K above the identity of H and vanishes everywhere else, and we identify μ with the a probability measure on $K \wr H$ supported on the obvious copy of H in $K \wr H$.

Thanks to [8, 14, 29, 34], quite a lot is known about the random walk invariant $\Phi_{K \wr H}$. Further, the results stated in Theorems 2.1.1-2.1.2 can in fact be rephrased as stating that

$$\Phi_{\Gamma_2(N)} \simeq \Phi_{\mathbb{Z}^a \wr \Gamma_1(N)}$$

for some/any integer $a \geq 1$. It is relevant to note here that for Γ of polynomial volume growth of degree $D > 0$ or Γ infinite polycyclic (and in many other cases as well), whe have $\Phi_{\mathbb{Z}^a \wr \Gamma} \simeq \Phi_{\mathbb{Z}^b \wr \Gamma}$ for any integers $a, b \geq 1$. Indeed, the proofs of Theorems 2.1.1–2.1.2–2.1.4 make use of the Magnus embedding which provides us with an injective homomorphism $\bar{\psi} : \Gamma_2(N) \hookrightarrow \mathbb{Z}^r \wr \Gamma_1(N)$. This embedding is use to prove a lower bound of the type

$$\Phi_{\Gamma_2(N)}(n) \geq c\Phi_{\mathbb{Z}^r \wr \Gamma_1(N)}(kn)$$

and an upper bound that can be stated as

$$\Phi_{\Gamma_2(N)}(Cn) \leq C\Phi_{\mathbb{Z} \wr \overline{\Gamma}}(n)$$

where $\overline{\Gamma} < \Gamma_1(N)$ is a subgroup which has a similar structure as $\Gamma_1(N)$. For instance, in the easiest cases including when $\Gamma_1(N)$ is nilpotent, $\overline{\Gamma}$ is a finite index subgroup of $\Gamma_1(N)$. The fact that the wreath product is taken with $\mathbb{Z}^r$ in the lower bound and with $\mathbb{Z}$ in the upper bound is not a typo. It reflects the nature of the arguments used for the proof. Hence, the fact that the lower and upper bounds that are produced by our arguments match up depends on the property that, under proper hypotheses on $\overline{\Gamma} < \Gamma_1(N)$ and $\Gamma_1(N)$,

$$\Phi_{\mathbb{Z}^a \wr \Gamma_1(N)} \simeq \Phi_{\mathbb{Z}^b \wr \overline{\Gamma}}$$

for any pair of positive integers a, b.

2.1.6 A short guide

Section 2 of the chapter is devoted to the algebraic structure of the group $\Gamma_2(N) = \mathbf{F}_r/[N, N]$. It describes the Magnus embedding as well as the interpretation of

$\Gamma_2(N)$ in terms of flows on $\Gamma_1(N)$. See [13, 28, 47]. The Magnus embedding and the flow representation play key parts in the proofs of our main results.

Section 3 describes two methods to obtain lower bounds on the probability of return of certain random walks on $\Gamma_2(N)$. The first method is based on a simple comparison argument and the notion of Følner couples introduced in [8] and already used in [14]. This method works for symmetric random walks driven by a finitely supported measure. The second method allows us to treat some measures that are not finitely supported, something that is of interest in the spirit of Theorem 2.1.4.

Section 4 focuses on upper bounds for the probability of return. This section also makes use of the Magnus embedding, but in a somewhat more subtle way. We introduce the notion of exclusive pair. These pairs are made of a subgroup Γ of $\Gamma_2(N)$ and an element $\boldsymbol{\rho}$ in the free group $\mathbf{F}_r$ that projects to a cycle on $\Gamma_1(N)$ with the property that the traces of Γ and $\boldsymbol{\rho}$ on $\Gamma_1(N)$ have, in a sense, minimal interaction. See Definition 2.4.3. Every upper bound we obtain is proved using this notion.

Section 5 presents a variety of applications of the results obtained in Sections 3 and 4. In particular, the statement regarding $\Phi_{\mathbf{S}_{2,r}}$ as well as Theorems 2.1.2–2.1.4 and assorted results are proved in Section 5.

Section 6 is devoted to the result concerning $\mathbf{S}_{d,r}$, $d \geq 3$. Both the lower bound and the upper bound methods are re-examined to allow iteration of the procedure.

Throughout this chapter, we will have to distinguish between convolutions in different groups. We will use $*$ to denote either convolution on a generic group G (when no confusion can possibly arise) or, more specifically, convolution on $\Gamma_2(N)$.

When $*$ is used to denote convolution on $\Gamma_2(N)$, we use e_* to denote the identity element in $\Gamma_2(N)$. We will use $\star$ to denote convolution on various wreath products such as $\mathbb{Z}^r \wr \Gamma_1(N)$. When this notation is used, $e_\star$ will denote the identity element in the corresponding group. When necessary, we will decorate $\star$ with a subscript to distinguish between different wreath products. So, if μ is a probability measure on $\Gamma_2(N)$ and ϕ a probability measure on $\mathbb{Z}^r \wr \Gamma_1(N)$, we will write $\mu^{*n}(e_*) = \phi^{\star n}(e_\star)$ to indicate that the n-fold convolution of μ on $\Gamma_2(N)$ evaluated at the identity element of $\Gamma_2(N)$ is equal to the n-fold convolution of ϕ on $\mathbb{Z} \wr \Gamma_1(N)$ evaluated at the identity element of $\mathbb{Z} \wr \Gamma_1(N)$.

2.2 $\Gamma_2(N)$ and the Magnus embedding

This chapter is concerned with random walks on the groups $\Gamma_\ell(N) = \mathbf{F}_r/N^{(\ell)}$ where $\mathbf{F}_r$ is the free group on r generators and N is a normal subgroup of $\mathbf{F}_r$. In fact, it is best to think of $\Gamma_\ell(N)$ as a marked group, that is, a group equipped with a generating tuple. In the case of $\Gamma_\ell(N)$, the generating r-tuple is always provided by the images of the free generators of $\mathbf{F}_r$. Ideally, one would like to obtain results based on hypotheses on the nature of $\Gamma_1(N)$ viewed as an unmarked group. However, as pointed out in Remark 2.2.8 below, the unmarked group $\Gamma_1(N)$ is not enough to determine either $\Gamma_2(N)$ or the random walk invariant $\Phi_{\Gamma_2(N)}$. That is, in general, one needs information about the pair $(\mathbf{F}_r, N)$ itself to obtain precise information about $\Phi_{\Gamma_2(N)}$. Note however that when $\Gamma_1(N)$ is nilpotent with volume growth of degree at least 2, Theorem 2.1.2 provides a result that does not require further information on N.

2.2.1 The Magnus embedding

In 1939, Magnus [24] introduced an embedding of $\Gamma_2(N) = \mathbf{F}_r/[N,N]$ into a matrix group with coefficients in a module over $\mathbb{Z}(\Gamma_1(N)) = \mathbb{Z}(\mathbf{F}_r/N)$. In particular, the Magnus embedding is used to embed free solvable groups into certain wreath products.

Let $\mathbf{F}_r$ be the free group on the generators $\mathbf{s}_i$, $1 \leq i \leq r$. Let N be a normal subgroup of $\mathbf{F}_r$ and let $\pi = \pi_N$ and $\pi_2 = \pi_{2,N}$ be the canonical projections

$$\pi : \mathbf{F}_r \to \mathbf{F}_r/N = \Gamma_1(N), \quad \pi_2 : \mathbf{F}_r \to \mathbf{F}_r/[N,N] = \Gamma_2(N).$$

We also let

$$\bar{\pi} : \Gamma_2(N) \to \Gamma_1(N)$$

the projection from $\Gamma_2(N)$ onto $\Gamma_1(N)$, whose kernel can be identified with $N/[N,N]$, has the property that $\pi = \bar{\pi} \circ \pi_2$. Set

$$s_i = \pi_2(\mathbf{s}_i), \quad \bar{s}_i = \pi(\mathbf{s}_i) = \bar{\pi}(s_i).$$

When it is necessary to distinguish between the identity element in $e \in \Gamma_2(N)$ and the identity element in $\Gamma_1(N)$, we write $\bar{e}$ for the latter.

Let $\mathbb{Z}(\mathbf{F}_r)$ be the integral group ring of the free group $\mathbf{F}_r$. By extension and with some abuse of notation, let π denote also the ring homomorphism

$$\pi : \mathbb{Z}(\mathbf{F_r}) \to \mathbb{Z}(\mathbf{F}_r/N)$$

determined by $\pi(\mathbf{s}_i) = \bar{s}_i$, $1 \leq i \leq r$.

Let Ω be the free left $\mathbb{Z}(\mathbf{F}_r/N)$-module of rank r with basis $(\lambda_{\mathbf{s}_i})_1^r$ and set

$$M = \begin{bmatrix} \mathbf{F}_r/N & \Omega \\ 0 & 1 \end{bmatrix}$$

which is a subgroup of the group of the 2×2 upper-triangular matrices over Ω. The map

$$\psi(\mathbf{s}_i) = \begin{bmatrix} \pi(\mathbf{s}_i) & \lambda_{\mathbf{s}_i} \\ 0 & 1 \end{bmatrix} \tag{2.1}$$

extends to a homomorphism ψ of $\mathbf{F}_r$ into M. We denote by $a(\mathbf{u})$, $\mathbf{u} \in \mathbf{F}_r$, the $(1,2)$-entry of the matrix $\psi(\mathbf{u})$, that is

$$\psi(\mathbf{u}) = \begin{bmatrix} \pi(\mathbf{u}) & a(\mathbf{u}) \\ 0 & 1 \end{bmatrix}. \tag{2.2}$$

Theorem 2.2.1 (Magnus [24])**.** *The kernel of the homomorphism $\psi : \mathbf{F}_r \to M$ defined as above is*

$$\ker(\psi) = [N, N].$$

Therefore ψ induces a monomorphism

$$\bar{\psi} : \mathbf{F}_r/[N, N] \hookrightarrow M.$$

It follows that $\mathbf{F}_r/[N, N]$ is isomorphic to the subgroup of M generated by

$$\begin{bmatrix} \pi(\mathbf{s}_i) & \lambda_{\mathbf{s}_i} \\ 0 & 1 \end{bmatrix}, \ i = 1, \ldots, r.$$

Remark 2.2.2. For $g \in \mathbf{F}_r/[N, N]$, we write

$$\bar{\psi}(g) = \begin{bmatrix} \bar{\pi}(g) & \bar{a}(g) \\ 0 & 1 \end{bmatrix} \tag{2.3}$$

where $\bar{a}(\pi_2(\mathbf{u})) = a(\mathbf{u})$, $\mathbf{u} \in \mathbf{F}_r$.

Remark 2.2.3. The free left $\mathbb{Z}(\mathbf{F}_r/N)$-module Ω with basis $\{\lambda_{\mathbf{s}_i}\}_{1 \le i \le d}$ is isomorphic to the direct sum $\sum_{x \in \mathbf{F}_r/N} (\mathbb{Z}^r)_x$. More precisely, if we regard the elements in

$\sum_{x\in\mathbf{F}_r/N}(\mathbb{Z}^r)_x$ as functions $f = (f_1, .., f_r) : \mathbf{F}_r/N \to \mathbb{Z}^r$ with finite support, the map

$$\begin{aligned} \sum_{x\in\mathbf{F}_r/N}(\mathbb{Z}^r)_x &\to \Omega : \\ f &\mapsto \left(\sum_{x\in\mathbf{F}_r/N} f_1(x)x\right)\lambda_{\mathbf{s}_1} + ... + \left(\sum_{x\in\mathbf{F}/N} f_r(x)x\right)\lambda_{\mathbf{s}_r} \end{aligned}$$

is a left $\mathbb{Z}(\mathbf{F}_r/N)$-module isomorphism. We will identity Ω with $\sum_{x\in\mathbf{F}_r/N}(\mathbb{Z}^r)_x$. Using the above interpretation, one can restate the Magnus embedding theorem as an injection from $\mathbf{F}_r/[N,N]$ into the wreath product $\mathbb{Z}^r \wr (\mathbf{F}_r/N)$.

The entry $a(g) \in \Omega$ under the Magnus embedding is given by Fox derivatives which we briefly review. Let G be a group and $\mathbb{Z}(G)$ be its integral group ring. Let M be a left $\mathbb{Z}(G)$-module. An additive map $d : \mathbb{Z}(G) \to M$ is called a *left derivation* if for all $x, y \in G$,

$$d(xy) = xd(y) + d(x).$$

As a consequence of the definition, we have $d(e) = 0$ and $d(g^{-1}) = -g^{-1}d(g)$.

For the following two theorems of Fox, we refer the reader to the discussion in [28, Sect. 2.3] and the references given there.

Theorem 2.2.4 (Fox)**.** *Let $\mathbf{F}_r$ be the free group on r generators $\mathbf{s}_i$, $1 \le i \le r$. For each i, there is a unique left derivation*

$$\partial_{\mathbf{s}_i} : \mathbb{Z}(\mathbf{F}_r) \to \mathbb{Z}(\mathbf{F}_r)$$

satisfying

$$\partial_{\mathbf{s}_i}(\mathbf{s}_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \ne j. \end{cases}$$

Further, if N is a normal subgroup of $\mathbf{F}_r$, then $\pi(\partial_{\mathbf{s}_i}\mathbf{u}) = 0$ in $\mathbb{Z}(\mathbf{F}_r/N)$ for all $1 \le i \le r$ if and only if $\mathbf{u} \in [N,N]$.

Example 2.2.1. For $\mathbf{g} = \mathbf{s}_{i_1}^{\varepsilon_1}...\mathbf{s}_{i_n}^{\varepsilon_n}$, $\varepsilon_j \in \{\pm 1\}$,

$$\begin{aligned}\partial_{\mathbf{s}_i}(\mathbf{g}) &= \sum_{j=1}^{n} \mathbf{s}_{i_1}^{\varepsilon_1}...\mathbf{s}_{i_{j-1}}^{\varepsilon_{j-1}} \partial_{\mathbf{s}_i}(\mathbf{s}_{i_j}^{\varepsilon_j}) \\ &= \sum_{j:i_j=i,\varepsilon_j=1} \mathbf{s}_{i_1}^{\varepsilon_1}...\mathbf{s}_{i_{j-1}}^{\varepsilon_{j-1}} - \sum_{j:i_j=i,\varepsilon_j=-1} \mathbf{s}_{i_1}^{\varepsilon_1}...\mathbf{s}_{i_{j-1}}^{\varepsilon_{j-1}} \mathbf{s}_{i_j}^{\varepsilon_j}.\end{aligned}$$

Theorem 2.2.5 (Fox)**.** *The Magnus embedding*

$$\bar{\psi} : \mathbf{F}_r/[N, N] \hookrightarrow M$$

is given by

$$\bar{\psi}(g) = \begin{bmatrix} \bar{\pi}(g) & \sum_{i=1}^{r} \pi(\partial_{\mathbf{s}_i}\mathbf{g})\lambda_{\mathbf{s}_i} \\ 0 & 1 \end{bmatrix} \tag{2.4}$$

where $\mathbf{g} \in \mathbf{F}_r$ *is any element such that* $\pi_2(\mathbf{g}) = g$*.*

Example 2.2.2. In the special case that $N = [\mathbf{F}_r, \mathbf{F}_r]$, we have $\mathbf{F}_r/N \simeq \mathbb{Z}^r$ and $\mathbb{Z}(\mathbf{F}_r/N)$ is the integral group ring over the free abelian group $\mathbb{Z}^r$. The integral group ring $\mathbb{Z}(\mathbb{Z}^r)$ is quite similar to the multivariate polynomial ring with integer coefficients, except that we allow negative powers like $Z_1^{-3}Z_2...Z_r^{-5}$. The monomials $\{Z_1^{x_1}Z_2^{x_2}...Z_r^{x_r} : x \in \mathbb{Z}^r\}$ are $\mathbb{Z}$-linear independent in $\mathbb{Z}(\mathbb{Z}^r)$.

2.2.2 Interpretation in terms of flows

Following [13, 28, 47], one can also think of elements of $\Gamma_2(N) = \mathbf{F}_r/[N, N]$ in terms of flows on the (labeled) Cayley graph of $\Gamma_1(N) = \mathbf{F}_r/N$. To be precise, Let $\mathbf{s}_1, \ldots, \mathbf{s}_k$ be the generators of $\mathbf{F}_r$ and $\bar{s}_1, \ldots, \bar{s}_k$ their images in $\Gamma_1(N)$. The Cayley graph of $\Gamma_1(N)$ is the marked graph with vertex set $V = \Gamma_1(N)$ and marked edge set $\mathfrak{E} \subset V \times V \times \{\mathbf{s}_1, \ldots, \mathbf{s}_k\}$ where $(x, y, \mathbf{s}_i) \in \mathfrak{E}$ if and only if $y = x\bar{s}_i$ in $\Gamma_1(N)$. Note that each edge $\mathfrak{e} = (x, y, \mathbf{s}_i)$ as an origin $o(\mathfrak{e}) = x$, an end (or terminus) $t(\mathfrak{e}) = y$ and a label or mark $\mathbf{s}_i$.

Given a function $\mathfrak{f}$ on the edge set $\mathfrak{E}$ and a vertex $v \in V$, define the net flow $\mathfrak{f}^*(v)$ of $\mathfrak{f}$ at v by

$$\mathfrak{f}^*(v) = \sum_{o(\mathfrak{e})=v} f(\mathfrak{e}) - \sum_{t(\mathfrak{e})=v} f(\mathfrak{e}).$$

A *flow* (or $\mathbb{Z}$-flow) with source s and sink t is a function $\mathfrak{f} : \mathfrak{E} \to \mathbb{Z}$ such that

$$\forall\, v \in V\backslash\{s,t\}, \;\; \mathfrak{f}^*(v) = 0,$$

$$\mathfrak{f}^*(s) = 1, \; \mathfrak{f}^*(t) = -1.$$

If $\mathfrak{f}^*(v) = 0$ holds for all $v \in V$, we say that $\mathfrak{f}$ is a *circulation.*

For each edge $\mathfrak{e} = (x, y, \mathbf{s}_i)$, introduce its formal inverse $(y, x, \mathbf{s}_i^{-1})$ and let $\mathfrak{E}^*$ be the set of all edges and their formal inverses. A finite path on the Cayley graph of $\Gamma_1(N)$ is a finite sequence $p = (\mathfrak{e}_1, \ldots, \mathfrak{e}_\ell)$ of edges in $\mathfrak{E}^*$ so that the origin of $\mathfrak{e}_{i+1}$ is the terminus of $\mathfrak{e}_i$. We call $o(\mathfrak{e}_1)$ (resp. $t(\mathfrak{e}_\ell)$) the origin (resp. terminus) of the path p and denote it by $o(p)$ (resp. $t(p)$). Note that reading the labels along the edge of a path determines a word in the generators of $\mathbf{F}_r$ and that, conversely, any finite word ω in the generators of $\mathbf{F}_r$ determines a path p_ω starting at the identity element in $\Gamma_1(N)$.

A (finite) path p determines a flow $\mathfrak{f}_p$ with source $o(p)$ and sink $t(p)$ by setting $\mathfrak{f}_p(e)$ to be the algebraic number of time the edge $\mathfrak{e} \in \mathfrak{E}$ is crossed positively or negatively along p. Here, the edge $\mathfrak{e} = (x, y, \mathbf{s}_\alpha) \in \mathfrak{E}$ is crossed positively at the i-step along p if $\mathfrak{e}_i = (x, y, \mathbf{s}_\alpha)$. It is crossed negatively if $\mathfrak{e}_i = (y, x, \mathbf{s}_\alpha^{-1})$. We note that $\mathfrak{f}_p$ has finite support and that either $o(p) = t(p)$ and $\mathfrak{f}_p$ is a circulation or $o(p) \neq t(p)$ and $\mathfrak{f}_p^*(o(p)) = 1$, $\mathfrak{f}_p^*(t(p)) = -1$.

Given a word $\omega = \mathbf{s}_{i_1}^{\varepsilon_1} \ldots \mathbf{s}_{i_n}^{\varepsilon_n}$ in the generators of $\mathbf{F}_r$, let $\mathfrak{f}_\omega$ denote the flow function on the Cayley graph of $\Gamma_1(N)$ defined by the corresponding path starting at the identity element in $\Gamma_1(N)$. We note that it is obvious from the definition that $\mathfrak{f}_\omega = \mathfrak{f}_{\omega'}$ if ω' is the reduced word in $\mathbf{F}_r$ associated with ω.

Theorem 2.2.6 ([28, Theorem 2.7])**.** *Two elements $\mathbf{u}, \mathbf{v} \in \mathbf{F}_r$ project to the same element in $\Gamma_2(N) = \mathbf{F}_r/[N, N]$ if and only if they induce the same flow on $\Gamma_1(N) = \mathbf{F}_r/N$. In other words,*

$$\mathbf{u} \equiv \mathbf{v} \bmod [N, N] \iff \mathfrak{f}_{\mathbf{u}} = \mathfrak{f}_{\mathbf{v}}.$$

This theorem shows that an element $g \in \Gamma_2(N)$ corresponds to a unique flow $\mathfrak{f}_\omega$ on $\mathbf{F}_r/N$, defined by the path p_ω associated with any word $\omega \in \mathbf{F}_r$ such that ω projects to g in $\Gamma_2(N)$. For $g \in \Gamma_2(N)$, $\mathfrak{f}_g := \mathfrak{f}_\omega$ is well defined (i.e., is independent of the word ω projecting to g, and we call $\mathfrak{f}_g$ the flow of g. Hence, in a certain sense, we can regard elements of $\Gamma_2(N)$ as flows on $\Gamma_1(N)$. In fact, the flow $\mathfrak{f}_\omega$ is directly related to the description of the image of the element $g = \omega \bmod [N, N]$ under the Magnus embedding through the following geometric interpretation of Fox derivatives.

Lemma 2.2.7 ([28, Lemma 2.6])**.** *Let $\omega \in \mathbf{F}_r$, then for any $g \in \mathbf{F}_r/N$ and $\mathbf{s}_i$, the value of $\mathfrak{f}_\omega$ on the edge $(g, gs_i, \mathbf{s}_i)$, is equal to coefficient in front of g in the Fox derivative $\pi(\partial_{\mathbf{s}_i}\omega) \in \mathbb{Z}(\mathbf{F}/N)$, i.e.*

$$\pi(\partial_{\mathbf{s}_i}\omega) = \sum_{g \in \mathbf{F}/N} \mathfrak{f}_\omega((g, gs_i, \mathbf{s}_i))g. \tag{2.5}$$

There is also a characterization of geodesics on $\Gamma_2(N)$ in terms of flows (see [28, Theorem 2.11]) which is closely related to the description of geodesics on wreath products. See [33, Theorem 2.6] where it is proved that the Magnus embedding is bi-Lipschitz with small explicit universal distortion.

Remark 2.2.8. In [15], it is asserted that the group $\Gamma_2(N)$ depends only of $\Gamma_1(N)$ (in [15], $\Gamma_1(N)$ is denoted by A and $\Gamma_2(N)$ by C_A). This assertion is correct only if one interprets $\Gamma_1(N)$ as a marked group, i.e., if information about $\pi : \mathbf{F}_r \to \Gamma_1(N)$ is retained. Indeed, $\Gamma_2(N)$ depends in some essential ways of the choice of the presentation $\Gamma_1(N) = \mathbf{F}_r/N$. We illustrate this fact by two examples that are very good to keep in mind.

Example 2.2.3. Consider two presentations of $\mathbb{Z}$, namely, $\mathbb{Z} = \mathbf{F}_1$ and $\mathbb{Z} = \langle a, b | b\rangle$. In the first presentation, the kernel N_1 is trivial, therefore $\mathbf{F}_1/[N_1, N_1] \simeq \mathbb{Z}$. In the second presentation, the kernel N_2 is the normal closure of $\langle b\rangle$ in the free group $\mathbf{F}_2$ on generators a, b. Hence, N_2 is generated by $\{a^i b a^{-i}, i \in \mathbb{Z}\}$. We can then write down a presentation of $\mathbf{F}_2/[N_2, N_2]$ in the form

$$\mathbf{F}_2/[N_2, N_2] = \left\langle a, b | [a^i b a^{-i}, a^j b a^{-j}], i, j \in \mathbb{Z}\right\rangle .$$

This is, actually, a presentation of the wreath product $\mathbb{Z} \wr \mathbb{Z}$. Therefore $\mathbf{F}_2/N_2' \simeq \mathbb{Z} \wr \mathbb{Z}$. We encourage the reader to recognize the structure of both $\mathbf{F}_1/[N_1, N_1] \simeq \mathbb{Z}$ and $\mathbf{F}_2/[N_2, N_2] \simeq \mathbb{Z} \wr \mathbb{Z}$ using flows on the labeled Cayley graphs associated with $\mathbf{F}_1/N_1$ and $\mathbf{F}_2/N_2$. The Cayley graph of $\mathbf{F}_2/N_2$ is the usual line graph of $\mathbb{Z}$ decorated with an oriented loop at each vertex. In the flow representation of an element of $\mathbf{F}_2/[N_2, N_2]$, the algebraic number of times the flow goes around each of these loops is recorded thereby creating the wreath product structure of $\mathbb{Z} \wr \mathbb{Z}$.

Example 2.2.4. Consider the following two presentations of $\mathbb{Z}^2$,

$$\begin{aligned} \mathbb{Z}^2 &= \langle a, b | [a, b]\rangle \\ \mathbb{Z}^2 &= \langle a, b, c | [a, b], c = ab\rangle . \end{aligned}$$

Call $N_1 \subset \mathbf{F}_2$ and $N_2 \subset \mathbf{F}_3$ be the associated normal subgroups. We claim that $\mathbf{F}_2/[N_1, N_1]$ is a proper quotient of $\mathbf{F}_3/[N_2, N_2]$. Let $\theta : \mathbf{F}_3 \to \mathbf{F}_2$ be the homomorphism determined by $\theta(a) = a$, $\theta(b) = b$, $\theta(c) = ab$. Obviously, $N_2 = \theta^{-1}(N_1)$, $[N_2, N_2] \subset \theta^{-1}([N_1, N_1])$, and θ induces a surjective homomorphism

$\theta' : \mathbf{F}_3/[N_2, N_2] \to \mathbf{F}_2/[N_1, N_1]$. The element abc^{-1} is nontrivial in $\mathbf{F}_3/[N_2, N_2]$, but $\theta'(abc^{-1}) = e$. A Hopfian group is a group that cannot be isomorphic to a proper quotient of itself. Finitely generated metabelian groups are Hopfian. Hence $\mathbf{F}_2/[N_1, N_1]$ is not isomorphic to $\mathbf{F}_3/[N_2, N_2]$.

2.3 Return probability lower bounds

2.3.1 Measures supported by the powers of the generators

The group $\Gamma_2(N) = \mathbf{F}_r/[N, N]$ comes equipped with the generators $(s_i)_1^r$ which are the images of the generators $(\mathbf{s}_i)_1^r$ of $\mathbf{F}_r$. Accordingly, we consider a special class of symmetric random walks defined as follows. Given probability measures p_i, $1 \le i \le r$ on $\mathbb{Z}$, we define a probability measure $\boldsymbol{\mu}$ on $\mathbf{F}_r$ by

$$\forall\, \mathbf{g} \in \mathbf{F}_r, \quad \boldsymbol{\mu}(\mathbf{g}) = \sum_{i=1}^{r} \frac{1}{r} \sum_{m \in \mathbb{Z}} p_i(m) \mathbf{1}_{\{\mathbf{s}_i^m\}}(\mathbf{g}). \tag{2.1}$$

This probability measure induces pushforward measures $\bar{\mu}$ and μ on $\Gamma_1(N) = \mathbf{F}_r/N$ and $\Gamma_2(N) = \mathbf{F}_r/[N, N]$, namely,

$$\begin{cases} \forall\, \bar{g} \in \Gamma_1(N), & \bar{\mu}(\bar{g}) = \boldsymbol{\mu}(\pi^{-1}(\bar{g})) \\ \forall\, g \in \Gamma_2(N), & \mu(g) = \boldsymbol{\mu}(\pi_2^{-1}(g)). \end{cases} \tag{2.2}$$

In fact, we will mainly consider two cases. In the first case, each p_i is the measure of the lazy random walk on $\mathbb{Z}$, that is $p_i(0) = 1/2$, $p_i(\pm 1) = 1/4$. In this case, $\boldsymbol{\mu}$ is the measure of the lazy simple random walk on $\mathbf{F}_r$, that is,

$$\boldsymbol{\mu}(e) = 1/2, \quad \boldsymbol{\mu}(\mathbf{s}_i^{\pm 1}) = 1/4r. \tag{2.3}$$

The second case can be viewed as a generalization of the first. Let $a = (\alpha_1)_1^r \in (0, \infty]^r$ be a r-tuple of extended positive reals. For each i, consider the symmetric probability measure p_{α_i} on $\mathbb{Z}$ with $p_{\alpha_i}(m) = c_i(1+|m|)^{-1-\alpha_i}$ (if $\alpha_i = \infty$, set $p_\infty(0) = 1/2$, $p_\infty(\pm 1) = 1/4$). Let $\boldsymbol{\mu}_a$ be the measure on $\mathbf{F}_r$ obtained by setting $p_i = p_{\alpha_i}$ in (2.1). When a is such that $\alpha_i = \infty$ for all i we recover (2.3). In particular, starting with (2.3), μ is given by

$$\forall\, g \in \Gamma_2(N), \quad \mu(g) = \frac{1}{2}\mathbf{1}_g(e) + \frac{1}{4r}\sum_1^r \mathbf{1}_{s_i}(g).$$

The formula for $\bar{\mu}$ is exactly similar. For any fixed $a \in (0,\infty]^r$, we let μ_a and $\bar{\mu}_a$ be the pushforward of $\boldsymbol{\mu}_a$ on $\Gamma_2(N)$ and $\Gamma_1(N)$, respectively.

2.3.2 Lower bound for simple random walk

In this section, we explain how, in the case of the lazy simple random walk measure μ on $\Gamma_2(N)$ associated with $\boldsymbol{\mu}$ at (2.3), one can obtained lower bounds for the probability of return $\mu^{(2n)}(e)$ by using well-known arguments and the notion of Følner couples introduced in [8].

Definition 2.3.1 (See [8, Definition 4.7] and [14, Proposition 2])**.** Let G be a finitely generated group equipped with a finite symmetric generating set T and the associated word distance d. Let $\mathcal{V}$ be a positive increasing function on $[1,\infty)$ whose inverse is defined on $[\mathcal{V}(1),\infty)$. We say that a sequence of pairs of nonempty sets $((\Omega_k,\Omega'_k))_1^\infty$ is a a sequence of Følner couples adapted to $\mathcal{V}$ if

1. $\Omega'_k \subset \Omega_k$, $\#\Omega'_k \geq c_0 \#\Omega_k$, $d(\Omega'_k, \Omega^c_k) \geq c_0 k$.
2. $v_k = \#\Omega_k \nearrow \infty$ and $v_k \leq \mathcal{V}(k)$.

Let ν be a symmetric finitely supported measure on G and $\lambda_1(\nu,\Omega)$ be the lowest Dirichlet eigenvalue in Ω for the convolution by $\delta_e - \nu$, namely,

$$\lambda_1(\nu, \Omega) = \inf \left\{ \sum_{x,y} |f(xy) - f(x)|^2 \nu(y) : \mathrm{supp}(f) \in \Omega, \sum |f|^2 = 1 \right\}.$$

If (Ω_k, Ω'_k) is a pair satisfying the first condition in Definition 2.3.1 then plugging $f = d(\cdot, \Omega_k^c)$ in the definition of of $\lambda_1(\nu, \Omega_k)$ immediately gives $\lambda_1(\nu, \Omega_k) \leq \frac{C}{k^2}$.

Given a function $\mathcal{V}$ as in Definition 2.3.1, let γ be defined implicitly by

$$\int_{\mathcal{V}(1)}^{\gamma(t)} ([\mathcal{V}^{-1}(s)]^2 \frac{ds}{s} = t. \tag{2.4}$$

This is the same as stating that γ is a solution of the differential equation

$$\frac{\gamma'}{\gamma} = \frac{1}{[\mathcal{V}^{-1} \circ \gamma]^2}, \ \ \gamma(0) = \mathcal{V}(1). \tag{2.5}$$

Following [14], we say that γ is δ-regular if $\gamma'(s)/\gamma(s) \geq \delta\gamma'(t)/\gamma(t)$ for all s, t with $0 < t < s < 2t$.

With this notation, Erschler [14, Proposition 2] gives a modified version of [8, Theorem 4.7] which contains the following statement.

Proposition 2.3.2. *If the group G admits a sequence of Følner couples adapted to the function $\mathcal{V}$ as in* Definition 2.3.1 *and the function γ associated to $\mathcal{V}$ by* (2.4) *is δ-regular for some $\delta > 0$ then there exist $c, C \in (0, \infty)$ such that*

$$\Phi_G(n) \geq \frac{c}{\gamma(Cn)}.$$

A key aspect of this statement is that it allows for very fast growing $\mathcal{V}$ as long as one can check that γ is δ-regular. Erschler [14] gives a variety of examples showing how this works in practice but it seems worth explaining why the δ-regularity of γ is a relatively mild assumption. Suppose first that $\mathcal{V}$ is regularly varying of positive finite index. Then the same is true for $\mathcal{V}^{-1}$ and $\int_{\mathcal{V}(1)}^{T} \mathcal{V}^{-1}(s)^2 \frac{ds}{s} \sim c\mathcal{V}^{-1}(T)^2$. In this case, it follows from (2.5) that $\gamma'(s)/\gamma(s) \simeq 1/s$. If instead we assume that $\log \mathcal{V}$ is of regular variation of positive index (resp. rapid variation) then $\mathcal{V}^{-1} \circ \exp$ is of regular variation of positve index (resp. slow variation) and we can show that

$$\int_{\mathcal{V}(1)}^{T} \mathcal{V}^{-1}(s)^2 \frac{ds}{s} \simeq \mathcal{V}^{-1}(T)^2 \log T.$$

In this case, it follows again that γ is δ-regular. All the examples treated in [14] and in the present chapter fall in these categories.

The following proposition regarding wreath products is key.

Proposition 2.3.3 (Proof of [14, Theorem 2]). *Assume that the group G is infinite, finitely generated, and admits a sequence of Følner couples adapted to the function $\mathcal{V}$ as in* Definition 2.3.1. *Set*

$$\begin{aligned} \Theta_k &= \{(f,x) \in \mathbb{Z}^r \wr G : x \in \Omega_k,\ \ \mathrm{supp}(f) \subset \Omega_k, |f|_\infty \leq k\#\Omega_k\}, \\ \Theta'_k &= \{(f,x) \in \mathbb{Z}^r \wr G : x \in \Omega'_k,\ \ \mathrm{supp}(f) \subset \Omega_k, |f|_\infty \leq k\#\Omega_k - k\}. \end{aligned}$$

Set

$$\mathcal{W}(v) := \exp\left(C\mathcal{V}(v) \log \mathcal{V}(v)\right).$$

Then (Θ_k, Θ'_k) is a sequence of Følner couples on $\mathbb{Z}^r \wr G$ adapted to $\mathcal{W}$ (for an appropriate choice of the constant C).

Proof. By construction (and with an obvious choice of generators in $\mathbb{Z}^r \wr G$ based on a given set of generators for G), the distance between Θ'_k and Θ^c_k in $\mathbb{Z}^r \wr G$ is greater or equal to the minimum of k and the distance between Ω'_k and Ω^c_k in G. Also, we have

$$\#\Theta_k = \#\Omega_k(k\#\Omega_k)^{r\#\Omega_k}, \ \ \#\Theta'_k = \#\Omega'_k(k\#\Omega_k - k)^{r\#\Omega_k}$$

so that

$$\frac{\#\Theta'_k}{\#\Theta_k} \geq \left(1 - (\#\Omega_k)^{-1}\right)^{r\#\Omega_k} \frac{\#\Omega'_k}{\#\Omega_k} \geq \frac{1}{e^r} \frac{\#\Omega'_k}{\#\Omega_k}$$

and

$$\#\Theta_k = \exp\left(\log \#\Omega_k + r\#\Omega_k(\log \#\Omega_k + \log k)\right) \leq \exp\left(C\mathcal{V}(k)\log \mathcal{V}(k)\right).$$

$\square$

Proposition 2.3.4 (Computations)**.** *Let $\mathcal{V}$ be given. Define $\mathcal{W}$ and $\gamma = \gamma_{\mathcal{W}}$ by*

$$\mathcal{W} = \exp(C\mathcal{V}\log\mathcal{V}) \text{ and } \gamma^{-1}(t) = \int_{\mathcal{W}(1)}^{t} [\mathcal{W}^{-1}(s)]^2 \frac{ds}{s}.$$

1. *Assume that $\mathcal{V}(t) \simeq t^D$. Then we have*

$$\gamma(t) \simeq \exp\left(t^{D/(2+D)}[\log t]^{2/(2+D)}\right).$$

2. *Assume that $\mathcal{V}(t) \simeq \exp(t^{\alpha}\ell(t))$, $\alpha > 0$, where $\ell(t)$ is slowly varying with $\ell(t^a) \simeq \ell(t)$ for any fixed $a > 0$. Then γ satisfies*

$$\gamma(t) \simeq \left(t\left(\frac{\ell(\log t)}{\log t}\right)^{2/\alpha}\right).$$

3. *Assume that $\mathcal{V}(t) \simeq \exp(\ell^{-1}(t))$ where $\ell(t)$ is slowly varying with $\ell(t^a) \simeq \ell(t)$ for any fixed $a > 0$. Then γ satisfies*

$$\gamma(t) \simeq \left(t/[\ell(\log t)]^2\right).$$

Note that if $\ell^{-1}(t) = \exp \circ \cdots \circ \exp(t\log t)$ with m exponentials then

$$\ell(t) \simeq \frac{\log_m t}{\log_{m+1}(t)}.$$

Theorem 2.3.5. *Let N be a normal subgroup of $\mathbf{F}_r$. Assume that the group $\Gamma_1(N) = \mathbf{F}_r/N$ admits a sequence of Følner couples adapted to the function $\mathcal{V}$ as in* Definition 2.3.1*. Let $\mathcal{W}$ and $\gamma = \gamma_{\mathcal{W}}$ be related to $\mathcal{V}$ as in* Proposition 2.3.4*. Then we have*

$$\Phi_{\Gamma_2(N)}(n) \geq \frac{c}{\gamma(Cn)}.$$

Proof. By the Magnus embedding, $\Gamma_2(N)$ is a subgroup of $\mathbb{Z}^r \wr \Gamma_1(N)$. By [30, Theorem 1.3], it follows that $\Phi_{\Gamma_2(N)} \geq \Phi_{\mathbb{Z}^r \wr \Gamma_1(N)}$. The conclusion then follows from Propositions 2.3.2–2.3.3. □

Example 2.3.1. Assume $\Gamma_1(N)$ has polynomial volume growth of degree D. Then $\Phi_{\Gamma_2(N)}(n) \geq \exp\left(-cn^{D/(2+D)}[\log n]^{2/(2+D)}\right)$.

Example 2.3.2. Assume $\Gamma_1(N)$ is either polycyclic or equal to the Baumslag-Solitar group $\mathrm{BS}(1,q) = \langle a, b | a^{-1}ba = b^q \rangle$, or equal to the lamplighter group $F \wr \mathbb{Z}$ with F finite. Then $\Phi_{\Gamma_2(N)}(n) \geq \exp\left(-cn/[\log n]^2\right)$.

Example 2.3.3. Assume $\Gamma_1(N) = F \wr \mathbb{Z}^D$ with F finite . Then

$$\Phi_{\Gamma_2(N)}(n) \geq \exp\left(-cn/[\log n]^{2/D}\right).$$

If instead $\Gamma_1(N) = \mathbb{Z}^b \wr \mathbb{Z}^D$ for some integer $b \geq 1$ then

$$\Phi_{\Gamma_2(N)}(n) \geq \exp\left(-cn\left(\frac{\log\log n}{\log n}\right)^{2/D}\right).$$

2.3.3 Another lower bound

The aim of this subsection is to provide lower bounds for the probability of return $\mu^{*n}(e_*)$ on $\Gamma_2(N)$ when μ at (2.2) is the pushforward of a measure $\boldsymbol{\mu}$ on $\mathbf{F}_r$ of the form (2.1), that is, supported on the powers of the generators $\mathbf{s}_i$, $1 \leq i \leq r$, possibly with unbounded support. Our approach is to construct symmetric probability measure ϕ on $\mathbb{Z}^r \wr \Gamma_1(N)$ such that the return probability $\phi^{\star n}(e_\star)$ of the random walk driven by ϕ coincides with $\mu^{*n}(e_*)$. Please note that we will use the notation $\star$ for convolution on the wreath product $\mathbb{Z}^r \wr \Gamma_1(N)$ and $*$ for convolution on $\Gamma_2(N)$. We also decorate the identity element e_* of $\Gamma_2(N)$ with a $*$ to distinguish it from the identity element $e_\star$ of $\mathbb{Z}^r \wr \Gamma_1(N)$. Recall that the identity element of $\Gamma_1(N)$ is denoted by $\bar{e}$. We will use $(\epsilon_i)_1^r$ for the canonical basis of $\mathbb{Z}^r$.

Fix r symmetric probability measures p_i, $1 \le i \le r$ on $\mathbb{Z}$. Recall that, by definition, μ is the pushforward of $\boldsymbol{\mu}$, the probability measure on $\mathbf{F}_r$ which gives probability $r^{-1}p_i(n)$ to $\mathbf{s}_i^n$, $1 \le i \le r$, $n \in \mathbb{Z}$. See (2.1)-(2.2).

On $\mathbb{Z}^r \wr \Gamma_1(N)$, consider the measures ϕ_i supported on elements of the form

$$g = (\delta^i, 0)(0, \overline{s}_i^m)(-\delta^i, 0),$$

where $\delta^i : \mathbf{F}_r/N = \Gamma_1(N) \to \mathbb{Z}^r$ is the function that's identically zero except that at identity e of $\Gamma_1(N)$, $\delta^i(e) = \epsilon_i \in \mathbb{Z}^r$. For such g, set (compare to (2.1))

$$\phi_i(g) = p_i(m).$$

Note that

$$g^{-1} = (\delta^i, 0)(0, \overline{s}_i^{-m})(-\delta^i, 0)$$

is an element of the same form, and $\phi_i(g^{-1}) = \phi_i(g) = p_i(m)$. Set

$$\phi = \frac{1}{r}\sum_{i=1}^r \phi_i.$$

More formally, ϕ can be written as

$$\forall\, g \in \mathbb{Z}^r \wr \Gamma_1(N), \quad \phi(g) = \sum_{1\le i\le r} \frac{1}{r} \sum_{m\in\mathbb{Z}} p_i(m) \mathbf{1}_{\{(\delta^i,0)(0,\overline{s}_i^m)(-\delta^i,0)\}}(g). \tag{2.6}$$

Let $(U_n)_1^\infty$ be a sequence of $\mathbf{F}_r$-valued i.i.d. random variables with distribution $\boldsymbol{\mu}$ and $Z_n = U_1 \cdots U_n$. Note that the projection of U_n to $\mathbf{F}_r/[N,N] = \Gamma_2(N)$ (resp. $\mathbf{F}_r/N = \Gamma_1(N)$) is an i.i.d. sequence of $\Gamma_2(N)$-valued (resp. $\Gamma_1(N)$-valued) random variables with distribution μ (resp. $\overline{\mu}$). Let X_i denote the projection of U_i on $\Gamma_1(N)$and $T_j = X_1 \cdots X_j$. Consider the $\mathbb{Z}^r \wr \Gamma_1(N)$-valued random variable defined by

$$V_n = (\delta^i, 0)(0, \overline{s}_i^m)(-\delta^i, 0) \text{ if } U_n = \mathbf{s}_i^m.$$

Then $(V_n)_1^\infty$ is a sequence of i.i.d. random variables on $\mathbb{Z}^r \wr \Gamma_1(N)$ with distribution ϕ. Write

$$W_n = V_1 ... V_n.$$

Then W_n is the random walk on $\mathbb{Z}^r \wr \Gamma_1(N)$ driven by ϕ.

The following proposition is based on Theorem 2.2.6, that is, [28, Theorem 2.7], which states that two words in $\mathbf{F}_r$ projects to the same element in $\Gamma_2(N)$ if and only if they induce the same flow on $\Gamma_1(N)$. In particular, the random walk on $\Gamma_2(N)$ returns to identity if and only if the path on $\Gamma_1(N)$ induces the zero flow function.

Proposition 2.3.6. *Fix a measure* $\boldsymbol{\mu}$ *on* $\mathbf{F}_r$ *of the form* (2.1)*. Suppose none of the* $\overline{s}_i$ *are torsion elements in* $\Gamma_1(N) = \mathbf{F}_r/N$*. Let* μ *be the probability measure on* $\Gamma_2(N)$ *defined at* (2.2)*. Let* ϕ *be the probability measure on* $\mathbb{Z}^r \wr \Gamma_1(N)$ *defined at* (2.6)*. It holds that*

$$\mu^{*n}(e_*) = \phi^{\star n}(e_\star).$$

Remark 2.3.7*.* It's important here that the probability measure μ is supported on powers of generators, so that each step is taken along one dimensional subgraphs $g\langle \overline{s}_i \rangle$. The statement is not true for arbitrary measure on $\mathbf{F}/N'$.

Proof. The random walk W_n on $\mathbb{Z}^r \wr (\mathbf{F}/N)$ driven by ϕ can be written as

$$W_n = (f_n, T_n) = ((f_n^1, ..., f_n^r), T_n).$$

By definition of W_n, for any $x \in \Gamma_1(N)$, $f_n^i(x)$ counts the algebraic sums of the i-arrivals and i-departures of the random walk T_n at x where by i-arrival (resp. i-departure) at x, we mean a time ℓ at which $T_\ell = x$ and $U_\ell \in \langle \mathbf{s}_i \rangle$ (resp. $U_{\ell+1} \in \langle \mathbf{s}_i \rangle$). The condition $T_n = x \neq \bar{e}$ implies that the vector $f_n(x)$ must have at least one non-zero component because the total number of arrivals and departures at x must be odd. Hence, we have

$$\phi^{\star n}(e_\star) = \mathbf{P}((f_n, T_n) = e_\star) = \mathbf{P}(f_n^i(x) = 0,\, 1 \le i \le r,\, x \in \Gamma_1(N)).$$

We also have

$$\mu^{*n}(e_*) = \mathbf{P}(\mathfrak{f}_{Z_n}(x, x\bar{s}_i, \mathbf{s}_i) = 0,\, 1 \le i \le r,\, x \in \Gamma_1(N)).$$

Given a flow $\mathfrak{f}$ on $\Gamma_1(N)$ (i.e., a function the edge set $\mathfrak{E} = \{(x, x\bar{s}_i, \mathbf{s}_i), x \in \Gamma_1(N), 1 \le i \le r\} \subset \Gamma_1(N) \times \Gamma_1(N) \times S$, for each $i, 1 \le i \le r$, introduce the i-partial total flow $\partial_i \mathfrak{f}(x)$ at $x \in \Gamma_1(N)$ by setting

$$\partial_i \mathfrak{f}(x) = \mathfrak{f}((x, x\bar{s}_i, \mathbf{s}_i)) - \mathfrak{f}(x\bar{s}_i^{-1}, x, \mathbf{s}_i).$$

It is easy to check (e.g., by induction on n) that

$$\forall\, x \in \Gamma_1(N), \quad f_n^i(x) = \partial_i \mathfrak{f}_{Z_n}(x). \tag{2.7}$$

Obviously, $\mathfrak{f}_{Z_n} \equiv 0$ implies $f_n^i \equiv 0$ for all $1 \le i \le r$ so

$$\phi^{\star n}(e_\star) \ge \mu^{*n}(e_*).$$

But, in fact, under the assumption that none of the $\bar{s}_i$ are torsion elements in $\Gamma_1(N)$, each edge $(x, x\bar{s}_i, \mathbf{s}_i)$ in the Cayley graph of $\Gamma_1(N)$ is contained in the one dimensional infinite linear subgraph $\{x\bar{s}_i^k : k \in \mathbb{Z}\}$ and, since f_n and $\mathfrak{f}_{Z_n}$ are finitely supported, the equation (2.7) shows that $f_n^i \equiv 0$ implies that $\mathfrak{f}_{Z_n}(x, x\bar{s}_i, \mathbf{s}_i) = 0$ for all $x \in \Gamma_1(N)$. In particular, if $f_n^i \equiv 0$ for all $1 \le i \le r$ then we must have $\mathfrak{f}_{Z_n} \equiv 0$. Hence, if none of the $\bar{s}_i$ is a torsion element in $\Gamma_1(N)$, we have $f_n^i \equiv 0$, $1 \le i \le r \iff \mathfrak{f}_{Z_n} \equiv 0$ and thus $\mu^{*n}(e_*) = \phi^{\star n}(e_\star)$. $\square$

In general, the probability measure ϕ on $\mathbb{Z}^r \wr \Gamma_1(N)$ does not have generating support because of the very specific and limited nature of the lamp moves and how they correlate to the base moves. To fix this problem, let η_r be the probability measure of the lazy random walk on $\mathbb{Z}^r$ so that $\eta_r(0) = 1/2$ and $\eta_r(\pm\epsilon_i) = 1/(4r)$, $1 \leq i \leq r$. With this notation, let

$$q = \eta_r \star \bar{\mu} \star \eta_r \tag{2.8}$$

be the probability measure of the switch-walk-switch random walk on the wreath product $\mathbb{Z}^r \wr \Gamma_1(N)$ associated with the walk-measure $\bar{\mu}$ on the base-group $\Gamma_1(N)$ and the switch-measure η_r on the lamp-group $\mathbb{Z}^r$. See [29,34] and Section 2.1.5 for further details.

Proposition 2.3.8. *Fix a measure* $\boldsymbol{\mu}$ *on* $\mathbf{F}_r$ *of the form* (2.1)*. Suppose that none of the* $\overline{s}_i$ *are torsion elements in* $\Gamma_1(N) = \mathbf{F}_r/N$*. Refering to the notation introduced above, there are* $c, N \in (0, \infty)$ *such that the probability measure* μ *on* $\Gamma_2(N)$ *defined by* (2.2) *and the measure* q *on* $\mathbb{Z}^r \wr \Gamma_1(N)$ *defined at* (2.8) *satisfy*

$$\mu^{*2n}(e_*) \geq c q^{\star 2Nn}(e_\star).$$

Proof. On a group G, the Dirichlet form associated with a symmetric measure p is defined by

$$\mathcal{E}_p(f, f) = \frac{1}{2} \sum_{x,y \in G} |f(xy) - f(x)|^2 p(y).$$

From the definition, it easily follows that $\mathbb{Z}^r \wr \Gamma_1(N)$, we have the comparison of Dirichlet forms

$$\mathcal{E}_\phi \leq (2r)^2 \mathcal{E}_{\eta_r \star \bar{\mu} \star \eta_r} = (2r)^2 \mathcal{E}_q.$$

Therefore, by [30, Theorem 2.3],

$$\phi^{\star 2n}(e_\star) \geq c q^{\star 2Nn}(e_\star).$$

From Proposition 2.3.6 we conclude that

$$\mu^{*2n}(e_*) = \phi^{\star 2n}(e_\star) \geq c q^{\star 2Nn}(e_\star).$$

□

Corollary 2.3.9. *Fix* $a = (\alpha_1, \dots, \alpha_r) \in (0,2)^r$ *and let* $\boldsymbol{\mu}_a$ *be defined by* (2.1) *with* $p_i(m) = c_i(1+|m|)^{-1-\alpha_i}$. *Let* $N = [\mathbf{F}_r, \mathbf{F}_r]$ *so that* $\Gamma_1(N) = \mathbb{Z}^r$ *and* $\Gamma_2(N) = \mathbf{S}_{2,r}$. *Let* μ_a *be the probability measure on* $\mathbf{S}_{2,r}$ *associated to* $\boldsymbol{\mu}_a$ *by* (2.2). *Then we have*

$$\mu_a^{*n}(e_*) \geq \exp\left(-Cn^{r/(r+\alpha)}[\log n]^{\alpha/(r+\alpha)}\right)$$

where

$$\frac{1}{\alpha} = \frac{1}{r}\left(\frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_r}\right).$$

Remark 2.3.10. Later we will prove a matching upper bound.

Proof. Proposition 2.3.8 yields

$$\mu_a^{*n}(e_*) \geq c[\eta_r \star \bar{\mu}_a \star \eta_r]^{\star n}(e_\star)$$

where the probability $\bar{\mu}_a$ on $\Gamma_1(N) = \mathbb{Z}^r$ is defined at (2.2) and is given explicitly by

$$\bar{\mu}_a(g) = \frac{1}{r}\sum_1^r p_i(m)\mathbf{1}_{\bar{s}_i{}^n}(g)$$

where $\bar{s}_i$ canonical generators of $\mathbb{Z}^r$ and we have retained the multiplicative notation so that $\bar{s}_1^n = (n, 0, \dots, 0), \dots, \bar{s}_r^n = (0, \dots, 0, n)$.

The behavior of the random walk on the wreath product $\mathbb{Z}^r \wr \mathbb{Z}^r$ associated with the switch-walk-switch measure $q = \eta_r \star \bar{\mu}_a \star \eta_r$ is studied in [34] where it is proved that

$$q^{\star 2n}(e_\star) \simeq \exp\left(-n^{r/(r+\alpha)}[\log n]^{\alpha/(r+\alpha)}\right).$$

Corollary 2.3.9 follows. □

2.4 Return probability upper bounds

This section explains how to use the Magnus embedding (defined at (2.1))

$$\bar{\psi} : \mathbf{F}_r/[N,N] = \Gamma_2(N) \hookrightarrow \mathbb{Z}^r \wr \Gamma_1(N), \ \ \Gamma_1(N) = \mathbf{F}_r/N,$$

to produce, in certain cases, an upper bound on the probability of return $\mu^{*2n}(e_*)$ on $\Gamma_2(N)$. Recall from (2.3) that the Magnus embedding ψ is described more concretely by

$$\begin{array}{rcl} \Gamma_2(N) & \hookrightarrow & \mathbb{Z}^r \wr \Gamma_1(N) \\ g & \mapsto & \bar{\psi}(g) = (\bar{a}(g), \overline{g}), \ \ \overline{g} = \bar{\pi}(g). \end{array}$$

Here $\bar{a}(g)$ is an element of $\sum_{x \in \Gamma_1(N)} \mathbb{Z}^r_x$, equivalently, a $\mathbb{Z}^r$-valued function with finite support defined on $\Gamma_1(N)$, equivalently, an element of the $\mathbb{Z}(\Gamma_1(N))$-module $\mathbb{Z}^r(\Gamma_1(N))$. In any group G, we let $\tau_g x = gx$ be the translation by $g \in G$ on the left as well as its extension to any $\mathbb{Z}(G)$ module. We will need the following lemma.

Lemma 2.4.1. *For any $g, h \in \Gamma_2(N)$ with $\bar{g} = \bar{\pi}(g) \in \Gamma_1(N)$, we have*

$$\bar{a}(gh) = \bar{a}(g) + \tau_{\bar{g}} \bar{a}(h).$$

In particular, if $g \in \Gamma_2(N)$ and $\boldsymbol{\rho} \in N$ with $\rho = \pi_2(\boldsymbol{\rho}) \in \Gamma_2(N)$, we have

$$\bar{a}(g \rho g^{-1}) = \tau_{\bar{g}} \bar{a}(\rho).$$

Proof. The first formula follows from the Magnus embedding by inspection. The second formula is an easy consequence of the first and the fact that $\pi(\boldsymbol{\rho})$ is the identity element in $\Gamma_1(N)$. $\square$

Remark 2.4.2. The identities stated in Lemma 2.4.1 can be equivalently written in terms of flows on $\Gamma_1(N)$. Namely, for $\mathbf{u}, \mathbf{v} \in \mathbf{F}_r$, we have

$$\mathfrak{f}_{\mathbf{uv}} = \mathfrak{f}_{\mathbf{u}} + \tau_{\pi(\mathbf{u})}\mathfrak{f}_{\mathbf{v}} \text{ and } \quad \mathfrak{f}_{\mathbf{uvu}^{-1}} = \tau_{\pi(\mathbf{u})}\mathfrak{f}_{\mathbf{v}}.$$

2.4.1 Exclusive pairs

Definition 2.4.3. Let Γ be a subgroup of $\Gamma_2(N)$ and $\boldsymbol{\rho}$ be a reduced word in $N \setminus [N, N] \subset \mathbf{F}_r$. Let $\overline{\Gamma} = \bar{\pi}(\Gamma)$. Set $\rho = \pi_2(\boldsymbol{\rho}) \in \Gamma_2(N)$. We say the pair $(\Gamma, \boldsymbol{\rho})$ is *exclusive* if the following two conditions are satisfied:

(i) The collection $\{\tau_{\overline{g}}(\bar{a}(\rho))\}_{\overline{g}\in\overline{\Gamma}}$ is $\mathbb{Z}$-independent in the $\mathbb{Z}$-module $\sum_{\Gamma_1(N)}(\mathbb{Z}^r)_x$.

(ii) In the $\mathbb{Z}$-module $\sum_{\Gamma_1(N)}(\mathbb{Z}^r)_x$, the $\mathbb{Z}$-submodule generated by $\{\tau_{\overline{g}}(\bar{a}(\rho))\}_{\overline{g}\in\overline{\Gamma}}$, call it $A = A(\Gamma, \boldsymbol{\rho})$, has trivial intersection with the subset $B = \{\bar{a}(g) : g \in \Gamma\}$, that is

$$A \cap B = \{\mathbf{0}\}.$$

Remark 2.4.4. Condition (i) implies that the $\mathbb{Z}$-submodule $A(\Gamma, \boldsymbol{\rho})$ of $\sum_{\Gamma_1(N)}(\mathbb{Z}^r)_x$ is isomorphic to $\sum_{\bar{g}\in\overline{\Gamma}}(\mathbb{Z})_{\bar{g}}$.

Example 2.4.1. In the free metabelian group $S_{2,r} = \mathbf{F}_r/[N, N]$, $N = [\mathbf{F}_r, \mathbf{F}_r]$, set $\Gamma = \langle s_1^2, ..., s_r^2\rangle$, and $\boldsymbol{\rho} = [\mathbf{s}_1, \mathbf{s}_2]$. Then $(\Gamma, \boldsymbol{\rho})$ is an exclusive pair. The conditions (i)–(ii) are easy to check because the monomials $\{Z_1^{x_1}Z_2^{x_2}...Z_r^{x_r} : x \in \mathbb{Z}^d\}$ are $\mathbb{Z}$-linear independent in $\mathbb{Z}(\mathbb{Z}^r)$. A similar idea was used in the proof of [15, Theorem 3.2].

We now formulate a sufficient condition for a pair $(\Gamma, \boldsymbol{\rho})$ to be exclusive. This sufficient condition is phrased in terms of the representation of the elements of

$\Gamma_2(N)$ as flows on $\Gamma_1(N)$. Recall that $\Gamma_1(N)$ come equipped with a marked Cayley graph structure as described in Section 2.2.2.

Lemma 2.4.5. *Fix* $\Gamma < \Gamma_2(N)$ *and* $\boldsymbol{\rho}$ *as in* Definition 2.4.3. *Set*

$$U = \bigcup_{g\in\Gamma} \mathrm{supp}(\mathfrak{f}_g),$$

that is the union of the support of the flows on $\Gamma_1(N)$ *induced by elements of* Γ. *Assume that* $\boldsymbol{\rho} = \mathbf{usv}$ *with* $\mathbf{s} \in \{\mathbf{s}_1, \dots, \mathbf{s}_r\}$ *and that:*

1. $\mathfrak{f}_{\boldsymbol{\rho}}((\overline{u}, \overline{us}, \mathbf{s})) \neq 0$*;*
2. *For all* $x \in \overline{\Gamma}\backslash\{\overline{e}\}$, $\mathfrak{f}_{\boldsymbol{\rho}}((x\overline{u}, x\overline{us}, \mathbf{s})) = 0$*;*
3. *The edge* $(\overline{u}, \overline{us}, \mathbf{s}))$ *is not in* U*.*

Then the pair $(\Gamma, \boldsymbol{\rho})$ *is exclusive.*

Remark 2.4.6. The first assumption insures that the given edge is really active in the loop associated with ρ on the Cayley graph $\Gamma_1(N)$. The proof given below shows that conditions 1-2 above imply condition (i) of Definition 2.4.3. All three assumptions above are used to obtain condition (ii) of Definition 2.4.3.

Proof. Condition (i). Suppose there is a nontrivial linear relation such that

$$c_1\tau_{\overline{g}_1}(\bar{a}(\rho)) + ... + c_n\tau_{\overline{g}_n}(\bar{a}(\rho)) = 0, \quad c_i \in \mathbb{Z},$$

where some c_j, say c_1, is not zero and the element $\overline{g}_j \in \overline{\Gamma}$ are pairwise distinct. Let $\mathbf{g}_j$ be representative of $\bar{g}_j$ in $\mathbf{F}_r$. Let b denote the coefficient of $\sum_{i=1}^n c_i\tau_{\overline{g}_i}(\bar{a}(\rho))$ in front of the term $\overline{g}_1\overline{u}\lambda_{\mathbf{s}}$. By formula (2.5),

$$b = \sum_{i=1}^{n} c_i \mathfrak{f}_{\mathbf{g}_i\boldsymbol{\rho}\mathbf{g}_i^{-1}}((\overline{g}_1\overline{u}, \overline{g}_1\overline{us}, \mathbf{s})).$$

Note that

$$\mathfrak{f}_{\mathbf{g}_i\boldsymbol{\rho}\mathbf{g}_i^{-1}}((\overline{g}_1\overline{u}, \overline{g}_1\overline{us}, \mathbf{s})) = \mathfrak{f}_{\boldsymbol{\rho}}((\overline{g}_i^{-1}\overline{g}_1\overline{u}, \overline{g}_i^{-1}\overline{g}_1\overline{us}, \mathbf{s})).$$

Therefore, since $\overline{g}_i^{-1}\overline{g}_1 \neq \overline{e}$ for all $i \neq 1$, the hypothesis stated in Lemma 2.4.5(2) gives

$$\forall\, i \neq 1, \;\; \mathfrak{f}_{\mathbf{g}_i\boldsymbol{\rho}\mathbf{g}_i^{-1}}((\overline{g}_1\overline{u}, \overline{g}_1\overline{us}, \mathbf{s})) = 0.$$

By hypothesis (1) of Lemma 2.4.5, this implies

$$b = c_1\mathfrak{f}_{\boldsymbol{\rho}}((\overline{u}, \overline{us}, \mathbf{s})) \neq 0$$

which provides a contradiction.

We now verify that Condition (ii) of Definition 2.4.3 holds. Fix $x \in A \cap B$ and assume that x is nontrivial. From Condition (i), x can be written uniquely as

$$x = c_1\tau_{\overline{g}_1}a(\rho) + ... + c_n\tau_{\overline{g}_n}a(\rho),$$

where $c_j \in \mathbb{Z}\backslash\{0\}$ and the elements $\overline{g}_j$ are pairwise distinct. On the other hand, since $x \in B$, there exists some $h \in \Gamma$ such that $x = \bar{a}(h)$. By formula (2.5), $\bar{a}(h) = \sum_{i=1}^{n} c_i\tau_{\overline{g}_i}a(\rho)$ is equivalent to

$$\mathfrak{f}_h = \sum_{i=1}^{n} c_i\mathfrak{f}_{g_i\rho g_i^{-1}}.$$

Therefore $\mathfrak{f}_{g_1^{-1}hg_1} = \sum_{i=1}^{n} c_i\mathfrak{f}_{g_1^{-1}g_i\rho g_i^{-1}g_1}$. By hypothesis (2), it follows that

$$\mathfrak{f}_{g_1^{-1}hg_1}((\overline{u}, \overline{us}, \mathbf{s})) = c_1\mathfrak{f}_{\rho}((\overline{u}, \overline{us}, \mathbf{s})) \neq 0.$$

However, since $g_1^{-1}hg_1 \in \Gamma$, this implies that $(\overline{u}, \overline{us}, \mathbf{s}_i)) \in U$, a conclusion which contradicts assumption (3). Hence $A \cap B = \{\mathbf{0}\}$ as desired.

$\square$

2.4.2 Existence of exclusive pairs

This section discuss algebraic conditions that allow us to produce appropriate exclusive pairs.

Lemma 2.4.7. *Assume* $\Gamma_1(N) = \mathbf{F}_r/N$ *is residually finite and* $r \geq 2$. *Fix an element* $\boldsymbol{\rho}$ *in* $N \setminus [N, N]$. *There exists a finite index normal subgroup* $K = K_{\boldsymbol{\rho}} \lhd \Gamma_1(N)$ *such that, for any edge* $(\mathbf{u}, \mathbf{us})$ *in* $\boldsymbol{\rho} = \mathbf{usv}$ *with* $\mathbf{s} \in \{\mathbf{s}_1, \ldots, \mathbf{s}_r\}$ *and any subgroup* $H < \Gamma_2(N)$ *with* $\overline{\pi}(H) < K$,

$$\forall\, x \in \overline{\pi}(H) \setminus \{\overline{e}\}, \;\; \mathfrak{f}_{\boldsymbol{\rho}}((x\overline{u}, x\overline{us}, \mathbf{s})) = 0.$$

Remark 2.4.8. Since $\boldsymbol{\rho} \notin [N, N]$, the flow induced by $\boldsymbol{\rho}$ is not identically zero. Therefore, after changing $\boldsymbol{\rho}$ to $\boldsymbol{\rho}^{-1}$ if necessary, there exists a reduced word $\mathbf{u}$ and $i \in \{1, \ldots, r\}$ such that $\boldsymbol{\rho} = \mathbf{us}_i\mathbf{u}'$ and $\mathfrak{f}_{\boldsymbol{\rho}}((\overline{u}, \overline{us}_i, \mathbf{s}_i)) \neq 0$. Hence Lemma 2.4.7 provides a way to verify conditions 1 and 2 of Lemma 2.4.5.

Proof. For any element $\boldsymbol{\rho}$ in $N \setminus [N, N]$, view $\boldsymbol{\rho}$ as a reduced word in $\mathbf{F}_r$. Let $B_{\boldsymbol{\rho}}$ be the collection of all proper subword $\mathbf{u}$ of $\boldsymbol{\rho}$ such that $\overline{\pi}(\mathbf{u})$ is not trivial in $\Gamma_1(N)$. Since $\Gamma_1(N)$ is residually finite, there exists a normal subgroup $K \lhd \Gamma_1(N)$ such that $\Gamma_1(N)/K$ is finite and $\overline{\pi}(B_\rho) \cap K = \emptyset$.

Suppose there exists $x \in \overline{\pi}(H)$ such that x is not trivial and $\mathfrak{f}_\rho((x\overline{u}, x\overline{us}, \mathbf{s})) \neq 0$. Therefore, there is a proper subword $\mathbf{v}$ of $\boldsymbol{\rho}$ such that $\boldsymbol{\rho} = \mathbf{vw}$ and $\overline{v} = x\overline{u}$. Since both $\mathbf{u}$ and $\mathbf{v}$ are prefixes of $\boldsymbol{\rho}$ and x is not trivial, $\mathbf{vu}^{-1}$ is the conjugate of a proper subword of $\boldsymbol{\rho}$ with non-trivial image in $\Gamma_1(N)$. By construction this implies that $\overline{\pi}(\mathbf{vu}^{-1}) \notin K$, a contradiction since $\overline{\pi}(\mathbf{vu}^{-1}) = x \in \overline{\pi}(H) < K$. □

Remark 2.4.9. By a classical result of Hirsch, polycyclic groups are residually finite, [32, 5.4.17]. By a result of P. Hall, a finitely generated group which is an extension of an abelian group by a nilpotent group is residually finite. In particular, all finitely generated metabelian groups are residually finite, [32, 15.4.1]. Gruenberg, [17], proves that free polynilpotent groups are residually finite. The free solvable groups $\mathbf{S}_{d,r}$ are examples of free polynilpotent groups. Note that all finitely generated residually finite groups are Hopfian, [32, 6.1.11].

Our next task is to find ways to verify condition 3 of Lemma 2.4.5. To this end, let A be the abelian group $\Gamma_1(N)/[\Gamma_1(N),\Gamma_1(N)]$. Fix $m=(m_1,\dots,m_r)\in\mathbb{N}^r$ and let A_m be the subgroup of A generated by the images of the elements $\bar{s}_i^{m_i}$, $1\le i\le r$, in A. Let T_m be the finite abelian group $T_m=A/A_m$. Let $\pi_{T_m}:\mathbf{F}_r\to T_m$ be the projection from $\mathbf{F}_r$ onto T_m. Set

$$H_m=\langle s_i^{m_i},1\le i\le r\rangle<\Gamma_2(N).$$

Lemma 2.4.10. *Fix a reduced word $\boldsymbol{\rho}\in N\backslash[N,N]$. Assume that $\boldsymbol{\rho}=\mathbf{usv}$, where $\mathbf{s}\in\{\mathbf{s}_1,\dots,\mathbf{s}_r\}$ and $\mathfrak{f}_\rho((\overline{u},\overline{us},\mathbf{s}))\neq 0$. Fix $m\in\mathbb{N}^r$ and assume that, in the finite abelian group T_m, $\pi_{T_m}(\mathbf{u})\notin\langle\pi_{T_m}(\mathbf{s})\rangle$. Then the edge $(\overline{u},\overline{us},\mathbf{s})$ is not in*

$$U(H_m)=\bigcup_{g\in H_m}\operatorname{supp}(\mathfrak{f}_g).$$

Proof. Assume that $(\overline{u},\overline{us},\mathbf{s})\in U(H_m)$. Then there must exist $x\in\pi(H_m)$ and $q\in\mathbb{Z}$ such that $x\bar{s}^q=\overline{u}$. But, projecting on T_m, this contradicts the assumption $\pi_{T_m}(\mathbf{u})\notin\langle\pi_{T_m}(\mathbf{s})\rangle$. □

We now put together these two lemmas and state a proposition that will allow us to produce exclusive pairs.

Proposition 2.4.11. *Fix $N\lhd\mathbf{F}_r$ and $\boldsymbol{\rho}\in N\setminus[N,N]$, in reduced form. Let $\mathbf{u}$ be a prefix of $\boldsymbol{\rho}$ such that $\boldsymbol{\rho}=\mathbf{usv}$, $\mathbf{s}\in\{\mathbf{s}_1,\dots,\mathbf{s}_r\}$ and $\mathfrak{f}_{\boldsymbol{\rho}}((\bar{u},\bar{u}\bar{s},\mathbf{s}))\neq 0$. Assume that the group $\Gamma_1(N)$ is residually finite and there exists an integer vector*

$m = (m_1, \ldots, m_r) \in \mathbb{N}^r$ such that, in the finite group T_m, $\pi_{T_m}(\mathbf{u}) \notin \langle \pi_{T_m}(\mathbf{s}) \rangle$. Then there is $m' = (m'_1, \ldots, m'_r)$ such that the pair $(H_{m'}, \boldsymbol{\rho})$ is an exclusive pair in $\Gamma_2(N)$.

Proof. Let $K_{\boldsymbol{\rho}}$ the the finite index normal subgroup of $\Gamma_1(N)$ given by Lemma 2.4.7. Since $K_{\boldsymbol{\rho}}$ is of finite index in $\Gamma_1(N)$, we can pick m'_i to be a multiple of m_i such that $\bar{s}_i^{m'_i} \in K_{\boldsymbol{\rho}}$. Observe that the assumption $\pi_{T_m}(\mathbf{u}) \notin \langle \pi_{T_m}(\mathbf{s}) \rangle$ implies the same property with m replaced by m'. Applying Lemmas 2.4.7, 2.4.10 and Lemma 2.4.5 yields that $(H_{m'}, \boldsymbol{\rho})$ is an exclusive pair in $\Gamma_2(N)$. □

We conclude this section with a concrete application of Proposition 2.4.11.

Proposition 2.4.12. *Assume that $\Gamma_1(N) = \mathbf{F}_r/N$ is an infinite nilpotent group and $r \geq 2$. Then there exists an exclusive pair (Γ, ρ) in $\Gamma_2(N)$ such that $\pi(\Gamma)$ is a subgroup of finite index in $\Gamma_1(N)$.*

Proof. First we construct an exclusive pair using Proposition 2.4.11. Suppose that $\Gamma_1(N)$ is not virtually $\mathbb{Z}$. Then the torsion-free rank of $\Gamma_1(N)/[\Gamma_1(N), \Gamma_1(N)]$ is at least 2. Choose two generators s_{i_1}, s_{i_2} such that their projections in the abelianization are $\mathbb{Z}$-independent. Choose $\boldsymbol{\rho}$ to be an element of minimal length in $N \cap \langle \mathbf{s}_{i_1}, \mathbf{s}_{i_2} \rangle$. Note that since $\Gamma_1(N)$ is nilpotent, this intersection contains commutators of $\mathbf{s}_{i_1}, \mathbf{s}_{i_2}$ with length greater than the nilpotency class, therefore it is non-empty. Proposition 2.4.11 applies and yields an integer m such that $(\Gamma = \langle s_1^m, ..., s_r^m \rangle, \boldsymbol{\rho})$ is an exclusive pair.

In the special case when $\Gamma_1(N)$ is virtually $\mathbb{Z}$, choose $\boldsymbol{\rho}$ to be an element of minimal length in N, and a generator $\mathbf{s}_{i_1}$ such that $\bar{s}_{i_1}$ is not a torsion element in

$\Gamma_1(N)$. Set $\Gamma = \langle s_{i_1}^m \rangle$ with $m = [\Gamma_1(N) : K_{\boldsymbol{\rho}}]$. Then by Lemmas 2.4.5, Lemma 2.4.7 and inspection, $(\Gamma, \boldsymbol{\rho})$ is an exclusive pair.

Next we use induction on nilpotency class c to show that, for any $m \in \mathbb{N}$, $\pi(\Gamma) = \langle \overline{s}_1^m, ..., \overline{s}_r^m \rangle$ is a subgroup of finite index in $\Gamma_1(N)$. When $c = 1$, observe that the statement is obviously true for finitely generated abelian groups. Suppose $\Gamma_1(N)$ is of nilpotency class c. Let $H = \gamma_c(\Gamma_1(N))$. Using the induction hypothesis, it suffices to prove that $H \cap \pi(\Gamma)$ is a finite index subgroup of H. Note that H is contained in the center of $\Gamma_1 1(N)$ and is generated by commutators of length c. Further,

$$[s_{i_c}[...[s_{i_2}, s_{i_1}]]]^{m^c} = [s_{i_c}^m[...[s_{i_2}^m, s_{i_1}^m]]].$$

Therefore $H/H \cap \pi(\Gamma)$ is a finitely generated torsion abelian group, hence finite, as desired. □

2.4.3 Random walks associated with exclusive pairs

The following result captures the main idea and construction of this section.

Theorem 2.4.13. *Let μ be a symmetric probability measure on $\Gamma_2(N)$. Let $\Gamma < \Gamma_2(N)$ and $\boldsymbol{\rho}$ be an exclusive pair as in* Definition 2.4.3. *Set $\rho = \pi_2(\boldsymbol{\rho}) \in \Gamma_2(N)$. Let ν be the probability measure on $\Gamma_2(N)$ such that*

$$\nu(\rho^{\pm 1}) = 1/2.$$

Let φ be a symmetric probability measure on Γ such that

$$\mathcal{E}_{\nu * \varphi * \nu} \leq C_0 \mathcal{E}_\mu. \tag{2.1}$$

Let $\bar{\varphi}$ be the symmetric probability on $\overline{\Gamma} = \bar{\pi}(\Gamma) < \Gamma_1(N)$ defined by

$$\forall\, \bar{g} \in \Gamma_1(N), \;\; \bar{\varphi}(\bar{g}) = \varphi(\bar{\pi}^{-1}(\bar{g})).$$

On the wreath product $\mathbb{Z} \wr \overline{\Gamma}$ (whose group law will be denoted here by $\star$), consider the switch-walk-switch measure $q = \eta \star \bar{\varphi} \star \eta$ with $\eta(\pm 1) = 1/2$ on $\mathbb{Z}$. Then there are constants $C, k \in (0, \infty)$ such that

$$\mu^{*2kn}(e_*) \leq C q^{\star 2n}(e_\star).$$

Proof. By [30, Theorem 2.3], the comparison assumption between the Dirichlet forms of μ and $\nu * \varphi * \nu$ implies that there is a constant C and an integer k such that

$$\forall\, n, \;\; \mu^{*kn}(e_*) \leq C[\nu * \varphi * \nu]^{*2n}(e_*).$$

Hence, the desired conclusion easily follows from the next proposition. □

Proposition 2.4.14. *Let $\Gamma < \Gamma_2(N)$ and $\boldsymbol{\rho}$ be an exclusive pair as in* Definition 2.4.3. *Let $\rho = \pi(\boldsymbol{\rho})$ and let ν be the probability measure on $\Gamma_2(N)$ such that $\nu(\rho) = \nu(\rho^{-1}) = \frac{1}{2}$. Let φ be a probability measure supported on Γ. Let $\bar{\varphi}$ be the pushforward of φ on $\bar{\pi}(\Gamma) = \overline{\Gamma} < \Gamma_1(N)$. Let η be the probability measure on $\mathbb{Z}$ such that $\eta(\pm 1) = 1/2$. Let $q = \eta \star \bar{\varphi} \star \eta$ be the switch-walk-switch measure on $\mathbb{Z} \wr \overline{\Gamma}$. Then*

$$(\nu * \varphi * \nu)^{*n}(e_*) \leq (\eta \star \overline{\varphi}' \star \eta)^{\star n}(e_\star) = q^{\star n}(e_\star).$$

To prove this proposition, we will use the following lemma.

Lemma 2.4.15. *Let φ be a probability measure on $\Gamma_2(N)$. Let ν be the uniform measure on $\{r_0^{\pm 1}\}$ where $r_0 \in \Gamma_2$ and $r_0 \neq r_0^{-1}$. Let $(Y_i)_1^\infty$ and $(\varepsilon_i)_1^\infty$ be i.i.d. sequence with law φ and ν respectively. Let $S_n = Y_1 \cdots Y_n$ and $\overline{S}_n = \bar{\pi}(S_n)$. Then we have*

$$\begin{aligned}&(\nu * \varphi * \nu)^{*n}(e_*)\\ &\qquad = \mathbf{P}\left(\overline{S}_n = \bar{e}, \sum_{j=1}^{n} (\varepsilon_{2j-1} + \varepsilon_{2j}) \tau_{\overline{S}_j} \bar{a}(r_0) + \bar{a}(S_n) = 0\right).\end{aligned}$$

Proof. The product $r_0^{\varepsilon_1} Y_1 r_0^{\varepsilon_2+\varepsilon_3} Y_2 r_0^{\varepsilon_4} \cdots r_0^{\varepsilon_{2n-1}} Y_n r_0^{\varepsilon_{2n}}$ has distribution

$$(\nu * \varphi * \nu)^{*n}.$$

Therefore, we have

$$\begin{aligned}
(\nu * \mu * \nu)^{*n}(e_*) &= \mathbf{P}(r_0^{\varepsilon_1} Y_1 r_0^{\varepsilon_2+\varepsilon_3} Y_2 \cdots Y_n r_0^{\varepsilon_n} = e_*) \\
&= \mathbf{P}(Y_1 r_0^{\varepsilon_2+\varepsilon_3} Y_2 \cdots Y_n r_0^{\varepsilon_{2n}+\varepsilon_1} = e_*).
\end{aligned}$$

Using the Magnus embedding

$$\psi : \mathbf{F}/[N,N] \hookrightarrow \mathbb{Z}^r \wr \Gamma_1(N)$$

(and re-indexing of the ε_i) this yields

$$(\nu * \mu * \nu)^{*n}(e_*) = \mathbf{P}(\overline{S}_n = \bar{e}, \bar{a}(Y_1 r_0^{\varepsilon_1+\varepsilon_2} Y_2 \cdots Y_n r_0^{\varepsilon_{2n-1}+\varepsilon_{2n}}) = 0).$$

However, we have

$$\begin{aligned}
&\bar{a}(Y_1 r_0^{\varepsilon_1+\varepsilon_2} Y_2 \cdots Y_n r_0^{\varepsilon_{2n-1}+\varepsilon_{2n}}) \\
&\qquad = \bar{a}(S_1 r_0^{\varepsilon_1+\varepsilon_2} S_1^{-1} \cdots S_n r_0^{\varepsilon_{2n-1}+\varepsilon_{2n}} S_n^{-1} S_n) \\
&\qquad = \sum_{j=1}^{n} (\varepsilon_{2j-1} + \varepsilon_{2j}) \bar{a}(S_j r_0 S_j^{-1}) + \bar{a}(S_n) \\
&\qquad = \sum_{j=1}^{n} (\varepsilon_{2j-1} + \varepsilon_{2j}) \tau_{\overline{S}_j} \bar{a}(r_0) + \bar{a}(S_n).
\end{aligned}$$

The last equality above from Lemma 2.4.1. □

Proof of Proposition 2.4.14. By Lemma 2.4.15,

$$\begin{aligned}
&(\nu * \mu * \nu)^{*n}(e_*) \\
&\qquad = \mathbf{P}\left(\overline{S}_n = \bar{e}, \sum_{j=1}^{n} (\varepsilon_{2j-1} + \varepsilon_{2j}) \tau_{\overline{S}_j} \bar{a}(r_0)) + \bar{a}(S_n) = 0\right).
\end{aligned}$$

Under the assumption that (Γ, ρ) is an exclusive pair, (ii) of Definition 2.4.3 gives

$$\left\{\sum_{j=1}^{n}(\varepsilon_{2j-1}+\varepsilon_{2j})\tau_{\overline{S}_j}\bar{a}(r_0)+\bar{a}(S_n)=0\right\} \\ =\left\{\sum_{j=1}^{n}(\varepsilon_{2j-1}+\varepsilon_{2j})\tau_{\overline{S}_j}\bar{a}(r_0)=0\right\}\cap\{\bar{a}(S_n)=0\} \tag{2.2}$$

Further, (i) of Definition 2.4.3 gives

$$\left\{\sum_{j=1}^{n}(\varepsilon_{2j-1}+\varepsilon_{2j})\tau_{\overline{S}_j}\bar{a}(r_0)=0\right\} \\ = \bigcap_{x\in\overline{\Gamma}}\left\{\sum_{j=1}^{n}(\varepsilon_{2j-1}+\varepsilon_{2j})\mathbf{1}_{\{x\}}(\overline{S}_j)=0\right\}.$$

Therefore, dropping $\{\bar{a}(S_n)=0\}$ in (2.2) yields

$$(\nu*\varphi*\nu)^{*n}(e_*) \\ \leq \mathbf{P}\left(\overline{S}_n=\bar{e},\sum_{j=1}^{n}(\varepsilon_{2j-1}+\varepsilon_{2j})\mathbf{1}_{\{x\}}(\overline{S}_j)=0 \text{ for all } x\in\overline{\Gamma}\right).$$

On the other hand, the return probability of the random walk on

$$\mathbb{Z}\wr\overline{\Gamma}<\mathbb{Z}\wr\Gamma_1(N)$$

driven by $\eta\star\overline{\varphi}'\star\eta$ is exactly

$$(\eta\star\overline{\varphi}\star\eta)^{\star n}(e_\star) \\ = \mathbf{P}\left(\overline{S}_n=\bar{e},\sum_{j=1}^{n}(\varepsilon_{2j-1}+\varepsilon_{2j})\mathbf{1}_{\{x\}}(\overline{S}_j)=0 \text{ for all } x\in\overline{\Gamma}\right).$$

□

2.5 Examples of two sided bounds on $\Phi_{\Gamma_2(N)}$

2.5.1 The case of nilpotent groups

Our first application of the techniques developed above yields the following Theorem.

Theorem 2.5.1. *Assume that $\Gamma_1(N) = \mathbf{F}_r/N$ is an infinite nilpotent group and $r \geq 2$. Let D be the degree of polynomial volume growth of $\Gamma_1(N)$. Then*

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n^{D/(2+D)}[\log n]^{2/(2+D)}\right).$$

Proof. Example 2.3.1 provides the desired lower bound. By Proposition 2.4.12, we have an exclusive pair $(\Gamma, \boldsymbol{\rho})$ in $\Gamma_2(N)$ such that $\overline{\Gamma} = \bar{\pi}(\Gamma)$ is of finite index in $\Gamma_1(N)$. Applying Theorem 2.4.13 gives

$$\Phi_{\Gamma_2(N)}(kn) \leq C\Phi_{\mathbb{Z}\wr\overline{\Gamma}}(n).$$

Since $\overline{\Gamma}$ has finite index in $\Gamma_1(N)$, it has the same volume growth degree D and, by [14, Theorem 2],

$$\Phi_{\mathbb{Z}\wr\overline{\Gamma}}(n) \leq \exp\left(-cn^{D/(2+D)}[\log n]^{2/(2+D)}\right).$$

$\square$

2.5.2 Application to the free metabelian groups

This section is devoted to the free metabelian group $\mathbf{S}_{2,r} = \mathbf{F}/[N, N]$, $N = [\mathbf{F}_r, \mathbf{F}_r]$.

Theorem 2.5.2. *The free metabelian group* $\mathbf{S}_{2,r}$ *satisfies*

$$\Phi_{\mathbf{S}_{2,r}}(n) \simeq \exp\left(-n^{r/(2+r)}[\log n]^{2/(2+r)}\right) \tag{2.1}$$

and, for any $\alpha \in (0,2)$,

$$\widetilde{\Phi}_{\mathbf{S}_{2,r},\rho_\alpha}(n) \simeq \exp\left(-n^{r/(\alpha+r)}[\log n]^{\alpha/(\alpha+r)}\right). \tag{2.2}$$

Further, for $a = (\alpha_1, \ldots, \alpha_r) \in (0,2)^r$, *let* $\boldsymbol{\mu}_a$ *be defined by* (2.1) *with* $p_i(m) = c_i(1+|m|)^{-1-\alpha_i}$. *Let* μ_a *be the probability measure on* $\mathbf{S}_{2,r}$ *associated to* $\boldsymbol{\mu}_a$ *by* (2.2). *Then we have*

$$\mu_a^n(e) \simeq \exp\left(-n^{r/(r+\alpha)}[\log n]^{\alpha/(r+\alpha)}\right) \tag{2.3}$$

where

$$\frac{1}{\alpha} = \frac{1}{r}\left(\frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_r}\right).$$

Proof. The lower bound in (2.1) follows from Theorem 2.3.5 (in particular, Example 2.3.1). The lower bound in (2.2) then follows from [5, Theorem 3.3]. The lower bound in (2.3) is Corollary 2.3.9. If we consider the measure μ_a with $a = (\alpha, \alpha, \ldots, \alpha)$, $\alpha \in (0,2)$, it is easy to check that this measure satisfies

$$\sup_{s>0}\{s\mu_a(g : (1+|g|)^\alpha > s)\} < \infty,$$

that is, has finite weak ρ_α-moment with $\rho_\alpha(g) = (1+|g|)^\alpha$. This implies that $\mu_a^{(2n)}(e)$ provides an upper bound for $\widetilde{\Phi}_{\mathbf{S}_{2,r},\rho_\alpha}(n)$. See the definition of $\widetilde{\Phi}_{G,\rho}$ in Section 2.1.4 and [5]. The upper bound in (2.2) is thus a consequence of the upper bound in (2.3).

We are left with proving the upper bounds contained in (2.1)-(2.3). The proofs follow the same line of reasoning and we focus on the upper bound (2.3).

Lemma 2.5.3. *Set* $\Gamma = \langle s_1^2, ..., s_r^2 \rangle < \mathbf{S}_{2,r}$ *and* $\boldsymbol{\rho} = [\mathbf{s}_1, \mathbf{s}_2] \in \mathbf{F}_r$. *The pair* $(\Gamma, \boldsymbol{\rho})$ *is an exclusive pair in the sense of* Definition 2.4.3

Proof. This was already observed in Example 2.4.1. □

In order to apply Proposition 2.4.14 to the pair $(\Gamma, \boldsymbol{\rho})$, we now define an appropriate measure φ on the subgroup $\Gamma = \langle s_1^2, \dots, s_r^2 \rangle$ of $\mathbf{S}_{2,r} = \mathbf{F}_r/[N, N] = \Gamma_2(N)$, $N = [\mathbf{F}_r, \mathbf{F}_r]$. In this context, $\overline{\Gamma} = (2\mathbb{Z})^r \subset \mathbb{Z}^r = \Gamma_1(N)$. The measure φ is simply given by

$$\varphi(g) = \sum_{i=1}^{r} \frac{1}{r} \sum_{m \in \mathbb{Z}} c_i (1 + |m|)^{-1-\alpha_i} \mathbf{1}_{\{s_i^{2m}\}}(g).$$

With this definition, it is clear that, on $\mathbf{S}_{2,r}$, we have the Dirichlet form comparison

$$\mathcal{E}_{\mu_a} \geq c\mathcal{E}_{\nu * \varphi * \nu}.$$

Then by Proposition 2.4.14,

$$\mu_a^{*n}(e_*) \ \preceq (\eta \star \overline{\varphi} \star \eta)^{\star n}(e_\star).$$

Here as in the previous section, $*$ denotes convolution in $\Gamma_2(N)$ and $\star$ denotes convolution on $\mathbb{Z} \wr \overline{\Gamma}$ (or $\mathbb{Z} \wr \Gamma_1(N)$). Here, $\overline{\Gamma} = (2\mathbb{Z})^r$ which is a subgroup of (but also isomorphic to) $\Gamma_1(N) = \mathbb{Z}^r$. The switch-walk-switch measure $q = \eta \star \overline{\varphi} \star \eta$ on $\mathbb{Z} \wr (2\mathbb{Z})^r$ has been studied by the authors in [34] where it is proved that

$$q^{*n}(e) \leq \exp\left(-cn^{\frac{r}{r+\alpha}} (\log n)^{\frac{\alpha}{r+\alpha}}\right).$$

The proof of this result given in [34] is based on an extension of the Donsker-Varadhan Theorem regarding the Laplace transform of the number of visited points. This extension treats random walks on $\mathbb{Z}^r$ driven by measures that are in the domain of normal attraction of an operator stable law. See [34, Theorem 1.3]. □

2.5.3 Miscellaneous applications

This section describes further examples of the results of Sections 2.4.1–2.4.3. Namely, we consider a number of examples consisting of a group $G = \Gamma_1(N) = \mathbf{F}_r/N$ given by an explicit presentation. We identify an exclusive pair (Γ, ρ) with the property that the subgroup $\overline{\Gamma}$ of $\Gamma_1(N)$ is either isomorphic to $\Gamma_1(N)$ or has a similar structure so that $\Phi_{\mathbb{Z}^d \wr \Gamma_1(N)} \simeq \Phi_{\mathbb{Z} \wr \overline{\Gamma}}$. In each of these examples, the results of Sections 2.3.2-2.3.3 and those of Section 2.4.1-2.4.3 provide matching lower and upper bounds for $\Phi_{\Gamma_2(N)}$ where $\Gamma_2(N) = \mathbf{F}_2/[N, N]$.

Example 2.5.1 (The lamplighter $\mathbb{Z}_2 \wr \mathbb{Z} = \langle a, t \,|\, a^2, [a, t^{-n} a t^n], n \in \mathbb{Z} \rangle$)**.** In the lamplighter description of $\mathbb{Z}_2 \wr \mathbb{Z}$, multiplying by t on the right produces a translation of the lamplighter by one unit. Multiplying by a on the right switch the light at the current position of the lamplighter. Let Γ be the subgroup of Γ_2 generated by the images of a and t^2 and note that $\overline{\Gamma}$ is, in fact, isomorphic to $\Gamma_1(N)$. Let $\boldsymbol{\rho} = [a, t^{-1}at] = a^{-1}t^{-1}a^{-1}tat^{-1}at$. In order to apply Lemma 2.4.5, set $\mathbf{u} = a^{-1}t^{-1}a^{-1}tat^{-1}$, $\mathbf{s} = a$ and $\mathbf{v} = t$ so that $\boldsymbol{\rho} = \mathbf{usv}$. By inspection, $\mathfrak{f}_{\boldsymbol{\rho}}((\bar{u}, \overline{us}, \mathbf{s})) \neq 0$ (condition (1) of Lemma 2.4.5). Also, because the elements of $\overline{\Gamma}$ can only have lamps on and the lamplighter at even positions, one checks that $\mathfrak{f}_{\boldsymbol{\rho}}((x\bar{u}, x\overline{us}, \mathbf{s})) = 0$ if $x \in \overline{\Gamma}$ (condition (2) of Lemma 2.4.5). For the same reason, it is clear that $\mathfrak{f}_x((\bar{u}, \overline{us}, \mathbf{s})) = 0$ if $x \in \overline{\Gamma}$, that is, $(\bar{u}, \overline{us}, \mathbf{s}) \notin U$ (condition (3) of Lemma 2.4.5). By the Magnus embedding and [30, Theorem 1.3], we have

$$\Phi_{\Gamma_2(N)}(n) \geq c\Phi_{\mathbb{Z}^r \wr \Gamma_1(N)}(kn).$$

Applying Lemma 2.4.5, Proposition 2.4.14, and the fact that $\overline{\Gamma} \simeq \Gamma_1(N)$, yields

$$\Phi_{\Gamma_2(N)}(kn) \leq C\Phi_{\mathbb{Z} \wr \Gamma_1(N)}(n).$$

The results of [14] gives

$$\Phi_{\mathbb{Z}^r \wr \Gamma_1(N)}(n) \simeq \Phi_{\mathbb{Z}\wr\Gamma_1(N)}(n) \simeq \exp\left(-n/[\log n]^2\right).$$

Hence we conclude that, in the present case where

$$\Gamma_1(N) = \mathbb{Z}_2 \wr \mathbb{Z} = \langle a, t \,|\, a^2, [a, t^{-n}at^n], n \in \mathbb{Z}\rangle,$$

we have

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n/[\log n]^2\right).$$

This extend immediately to $\mathbb{Z}_q \wr \mathbb{Z} = \langle a, t \,|\, a^q, [a, t^{-n}at^n], n \in \mathbb{Z}\rangle$. It also extend to similar presentations of $F \wr \mathbb{Z}$ with F finite. See the next class of examples.

Example 2.5.2 (Examples of the type $K \wr \mathbb{Z}^d$)**.** Let $K = \langle \mathbf{k}_1, \ldots, \mathbf{k}_m \,|\, N_K\rangle$ be a m generated group. The wreath product $K \wr \mathbb{Z}^d$ admits the presentation $\mathbf{F}_r/N$ with $r = m + d$ generators denoted by

$$\mathbf{k}_1, \ldots, \mathbf{k}_m, \mathbf{t}_1, \ldots, \mathbf{t}_d$$

and relations $[\mathbf{t}_i, \mathbf{t}_j], 1 \le i, j \le d$, N_K and

$$[\mathbf{k}', \mathbf{t}^{-1}\mathbf{k}\mathbf{t}], \mathbf{k}, \mathbf{k}' \in \mathbf{F}(\mathbf{k}_1, \ldots, \mathbf{k}_m), \mathbf{t} = \mathbf{t}_1^{x_1} \cdots \mathbf{t}_d^{x_d},\ (x_1, \ldots, x_d) \neq 0.$$

Without loss of generality, we can assume that the image of $\mathbf{k}_1$ in K is not trivial. Let Γ be the subgroup of $\Gamma_2(N)$ generated by the images of $\mathbf{t}_i^2$, $1 \le i \le d$. Let

$$\boldsymbol{\rho} = [\mathbf{k}_1, \mathbf{t}_1^{-1}\mathbf{k}_1\mathbf{t}_1]$$

and write

$$\boldsymbol{\rho} = \mathbf{usv} \text{ with } \mathbf{u} = \boldsymbol{\rho}\mathbf{t}_1^{-1}\mathbf{k}_1^{-1}, \mathbf{s} = \mathbf{k}_1, \mathbf{v} = \mathbf{t}_1.$$

As in the previous example, $(\Gamma, \boldsymbol{\rho})$ is an exclusive pair and $\overline{\Gamma}$ is in fact isomorphic to $\Gamma_1(N)$. By the same token, it follows that

$$\Phi_{\Gamma_2(N)}(n) \ge c\Phi_{\mathbb{Z}^r\wr\Gamma_1(N)}(kn) \text{ and } \Phi_{\Gamma_2(N)}(kn) \le C\Phi_{\mathbb{Z}\wr\Gamma_1(N)}(n).$$

Now, thanks to the results of [14] concerning wreath products, we obtain

- If K is a non-trivial finite group then

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n/[\log n]^{2/d}\right).$$

- If K is not finite but has polynomial volume growth then

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n\left(\frac{\log\log n}{\log n}\right)^{2/d}\right).$$

- If K is polycyclic with exponential volume growth then

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n/[\log\log n]^{2/(d+1)}\right)$$

In particular, when $\Gamma_1(N) = \mathbb{Z} \wr \mathbb{Z}$ with presentation $\langle a, t \mid [a, t^{-n}at^n], n \in \mathbb{Z}\rangle$ we obtain that

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n\left(\frac{\log\log n}{\log n}\right)^{2}\right).$$

Example 2.5.3 (The Baumslag-Solitar group $\mathrm{BS}(1,q)$)**.** Consider the presentation

$$\mathrm{BS}(1,q) = \Gamma_1(N) = \mathbf{F}_2/N =< a, b \mid a^{-1}ba = b^q >$$

with $q > 1$. In order to apply Proposition 2.4.14, let Γ be the group generated by the image of a^2 and b in $\Gamma_2(N)$. Let $\boldsymbol{\rho} = b^{-q}a^{-1}ba$, $\mathbf{u} = b^{-q}a^{-1}$, $\mathbf{s} = b$, $\mathbf{v} = a$. One checks that $(\Gamma, \boldsymbol{\rho})$ is an exclusive pair and that $\overline{\Gamma} \simeq \mathrm{BS}(1, q^2)$. After some computation, we obtain

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n/[\log n]^2\right).$$

Example 2.5.4 (Polycyclic groups)**.** Let G be a polycyclic group with polycyclic series $G = G_1 \triangleright G_2 \triangleright \cdots \triangleright G_{r+1} = \{e\}$, $r \geq 2$. For each i, $1 \leq i \leq r$, let a_i be an element in G_i whose projection in G_i/G_{i+1} generates that group. Write $G = \mathbf{F}_r/N$

where $\mathbf{s}_i$ is sent to a_i. This corresponds to the standard polycyclic presentation of G relative to $a_1, \ldots, a_n$ and N contains a word of the form

$$\boldsymbol{\rho} = \mathbf{s}_1^{-1}\mathbf{s}_2\mathbf{s}_1\mathbf{s}_r^{\alpha_r} \cdots \mathbf{s}_2^{\alpha_2}$$

where α_ℓ, $2 \leq \ell \leq r$ are integers. See [40, page 395].

Theorem 2.5.4. *Let $G = \Gamma_1(N)$ be an infinite polycyclic group equipped with a polycyclic presentation as above with at least two generators.*

- *If G has polynomial volume growth of degree D, then*

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n^{D/(2+D)}[\log n]^{2/(2+D)}\right).$$

- *If G has exponential volume growth then*

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n/[\log n]^2\right).$$

Proof. Our first step is to construct an exclusive pair $(\Gamma, \boldsymbol{\rho})$ with $\overline{\Gamma} = \bar{\pi}(\Gamma)$ of finite index in $\Gamma_1(N)$.

Assume first that G_1/G_2 is finite. In this case, let $\Gamma = \langle s_2, \ldots, s_r \rangle$. Assume that $x \in \Gamma$ is such that $\mathfrak{f}_{\boldsymbol{\rho}}((\bar{x}\bar{s}_1^{-1}, \bar{x}\bar{s}_1^{-1}\bar{s}_2, \mathbf{s}_2)) \neq 0$. Then there must be a prefix $\mathbf{u}$ of $\boldsymbol{\rho}$ such that $\pi(\mathbf{u}) = \bar{x}\bar{s}_1^{-1}$. Computing modulo $\pi(\Gamma) = G_2$, the only prefixes of $\boldsymbol{\rho}$ that can have this property are $\mathbf{s}_1^{-1}$ and $\mathbf{s}_1^{-1}\mathbf{s}_2$. If $\mathbf{u} = \mathbf{s}_1^{-1}$ then $\bar{x}$ is the identity. If $\mathbf{u} = \mathbf{s}_1^{-1}\mathbf{s}_2$ then $\mathbf{s}_1^{-1}\mathbf{s}_2^2$ is not a prefix of $\boldsymbol{\rho}$ and $\mathfrak{f}_{\boldsymbol{\rho}}((\bar{x}\bar{s}_1^{-1}, \bar{x}\bar{s}_1^{-1}\bar{s}_2, \mathbf{s}_2)) = 0$, a contradiction. It follows that condition 2 of Lemma 2.4.5 is satisfied. In this case, it is obvious that condition 3 holds as well. Further, $\pi(\Gamma) = G_2$ is a subgroup of finite index in $G = \Gamma_1(N)$.

In the case when $G_1/G_2 \simeq \mathbb{Z}$, set

$$\Gamma = \langle s_1^2, s_2, \ldots, s_r \rangle < \Gamma_2(N).$$

The same argument as used in the case when G_1/G_2 is finite apply to see that condition 2 of Lemma 2.4.5 is satisfied. To check that condition 3 of Lemma 2.4.5 is satisfied, observe that, if $\mathfrak{f}_g((\bar{y}, \bar{y}\bar{s}_i, \mathbf{s}_i)) \neq 0$ with $2 \leq i \leq r$ and $g \in \Gamma$ then $\bar{y}$ must belong to $\overline{\Gamma}$. But, by construction $\bar{s}_1^{-1} \notin \overline{\Gamma}$. Therefore $\mathfrak{f}_g(\bar{s}_1^{-1}, \bar{s}_1^{-1}\bar{s}_2, \mathbf{s}_2) = 0$ for every $g \in \Gamma$. Finally, $\overline{\Gamma}$ is obviously of index 2 in $\Gamma_1(N)$.

By the Magnus embedding we have $c\Phi_{\mathbb{Z}^r \wr \Gamma_1(N)}(kn) \leq \Phi_{\Gamma_2(N)}(n)$. By Theorem 2.4.14 and the existence of the exclusive pair (Γ, ρ) exhibited above, we also have $c\Phi_{\Gamma_2(N)}(kn) \leq \Phi_{\mathbb{Z} \wr \overline{\Gamma}}(n)$ with $\overline{\Gamma}$ of finite index in $\Gamma_1(N)$. Because $\Gamma_1(N)$ is infinite polycyclic, the desired result follows from the known results about wreath products. See [14].

□

2.6 Iterated comparison and $\mathbf{S}_{d,r}$ with $d > 2$

Let $\mathbf{F}_r/N = \Gamma_1(N)$ be a given presentation. Write $N^{(2)} = [N, N]$ and $N^{(\ell)} = [N^{(\ell-1)}, N^{(\ell-1)}]$, $\ell > 2$. The goal of this section is to obtain bounds on the probability of return for random walks on $\Gamma_\ell(N) = \mathbf{F}_r/N^{(\ell)}$. Our approach is to iterate the method developed in the previous sections in the study of random walks on $\Gamma_2(N)$.

We need to fix some notation. We will use $* = *_\ell$ to denote convolution in $\Gamma_\ell(N)$. In general, ℓ will be fixed so that there will be no need to distinguish between different $*_\ell$. We will consider several wreath products $A \wr G$ as well as iterated wreath products

$$A \wr (A \wr (\cdots (A \wr G) \cdots))$$

where A and G are given with A abelian (in fact, A will be either $\mathbb{Z}$ or $\mathbb{Z}^r$). Set $W(A,G) = W_1(A,G) = A \wr G$ and $W_k(A,G) = W(A, W_{k-1}(A,G))$. Depending on the context, we will denote convolution in $W_k(A,G)$ by

$$\star_k \text{ or } \star_{W_k} \text{ or } \star_{W_k(A,G)} .$$

Let μ be a probability measure on G and η be a probability measure on A. Note that the measures μ and η can also be viewed, in a natural way, as measures on $W(A,G)$ with η being supported by the copy of A that sits above the identity element of G in $A \wr G$. The associated switch-walk-switch measure on $W = W_1(A,G)$ is the measure

$$q = q_1(\eta, \mu) = \eta \star_1 \mu \star_1 \eta.$$

Iterating this construction, we define the probability measure q_k on $W_k(A,G)$ by the iterative formula

$$q_k = q_k(\eta, \mu) = \eta \star_k q_{k-1} \star_k \eta.$$

We refer to q_k as the iterated switch-walk-switch measure on W_k associated with the initial pair η, μ. We will make repeated use of the following simple lemma.

Lemma 2.6.1. *Let A, G, H be finitely generated groups with A abelian. Let $\theta : G \to H$ be a group homomorphism. Define $\theta_1 : W_1(A,G) \to W_1(A,H)$ by*

$$\theta_1 : (f, x) \mapsto (\overline{f}, \theta(x)), \text{ where } \overline{f}(h) = \sum_{g:\theta(g)=h} f(g)$$

with the convention that sum over empty set is 0. Then θ_1 is group homomorphism.

Define $\theta_k : W_k(A,G) \to W_k(A,H)$ by iterating the previous construction so

$$\theta_k = (\theta_{k-1})_1 : W_1(A, W_{k-1}(A,G)) \to W_1(A, W_{k-1}(A,H)).$$

Then θ_k is group homomorphism. Moreover, if θ is injective (resp., surjective), then θ_k is also injective (resp., surjective).

Proof. The stated conclusions follow by inspection. □

Lemma 2.6.2. *Let A, G, H be finitely generated groups with A abelian. Let μ and η be a probability measures on G and A, respectively. Let $\theta : G \to H$ be a homomorphism and $\theta_k : W_k(A,G) \to W_k(A,H)$ be as in* Lemma 2.6.1. *Let $\theta_k(q_k(\eta,\mu))$ be the pushforward of the iterated switch-walk-switch measure $q_k(\eta,\mu)$ on $W_k(A,G)$ under θ_k. Then we have*

$$\theta_k(q_k(\eta,\mu)) = q_k(\eta,\theta(\mu)).$$

Proof. It suffices to check the case $k = 1$ where the desired conclusion reads

$$\theta_1(\eta \star_{A\wr G} \mu \star_{A\wr G} \eta) = \eta \star_{A\wr H} \theta(\mu) \star_{A\wr H} \eta.$$

This equality follows from the three identities

$$\theta_1(\eta \star_{A\wr G} \mu \star_{A\wr G} \eta) = \theta_1(\eta) \star_{A\wr H} \theta_1(\mu) \star_{A\wr G} \theta_1(\eta),$$

$$\theta_1(\mu) = \theta(\mu) \text{ and } \quad \theta_1(\eta) = \eta.$$

The first identity holds because θ_1 is an homomorphism. The other two identities hold by inspection (with some slight abuse of notation). □

2.6.1 Iterated lower bounds

This section develops lower bounds for the probability of return of symmetric finitely supported random walks on $\Gamma_\ell(N) = \mathbf{F}_r/N^{(\ell)}$. By Dirichlet form comparison techniques (see [30]), it suffices to consider the case of the measure μ_ℓ on $\Gamma_\ell(N)$ which is the image under the projection $\pi_\ell : \mathbf{F}_r \to \mathbf{F}_r/N^{(\ell)}$ of the lazy symmetric simple random walk measure $\boldsymbol{\mu}$ on $\mathbf{F}_r$ defined at (2.3), that is $\mu_\ell = \pi_\ell(\boldsymbol{\mu})$. On $\mathbb{Z}^r$, let the probability η be defined by $\eta(\pm\epsilon_i) = 1/2r$ where $(\epsilon_i)_1^r$ is the canonical basis for $\mathbb{Z}^r$. Let $q_{\ell,j}$ be the j-th iterated switch-walk-switch measure on $W_j(\mathbb{Z}^r, \Gamma_{\ell-j}(N))$ based on the probability measures η on $\mathbb{Z}^r$ and $\mu_{\ell-j}$ on $\Gamma_{\ell-j}(N)$.

Theorem 2.6.3. *Let the presentation $\Gamma_1(N) = \mathbf{F}_r/N$ be given. Fix an integer ℓ and let $\mu_\ell = \pi_\ell(\boldsymbol{\mu})$ be the probability measure on $\Gamma_\ell(N)$ describe above. Let $*$ denote convolution on $\Gamma_\ell(N)$ and $\star$ denote convolution on $W_{\ell-1}(\mathbb{Z}^r, \Gamma_1(N))$. Then there exist $c, k \in (0,\infty)$ such that*

$$\forall\, n, \quad \mu_\ell^{*2n}(e_*) \geq c q_{\ell,\ell-1}^{\star 2kn}(e_\star).$$

Proof. The proof is obtain by an iterative procedure based on repeated use of the Magnus embedding $\Gamma_m(N) \hookrightarrow \mathbb{Z}^r \wr \Gamma_{m-1}(N)$ and comparison of Dirichlet forms. The desired conclusion follows immediately from the following two lemmas. □

Lemma 2.6.4. *Let $*$ denotes convolution on $\Gamma_\ell(N)$. Let $\star$ denote convolution on $W(\mathbb{Z}^r, \Gamma_{\ell-1}(N)) = \mathbb{Z}^r \wr \Gamma_{\ell-1}(N)$. Let μ_ℓ and $\mu_{\ell-1}$ be as defined above. Then*

$$\mu_\ell^{*2n}(e_*) \geq c[\eta \star \mu_{\ell-1} \star \eta]^{\star 2kn}(e_\star).$$

Proof. Let $\bar{\psi}_\ell : \Gamma_\ell(N) \hookrightarrow \mathbb{Z}^r \wr \Gamma_{\ell-1}(N)$ be the Magnus embedding. Then $\mu_\ell^{*n}(e_*) = [\bar{\psi}_\ell(\mu)]^{\star n}(e_\star)$ and, by a simple Dirichlet form comparison argument,

$$[\bar{\psi}_\ell(\mu)]^{\star 2n}(e_\star) \geq c[\eta \star \mu_{\ell-1} \star \eta]^{\star 2kn}(e_\star).$$

□

Lemma 2.6.5. *Fix two integers $0 < j < \ell$. Let $\star_j$ denote convolution on $W_j(\mathbb{Z}^r, \Gamma_{\ell-j}(N))$. Then, for $2 \leq j < \ell$,*

$$q_{\ell,j-1}^{\star_{j-1} 2n}(e_{\star_{j-1}}) \geq c q_{\ell,j}^{\star_j 2kn}(e_{\star_j}).$$

Proof. By definition, we have

$$q_{\ell,j-1} = \eta \star_{j-1} q_{\ell-1,j-2} \star_{j-1} \eta.$$

where $q_{\ell-1,j-2}$ is the switch-walk-switch measure on $W_{j-1}(\mathbb{Z}^r, \Gamma_{\ell-j+1}(N))$. Let $\bar{\psi}$ denote the Magnus embedding

$$\bar{\psi} : \Gamma_{\ell-j+1}(N) \hookrightarrow \mathbb{Z}^r \wr \Gamma_{\ell-j}(N).$$

Let

$$\widetilde{\psi} : W_{j-1}(\mathbb{Z}^r, \Gamma_{\ell-j+1}(N)) \hookrightarrow W_{j-1}(\mathbb{Z}^r, \mathbb{Z}^r \wr \Gamma_{\ell-j}(N)) = W_j(\mathbb{Z}^r, \Gamma_{\ell-j}(N))$$

its natural extension as in Lemma 2.6.1. Observe that

$$\begin{aligned} q_{\ell,j-1}^{\star_{j-1}2n}(e_{\star_{j-1}}) &= \widetilde{\psi}(\eta \star_{j-1} q_{\ell-1,j-2} \star_{j-1} \eta)^{\star_j 2n}(e_{\star_j}) \\ &= [\eta \star_j \widetilde{\psi}(q_{\ell-1,j-2}) \star_j \eta]^{\star_j 2n}(e_{\star_j}) \end{aligned}$$

where we used Lemma 2.6.2 to obtain the second identity. Again, by a simple Dirichlet form comparison argument,

$$\begin{aligned} [\eta \star_j \widetilde{\psi}(q_{\ell-1,j-2}) \star_j \eta]^{\star_j 2n}(e_{\star_j}) &\geq c[\eta \star_j q_{\ell-1,j-1} \star_j \eta]^{\star_j 2kn}(e_{\star_j}) \\ &= c q_{\ell,j}^{\star_j 2kn}(e_{\star_j}). \end{aligned}$$

□

Propositions 2.3.2–2.3.3–2.3.4 (which are based on the results in [8,14]) provide us with good lower bounds for the probability of return on iterated wreath product. Namely,

- Assume that $A = \mathbb{Z}^b$ with $b \geq 1$ and G has polynomial volume growth of degree D. Then, for $\ell \geq 2$,

$$\Phi_{W_\ell(A,G)}(n) \simeq \exp\left(-n\left(\frac{\log_{[\ell]} n}{\log_{[\ell-1]} n}\right)^{2/D}\right).$$

- Assume that $A = \mathbb{Z}^b$ with $b \geq 1$ and G is polycyclic with exponential volume growth. Then, for $\ell \geq 1$,

$$\Phi_{W_\ell(A,G)}(n) \simeq \exp\left(-n/[\log_{[\ell]} n]^2\right).$$

  This applies, for instance, when G is the Baumslag-Solitar group $\mathrm{BS}(1,q)$, $q > 1$. Further, the same result holds for the wreath product $G = \mathbb{Z}_q \wr \mathbb{Z}$, $q > 1$, (even so it is not polycyclic).

Together with Theorem 2.6.3, these computations yield the following results.

Corollary 2.6.6. *Let* $\Gamma_\ell(N) = \mathbf{F}_r/N^{(\ell)}$.

- *Assume that* $\Gamma_1(N)$ *has polynomial volume growth of degree* D. *Then, for* $\ell \geq 3$,

$$\Phi_{\Gamma_\ell(N)}(n) \geq \exp\left(-Cn\left(\frac{\log_{[\ell-1]} n}{\log_{[\ell-2]} n}\right)^{2/D}\right).$$

- *Assume that* $\Gamma_1(N)$ *is* $\mathrm{BS}(1,q)$ *with* $q > 1$, *or* $\mathbb{Z}_2 \wr \mathbb{Z}$, *or polycyclic of exponential volume growth volume growth. Then, for* $\ell \geq 2$,

$$\Phi_{\Gamma_\ell(N)}(n) \geq \exp\left(-Cn/[\log_{[\ell-1]} n]^2\right).$$

- *Assume that* $\Gamma_1(N) = K \wr \mathbb{Z}^D$, $D \geq 1$ *and* K *finite. Then, for* $\ell \geq 2$,

$$\Phi_{\Gamma_\ell(N)}(n) \geq \exp\left(-Cn/[\log_{[\ell-1]} n]^{2/D}\right).$$

- *Assume that* $\Gamma_1(N) = \mathbb{Z}^a \wr \mathbb{Z}^D$, $a, D \geq 1$. *Then, for* $\ell \geq 2$,

$$\Phi_{\Gamma_\ell(N)}(n) \geq \exp\left(-Cn\left(\frac{\log_{[\ell]} n}{\log_{[\ell-1]} n}\right)^{2/D}\right).$$

2.6.2 Iterated upper bounds

We now present an iterative approach to obtain upper bounds on $\Phi_{\Gamma_\ell(N)}$. Although similar in spirit to the iterated lower bound technique developed in the previous section, the iterative upper bound method is both more difficult and much less flexible. In the end, we will be able to apply it only in the case of the free solvable groups $\mathbf{S}_{d,r}$, that is, when $N = [\mathbf{F}_r, \mathbf{F}_r]$.

Our first task is to formalize algebraically the content of Proposition 2.4.14. Recall once more that the Magnus embedding provides an injective homomorphism $\bar{\psi} : \mathbf{F}_r/[N,N] \hookrightarrow \left(\sum_{x\in\mathbf{F}_r/N} \mathbb{Z}^r_x\right) \rtimes \mathbf{F}_r/N$ with $\bar{\psi}(g) = (\bar{a}(g), \bar{\pi}(g))$. Let Γ be a subgroup of $\mathbf{F}_r/[N,N]$ and $\boldsymbol{\rho} \in N \setminus [N,N] \subset \mathbf{F}_r$. Set $\rho = \pi_2(\boldsymbol{\rho})$ and $\overline{\Gamma} = \pi(\Gamma) \subset \mathbf{F}_r/N$.

Assume that $(\Gamma, \boldsymbol{\rho})$ is an exclusive pair as in Definition 2.4.3. We are going to construct a surjective homomorphism

$$\vartheta : \langle \Gamma, \rho \rangle \to \mathbb{Z} \wr \overline{\Gamma}.$$

Let $g \in \langle \Gamma, \rho \rangle$. Consider two decompositions of g as products

$$g = \gamma_1 \rho^{x_1} \gamma_2 \rho^{x_2} \cdots \gamma_p \rho^{x_p} \gamma_{p+1} = \gamma'_1 \rho^{x'_1} \gamma'_2 \rho^{x'_2} \cdots \gamma'_q \rho^{x'_q} \gamma'_{q+1}$$

with $\gamma_i \in \Gamma$, $1 \le i \le p+1$, $\gamma'_i \in \Gamma$, $1 \le i \le q+1$. Set $\sigma_i = \gamma_1 \dots \gamma_i$, $1 \le i \le p+1$, and $\sigma'_i = \gamma_1 \dots \gamma_i$, $1 \le i \le q+1$. Observe that

$$g = \sigma_1 \rho^{x_1} \sigma_1^{-1} \sigma_2 \rho^{x_2} \sigma_2^{-1} \cdots \sigma_p \rho^{x_p} \sigma_p^{-1} \sigma_{p+1} = \alpha \sigma_{p+1}$$

where

$$\alpha = \sigma_1 \rho^{x_1} \sigma_1^{-1} \sigma_2 \rho^{x_2} \sigma_2^{-1} \cdots \sigma_p \rho^{x_p} \sigma_p^{-1}.$$

Similarly $g = \alpha' \sigma'_{q+1}$ and we have

$$(\alpha'^{-1} \alpha = \sigma'_{q+1} (\sigma_{p+1})^{-1}.$$

By Lemma 2.4.1, we have

$$\begin{aligned}\bar{a}(g) &= \sum_1^p x_i \tau_{\bar{\sigma}_i} \bar{a}(\rho) + \bar{a}(\sigma_{p+1}) \\ &= \sum_1^q x'_i \tau_{\bar{\sigma}'_i} \bar{a}(\rho) + \bar{a}(\sigma'_{q+1}).\end{aligned}$$

and

$$\sum_1^p x_i \tau_{\bar{\sigma}_i} \bar{a}(\rho) - \sum_1^q x'_i \tau_{\bar{\sigma}'_i} \bar{a}(\rho) = \bar{a}(\sigma'_{q+1}\sigma_{p+1}^{-1}).$$

As $(\Gamma, \boldsymbol{\rho})$ is an exclusive pair, condition (ii) of Definition 2.4.3 implies that

$$\sum_1^p x_i \tau_{\bar{\sigma}_i} \bar{a}(\rho) - \sum_1^q x'_i \tau_{\bar{\sigma}'_i} \bar{a}(\rho) = \bar{a}(\sigma'_{q+1}\sigma_{p+1}^{-1}) = 0.$$

Hence

$$\sum_1^p x_i \tau_{\bar{\sigma}_i} \bar{a}(\rho) = \sum_1^q x'_i \tau_{\bar{\sigma}'_i} \bar{a}(\rho)$$

in $\sum_{x \in \Gamma_1(N)} \mathbb{Z}_x^r$. This also implies that $a(\sigma_{p+1}) = a(\sigma'_{q+1})$. By construction, we also have $\bar{\pi}(\sigma_{p+1}) = \bar{\pi}(\sigma'_{q+1})$. Hence, $\sigma_{p+1} = \sigma'_{q+1}$ in Γ.

By condition (i) of Definition 2.4.3 (see Remark 2.4.4), we can identify

$$\sum_1^p x_i \tau_{\bar{\sigma}_i} \bar{a}(\rho) = \sum_1^q x'_i \tau_{\bar{\sigma}'_i} \bar{a}(\rho)$$

with the element

$$\left(\sum_1^p x_i \mathbf{1}_h(\sigma_i)\right)_{h \in \overline{\Gamma}} \quad \text{of} \quad \sum_{h \in \overline{\Gamma}} \mathbb{Z}_h.$$

This preparatory work allows us to define a map

$$\begin{aligned}\vartheta \quad &: \quad \langle \Gamma, \rho \rangle \to \mathbb{Z} \wr \overline{\Gamma} \\ g \quad &= \quad \gamma_1 \rho^{x_1} \gamma_2 \rho^{x_2} \cdots \gamma_p \rho^{x_p} \gamma_{p+1} \mapsto \left(\left(\sum_1^p x_i \mathbf{1}_h(\sigma_i)\right)_{h \in \overline{\Gamma}}, \bar{\pi}(g) \right).\end{aligned}$$

Lemma 2.6.7. *The map $\vartheta : \langle \Gamma, \rho \rangle \to \mathbb{Z} \wr \overline{\Gamma}$ is a surjective homomorphism.*

Proof. Note that $\vartheta(e)$ is the identity element in $\mathbb{Z} \wr \overline{\Gamma}$. To show that ϑ is an homomorphism, it suffices to check that, for any $g \in \langle \Gamma, \rho \rangle$ and $\gamma \in \Gamma$

$$\vartheta(g\gamma) = \vartheta(g)\vartheta(\gamma), \;\; \vartheta(g\rho^{\pm 1}) = \vartheta(g)\vartheta(\rho^{\pm 1}).$$

These identities follow by inspection. One easily check that ϑ is surjective. $\square$

Lemma 2.6.8. *Let μ be a probability measure supported on Γ and ν be the probability measure defined by $\nu(\rho^{\pm 1}) = 1/2$. Let η be the probability measure on $\mathbb{Z}$ defined by $\eta(\pm 1) = 1/2$. Let $*$ be convolution on $\langle \Gamma, \rho \rangle < \Gamma_2(N)$ and $\star$ be convolution on $\mathbb{Z} \wr \overline{\Gamma}$ Then we have*

$$\vartheta(\nu * \mu * \nu) = \eta \star \bar{\pi}(\mu) \star \eta.$$

Proof. This follows from the fact that ϑ is an homomorphism, $\vartheta|_\Gamma = \bar{\pi}$ and $\vartheta(\nu) = \eta$. $\square$

In addition to the canonical projections $\pi_j : \mathbf{F}_r \to \mathbf{F}_r/N^{(j)} = \Gamma_j(N)$, for $1 \leq j \leq i$, we also consider the projection $\pi_j^i : \Gamma_i(N) \to \Gamma_j(N)$.

Definition 2.6.9. Fix a presentation $\Gamma_1(N) = \mathbf{F}_r/N$ and an integer ℓ. Let Γ_i be a finitely generated subgroup of $\Gamma_i(N)$, $2 \leq i \leq \ell$. Set

$$\Gamma'_{i-1} = \pi_{i-1}^i(\Gamma_i), \;\; 2 \leq i \leq \ell.$$

Let $\boldsymbol{\rho}_i \in \mathbf{F}_r$, $2 \leq i \leq \ell$. Set $\rho_\ell = \pi_\ell(\boldsymbol{\rho}_\ell)$. We say that $(\Gamma_i, \boldsymbol{\rho}_i)_2^\ell$ is an exclusive sequence (adapted to $(\Gamma_i(N))_1^\ell$) if the following properties hold:

1. $\Gamma_\ell < \Gamma_\ell(N)$ and $\pi_{\ell-1}^\ell(\rho_\ell)$ is trivial.
2. For $2 \leq j \leq \ell - 1$, $\Gamma_j < \Gamma'_j$, $\rho_j \in \Gamma'_j$ and $\pi_{j-1}^j(\rho_j)$ is trivial.
3. For each $2 \leq i \leq \ell$, $(\Gamma_i, \boldsymbol{\rho}_i)$ is an exclusive pair in $\Gamma_2(N^{(i-1)}) = \Gamma_i(N)$.

Theorem 2.6.10. *Fix a presentation $\Gamma_1(N) = \mathbf{F}_r/N$ and an integer $\ell \geq 2$. Assume that there exists an exclusive sequence $((\Gamma_i, \boldsymbol{\rho}_i))_2^\ell$ adapted to $(\Gamma_j(N))_1^\ell$. Then there exists $k, C \in (0, \infty)$ such that*

$$\Phi_{\Gamma_\ell(N)}(kn) \leq C\Phi_{W_{\ell-1}(\mathbb{Z},\Gamma_1')}(n)$$

where $\Gamma_1' = \pi_1^2(\Gamma_2) < \Gamma_1(N)$.

Remark 2.6.11. The technique and results of [14] provides good upper bounds on Φ_G when G is an iterated wreath product such as $W_{\ell-1}(\mathbb{Z}, \Gamma_1')$ and we have some information on Γ_1'. The real difficulty in applying the theorem above lies in finding an exclusive sequence.

Proof. The Theorem follows immediately from the following two lemmas. □

We will need the following notation. For each $1 \leq i \leq \ell$, let ϕ_i be a symmetric finitely supported probability measure on Γ_i with generating support. Let $\mu_\ell = \pi(\boldsymbol{\mu})$ be the projection on $\Gamma_\ell(N)$ of the lazy symmetric simple random walk probability measure on $\mathbf{F}_r$. Let ν_i be the probability measure on $\Gamma_i(N)$ given by $\nu_i(\rho_i^{\pm 1}) = 1/2$.

Lemma 2.6.12. *Under the hypothesis of* Theorem 2.6.10*, there are $k, C \in (0, +\infty)$ such that*

$$\Phi_{\Gamma_\ell(N)}(kn) \leq C\Phi_{W_1(\mathbb{Z},\Gamma_{\ell-1}')}(n).$$

Proof. For this proof, let $*$ be convolution on $\Gamma_\ell(N)$ and $\star$ be convolution on $\mathbb{Z} \wr \Gamma_{\ell-1}' = W_1(\mathbb{Z}, \Gamma_{\ell-1}')$, $\Gamma_{\ell-1}' = \pi_{\ell-1}^\ell(\Gamma_\ell)$. Since $\nu_\ell * \phi_\ell * \nu_\ell$ is symmetric and finitely supported on $\Gamma_\ell(N)$, we have the Dirichlet form comparison

$$\mathcal{E}_{\mu_\ell} \geq c\mathcal{E}_{\nu_\ell * \phi_\ell * \nu_\ell}.$$

Hence

$$\mu_\ell^{*2kn}(e_*) \le C[\nu_\ell * \phi_\ell * \nu_\ell]^{*2n}(e_*).$$

Note that the measure $\nu_\ell * \phi_\ell * \nu_\ell$ lives on $\langle \Gamma_\ell, \rho_\ell \rangle < \Gamma_\ell(N)$. By Lemma 2.6.7, we have the surjective homomorphism $\vartheta_\ell : \langle \Gamma_\ell, \rho_\ell \rangle \to \mathbb{Z} \wr \Gamma'_{\ell-1} = W_1(\mathbb{Z}, \Gamma'_{\ell-1})$. Hence

$$[\nu_\ell * \phi_\ell * \nu_\ell]^{*2n}(e_*) \le [\vartheta_\ell(\nu_\ell * \phi_\ell * \nu_\ell)]^{\star 2n}(e_\star)$$

where $e_\star$ is the identity element in $W_1(\mathbb{Z}, \Gamma'_{\ell-1})$. By Lemma 2.6.8,

$$[\vartheta_\ell(\nu_\ell * \phi_\ell * \nu_\ell)]^{\star 2n}(e_\star) = (\eta * \pi^\ell_{\ell-1}(\phi_\ell) * \eta]^{\star 2n}(e_\star).$$

This shows that $\Phi_{\Gamma_\ell(N)}(kn) \le C\Phi_{W_1(\mathbb{Z},\Gamma'_{\ell-1})}(n)$. $\square$

Lemma 2.6.13. *Under the hypothesis of* Theorem 2.6.10, *for each* j, $1 \le j \le \ell - 2$, *there are* $k, C \in (0, +\infty)$ *such that*

$$\Phi_{W_j(\mathbb{Z},\Gamma'_{\ell-j})}(kn) \le C\Phi_{W_{j+1}(\mathbb{Z},\Gamma'_{\ell-j-1})}(n).$$

Proof. For this proof, we let $\star_j$ denote convolution on the iterated wreath product $W_j(\mathbb{Z}, \Gamma'_{\ell-j})$. To control $\Phi_{W_j(\mathbb{Z},\Gamma'_{\ell-j})}$ from above, it suffices to control from above the probability of return $n \mapsto q_j^{\star_j 2n}(e_{\star_j})$, for the iterated switch-walk-switch measure q_j based on η and $\pi^{\ell-j+1}_{\ell-j}(\phi_{\ell-j+1})$.

By a simple comparison of Dirichlet forms on the group $W_j(\mathbb{Z}, \Gamma'_{\ell-j})$, we have

$$q_j^{\star_j 2kn}(e_{\star_j}) \le C\widetilde{q}_j^{\star_j 2n}(e_{\star_j}) \tag{2.1}$$

where $\widetilde{q}_j$ is the iterated switch-walk-switch measure based on η and

$$\nu_{\ell-j} \star_j \phi_{\ell-j} \star_j \nu_{\ell-j}$$

supported on $\langle \Gamma_{\ell-j}, \rho_{\ell-j} \rangle < \Gamma'_{\ell-j}$. Consider the surjective homomorphism

$$\vartheta_{\ell-j} : \langle \Gamma_{\ell-j}, \rho_{\ell-j} \rangle \to \mathbb{Z} \wr \Gamma'_{\ell-j-1}.$$

By Lemma 2.6.1, this homomorphism can be extended to a surjective homomorphism

$$\vartheta_{\ell-j,j} : W_j(\mathbb{Z}, \langle \Gamma_{\ell-j}, \rho_{\ell-j} \rangle) \to W_j(\mathbb{Z}, \mathbb{Z} \wr \Gamma'_{\ell-j-1}) = W_{j+1}(\mathbb{Z}, \Gamma'_{\ell-j-1}).$$

Further, by Lemmas 2.6.2 and 2.6.8, we have

$$\vartheta_{\ell-j,j}(\widetilde{q}_j) = q_{j+1}$$

since q_{j+1} is the iterated switch-walk-switch measure on $W_{j+1}(\mathbb{Z}, \Gamma'_{\ell-j-1})$ based on η and $\pi^{\ell-j}_{\ell-j-1}(\phi_{\ell-j})$. This yields

$$\widetilde{q}_j^{\star_j 2n}(e_{\star_j}) \leq q_{j+1}^{\star_{j+1} 2n}(e_{\star_{j+1}}).$$

This, together with (2.1), proves the desired relation between $\Phi_{W_j(\mathbb{Z},\Gamma'_{\ell-j})}$ and $\Phi_{W_{j+1}(\mathbb{Z},\Gamma'_{\ell-j-1})}$. □

2.6.3 Free solvable groups

In this section, we conclude the proof of Theorem 2.1.1 by proving that, for $d \geq 3$,

$$\Phi_{\mathbf{S}_{d,r}}(n) \simeq \exp\left(-n\left(\frac{\log_{[d-1]} n}{\log_{[d-2]} n}\right)^{2/r}\right).$$

The lower bound follows from Corollary 2.6.6. By Theorem 2.6.10, it suffices to construct an exclusive sequence $(\Gamma_i, \boldsymbol{\rho}_i)_2^d$ adapted to $(\Gamma_i(N))_1^d$, when $N = [\mathbf{F}_r, \mathbf{F}_r]$ with the property that Γ'_1 is isomorphic to $\mathbb{Z}^r$. The technique developed in Section 2.4.2 is the key to constructing such an exclusive sequence.

In fact, we are able to deal with a class of groups that is more general than the family $\mathbf{S}_{d,r}$. Observe that $\mathbf{S}_{d,r} = \Gamma_d(\gamma_2(\mathbf{F}_r))$. More generally, define

$$\mathbf{S}^c_{d,r} = \Gamma_d(\gamma_{c+1}(\mathbf{F}_r)) = \mathbf{F}_r/(\gamma_{c+1}(\mathbf{F}_r))^{(d)}.$$

Note that $\Gamma_1(\gamma_{c+1}(\mathbf{F}_r)) = \mathbf{F}_r/\gamma_{c+1}(\mathbf{F}_r)$ is the free nilpotent group of nilpotent class c on r generators. The groups $\mathbf{S}^c_{d,r}$ are examples of (finite rank) free polynilpotent groups. These groups are studied in [17] where it is proved that they are residually finite. In the notation of [17], $\mathbf{S}^c_{d,r}$ is a free polynilpotent group of class row $(c, 1, \ldots, 1)$ with $d-1$ ones following c.

Let

$$D(r, c) = \sum_1^c \sum_{k|m} \mu(k) r^{m/k}$$

where μ is the Möbius function. The integer $D(r, c)$ is the exponent of polynomial volume growth of the free nilpotent group $\mathbf{F}_r/\gamma_{c+1}(\mathbf{F}_r)$. See [18, Theorem 11.2.2] and [12]. Note that $D(r, 1) = r$.

Theorem 2.6.14. *Fix $c \geq 1$, $r \geq 2$ and $d \geq 3$. Let $D = D(r, c)$. We have*

$$\Phi_{\mathbf{S}^c_{d,r}}(n) \simeq \exp\left(-n\left(\frac{\log_{[d-1]} n}{\log_{[d-2]} n)}\right)^{2/D}\right).$$

Remark 2.6.15. The case $d = 2$ is covered by Theorem 2.5.1.

For the proof of Theorem 2.6.14, we will use a result concerning the subgroup of $\Gamma_\ell(N)$ generated by the images of a fix power $\mathbf{s}_i^m$ of the generators $\mathbf{s}_i$, $1 \leq i \leq r$. Let $\delta_m : \mathbf{F}_r \to \mathbf{F}_r$ be the homomorphism from the free group to itself determined by $\delta_m(\mathbf{s}_i) = \mathbf{s}_i^m$, $1 \leq i \leq r$.

Lemma 2.6.16. *Suppose δ_m induces an injective homomorphism $\mathbf{F}_r/N \to \mathbf{F}_r/N$, and $\pi(\mathbf{s}_i^q) \notin \delta_m(\mathbf{F}_r/N)$, $1 \leq q \leq m-1$, $1 \leq i \leq r$. Then δ_m induces an injective homomorphism $\mathbf{F}_r/[N, N] \to \mathbf{F}_r/[N, N]$.*

Proof. The proof is based on the representation of the elements of $\Gamma_2(N) = \mathbf{F}_r/[N, N]$ using flows on the labeled Cayley graph of $\Gamma_1(N) = \mathbf{F}_r/N$.

Let δ_m also denote the induced injective homomorphism on $\Gamma_1(N)$. Let $\mathfrak{f}$ be a flow function defined on edge set $\mathfrak{E}$ of Cayley graph of $\Gamma_1(N)$. Let $\mathfrak{E}_m$ be a subset of $\mathfrak{E}$ given by

$$\mathfrak{E}_m = \{(\delta_m(x)s_i^j, \delta_m(x)s_i^{j+1}, \mathbf{s}_i) : x \in \Gamma_1(N), 0 \leq j \leq m-1, 1 \leq i \leq r\}.$$

Let $t_m : \mathfrak{f} \mapsto t_m\mathfrak{f}$ be the map on flows defined by

$$t_m\mathfrak{f}((\delta_m(x)s_i^j, \delta_m(x)s_i^{j+1}, \mathbf{s}_i)) = \mathfrak{f}((x, xs_i, \mathbf{s}_i)), \quad 0 \leq j \leq m-1,$$

and $t_m\mathfrak{f}$ is 0 on edges not in $\mathfrak{E}_m$. This map is well-defined. Indeed, if two pairs (x, j) and (y, j') in $\Gamma_1(N) \times \{0, \cdots, m-1\}$ correspond to a common edge, that is,

$$(\delta_m(x)s_i^j, \delta_m(x)s_i^{j+1}, \mathbf{s}_i) = (\delta_m(y)s_i^{j'}, \delta_m(y)s_i^{j'+1}, \mathbf{s}_i),$$

then $\delta_m(x)s_i^j = \delta_m(y)s_i^{j'}$, $\delta_m(y^{-1}x) = s_i^{j'-j}$. Since $|j'-j| \leq m-1$, from the assumption $\pi(\mathbf{s}_i^q) \notin \delta_m(\mathbf{F}_r/N)$, $1 \leq q \leq m-1$ it follows that $j' = j$. Then $\delta_m(y^{-1}x) = \bar{e}$ and, since δ_m is injective, we must have $x = y$.

By definition, t_m is additive in the sense that

$$t_m(\mathfrak{f}_1 + \mathfrak{f}_2) = t_m\mathfrak{f}_1 + t_m\mathfrak{f}_2.$$

Also, regarding translations in $\Gamma_1(N)$, we have

$$t_m\tau_y\mathfrak{f} = \tau_{\delta_m(y)}t_m\mathfrak{f}.$$

Therefore the identity $\mathfrak{f}_{\mathbf{uv}} = \mathfrak{f}_{\mathbf{u}} + \tau_{\pi(\mathbf{u})}\mathfrak{f}_{\mathbf{v}}$, of Remark 2.4.2 yields

$$t_m\mathfrak{f}_{\mathbf{uv}} = t_m\mathfrak{f}_{\mathbf{u}} + \tau_{\delta_m(\pi(\mathbf{u}))}t_m\mathfrak{f}_{\mathbf{v}}.$$

By assumption $\pi(\delta_m(\mathbf{u})) = \delta_m(\pi(\mathbf{u}))$, therefore

$$t_m\mathfrak{f}_{\mathbf{uv}} = t_m\mathfrak{f}_{\mathbf{u}} + \tau_{\pi(\delta_m(\mathbf{u}))}t_m\mathfrak{f}_{\mathbf{v}}.$$

This identity allows us to check that the definition of t_m acting on flows is consistent with $\delta_m : \mathbf{F}_r \to \mathbf{F}_r$. More precisely, for any $\mathbf{g} \in \mathbf{F}_r$, we have

$$\mathfrak{f}_{\delta_m(\mathbf{g})} = t_m \mathfrak{f}_{\mathbf{g}}.$$

To see this, first note that this formula holds true on the generators and their inverses and proceed by induction on the word length of $\mathbf{g} \in \mathbf{F}_r$.

Given $g \in \Gamma_2(N)$, pick a representative $\mathbf{g} \in \mathbf{F}_r$ so that g corresponds to the flow $\mathfrak{f}_{\mathbf{g}}$ on $\Gamma_1(N)$. Define $\widetilde{\delta}_m(g)$ to be the element of $\Gamma_2(N)$ that corresponds to the flow $t_m \mathfrak{f}_{\mathbf{g}} = \mathfrak{f}_{\delta_m(\mathbf{g})}$. This map is well defined and satisfies

$$\widetilde{\delta}_m \circ \pi_2 = \pi_2 \circ \delta_m.$$

This implies that $\widetilde{\delta}_m : \Gamma_2(N) \to \Gamma_2(N)$ is an injective homomorphism. Abusing notation, we will drop the $\widetilde{}$ and use the same name, δ_m, for the injective homomorphisms $\Gamma_1(N) \to \Gamma_1(N)$ and $\Gamma_2(N) \to \Gamma_2(N)$ induced by $\delta_m : \mathbf{F}_r \to \mathbf{F}_r$. $\square$

Proof of Theorem 2.6.14. The lower bound follows from Corollary 2.6.6. By Theorem 2.6.10, in order to prove the upper bound, it suffices to construct an exclusive sequence $(\Gamma_\ell, \boldsymbol{\rho}_\ell)_2^d$ adapted to $(\Gamma_\ell(N))_1^d$, $N = \gamma_{c+1}(F_r)$, and with the property that Γ_1' is isomorphic to $\Gamma_1(N) = \mathbf{F}_r/\gamma_{c+1}(\mathbf{F}_r)$, the free nilpotent group of class c on r generators.

The work of Gruenberg, [17, Theorem 7.1] implies that $\Gamma_\ell(N)$ is residually finite. Hence the technique developed in Section 2.4.2 apply easily to this situation. We are going to use repeatedly Proposition 2.4.11.

To start, for each ℓ, we construct an exclusive pair $(H_\ell, \boldsymbol{\sigma}_\ell)$ in $\Gamma_\ell(N)$. Namely, let $\boldsymbol{\sigma}_\ell$ be an element in $N^{(\ell-1)} \setminus N^{(\ell)}$ in reduced form in $\mathbf{F}_r$ and such that it projects to a non-self-intersecting loop in $\Gamma_{\ell-1}(N)$. Let $\mathbf{s}_{i_1}^k \mathbf{s}_{i_2}^\epsilon$, $\epsilon = \pm 1$, $k \neq 0$, be beginning of $\boldsymbol{\sigma}_\ell$. Without loss of generality, we assume that $i_1 = 1$, $i_2 = 2$ and $\epsilon = 1$. Let also s_i and $\bar{s}_i$ be the projections of $\mathbf{s}_i$ onto $\Gamma_\ell(N)$ and $\Gamma_{\ell-1}(N)$, respectively. Let $(\bar{s}_1^k, \bar{s}_1^k\bar{s}_2, \mathbf{s}_2)$ be the corresponding edge in $\Gamma_{\ell-1}(N)$. Since $\boldsymbol{\sigma}_\ell$ projects to a simple loop in $\Gamma_{\ell-1}(N)$, we must have

$$\mathfrak{f}_{\boldsymbol{\sigma}_\ell}((\bar{s}_1^k, \bar{s}_1^k\bar{s}_2, \mathbf{s}_2)) \neq 0.$$

Since $\Gamma_{\ell-1}(N)$ is residually finite, there exists a finite index normal subgroup $K_{\boldsymbol{\sigma}_\ell} \lhd \Gamma_{\ell-1}(N)$ as in Lemma 2.4.7. Pick an integer m_ℓ such that

$$[\Gamma_{\ell-1}(N) : K_{\sigma_\ell}] \mid m_\ell \quad \text{and} \quad |k| < m_\ell$$

and set

$$H_\ell = \langle s_i^{m_\ell}, 1 \leq i \leq r \rangle < \Gamma_\ell(N).$$

Thinking of $\Gamma_\ell(N)$ and $\Gamma_{\ell-1}(N)$ as $\Gamma_2(N^{(\ell-1)})$ and $\Gamma_1(N^{(\ell-1)})$, respectively, Proposition 2.4.11 shows that (H_ℓ, σ_ℓ) is an exclusive pair in $\Gamma_\ell(N)$.

Next, by Lemma 2.6.16, for each integer m and each ℓ, the injective homomorphism $\delta_m : \mathbf{F}_r \to \mathbf{F}_r$ induces on $\Gamma_\ell(N)$ an injective homomorphism still denoted by $\delta_m : \Gamma_\ell(N) \to \Gamma_\ell(N)$. For each $1 \leq \ell \leq d$, set

$$M_d = 1, \quad M_\ell = m_{\ell+1} \cdots m_d,$$

and, for $2 \leq \ell \leq d$,

$$\Gamma_\ell = \delta_{M_\ell}(H_\ell) < \Gamma_\ell(N), \quad \boldsymbol{\rho}_\ell = \delta_{M_\ell}(\boldsymbol{\sigma}_\ell).$$

By construction, $((\Gamma_\ell, \rho_\ell))_2^d$ is an exclusive sequence in $(\Gamma_\ell(N))_1^d$ and

$$\Gamma_1' = \pi_1^2(\Gamma_2) = \langle \bar{s_1}^{M_1}, \ldots, \bar{s}_r^{M_1} \rangle < \Gamma_1(N)$$

is isomorphic to $\Gamma_1(N)$ because $\Gamma_1(N)$ is the free nilpotent group on $\bar{s}_1, \ldots, \bar{s}_r$ of nilpotent class c. $\square$

CHAPTER 3

RANDOM WALKS ON NILPOTENT GROUPS DRIVEN BY MEASURES SUPPORTED ON POWERS OF GENERATORS

3.1 Introduction

3.1.1 The measures $\mu_{S,a}$

Generating sets play an essential role in the theory of countable groups. This is obvious when a group is defined by generators and relations or when a group is defined as the subgroup generated by a given finite subset of elements in a much larger group. In this context, the larger ambient group serves as a sort of "black box" that encodes the law of the group.

In this chapter we study a natural family of random walks driven by measures $\mu_{S,a}$ which are defined as follows. The letter S represents a finite generating tuple, i.e., a list $S = (s_1, s_2, \dots, s_k)$ of generators (repetitions are permitted). In addition, we are given a k-tuple a of (extended) positive reals $a = (\alpha_1, \alpha_2, \dots, \alpha_k)$, $\alpha_i \in (0, \infty]$. The measure $\mu_{S,a}$ allows long steps along any of the one-parameter group $\langle s_i \rangle = \{s_i^n : n \in \mathbb{Z}\}$, $1 \leq i \leq k$. The probability of such a long step along $\langle s_i \rangle$ is given by a power law whose exponent α_i is the i-th entry of the tuple a. Namely, we set,

$$\mu_{S,a}(g) = \frac{1}{k} \sum_{i=1}^{k} c(\alpha_i) \sum_{m \in \mathbb{Z}} (1 + |m|)^{-\alpha_i - 1} \mathbf{1}_{s_i^m}(g) \tag{3.1}$$

where

$$c(\alpha)^{-1} = \sum_{\mathbb{Z}} (1 + |m|)^{-\alpha - 1}.$$

We make the somewhat arbitrary convention that if $\alpha = \infty$ then $(1+|m|)^{-\alpha-1} = 0$ unless $m = 0, \pm 1$ in which case $(1+|m|)^{-\alpha-1} = 1$. Note that $\mu_{S,a}$ is symmetric, that is, satisfies $\mu_{S,a}(g^{-1}) = \mu_{S,a}(g)$. We can also describe $\mu_{S,a}$ as the push-forward of the probability measure μ_a on the free group $\mathbf{F}_k$ on k generators $\mathbf{s}_i$, $1 \leq i \leq k$, which gives probability

$$\mu_a(\mathbf{s}_i^{\pm m}) = k^{-1}c(\alpha_i)(1+|m|)^{-\alpha_i-1} \text{ to } \mathbf{s}_i^{\pm m}.$$

Indeed, if π is the projection from $\mathbf{F}_k$ onto G which sends $\mathbf{s}_i$ to s_i,

$$\mu_{S,a}(g) = \mu_a(\pi^{-1}(g)).$$

On $\mathbb{Z}$, the power laws $\mu_\alpha(\pm k) = c(\alpha)(1+|k|)^{-\alpha-1}$ are very natural probability measures. For $\alpha \in (0,2)$, μ_α can be viewed as a discrete version of the symmetric stable laws which is the probability distribution on $\mathbb{R}$ whose Laplace transform is $e^{-|y|^\alpha}$.

The main result of this chapter, Theorem 3.1.2 below, describes the behavior of

$$n \mapsto \mu_{S,a}^{(n)}(e)$$

when G is any given finitely generated nilpotent group, S is any given finite generating tuple of elements of G and the entries of the tuple a are in $(0,2)$. What makes this problem interesting is the interaction between the nature of the long jumps allowed in the directions of each generators and the non-commutative structure of the group. As we shall see, the behaviors of the random walks driven by the measures $\mu_{S,a}$ capture a wealth of information on the algebraic structure of G.

Because of the results of [30] — in particular, Theorem 3.1.9 stated below — the very precise form of the measure $\mu_{S,a}$ defined at (3.1) is not really essential in determining the behavior of $n \mapsto \mu_{S,a}^{(n)}(e)$. Indeed, any symmetric measure ν on G such that $c\nu \le \mu_{S,a} \le C\nu$ will satisfy

$$\nu^{(kn)}(e) \le K\mu_{S,a}^{(n)}(e) \text{ and } \mu_{S,a}^{(kn)}(e) \le K\nu^{(n)}(e)$$

for some k, K independent of n.

3.1.2 The case of $\mathbb{Z}^d$

In the simplest non-trivial case where $G = \mathbb{Z}^2 = \{(x, y) : x, y \in \mathbb{Z}\}$, $S = \{(1,0),(0,1)\}$ and $a = (\alpha_1, \alpha_2) \in (0,\infty]^2$, it is not hard to see that $\mu_{S,a}^{(n)}(e)$, $e = (0,0)$, behaves as follows. Set

$$\tilde{\alpha} = \min\{\alpha, 2\}, \quad \frac{1}{\beta} = \frac{1}{\tilde{\alpha}_1} + \frac{1}{\tilde{\alpha}_2} \text{ and } \gamma = \#\{i : \alpha_i = 2\}.$$

1. If $2 \notin \{\alpha_1, \alpha_2\}$, $\mu_{S,a}^{(n)}(e) \sim c(\alpha_1, \alpha_2) n^{-1/\beta}$;
2. If $2 \in \{\alpha_1, \alpha_2\}$, $\mu_{S,a}^{(n)}(e) \simeq n^{-1/\beta}(\log n)^{-\gamma/2}$.

Here and in the rest of this chapter $\sim$ and $\simeq$ are used with the following meaning. For two functions f, g defined either over the positive reals or the natural numbers, we say that $f \sim g$ (usually, at 0 or infinity), if $\lim f/g = 1$. We say that $f \simeq g$ if there are constants c_1 such that

$$c_1 f(c_2 t) \le g(t) \le c_3 f(c_4 t)$$

(in a neighborhood of the relevant value, usually 0 or infinity). We recommend to restrict the use of $\simeq$ to cases where one of the two functions f or g is monotone.

Next, let us review briefly what happens when $G = \mathbb{Z}^d$ and $S = (s_1, \dots, s_k)$, $k \ge d$. By hypothesis, S is generating. Given $a = (\alpha_1, \dots, \alpha_k)$, we extract from

S a d-tuple $\Sigma = (\sigma_1, \ldots, \sigma_d)$ using the following algorithm. Set $\Sigma_1 = \{\sigma_1 = s_{i_1}\}$ where $\alpha_{i_1} = \min\{\alpha_i : 1 \le i \le k\}$. For $t \ge 1$, if

$$\Sigma_t = (\sigma_1, \ldots, \sigma_t), \quad \sigma_1 = s_{i_1}, \ldots, \sigma_t = s_{i_t}$$

have been chosen, pick $\sigma_{t+1} = s_{i_{t+1}}$ in $\{s_i : 1 \le i \le k\}$ with the properties that $\alpha_{i_{t+1}} = \min\{\alpha_j : j \notin \{i_1, \ldots, i_t\}\}$ and the rank of the lattice generated by $\Sigma_{t+1} = \Sigma_t \cup \{\sigma_{t+1}\}$ is (strictly) greater than the rank of the lattice generated by Σ_t. Note that the final d-tuple Σ might not generates $\mathbb{Z}^d$ but does generate a lattice of finite index in $\mathbb{Z}^d$. Set $a(\Sigma) = (\alpha_{i_1}, \ldots, \alpha_{i_d})$.

Theorem 3.1.1. *Let $G = \mathbb{Z}^d$. Let $S = (s_i)_1^k$ be a generating k-tuple. Let $a = (\alpha_i)_1^k \in (0, \infty]^k$. Let $\Sigma = (\sigma_i)_1^d$ and $a(\Sigma)$ be obtained from (S, a) by the algorithm described above. Set*

$$\gamma = \#\{j \in \{1, \ldots, d\} : \alpha_{i_j} = 2\} \text{ and } \frac{1}{\beta} = \sum_{s=1}^{d} \frac{1}{\tilde{\alpha}_{i_s}}$$

where $\tilde{\alpha} = \min\{\alpha, 2\}$. Then we have

$$\mu_{S,a}^{(n)}(e) \simeq \mu_{\Sigma,a(\Sigma)}^{(n)}(e) \simeq n^{-1/\beta}[\log n]^{-\gamma/2}$$

With some work, this result can be extracted from [16].

3.1.3 The main result in its simplest form

The goal of this chapter is to prove the following theorem together with more sophisticated assorted results.

Theorem 3.1.2. *Let G be a nilpotent group equipped with a generating k-tuple $S = (s_i)_1^k$ and $a = (\alpha_i)_1^k \in (0, \infty]^k$. Assume that the subgroup generated by $\{s_i :$*

$\alpha_i < 2\}$ *is of finite index in* G. *Then there exists a real* $D \geq 0$ *depending on* (G, S, a) *such that*

$$\mu_{S,a}^{(n)}(e) \simeq n^{-D}.$$

This statement suggests further questions including the following three:

- Can we compute D? how does it depends on S, a and G?
- What happen if the subgroup generated by $\{s_i : \alpha_i < 2\}$ is not of finite index in G?
- What happens on other groups? In particular, how does Theorem 3.1.2 generalize to finitely generated groups of polynomial volume growth?

The first question will be answered completely in this chapter. Indeed, we would not be able to prove the above theorem without a detailed understanding of how to compute the real D. The exact value of D depends in an intricate and interesting way on (a) the commutator structure of G, (b) the position of the generators s_i in the commutator structure of G and (c) the values of the parameters α_i. See Theorem 3.1.8 in the next subsection.

The second question is rather subtle and will not be completely elucidated in this chapter although some partial results will be obtain in this direction.

In its full generality, the third question is too wide ranging to be discussed here in details. The question regarding groups of polynomial growth is tantalizing but appears surprisingly difficult to attack.

3.1.4 Weight systems and the value of D

The goal of this section is to give the reader a clear idea of the key ingredients that enter the exact computation of the real D governing the behavior of $\mu^{(n)}_{S,a}(e)$ in Theorem 3.1.2.

Consider $S = (s_1, \ldots, s_k)$ as a formal alphabet equipped with a weight system $\mathfrak{w}$ which assigns weight $w_i \in (0, \infty)$ to the letter s_i, $1 \leq i \leq k$. We extend our alphabet by adjoining to each s_i its formal inverse s_i^{-1}. Using this alphabet, we build the set $\mathfrak{C}(S, m)$ of all formal commutators of length m by induction on m. Commutators of length 1 are the letters in $S^{\pm 1}$. Commutators of length m are the formal expression c of the form $c = [c_1, c_2]$ where c_1, c_2 are commutators of length $m_1, m_2 \geq 1$ with $m_1 + m_2 = m$.

The commutators of length 2 are (the ± 1 must be understood here as independent of each other)

$$[s_i^{\pm 1}, s_j^{\pm 1}], \;\; 1 \leq i, j \leq k.$$

The commutators of length 3 are

$$[[s_i^{\pm 1}, s_j^{\pm 1}], s_\ell^{\pm 1}], \;\; [s_i^{\pm 1}, [s_j^{\pm 1}, s_\ell^{\pm 1}]], \;\; 1 \leq i, j, \ell \leq k.$$

For $1 \leq i_1, i_2, i_3, i_4 \leq k$, the commutators of length 4 are

$$[[[s_{i_1}^{\pm 1}, s_{i_2}^{\pm 1}], s_{i_3}^{\pm 1}], s_{i_4}^{\pm 1}], \;\; [[s_{i_1}^{\pm 1}, [s_{i_2}^{\pm 1}, s_{i_3}^{\pm 1}]], s_{i_4}^{\pm 1}], \;\; [[s_{i_1}^{\pm 1}, s_{i_2}^{\pm 1}], [s_{i_3}^{\pm 1}, s_{i_4}^{\pm 1}]]$$

$$[s_{i_1}^{\pm 1}, [[s_{i_2}^{\pm 1}, s_{i_3}^{\pm 1}], s_{i_4}^{\pm 1}]], \;\; [s_{i_1}^{\pm 1}, [s_{i_2}^{\pm 1}, [s_{i_3}^{\pm 1}, s_{i_4}^{\pm 1}]]].$$

To any formal commutators we can associate its build-word and its group-word. The build-word of a commutator c is the word over S that list the entries of c in order after one removes brackets and ± 1. So, the build-word of

$c = [[s_{i_1}^{\pm 1}, s_{i_2}^{\pm 1}], [s_{i_3}^{\pm 1}, s_{i_4}^{\pm 1}]]$ is $s_{i_1}s_{i_2}s_{i_3}s_{i_4}$. The group word is the word on $S^{\pm 1}$ obtained by applying repeatedly the group rules

$$[c_1, c_2]^{-1} = [c_2, c_1] \text{ and } [c_1, c_2] = c_1^{-1}c_2^{-1}c_1c_2.$$

So the group-word of $c = [[s_i, s_j^{-1}], s_\ell]$ is $s_js_i^{-1}s_j^{-1}s_is_\ell^{-1}s_i^{-1}s_js_is_j^{-1}s_\ell$.

Definition 3.1.3 (Power weight systems)**.** Given a k-tuple $(s_1, \ldots, s_k)$ of formal letters and a k-tuple $(w_1, \ldots, w_k)$ of positive reals, define the weight system $\mathfrak{w}$ on $\mathfrak{C}(S)$ by setting (inductively)

$$w(c) = w(c_1) + w(c_2) \text{ if } c = [c_1, c_2].$$

Let

$$\bar{w}_1 < \bar{w}_2 < \cdots < \bar{w}_j < \cdots$$

be the increasing sequence of the weight values of the weight system $\mathfrak{w}$. For $j = 1, 2, \ldots$, let $\mathfrak{C}_j^{\mathfrak{w}}$ be the set of all commutators c with $w(c) \geq \bar{w}_j$.

Clearly, the weight of a formal commutator is the sum of the weights of the letters appearing in its build-word. If $S = (s_1, s_2)$ and $w_1 = 3, w_2 = 13/2$, then the weight-value sequence is

$$\bar{w}_1 = 3, \bar{w}_2 = 6, \bar{w}_3 = 13/2, \bar{w}_4 = 9, \bar{w}_5 = 12, \bar{w}_6 = 25/2, \bar{w}_7 = 13, \ldots$$

Given a group G generated by a k-tuple $S = (s_1, \ldots, s_k)$, any finite word ω on the alphabet $S^{\pm 1}$ has a well defined image $\pi_G(\omega)$ in G. Similarly, any formal commutator c on the alphabet $S^{\pm 1}$ has an image in G given by its group-word representation.

Definition 3.1.4 (Group filtration associated to $\mathfrak{w}$)**.** Let G be a nilpotent group equipped with a generating k-tuple $S = (s_1, \ldots, s_k)$ and a weight system $\mathfrak{w}$ generated by $(w_1, \dots, w_k) \in (0, \infty)^k$. Set

$$G_j^{\mathfrak{w}} = \langle \mathfrak{C}_j^{\mathfrak{w}} \rangle.$$

That is, $G_j^{\mathfrak{w}}$ is the subgroup of G generated by the images of all formal commutators of weight greater or equal to $\bar{w}_j$. Let $j_* = j_*(G, S, \mathfrak{w})$ be the smallest integer such that $G_{j_*+1}^{\mathfrak{w}} = \{e\}$.

Example 3.1.1. Let G be the discrete Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}.$$

Let

$$s_1 = X = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; s_2 = Y = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \; s_3 = Z^5 = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$w_1 = 1, \; w_2 = 3/2, \; w_3 = 3.$$

In this case, the increasing sequence $\bar{w}_j$ is given by $\bar{w}_1 = 1, \bar{w}_2 = 3/2, \bar{w}_3 = 2, \bar{w}_4 = 5/2, \bar{w}_5 = 3, \bar{w}_6 = 7/2, \dots$ and we have

$$G_6^{\mathfrak{w}} = \{e\}, \; G_5^{\mathfrak{w}} = \{s_3^k : k \in \mathbb{Z}\}, \; G_4^{\mathfrak{w}} = G_3^{\mathfrak{w}} = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{Z} \right\},$$

$$G_2^{\mathfrak{w}} = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : y, z \in \mathbb{Z} \right\}, \;\; G_1^{\mathfrak{w}} = G.$$

Proposition 3.1.5. *Referring to the setting and notation of* Definition 3.1.4, *for all* $j = 1, 2\ldots,$ *we have* $G_j^{\mathfrak{w}} \subset G_{j+1}^{\mathfrak{w}}$ *and* $[G, G_j^{\mathfrak{w}}] \subset G_{j+1}^{\mathfrak{w}}$. *In particular,*

$$G = G_1^{\mathfrak{w}} \supseteq G_2^{\mathfrak{w}} \supseteq \cdots \supseteq G_j^{\mathfrak{w}} \supseteq \cdots \supseteq G_{j_*}^{\mathfrak{w}} \supset G_{j_*+1}^{\mathfrak{w}} = \{e\}$$

is a descending normal series with $[G_j^{\mathfrak{w}}, G_j^{\mathfrak{w}}] \subset G_{j+1}^{\mathfrak{w}}$.

Proof. Recall that if X, Y are subsets of G, $[X, Y]$ denotes the subgroup generated by $\{[x, y] : x \in X, y \in Y\}$. Recall further that

$$[< X >, < Y >] = [X, Y]^{<X><Y>}$$

where the right-hand side is the group generated by all conjugates of $[X, Y]$ by elements of the form $g = xy$, $x \in< X >, y \in< Y >$. Since $[f_1, f_j] \in \mathfrak{C}_{j+1}^{\mathfrak{w}}$ for all $f_1 \in \mathfrak{C}_1^{\mathfrak{w}}$, $f_j \in \mathfrak{C}_j^{\mathfrak{w}}$ and

$$[G, G_j^{\mathfrak{w}}] = [\mathfrak{C}_1^{\mathfrak{w}}, \mathfrak{C}_j^{\mathfrak{w}}]^G$$

it follows that

$$[G, G_j^{\mathfrak{w}}] \subset (G_{j+1}^{\mathfrak{w}})^G$$

Thus a descending induction on j shows that the groups $G_j^{\mathfrak{w}}$ are all normal subgroups of G and that

$$[G, G_j^{\mathfrak{w}}] \subset G_{j+1}^{\mathfrak{w}}.$$

Note that it may happen that $G_j^{\mathfrak{w}} = G_{j+1}^{\mathfrak{w}}$ for some values of j, $1 < j < j_*$. For instance, it may happen that all formal commutators of a certain weight are trivial in G. In Example 3.1.1, $G_3^{\mathfrak{w}} = G_4^{\mathfrak{w}}$ because all commutators of weight $\bar{w}_3 = 2$ are obviously trivial. □

Definition 3.1.6. Referring to the setting and notation of Definition 3.1.4, let

$$R_j^{\mathfrak{w}} = \text{rank}(G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}})$$

be the torsion free rank of the abelian group $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$.

By construction, the images of the formal commutators of weight $\bar{w}_j$ form a generating subset of $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$, $\quad j = 1, 2, \dots, j_*$. By definition, the torsion free rank of this abelian group is the minimal number of elements needed to generates $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$ modulo torsion.

Definition 3.1.7. Referring to the setup and notation of Definition 3.1.4, set

$$D(S, \mathfrak{w}) = \sum_1^{j_*} \bar{w}_j \operatorname{rank}(G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}).$$

Note that $D(S, \mathfrak{w})$ depends on the weights values $\bar{w}_j$ as well as on the algebraic relations between elements of S in G (via the rank of the group $G_j^{\mathfrak{w}}$).

Example 1.1(continued) In Example 3.1.1, we have $j_* = 5$,

$$G_5^{\mathfrak{w}}/G_6^{\mathfrak{w}} = \mathbb{Z},\ G_4^{\mathfrak{w}}/G_5^{\mathfrak{w}} = \mathbb{Z}/5\mathbb{Z},\ G_3^{\mathfrak{w}}/G_4^{\mathfrak{w}} = \{0\},\ G_2^{\mathfrak{w}}/G_3^{\mathfrak{w}} = \mathbb{Z} \text{ and } G_1^{\mathfrak{w}}/G_2^{\mathfrak{w}} = \mathbb{Z}.$$

Hence $\operatorname{rank}(G_5^{\mathfrak{w}}/G_6^{\mathfrak{w}}) = 1$, $\operatorname{rank}(G_4^{\mathfrak{w}}/G_5^{\mathfrak{w}}) = \operatorname{rank}(G_3^{\mathfrak{w}}/G_4^{\mathfrak{w}}) = 0$, $\operatorname{rank}(G_2^{\mathfrak{w}}/G_3^{\mathfrak{w}}) = \operatorname{rank}(G_1^{\mathfrak{w}}/G_2^{\mathfrak{w}}) = 1$ and $D(S, \mathfrak{w}) = 1 + 3/2 + 3 = 11/2$ since $\bar{w}_1 = 1, \bar{w}_2 = 3/2, \bar{w}_3 = 2, \bar{w}_4 = 5/2, \bar{w}_5 = 3, \bar{w}_6 = 7/2, \dots$.

Example 3.1.2. Assume that the weight w_i are all equal, namely, $w_i = v$, $i = 1, \dots, k$. Then the weight-value sequence is given by $\bar{w}_j = jv$ and j_* is equal to the nilpotency class of G. In this case, the descending normal series $G_j^{\mathfrak{w}}$ is the lower central series defined inductively by $G_1 = G$, $G_j = [G, G_{j-1}]$, $j \geq 2$, and $D(S, \mathfrak{w}) = vD(G)$ where

$$D(G) = \sum_1^{j_*} j \operatorname{rank}(G_j/G_{j+1}). \tag{3.2}$$

Theorem 3.1.8. *Let G be a nilpotent group equipped with a generating k-tuple $S = (s_i)_1^k$ and $a = (\alpha_i)_1^k \in (0, \infty]^k$. Assume that the subgroup generated by $\{s_i :$*

$\alpha_i < 2\}$ *is of finite index in* G. *Consider the weight system* $\mathfrak{w}(a) = \mathfrak{w}$ *induced by setting* $w_i = 1/\widetilde{\alpha}_i$ *where* $\widetilde{\alpha} = \min\{2, \alpha\}$. *Then*

$$\mu_{S,a}^{(n)}(e) \simeq n^{-D(S,\mathfrak{w})}$$

with $D(S, \mathfrak{w})$ *as in* Definition 3.1.7.

Example 3.1.3. Let G be the discrete Heisenberg group equipped with the generating triple $S = (s_i)_1^3$ has in Example 3.1.1. Let $a = (\alpha_i)_1^3$. In this case, the condition that $\{s_i : \alpha_i < 2\}$ generates a subgroup of finite index is equivalent to $\alpha_1, \alpha_2 \in (0, 2)$. Let $\mathfrak{w}$ be as defined in Theorem 3.1.8. Then

$$D(S, \mathfrak{w}) = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \max\left\{\frac{1}{\alpha_1} + \frac{1}{\alpha_2}, \frac{1}{\alpha_3}\right\}.$$

3.1.5 Some background on random walks

Given a finite symmetric generating set A, we set $|x|_A = \inf\{k : x \in A^k\}$ (since $A^0 = \{e\}$, by convention, $|e| = 0$). This is called the word-length of x (w.r.t. the generating set A). With some abuse of notation, if $S = (s_1, \ldots, s_k)$ is a generating k-tuple, we write $|\cdot|_S$ for the word-length associated with the symmetric generating set $\{s_i^{\pm 1}, 1 \leq i \leq k\}$. The volume growth of G (with respect to A) is the function $V_A(r) = \#\{g : |g|_A \leq r\}$. The $\simeq$-equivalence class of the function V_A is independent of the choice of A. It is a group invariant called the growth function of G.

We say that a probability measure ϕ is symmetric if $\check{\phi} = \phi$ where $\check{\phi}(x) = \phi(x^{-1})$, $x \in G$. The Dirichlet form associated with ϕ is the quadratic form

$$\mathcal{E}_\phi(f, f) = \frac{1}{2} \sum_{x,y \in G} |f(xy) - f(x)|^2 \phi(y).$$

This form is fundamental in the study of random walks because of the following basic result.

Theorem 3.1.9. *[30] Assume that ϕ, ψ are two symmetric probability measures on a countable group G. If $\mathcal{E}_\phi \leq C\mathcal{E}_\psi$ then*

$$\psi^{(2kn)}(e) \leq 2\phi^{(2n)}(e) + 2e^{-2kn},\ k = [C] + 2.$$

This theorem will be use extensively in the present chapter. In [30], it is used to prove that the long time asymptotic behavior of the probability of return is roughly the same for all random walks driven by symmetric measures with generating support and finite second moment.

Theorem 3.1.10 ([30]). *Assume that ϕ is a symmetric probability measure on a finitely generated group G with finite symmetric generating set A. Let u_A be the uniform probability measure on A. If ϕ satisfies*

$$\sum_{g \in G} |g|_A^2 \phi(g) < \infty \tag{3.3}$$

then there are constants k, C such that

$$u_A^{(2kn)}(e) \leq C\phi^{(2n)}(e).$$

Further, if ϕ satisfies (3.3) *and $\phi > 0$ on a finite generating set then*

$$\phi^{(2n)}(e) \simeq u_A^{(2n)}(e).$$

This theorem implies that, if A and B are two symmetric finite generating sets of the group G, we have $u_A^{(2n)}(e) \simeq u_B^{(2n)}(e)$. Further, for any symmetric ϕ with finite second moment and generating support, $\phi^{(2n)}(e) \simeq u_A^{(n)}(e)$. In this sense, the equivalence class of the function $n \mapsto u_A^{(2n)}(e)$ under the equivalence relation $\simeq$ is a group invariant. This group invariant, which we denote by Φ_G, i.e.,

$$\Phi_G(n) \simeq u_A^{(2n)}(e), \tag{3.4}$$

has been studied extensively ([30] shows that Φ_G is invariant under quasi-isometries). In particular,

$$\Phi_G(n) \simeq \begin{cases} n^{-D/2} & \text{if } G \text{ has volume growth } V(r) \simeq r^D, \\ \exp(-n^{1/3}) & \text{if } G \text{ is polycyclic with exponential volume growth,} \\ \exp(-n) & \text{if } G \text{ is non-amenable.} \end{cases}$$

Nilpotent groups belong to the first category and have $D = D(G)$ given explicitly by (3.2). Many other behaviors beyond the three mentioned above are known to occurs and their are many groups for which Φ_G is unknown. See, e.g., [38, 39] and the references therein.

To explain how Theorem 3.1.10 applies to the measures $\mu_{S,a}$ defined at (3.1), we need the following definition.

Definition 3.1.11. Let G be a nilpotent group with descending lower central series G_j. The commutator length $\ell(g)$ of an element g of G is the supremum of the integers ℓ such that $g^m \in G_\ell$ for some integer m. In particular, by definition, torsion elements have infinite commutator length.

Corollary 3.1.12. *On any finitely generated group G equipped with a generating k-tuple S, we have*

$$\mu_{S,a}^{(n)}(e) \simeq \Phi_G(n) \simeq n^{-D(G)/2}$$

for all k-tuple $a = (\alpha_1, \dots, \alpha_k)$ such that $\alpha_i \ell(s_i) > 2$ for all $i = 1, \dots, k$.

Proof. It is well known that for any fixed $g \in G$, we have $|g^n|_S \simeq n^{1/\ell(g)}$ (see also Proposition 3.2.17 where a more general version of this fact is proved). It follows that, as long as the k-tuple a satisfies the condition stated in the corollary, $\mu_{S,a}$ has finite second moment. Hence, Theorem 3.1.10 implies $\mu_{S,a}^{(n)}(e) \simeq \Phi_G(n)$ as desired. $\square$

As a consequence of the more detailed results proved in this chapter, we can state the following complementary result.

Theorem 3.1.13. *Let G be a nilpotent group equipped with a generating k-tuple S. Let $a \in (0, \infty]^k$. If there exists $i \in \{1, \ldots, k\}$ such that $(\alpha_i, \ell(s_i)) = (2, 1)$ or $\alpha_i \ell(s_i) < 2$ then we have*

$$\lim_{n \to \infty} [n^{D(G)/2} \mu_{S,a}^{(n)}(e)] = 0. \tag{3.5}$$

Regarding (3.5), we conjecture but are not able to prove that the sufficient condition provided by Theorem 3.1.13 is also necessary. See Theorems 3.5.11–3.5.12.

3.1.6 Radial stable laws

Let G be a finitely generated group with symmetric finite generating set A. Set $B_m = \{g : |g|_A \leq m\}$. Define the radially symmetric "stable law" on G with index $\alpha \in (0, 2)$ to be probability measure

$$\mu_\alpha(g) = c_\alpha \sum_{m=0}^{\infty} (1 + m)^{-\alpha - 1} \frac{\mathbf{1}_{B_m}(g)}{V_A(m)}, \quad c_\alpha^{-1} = \sum_{0}^{\infty} (1 + m)^{-\alpha - 1}$$

Note that μ_α is well defined for all $\alpha > 0$ and that

$$\forall\, 0 < \beta < \alpha < \infty, \quad \sum_g |g|_A^\beta \mu_\alpha(g) < \infty.$$

It is observed in [36, 37, 43] that

$$\forall\, n,\;\; V_A(n) \geq cn^D \Longrightarrow \forall\, n,\;\; \mu_\alpha^{(n)}(e) \leq Cn^{-D/\alpha}.$$

In addition, by [5, 20], for a given group G and for some/any $\alpha \neq 2$,

$$V_A(n) \simeq cn^D \Longleftrightarrow \mu_\alpha^{(n)}(e) \simeq Cn^{-D/\tilde{\alpha}},\;\; \tilde{\alpha} = \min\{2, \alpha\}. \tag{3.6}$$

In fact, if we assume that the group G has polynomial volume growth $V(n) \simeq n^D$ then

$$\mu_\alpha(g) \simeq (1 + |g|_A)^{-D-\alpha}.$$

Further, it follows from [20] that, for any $\alpha \in (0, 2)$, there are constants $c_1(\alpha), c_2(\alpha)$ such that

$$c_1(\alpha)\mu_\alpha \leq \nu_\alpha \leq c_2(\alpha)\mu_\alpha$$

where ν_α denotes the measure that is α-subordinated to u_A in the sense of ([4]), that is,

$$\nu_\alpha = \sum_1^\infty \frac{\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)} u_A^{(n)}.$$

Moreover, for any $\alpha \in (0, 2)$,

$$\forall\, n \in \mathbb{N}, \quad \mu_\alpha^{(n)}(e) \simeq \nu_\alpha^{(n)}(e) \simeq n^{-D/\alpha}.$$

In Chapter 4, motivated by applications given below, we prove the following complementary statement regarding the behavior of μ_2.

Proposition 3.1.14 (see Theorem 4.2.10)**.** *Assume that G has polynomial volume growth $V_S(n) \simeq n^D$. Then we have*

$$\mu_2^{(n)}(e) \simeq (n \log n)^{-D/2}.$$

The lower bounds on $\mu_{S,a}^{(n)}(e)$ obtained in this chapter are proved by establishing Dirichlet form comparisons involving appropriate generalization of the above radially symmetric stable measures and using Theorem 3.1.9. See Section 4.3 for a different method to obtain the lower bounds.

3.1.7 Background on nilpotent groups

The classical setting for the study of random walks is the lattice $\mathbb{Z}^d$. See [41]. Since this work is concerned with random walks on nilpotent groups, we briefly discuss some of the similarities and differences between the lattice $\mathbb{Z}^d$ and finitely generated nilpotent groups. We also describe three basic examples.

The most fundamental similarity between a finitely generated nilpotent group G and the lattice $\mathbb{Z}^d$ is that, assuming that G is torsion free, there exists a real nilpotent Lie group $\mathbb{G}$ such that G can be identified with a discrete subgroup of $\mathbb{G}$ with compact quotient $\mathbb{G}/G$. In other words, G is a (co-compact) lattice in $\mathbb{G}$ in exactly the same way that $\mathbb{Z}^d$ is a lattice in $\mathbb{R}^d$ (except that the quotient is not a group, in general). This is a fundamental result of Malcev. See, e.g., Philip Hall famous notes [19]. However, simply connected real nilpotent Lie groups and their lattices are classified only in very small dimensions. See [11]. For instance, there are essentially 5 distinct "irreducible" simply connected real nilpotent Lie groups of dimension 5. In dimension 6, there are 34. No one knows the list of all simply connected nilpotent real Lie groups of dimension 9, let alone higher dimensions.

From a technical viewpoint, the study of random walks on abelian groups is mostly based on the use of the Fourier transform (see [41]). Although the representation theory of (real) nilpotent Lie groups is well developed, it has proved very hard to use this theory to study random walks (except in some very particular cases). For these reasons, the study of random walks on nilpotent groups is often based on techniques that are rather different from the classical techniques used in the abelian case. This is certainly the case for the present work.

Example 3.1.4. Let $U(d)$ be the group of all upper triangular $d \times d$ matrices over $\mathbb{Z}$ with diagonal entries equal to 1. This group is a lattice in the nilpotent real

Lie group $\mathbb{U}(d)$ of all upper triangular $d \times d$ matrices over the reals with diagonal entries equal to 1. Let $E_{i,j}$, $1 \leq i < j \leq d$, be the matrix in $\mathbb{U}(d)$ with all non-diagonal entries equal to 0 except for the entry in the i-th row and j-th column which equals 1. These elements are related by $E_{i,j}E_{\ell,m} = \delta_{j,\ell}E_{i,m}$. Further,

$$E_{i,j} = [E_{i,i+1}, [E_{i+1,i+2}, \dots, [E_{j-2,j-1}, E_{j-1,j}] \cdots]].$$

In particular, the $(d-1)$-tuple $S = (E_{i,i+1})_1^{d-1}$ is generating. For any $m = 1, \dots, d-1$, the elements $\{E_{i,i+m} : 1 \leq i \leq d - m\}$ can be expressed as commutators of length m on $S^{\pm 1}$ and form a minimal generating set for the subgroup $U(d)_m = [U(d), U(d)_{m-1}]$ in the lower central series of $U(d)$. The nilpotency class of $U(d)$ is $d-1$, that is, any commutator of length greater than $d-1$ equals the identity in $U(d)$.

Any matrix $M = (m_{i,j})$ in $U(d)$ can (obviously) be written uniquely (order matters!)

$$M = \prod_{k=1}^{d-1} \left(\prod_{i=0}^{k-1} E_{k-i,d-i}^{m_{k-i,d-i}} \right)$$

where the $m_{i,j}$ are simply the entry of the matrix M. Much less trivially, there is also a unique expression of the form

$$M = \prod_{k=1}^{d-1} \left(\prod_{i=k}^{d-1} E_{i-k+1,i+1}^{m'_{i-k+1,i+1}} \right)$$

where $(m'_{i,j})_{1 \leq i < j \leq n}$ is obtained from $(m_{i,j})_{1 \leq i < j \leq n}$ by a polynomial bijective transformation with polynomial inverse.

Since $A = \{E_{i,i+1}^{\pm 1}, 1 \leq i \leq d-1\}$ generates $U(d)$, it is of great interest to describe the word length $|M|_A$ of a matrix $M \in U(d)$ in terms of the coordinate systems $(m_{i,j})_{1 \leq i < j \leq d}$ and $(m'_{i,j})_{1 \leq i < j \leq d}$. The answer is essentially the same in both cases, namely,

$$|M|_A \simeq \sum_{1 \leq i < j \leq d} |m_{i,j}|^{1/|j-i|} \simeq \sum_{1 \leq i < j \leq d} |m'_{i,j}|^{1/|j-i|}.$$

This well known (but non-trivial) result is the key to the volume growth estimate

$$V_{U(d),A}(r) \simeq r^{D(U(d))}, \quad D(U(d)) = \sum_{i=1}^{d-1} i(d-i)$$

and to the assorted random walk result (see, e.g., [45]) $\Phi_{U(d)}(n) \simeq n^{-D(U(d))/2}$. If we set $S = (s_i = E_{i,i+1})_1^{d-1}$ then for any $a = (\alpha_i)_1^{d-1} \in (0,2)^{d-1}$ our main result yields

$$\mu_{S,a}^{(n)}(e) \simeq n^{-D}, \quad D = \sum_{1 \le i < j \le d} \sum_{m=i}^{j-1} \frac{1}{\alpha_m}.$$

Example 3.1.5. The free nilpotent group of nilpotency class ℓ on k generators , $N(k,\ell)$, can be defined as the quotient of the free group on k generators by the normal subgroup generated by the images of all formal commutators of length greater than ℓ. This group has the (universal) property that it covers any k generated nilpotent group G of nilpotency class ℓ with a covering homomorphism sending the canonical generating k-tuple of $N(k,\ell)$ to the given generating k-tuple of G.

Marshal Hall gave a description of $N(k,\ell)$ in terms of the so-called "basic commutators". See [18, Chapter 11]. Let $(s_1, \dots, s_k)$ be the canonical generators of $N(k,\ell)$. Define the ordered set of all basic commutators $c_1 < \dots < c_t$ using the following inductive procedure.

(1) $s_1, \dots, s_k$ are the basic commutators of length 1 and, by definition $s_1 < s_2 < \dots < s_k$; (2) for each m the basic commutators of length m are all commutators of the form $c = [c', c'']$ with c', c'' basic commutators of length m', m'' with $m' + m'' = m$ such that $c' > c''$ and, if $c' = [d', d'']$ (d, d' basic commutators) then $c'' \ge d''$; (3) commutators of length m come after commutators of length $m-1$ and are ordered arbitrary with respect to each other. By a theorem of Witt (e.g., [18, Theorem 11.2.2]), the number of basic commutators of length m on k generators is $M_k(m) = m^{-1} \sum_{d|m} \mu(d) k^{m/d}$ where μ denotes the classical Möbius function.

Marshall Hall proved that the basic commutators of length m form a basis of the abelian group $N(k,\ell)_m/N(k,\ell)_{m+1}$ for $1 \le m \le \ell$ and that any element g of $N(k,\ell)$ can be written uniquely

$$g = \prod_1^t c_i^{x_i}, \quad x_i \in \mathbb{Z}.$$

Moreover, the length of g with respect to the generating set $A = \{s_i^{\pm 1}\}$ satisfies $|g|_A \simeq \sum_1^t |x_i|^{1/m_i}$ where m_i is the commutator length of c_i. This gives the volume group estimate

$$V_A(r) \simeq r^{D(N(k,\ell))}, \quad D(N(k,\ell)) = \sum_{m=1}^{\ell} m M_k(m) = \sum_{m=1}^{\ell} \sum_{d|m} \mu(d) k^{m/d}$$

and the assorted random walk estimate $\Phi_{N(k,\ell)}(n) \simeq n^{-D(N(k,\ell))/2}$.

In this case, the main result of the present work, together with Witt's theorem (e.g., [18, Theorem 11.2.2]), gives that for any k-tuple $a = (\alpha_i)_1^k \in (0,2)^k$, we have

$$\mu_{S,a}^{(n)}(e) \simeq n^{-D}$$

where

$$D = \sum_{m=1}^{\ell} \sum_{(m_1,\ldots,m_k)\vdash m} \frac{1}{m} \left(\sum_1^k \frac{m_i}{\alpha_i} \right) \sum_{d|m_1,\ldots,m_k} \mu(d) \binom{m/d}{m_1/d, \cdots, m_k/d}.$$

Example 3.1.6. Let G be the group

$$G = \langle u_1, \ldots, u_\ell, t | [u_i, u_j] = 1; [u_i, t] = u_{i+1}, i < \ell; [u_\ell, t] = 1 \rangle$$

defined by generators and relations. This group is nilpotent of nilpotency class ℓ and it is generated by $S = (s_1 = u_1, s_2 = t)$ with G_m generated by $\{u_i : i \ge m\}$. In this case, we have $\Phi_G(n) \simeq n^{-D(G)/2}$ with $D(G) = 1 + \ell(\ell+1)/2$. If we let $a = (\alpha_1, \alpha_2) \in (0,2)^2$, our main result yields $\mu_{S,a}^{(n)}(e) \simeq n^{-D}$ with

$$D = \frac{\ell}{\alpha_1} + \frac{1 + (\ell-1)\ell/2}{\alpha_2}.$$

In any of the above examples, we can also consider other choices of generating tuples. For instance, in the current example, we can fix $j \in \{1, \ldots, \ell - 1\}$ and consider the generating 3-tuple $S_j = (s_1 = u_1, s_2 = t, s_3 = u_{j+1})$ with $a' = (\alpha_1', \alpha_2', \alpha_3') \in (0,2)^3$. In this case, our main result yields $\mu^{(n)}_{S_j,a'}(e) \simeq n^{-D}$ with

$$D = \begin{cases} \frac{\ell}{\alpha_1'} + \frac{1+(\ell-1)\ell/2}{\alpha_2'} & \text{if } \frac{1}{\alpha_3'} \leq \frac{1}{\alpha_1'} + \frac{j}{\alpha_2'} \\ \frac{j}{\alpha_1'} + \frac{1+j(j+1)/2}{\alpha_2'} + \frac{\ell-j}{\alpha_3'} + \frac{(\ell-j)(\ell-j+1)/2}{\alpha_2'} & \text{if } \frac{1}{\alpha_3'} > \frac{1}{\alpha_1'} + \frac{j}{\alpha_2'}. \end{cases}$$

3.2 Quasi-norms and approximate coordinates

This section describes results of an algebraic and geometric nature that play a key role in our study to the random walks driven by the measures $\mu_{S,a}$ defined at (3.1). One of the basic idea in the study of simple random walks on groups (i.e., the collection of random walks driven by the uniform probability measures u_A where A is a finite symmetric generating set) is that the notion of "volume growth" of the group leads to basic upper bounds on $u_A^{(2n)}(e)$: the faster the volume growth, the faster the decay of the probability of return. In the case of nilpotent group, this heuristic leads to sharp bounds. Indeed, for any given $D \geq 0$, $V_A(n) \simeq n^D$ if and only if $u_A^{(2n)}(e) \simeq n^{-D/2}$. See [45].

The estimates of $\mu^{(n)}_{S,a}(e)$ obtained in this chapter are based on a similar heuristic which requires us to define appropriate geometries associated with the different choices of S and a. This section defines these geometries and develop the needed key results.

3.2.1 Weight systems and weight-functions systems

We refer the reader to subsection 3.1.4 for notation regarding words and formal commutators over a finite alphabet $S^{\pm 1}$, $S = (s_1, \dots, s_k)$.

Definition 3.2.1 (Multidimensional weight system)**.** Given a k-tuple $(w_1, \dots, w_k)$ with $w_i \in (0, \infty) \times \mathbb{R}^{d-1}$, $1 \le i \le k$, let $\mathfrak{w}$ be the weight system

$$\mathfrak{w} : \mathfrak{C}(S) \ni c \mapsto w(c) \in (0, \infty) \times \mathbb{R}^{d-1}$$

on the set $\mathfrak{C}(S)$ of all formal commutators on $S^{\pm 1}$ defined by $w(s_i^{\pm 1}) = w_i$ and $w(c) = w(c_1) + w(c_2)$ if $c = [c_1, c_2]$. Let

$$\bar{w}_1 < \bar{w}_2 < \cdots < \bar{w}_j < \dots$$

be the ordered sequence of the values $w(c)$ when c runs over all formal commutators and $(0, \infty) \times \mathbb{R}^{d-1}$ is given the usual lexicographic order.

Note that we always have $w([c_1, c_2]) > \max\{w(c_1), w(c_2)\}$.

Definition 3.2.2. For each $j = 1, \dots$, let $\mathfrak{C}_j(S)$ be the set of all formal commutators of weight at least $\bar{w}_j$. If G is a group generated by a k-tuple $S = (s_1, \dots, s_k)$, let $G_j^{\mathfrak{w}} = \langle \mathfrak{C}_j(S) \rangle$ be the subgroup of G generated by the image in G of $\mathfrak{C}_j(S)$. Assuming that G is nilpotent, let $j_* = j_*(\mathfrak{w})$ be the smallest integer such that $G_{j_*+1}^{\mathfrak{w}} = \{e\}$.

The proof of the following proposition is the same as that of Proposition 3.1.5.

Proposition 3.2.3. *Referring to the setting and notation of* Definition 3.2.2*, assume that G is nilpotent. Then, for all $j = 1, 2 \dots$, we have $G_j^{\mathfrak{w}} \subset G_{j+1}^{\mathfrak{w}}$ and $[G, G^{\mathfrak{w}}] \subset G_{j+1}^{\mathfrak{w}}$. In particular,*

$$G = G_1^{\mathfrak{w}} \supseteq G_2^{\mathfrak{w}} \supseteq \cdots \supseteq G_j^{\mathfrak{w}} \supseteq \cdots \supseteq G_{j_*}^{\mathfrak{w}} \supset G_{j_*+1}^{\mathfrak{w}} = \{e\}$$

is a descending normal series with $[G_j^{\mathfrak{w}}, G_j^{\mathfrak{w}}] \subset G_{j+1}^{\mathfrak{w}}$. *We let* $R_j^{\mathfrak{w}}$ *be the torsion free rank of the abelian group* $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$.

Definition 3.2.4 (Weight-function system)**.** Given increasing functions

$$F_i : [1, \infty) \to [1, \infty),$$

we define the weight-function system $\mathfrak{F}$ to be the collection of functions

$$F_c : [1, \infty) \to [1, \infty), \ c \in \mathfrak{C}(S),$$

by setting inductively $F_{s_i^{\pm 1}} = F_i$, $1 \le i \le k$, and $F_c = F_{c_1} F_{c_2}$ if $c = [c_1, c_2]$.

Remark 3.2.5. According to Definitions 3.2.1-3.2.4, if the build-sequence of the commutators c of length ℓ is $(u_1, \dots, u_\ell) \in S^\ell$ then

$$w(c) = \sum_1^\ell w_i, \quad F_c(r) = \prod_1^\ell F_i(r).$$

Remark 3.2.6. A key collection of examples of weight systems are the (one-dimensional) power-weight systems introduced in 3.1.3 where $w_i \in (0, \infty)$. Such a weight system is naturally associated with the weight-function system of power functions where $F_i(r) = r^{w_i}$. In the context of the study of the random walks driven by the measures $\mu_{S,a}$, these power weight systems and associated power function systems are relevant to the case when $a = (\alpha_i)_1^k \in (0, 2)^k$.

Example 3.2.1. In order to study the measures $\mu_{S,a}$ with tuples a with $\alpha_j = 2$ for some j, it is necessary to introduce weight functions of the type $r^2 \log r$. To allow for such functions, one can consider the two-dimensional weight systems build on

$$w_i = (u_i, v_i) \text{ with } u_i > 0 \text{ and } v_i \in \mathbb{R}, 1 \le i \le k.$$

In this case a natural compatible weight-function system would be

$$F_i(r) = r^{u_i} [\log(e + r)]^{v_i}, 1 \le i \le k.$$

Example 3.2.2. When dealing with more general measures than $\mu_{S,a}$, it makes sense to consider multiparameter weight functions such that

$$f_{v_1,v_2,v_3}(r) = r^{v_1}[\log(e+r)]^{v_2}[\log(e+\log(e+r))]^{v_3}, \quad v_1 \in (0,\infty), \quad v_2, v_3 \in \mathbb{R},$$

together with the natural associated lexicographical order on the parameter space (v_1, v_2, v_3).

In what follows we will mostly use weight-function systems $\mathfrak{F}$ such that

$$\exists\, C \geq 1, \forall i \in \{1,\ldots,k\},\ \forall\, r \geq 1,\ \ 2F_i(r) \leq F_i(Cr),\ F(2r) \leq CF(r). \tag{3.1}$$

Further, we will often make the assumption that we are given a weight system $\mathfrak{w}$ and a weight-function system $\mathfrak{F}$ that are compatible in the sense that

$$\exists\, C \geq 1,\ \forall\, c, c',\ \ w(c) \preceq w(c') \Longleftrightarrow \forall r,\ \ F_c(r) \leq CF_{c'}(r). \tag{3.2}$$

Note that under these two hypotheses, $w(c) = w(c')$ is equivalent to $F_c \simeq F_{c'}$. In this case, except for notational convenience, it is obviously somewhat redundant to use both $\mathfrak{w}$ and $\mathfrak{F}$ since they contain more or less the same information.

Definition 3.2.7. Referring to the setting and notation introduced above, assume that the weight-function system $\mathfrak{F}$ and the weight system $\mathfrak{w}$ satisfy (3.1)-(3.2). For any $j = 1, \ldots, j_*$, let $\mathbf{F}_j$ be a function such that for any commutator c with $w(c) = \bar{w}_j$, we have

$$\mathbf{F}_j \simeq F_c.$$

(The function $\mathbf{F}_j$ corresponding to commutators c with $w(c) = \bar{w}_j$ should not be confused $F_i = F_{s_i}$).

In the following definition, given a finite tuple Σ of elements of a nilpotent group G, we let $\Omega(\Sigma)$ be the set of all finite words with formal letters in $\Sigma \cup \Sigma^{-1}$.

For $\omega \in \Omega(\Sigma)$, we write $\pi(\omega)$ to denote the corresponding element of G. For $\omega \in \Omega(\Sigma)$ and $\sigma \in \Sigma$, let $\deg_\sigma(\omega)$ is the number of occurrences of $\sigma^{\pm 1}$ in ω.

Definition 3.2.8. Let G be a nilpotent group generated by the k-tuple $S = (s_1, \ldots, s_k)$. Let $\mathfrak{w}, \mathfrak{F}$ be a weight system and associated weight function system on a generating k-tuple S which satisfy (3.1)-(3.2). For any tuple Σ of elements in $\mathfrak{C}(S)$, set $F_\Sigma = F_c$ where $w(c) = \min\{w(\sigma) : \sigma \in \Sigma\}$. For $g \neq e$, set

$$\|g\|_{\Sigma,\mathfrak{F}} = \min\{r \geq 1 : g = \pi(\omega) : \omega \in \Omega(\Sigma),\ \deg_c(\omega) \leq F_c \circ F_\Sigma^{-1}(r), c \in \Sigma\}.$$

By convention, $\|e\|_{\Sigma,\mathfrak{F}} = 0$. Set also

$$Q(\Sigma, \mathfrak{F}, r) = \{g \in G : F_\Sigma^{-1}(\|g\|_{\Sigma,\mathfrak{F}}) \leq r\}.$$

Further, when S and $\mathfrak{w}, \mathfrak{F}$ are fixed, set

$$\|g\|_{\text{com}} = \|g\|_{\mathfrak{F},\text{com}} = \|g\|_{\mathfrak{C}(S),\mathfrak{F}}, \quad \|g\|_{\text{gen}} = \|g\|_{\mathfrak{F},\text{gen}} = \|g\|_{S,\mathfrak{F}}$$

and

$$Q_{\text{com}}(r) = Q(\mathfrak{C}(S), \mathfrak{F}, r), \quad Q_{\text{gen}}(r) = Q(S, \mathfrak{F}, r).$$

Note that $F_S = F_{\mathfrak{C}(S)}$.

Remark 3.2.9. If Σ generates G then $\|\cdot\|_{\Sigma,\mathfrak{F}}$ is a quasi-norm on G (see 3.5.1 below for a precise definition). It is a norm on G (i.e., satisfies the triangle inequality) if each of the functions $\{F_c \circ F_\Sigma^{-1}\ ,\ c \in \Sigma\}$, defined on $[1, \infty)$ can be extended to a convex function on $[0, \infty)$ that vanishes at 0.

Example 3.2.3. The simplest example is when the weight system $\mathfrak{w}$ is one dimensional, generated by $w(s_i) = w_i \in [2, \infty)$, and the associated weight function system $\mathfrak{F}$ is generated by $F_i(r) = r^{w_i}$. In this case, it will sometimes be convenient to write $\|\cdot\|_{S,\mathfrak{w}}$ for $\|\cdot\|_{S,\mathfrak{F}}$ (resp. $\|\cdot\|_{\Sigma,\mathfrak{w}}$ for $\|\cdot\|_{\Sigma,\mathfrak{F}}$).

Example 3.2.4. For further illustration, consider the groups $\mathbb{Z}^3$ equipped with its natural generating 3-tuple $S = (s_i)_1^3$ and the discrete Heisenberg group (see Example 3.1.1) equipped with the generating 3-tuple $S = (s_1 = X, s_2 = Y, s_3 = Z)$ where X is the matrix with $x = 1, y = z = 0$ and Y, Z are defined similarly. Set $F_1(r) = r^{3/2}$, $F_2(r) = r^2 \log(e+r)$, $F_3(r) = r^\gamma$, $\gamma > 3/2$, and let $\mathfrak{F}$ be the associated weight-function system (we let the reader define the natural 2-dimensional weight system $\mathfrak{w}$ that is compatible with $\mathfrak{F}$).

On $\mathbb{Z}^3$, it is clear from the definition that

$$\|(x, y, x)\|_{\mathfrak{F},\mathrm{gen}} \simeq \max\left\{|x|, \frac{|y|^{3/4}}{\log(e+|y|)^{3/4}}, |z|^{3/(2\gamma)}\right\}.$$

On the Heisenberg group, it is not immediately obvious how to compute the $\|\cdot\|_{\mathfrak{F},\mathrm{gen}}$-norm of the element

$$g_{x,y,z} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem 3.2.10 below (and the fact that the matrix representation of $g_{x,y,z}$ is unique) leads to the conclusion that

$$\|g_{x,y,z}\|_{\mathfrak{F},\mathrm{gen}} \simeq \max\left\{|x|, \frac{|y|^{3/4}}{\log(e+|y|)^{3/4}}, |z|^{3/(2\gamma)}\right\} \text{ if } \gamma > 7/2$$

and

$$\|g_{x,y,z}\|_{\mathfrak{F},\mathrm{gen}} \simeq \max\left\{|x|, \frac{|y|^{3/4}}{\log(e+|y|)^{3/4}}, \frac{|z|^{3/7}}{[\log(e+|z|)]^{3/7}}\right\} \text{ if } 3/2 \leq \gamma \leq 7/2.$$

One can check (without much trouble) that $\|\cdot\|_{\mathfrak{F},\mathrm{gen}}$ satisfies the triangle inequality in this case (on either $\mathbb{Z}$ or the Heisenberg group). We shall see that this choice of weight-function system is relevant to the study of the probability measure

μ on G such that

$$\mu(s_i^n) \text{ is proportional to } \frac{1}{1+|n|F_i^{-1}(|n|)},\ n \in \mathbb{Z}.$$

We will use this example to illustrate some of our main results in the rest of the paper.

The following theorem contains some of the key geometric results we will need to study the walk driven by measures of the type $\mu_{S,a}$.

Theorem 3.2.10 ($\mathfrak{w}$-F-adapted coordinates)**.** *Let G be a nilpotent group equipped with a generating k-tuple $S = (s_1, \dots, s_k)$. Let $\mathfrak{w}$, $\mathfrak{F}$ be weight and weight-function systems on S satisfying* (3.1)-(3.2).

Let $\Sigma = (c_1, \dots, c_t)$ be a tuple of formal commutators in $\mathfrak{C}(S)$ with non-decreasing weights $w(c_1) \preceq \cdots \preceq w(c_t)$. Let m_j, $j = 0, \dots, j_*$ be defined by

$$\{c_i : w(c_i) = \bar{w}_j\} = \{c_i : m_{j-1} < i \leq m_j\}.$$

Assume that (the image of) $\{c_i : w(c_i) = \bar{w}_j\}$ generates $G_j^{\mathfrak{w}}$ modulo $G_{j+1}^{\mathfrak{w}}$ and that $\{c_i : m_{j-1} < i \leq m_{j-1} + R_j^{\mathfrak{w}}\}$ is free in $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$. Then the following properties hold:

- *There exists a constant $C = C(G, S, \mathfrak{F})$ such that for any $r \geq 1$, if $g \in G$ can be expressed as a word ω over $\mathfrak{C}(S)$ with $\deg_c(\omega) \leq F_c(r)$ for all $c \in \mathfrak{C}(S)$ then g can be expressed in the form*

$$g = \prod_{i=1}^{t} c_i^{x_i} \ \textit{with}\ |x_i| \leq C \times \begin{cases} \mathbf{F}_j(r) & \textit{if } m_{j-1}+1 \leq i \leq R_j^{\mathfrak{w}} \\ 1 & \textit{if } R_j^{\mathfrak{w}}+1 \leq i \leq m_j. \end{cases}$$

- *There exist an integer $p = p(G, S, \mathfrak{F})$, a constant $C = C(G, S, \mathfrak{F})$ and a sequence $(i_1, \dots, i_p) \in \{1, \dots, k\}^p$ such that if g can be expressed as a word*

ω over $\mathfrak{C}(S)$ with $\deg_c(\omega) \le F_c(r)$ for some $r \ge 1$ and all $c \in \mathfrak{C}(S)$ then g can be expressed in the form

$$g = \prod_{j=1}^{p} s_{i_j}^{x_j} \text{ with } |x_j| \le CF_{i_j}(r).$$

This important theorem will be proved in the last section of this article. See also Theorem 3.6.22 for an additional improvement of the the last statement of Theorem 3.2.10. Note that in the decomposition $g = \prod_{j=1}^{p} s_{i_j}^{x_j}$, the sequence $(i_j)_1^p$ is independent of the group element g.

The proof of the following simple corollary is omitted.

Corollary 3.2.11. *Referring to* Definition 3.2.8*, the quasi-norms $\|\cdot\|_{\text{com}}$ and $\|\cdot\|_{\text{gen}}$ defined on G satisfy*

$$\|\cdot\|_{\text{gen}} \simeq \|\cdot\|_{\text{com}} \text{ over } G.$$

Further, referring to the t-tuple $\Sigma = (c_1, \ldots, c_t)$ of Theorem 3.2.10*, we have*

$$F_{\Sigma}^{-1}(\|\cdot\|_{\Sigma,\mathfrak{F}}) \simeq F_S^{-1}(\|\cdot\|_{\text{com}}) \text{ over } G.$$

Remark 3.2.12. In the case when the generators s_i are given equal weight-functions, i.e., $F_i = F_j$, $1 \le i \le j \le k$, the quasi-norms $\|\cdot\|_{S,\mathfrak{F}}$, $\|\cdot\|_{\Sigma,\mathfrak{F}}$ and $\|\cdot\|_{\mathfrak{C}(S),\mathfrak{F}}$ are all comparable to the usual word-norm $|\cdot|_S$.

3.2.2 Norm equivalences

In this section, we briefly discuss how changing weight functions affect the quasi-norms $\|\cdot\|_{\text{com}}$ and $\|\cdot\|_{\text{gen}}$ introduced in Definition 3.2.8.

Definition 3.2.13. Let G be a countable nilpotent group equipped with a generating k-tuple $S = (s_1, \ldots, s_k)$ and a (possibly multidimensional) weight system $\mathfrak{w}$

as above. For each $g \in G$, let

$$j_{\mathfrak{w}}(g) = \max\{j : \exists u \in \mathbb{N},\ g^u \in G_j^{\mathfrak{w}}\}.$$

Let $\mathrm{core}(\mathfrak{w}, S)$ be the sub-sequence of S obtained by keeping only those s_i such that $w(s_i) = \overline{w}_{j_{\mathfrak{w}}(s)}$.

By construction, we always have $w(s) \leq \bar{w}_{j_{\mathfrak{w}}(s)}$. Those generators $s \in S$ with $w(s) < \bar{w}_{j(s)}$ are, in some sense, inefficient. The following proposition makes this precise and motivates this definition.

Proposition 3.2.14. *Any formal commutator $c \in \mathfrak{C}(S)$ whose image in G is free in $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$ must only use letters in* $\mathrm{core}(\mathfrak{w}, S)$*. In particular, referring to the sequence of commutators $c_1, \ldots, c_t$ in* Theorem 3.2.10*, any formal commutator c_i with $i \in m_{j-1}+1, \ldots, m_{j-1}+R_j^{\mathfrak{w}}$ must only use letters in* $\mathrm{core}(\mathfrak{w}, S)$*.*

Proof. Assume that the image of c is in the torsion free part of $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$ and involves $s \notin \mathrm{core}(S)$, say $c = [c', [s, c'']]$. Then $\exists u \in \mathbb{N},\ s^u \in G_{j(s)}^{\mathfrak{w}}$ with $\overline{w}_{j(s)} > w(s)$ (where we write $j(s) = j_{\mathfrak{w}}(s)$). From the linearity of brackets, we have

$$c^u \equiv [c', [s^u, c'']] \mod G_{j+1}^{\mathfrak{w}}$$

while $[c', [s^u, c'']] \in G_{j+1}^{\mathfrak{w}}$ since $s^u \in G_{j(s)}^{\mathfrak{w}}$ with $\overline{w}_{j(s)} > w(s)$. Therefore

$$c^u \equiv 0 \mod G_{j+1}^{\mathfrak{w}}.$$

This contradicts the assumption that c is free in $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$. The proposition follows. $\square$

Definition 3.2.15. Let G be a countable nilpotent group equipped with a generating k-tuple $S = (s_1, \ldots, s_k)$ and a (possibly multidimensional) weight system

$\mathfrak{w}$ as above. Let $\Sigma = (c_1, \dots, c_t)$ be a sequence of formal commutators as in Theorem 3.2.10. Let $\mathrm{core}(\mathfrak{w}, S, \Sigma)$ be the sub-sequence of S of those letters s_δ that appear in the build-sequence of one or more of the formal commutators $c_i \in \Sigma$ with $i \in \cup_{j=1}^{q+1}\{m_{j-1}+1, \dots, m_{j-1}+R_j^{\mathfrak{w}}\}$.

Remark 3.2.16. Proposition 3.2.14 shows that, for any sequence Σ of formal commutators as in Theorem 3.2.10, we have

$$\mathrm{core}(\mathfrak{w}, S, \Sigma) \subset \mathrm{core}(\mathfrak{w}, S).$$

In what follows, given two tuples $S = \{s_1, \dots, s_k)$, $\Theta = (\theta_1, \dots, \theta_\kappa)$ of elements of G (possibly of different length k, κ), we write $S \subset \Theta$ if there is a one to one map $J : \{1, \dots, k\} \to \{1, \dots, \kappa\}$ such that $s_{J(i)} = \theta_i$ in G. This applies, for instance, to the "inclusion" $\mathrm{core}(\mathfrak{w}, S, \Sigma) \subset \mathrm{core}(\mathfrak{w}, S)$ in the previous remark. Abusing notation, we will sometimes use the same letter s to denote an element of S and the associated element in Θ.

Proposition 3.2.17. *Referring to the setting and notation of* Theorem 3.2.10*, for each $g \in G$ either G is a torsion element and $\|g^n\|_{\mathrm{com}} \simeq 1$ for all n or*

$$\forall\, n, \quad \|g^n\|_{\mathrm{com}} \simeq F_S \circ \mathbf{F}_j^{-1}(n) \ \textit{where } j = j_{\mathfrak{w}}(g). \tag{3.3}$$

Proof. The upper bound is very easy. Let κ be such that $g^\kappa \in G_j^{\mathfrak{w}}$, $j = j_{\mathfrak{w}}(g)$. Since g^κ is in $G_j^{\mathfrak{w}}$ it can be written as word ω using formal commutators of weight at least $\bar{w}_j$. Hence, $g^{\kappa n}$ can be written as a word ω_n, namely, ω repeated n times. Obviously, if $w(c) \geq \bar{w}_j$, $\deg_c(\omega_n) \leq \deg_c(\omega)n$. By definition, this implies $\|g^{\kappa n}\|_{\mathrm{com}} \leq CF_S \circ \mathbf{F}_j^{-1}(n)$. The estimate $\|g^n\|_{\mathrm{com}} \leq C' F_S \circ \mathbf{F}_j^{-1}(n)$ easily follows.

The lower bound is more involved. Using Theorem 3.2.10, it suffices to show that any writing of $g^{\kappa n}$ as a product

$$g^{\kappa n} = \prod_1^t c_i^{x_i} \text{ with } |x_i| \leq C \text{ for } i \in \cup_h \{m_{h-1} + R_h^{\mathfrak{w}} + 1, \ldots, m_h\} \tag{3.4}$$

must have $\max_{i \in \{m_{j-1}+1, \ldots, m_{j-1}+R_j^{\mathfrak{w}}\}} \{|x_i|\} \geq cn$. First, we claim that there exists a constant T (independent of g but depending on the structure of G, S, the weight system $\mathfrak{w}$ and the constant C appearing in the above displayed equation) such that for any n and any writing of $g^{\kappa n}$ as above we have

$$|x_i| \leq T \text{ for all } i \leq m_{h-1}, h \leq j. \tag{3.5}$$

The proof is by induction on $h \leq j$. There is nothing to prove for $h = 1$. Assume that $h + 1 \leq j$ and that we have proved that $|x_i| \leq T$ for all $i \leq m_{h-1}$. Since $g^{\kappa}, g^{\kappa n} \in G_h^{\mathfrak{w}}$, the product $\sigma = \prod_1^{m_{h-1}} c_i^{x_i}$ is in $G_h^{\mathfrak{w}}$. Since $|x_i| \leq T$, $i \leq m_{h-1}$, $\sigma = \prod_{i > m_{h-1}} c_i^{z_i}$ with $|z_i| \leq T'$ where T' depends only on $G, S, \mathfrak{w}, T$ but not on g, n. Computing in $G_h^{\mathfrak{w}}$ modulo $G_{h+1}^{\mathfrak{w}}$, we have

$$g^{\kappa n} = \prod_{m_{h-1}+1}^{m_h} c_i^{x_i + z_i} = e \mod G_{h+1}^{\mathfrak{w}}.$$

The last equality holds because $g^{\kappa n} \in G_h^{\mathfrak{w}}$ and $h + 1 \leq j$. Since

$$\{c_{m_{h-1}+1}, \ldots, c_{m_{h-1}+R_h^{\mathfrak{w}}}\}$$

is free in $G_h^{\mathfrak{w}}/G_{h+1}^{\mathfrak{w}}$ and $\sup_i |z_i| \leq T'$, $\sup\{|x_i| : m_{h-1} + R_h^{\mathfrak{w}} + 1 \leq i \leq m_h\} \leq C$, there is a constant T'' depending only on $G, S, \mathfrak{w}, C$ and T' such that $|x_i| \leq T''$ for $i \in \{m_{h-1} + 1, \ldots, m_{h-1} + R_h^{\mathfrak{w}}\}$. This proves (3.5).

On the one hand, since j is the largest integer such that $g^u \in G_j^{\mathfrak{w}}$ for some u, it follows that for any n we can write

$$g^{\kappa n} = \prod_{i=m_{j-1}+1}^{m_j} c_i^{y_i} \mod G_{j+1}^{\mathfrak{w}} \text{ with } \sum_{i=m_{j-1}+1}^{m_{j-1}+R_j^{\mathfrak{w}}} |y_i| \geq cn$$

and

$$\max\{|y_i| : m_{j-1} + R_j^{\mathfrak{w}} + 1 \leq i \leq m_j\} \leq C'.$$

On the other hand, since any writing of $g^{\kappa n}$ as in (3.4) satisfies (3.5), the same reasoning as in the induction step for (3.5) gives

$$g^{\kappa n} = \prod_{m_{j-1}+1}^{m_j} c_i^{y_i - x_i - z_i} = e \mod G_{j+1}^{\mathfrak{w}}$$

with $|z_i| \leq T$. Since $\{c_i : m_{j-1} + 1 \leq i \leq m_{j-1} + R_j^{\mathfrak{w}}\}$ is free, the facts that

$$\sum_{i=m_{j-1}+1}^{m_{j-1}+R_j^{\mathfrak{w}}} |y_i| \geq cn, \;\; \max\{|y_i| : m_{j-1} + R_j^{\mathfrak{w}} + 1 \leq i \leq m_j\} \leq C'$$

and $|z_i| \leq T$ together imply that

$$\sum_{i=m_{j-1}+1}^{m_{j-1}+R_j^{\mathfrak{w}}} |x_i| \geq c'n.$$

Hence, $\|g^{\kappa n}\|_{\text{com}} \simeq F_S \circ \mathbf{F}_j^{-1}(n)$. □

Theorem 3.2.18. *Let G be a countable nilpotent group equipped with two generating tuples S, S' and associated multidimensional weight systems $\mathfrak{w}, \mathfrak{w}'$ as well as weight function systems $\mathfrak{F}, \mathfrak{F}'$ satisfying* (3.1)-(3.2)*. By definition, F_S and $F'_{S'}$ are the weight functions associated with the smallest weights in $\mathfrak{w}$ and $\mathfrak{w}'$, respectively. Let $\Sigma = (c_1, \ldots, c_t)$ be a sequence of formal commutators as in* Theorem 3.2.10 *applied to $(S, \mathfrak{w}, \mathfrak{F})$.*

1. *Assume that $S' \supset \mathrm{core}(\mathfrak{w}, S, \Sigma)$ and $F'_s \geq F_s$ for all $s \in \mathrm{core}\,(\mathfrak{w}, S, \Sigma)$. Then*

$$\forall\, g \in G, \;\; (F'_{S'})^{-1}(\|g\|_{S',\mathfrak{F}'}) \leq C F_S^{-1}(\|g\|_{S,\mathfrak{F}})$$

2. *Assume that, for all $s \in S'$, $F'_s \leq \mathbf{F}_{j_{\mathfrak{w}}(s)}$. Then*

$$\forall\, g \in G, \;\; (F'_{S'})^{-1}(\|g\|_{S',\mathfrak{F}'}) \geq c F_S^{-1}(\|g\|_{S,\mathfrak{F}})$$

Proof. To prove the first statement, referring to the notation used in Theorem 3.2.10, Set

$$I_1 = \cup_j \{m_{j-1}+1, \dots, m_{j-1}+R_j^{\mathfrak{w}}\}, \quad I_2 = \{1,\dots,t\} \setminus I_1$$

and recall that any any $g \in G$ can be written as

$$g = \prod_1^t c_i^{x_i}, \quad |x_i| \le C \begin{cases} F_{c_i}(F_S^{-1}(\|g\|_{\mathrm{com}})) & \text{if } i \in I_1 \\ 1 & \text{if } i \in I_2. \end{cases}$$

By hypothesis, $F'_{c_i} \ge F_{c_i}$ for $i \in I_1$. Further, each c_i, $i \in I_2$, is a product of elements in S'. Hence, we obtain an expression for g as a word ω on formal commutators on S' with $\deg_c(\omega) \le CF'_c(F_S^{-1}(\|g\|_{\mathrm{com}}))$. This proves that $(F'_{S'})^{-1}(\|g\|_{S',\mathfrak{F}'}) \le CF_S^{-1}(\|g\|_{S,\mathfrak{F}})$ as desired.

To prove the second statement, apply Theorem 3.2.10(iii) to $(S', \mathfrak{w}', \mathfrak{F}')$ to write any $g \in G$ as a product

$$g = \prod_1^p (s'_{i_j})^{x_j} \text{ with } |x_j| \le F'_{s'_{i_j}} \circ (F'_{S'})^{-1}(\|g\|_{S',\mathfrak{F}'})$$

where $s'_{i,j} \in S'$ (note that the sequence (i_j) and the integer p are fixed and independent of g). By Proposition 3.2.17 and the hypothesis $\mathbf{F}_{j_{\mathfrak{w}}(s)} \ge F'_s$ for all $s \in S'$, we obtain that $F_S^{-1}(\|g\|_{S,\mathfrak{F}}) \le C(F'_{S'})^{-1}(\|g\|_{S',\mathfrak{F}'})$ as desired. $\square$

Corollary 3.2.19. *Let G be a countable nilpotent group equipped with two generating tuple S, S' and associated multidimensional weight systems $\mathfrak{w}, \mathfrak{w}'$ with function systems $\mathfrak{F}, \mathfrak{F}'$ satisfying* (3.1)-(3.2)*. Let $\Sigma = (c_1, \dots, c_t)$ be a sequence of formal commutators as in* Theorem 3.2.10 *applied to $(S, \mathfrak{w}, \mathfrak{F})$. Assume that there exists $C \in (0, \infty)$ such that the following two conditions are satisfied:*

(i) $\mathrm{core}(\mathfrak{w}, S, \Sigma) \subset S'$ *and,* $\forall\, s \in \mathrm{core}(\mathfrak{w}, S, \Sigma)$, $CF'_s \ge F_s$.

(ii) $\forall\, s \in S',\ \ F'_s \leq C\mathbf{F}_{j_{\mathfrak{w}}(s)}$.

Then

$$\forall\, g \in G,\ \ (F'_{S'})^{-1}(\|g\|_{S',\mathfrak{F}'}) \simeq F_S^{-1}(\|g\|_{S,\mathfrak{F}}).$$

In particular,

$$\forall\, r > 0,\ \ \#Q(S', \mathfrak{F}', r) \simeq \#Q(S, \mathfrak{F}, r).$$

Example 3.2.5 (Continuation of Example 3.2.4)**.** Consider the discrete Heisenberg group as in Example 3.2.4 equipped with the generating 3-tuple $S = (s_1 = X, s_2 = Y, s_3 = Z)$ and $S' = (s'_i = X, s'_2 = Y)$. Set $F_1(r) = F'_1(r) = r^{3/2}$, $F_2(r) = F'_2(r) = r^2 \log(e + r)$, $F_3(r) = r^\gamma$, $\gamma > 3/2$, and let $\mathfrak{F}, \mathfrak{F}'$ be the associated weight-function systems. The natural 2 dimensional weight systems $\mathfrak{w}, \mathfrak{w}'$ are generated by $w_1 = w'_1 = (3/2, 0)$, $w_2 = w'_2 = (2, 1)$, $w_3 = (\gamma, 0)$. The first observation is that $\mathrm{core}(\mathfrak{w}, S) = (s_1, s_2, s_3)$ is $\gamma > 7/2$ and $\mathrm{core}(\mathfrak{w}, S) = (s_1, s_2)$ if $3/2 < \gamma \leq 7/2$. It follows that, $\forall\, g \in G$, $\|g\|_{S',\mathfrak{F}'} \simeq \|g\|_{S,\mathfrak{F}}$ if $\gamma \in (3/2, 7/2]$ whereas these norms are not equivalent if $\gamma > 7/2$.

3.3 Volume estimates

This section gathers some of the main results we will need regrading volume estimates for the balls $Q(S, \mathfrak{F}, r)$ introduced in Definition 3.2.8. It also addresses the question of how changes in the weight-function system affect these volume estimates.

We start with a general and very flexible result which admits a rather simple proof. In this theorem, the weight-function system $\mathfrak{F}$ is not necessarily tightly related to the weight system $\mathfrak{w}$. The proof of this theorem will be given in the last section of this chapter.

Theorem 3.3.1. *Let $\mathfrak{w}$ be a multidimensional weight system as in* Section 3.2.1. *Assume that we are given weight functions F_i, $1 \le i \le k$ satisfying* (3.1). *Let $\Sigma = (c_1, \dots, c_s)$ be a s-tuple of formal commutators on $\{s_i^{\pm 1} : 1 \le i \le k\}$. Assume that, for any h, the family $\{c_i : w(c_i) = \bar{w}_h\}$ projects to a free family in the abelian group $G_h^{\mathfrak{w}}/G_{h+1}^{\mathfrak{w}}$. Then there exist an integer $M = M_\Sigma$ and a sequence $(i_1, \dots, i_M) \in \{1, \dots, k\}^M$, depending on Σ such that for any $r > 0$ there exists a subset $K_\Sigma(r) \subset G$ satisfying the following two properties:*

1. $\#K_\Sigma(r) \ge \prod_{i=1}^{s}(2F_{c_i}(r) + 1)$
2. $g \in K_\Sigma(r) \Longrightarrow g = \prod_{j=1}^{M} s_{i_j}^{x_j}$, $|x_j| \le F_{i_j}(r)$.

Further, every s_{i_j}, $1 \le j \le M$, belongs to the build-sequence of at least one $c_h \in \Sigma$.

Theorem 3.3.1 is very useful for comparing the volume growth associated with different "weight-function systems". See the proof of Theorem 3.3.4 below.

Next we state and prove sharp volume estimates related to Theorem 3.2.10.

Theorem 3.3.2. *Referring the setting and notation of* Theorem 3.2.10, *we have*

$$\#Q(\mathfrak{C}(S), \mathfrak{F}, r) \simeq \#Q(\Sigma, \mathfrak{F}, r) \simeq \#Q(S, \mathfrak{F}, r) \simeq \prod_{j=1}^{j_*} \mathbf{F}_j(r)^{R_j^{\mathfrak{w}}}.$$

Remark 3.3.3. Assume that the weight system $\mathfrak{w}$ is unidimensional, generated by $(w_i)_1^k \in (0, \infty)^k$, and the weight-functions F_i are power functions $F_i(r) = r^{\mathfrak{w}_i}$, $i = 1, \dots, k$. Then

$$Q(S, \mathfrak{F}, r) \simeq r^{D(S,\mathfrak{w})}$$

with $D(S, \mathfrak{w})$ as in Definition 3.1.7.

Proof. The equivalences $\#Q(\mathfrak{C}(S), \mathfrak{F}, r) \simeq \#Q(\Sigma, \mathfrak{F}, r) \simeq \#Q(S, \mathfrak{F}, r)$ and the upper bound $\#Q(\Sigma, \mathfrak{F}, r) \leq C \prod_{j=1}^{j_*} \mathbf{F}_j(r)^{R_j^{\mathfrak{w}}}$ follows immediately from Theorem 3.2.10 and inspection.

The lower bound $\#Q(\Sigma, \mathfrak{F}, r) \geq c \prod_{j=1}^{j_*} \mathbf{F}_j(r)^{R_j^{\mathfrak{w}}}$ requires an additional argument. Note that $Q(\Sigma, \mathfrak{F}, r)$ contains the image in G of

$$\prod_{j=1}^{j_*} \prod_{i=m_{j-1}+1}^{m_{j-1}+R_j} c_i^{x_i}, \quad |x_i| \leq F_{c_i}(r).$$

Further, it is not hard to check that

$$\prod_j \prod_{i=m_{j-1}+1}^{m_{j-1}+R_j} c_i^{x_i} = \prod_j \prod_{i=m_{j-1}+1}^{m_{j-1}+R_j} c_i^{y_i}$$

implies

$$x_i = y_i, \quad i \in \bigcup_{j=1}^{j_*} \{m_{j-1}+1, \ldots, m_{j-1}+R_j\}.$$

The desired lower bound follows. $\square$

Theorem 3.3.4. *Let G be a countable nilpotent group equipped with two generating tuples S, S' and associated multidimensional weight systems $\mathfrak{w}, \mathfrak{w}'$ as well as weight function systems $\mathfrak{F}, \mathfrak{F}'$ satisfying* (3.1)-(3.2). *Let $\Sigma = (c_1, \ldots, c_t)$ be a sequence of formal commutators as in* Theorem 3.2.10 *applied to $(S, \mathfrak{w}, \mathfrak{F})$. Assume that $S' \supset \mathrm{core}(\mathfrak{w}, S, \Sigma)$ and that*

$$F'_s \geq F_s \text{ for all } s \in \mathrm{core}(\mathfrak{w}, S, \Sigma).$$

Then

$$\#Q(S', \mathfrak{F}', r) \simeq \prod_{j=1}^{j_*(\mathfrak{w}')} \mathbf{F}'_j(r)^{R_j^{\mathfrak{w}'}} \geq \#Q(S, \mathfrak{F}, r) \simeq \prod_{j=1}^{j_*(\mathfrak{w})} \mathbf{F}_j(r)^{R_j^{\mathfrak{w}}}.$$

Assume further that there exists $\sigma \in S'$ such that $F'_\sigma \geq \mathbf{F}_{j_{\mathfrak{w}}(\sigma)}$. Then

$$\#Q(S', \mathfrak{F}', r) \geq c \left(\frac{F'_\sigma(r)}{\mathbf{F}_{j_{\mathfrak{w}}(\sigma)}(r)} \right) \#Q(S, \mathfrak{F}, r).$$

Proof. Since $\operatorname{core}(\mathfrak{w}, S, \Sigma) \subset S'$ it follows that, for any $c_i \in \Sigma$, F'_{c_i} is well defined as the product of F'_s with $s \in \operatorname{core}(\mathfrak{w}, S, \Sigma) \subset S'$. Use the collection of commutators c_i, $i \in \{m_{j-1}+1, \ldots, m_{j-1}+R_j^{\mathfrak{w}}\}$, $j = 1, \ldots, j_*$ in Theorem 3.2.10 with the weight system $\mathfrak{w}$ and weight-function system $\mathfrak{F}'$. For each r, Theorem 3.3.1 provides a set $K(r) \in G$ such that

$$\#K(r) \geq \prod_{j=1}^{j_*(\mathfrak{w})} \prod_{i=m_{j-1}+1}^{m_{j-1}+R_i^{\mathfrak{w}}} F'_{c_i}(r) \tag{3.6}$$

and, by Theorem 3.2.10 , Theorem 3.3.1 and the definition of $\operatorname{core}(\mathfrak{w}, S, \Sigma)$,

$$K(r) \subset \{g \in G : \|g\|_{S', \mathfrak{F}'} \leq F'_{S'}(r)\}.$$

By Theorem 3.3.2, it follows that

$$\forall\, r, \;\; \#K(r) \leq \#Q(S', \mathfrak{F}', r).$$

By hypothesis, $F'_s \geq F_s$ if $s \in \operatorname{core}(\mathfrak{w}, S, \Sigma)$. Hence $F'_{c_i} \geq F_{c_i}$ (i.e., $w'(c_i) \geq w(c_i)$). By (3.6) and Theorem 3.3.2, this implies $\#K(r) \geq c \prod_{j=1}^{j_*(\mathfrak{w})} \mathbf{F}_j^{R_j^{\mathfrak{w}}}$. This proves the first statement.

Suppose now that there exists $\sigma \in S'$ such that $w'(s) > \bar{w}_{j_{\mathfrak{w}}(\sigma)}$. Set $j_0 = j_{\mathfrak{w}}(\sigma)$. In the sequence of commutators $c_1, \ldots, c_t$ used above, consider the the free family

$$\{c_i : i \in \{m_{j_0-1}+1, \ldots, m_{j_0-1}+R_{j_0}^{\mathfrak{w}}\}\} \text{ in } G_{j_0}^{\mathfrak{w}}/G_{j_0+1}^{\mathfrak{w}}.$$

By hypothesis, there exists an integer u such that $\sigma^u \in G_{j_0}^{\mathfrak{w}}$ is free in $G_{j_0}^{\mathfrak{w}}/G_{j_0+1}^{\mathfrak{w}}$. Since a maximal free subset of $\{\sigma^u\} \cup \{c_i : i \in \{m_{j_0-1}+1, \ldots, m_{j_0-1}+R_{j_0}^{\mathfrak{w}}\}\}$ in $G_{j_0}^{\mathfrak{w}}/G_{j_0+1}^{\mathfrak{w}}$ containing σ^u must contain $R_{j_0}^{\mathfrak{w}}$ elements, we can replace one of the c_i, say c_{i_*} by σ^u so that the $R_{j_0}^{\mathfrak{w}}$-tuple so obtained is free in $G_{j_0}^{\mathfrak{w}}/G_{j_0+1}^{\mathfrak{w}}$. Let $b_i = c_i$ if $i \neq i_*$, $b_{i_*} = \sigma^u$, $\widetilde{F}^i = F'_{c_i}$ if $i \neq i_*$, $\widetilde{F}^{i_*}(r) = F'_\sigma(r/|u|)$, and apply Theorem 3.6.4. The desired result follows. $\square$

3.4 Random walk upper bounds

This section is devoted to obtaining upper bounds on the return probability of a large collection of random walks including those driven by the measures $\mu_{S,a}$. Generalizing one of the approaches developed in [45] for simple random walks, we will make use of appropriate volume growth estimates and of the notion of pseudo-Poincaré inequality.

3.4.1 Pseudo-Poincaré inequality

Let G be a group generated by a finite symmetric set A. Then it holds that for any finitely supported function f on G,

$$\|f_g - f\|_2^2 \leq C_A |g|_A^2 \mathcal{E}_A(f,f) \tag{3.1}$$

where

$$\mathcal{E}_A(f,f) = \frac{1}{2|A|} \sum_{x \in G, y \in A} |f(xy) - f(x)|^2.$$

This expression is the Dirichlet form associated with the simple random walk based on A. Inequality (3.1) captures a fundamental universal property of Cayley graphs. In [45], it is proved that this simple property implies interesting upper-bounds on $u_A^{(2n)}(e)$ in terms of the volume growth function V_A.

The main result of this section is a pseudo-Poincaré inequality adapted to probability measure of the form

$$\mu(g) = k^{-1} \sum_{j=1}^{k} \sum_{n \in \mathbb{Z}} \mu_i(n) \mathbf{1}_{s_i^n}(g). \tag{3.2}$$

where $(s_1, \dots, s_k)$ is a generating k-tuple in G and the μ_i's are probability measures on $\mathbb{Z}$ with truncated second moment

$$\mathcal{G}_i(n) := \sum_{|m| \le n} m^2 \mu_i(n) \tag{3.3}$$

satisfying

$$\mathcal{G}_i(n) \ge c n^{2-\tilde{\alpha}_i} L_i(n), \quad \tilde{\alpha}_i \in (0, 2], \tag{3.4}$$

for some slowly positive varying functions L_i, $1 \le i \le k$. Under these circumstances, we let F_i denote the inverse function of $n \mapsto n^{\tilde{\alpha}_i}/L_i(n)$. The function F_i is a regularly varying function of positive index $1/\tilde{\alpha}_i \in [2, \infty)$. In addition, we assume that the μ_i's are essentially decreasing in the sense that there is a constant C_1 such that

$$\forall\, i = 1, \dots, k, 0 \le m \le n, \quad \mu_i(n) \le C_1 \mu_i(m). \tag{3.5}$$

Example 3.4.1. The measure $\mu_{S,a}$ with $a = (\alpha_i)_1^k \in (0, \infty)^k$ satisfies

$$\mathcal{G}_i(n) \simeq \begin{cases} n^{2-\alpha_i} & \text{if } \alpha_i \in (0, 2), \\ \log n & \text{if } \alpha_i = 2, \\ 1 & \text{if } \alpha_i > 2. \end{cases}$$

Hence, in this case, we have $\tilde{\alpha}_i = \min\{\alpha_i, 2\}$ and $L_i = 1$ except if $\alpha_i = 2$ in which case $L_i(n) = \log n$.

We will make use of the following general result (which is essentially well-known). We let $\mathcal{C}_c(G)$ be the set of all finitely supported function on G and set $f_g(x) = f(xg)$.

Theorem 3.4.1. *Let G be a finitely generated group. Let μ be a symmetric probability measure on G. Assume that for each $r \ge 1$ there is a subset $K(r)$ of G such that*

$$\forall\, g \in K(r), \quad \|f_g - f\|_2^2 \le C_0\, r \mathcal{E}_\mu(f, f). \tag{3.6}$$

and

$$\forall\, r \geq 1, \quad \#K(r) \geq v(r) \tag{3.7}$$

where v is increasing and regularly varying of positive index. Let ψ be the right-continuous inverse of v. Then there is a function $\Psi \simeq \psi$ such that the Nash inequality

$$\forall\, f \in \ell^1(G), \quad \|f\|_2^2 \leq \Psi(\|f\|_1^2/\|f\|_2^2)\mathcal{E}_\mu(f,f) \tag{3.8}$$

is satisfied. Moreover

$$\mu^{(2n)}(e) \leq C_1\eta(n) \tag{3.9}$$

where η is defined implicitly by

$$\tau = \int_1^{1/\eta(\tau)} \Psi(s)\frac{ds}{s}, \quad \tau > 0.$$

Proof. Assuming (3.6) and $\#K(r) \geq v(r)$, the Nash inequality (3.8) easily follows from writing

$$\|f\|_2 \leq \|f - f_{K(r)}\|_2 + \|f_{K(r)}\|_2 \leq (C_0 r\mathcal{E}(f,f))^{1/2} + v(r)^{-1/2}\|f\|_1$$

and optimizing in r. Here $f_{K(r)}(x)$ is the average of f over $xK(r)$ so that, obviously, $\|f_{K(r)}\|_2 \leq (\#K(r))^{-1/2}\|f\|_1$ and (3.11) gives $\|f - f_{K(r)}\|_2^2 \leq C_0 r\mathcal{E}_\mu(f,f)$ with $C_0 = CMk$. The return probability estimate (3.9) is a well-known consequence of (3.8). See [9, 31]. □

Remark 3.4.2. In this theorem, the parametrization of the set $K(r)$ is chosen so that r appears on the right-hand side of (3.6) instead of r^2.

Theorem 3.4.3. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1, \ldots, s_k)$. Let μ be as in* (3.2) *with $(\tilde{\alpha}_i)_1^k$, L_i and F_i be as in* (3.4). *Assume that* (3.5) *holds. Assume that there exists an integer M and a* *sequence* $(i_j)_1^M \in \{1,\dots,k\}^M$ *such that for each* $r \geq 1$ *there is a subset* $K(r)$ *of* G *with the property that*

$$g \in K(r) \Longrightarrow g = \prod_1^M s_{i_j}^{x_j} \quad \textit{with } |x_j| \leq F_{i_j}(r). \tag{3.10}$$

Then there exists a constant $C = C(\mu)$ *such that*

$$\forall\, g \in K(r), \;\; \|f_g - f\|_2 \leq CM^2\, r\mathcal{E}_\mu(f,f). \tag{3.11}$$

Proof. Because we assume (3.10), the proof boils down to a collection of one dimensional inequalities, one for each of the measures μ_i on $\mathbb{Z}$ that appear in the definition (3.2) of μ. Indeed, Lemma 3.4.4 stated below shows that there exists a constant C such that, for each $i \in \{1,\dots,k\}$ and $y \in \mathbb{Z}$ with $|y| \leq F_i(r)$ we have

$$\|f_{s_i^y} - f\|_2^2 \leq C\, r\, \mathcal{E}_\mu(f,f) \tag{3.12}$$

for any finitely supported function f on G. Together, (3.10) and (3.12) imply (3.11). Since there exists a constant C such that, for all $i \in \{1,\dots,k\}$,

$$|y| \leq F_i(r) \text{ implies } \mathcal{G}_i(|y|)^{-1}|y|^2 \leq Cr,$$

the claim (3.12) follows from Lemma 3.4.4. □

Lemma 3.4.4. *Let* ν *be a symmetric probability measure on* $\mathbb{Z}$ *satisfying*

$$\exists\, C_1, \;\; \forall\, 0 \leq m \leq n, \;\; \nu(n) \leq C_1 \nu(m).$$

Let G *be a finitely generated group equipped with a distinguished element* s*. Set*

$$\mathcal{E}_{s,\nu}(f,f) = \frac{1}{2} \sum_{x \in G, z \in \mathbb{Z}} |f(xs^z) - f(x)|^2 \nu(z) \;\; \textit{and} \;\; \mathcal{G}_\nu(m) = \sum_{|n| \leq m} |n|^2 \nu(n).$$

(i) *For any finitely supported function* f *on* G *we have*

$$\forall\, y \in \mathbb{Z}, \;\; \|f_{s^y} - f\|_2^2 \leq C_\nu \left(\mathcal{G}_\nu(|y|)\right)^{-1} |y|^2 \mathcal{E}_{s,\nu}(f,f).$$

(ii) Further, for any two finitely supported functions f, g we have

$$\forall x \in G,\ y \in \mathbb{Z},\ |f * g(xs^y) - f * g(x)|^2 \leq C_\nu(\mathcal{G}_\nu(|y|))^{-1} |y|^2 \mathcal{E}_{s,\nu}(f,f) \|g\|_2^2 .$$

Proof of (i). For any pair of integers $0 < m \leq n$, write $n = a_m m + b_m$ with $0 \leq b_m < m$ and

$$\begin{aligned}
\|f - f_{s^n}\|_2^2 &= \sum_{x\in G}(f(xs^n) - f(x))^2 \\
&\leq 2\sum_{x\in G}(f(xs^{a_m m}) - f(x))^2 + 2\sum_{x\in G}(f(xs^{b_m}) - f(x))^2 \\
&\leq 2a_m^2\sum_{x\in G}(f(xs^m) - f(x))^2 + 2\sum_{x\in G}(f(xs^{b_m}) - f(x))^2.
\end{aligned}$$

This yields

$$\begin{aligned}
\|f - f_{s^n}\|_2^2 \left(\sum_{m=1}^n m^2\nu(m)\right) &\leq 2\sum_{x\in G}\sum_{m=1}^n (f(xs^m) - f(x))^2 a_m^2 m^2 \nu(m) \\
&\quad +2\sum_{x\in G}\sum_{m=1}^n (f(xs^{b_m}) - f(x))^2 m^2\nu(m).
\end{aligned}$$

Next, observe that

$$\begin{aligned}
&\sum_{x\in G}\sum_{m=1}^n (f(xs^m) - f(x))^2 (a_m m)^2 \nu(m) \\
&\qquad\leq n^2 \sum_{x\in G}\sum_{m=1}^n (f(xs^m) - f(x))^2 \nu(m) \leq n^2 \mathcal{E}_{s,\nu}(f,f).
\end{aligned}$$

Further, using the hypothesis that ν is essentially decreasing, i.e., $\nu(m) \leq C_1\nu(b)$ is $0 \leq b \leq m$, write

$$\begin{aligned}
&\sum_{x\in G}\sum_{m=1}^n (f(xs^{b_m}) - f(x))^2 m^2\nu(m) \\
&\qquad= \sum_{x\in G}\sum_{b=1}^{n/2}\sum_{\substack{m|n-b \\ b<m\leq n}} (f(xs^b) - f(x))^2 m^2 \nu(m) \\
&\qquad\leq C_1 \sum_{x\in G}\sum_{b=1}^{n/2}\left(\sum_{\substack{m|n-b \\ b<m\leq n}} m^2\right)(f(xs^b) - f(x))^2 \nu(b).
\end{aligned}$$

As

$$\sum_{\substack{m|n-b \\ b<m\leq n}} m^2 \leq (\sum_1^\infty i^{-2})n^2,$$

we obtain

$$\sum_{x\in G}\sum_{m=1}^{n}(f(xs^{b_m})-f(x))^2 m^2\nu(m) \leq C_2 n^2 \mathcal{E}_{s,\nu}(f,f).$$

It follows that, for both $n>0$ and $n<0$,

$$\|f-f_{s^n}\|_2^2 \left(\sum_{0<m\leq|n|} m^2\nu(m)\right) \leq 2(1+C_2)n^2\mathcal{E}_{s,\nu}(f,f).$$

□

Proof of (ii). By Cauchy-Schwarz

$$\begin{aligned}
|f*g(xs^y)-f*g(x)| &= \left|\sum_{z\in G}(f(z^{-1}xs^y)-f(z^{-1}x))g(z)\right| \\
&\leq \left(\sum_{z\in G}(f(z^{-1}xs^y)-f(z^{-1}x))^2\right)^{\frac{1}{2}} \left(\sum_{z\in G}|g(z)|^2\right)^{\frac{1}{2}} \\
&= \|f-f_{s^y}\|_2 \|g\|_2 .
\end{aligned}$$

Applying part (i) to $\|f-f_{s^y}\|_2$ yields the desired inequality. □

Remark 3.4.5. When $G=\mathbb{Z}$, Lemma 3.4.4 provides an interesting and new pseudo-Poincaré inequality for probability measure ν satisfying (3.5) (i.e., which are essentially decreasing) in terms of the truncated second moment $\mathcal{G}_\nu$. Namely, assuming (3.5), we have

$$\sum_{x\in\mathbb{Z}}|f(x+y)-f(x)|^2 \leq C_\nu \frac{|y|^2}{\mathcal{G}_\nu(|y|)}\mathcal{E}_\nu(f,f)$$

where

$$\mathcal{E}_\nu(f,f)=\frac{1}{2}\sum_{x,z\in\mathbb{Z}}|f(x+z)-f(x)|^2\nu(z).$$

Together with the trivial fact that $\#\{y:|y|\leq r\}=2r+1$, this pseudo-Poincaré inequality and Theorem 3.4.1 provide a sharp Nash inequality satisfied by $\mathcal{E}_\nu$.

3.4.2 Assorted return probability upper bounds

This section describes direct applications of Theorem 3.3.1 together with Theorems 3.4.1-3.4.3. We use the notation introduced in Sections 3.1.4 and 3.2.1.

Theorem 3.4.6. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1, \dots, s_k)$ and a k-tuple of positive reals $a = (\alpha_1, \dots, \alpha_k)$. Let $\mathfrak{w}$ be the weight system which assigns weight $w_i = 1/\tilde{\alpha}_i$ to s_i where $\tilde{\alpha}_i = \min\{2, \alpha_i\}$. Then*

$$\mu_{S,a}^{(n)}(e) \leq C_{S,a} n^{-D(S,\mathfrak{w})}$$

where $D(S,\mathfrak{w}) = \sum_h \bar{w}_h \operatorname{rank}(G_h^{\mathfrak{w}}/G_{h+1}^{\mathfrak{w}})$.

Proof. By Theorem 3.3.1, for each $r \geq 1$ we can find a subset $K(r)$ of G such that $\#K(r) \geq r^{D(S,\mathfrak{w})}$ and $g \in K(r)$ implies $g = \prod_1^M s_{i_j}^{x_j}$ with $|x_i| \leq r^{w(s_{i_j})}$. The result then follows from Theorems 3.4.1-3.4.3 $\square$

Remark 3.4.7. If all the α_i's are in $(0,2)$ or, more generally, if $R_h^{\mathfrak{w}} > 0$ implies $\bar{w}_h > 1/2$, the upper bound given in Theorem 3.4.6 is sharp. Indeed, we will prove a matching lower bound in the next section.

If all the α_i's are greater than 2 the measure $\mu_{S,a}$ has finite second moment and $D(S,\mathfrak{w}) = \frac{1}{2}\sum h \operatorname{rank}(G_h/G_{h+1})$. In this case the upper bound of Theorem 3.4.6 is also sharp. It coincides with the bound provided by Corollary 3.1.12.

We conjecture that this upper bound is sharp when $\alpha_i \neq 2$ for all $i \in \{1, \dots, k\}$ but we have not been able to prove this conjecture when there exists i, j such that $\alpha_i < 2$ and $\alpha_j > 2$.

The next result shows that Theorem 3.4.6 is not always sharp when some of the α_i's are equal to 2.

Theorem 3.4.8. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1,\dots,s_k)$ and a k-tuple of positive reals $a=(\alpha_1,\dots,\alpha_k)\in(0,\infty]^k$. Let $\mathfrak{w}=\mathfrak{w}(a)$ be the two-dimensional weight system which assigns weight $w_i=(v_{i,1},v_{i,2})$ to s_i where*

$$v_{i,1}=\frac{1}{\tilde{\alpha}_i},\ \ \tilde{\alpha}_i=\min\{2,\alpha_i\}$$

and

$$v_{i,2}=0 \text{ unless } \alpha_i=2 \text{ in which case } v_{i,2}=1/2.$$

Then

$$\mu_{S,a}^{(n)}(e)\le C_{S,a}n^{-D_1(S,\mathfrak{w})}[\log(e+n)]^{-D_2(S,\mathfrak{w})}$$

where

$$D_i(S,\mathfrak{w})=\sum_h \bar{v}_{h,i}\ \mathrm{rank}(G_h^{\mathfrak{w}}/G_{h+1}^{\mathfrak{w}}),\ \ \bar{w}_h=(\bar{v}_{h,1},\bar{v}_{h,2}).$$

Proof. The proof is the same as for Theorem 3.4.6 but uses a refined weight system and the associated weight function system $\mathfrak{F}(a)$ where the function F_c associated to a commutator of weight $v(c)=(v_1,v_2)$ is $F_c(r)=r^{v_1}[\log(e+r)]^{v_2}$. □

Remark 3.4.9. Referring to Theorem 3.4.8, let Σ be a sequence of formal commutators as in Theorem 3.2.10 applied to $S,\mathfrak{w},\mathfrak{F}(a)$. Assume that for any i such that $s_i\in\mathrm{core}(\mathfrak{w},S,\Sigma)$, we have $\alpha_i=2$. Then $D_1(S,\mathfrak{w})=D_2(S,\mathfrak{w})=D(G)/2$ and

$$\mu_{S,a}^{(n)}(e)\le C_{S,a}[n\log n]^{-D(G)/2}.$$

Example 3.4.2. Let G be the group of 4 by 4 unipotent upper-triangular matrices

$$G=\left\{\begin{pmatrix}1 & x_{1,2} & x_{1,3} & x_{1,4}\\ 0 & 1 & x_{2,3} & x_{2,4}\\ 0 & 0 & 1 & x_{3,4}\\ 0 & 0 & 0 & 1\end{pmatrix} : x_{i,j}\in\mathbb{Z}\right\}.$$

With obvious notation, let $X_{i,j}$ be the matrix in G with a 1 in position i,j and all other non-diagonal entries equal to 0. Consider the generating 4-tuple

$$S = (s_1 = X_{1,2}, s_2 = X_{2,3}, s_3 = X_{3,4}, s_4 = X_{1,4}).$$

The non-trivial brackets are

$$[X_{1,2}, X_{2,3}] = X_{1,3}, [X_{2,3}, X_{3,4}] = X_{2,4}, [X_{1,2}, X_{2,4}] = [X_{1,3}, X_{3,4}] = X_{1,4}.$$

Let $a = (1, 2, 5, 1/3)$. The 2-dimensional weight system $\mathfrak{w}$ is generated by $w(s_1) = (1, 0), w(s_2) = (\frac{1}{2}, \frac{1}{2}), w(s_3) = (\frac{1}{2}, 0), w(s_4) = (3, 0)$. This implies

$$w([X_{1,2}, X_{2,3}]) = (\tfrac{3}{2}, \tfrac{1}{2}), w([X_{2,3}, X_{3,4}]) = (1, \tfrac{1}{2}),$$

$$w([X_{1,2}, [X_{2,3}, X_{3,4}]]) = (2, \tfrac{1}{2}), w([[X_{1,2}, X_{2,3}], X_{3,4}]) = (2, \tfrac{1}{2}).$$

Ignoring (as we may) the weight values that would obviously lead to trivial quotients $G_h^{\mathfrak{w}}/G_{h+1}^{\mathfrak{w}}$, we have $\bar{w}_1 = (\frac{1}{2}, 0), \bar{w}_2 = (\frac{1}{2}, \frac{1}{2})$, $\bar{w}_3 = (1, 0)$, $\bar{w}_4 = (1, \frac{1}{2})$, $\bar{w}_5 = (\frac{3}{2}, \frac{1}{2})$, $\bar{w}_6 = (2, \frac{1}{2})$ and $\bar{w}_7 = (3, 0)$. Next we compute the groups $G_i^{\mathfrak{w}}$. We have

$$\begin{aligned} G_7^{\mathfrak{w}} = G_6^{\mathfrak{w}} =< X_{1,4} > \quad &\subset \quad G_5^{\mathfrak{w}} =< X_{1,4}, X_{1,3} > \\ &\subset \quad G_4^{\mathfrak{w}} =< X_{1,4}, X_{1,3}, X_{2,4} > \\ &\subset \quad G_3^{\mathfrak{w}} =< X_{1,4}, X_{1,3}, X_{2,4}, X_{1,2} > \\ &\subset \quad G_2^{\mathfrak{w}} =< X_{1,4}, X_{1,3}, X_{2,4}, X_{1,2}, X_{2,3} > \\ &\subset \quad G_1^{\mathfrak{w}} =< X_{1,4}, X_{1,3}, X_{2,4}, X_{1,2}, X_{2,3}, X_{3,4} >= G. \end{aligned}$$

This gives

$$D_1(S, \mathfrak{w}) = \frac{1}{2} + \frac{1}{2} + 1 + 1 + \frac{3}{2} + 3 = \frac{15}{2}$$

and

$$D_2(S, \mathfrak{w}) = 0 + \frac{1}{2} + 0 + \frac{1}{2} + \frac{1}{2} + 0 = \frac{3}{2}.$$

We believe that the associated upper bound $\mu_{S,a}^{(n)}(e) \leq Cn^{-15/2}[\log n]^{-3/2}$ is sharp but, at this writing, we are not able to obtain a matching lower bound.

As a corollary of Theorem 3.4.8, we can prove Theorem 3.1.13. The bracket length $\ell(g)$ of an element of G is defined just before Theorem 3.1.13.

Corollary 3.4.10. *Referring to* Theorem 3.4.8*, assume that S and a are such that there exists $i \in \{1, \dots, k\}$ with the property that*

$$(\alpha_i, \ell(s_i)) = (2, 1) \text{ or } \quad \alpha_i \ell(s_i) < 2.$$

Then

$$\lim_{n\to\infty} n^{D(G)/2} \mu_{S,a}^{(n)}(e) = 0 \tag{3.13}$$

where $D(G) = \sum j \operatorname{rank}(G_j/G_{j+1})$ where G_j is the lower central series of G.

Proof. Pick i_0 among those $i \in \{1, \dots, k\}$ such that $(\alpha_i, \ell(s_i)) = (2, 1)$ or $\alpha_i \ell(s_i) < 2$ so that α_{i_0} is smallest possible. Let $\mathfrak{w}' = \mathfrak{w}(a)$ be the 2-dimensional weight system introduced in Theorem 3.4.8 and let $\mathfrak{F}' = \mathfrak{F}(a)$ be the weight function system appearing in the proof of Theorem 3.4.8. Let $\mathfrak{w}$ be the weight system that assigns weight $(1/2, 0)$ to every $s_i \in S$ with weight function $F_{s_i} = (1 + r)^{\frac{1}{2}}$.

If $\alpha_{i_0} < 2/\ell(s_{i_0})$ then by Theorem 3.3.4 shows that $D_1(S, \mathfrak{w}') > D(S, \mathfrak{w}) = D(G)/2$. If $\alpha_{i_0} = 2$ then we must have $\ell(s_{i_0}) = 1$. This time, it follows that $D_2(S, \mathfrak{w}') \geq 1/2 > D_2(S, \mathfrak{w}) = 0$. In both case, Theorem 3.4.8 show that $\mu_{S,a}^{(n)}(e) = o(n^{-D(G)/2})$ as desired. □

The next statement illustrates the use of a weight system $\mathfrak{w}$ and weight-functions system $\mathfrak{F}$ that are not tightly connected to each other (including cases when the weight functions F_c cannot be order in a useful way).

Theorem 3.4.11. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1, \dots, s_k)$. Assume that μ is a probability measure on G of the form (3.2) with*

$$\mu_i(n) = \kappa_i(1 + |n|)^{-\alpha_i - 1}\ell_i(|n|), \quad 1 \le i \le k,$$

where each ℓ_i is a positive slowly varying function satisfying $\ell_i(t^b) \simeq \ell_i(t)$ for all $b > 0$ and $\alpha_i \in (0,2)$. Let $\mathfrak{w}$ be the power weight system associated with $a = (\alpha_1, \dots, \alpha_k)$ by setting $w_i = 1/\alpha_i$. Let $(c_i)_1^t$ be a t-tuple of formal commutators such that for each h, the family $\{c_i : w(c_i) = \bar{w}_h\}$ projects to a linearly independent family in $G_h^{\mathfrak{w}}/G_{h+1}^{\mathfrak{w}}$. Let $(s_{i_j}^{\pm 1})_{j=1}^N$ be the list of all the letters (with multiplicity) used in the build-words for the commutators c_i, $1 \le i \le t$. Then

$$\mu^{(n)}(e) \le Cn^{-D(S,\mathfrak{w})}L(n)^{-1}$$

where

$$D(S,\mathfrak{w}) = \sum_h \bar{w}_h \mathrm{rank}(G_h^{\mathfrak{w}}/G_{h+1}^{\mathfrak{w}}) \text{ and } \quad L(n) = \prod_1^N \ell_{i_j}(n)^{1/\alpha_{i_j}}.$$

Note that this theorem does not offer one but many upper bounds. For each n, one can choose the commutator sequence $(c_i)_1^t$ so as to maximize the size of the resulting $L(n)$.

Example 3.4.3. Consider the Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\},$$

with generating 3-tuple $S = (X, Y, Z)$ where X is the matrix with $x = 1, y = z = 0$ and Y, Z a defined similarly. Let $a = (\alpha_1, \alpha_2, \alpha_3) \in (0,2)$ and let $\ell_1 \equiv 1, \ell_2, \ell_3$ be slowly varying functions such that $\ell_2 \le \ell_3$ if and only if $n \in \cup_k[n_{2k}, n_{2k+1}]$ for some increasing sequence n_k tending to infinity. We also assume that ℓ_2, ℓ_3 satisfy $\ell_i(t^b) \simeq \ell_i(t)$ for all $b > 0$. Applying Theorem 3.4.11, we obtain:

- If $\frac{1}{\alpha_3} < \frac{1}{\alpha_1} + \frac{1}{\alpha_3}$ then we have

$$\mu^{(n)}(e) \leq Cn^{-2(\frac{1}{\alpha_1}+\frac{1}{\alpha_2})}\ell_2(n)^{-\frac{2}{\alpha_2}}.$$

- If $\frac{1}{\alpha_3} > \frac{1}{\alpha_1} + \frac{1}{\alpha_3}$ then we have

$$\mu^{(n)}(e) \leq Cn^{-\sum_1^3 \frac{1}{\alpha_i}}\ell_2(n)^{-\frac{1}{\alpha_2}}\ell_3(n)^{-\frac{1}{\alpha_3}}.$$

- Finally, if $\frac{1}{\alpha_3} = \frac{1}{\alpha_1} + \frac{1}{\alpha_3}$, we have

$$\mu^{(n)}(e) \leq Cn^{-\frac{2}{\alpha_3}} \begin{cases} \ell_2(n)^{-\frac{2}{\alpha_2}} & \text{if } n \in \cup_k [n_{2k-1}, n_{2k}] \\ \ell_2(n)^{-\frac{1}{\alpha_2}}\ell_3(n)^{\frac{1}{\alpha_3}} & \text{if } n \in \cup_k [n_2k, n_{2k+1}]. \\ \end{cases}$$

Example 3.4.4 (continuation of Example 3.2.4-3.2.5)**.** Consider again the Heisenberg group with $S = (s_1 = X, s_2 = Y, s_3 = Z)$. Set $F_1(r) = r^{3/2}$, $F_2(r) = r^2 \log(e+r), F_3(r) = r^\gamma$ with $\gamma > 3/2$.Let μ be the probability measure which assigns to s_i^n, $i = 1, 2, 3$, $n \in \mathbb{Z}$ a probability proportional to $\frac{1}{(1+|n|F_i^{-1}(|n|))}$. Namely,

$$\mu(g) = \frac{1}{3}\sum_{i=1}^{3}\sum_{n\in\mathbb{Z}} \mu_i(n)\mathbf{1}_{s_i^n}(g), \quad \mu_i(n) = \frac{c}{1+|n|F_i^{-1}(|n|)}.$$

Referring to the notation (3.3)(3.4), we have

$$\begin{aligned} \mathcal{G}_1(n) &\simeq (1+n)^{2-(2/3)}, \;\; \tilde{\alpha}_1 = 2/3, \;\; L_1 \equiv 1, \\ \mathcal{G}_2(n) &\simeq (1+n)^{2-(1/2)}[\log(e+n)]^{-1/2}, \;\; \tilde{\alpha}_2 = 1/2, \;\; L_2(n) \simeq [\log(e+n)]^{-1/2} \\ \mathcal{G}_3(n) &\simeq (1+n)^{2-1/\gamma}, \;\; \tilde{\alpha}_3 = 1/\gamma, \;\; L_3 \equiv 1. \end{aligned}$$

Apply Theorem 3.4.11 with $\alpha_i = \tilde{\alpha}_i$, $\ell_i = L_i$. If $\gamma \in (3/2, 7/2]$, use the sequence of formal commutators $(c_1 = s_1, c_2 = s_2, c_3 = [s_1, s_2])$. If $\gamma > 7/2$, use the sequence of formal commutators ($c_1 = s_1, c_2 = s_2, c_3 = s_3$) instead. This gives

$$\mu^{(n)}(e) \leq C \begin{cases} (1+n)^{-7}[\log(e+n)]^{-2} & \text{if } \gamma \in (3/2, 7/2] \\ (1+n)^{-(7/2)-\gamma}[\log(e+n)]^{-1} & \text{if } \gamma > 7/2. \end{cases}$$

Below, we will prove a matching lower bound.

3.5 Norm-radial measures and return probability lower bounds

The aim of this section is to provide lower bounds for the return probability for the random walk driven by the measure $\mu_{S,a}$ on a nilpotent group G, that is, lower bounds on $\mu_{S,a}^{(n)}(e)$. These lower bounds are obtained via comparison with appropriate norm-radial measures.

3.5.1 Norm-radial measures

A (proper) norm $\|\cdot\|$ on a countable group G is a function $g \mapsto \|g\| \in [0, \infty)$ such that $\|g\| = 0$ if and only if $g = e$, $\#\{g\| \leq r\}$ is finite for all $r > 0$, $\|g\| = \|g^{-1}\|$ and $\|g_1 g_2\| \leq \|g_1\|\|g_2\|$. If the triangle inequality is replaced by the weaker property that there exists K such that $\|g_1 g_2\| \leq K\|g_1\|\|g_2\|$, we say that $\|\cdot\|$ is a quasi-norm.

The associated left-invariant distance is obtained by setting $d(g_1, g_2) = \|g_1^{-1} g_2\|$. A norm is κ-geodesic if for any element $g \in G$ there is a sequence $g_1, \ldots, g_N$ with $N \leq \kappa\|g\|$ such that $\|g_i^{-1} g_{i+1}\| \leq \kappa$.

A simple observation is that any two κ-geodesic proper norms $\|\cdot\|_1, \|\cdot\|_2$ are comparable in the sense that there is a constant $C \in (0, \infty)$ such that

$$C^{-1}\|g\|_1 \leq \|g\|_2 \leq C\|g\|_1.$$

The word-length norm associated to any finite symmetric generating set is a proper 1-geodesic norm. Most of the quasi-norms that we will consider below are not κ-geodesic. In general, they are not norms but only quasi-norms.

Theorem 3.5.1. *Let G be a countable group. Let $\|\cdot\|$ be a norm on G such that*

$$\forall r \geq 1, \;\; V(r) = \#\{g : \|g\| \leq r\} \simeq r^D$$

for some $d > 0$. Fix $\gamma \in (0, 2)$ and set

$$\nu_\gamma(g) = \frac{C_\gamma}{(1 + \|g\|)^\gamma V(\|g\|)}, \;\; C_\gamma^{-1} = \sum_g \frac{1}{(1 + \|g\|)^\gamma V(\|g\|)}.$$

Then we have

$$\forall\, n \in \mathbb{N}, \;\; \nu_\gamma^{(n)}(e) \simeq cn^{-D/\gamma}. \tag{3.1}$$

Remark 3.5.2. This is a subtle result in that, as stated, it depends very much on the fact that $\|\cdot\|$ is norm versus a quasi-norm. Indeed, the lower bound in (3.1) is false if $\gamma \geq 2$ and the only thing that prevents us to apply the result to $\|\cdot\|^\theta$ with $\theta > 1$ is that, in general, $\|\cdot\|^\theta$ is only a quasi-norm when $\theta > 1$. However, by Theorem 3.1.9, (3.1) holds true for any measure ν such that $\nu \simeq \nu_\gamma$.

Remark 3.5.3. Definition 3.2.8 provides a great variety of examples of norms to which Theorem 3.5.1 applies.

Proof. The probability of return $\nu_\gamma^{(n)}(e)$ behaves in the same way as the probability of return of the associated the continuous time jump process. For the continuous time jump process, the result follows from [2]. $\square$

3.5.2 Comparisons between $\mu_{S,a}$ and radial measures

Let G be a countable group. Let $\|\cdot\|$ be a quasi-norm on G. Set

$$\forall r \geq 1, \;\; V(r) = \#\{g : \|g\| \leq r\}.$$

Let $\phi : [0,\infty) \to (0,\infty)$ be continuous. Consider the following hypotheses:

$$\exists\, C, \;\; \forall\, r \geq 0, \;\; V(2r) \leq CV(r); \tag{3.2}$$

$$\exists\, C, \;\; \forall\, \lambda \in (1/2, 2), \; t \in (0,\infty), \;\; \phi(t) \leq C\phi(\lambda t); \tag{3.3}$$

and

$$\sum_g \frac{1}{\phi(\|g\|)V(\|g\|)} < \infty. \tag{3.4}$$

Lemma 3.5.4. *Assume* (3.2)-(3.3)-(3.4). *For each $n \in \mathbb{Z}$, let $g_n \in G$ and $\Lambda_n \subset G$ be such that:*

1. $g \in \Lambda_n \Longrightarrow \|g^{-1}g_n\| \leq C\|g_n\|$ *and* $\|g\| \leq C\|g_n\|$
2. $V(\|g_n\|) \leq Cn\#\Lambda_n$
3. $\forall\, g \in G, \;\; \#\{n : g \in \Lambda_n\} \leq C$ *and* $\#\{n : g \in g_n^{-1}\Lambda_n\} \leq C$.

Then there is a constant C_1 such that

$$\sum_{n\in\mathbb{Z}} \sum_{x\in G} \frac{|f(xg_n) - f(x)|^2}{(1+n)\phi(\|g_n\|)} \leq C_1 \sum_{x,g\in G} \frac{|f(xg) - f(x)|^2}{\phi(\|g\|)V(\|g\|)}.$$

Proof. Using 2,1 and 3 successively, write

$$\begin{aligned}
\sum_n \sum_x \frac{|f(xg_n)-f(x)|^2}{(1+n)\phi(\|g_n\|)} &\le C \sum_n \sum_x \frac{|f(xg_n)-f(x)|^2 \# \Lambda_n}{\phi(\|g_n\|)V(\|g_n\|)} \\
&\le \; 2C \sum_n \sum_{g\in\Lambda_n} \sum_x (|f(xg_n)-f(xg)|^2 + |f(xg)-f(x)|^2) \frac{1}{\phi(\|g_n\|)V(\|g_n\|)} \\
&\le \; C' \sum_n \sum_{g\in\Lambda_n} \sum_x \left(\frac{|f(xg^{-1}g_n)-f(x)|^2}{\phi(\|g^{-1}g_n\|)V(\|g^{-1}g_n\|)} + \frac{|f(xg)-f(x)|^2}{\phi(\|g\|)V(\|g\|)} \right) \\
&\le \; C'' \sum_{x,g} \frac{|f(xg)-f(x)|^2}{\phi(\|g\|)V(\|g\|)}.
\end{aligned}$$

□

Remark 3.5.5. Note that under the hypotheses of Lemma 3.5.4, we have

$$\sum \frac{1}{(1+n)\phi(\|g_n\|)} < \infty.$$

The next lemma will allow us to apply Lemma 3.5.4 in the context of Theorem 3.2.10. Assume that G is a nilpotent group generated by the k -tuple $(s_1,\dots,s_k)$. In addition, we are given a weight system $\mathfrak{w}$ and weight functions F_c such that (3.1)-(3.2) holds. Observe that for any commutators c, c', we have

$$\forall\, r_1, r_2 \ge 1, \quad F_{c'} \circ F_c^{-1}(r_1+r_2) \simeq F_{c'} \circ F_c^{-1}(r_1) + F_{c'} \circ F_c^{-1}(r_2). \tag{3.5}$$

Indeed, it follows from our hypotheses that $F_{c'} \circ F_c^{-1}$ is an increasing doubling function.

Lemma 3.5.6. *Referring to the setting of* Theorem 3.2.10*, fix* $h \in \{1,\dots,q\}$*,* $i \in \{m_{h-1}+1, \dots m_{h-1}+R_h\}$ *and an integer* u*. For each* $n \in \mathbb{Z}$*, let* $z_n \in G^{\mathfrak{w}}_{h+1}$ *with* $\|z_n\|_{\mathfrak{F},\mathrm{com}} \le F_{c_1} \circ F_{c_i}^{-1}(n)$*. Set*

$$g_n = \pi(c_i^{un}) z_n \in G$$

and

$$\Lambda_n = \left\{ g = \pi\left(\prod_1^q \prod_{m_{h-1}+1}^{m_{h-1}+R_h} c_j^{x_j} \right) : |x_j| \le F_{c_j} \circ F_{c_i}^{-1}(n),\ x_i = \lfloor \frac{un}{2} \rfloor. \right\}.$$

Then (g_n) *and* (Λ_n) *satisfy the hypotheses 1,2 and 3 of* Lemma 3.5.4.

Proof. By Proposition 3.2.17 and Theorem 3.2.10, $\|g_n\|_{\mathfrak{F},\mathrm{com}} \simeq F_{c_1} \circ F_{c_i}^{-1}(n)$ and $g \in \Lambda_n$ implies

$$\|g\|_{\mathfrak{F},\mathrm{com}} \leq CF_{c_1} \circ F_{c_i}^{-1}(n),$$

so, Property 1 in Lemma 3.5.4 is satisfied. Property 2 also follows from Theorem 3.2.10 and the proof of Theorem 3.3.2.

Suppose that $g \in \Lambda_n \cap \Lambda_m$. Then, computing modulo $G_{h+1}^{\mathfrak{w}}$ and using the fact that $[G_h^{\mathfrak{w}}, G_h^{\mathfrak{w}}] \subset G_{h+1}^{\mathfrak{w}}$ we obtain that $\lfloor un/2 \rfloor = \lfloor um/2 \rfloor$. Similarly, $g \in g_n^{-1}\Lambda_n \cap g_m^{-1}\Lambda_m$ implies $n + \lfloor un/2 \rfloor = m + \lfloor um/2 \rfloor$. In both cases we must have $|n - m| \leq 1$. This shows that Property 3 of Lemma 3.5.4 is satisfied. $\square$

The main result of this section is the following theorem.

Theorem 3.5.7. *Let* G *be a nilpotent group with generating the* k*-tuple* $S = (s_1, \ldots, s_k)$*. Let* $I_{\mathrm{tor}} = \{i \in \{1, \ldots, k\} : s_i \text{ is torsion in } G\}$*. Fix a weight system* $\mathfrak{w}$ *and a weight-function system* $\mathfrak{F}$ *such that* (3.1)-(3.2) *are satisfied. Let* $\|\cdot\| = \|\cdot\|_{\mathfrak{F},\mathrm{com}}$ *be the associated quasi-norm introduced in* Definition 3.2.8. *For each* $i \in \{1, \ldots, k\} \setminus I_{\mathrm{tor}}$*, let*

$$h_i = j_{\mathfrak{w}}(s_i).$$

Let ϕ *be such that* (3.3)-(3.4) *are satisfied.*

Let μ *be a probability measure on* G *of the form*

$$\mu(g) = \frac{1}{k} \sum_{j=1}^{k} \sum_{n \in \mathbb{Z}} \mu_i(n) \mathbf{1}_{s_i^n}(g)$$

where μ_i *is an arbitrary symmetric probability measure on* $\mathbb{Z}$ *if* $i \in I_{\mathrm{tor}}$ *and*

$$\mu_i(n) = \frac{C_i}{(1+n)\phi(F_{c_1} \circ \mathbf{F}_{h_i}^{-1}(n))}, \quad C_i^{-1} = \sum_n \frac{1}{(1+n)\phi(F_{c_1} \circ \mathbf{F}_{h_i}^{-1}(n))},$$

for $i \in \{1,\dots,k\} \setminus I_{\text{tor}}$. Then there exists C such that

$$\mathcal{E}_\mu(f,f) \le C\mathcal{E}_\nu(f,f)$$

where

$$\nu(g) = \frac{C_\phi}{\phi(\|g\|)V(\|g\|)}, \quad C_\phi^{-1} = \sum_g \frac{1}{\phi(\|g\|)V(\|g\|)}.$$

In particular, there are constants $c > 0$ and N such that

$$\mu^{(2n)}(e) \ge c\nu^{(2Nn)}(e).$$

Proof. Fix i and write $s = s_i$. By Definition 3.2.13, either s is a torsion element and $s^\kappa = e$ for some κ or $j_{\mathfrak{w}}(s) = h < \infty$. In the second case we can find κ such that

$$s^\kappa = \pi\left(\prod_{m_{h-1}+1}^{m_{h-1}+\rho} c_i^{x_i}\right) z, \ x_{m_{h-1}+\rho} \neq 0, \quad z \in G_{h+1}^{\mathfrak{w}}.$$

If s is torsion, it is very easy to see that $\mathcal{E}_{s,\mu_i}(f,f) \le C\nu(f,f)$. In the course of this proof, C denotes a generic constant that may change from line to line. If s is not torsion and

$$s^\kappa = \pi\left(\prod_{m_{h-1}+1}^{m_{h-1}+\rho} c_i^{x_i}\right) z, \ x_{m_{h-1}+\rho} \neq 0, \quad z \in G_{h+1}^{\mathfrak{w}},$$

set $F = F_{c_{m_{h-1}+1}}$ (we have $F \simeq F_{c_j}$, $j \in \{m_{h-1}+1, m_h\}$). Then, for any n, we have

$$s^{\kappa n} = \pi\left(\prod_{m_{h-1}+1}^{m_{h-1}+\rho} c_i^{x_i n}\right) z_n \text{ with } \|z_n\| \le CF_{c_1} \circ F^{-1}(|n|), \quad z_n \in G_{h+1}^{\mathfrak{w}}.$$

Now, write $n = \kappa u_n + v_n$ with $|v_n| < \kappa$ and

$$\sum_g |f(gs^n) - f(g)|^2 \le 2\left(\sum_g |f(gs^{\kappa u_n}) - f(g)|^2 + \sum_g |f(gs^{v_n}) - f(g)|^2\right).$$

By Lemma 3.5.6 and Remark 3.5.5, the hypotheses of Theorem 3.5.7 imply that $\sum((1+n)\phi(\|s^n\|))^{-1} < \infty$. Hence, it is is easy to check that

$$\sum_g \sum_n \frac{|f(gs^{v_n}) - f(g)|^2}{(1+n)\phi(\|s^n\|)} \leq C\mathcal{E}_\nu(f,f). \tag{3.6}$$

Consequently, it suffices to show that

$$\sum_g \sum_n \frac{|f(gs^{\kappa u_n}) - f(g)|^2}{(1+n)\phi(\|s^n\|)} \leq C\mathcal{E}_\nu(f,f).$$

We have $\|s^n\| \simeq \|s^{\kappa u_n}\| \simeq F_{c_1} \circ F^{-1}(\kappa u_n)$. Hence

$$\sum_g \sum_n \frac{|f(gs^{\kappa u_n}) - f(g)|^2}{(1+n)\phi(\|s^n\|)} \leq C \sum_g \sum_\ell \frac{|f(gs^{\kappa \ell}) - f(g)|^2}{\ell\phi(F_{c_1} \circ F^{-1}(\ell))}. \tag{3.7}$$

Next, set $i_1 = m_{h-1}+1, i_2 = m_{h-1}+\rho$ and write

$$\begin{aligned}
&\sum_g \sum_\ell |f(gs^{\kappa\ell}) - f(g)|^2 \\
&\leq \;\; \rho\left(\sum_g \sum_\ell \sum_{i=i_1}^{i_2-1} |f(g\pi(c_i^{x_i\ell})) - f(g)|^2 + \sum_g \sum_\ell |f(g\pi(c_{i_2}^{x_{i_2}\ell})z_\ell) - f(g)|^2\right).
\end{aligned}$$

By Lemmas 3.5.4-3.5.6, for each $i = i_1, \ldots, i_2 - 1$, we have

$$\sum_g \sum_\ell \frac{|f(g\pi(c_i^{x_i\ell})) - f(g)|^2}{(1+\ell)\phi(\|\pi(c_i^{x_i\ell})\|)} \leq C\mathcal{E}_\nu(f,f)$$

and, since $z_\ell \in G_{h+1}^{\mathfrak{w}}$ and $\|z_\ell\| \leq CF_{c_1} \circ F^{-1}(\ell)$,

$$\sum_g \sum_\ell \frac{|f(g\pi(c_{i_2}^{x_{i_2}\ell})z_\ell) - f(g)|^2}{(1+\ell)\phi(\|\pi(c_{i2}^{x_{i_2}\ell})z_\ell\|)} \leq C\mathcal{E}_\nu(f,f).$$

Further, for each $i = i_1, \ldots, i_2$ with $x_i \neq 0$, we have

$$\|\pi(c_i^{x_i\ell})\| \simeq F_{c_1} \circ F^{-1}(\ell)$$

as well as $\|\pi(c_i^{x_{i_2}\ell})z_\ell\| \simeq F_{c_1} \circ F^{-1}(\ell)$. Hence (3.7) and the above estimates give

$$\sum_g \sum_n \frac{|f(gs^{\kappa u_n}) - f(g)|^2}{(1+n)\phi(\|s^n\|)} \leq C\mathcal{E}_\nu(f,f).$$

Together with (3.6), this gives

$$\sum_{g\in G} \sum_{n\in\mathbb{Z}} \frac{|f(gs^n) - f(g)|^2}{(1+n)\phi(\|s^n\|)} \leq C\mathcal{E}_\nu(f,f).$$

Since this holds true for each $s = s_i$, $i = 1, \ldots, k$, the desired result follows. $\square$

3.5.3 Assorted corollaries: return probability lower bounds

In this section we use the comparison with norm-radial measures to obtain explicit lower estimates on $\mu_{S,a}^{(n)}(e)$. The simplest and most important result of this type is as follows.

Theorem 3.5.8. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1, \ldots, s_k)$ and a k-tuple of positive reals $a = (\alpha_1, \ldots, \alpha_k) \in (0, 2)^k$. Let $\mathfrak{w}$ be the weight system which assigns weight $w_i = 1/\alpha_i$ to s_i. Then*

$$\mu_{S,a}^{(n)}(e) \geq c_{S,a} n^{-D(S,\mathfrak{w})}$$

where $D(S, \mathfrak{w}) = \sum_h \bar{w}_h \operatorname{rank}(G_h^{\mathfrak{w}}/G_{h+1}^{\mathfrak{w}})$.

Remark 3.5.9. This lower bound matches precisely the upper bound given by Theorem 3.4.6. Thus, as stated in Theorems 3.1.2-3.1.8, for any $a \in (0, 2)^k$,

$$\mu_{S,a}^{(n)}(e) \simeq n^{-D(S,\mathfrak{w})}.$$

Note however that, in Theorems 3.1.2-3.1.8, the constraints on the α_i's is weaker. This more general case will be treated below.

Proof. Fix a sequence $\Sigma = (c_i)_1^t$ of commutators as in Theorem 3.2.10 and let $\|\cdot\|$ be the associated norm $\|\cdot\| = \|\cdot\|_\Sigma$ introduced in Definition 3.2.8. Note that, by Remark 3.2.9, $\|\cdot\|$ is indeed not only a quasi-norm but a norm. By hypothesis, $1/w(c_1) < 2$. Hence Theorem 3.5.1, together with Theorem 3.3.2, shows that the norm-radial measure

$$\nu(g) = \frac{C}{(1 + \|g\|)^{1/w(c_1)} V(\|g\|)}$$

satisfies

$$\nu^{(n)}(e) \geq cn^{-w(c_1)D(S,\mathfrak{w})/w(c_1)} = cn^{-D(S,\mathfrak{w})}. \tag{3.8}$$

Theorem 3.5.7 produces a symmetric measure μ such that $\mathcal{E}_\mu \leq C\mathcal{E}_\nu$. This measure μ is given by

$$\mu(g) = \frac{1}{k}\sum_{j=1}^{k}\sum_{n\in\mathbb{Z}} \mu_i(n)\mathbf{1}_{s_i^n}(g)$$

where μ_i is an arbitrary symmetric probability measure on $\mathbb{Z}$ if $i \in I_{\text{tor}}$ and

$$\mu_i(n) = \frac{C_i}{(1+n)(1+F_{c_1}\circ \mathbf{F}_{h_i}^{-1}(n))^{1/w(c_1)}}$$

with

$$C_i^{-1} = \sum_n \frac{1}{(1+n)(1+F_{c_1}\circ \mathbf{F}_{h_i}^{-1}(n))^{1/w(c_1)}}$$

for $i \in \{1,\ldots,k\}\setminus I_{\text{tor}}$. In the latter case, we have $\mathbf{F}_{h_i}(t) = t^{\bar{w}_{h_i}}$ with $\bar{w}_{h_i} \geq w(s_i) = 1/\alpha_i$ and $F_{c_1}(t) = t^{w(c_1)}$. Hence

$$\mu_i(n) \simeq \frac{C_i}{(1+n)^{1+1/\bar{w}_{h_i}}} \geq \frac{C_i'}{(1+n)^{1+\alpha_i}}.$$

It follows that if we pick μ_i to be given by $\mu_i(n) = c_i(1+n)^{-(1+\alpha_i)}$ for $i \in I_{\text{tor}}$, and $\mu_i = c_i(1+n)^{1+1/\bar{w}_{h_i}}$ if $i \in I \setminus I_{\text{tor}}$, we obtain a measure μ such that

$$\mathcal{E}_{\mu_{S,a}} \leq C\mathcal{E}_\mu \leq C'\mathcal{E}_\nu.$$

By Theorem 3.1.9, this implies that there are $c, N \in (0,\infty)$ such that

$$\mu_{S,a}^{(2n)}(e) \geq c\nu^{(2nN)}(e).$$

Thus the lower bound stated in Theorem 3.5.8 follows from (3.8). □

The following theorem extends the range of applicability of the previous result. In particular, the statement is different but equivalent to the statement recorded in Theorem 3.1.8. See also Theorem 3.5.13 below.

Theorem 3.5.10. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1, \ldots, s_k)$ and a k-tuple of positive reals $a = (\alpha_1, \ldots, \alpha_k) \in (0, \infty]^k$. Set $\tilde{\alpha}_i = \min\{\alpha_i, 2\}$. Let $\mathfrak{w}$ be the weight system which assigns weight $1/\tilde{\alpha}_i$ to $s_i \in S$. Let Σ be a sequence of formal commutators as in* Theorem 3.2.10. *Assume that $w(s) > 1/2$ for all $s \in \mathrm{core}(\mathfrak{w}, S, \Sigma)$. Then*

$$\mu_{S,a}^{(n)}(e) \simeq n^{-D(S,\mathfrak{w})}.$$

Proof. The upper bound follows from Theorem 3.4.6. The lower bound is more subtle. Consider any $s \in S$ such that $w(s) = 1/2$ (i.e., $s = s_i$ with $\alpha_i \geq 2$). Observe that $1/2$ is the lowest possible value for weights in $\mathfrak{w}$ and that the hypothesis that $w > 1/2$ on $\mathrm{core}(\mathfrak{w}, S, \Sigma)$ implies that $G_1^{\mathfrak{w}}/G_2^{\mathfrak{w}}$ is a torsion group. In particular, this implies that $\bar{w}_{j_{\mathfrak{w}}(s)} > 1/2 = w(s)$. By Corollary 3.2.19, the weight system $\mathfrak{w}'$ generated by

$$w'(s) = \begin{cases} w(s) & \text{if } w(s) \neq 1/2 \\ \bar{w}_2 & \text{if } w(s) = 1/2 \end{cases}$$

is such that $w(s) \leq w'(s) \leq \bar{w}_{j_{\mathfrak{w}}(s)}$ for all $s \in S$ and $w'(s) > 1/2$ for all $s \in S$. Now, Theorem 3.5.7 gives the comparison $\mathcal{E}_{\mu_{S,a}} \leq C\mathcal{E}_\nu$ with

$$\nu(g) \simeq \frac{1}{(1 + \|g\|_{\Sigma,\mathfrak{w}})^{1/w_\Sigma} V_{\Sigma,\mathfrak{w}}(\|g\|_{\Sigma,\mathfrak{w}})}.$$

However, since the minimum weight value w_Σ may be equal to $1/2$, we cannot apply Theorem 3.5.1 directly. We proceed as follows. By the definition of w' and Corollary 3.2.19, we have

$$\forall\, g \in G, \;\; \|g\|_{\Sigma,\mathfrak{w}}^{1/w_\Sigma} \simeq \|g\|_{S,\mathfrak{w}'}^{1/w'_S}.$$

Note that this implies that

$$V_{\Sigma,\mathfrak{w}}(\|g\|_{\Sigma,\mathfrak{w}}) = \#\{g' \in G : \|g'\|_{\Sigma,\mathfrak{w}} \leq \|g\|_{\Sigma,\mathfrak{w}}\} \simeq V_{S,\mathfrak{w}'}(\|g\|_{S,\mathfrak{w}'}).$$

Hence we have

$$\mathcal{E}_\nu \simeq \mathcal{E}_{\nu'}$$

where

$$\nu'(g) \simeq \frac{1}{(1+\|g\|_{S,\mathfrak{w}'})^{1/w'_S} V_{S,\mathfrak{w}'}(\|g\|_{S,\mathfrak{w}'})}.$$

Now, since by construction $w'_S > 1/2$, we can apply Theorem 3.5.1 which gives $(\nu')^{(n)}(e) \simeq n^{-D(S,\mathfrak{w}')} = n^{-D(S,\mathfrak{w})}$. Also, we have $\mathcal{E}_{\mu_{S,a}} \leq C\mathcal{E}_\nu \simeq \mathcal{E}_{\nu'}$. Hence

$$\mu_{S,a}^{(n)}(e) \geq cn^{-D(S,\mathfrak{w})}.$$

This ends the proof of Theorem 3.5.10. □

Our next results provides a comparison between the behaviors of two measures $\mu_{S,a}$ and $\mu_{S',a'}$. Compare to Corollary 3.1.12 and Theorem 3.1.13 which treats comparison with $\mu_{S',a'}$ when $a' = (\alpha'_i)_1^{k'} \in (2,\infty]^{k'}$, a case that is excluded in Theorem 3.5.11.

Theorem 3.5.11. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1,\dots,s_k)$ and a k-tuple of positive reals $a = (\alpha_1,\dots,\alpha_k) \in (0,\infty]^k$. Set $\tilde{\alpha}_i = \min\{\alpha_i, 2\}$. Let $\mathfrak{w}$ be the weight system which assigns weight $1/\tilde{\alpha}_i$ to $s_i \in S$. Fix another weight system $\mathfrak{w}' = (w'_1,\dots,w'_k)$ with minimal weight $w'_S > 1/2$. Let Σ be a sequence of formal commutators as in* Theorem 3.2.10 *for $(S,\mathfrak{w}')$. Assume that $w(s) \geq w'(s)$ for all $s \in \mathrm{core}(\mathfrak{w}',S,\Sigma)$. Then*

$$\mu_{S,a}^{(n)}(e) = o(n^{-D(S,\mathfrak{w}')})$$

if and only if there exists $s \in S$ such that $w(s) > \bar{w}'_{j_{\mathfrak{w}'}(s)}$.

Proof. Apply Theorems 3.4.6 and 3.5.10 together with Corollary 3.2.19 and Theorem 3.3.4. □

Theorem 3.5.12. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1, \ldots, s_k)$ and a k-tuple of positive reals $a = (\alpha_1, \ldots, \alpha_k) \in (0, \infty]^k$. Set $\tilde{\alpha}_i = \min\{\alpha_i, 2\}$. Let $\mathfrak{w}$ be the weight system which assigns weight $w_i = 1/\tilde{\alpha}_i$ to s_i. Then there exists $A \geq 0$ such that*

$$\mu_{S,a}^{(n)}(e) \geq c_{S,a} n^{-D(S,\mathfrak{w})} [\log n]^{-A}.$$

Further, let Σ be as in Theorem 3.2.10 *applied to $(S, \mathfrak{w})$ and assume that $\alpha_i = 2$ for all $i \in \{1, \ldots, k\}$ such that $s_i \in \mathrm{core}(S, \mathfrak{w}, \Sigma)$. Then*

$$\mu_{S,a}^{(n)}(e) \simeq [n \log n]^{-D(G)/2}.$$

Proof. The proof of the general lower bound is essentially the same as for Theorem 3.5.8, except that we cannot rule out the possibility that $w(c_1) = 1/2$. If $w(c_1) > 1/2$ then the previous proof applies and we obtain $\mu_{S,a}^{(n)}(e) \geq cn^{-D(S,\mathfrak{w})}$ which is better than the statement we need to prove. If $w(c_1) = 1/2$ then we have a comparison

$$\mathcal{E}_{\mu_{S,a}} \leq C \mathcal{E}_\nu \tag{3.9}$$

with

$$\nu(g) = \frac{C}{(1 + \|g\|)^2 V(\|g\|)}.$$

To conclude, we need a lower bound on $\nu^{(n)}(e)$. This turns out to be rather subtle and difficult question in the present generality. In Chapter 4, we show that there exists $A \geq 0$ such that

$$\nu^{(n)}(e) \geq cn^{-D(S,\mathfrak{w})} [\log n]^{-A}. \tag{3.10}$$

See Theorem 4.4.6. This proves the desired lower bound on $\mu_{S,a}^{(n)}(e)$.

When $\alpha_i = 2$ for all $i \in \mathrm{core}(S, \mathfrak{w}, \Sigma)$, it follows that

$$D(S, \mathfrak{w}) = D(G)/2 \quad \text{and} \quad \|g\| \simeq |g|_S$$

where $|g|_S$ denotes the usual word-length of g over the symmetric generating set $\{s_i^{\pm 1} : 1 \le i \le k\}$. Theorem 3.4.8 provides the upper bound

$$\mu_{S,a}^{(n)}(e) \le C[n \log n]^{-D(G)/2}.$$

For the lower bound, by the Dirichlet form inequality (3.9), it suffices to bound $\nu^{(n)}(e)$ from below. Using the fact that $\|g\| \simeq |g|_S$, we prove in Theorem 4.3.5 that, in this special case, (3.10) holds with $A = D(G)/2$. This provides the desired matching lower bounds

$$\mu_{S,a}^{(n)}(e) \ge c[n \log n]^{-D(G)/2}.$$

□

Theorem 3.5.13. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1, \dots, s_k)$ and a k-tuple of positive reals $a = (\alpha_1, \dots, \alpha_k) \in (0, \infty]^k$. Set $\tilde{\alpha}_i = \min\{\alpha_i, 2\}$ and $w_i = 1/\tilde{\alpha}_i$. Let $\mathfrak{w}$ be the associated weight system. Let Σ be as in* Theorem 3.2.10 *applied to $(S, \mathfrak{w})$. Let*

$$\Theta = (\theta_1 = s_{i_1}, \dots, \theta_\kappa = s_{i,\kappa}) = \mathrm{core}(S, \mathfrak{w}, \Sigma).$$

Let H be the subgroup of G generated by Θ. Set $b = (\beta_1 = \alpha_{i_1}, \cdots, \beta_\kappa = \alpha_{i_\kappa})$, $\tilde{\beta}_i = \tilde{\alpha}_{i_j}$, $v(\theta_i) = w(s_{i_j})$. Let $\mathfrak{v}$ be the weight system associated to v on (H, Θ), respectively. Then

$$D(\Theta, \mathfrak{v}) = D(S, \mathfrak{w}).$$

In particular, letting e_H, e_G be the identity elements in H and G, respectively, we have:

- *if $\alpha_i \in (0, 2)$ for all i such that $s_i \in \mathrm{core}(S, \mathfrak{w}, \Sigma)$ then*

$$\mu_{S,a}^{(n)}(e_G) \simeq \mu_{\Theta,b}^{(n)}(e_H) \simeq n^{-D(\Theta,\mathfrak{v})}.$$

- *if* $\alpha_i = 2$ *for all* i *such that* $s_i \in \operatorname{core}(S, \mathfrak{w}, \Sigma)$ *then*

$$\mu_{S,a}^{(n)}(e_G) \simeq \mu_{\Theta,b}^{(n)}(e_H) \simeq [n \log n]^{-D(H)/2}.$$

Remark 3.5.14. One can easily prove that H is a subgroup of finite index in G. It is also easy to prove by the direct comparison techniques of [30] that

$$\forall\, n,\ \ \mu_{S,a}^{(2Kn)}(e_G) \leq C\mu_{\Theta,b}^{(2n)}(e_H)$$

for some integer K and constant C and for each $a = (\alpha_1, \dots, \alpha_k)$. The converse inequality seems significantly harder to prove although we conjecture it does hold true.

Proof. First we observe that $D(\Theta, \mathfrak{v}) \leq D(S, \mathfrak{w})$. Indeed, this follows immediately from the obvious fact that

$$\{g \in H : \|g\|_{\Theta,\mathfrak{v}}^{1/v_\Theta} \leq r\} \subset \{g \in G : \|g\|_{S,\mathfrak{w}}^{1/w_S} \leq r\}.$$

To prove that $D(\Theta, \mathfrak{v}) \geq D(S, \mathfrak{w})$, it is convenient to introduce the generating k-tuple $S^* = (s_i^*)_1^k$ of H such that $s_{i,j}^* = s_{i_j}$ if $s_{i_j} = \theta_j \in \Theta$, and $s_{i_j}^* = e$ otherwise. Both S and S^* are equipped with the weight system $\mathfrak{w}$. Obviously, the non-decreasing sequence of subgroups $(H_j^{\mathfrak{w}})$ is a trivial refinement of the sequence $(H_j^{\mathfrak{v}})$ in the sense that the two sequences differ only by insertion of some repetitions. For instance, A, B, C may become A, A, B, B, B, B, C. It follows that $D(\Theta, \mathfrak{v}) = D(S^*, \mathfrak{w})$. The notational advantage is that the weight system $\mathfrak{w}$ with increasing weight-value sequence $\bar{w}_j$ is now shared by S and S^*. We wish to prove that

$$\operatorname{rank}(H_j^{\mathfrak{w}}/H_{j+1}^{\mathfrak{w}}) \geq \operatorname{rank}(G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}).$$

The (torsion free) rank of an abelian group can be computed as the cardinality of a maximal free subset. Set $R = R_j^{\mathfrak{w}}$ be the torsion free rank of $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$. Let

$(c_{m_{j-1}+1}, \ldots, c_{m_{j-1}+R})$ be the formal commutators given by Theorem 3.2.10 which form a maximal free subset of $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$. By definition of core$(S, \mathfrak{w}, \Sigma)$, the images of these formal commutators in G belong to H. In fact, they clearly belong to $H_j^{\mathfrak{w}} \subset G_j^{\mathfrak{w}}$. Now, we also have $H_{j+1}^{\mathfrak{w}} \subset G_{j+1}^{\mathfrak{w}}$. Assume that $\prod_{m_{j-1}+1}^{m_{j-1}+R} c_i^{x_i} = e$ in $H_j^{\mathfrak{w}}/H_{j+1}^{\mathfrak{w}}$. Then, a fortiori, this product is trivial in

$$H_j^{\mathfrak{w}} G_{j+1}^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}} \simeq H_j^{\mathfrak{w}}/(H_j^{\mathfrak{w}} \cap G_{j+1}^{\mathfrak{w}})$$

since $(H_j^{\mathfrak{w}} \cap G_{j+1}^{\mathfrak{w}}) \subset H_{j+1}^{\mathfrak{w}}$. In particular, this product must be trivial in $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$. This implies that $x_i = 0$ for all i so that $H_j^{\mathfrak{w}}/H_{j+1}^{\mathfrak{w}}$ admits a free subset of size R. It follows that $\operatorname{rank}(H_j^{\mathfrak{w}}/H_{j+1}^{\mathfrak{w}}) \geq R$ as desired. $\square$

To state the final result of this section, we need some preparation. Consider the class of measure μ of the form (3.2) with

$$\mu_i(n) = \kappa_i(1+|n|)^{-\alpha_i-1}\ell_i(|n|), \quad 1 \leq i \leq k, \tag{3.11}$$

where each ℓ_i is a positive slowly varying function satisfying $\ell_i(t^b) \simeq \ell_i(t)$ for all $b > 0$ and $\alpha_i \in (0,2)$. Consider the weight-function system $\mathfrak{F}$ generated by letting F_i be the inverse function of $r \mapsto r^{\alpha_i}/\ell_i(r)$. Note that F_i is regularly varying of order $1/\alpha_i$ and that $F_i(r) \simeq [r\ell_i(r)]^{1/\alpha_i}$, $r \geq 1$, $i = 1, \ldots, k$. We make the fundamental assumption that the functions F_i have the property that for any $1 \leq i, j \leq k$, either $F_i(r) \leq CF_j(r)$ of $F_j(r) \leq CF_i(r)$. For instance, this is clearly the case if all α_i are distinct. Without loss of generality, we can assume that there exists a multidimensional weight system $\mathfrak{w}$, say of dimension d, with

$$w_i = (v_i^1, \ldots, v_i^d), \quad v_i^1 = 1/\alpha_i, \quad 1 \leq i \leq k,$$

and such that $\mathfrak{w}$ and $\mathfrak{F}$ are compatible in the sense that (3.1)-(3.2) hold true. Separately, consider also the one-dimensional weight system $\mathfrak{v}$ generated by $v_i =$

$1/\alpha_i$, $1 \leq i \leq k$. Note that one can check that

$$D(S, \mathfrak{v}) = \sum_j \bar{v}_j R_j^{\mathfrak{v}} = \sum_j \bar{v}_i^1 R_j^{\mathfrak{w}}$$

where, by definition, $\bar{w}_j = (\bar{v}_j^1, \dots, \bar{v}_j^d)$. Fix $\alpha_0 \in (0, 2)$ such that

$$\alpha_0 > \max\{\alpha_i : 1 \leq i \leq k\}$$

and $\alpha_0/\alpha_i \notin \mathbb{N}$, $i = 1, \dots, k$. Observe that there are convex functions $K_i \geq 0$, $i = 0, \dots, k$, such that $K_i(0) = 0$ and

$$\forall r \geq 1, \quad F_i(r^{\alpha_0}) \simeq K_i(r). \tag{3.12}$$

Indeed, $r \mapsto F_i(r^{\alpha_0})$ is regularly varying of index α_0/α_i with $1 < \alpha_0/\alpha_i \notin \mathbb{N}$. By [6, Theorems 1.8.2-1.8.3] there are smooth positive convex functions $\tilde{K}_i$ such that $\tilde{K}_i(r) \sim F_i(r^{\alpha_0})$. If $\tilde{K}_i(0) > 0$, it is easy to construct a convex function $K_i : [0, \infty) \to [0, \infty)$ such that $K_i \simeq \widetilde{K}_i$ on $[1, \infty)$ and $K_i(0) = 0$.

Theorem 3.5.15. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1, \dots, s_k)$. Assume that μ is a probability measure on G of the form* (3.2) *with μ_i as in* (3.11)*. Let ℓ_i, F_i, $\mathfrak{F}, \mathfrak{w}, \mathfrak{v}$ be as described above. Let $(c_i)_1^t$ be a t-tuple of formal commutators as in* Theorem 3.2.10 *applied to $G, S, \mathfrak{w}, \mathfrak{F}$. Let $(s_{i_j}^{\pm 1})_{j=1}^N$ be the list of all the letters (repeated according to multiplicity) used in the build-words for the commutators c_i with $i \in \bigcup_j \{m_{j-1} + 1, \dots, m_{j-1} + R_j^{\mathfrak{w}}\}$. Then*

$$\mu^{(n)}(e) \simeq n^{-D(S,\mathfrak{v})} L(n)^{-1}$$

where

$$L(n) = \prod_1^N \ell_{i_j}(n)^{1/\alpha_{i_j}}.$$

Proof. The upper bound follows immediately from Theorem 3.4.11. For the lower bound, it is technically convenient to adjoint to S the dummy generator $s_0 = e$

with associated weight function $F_0(r) = r^{1/\alpha_0}$. Let $\mathfrak{W}_0$, $\mathfrak{F}_0$ we the weight systems induced by $S_0 = (e, s_1, \dots, s_k)$, $F_0, F_1, \dots, F_k$.

Apply Theorem 3.5.7 to $G, S, \mathfrak{w}_0, \mathfrak{F}_0$ to obtain that $\mathcal{E}_\mu \leq C\mathcal{E}_\nu$ where

$$\nu(g) \simeq \frac{1}{(1 + \|g\|_{\mathfrak{F}_0,\mathrm{com}})^{\alpha_0} V_{\mathfrak{F}_0,\mathrm{com}}(\|g\|_{\mathfrak{F}_0,\mathrm{com}})}$$

with $V_{\mathfrak{F}_0,\mathrm{com}}(r) = \#\{g \in G : \|g\|_{\mathfrak{F}_0,\mathrm{com}} \leq r\}$. By construction, we can also write

$$\nu(g) \simeq \frac{1}{(1 + \|g\|)^{\alpha_0} V(\|g\|)}$$

where $\|\cdot\|$ is the norm $\|\cdot\|_{\mathfrak{K},\mathrm{com}}$ based on the convex function $K_i \simeq F_i(r^{\alpha_0})$, $0 \leq i \leq k$, provided by (3.12) and V denotes the associated volume function. Indeed, by construction we have $\|\cdot\| \simeq \|\cdot\|_{\mathfrak{F}_0,\mathrm{com}}$. As $\|\cdot\|$ is a norm and $\alpha_0 \in (0, 2)$, an extension of Theorem 3.5.1 obtained in Chapter 5, see Theorem 4.4.3, which allows volume growth of regular variation with positive index gives

$$\nu^{(n)}(e) \simeq \frac{1}{V(n^{1/\alpha_0})} \simeq \frac{1}{V_{\mathfrak{F}_0,\mathrm{com}}(n^{1/\alpha_0})} \simeq \frac{1}{\#Q(S_0, \mathfrak{F}_0, n)} \simeq \frac{1}{\#Q(S, \mathfrak{F}, n)}.$$

Using the notation introduced in Theorem 3.5.15, we have

$$\#Q(S, \mathfrak{F}, r) \simeq n^{D(S,\mathfrak{v})} L(n)$$

which yields the desired result. $\square$

3.5.4 Near diagonal lower bounds

In this section we use Lemma 3.4.4(ii) to turn the sharp *on diagonal lower bounds* of the previous section into *near diagonal lower bounds.* The key tool is the following lemma.

Lemma 3.5.16. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1,\dots,s_k)$ and a k-tuple of positive reals $a = (\alpha_1,\dots,\alpha_k) \in (0,\infty]^k$. Let $\mathfrak{w} = \mathfrak{w}(a)$ be the two-dimensional weight system which assigns weight $w_i = (v_{i,1}, v_{i,2})$ to s_i where*

$$v_{i,1} = \frac{1}{\tilde{\alpha}_i}, \quad \tilde{\alpha}_i = \min\{2, \alpha_i\}$$

and

$$v_{i,2} = 0 \text{ unless } \alpha_i = 2 \text{ in which case } v_{i,2} = 1/2.$$

Let $\mathfrak{F}$ be the associated weight function system generated by

$$F_i(r) = r^{v_{i,1}}[\log(1+r)]^{v_{i,2}},\ 1 \le i \le k.$$

Then

$$\left|\mu_{S,a}^{(2n+m)}(xg) - \mu_{S,a}^{(2n+m)}(x)\right| \le C\left(F_S^{-1}(\|g\|_{\Sigma,\mathfrak{F}})/m\right)^{1/2}\ \mu_{S,a}^{(2n)}(e).$$

Proof. By Theorem 3.2.10, there is an integer $p = p(G,S,\mathfrak{w})$ such that any g with $F_S^{-1}(\|y\|_{S,\mathfrak{F}}) = r$ can be expressed as

$$g = \prod_{j=1}^{p} s_{i_j}^{x_j} \text{ with } |x_j| \le CF_{i_j}(r).$$

Write $\mu_{S,a}^{(2n+m)} = \mu_{S,a}^{(n+m)} * \mu_{S,a}^{(n)}$ and, for each step $s_{i_j}^{x_j}$, apply Lemma 3.4.4(ii) to obtain

$$\begin{aligned}
&\left|\mu_{S,a}^{(2n+m)}(zs_{i_j}^{x_j}) - \mu_{S,a}^{(2n+m)}(z)\right| \\
&\qquad \le \quad C\mathcal{G}_{i_j}(|x_j|)^{-1/2}\,|x_j|\,\mathcal{E}_{\mu_{S,a}}(\mu_{S,a}^{(n+m)},\mu_{S,a}^{(n+m)})^{1/2}\left\|\mu_{S,a}^{(n)}\right\|_2 \\
&\qquad \le \quad Cr^{1/2}\mathcal{E}_{\mu_{S,a}}(\mu_{S,a}^{(n+m)},\mu_{S,a}^{(n+m)})^{1/2}\left\|\mu_{S,a}^{(n)}\right\|_2.
\end{aligned}$$

Here, according to Lemma 3.4.4, $\mathcal{G}_i(r) = r^{2-\tilde{\alpha}_i}$ if $v_i, 2 = 0$ and $\mathcal{G}_i(r) = \log(1+r)$ if $v_{i,2} = 1/2$ (i.e., if $\alpha_i = 2$). Hence, $s^2/\mathcal{G}_i(s) \simeq F_i^{-1}(s)$, which gives the last inequality.

By [20, Lemma 3.2], we also have

$$\mathcal{E}_{\mu_{S,a}}(\mu_{S,a}^{(n+m)},\mu_{S,a}^{(n+m)})^{1/2} \le Cm^{-1/2}\left\|\mu_{S,a}^{(n)}\right\|_2 = Cm^{-1/2}\mu_{S,a}^{(2n)}(e)^{1/2}.$$

This gives the desired inequality. $\square$

Theorem 3.5.17. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1,\dots,s_k)$ and a k-tuple of positive reals $a=(\alpha_1,\dots,\alpha_k)\in(0,\infty]^k$. Set $\tilde{\alpha}_i=\min\{\alpha_i,2\}$. Let $\mathfrak{w}$ be the weight system which assigns weight $1/\tilde{\alpha}_i$ to $s_i\in S$. Let Σ be a sequence of formal commutators as in* Theorem 3.2.10. *Assume that $w(s)>1/2$ for all $s\in\mathrm{core}(\mathfrak{w},S,\Sigma)$. Then, there exists $\epsilon>0$ such that, uniformly over the region $\{x\in G:\|x\|_{S,\mathfrak{w}}\le F_S(\epsilon n)\}$, we have*

$$\mu_{S,a}^{(n)}(x)\simeq n^{-D(S,\mathfrak{w})}.$$

Proof. Theorem 3.5.10 gives $\mu_{S,a}^{(n)}(e)\simeq n^{-D(S,\mathfrak{w})}$. This, together with Lemma 3.5.16, yields the desired lower bound. $\square$

Theorem 3.5.18. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1,\dots,s_k)$ and a k-tuple of positive reals $a=(\alpha_1,\dots,\alpha_k)\in(0,\infty]^k$. Set $\tilde{\alpha}_i=\min\{\alpha_i,2\}$. Let $\widetilde{\mathfrak{w}}$ be the weight system which assigns weight $\widetilde{w}_i=1/\tilde{\alpha}_i$ to s_i. Let Σ be as in* Theorem 3.2.10 *applied to $(S,\widetilde{\mathfrak{w}})$ and assume that $\alpha_i=2$ for all $i\in\{1,\dots,k\}$ such that $s_i\in\mathrm{core}(S,\widetilde{\mathfrak{w}},\Sigma)$. Then there exists $\epsilon>0$ such that, uniformly over the region*

$$\{x\in G:|x|_S^2[\log|x|_S]^{-1}\le\epsilon n\},$$

we have

$$\mu_{S,a}^{(n)}(x)\simeq[n\log n]^{-D(G)/2}.$$

Proof. By Theorem 3.5.12, we have $\mu_{S,a}^{(n)}(e)\simeq[n\log n]^{-D(G)/2}$. Let $\mathfrak{w},\mathfrak{F}$ be the two dimensional weight system and weight function system introduced above in

Lemma 3.5.16. It follows from Theorems 3.2.10-3.6.22 and Corollary 3.2.19 that $F_S^{-1}(\|\cdot\|_{S,\mathfrak{F}}) \simeq |\cdot|_S^2/\log|\cdot|_S$. The result follows.

$\square$

3.6 Proofs regarding approximate coordinate systems

This section contains the proofs of the key results stated in Sections 3.2.1-3.3, namely, Theorems 3.2.10-3.3.1. Throughout this section, G is a finitely generated nilpotent group equipped with a generating k-tuple $(s_1,\dots,s_k)$. Formal commutators refer to commutators on the alphabet $\{s_i^{\pm 1} : 1 \le i \le k\}$.

3.6.1 Proof of Theorem 3.3.1 and assorted results

Theorem 3.3.1 is one of the keys to the random walk upper bounds of Section 3.4. It can be understood as providing a volume lower bound for the volume of certain balls together with some additional "structural information" on the balls in question.

Fix a weight system $\mathfrak{w}$ and weight functions F_c as in Theorem 3.3.1. Let $G_h^{\mathfrak{w}}$ be the associated descending normal series in G. By construction, $G_h^{\mathfrak{w}}$ is normal in G and, for all p,q,j such that $\bar{w}_p + \bar{w}_q \ge \bar{w}_j$, we have (see Section 3.1.3)

$$[G_p^{\mathfrak{w}}, G_q^{\mathfrak{w}}] \subset G_j^{\mathfrak{w}}.$$

It follows that the commutators map

$$G_p^{\mathfrak{w}} \times G_q^{\mathfrak{w}} : (u,v) \mapsto [u,v] \in G_j^{\mathfrak{w}}$$

induces a group homomorphism

$$G_p^{\mathfrak{w}}/G_{p+1}^{\mathfrak{w}} \otimes G_q^{\mathfrak{w}}/G_{q+1}^{\mathfrak{w}} \to G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}.$$

This yields the following lemma.

Lemma 3.6.1 (Similar to [3, Lemma 3]). *Let c be a formal commutator of weight $\bar{w}_j$ and let g_c be its image in G. There is an integer $\ell = \ell(c) \leq 8^j$ and a sequence $(i_1, \ldots, i_\ell) \in \{1, \ldots, k\}^\ell$ such that, for any $r \geq 1$ and $n \in \mathbb{Z}$ satisfying $|n| \leq F_c(r)$, we have*

$$g_c^n = s_{i_1}^{n_1} s_{i_2}^{n_2} \cdots s_{i_\ell}^{n_\ell} \quad \textit{mod } G_{j+1}^{\mathfrak{w}}$$

for some $n_{i_j} \in \mathbb{Z}$ with $|n_j| \leq F_{s_{i_j}}(r)$.

Proof. The proof is by induction on j. For $j = 1$, c must have length 1 and $g_c^n = s_i^n$ for some $i \in \{1, \ldots, k\}$. Assume the result holds true for all $h < j$ and let c be a commutator of weight $\bar{w}_j$. Either c has length 1 and the result is trivial or $c = [u, v]$ where u, v are commutators of weights $\bar{w}_p, \bar{w}_p$, $\bar{w}_p + \bar{w}_q = \bar{w}_j$. Since $F_c = F_u F_v$, for all $|n| \leq F_c(r)$ we can write $n = ab + d$ with $|a|, |d| \leq F_u(r)$, $0 \leq d \leq F_v(r)$. Then

$$g_c^n = [u, v]^{ab}[u, v]^d = [u^a, v^b][u^d, v] \bmod G_{j+1}^{\mathfrak{w}}.$$

The desired result follows from the induction hypothesis. $\square$

Definition 3.6.2. Given c, $\ell = \ell(c)$ and $(i_1, \ldots, i_\ell)$ as in Lemma 3.6.1, for any $\mathbf{x} = (x_1, \ldots, x_\ell) \in \mathbb{Z}^\ell$, set

$$\mathbf{g}_c(\mathbf{x}) = \mathbf{g}_c(x_1, \ldots, x_\ell) = s_{i_1}^{x_1} s_{i_2}^{x_2} \cdots s_{i_\ell}^{x_\ell} \in G.$$

Set

$$F_j^c = F_{s_{i_j}} = F_{i_j}, \ \ 1 \leq j \leq \ell.$$

By Lemma 3.6.1, if $w(c) = \bar{w}_j$ and $|n| \leq F_c(r)$ then

$$g_c^n = \mathbf{g}_c(\mathbf{n}(c)) \bmod G_{j+1}^{\mathfrak{w}}$$

for some $\mathbf{n}(c) = (n_1(c), \dots, n_\ell(c))$ with $|n_j(c)| \leq F_{s_{i_j}}(r) = F_j^c(r)$.

Theorem 3.6.3. *Let $c_1, \dots c_t$ be a sequence of formal commutators with non-decreasing $\mathfrak{w}$-weights and such that, for each h, the image in $G_h^{\mathfrak{w}}/G_{h+1}^{\mathfrak{w}}$ of the family $\{c_i : w(c_i) = \bar{w}_h\}$ is a linearly independent family. Set*

$$K(r) = \{g \in G : g = \prod_{i=1}^{t} \mathbf{g}_{c_i}(\mathbf{x}_i), \ \ \mathbf{x}_i = (x_1^i, \dots, x_{\ell(c_i)}^i) \in \mathbb{Z}^{\ell(c_i)}, |x_j^i| \leq F_j^{c_i}(r)\}.$$

Then

$$\# K(r) \geq \prod_1^t (2F_{c_i}(r) + 1) \geq \prod_{i=1}^{t} \prod_{j=1}^{\ell(c_i)} F_j^c(r).$$

Proof. For each $(y_i)_1^t \in \mathbb{Z}^t$ with $|y_i| \leq F_{c_i}(r)$, let $\mathbf{y}_i = (y_j^i)_1^{\ell(c_i)}$, $1 \leq i \leq t$, be such that

$$g_{c_i}^{y_i} = \mathbf{g}_{c_i}(\mathbf{y}_i) \ \bmod G_{j+1}^{\mathfrak{w}}, \ \ w(c_i) = \bar{w}_j, \ 1 \leq i \leq t.$$

Such a $(\mathbf{y}^i)_1^t$ is given by Lemma 3.6.1. Assume that two sequences $(y_i)_1^t$ and $(\tilde{y}_i)_1^t$ are such that $\prod_{i=1}^t \mathbf{g}_{c_i}(\mathbf{y}_i) = \prod_{i=1}^t \mathbf{g}_{c_i}(\widetilde{\mathbf{y}}_i)$. Then by projecting on $G_1^{\mathfrak{w}}/G_2^{\mathfrak{w}}$ and using the assumed linear independence of the collection of the c_i's with $w(c_i) = \bar{w}_1$ in $G_1^{\mathfrak{w}}/G_2^{\mathfrak{w}}$ and the fact that $g_{c_i}^{y_i} = \mathbf{g}_{c_i}(\mathbf{y}^i)$ in $G_1^{\mathfrak{w}}/G_2^{\mathfrak{w}}$,we find that $y_i = \tilde{y}_i$ for those i with $w(c_i) = \bar{w}_1$. This implies that $\mathbf{y}_1 = \widetilde{\mathbf{y}}_1$. Proceeding further up in the weight filtration shows that we must have $\mathbf{y}_i = \tilde{\mathbf{y}}_i$ for all $1 \leq i \leq t$. This shows that there are at least $\prod_1^t (2F_{c_i}(r) + 1)$ distinct elements in $K(r)$ which is the desired result. □

Theorem 3.6.4. *Fix a weight system $\mathfrak{w}$ and weight functions F_c as in* Theorem 3.3.1. *Let $b_1, \dots b_t$ be a sequence of elements in G. Assume that :*

1. *For each $i = 1, \dots, t$, there exists an integer $h(i)$ such that $b_i \in G^{\mathfrak{w}}_{h(i)}$ and b_i is torsion free in $G^{\mathfrak{w}}_{h(i)}/G^{\mathfrak{w}}_{h(i)+1}$. Further, for each h, the system $\{b_i : h(i) = h\}$ is free in $G^{\mathfrak{w}}_{h(i)}/G^{\mathfrak{w}}_{h(i)+1}$.*

2. *For each $i = 1, \dots, t$, there exists and increasing function $\widetilde{F}^i$, a positive integer $\ell(i)$ and a sequence $j^i_1, \dots, j^i_{\ell(i)}$ such that, for any $r > 0$ and any integer n with $|n| \leq \widetilde{F}^i(r)$, there exists $\mathbf{n}^i = (n^i_1, \dots, n^i_{\ell(i)})$ with $|n^i_q| \leq F_{j^i_q}(r)$ satisfying*

$$b_i^n = \prod_{q=1}^{\ell(i)} s_{j^i_q}^{n^i_q} \mod G^{\mathfrak{w}}_{h(i)+1}.$$

For $\mathbf{x} = (x_1, \dots, x_{\ell(i)}) \in \mathbb{Z}^{\ell(i)}$, set $\mathbf{b}_i(\mathbf{x}) = \prod_{q=1}^{\ell(i)} s_{j^i_q}^{x_q} \in G$ and

$$K(r) = \{g \in G : g = \prod_{i=1}^{t} \mathbf{b}_i(\mathbf{x}_i), \ \ \mathbf{x}_i = (x^i_1, \dots, x^i_{\ell(i)}) \in \mathbb{Z}^{\ell(i)}, |x^i_q| \leq F_{j^i_q}(r)\}.$$

Then

$$\#K(r) \geq \prod_{1}^{t}(2\widetilde{F}_i(r) + 1).$$

Proof. This a straightforward generalization of Theorem 3.6.3. Instead of considering commutators and their natural weight function F_c, we consider arbitrary group elements b with associated weight function $\widetilde{F}$ with the property that b is free in $G^{\mathfrak{w}}_h/G^{\mathfrak{w}}_{h+1}$, for some u, h, and b^n, $|n| \leq \widetilde{F}(r)$, can be express modulo $G^{\mathfrak{w}}_{h+1}$ as a fixed product of powers of generators with properly controlled exponents. The proof is essentially the same as that of Theorem 3.6.3. Namely, for each $(y_i)_1^t \in \mathbb{Z}^t$ with $|y_i| \leq \widetilde{F}^i(r)$, let $\mathbf{y}_i = (y^i_j)_1^{\ell(i)}$, $1 \leq i \leq t$, be such that

$$b_i^{u_i y_i} = \mathbf{b}_i(\mathbf{y}_i) \mod G^{\mathfrak{w}}_{h(i)+1}, \quad 1 \leq i \leq t.$$

Such a $(\mathbf{y}^i)_1^t$ exists by hypothesis. Assume that two sequences $(y_i)_1^t$ and $(\tilde{y}_i)_1^t$ are such that $\prod_{i=1}^t \mathbf{b}^i(\mathbf{y}_i) = \prod_{i=1}^t \mathbf{b}^i(\widetilde{\mathbf{y}}_i)$. Then by projecting on $G^{\mathfrak{w}}_1/G^{\mathfrak{w}}_2$ and using the assumed freeness of the collection of the b_i's with $h(i) = 1$ in $G_1^{\mathfrak{w}}/G_2^{\mathfrak{w}}$ and the fact that $b_i^{u_i y_i} = \mathbf{b}^i(\mathbf{y}^i)$ in $G_1^{\mathfrak{w}}/G_2^{\mathfrak{w}}$,we find that $y_i = \tilde{y}_i$ for those i with $h(i) = 1$. This implies $\mathbf{y}_1 = \widetilde{\mathbf{y}}_1$. Proceeding further up in the weight filtration shows that we must have $y_i = \tilde{y}_i$ for all $1 \leq i \leq t$. This shows that there are at least $\prod_1^t (2\widetilde{F}^i(r) + 1)$ distinct elements in $K(r)$, as desired. $\square$

Remark 3.6.5. Theorem 3.6.4 allows for much more freedom than Theorem 3.6.3. This freedom is used in the proof of Theorem 3.3.4.

3.6.2 Commutator collection on free nilpotent groups

In this section, we prove the following weak version of Theorem 3.2.10.

Theorem 3.6.6. *Referring to the setting and notation of* Theorem 3.2.10*, assume that* (3.1)-(3.2) *hold true. Then there exist an integer $t = t(G, S, \mathfrak{w})$, a constant $C = C(G, S, \mathfrak{w}) \geq 1$, and a sequence Σ of commutators (depending on $G, S, \mathfrak{w}$)*

$$c_1, \ldots, c_t \text{ with non-decreasing weights } w(c_1) \preceq \cdots \preceq w(c_t)$$

such that

(i) For any $r > 0$, if $g \in G$ can be expressed as a word ω over $\mathfrak{C}(S)^{\pm 1}$ with $\deg_c(\omega) \leq F_c(r)$ for all $c \in \mathfrak{C}(S)$ then g can be expressed in the form

$$g = \prod_{i=1}^{t} c_i^{x_i} \text{ with } |x_i| \leq F_{c_i}(Cr) \text{ for all } i \in \{1, \ldots, t\}.$$

(ii) There exist an integer $p = p(G, S, \mathfrak{w})$ and $(i_j)_1^p \in \{1, \ldots, k\}^p$ (also depending on $(G, S, \mathfrak{w})$ such that, if g can be expressed as a word ω over $\{c_i^{\pm 1} : 1 \leq i \leq t\}$ with $\deg_{c_i}(\omega) \leq F_{c_i}(r)$ for some $r > 0$ then g can be expressed in the form

$$g = \prod_{j=1}^{p} s_{i_j}^{x_j} \text{ with } |x_j| \leq F_{i_j}(Cr).$$

Remark 3.6.7. Note that it must be the case that, for any j, the image of $\{c_i : w(c_i) = \bar{w}_j\}$ in $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$ generates $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$. The key difference with Theorem 3.2.10 is that Theorem 3.6.6 does not identify a maximal subset of $\{c_i : w(c_i) = \bar{w}_j\}$ that is free in $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$.

The proof of Theorem 3.6.6 requires a number of steps. The first observation is that it is enough to prove Theorem 3.6.6 in the case of the free nilpotent group $N(k,\ell)$ on k generators $s_1, \dots, s_k$ and of nilpotency class ℓ. Indeed, once Theorem 3.6.6 is proved on $N(k,\ell)$, the same statement holds on any nilpotent G of nilpotency class ℓ equipped with a generating k-tuple S via the canonical projection from $N(k,\ell)$ to G (by definition, the canonical projection is the group homeomorphism from $N(k,\ell)$ onto G which sends the canonical k generators of $N(k,\ell)$ to the given k generators of G).

Notation 3.6.8. For the rest of this section, we assume that $G = N(k,\ell)$ is the free nilpotent group $N(k,\ell)$ equipped with its canonical generating set $S = (s_1, \dots, s_k)$ and the multidimensional weight-system $\mathfrak{w}$ generated by the $(w_1, \dots, w_k)$. Without loss of generality, we assume that the commutator set $\mathfrak{C}(S)$ is equipped with a total order $\prec$ such that the function

$$w : \mathfrak{C}(S) \ni c \mapsto w(c) \in (0,\infty) \times \mathbb{R}^{d-1}$$

associated with the given weight system $\mathfrak{w}$ is non-decreasing. Hence, $c \prec c'$ implies $w(c) \preceq w(c')$. In addition, we let $\mathfrak{F}$ be a weight function system that is compatible with $\mathfrak{w}$ in the sense that (3.1)-(3.2) hold true.

Notation 3.6.9. Recall that $\deg_c(\omega)$ denotes the number of occurrences of $c^{\pm 1}$ in the word ω over $\mathfrak{C}(S)$. Similarly, we define $\deg_c^*(\omega)$ to be the number of occurrences of c minus the number of occurrences of c^{-1} in a word over $\mathfrak{C}(S)$.

On $\mathfrak{C}(S)$, consider the map J such that $J(s_i^{\pm 1}) = s_i^{\mp 1}$ and $J([a,b]) = [b,a]$. Abusing notation, we also write $J(c) = c^{-1}$. Note that J^2 is the identity. Restrict J to $\mathfrak{C}^*(S) = \{c : J(c) \neq c\}$ (where $J(c) = c$ is understood as equality as formal commutator so that $J(s_i) \neq s_i$ and $J([a,b]) = [a,b]$ if and only if $a = b$). Let $\mathfrak{C}^*_+$ be the set of representative of $\mathfrak{C}^*(S)/J$ given by $c \in \mathfrak{C}^*_+(S)$ if and only if $c = s_i$ or $c = [a,b]$ with $a \succ b$.

It is convenient to enumerate all formal commutators in $\mathfrak{C}^*_+(S,\ell)$ and write

$$\mathfrak{C}^*_+(S,\ell) = \{c_1, \dots, c_t\}, \ \ t = \#\mathfrak{C}^*_+(S,\ell).$$

Since ℓ is fixed throughout, we write

$$\mathfrak{C}^*_+(S) = \mathfrak{C}^*_+(S,\ell).$$

Note that, a priori, this list contains commutators that are trivial in $N(k,\ell)$. This does not matter although these formal commutators can be omitted if desired. Let us describe the basic collecting process on $N_{k,\ell}$.

Commutator collecting algorithm

- Given a word $\omega = c_{i_1}^{\epsilon_{i_1}} c_{i_2}^{\epsilon_{i_2}} ... c_{i_m}^{\epsilon_{i_m}}$ in $\mathfrak{C}^*_+(S) \cup \mathfrak{C}^*_+(S)^{-1}$, first identify the commutator of lowest order with respect to $\prec$, say it is commutator c_{i_j}, mark all the contributions of c_{i_j} to ω from left to right in order: $\{y_1, ..., y_q\}$, $y_j \in \{c_{i_j}^{\pm 1}\}$.
- Starting with y_1, move $y_1, ..., y_q$ to the left one by one by successive commutation. Note that every time c_{i_j} jumps backward over a commutator c, the jump produces the sequence $...c_{i_j} c [c, c_{i_j}]...$. It follows that all commutators that are created in this process belong to $\mathfrak{C}^*_+(S)$ and have weight $\succeq 2w(c_{i_j}) \succ w(c_{i_j})$.
- After $y_1, ..., y_q$ have been moved to the left, we obtain a word $y_1...y_q\omega'$ with the same image as ω, and where ω' is a word in commutators $\succ c_{i_j}$.

- Apply the previous steps to ω', producing ω'' and continue until the process terminates after at most $\#\mathfrak{C}^*_+(S)$ steps.

This proves the following weak version of M. Hall basis theorem [18, Theorem 11.2.3] (in Hall's more sophisticated version, only the so called "basic" commutators are used and this results in a unique representation of any element of $N(k,\ell)$).

Proposition 3.6.10. *Any element $g \in N(k,\ell)$ has a representation*

$$g = c_1^{x_1} c_2^{x_2} ... c_t^{x_t}, \quad x_i \in \mathbb{Z}.$$

Next we want to have some control over $\{x_i,\ 1 \le i \le t\}$. Let's start with a simple binomial counting lemma adapted from [18, page 173] and [42]. We will use the following notation. For any two commutators $c_j \succ c_i$, let $C_{n-1}(i,j)$ be the sets of all commutators $c \in \mathfrak{C}^*_+(S)$ such that there exist $\epsilon_0, \dots, \epsilon_n \in \{-1, 1\}$ such that $c_j^{\epsilon_n} = [\cdots[c^{\epsilon_0}, c_i^{\epsilon_1}], \dots, c_i^{\epsilon_{n-1}}]$ (as formal commutators in $\mathfrak{C}(S)$).

Lemma 3.6.11. *Consider a word ω in $\{c_j : c_j \succeq c_i\}^{\pm 1}$. Let $m = \deg_{c_i} \omega$, and let $\{y_1, ..., y_m\}$, $y_j \in \{\ c_i^{\pm 1}\}$, be the left to right contribution of c_i to ω. For $0 \le q \le m$, there is a word ω_q in $\{c_j : c_j \succeq c_i\}^{\pm 1}$ which starts with $y_1 ... y_q$, whose left to right contribution of $c_i^{\pm 1}$ is $y_1, \dots, y_m$, and in which, for all $c_j \succ c_i$,*

$$\begin{aligned}\deg_{c_j}(\omega_q) \ &\le \ \deg_{c_j}(\omega) + q \sum_{c \in C_1(i,j)} \deg_c(\omega) + \binom{q}{2} \sum_{c \in C_2(i,j)} \deg_c(\omega) \\ &\quad + ... + \binom{q}{\ell} \sum_{c \in C_\ell(i,j)} \deg_c(\omega)\end{aligned}$$

Further, if c' denotes the lowest commutator in ω with $c' \succ c_i$ then contributions of commutators c with $w(c) \prec w(c') + w(c_i)$ remain unchanged in ω_q.

Remark 3.6.12. Note that, after we move all contributions of c_i to ω to the left, we obtain a word ω_m with same image as ω of the form

$$\omega_m = c_i^x \omega'_m$$

where $x = \deg^*_{c_i}(\omega)$, ω'_m is a word in $[\mathfrak{C}^*_+(S) \cap \{c \succ c_i\}]^{\pm 1}$, and in which the contributions of commutators c with $w(c) \prec w(c') + w(c_i)$ remain the same than in ω.

Proof. The proof is by induction on q. It holds trivially for $q = 0$. The induction hypothesis gives us a word ω_{q-1} with

$$\begin{aligned}\deg_{c_j}(\omega_{q-1}) \;\; &\leq \;\; \deg_{c_j}(\omega) + (q-1) \sum_{c \in C_1(i.j)} \deg_c(\omega) + \binom{q-1}{2} \sum_{c \in C_2(i,j)} \deg_c(\omega) \\ &\quad + ... + \binom{q-1}{\ell} \sum_{c \in C_\ell(i,j)} \deg_c(\omega).\end{aligned}$$

Now, we move y_q to the left as in the collecting process by successive commutations. To keep track of contribution of c_j, notice that a new contribution of c_j is produced only if y_q jumps over a commutator $c^{\pm 1}$ such that $[c^{\pm 1}, y_q] = c_j^{\pm 1}$. Further, $w([c^{\pm 1}, y_q]) = w(c) + w(c_i) \succeq w(c') + w(c_i)$. Hence, c_j must satisfies $w(c_j) \succeq w(c') + w(c_i)$. Therefore we eventually get a word ω_q in $[\mathfrak{C}^*_+(S) \cap \{c \succeq c_i\}]^{\pm 1}$ with $\pi(\omega_q) = \pi(\omega)$, in which the left to right contribution of c_i is the same as in ω, which starts with $y_1 ... y_q$, and such that

$$\deg_{c_j}(\omega_q) \leq \deg_{c_j}(\omega_{q-1}) + \sum_{c \in C_1(i,j)} \deg_c(\omega_{q-1}).$$

Using the induction hypothesis on ω_{q-1} and the fact that all brackets of length at least $\ell + 1$ drop out,

$$\begin{aligned}\sum_{c \in C_1(i,j)} \deg_c(\omega_{q-1}) \;\; &= \;\; \sum_{c = c_\alpha \in C_2(i,j)} \sum_{p=0}^{\ell} \binom{q-1}{p} \sum_{\tilde{c} \in C_p(i,\alpha)} \deg_{\tilde{c}}(\omega) \\ &\leq \;\; \sum_{p=1}^{\ell} \binom{q-1}{p-1} \sum_{\tilde{c} \in C_p(i,j)} \deg_{\tilde{c}}(\omega).\end{aligned}$$

Hence, we have

$$\begin{aligned}
\deg_{c_j}(\omega_q) &\leq \deg_{c_j}(\omega_{q-1}) + \sum_{c \in C_2(i,j)} \deg_c(\omega_{q-1}) \\
&\leq \sum_{p=0}^{\ell} \left(\binom{q-1}{p} + \binom{q-1}{p-1} \right) \sum_{\widetilde{c} \in C_p(i,j)} \deg_{\widetilde{c}}(\omega) \\
&= \sum_{p=0}^{\ell} \binom{q}{p} \sum_{\widetilde{c} \in C_p(i,j)} \deg_{\widetilde{c}}(\omega).
\end{aligned}$$

□

Lemma 3.6.13. *There exists a constant $C > 0$ such that for any word ω in $[\mathfrak{C}_+^*(S) \cap \{c \succeq c_i\}]^{\pm 1}$ with $\deg_c \omega \leq F_c(d)$ for all $c \succeq c_i$, there exists a word ω' in $[\mathfrak{C}_+^*(S) \cap \{c \succeq c_i\}]^{\pm 1}$ in collected form:*

$$\omega' = \prod_{j=i}^{t} c_j^{x_j}$$

such that $\pi(\omega') = \pi(\omega)$, $x_j = \deg^*_{c_j} \omega$ for those j such that $w(c_j) \prec 2w(c_i)$ and $|x_j| \leq F_{c_j}(Cd)$ for all $i \leq j \leq t$.

Proof. The proof is by backward induction on i. For $i = t$, the statement holds trivially since commutators with $c \succeq c_t$ commute.

Suppose the assertion holds for $i+1$. Consider a word ω on $[\mathfrak{C}_+^*(S) \cap \{c \succeq c_i\}]^{\pm 1}$ as in the lemma. Let $\{y_1, ..., y_q\}$ be the contribution of c_i to ω, $q = \deg_{c_i} \omega$. The previous lemma yields $\omega_q = y_1 ... y_q \omega'_q$, where ω'_q is a word in $[\mathfrak{C}_+^*(S) \cap \{c \succeq c_{i+1}\}]^{\pm 1}$. From the hypothesis on the degrees of ω,

$$\deg_{c_j}(\omega_k) \leq \sum_{p=0}^{\ell} \binom{k}{p} \sum_{c \in C_p(i,j)} F_c(d)$$

From definition of weight functions, if $c \in C_p(i,j)$ then $F_c F_{c_i}^p = F_{c_j}$. Further,$\# C_p(i,j) \leq t = \# \mathfrak{C}_+^*(S)$ and $q = \deg_{c_i} \omega \leq F_{c_i}(d)$. Therefore, we obtain

$$\begin{aligned}\deg_{c_j}(\omega_q) &\leq tF_{c_j}(d)\left(\sum_{p=0}^{\ell}\binom{q}{p}F_{c_i}(d)^{-p}\right)\\ &\leq tF_{c_j}(d)\left(\sum_{p=0}^{\ell}q^pF_{c_i}(d)^{-p}\right)\leq t(1+\ell)F_{c_j}(d).\end{aligned}$$

By assumption (3.1), there exists a constant C_1 such that

$$t(1+\ell)F_c(d)\leq F_c(C_1d)$$

for all c and $d \geq 1$. $\square$

Lemma 3.6.13 with $i=1$ proves Theorem 3.6.6(i). Next we work on improving Theorem 3.6.6(i) in the special case of the free nilpotent group $N(k,\ell)$. This improvement will be instrumental in proving Theorem 3.6.6(ii). It is based on the following important Lemma.

Lemma 3.6.14. *For each j, $N(k,\ell)_j^{\mathfrak{w}}/N(k,\ell)_{j+1}^{\mathfrak{w}}$ is a finitely generated free abelian group.*

Proof. The proof is by a backward induction on ℓ. If $\ell=1$, $N(k,1)$ is the free abelian group on k generators and the desired result holds by inspection. Let $g\in N(k,\ell)_j^{\mathfrak{w}}$ such that $g\notin N(k,\ell)_{j+1}^{\mathfrak{w}}$. Let $N_\ell = N(k,\ell)_\ell$ be the center of $N(k,\ell)$ (i.e., the subgroup generated by commutators of length ℓ). Assume first that $g\in N(k,\ell)_{j+1}^{\mathfrak{w}}N_\ell$. Since

$$N(k,\ell)_{j+1}^{\mathfrak{w}}N_\ell/N(k,\ell)_{j+1}^{\mathfrak{w}}\simeq N_\ell/[N(k,\ell)_{j+1}^{\mathfrak{w}}\cap N_\ell],$$

and $N(k,\ell)_{j+1}^{\mathfrak{w}}\cap N_\ell$ is generated by the basic commutators of weight $\bar{w}_j$ and length ℓ, $N_\ell/[N(k,\ell)_{j+1}^{\mathfrak{w}}\cap N_\ell]$ is torsion free. It thus follows that g is not torsion in $N(k,\ell)_j^{\mathfrak{w}}/N(k,\ell)_{j+1}^{\mathfrak{w}}$.

Now, consider the case when $g \notin N(k,\ell)_j^{\mathfrak{w}} N_\ell$. Let g' be the projection of g in $N(K,\ell)/N_\ell = N(k,\ell-1)$. Clearly $g' \in N(k,\ell-1)_j^{\mathfrak{w}}$ and $g' \notin N(k,\ell-1)_{j+1}^{\mathfrak{w}}$ because the inverse image of $N(k,\ell-1)_{j+1}^{\mathfrak{w}}$ under this projection is $N(k,\ell)_{j+1}^{\mathfrak{w}} N_\ell$. Further,

$$N(k,\ell)_j^{\mathfrak{w}} N_\ell / N(k,\ell)_{j+1}^{\mathfrak{w}} N_\ell \simeq N(k,\ell-1)_j^{\mathfrak{w}} / N(k,\ell-1)_{j+1}^{\mathfrak{w}}.$$

By the induction hypothesis, g' is not torsion in $N(k,\ell-1)_j^{\mathfrak{w}} / N(k,\ell-1)_{j+1}^{\mathfrak{w}}$. It follows that g is not torsion in $N(k,\ell)_j^{\mathfrak{w}} / N(k,\ell)_{j+1}^{\mathfrak{w}}$. □

Next, let $(b_i)_1^\tau$ be a sequence of elements of $\mathfrak{C}_+^*(S)$ such that $\{b_i : w(b_i) = \bar{w}_j\}$ projects to a basis of $N(k,\ell)_j^{\mathfrak{w}} / N(k,\ell)_{j+1}^{\mathfrak{w}}$. Let $R_j^{\mathfrak{w}}$ be the rank of this torsion free abelian group and set $m'_j = \sum_1^j R_i^{\mathfrak{w}}$ so that $\tau = m'_{j_*}$. Set also $m_j = \max\{i : w(c_i) = \bar{w}_j\}$. Without loss of generality, we can assume that our ordering on $\mathfrak{C}_+^*(S)$ is such that

$$(b_i)_{m'_{j-1}+1}^{m'_j} = (c_j)_{m_{j-1}+1}^{m_{j-1}+R_j^{\mathfrak{w}}}.$$

Lemma 3.6.15. *Referring o the above setup and notation, there exists a constant $C > 0$ such that for any word ω in $\{c_i : w(c_i) \succeq \overline{w}_h\}^{\pm 1}$ with $\deg_{c_j} \omega \le F_{c_j}(d)$ for all j, there is a word ω_h*

$$\omega_h = \prod_{j=m'_{h-1}+1}^{\tau} b_j^{x_j}$$

such that $\pi(\omega_h) = \pi(\omega)$ and $|x_j| \le C F_{c_j}(Cd)$, $m'_{h-1} + 1 \le j \le m'_h$.

Proof. The proof is by backward induction on h. When $h = j_*$, $N(k,\ell)_{j_*}^{\mathfrak{w}}$ is abelian and this is just linear algebra.

For a word ω as in the lemma, Lemma 3.6.13 gives a word

$$\omega' = \prod_{i \ge m_{h-1}+1} c_i^{x_i}, \quad |x_i| \le F_{c_i}(Cd)$$

with the same image as ω. Set

$$I_1(h) = \{m_{h-1}+1, \ldots, m_{h-1}+R_h^{\mathfrak{w}}\}, \quad I_2(h) = \{m_{h-1}+R_h^{\mathfrak{w}}+1, \ldots, m_h\}$$

For $i \in I_2(h)$, c_i has the same image than

$$\prod_{j \in I_1(h)} c_j^{z_{j,i}} v_i$$

with v_i a word in $\{c_p : w(c_p) \succeq \bar{w}_{h+1}\}^{\pm 1}$. Hence

$$\omega'' = \prod_{j \in I_1(h)} c_j^{x_j} \prod_{i \in I_2(h)} \left(\prod_{j \in I_1(h)} c_j^{z_{i,j}} v_i \right)^{x_i} \prod_{p > m_h} c_p^{x_p}$$

has the same image than ω. Applying Lemma 3.6.13 to this word ω'' gives

$$\omega_h' = \prod_{j \in I_1(h)} c_j^{x_j + \sum_{i \in I_2 h} z_{i,j} x_i} \prod_{p > m_h} c_p^{x_p'}$$

with the same image than ω'' and $|x_p'| \leq F_{c_p}(Cd)$ for $p > m_h$. Further, since $F_{c_i} \simeq F_{c_j} \simeq \mathbf{F}_h$, for $i \in I_1(h), j \in I_2(h)$, we have

$$|x_j + \sum_{i \in I_2(h)} z_{i,j} x_i| \leq F_{c_j}(Cd).$$

Applying the induction hypothesis to rewrite $\prod_{p > m_h} c_p^{x_p'}$ finishes the proof. $\square$

Theorem 3.6.16. *Assume that the free nilpotent group $N(k,\ell)$ is equipped with its canonical generating k-tuple $S = (s_1, \ldots, s_k)$ and a weight system $\mathfrak{w}$ and weight-function system $\mathfrak{F}$ such that (3.1)-(3.2) hold true. Let b_i, $1 \leq i \leq \tau$, be a sequence of elements of $C_+^*(S)$ with $w(b_i) \preceq w(b_{i+1})$, $1 \leq i \leq \tau - 1$ and such that, for each j, $\{b_i : w(b_i) = \bar{w}_j\}$ is a basis of the free abelian group $N(k,\ell)_j^{\mathfrak{w}} / N(k,\ell)_{j+1}^{\mathfrak{w}}$. Then*

(i) Any element $g \in N(k,\ell)$ can be expressed uniquely in the form

$$g = \prod_{i=1}^{\tau} b_i^{x_i}, \quad x_i \in \mathbb{Z}, i \in \{1, \ldots, \tau\}.$$

Further,

$$F_S^{-1}(\|g\|_{\mathfrak{C}(S),\mathfrak{F}}) \simeq \max_{1 \leq i \leq \tau} \{F_{b_i}^{-1}(|x_i|)\}.$$

(ii) There exist an integer p and $(i_j)_1^p \in \{1,\dots,k\}^p$ such that any $g \in N(k,\ell)$ with $\|g\|_{\mathfrak{C}(S),\mathfrak{F}} \le F_S(r)$, $r > 0$, can be expressed in the form

$$g = \prod_{j=1}^{p} s_{i_j}^{y_j} \text{ with } |y_j| \le F_{i_j}(Cr),\ j \in \{1,\dots,p\}.$$

Remark 3.6.17. This result is a strong version of Theorem 3.2.10 in the special case when $G = N(k,\ell)$.

Proof of (i). The first assertion follows from Lemma 3.6.15. Uniqueness is clear if one considers the projections of g onto the successive free abelian groups $N(k,\ell)_j^{\mathfrak{w}}/N(k,\ell)_{j+1}^{\mathfrak{w}}$. □

The proof of the the second assertion requires some preparation. Given a commutator c with length $m \le \ell$, let $\sigma = \sigma_1 ... \sigma_m$ be the formal word on the alphabet S obtained from c by removing brackets and inverses. For $\overrightarrow{a} = (a_1, ..., a_\ell) \in \mathbb{Z}^\ell$, $\Theta(\overrightarrow{a}, c)$ is defined as the expression we get by substituting in c each σ_i by $\sigma_i^{a_i}$, while keeping all the brackets and signs unchanged. For example, if $c = [[s_{i_1}, s_{i_2}^{-1}], s_{i_3}^{-1}]$, and $\overrightarrow{a} = (a_1, a_2, a_3, 0, ..., 0)$, we have

$$\Theta(\overrightarrow{a}, c) = [[s_{i_1}^{a_1}, s_{i_2}^{-a_2}], s_{i_3}^{-a_3}].$$

Lemma 3.6.18. *For a commutator c with length $m \le \ell$, let $\sigma = \sigma_1 ... \sigma_m$ be the formal word associated with it. Suppose $a_1, ..., a_m \in \mathbb{Z}$ are such that $|a_j| \le F_{\sigma_j}(d)$ for all $1 \le j \le m$, $d > 0$. Set $\overrightarrow{a} = (a_1, ..., a_m, 0, ..., 0) \in \mathbb{Z}^\ell$ and consider the element $u \in N(k,\ell)$ such that*

$$uc^{a_1 ... a_k} = \Theta(\overrightarrow{a}, c).$$

Then u can be represented by a word ω on $\{c_j : w(c_j) \succ w(c)\}^{\pm 1}$ with $\deg_{c_j}(\omega) \le F_{c_j}(Cd)$ for all c_j with $w(c_j) \succ w(c)$.

Proof. The proof is by induction on the length m of the commutator c. When $m = 1$, the statement is trivial.

Suppose the statement is true for commutators of length $\leq m-1$. Let c be a commutator with length m, say $c = [f_1, f_2]$, where f_1, f_2 are commutators of length $m_1,\ m_2 < m$. Write $\overrightarrow{a}_1 = (a_1, ..., a_{m_1}, 0, ..., 0)$ and $\overrightarrow{a}_2 = (a_{m_1+1}, ..., a_{m_1+m_2}, 0, ..., 0)$, then by definition

$$\Theta(\overrightarrow{a}, c) = [\Theta(\overrightarrow{a}_1, f_1), \Theta(\overrightarrow{a}_2, f_2)].$$

By the induction hypothesis,

$$\Theta(\overrightarrow{a}_1, f_1) = u_1 f_1^{a_1 \ldots a_{m_1}},\ \ \Theta(\overrightarrow{a}_2, f_2) = u_2 f_2^{a_{m_1+1} \ldots a_{m_1+m_2}}$$

where u_1 can be represented by a word ω_1 in commutators c_p with $w(c_p) \succ w(f_1)$ and $\deg_{c_p}(\omega) \leq F_{c_p}(Cd)$. Similarly, u_2 can be represented by a word ω_2 in commutators c_p with $w(c_p) \succ w(f_2)$ and $\deg_{c_p}(\omega) \leq F_{c_p}(Cd)$.

Suppose $w(f_1) = \overline{w}_{h_1}$, $w(f_2) = \overline{w}_{h_2}$, and $w([f_1, f_2]) = \overline{w}_h$. By the natural group homomorphism

$$N^{\mathfrak{w}}_{h_1}/N^{\mathfrak{w}}_{h_1+1} \otimes N^{\mathfrak{w}}_{h_2}/N^{\mathfrak{w}}_{h_2+1} \to N^{\mathfrak{w}}_{h}/N^{\mathfrak{w}}_{h+1},$$

we have that

$$\begin{aligned}
[\Theta(\overrightarrow{a}_1, f_1), \Theta(\overrightarrow{a}_2, f_2)] &\equiv [f_1^{a_1 \ldots a_{m_1}}, f_2^{a_{m_1+1} \ldots a_{m_1+m_2}}] \bmod N^{\mathfrak{w}}_{h+1} \\
&\equiv [f_1, f_2]^{a_1 \ldots a_{m_1+m_2}} \bmod N^{\mathfrak{w}}_{h+1} \\
&\equiv c^{a_1 \ldots a_m} \bmod N^{\mathfrak{w}}_{h+1}.
\end{aligned}$$

Therefore $u = \Theta(\overrightarrow{a}, c) c^{-a_1 \ldots a_m} \in N^{\mathfrak{w}}_{h+1}$, and since

$$u = [u_1 f_1^{a_1 \ldots a_{k_1}}, u_2 f_2^{a_{k_1+1} \ldots a_{k_1+k_2}}] c^{-a_1 \ldots a_k},$$

it can be represented by a word ω such that $\deg_{c_i} \omega \leq 5F_{c_i}(Cd)$ for all i. Then by Theorem 3.6.16(i), we have

$$u = \prod_{j:w(b_j)\succeq \bar{w}_h} b_j^{x_j}.$$

with $|x_j| \leq F_{b_j}(C'd)$. $\square$

Lemma 3.6.19. *For any h, there exist constants $M_h > 0$ and $C_h > 0$ such that, for any $c \in \mathfrak{C}_+^*(S)$ with $w(c) \succeq \bar{w}_h$, there a integer $p = p(c)$ with $0 \leq p \leq M_h$ and a p-tuple $(i_1, \ldots, i_p) \in \{1, \ldots, k\}^p$, such that for any $x \in \mathbb{Z}$ with $|x| \leq F_c(d)$, $d > 0$, we have*

$$c^x = s_{i_1}^{x_1} s_{i_2}^{x_2} ... s_{i_p}^{x_p} \text{ with } x_j \in \mathbb{Z}, \ \ |x_j| \leq F_{i_j}(Cd), j = 1, \ldots, p.$$

Proof. The proof is by backward induction on h. When $h = j_*$ and c is a commutator with $w(c) = \overline{w}_{j^*}$, let $\sigma = \sigma_1 ... \sigma_m$, $\sigma_i \in \{s_1, \ldots, s_k\}$ be the formal word associated with c (by forgetting brackets and inverses). Write

$$x = a_0 \prod_{1\leq j\leq m} \left\lfloor F_{\sigma_j}(d) \right\rfloor + a_1 \prod_{2\leq j\leq m} \left\lfloor F_{\sigma_j}(d) \right\rfloor + ... + a_{m-1} \left\lfloor F_{\sigma_m}(d) \right\rfloor + a_m$$

with $a_j \in \mathbb{Z}$, $|a_0| \leq C$ and $|a_j| \leq F_{\sigma_j}(d)$. Write

$$\overrightarrow{a}_0 = (a_0 \left\lfloor F_{\sigma_1}(d) \right\rfloor, \left\lfloor F_{\sigma_2}(d) \right\rfloor ..., \left\lfloor F_{\sigma_m}(d) \right\rfloor),$$

$$\overrightarrow{a}_j = (\underbrace{1, ..., 1}_{j-1}, a_j, \left\lfloor F_{\sigma_{j+1}}(d) \right\rfloor, ..., \left\lfloor F_{\sigma_m}(d) \right\rfloor),$$

then

$$c^x \equiv \Theta(\overrightarrow{a}_1, c) ... \Theta(\overrightarrow{a}_k, c) \bmod N(k, \ell)^{\mathfrak{w}}_{j_*+1}.$$

Since $N(k, \ell)^{\mathfrak{w}}_{j_*+1} = \{e\}$, we actually have equality. Unraveling the brackets in $\Theta(\overrightarrow{a}_j, c)$ we get an expression in the powers of the generators satisfying the desired conditions.

Suppose the claim holds for $h+1$. Given a commutator c with $w(c) = \bar{w}_h$, let again $\sigma_1, \dots \sigma_m$ (m depends on c) be the formal word on the generators associated with c. For $x \in \mathbb{Z}$, $|x| \leq F_c(d)$, decompose x as above and use Lemma 3.6.18 to write

$$c^x = u_0^{-1}\Theta(\overrightarrow{a}_0, c)...u_m^{-1}\Theta(\overrightarrow{a}_m, c)$$

where $u_i \in N(k,\ell)^{\mathfrak{w}}_{h+1}$ can be represented by a word ω_i with $\deg_{c_j} \upsilon_i \leq F_{c_j}(Cd)$ for all j. By Lemma 3.6.15, u_i can also be represented in the form $\prod_{j \geq h+1} b_j^{y_{i,j}}$ with $|y_{i,j}| \leq F_{b_j}(Cd)$. Applying the induction hypothesis to each terms of these products we can now write c^x in the desired form $c^x = s_{i_1}^{x_1} s_{i_2}^{x_2} ... s_{i_p}^{x_p}$. □

Proof of Assertion (ii) in Theorem 3.6.16. By Theorem 3.6.16(i), any $g \in N(k,\ell)$ with $\|g\|_{S,\mathfrak{F}} \leq F_S^{-1}(r)$, $r > 0$, as a unique representation of the form $g = \prod_1^\tau b_j^{x_j}$ with $|x_j| \leq F_{b_j}(Cr)$. Applying Lemma 3.6.19 with $c = b_j, x = x_j$ for each $j = 1, \dots, \tau$ produces a sequence $((i_n)_1^p$ (independent of g) and a sequence $(x_n') \in \mathbb{Z}^p$ (depending on g) with $|x_n'| \leq F_{s_{i_n}}(Cr)$ for all $n \in \{1, \dots, p\}$ and such that

$$g = \prod_1^p s_{i_n}^{x_n'}.$$

□

3.6.3 End of the proof of Theorem 3.2.10

In order to finish the proof of Theorem 3.2.10 for a general finitely generated nilpotent group G, we simply need to improve upon Theorem 3.6.6(i). Namely, Theorem 3.6.6(i) provide a decomposition of any element g with $\|f\|_{\mathfrak{C}(S),\mathfrak{F}} \leq F_S(r)$ in the form

$$g = \prod_1^t c_i^{x_i}, \quad |x_i| \leq F_{c_i}(Cr).$$

Here $(c_i)_1^t$ is an enumeration of $\mathfrak{C}_+^*(S)$ so that $w(c_i) \preceq w(c_{i+1})$.

Now, let $(b_i)_1^\tau$ be a collection of formal commutators with $w(b_i) \preceq w(b_{i+1})$. For $j \in \{1, \dots, j_*\}$, let

$$m_j = \max\{i : w(b_i) = \bar{w}_j\}.$$

Clearly, $w(b_i) = \bar{w}_j$ if and only if $m_{j-1} + 1 \leq i \leq m_j$. Recall that $R_j^{\mathfrak{w}}$ is the torsion free rank of the abelian group $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$. We make two natural assumptions on the sequence (b_i):

(A1) For each j, $\{b_i' : m_{j-1} < i \leq m_j\}$ generates $G_j^{\mathfrak{w}}$ modulo $G_{j+1}^{\mathfrak{w}}$.

(A2) For each j, $\{b_i' : m_{j-1} < i \leq m_{j-1} + R_j^{\mathfrak{w}}\}$ is free in $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$.

Note that, since $R_j^{\mathfrak{w}}$ is the torsion free rank of $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$, (A2) implies that (the image of) $\{b_i' : m_{j-1} < i \leq m_{j-1} + R_j^{\mathfrak{w}}\}$ generates a subgroup of finite index in $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$.

Lemma 3.6.20. *Referring to the notion introduce above, assume that $(b_i)_1^\tau$ satisfies* (A1). *Then there exists $C \in (0, \infty)$ such that, for any $h = 1, \dots, j_*$, any $g \in G$ that can be written in the form*

$$g = \prod_{i : w(c_i) \succeq \bar{w}_h} c_i^{x_i}, \;\; |x_i| \leq F_{c_i}(r)$$

can also be written in the from

$$g = \prod_{i : w(b_i) \succeq \bar{w}_h} b_i^{y_i}, \;\; |x_i| \leq F_{b_i}(Cr).$$

Proof. The proof is by backward induction on h and is similar to the proof of Lemma 3.6.15. The details are omitted. □

Proposition 3.6.21. *Assume that, for each j, the image of*

$$\{b_i : m_{j-1} + 1 \le i \le m_{j-1} + R_j\}$$

in $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$ generates a subgroup of finite index in $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$. Then there exists a constant $C > 0$ such that for any word ω in $\{b_i : w(b_i) \succeq \bar{w}_h\}^{\pm 1}$ with $\deg_{b_i} \omega \le F_{b_i}(r)$ for all i, there is a word ω' of the form

$$\omega' = \prod_{i=m_{h-1}+1}^{\tau} b_i^{x_i}$$

with

$$|x_i| \le \begin{cases} F_{b_i}(Cr) & \text{for } m_{j-1} + 1 \le i \le m_{j-1} + R_j^{\mathfrak{w}} \\ C & \text{for } m_{j-1} + R_j^{\mathfrak{w}} + 1 \le i \le m_j \end{cases}$$

for $j \in \{h, \dots, j_*\}$ and such that $\pi(\omega') = \pi(\omega)$.

Proof. The proof is by backward induction on h. When $h = j_*$,$G_{j_*}^{\mathfrak{w}}$ is abelian and the desired result holds.

In general, let ω as in the proposition. By an application of Lemmas 3.6.13-3.6.20, we obtain a word $\omega_1 = \prod_{j=m_{h-1}+1}^{t} b_j^{x_j}$ with $|x_j| \le F_{b_j}(Cr)$ for all $j \ge m_{h-1} + 1$ and such that $\pi(\omega) = \pi(\omega_1)$.

By hypothesis, the images of the commutators $b_j, m_{h-1} + 1 \le j \le m_{h-1} + R_h^{\mathfrak{w}}$, generates a subgroup of finite index in $G_h^{\mathfrak{w}}/G_{h+1}^{\mathfrak{w}}$. Let N_h denote the index. Then for $m_{h-1} + R_h^{\mathfrak{w}} + 1 \le j \le m_h$, there exists $a_1^{(j)}, ..., a_{R_h^{\mathfrak{w}}}^{(j)} \in \mathbb{Z}$ such that

$$b_j^{N_h} = b_{m_{h-1}+1}^{a_1^{(j)}} ... b_{m_{h-1}+R_h^{\mathfrak{w}}}^{a_{R_h^{\mathfrak{w}}}^{(j)}} \bmod G_{h+1}^{\mathfrak{w}},$$

that is

$$\pi(b_j^{N_h}) = \pi(b_{m_{h-1}+1}^{a_1^{(j)}} ... b_{m_{h-1}+R_h^{\mathfrak{w}}}^{a_{R_h^{\mathfrak{w}}}^{(j)}} v_j),$$

where v_j is a word in $\{c_i : w(c) \succeq \overline{w}_{h+1}\}^{\pm 1}$. In

$$\omega_1 = \prod_{j=m_{h-1}+1}^{t} b_j^{x_j},$$

for each $j \in \{m_{h-1} + R_h^{\mathfrak{w}} + 1, \dots, m_h\}$, write $x_j = z_j N_h + y_j$ with $0 \leq y_j < N_h$ and replace $b_j^{N_h}$ by the word

$$\omega_j = b_{m_{h-1}+1}^{a_1^{(j)}} \dots b_{m_{h-1}+R_h^{\mathfrak{w}}}^{a_{R_h^{\mathfrak{w}}}^{(j)}} v_j.$$

This produce a new word

$$\omega_1' = \prod_{j=m_{h-1}+1}^{m_{h-1}+R_h^{\mathfrak{w}}} b_j^{x_j} \cdot \prod_{j=m_{h-1}+1+R_h^{\mathfrak{w}}}^{m_h} \omega_j^{z_j} b_j^{y_j} \cdot \prod_{j=m_h+1}^{t} b_j^{x_j}$$

satisfying $\pi(\omega_1') = \pi(\omega_1)$. For $m_{h-1} + 1 \leq j \leq m_{h-1} + R_h^{\mathfrak{w}}$,

$$\deg_{b_j} \omega_1' \leq |x_j| + \sum_{m_{h-1}+R_h^{\mathfrak{w}}+1 \leq i \leq m_h} |a_{j-m_{h-1}}^{(i)}| \, |x_i| \, ,$$

By hypothesis, $\deg_{b_j} \omega \leq F_{b_j}(Cd) \leq \mathbf{F}_h(C_1 d)$ for all $m_{h-1} + 1 \leq j \leq m_h$ and

$$\max\{|a_n^{(i)}| : m_{h-1} + R_h^{\mathfrak{w}} + 1 \leq i \leq m_h, 1 \leq n \leq R_h^{\mathfrak{w}}\} = C_h < \infty.$$

Hence, for $m_{h-1} + 1 \leq j \leq m_{h-1} + R_h^{\mathfrak{w}}$, we obtain

$$\deg_{b_j} \omega_1' \leq C_1(m_h - m_{h-1})\mathbf{F}_h(Cd) \leq \mathbf{F}_h(C_2 d).$$

For $m_{h-1} + R_h^{\mathfrak{w}} + 1 \leq j \leq m_h$, $\deg_{b_j} \omega \leq N_h$. Finally, for any $c \in \{c_i : 1 \leq i \leq t\}$ with $w(c) \succ \bar{w}_h$, we have $F_c \succ \mathbf{F}_h$ and

$$\begin{aligned} \deg_c \omega_1' &\leq \deg_c \omega_1 + \sum_{m_{h-1}+R_h^{\mathfrak{w}}+1 \leq k \leq m_h} |z_k| \deg_c v_k \\ &\leq F_c(C_3 d). \end{aligned}$$

Applying Lemmas 3.6.13-3.6.20 to ω_1', we obtain a word ω' with $\pi(\omega) = \pi(\omega')$ and

$$\omega_2 = \prod_{j=m_{h-1}+1}^{m_{h-1}+R_h^{\mathfrak{w}}} b_j^{\widetilde{x_j}} \prod_{j=m_{h-1}+1+R_h^{\mathfrak{w}}}^{m_h} b_j^{y_j} \prod_{j>m_h} b_j^{\widetilde{x_j}}$$

where $|\widetilde{x_j}| \leq \mathbf{F}_h(C_1 d)$ for $m_{h-1}+1 \leq j \leq m_{h-1}+R_h^{\mathfrak{w}}$; $0 \leq y_j < N_h$ for $m_{h-1}+R_h^{\mathfrak{w}}+1 \leq j \leq m_h$, and $|\widetilde{x_j}| \leq F_{c_j}(C_2' d)$ for all $j > m_h$. Now, apply the induction hypothesis to $\prod_{j=m_h+1}^{t} b_j^{\widetilde{x_j}}$, to obtain the desired conclusion. □

We end with the following simple improvement of the last statement in Theorem 3.2.10. The proof is a simple combination of the previous proposition together with Lemma 3.6.19.

Theorem 3.6.22. *Let G be a nilpotent group equipped with a generating k-tuple $S=(s_1,\dots,s_k)$. Let $\mathfrak{w}$, $\mathfrak{F}$ be weight and weight-function systems on S satisfying (3.1)-(3.2). Let $\Sigma=(c_1,\dots,c_t)$ be a tuple of formal commutators in $\mathfrak{C}(S)$ with non-decreasing weights $w(c_1)\preceq\cdots\preceq w(c_t)$. Let m_j, $j=0,\dots,j_*$ be defined by*

$$\{c_i : w(c_i)=\bar{w}_j\} = \{c_i : m_{j-1} < i \leq m_j\}.$$

Assume that (the image of) $\{c_i : w(c_i)=\bar{w}_j\}$ generates $G_j^{\mathfrak{w}}$ modulo $G_{j+1}^{\mathfrak{w}}$ and that $\{c_i : m_{j-1} < i \leq m_{j-1}+R_j^{\mathfrak{w}}\}$ is free in $G_j^{\mathfrak{w}}/G_{j+1}^{\mathfrak{w}}$.

There exist an integer $p=p(G,S,\mathfrak{F})$, a constant $C=C(G,S,\mathfrak{F})$ and a sequence $(i_1,\dots,i_p)\in\{1,\dots,k\}^p$ such that if g can be expressed as a word ω over $\mathfrak{C}(S)$ with $\deg_c(\omega)\leq F_c(r)$ for some $r\geq 1$ and all $c\in\mathfrak{C}(S)$ then g can be expressed in the form

$$g=\prod_{j=1}^{p} s_{i_j}^{x_j} \text{ with } |x_j| \leq C \begin{cases} F_{i_j}(r) & \text{if } s_{i_j}\in \operatorname{core}(S,\mathfrak{w},\Sigma) \\ 1 & \text{if } s_{i_j}\notin \operatorname{core}(S,\mathfrak{w},\Sigma). \end{cases}$$

CHAPTER 4

ON SOME RANDOM WALKS DRIVEN BY SPREAD-OUT MEASURES

4.1 Introduction

This chapter is concerned with random walks on groups, mostly nilpotent groups and groups of polynomial volume growth, associated with various type of spread-out probability measures. Here, spread-out is used to convey the idea that these measures do not have finite support.

Definition 4.1.1. We say that $\|\cdot\| : G \to [0,\infty)$ is a norm on G if $\|g\| = 0$ if and only if $g = e$ and, for all $g, h \in G$, $\|gh\| \leq \|g\| + \|h\|$. Given a norm $\|\cdot\|$, we say that $V(r) = \#\{g \in G : \|g\| \leq r\}$ is the associated volume function.

The key notions studied here are the following.

Definition 4.1.2. Let μ be a symmetric probability measure on a group G. Let $\|\cdot\|$ be a norm with volume function V. Let $r : (0,\infty) \to (0,\infty),\ t \mapsto r(t)$, be an non-decreasing function. Let $(X_n)_0^\infty$ denote the random walk on G driven by μ. We say that μ is $(\|\cdot\|, r)$-controlled if the following properties are satisfied:

1. For all n, $\mu^{(2n)}(e) \simeq V(r(n))^{-1}$.
2. For all $\epsilon > 0$ there exists $\gamma \in (0,\infty)$ such that

$$\mathbf{P}_e\left(\sup_{0\leq k\leq n}\{\|X_k\|\} \geq \gamma r(n)\right) \leq \epsilon.$$

Definition 4.1.3. Let μ be a symmetric probability measure on a group G. Let $\|\cdot\|$ be a norm with volume function V. Let $r : (0,\infty) \to (0,\infty),\ t \mapsto r(t)$, be an non-decreasing function with inverse ρ. Let $(X_n)_0^\infty$ denote the random walk on G driven by μ. We say that μ is strongly $(\|\cdot\|, r)$-controlled if the following properties are satisfied:

1. There exists $C \in (0, \infty)$ and, for any $\kappa > 0$, there exists $c(\kappa) > 0$ such that, for all $n \geq 1$ and g with $\|g\| \leq \kappa r(n)$,
$$c(\kappa)V(r(n)))^{-1} \leq \mu^{(2n)}(g) \leq CV(r(n))^{-1}.$$

2. There exists $\epsilon, \gamma_1, \gamma_2 \in (0, \infty)$, $\gamma_2 \geq 1$, such that, for all n, τ such that $\frac{1}{2}\rho(\tau/\gamma_1) \leq n \leq \rho(\tau/\gamma_1)$
$$\inf_{x:\|x\|\leq\tau} \left\{ \mathbf{P}_x \left(\sup_{0\leq k\leq n} \{\|X_k\|\} \leq \gamma_2\tau; \|X_n\| \leq \tau \right) \right\} \geq \epsilon. \tag{4.1}$$

Strong control implies the following useful estimate.

Proposition 4.1.4. *Assume that r is continuous increasing with inverse ρ and that the symmetric probability measure μ is strongly $(\|\cdot\|, r)$-controlled. Then, for any n and τ such that $\gamma_1 r(2n) \geq \tau$, we have*
$$\inf_{x:\|x\|\leq\tau} \left\{ \mathbf{P}_x \left(\sup_{0\leq k\leq n} \{\|X_k\| \leq \gamma_2\tau; \|X_n\| \leq \tau \right) \right\} \geq \epsilon^{1+2n/\rho(\tau/\gamma_1)}. \tag{4.2}$$

Proof. By induction on $\ell \geq 1$ such that $1 \leq 2n/\rho(\tau/\gamma_1) < (\ell+1)$, we are going to prove that
$$\inf_{x:\|x\|\leq\tau} \left\{ \mathbf{P}_x \left(\sup_{0\leq k\leq n} \{\|X_k\| \leq \gamma_2\tau; \|X_n\| \leq \tau \right) \right\} \geq \epsilon^{1+\ell}.$$

This easily yields the desired result. For $\ell = 1$, the inequality follows from the strong control assumption. Assume the property holds for some $\ell \geq 1$. Let n, τ be such that $(\ell+1) \leq 2n/\rho(\tau/\gamma_1) < (\ell+2)$. Choose n' such that $n - n' = \lceil \rho(\tau/\gamma_1)/2 \rceil$

and note that $2n' \in [1, (\ell+1)\rho(\tau/\gamma_1))$. Write $Z_n = \sup_{k\le n}\{\|X_k\|\}$ and, for any x such that $\|x\| \le \tau$,

$$\begin{aligned}
&\mathbf{P}_x\left(Z_n \le \gamma_2\tau; \|X_n\| \le \tau\right)\\
&\quad\ge \mathbf{P}_x\left(Z_n \le \gamma_2\tau; \|X_{n'}\| \le \tau; \|X_n\| \le \tau\right)\\
&\quad\ge \mathbf{P}_x\left(Z_{n'} \le \gamma_2\tau; \|X_{n'}\| \le \tau; \sup_{n'\le k\le n}\{\|X_k\|\} \le \gamma_2\tau; \|X_n\| \le \tau\right)\\
&\quad= \mathbf{E}_x\left(\mathbf{1}_{\{Z_{n'}\le\gamma_2\tau;\|X_{n'}\|\le\tau\}}\mathbf{P}_{X_{n'}}\left(Z_{n-n'} \le \gamma_2\tau; \|X_{n-n'}\| \le \tau\right)\right)\\
&\quad\ge \epsilon\mathbf{P}_x\left(Z_{n'} \le \gamma_2\tau; \|X_{n'}\| \le \tau\right) \ge \epsilon^{2+\ell}.
\end{aligned}$$

This gives the desired property for $\ell+1$. $\square$

Let the group G be equipped with a generating k-tuple

$$S = (s_1, \ldots, s_k)$$

and the associated finite symmetric set of generators $\mathcal{S} = \{s_1^{\pm1}, \ldots, s_k^{\pm1}\}$. Let $|g|$ be the associated word length, that is, the minimal k such that $g = u_1 \ldots u_k$ with $u_i \in S$, $1 \le i \le k$. By definition, the identity element e has length 0. Hence, $|\cdot|$ is a norm and $(x,y) \to |x^{-1}y|$ is a left-invariant distance function on G. Let

$$V_S(r) = \#\{G : |g| \le r\}$$

be the volume of the ball of radius r. We say that G has polynomial volume growth of degree D if there exists an integer D such that $V_S(r) \simeq r^D$ in the sense that the ratio $V_S(r)/r^D$ is bounded away from 0 and ∞ for $r \ge 1$. Finitely generated nilpotent groups have polynomial volume growth and, by Gromov's theorem, any finitely generated group with polynomial volume growth contains a nilpotent subgroup of finite index.

Example 4.1.1. Let G be equipped with a word-length function $|\cdot|$ associated with a symmetric finite generating subset. Assume that G has polynomial volume growth. The main results of [20] imply that, for any symmetric probability measure μ with finite generating support, μ is strongly $(|\cdot|, t \mapsto \sqrt{t})$-controlled. The main results of [2, 27] show that, if ν_β is symmetric and satisfies $\nu_\beta(g) \simeq [(1+|g|)^\beta V(|g|)]^{-1}$ with $\beta \in (0,2)$, then ν_β is strongly $(|\cdot|, t \mapsto t^{1/\beta})$-controlled. See also [5, 27].

We prove a number of complementary results including the following theorem.

Theorem 4.1.5. *Let G be equipped with a word-length function $|\cdot|$ associated with a symmetric finite generating subset. Let V be the associated volume function and assume that G has polynomial volume growth. Let $\phi : [0,\infty) \to [1,\infty)$ be a continuous regularly varying function of positive index. Let r be the inverse function of*

$$t \mapsto t^2 / \int_0^t \frac{tdt}{\phi(t)}.$$

Let ν_ϕ be a symmetric probability measure such that

$$\nu_\phi(g) \simeq \frac{1}{\phi(|g|)V(|g|)}. \tag{4.3}$$

Then ν_ϕ is strongly $(|\cdot|, r)$-controlled.

Example 4.1.2. Assume that $\phi(t) = (1+t)^\beta \ell(t)$ with ℓ positive continuous and slowly varying. The scaling function r of Theorem 4.1.5 can be described more explicitly as follows.

- If $\beta > 2$, $r(t) \simeq t^{1/2}$.
- If $\beta < 2$, we have $t^2 / \int_0^t \frac{tdt}{\phi(t)} \simeq c_\phi\, \phi(t)$ and r is essentially the inverse of ϕ, namely,

  $$r(t) \simeq t^{1/\beta} \ell_\#^{1/\beta}(t^\beta)$$

  where $\ell_\#$ is the de Bruijn conjugate of ℓ. See [6, Prop. 1.5.15]. For instance, if ℓ has the property that $\ell(t^a) \simeq \ell(t)$ for all $a > 0$ then $\ell_\# \simeq 1/\ell$.

- The case $\beta = 2$ is more subtle. The function $\psi : t \mapsto \int_0^t \frac{tdt}{\phi(t)}$ is slowly varying and satisfies $\psi(t) \geq \frac{c_1}{\ell(t)}$. For instance, if $\ell \equiv 1$, we have $\psi(t) \simeq \log t$ and $r(t) \simeq (t \log t)^{1/2}$. When $\ell(t) = (\log t)^\gamma$ with $\gamma \in R$ then
  - If $\gamma > 1$, $\psi(t) \simeq 1$ and $r(t) \simeq t^{1/2}$;
  - i $\gamma = 1$, $\psi(t) \simeq \log \log t$ and $r(t) \simeq (t \log \log t)^{1/2}$;
  - If $\gamma < 1$, $\psi(t) \simeq (\log t)^{1-\gamma}$ and $r(t) \simeq (t(\log t)^{1-\gamma})^{1/2}$;

In case $\beta = 2$, the result of Theorem 4.1.5 is quite subtle and difficult. The proof makes essential use of some of the results from [35] which are related to variations on the following class of examples. Next, recall that G is equipped with the generating k-tuple $S = (s_1, \ldots, s_k)$. For any k-tuple $a = (\alpha_1, \ldots, \alpha_k) \in (0, \infty)^k$, and consider the probability measure $\mu_{S,a}$ supported on the powers of the generators $s_1, \ldots, s_k$ and defined by

$$\mu_{S,a}(g) = \frac{1}{k} \sum_1^k \sum_{m \in \mathbb{Z}} \frac{\kappa_i}{(1 + |m|)^{1+\alpha_i}} \mathbf{1}_{s_i^m}(g). \tag{4.4}$$

Set

$$\widetilde{\alpha}_i = \min\{\alpha_i, 2\} \quad \text{and} \quad \alpha_* = \max\{\widetilde{\alpha}_i, 1 \leq i \leq k\}.$$

Define

$$\|g\|_{S,a} = \min \left\{ r : g = \prod_{j=1}^m s_{i_j}^{\epsilon_j} : \epsilon_j = \pm 1, \;\; \#\{j : i_j = i\} \leq r^{\alpha_*/\widetilde{\alpha}_i} \right\}. \tag{4.5}$$

Note that $g \mapsto \|g\|_{S,a} : G \to [0, \infty)$ is a norm. Consider also the measure

$$\nu_{S,a,\beta}(g) = \frac{c(G, a, \beta)}{(1 + \|g\|_{S,a})^\beta V_{S,a}(\|g\|_{S,a})} \tag{4.6}$$

with $\beta \in (0, 2)$.

Under the key assumption that G is nilpotent and $\{s_i : \alpha_i \in (0,2)\}$ generates a subgroup of finite index in G, it is proved in Chapter 3 that there exists a positive real $D_{S,a}$ such that

$$Q_{S,a}(r) = \#\{\|g\|_{S,a} \leq r^{1/\alpha_*}\} \simeq r^{D_{S,a}}$$

and

$$\mu_a^{(n)}(e) \leq C_{S,a} n^{-D_{S,a}}, \;\; \nu_{S,a,\beta}^{(n)}(e) \leq C_{S,a,\beta} n^{-\alpha_* D(S,a)/\beta}.$$

Here we prove the following complementary result.

Theorem 4.1.6. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $S = (s_1, \dots, s_k)$. Referring to the notation introduced above, fix $a \in (0,\infty)^k$ and assume that $\{s_i : \alpha_i \in (0,2)\}$ generates a subgroup of finite index in G.*

- *The probability measure $\mu_{S,a}$ is strongly $(\|\cdot\|_{S,a}, t \mapsto t^{1/\alpha_*})$-controlled.*
- *For any $\beta \in (0,2)$, $\nu_{S,a,\beta}$ is strongly $(\|\cdot\|_{S,a}, t \mapsto t^{1/\beta})$-controlled.*

Remark 4.1.7. In Chapter 3, a detailed analysis of the sub-additive function $\|\cdot\|_{S,a}$ and the associated geometry is given. This analysis is key to the above result and to its proper understanding. For instance, it is important to understand that the parameter α_* is not necessarily a key parameter. It is the quantity $\|\cdot\|_{S,a}^{\alpha_*}$ that is the key expression. Indeed, for any given nilpotent group G, in Chapter 3, we described conditions on two pairs of tuples (S,a), (S',a'),

$$S = (s_i)_1^k \in G^k, a = (\alpha_i)_1^k \in (0,\infty)^k, S' = (s'_i)_1^{k'} \in G^{k'}, a' = (\alpha'_i)_1^{k'} \in (0,\infty)^{k'},$$

such that $\|\cdot\|_{S,a}^{\alpha_*} \simeq \|\cdot\|_{S',a'}^{\alpha'_*}$. Since the geometry $\|g\|_{S,a}$ is studied and described rather explicitly in Chapter 3, the above results give rather concrete controls of the random walks driven with $\mu_{S,a}$ or $\nu_{S,a,\beta}$.

With more work, it is possible to prove sharper results concerning $\nu_{S,a,\beta}$. Indeed, based on the results of Chapter 3, the method developed in [27] shows that, for all $g \in G$ and $n \geq 1$,

$$\nu^{(n)}_{S,a,\beta}(g) \simeq \frac{n}{(n^{1/\beta} + \|g\|_{S,a})^{\alpha_* D_{S,a}+\beta}} \simeq \min\left\{\frac{1}{n^{\alpha_* D_{S,a}/\beta}}, \frac{n}{\|g\|_{S,a}^{\alpha_* D_{S,a}+\beta}}\right\}.$$

The measures $\mu_{S,a}$ are good examples of measure for which no matching two-sided global bounds are known.

4.2 Davies method, tightness and control

4.2.1 Davies method for the truncated process

In this section, we review how Davies' method applies to the continuous time process associated with truncated jumping kernels. We follow [26, Section 5] rather closely even so our setup is somewhat different. The first paper treating jump kernels by Davies method is [7].

Throughout this section G is a discrete group equipped with its counting measure. Fix a norm $g \mapsto \|g\|$ with volume function V and set $d(x,y) = \|x^{-1}y\|$. Note that d is a distance function on G. Consider the left-invariant symmetric jumping kernel

$$J(x,y) = \nu(x^{-1}y)$$

associated to a given symmetric probability measure ν on G. For $R > 0$, define

$$\delta_R := \sum_{\|x\|>R} \nu(x) \text{ and } \mathcal{G}(R) = \sum_{\|x\|\leq R} \|x\|^2 \nu(x), \tag{4.1}$$

and

$$J_R(x,y) := J(x,y)\mathbf{1}_{\{d(x,y)\le R\}},\quad J'_R(x,y) := J(x,y)\mathbf{1}_{\{d(x,y)>R\}}.$$

Denote by $p(t,x,y)$ and $p_R(t,x,y)$ the transition densities of the continuous time processes associated to J and J_R, respectively. In particular,

$$p(t,x,y) = e^{-t}\sum_{0}^{\infty}\frac{t^n}{n!}\nu^{(n)}(x^{-1}y).$$

Let

$$\mathcal{E}(f,f) = \frac{1}{2}\sum_{x,y}(f(x)-f(y))^2 J(x,y),\ \mathcal{E}_R(f,f) = \frac{1}{2}\sum_{x,y}(f(x)-f(y))^2 J_R(x,y)$$

be the corresponding Dirichlet forms and note that

$$\begin{aligned}\mathcal{E}(f,f)-\mathcal{E}_R(f,f) &= \frac{1}{2}\sum_{x,y:d(x,y)>R}|f(x)-f(y)|^2 J(x,y)\\ &\le \sum_{x,y:d(x,y)>R}(f(x)^2+f(y)^2)J(x,y)\le 2\|f\|_2^2\,\delta_R\end{aligned}$$

Consider the on-diagonal upper bound given by

$$\forall\, x\in G,\ t\ge 0,\ \ p(t,x,x)\le m(t), \tag{4.2}$$

where m is continuous regularly varying function of index $-D$ at infinity and $m(0)<\infty$. Since the function $t\mapsto m(t)$ may present a slowly varying factor, we follow [26]. The starting point is the log-Sobolev inequality

$$\sum f^2\log f \le \epsilon\mathcal{E}_R(f,f) + (2\epsilon\delta_R + \log m(\epsilon))\|f\|_2^2 + \|f\|_2^2\log\|f\|_2 \tag{4.3}$$

with $\epsilon>0$ which follows from (4.2) by [10, Theorem 2.2.3]. The following technical proposition is the key to most of the results obtained in later sections.

Proposition 4.2.1. *Assume that the on-diagonal upper bound* (4.2) *holds with* m *regularly varying of negative index. Then there is a constant* C *such that, for all* $R,t>0$ *and* $x\in G$ *we have*

$$p_R(t,e,x)\le Ce^{4\delta_R t}m(t)\left(\frac{t}{R^2/\mathcal{G}(R)}\right)^{\|x\|/3R}.$$

Proof. It suffices to consider the case $t < R^2/\mathcal{G}(R)$. Starting with (4.3), we apply Davies method, as described in [26, Section 5.1] to estimate $p_R(t,e,x)$. Let

$$\Lambda_R(\psi) = \max\left\{\|e^{-2\psi}\Gamma_R(e^{\psi},e^{\psi})\|_\infty, \|e^{\psi}\Gamma_R(e^{-\psi},e^{-\psi})\|_\infty\right\}$$

with

$$\Gamma_R(\psi)(x) = \sum_y |\psi(x)-\psi_y)|^2 J_R(x,y).$$

then by [26, Corollary 5.3],

$$p_R(t,x,y) \le Cm(t)\exp\left(4\delta_R t + 72\Lambda_R(\psi)^2 t - \psi(y) + \psi(x)\right).$$

Consider the case $x = x_0$ and $y = e$. For $\lambda > 0$, set $\psi(z) = \lambda(\|x_0\| - \|z\|)^+$ and write

$$\begin{aligned} e^{-2\psi(z)}\Gamma_R(e^{\psi},e^{\psi})(z) &= \sum_y (e^{\psi(z)-\psi(y)}-1)^2 J_R(z,y) \\ &\le e^{2\lambda R}\sum_y (\psi(z)-\psi(y))^2 J_R(z,y) \\ &\le \lambda^2 e^{2\lambda R}\sum_{\|y\|\le R} \|y\|^2\, d\nu \le R^{-2}e^{3\lambda R}\mathcal{G}(R). \end{aligned}$$

Since $\psi(e) = \lambda\|x_0\|$, we obtain

$$p_R(t,e,x_0) \le Cm(t)\exp\left(4\delta_R t + 72tR^{-2}e^{3\lambda R}\mathcal{G}(R) - \lambda\|x_0\|\right).$$

Since $t < R^2/\mathcal{G}(R)$, we can set

$$\lambda = \frac{1}{3R}\log\frac{R^2}{t\mathcal{G}(R)}$$

so that the second term $72tR^{-2}e^{3\lambda R}\mathcal{G}(R)$ is a constant, then we obtain the upper bound as stated. $\square$

4.2.2 Control

In this section, we combine the off-diagonal upper bound in Proposition 4.2.1 with Meyer's construction to derive control type results for the process with jumping kernel J. Our goal is to show that there is a certain choice of continuous increasing function $r(t)$, for any $\varepsilon > 0$, there exists constant $\gamma > 1$ such that

$$\mathbf{P}_e\left(\sup_{s\le t}\|X_s\| \ge \gamma r(t)\right) \le \varepsilon.$$

Let X_s^R denote the process with truncated kernel J_R. Meyer's construction is a useful technique to construct process X_s by adding big jumps to X_s^R. See, e.g., [25] and [2, Lemma 3.1]. It follows from the construction that

$$\mathbf{P}_e(X_s \ne X_s^R \text{ for some } s \le t) \le t\delta_R.$$

For any $r > 0$, $\gamma > 1$, both to be specified later, we have

$$\begin{aligned}
&\mathbf{P}_e\left(\sup_{s\le t}\|X_s\| \ge \gamma r\right)\\
&\qquad \le \mathbf{P}_e\left(\sup_{s\le t}\left\|X_s^R\right\| \ge \gamma r\right) + \mathbf{P}_e\left(X_s \ne X_s^R \text{ for some } s \le t\right)\\
&\qquad \le \mathbf{P}_e\left(\sup_{s\le t}\left\|X_s^R\right\| \ge \gamma r\right) + t\delta_R\\
&\qquad \le 2\sup_{s\le t}\left\{\mathbf{P}_e\left(\left\|X_s^R\right\| \ge \frac{\gamma}{2} r\right)\right\} + t\delta_R \qquad (4.4)
\end{aligned}$$

This will be helpful in deriving the following result.

Proposition 4.2.2. *Assume that for all $\rho > 0$, $V(2\rho) \le C_{VD}V(\rho)$. Assume also that ν is such that (4.2) holds where m is regularly varying of negative index. For $\varepsilon > 0$, fix a function $R(t)$ such that*

$$2t\delta_{R(t)} < \varepsilon \text{ and } \quad \frac{t}{R(t)^2/\mathcal{G}(R(t))} < e^{-1}.$$

Let $r(t) \ge R(t)$ be a positive continuous increasing function such that

$$\sup_{t>0}\left\{m(t)V(r(t))e^{-r(t)/6R(t)}\right\} < \infty.$$

Then, for any $\epsilon > 0$ there exists a constant $\gamma \ge 1$ such that

$$\mathbf{P}_e\left(\sup_{s\le t}\|X_s\| \ge \gamma r(t)\right) < \varepsilon$$

In particular, we have

$$p(t,e,e) \geq \frac{1-\varepsilon}{V(\gamma r(t))}$$

and the measure ν is $(\|\cdot\|, r)$-controlled.

Proof. Proposition 4.2.1 implies that for $s \leq t$,

$$\begin{aligned} p_R(s,e,x) &\leq Cm(s)\left(\frac{s}{R^2/\mathcal{G}(R)}\right)^{\|x\|/3R} \\ &= Cm(s)\left(\frac{s}{t}\right)^{\|x\|/3R}\left(\frac{t}{R^2/\mathcal{G}(R)}\right)^{\|x\|/3R}. \end{aligned}$$

Fix $R = R(t)$, $r = r(t) \geq R$, decompose $\{x : \|x\| \geq \frac{\gamma}{2}r\}$ into dyadic annuli $\{x : \|x\| \simeq 2^i\gamma r\}$ and write

$$\begin{aligned} \mathbf{P}_e\left(\|X_s^R\| \geq \frac{\gamma r}{2}\right) &\leq C\sum_{i=0}^{\infty} m(s)\left(\frac{s}{t}\right)^{2^{i-1}\gamma/3} e^{-2^{i-1}\gamma r/3R} V(2^i\gamma r(t)) \\ &= Cm(t)V(\gamma r)\sum_{i=0}^{\infty}\frac{m(s)}{m(t)}\left(\frac{s}{t}\right)^{2^{i-1}\gamma/3} e^{-2^{i-1}\gamma r/3R}\left(\frac{V(2^i\gamma r)}{V(\gamma r)}\right). \end{aligned}$$

Let C_{VD} denotes the volume doubling constant of (G,d), then

$$V(\gamma r) \leq C_{VD}^{1+\log\gamma} V(r), \quad \frac{V(2^i\gamma r)}{V(\gamma r)} \leq C_{VD}^i.$$

Recall that $m(t)$ is a regularly varying function with negative index. Hence, for γ large enough, we have

$$M = \sup_{0<s\leq t, i\in\mathbb{N}}\left\{\frac{m(s)}{m(t)}\left(\frac{s}{t}\right)^{2^{i-1}\gamma/3}\right\} < \infty.$$

Therefore

$$\mathbf{P}_e\left(\|X_s^R\| \geq \frac{\gamma r}{2}\right) \leq C_1 m(t)V(r)e^{-r/6R}\sum_{i=0}^{\infty} e^{-2^i\gamma/12}C_{VD}^i$$

By assumption, $r = r(t)$ and $R = R(t)$ satisfy

$$\sup_{t>0}\left\{m(t)V(r(t))e^{-r(t)/3R(t)}\right\} < \infty.$$

It follows that for γ sufficiently large, we have

$$\mathbf{P}_e\left(\|X_s^R\| \geq \frac{\gamma}{2}r(t)\right) < \frac{\varepsilon}{4}.$$

Plugging this estimate into (4.4), we obtain $P\left(\|X_s\| \geq \gamma r(t)\right) < \varepsilon$. $\square$

Corollary 4.2.3. *Under the hypotheses of* Proposition 4.2.2*, for any $\epsilon > 0$ there exists $\gamma > 0$ such that*

$$\mathbf{P}_e\left(\sup_{s\leq t}\|X_s\| \leq 2\gamma r(t), \|X_t\| \leq \gamma r(t)\right) \geq 1-\varepsilon.$$

Proof. Write

$$\begin{aligned}
&\mathbf{P}_e\left(\sup_{s\leq t}\|X_s\| \leq 2\gamma r(t), \|X_t\| \leq \gamma r(t)\right)\\
&\quad = \mathbf{P}_e\left(\|X_t\| \leq \gamma r(t)\right) - \mathbf{P}_e\left(\sup_{s\leq t}\|X_s\| \geq 2\gamma r(t), \|X_t\| \leq \gamma r(t)\right)\\
&\quad \geq 1 - 2\mathbf{P}_e\left(\sup_{s\leq t}\|X_s\| \geq \gamma r(t)\right).
\end{aligned}$$

Note that we have used the fact that, because of space homogeneity (i.e., group invariance), X_t cannot escape to infinity in finite time. $\square$

Remark 4.2.4. The conclusions of Proposition 4.2.2 and Corollary 4.2.3 apply to the associated discrete time random walk. To see this, fix a regularly varying function m and note that (up to changing m to cm for some constant c), (4.2) is equivalent to $\nu^{(2n)}(e) \leq m(n)$. Further, it is easy to control the difference between $\mathbf{P}_e(\|X_t\| \geq r)$ and $\mathbf{P}_e(\|X_n\| \geq r)$ with $n = \lfloor t \rfloor$ as long as n is large enough. It follows that the proof above applies the discrete random walk result as well.

4.2.3 Pseudo-Poincaré inequality

With some work, the result of the previous section can be extended to the more general context of graphs and discrete spaces. The results of this section make a more significant use of the group structure.

Definition 4.2.5. Let G be a discrete group equipped with a symmetric probability measure ν, a sub-additive function $\|\cdot\|$ and a positive continuous increasing function r with inverse ρ. We say that ν satisfies a point-wise $(\|\cdot\|, r)$-pseudo-Poincaré inequality if, for any f with finite support on G,

$$\forall\, g \in G, \ \sum_{x\in G} |f(xg) - f(x)|^2 \leq C\rho(\|g\|)\mathcal{E}(f,f) \tag{4.5}$$

Theorem 4.2.6. *Assume that $(G, \|\cdot\|)$ is such that V is doubling. Let ν be a symmetric probability measure such that $\nu(e) > 0$. Assume that r is a positive doubling continuous increasing function such that*

$$\nu^{(2n)}(e) \simeq V(r(n))^{-1}.$$

Assume further that ν satisfies the $(\|\cdot\|, r)$-pseudo-Poincaré inequality. Then there exists $\eta > 0$ such that for all n and g with $\|g\| \leq \eta r(n)$ we have

$$\nu^{(n)}(g) \simeq V(r(n))^{-1}.$$

Proof. The hypothesis (4.5) and the argument of [20, Theorem 4.2] gives

$$|\nu^{(2n+N)}(x) - \nu^{(2n+N)}(e)| \leq C\left(\frac{\rho(\|x\|)}{N}\right)^{1/2} \nu^{(2n)}(e).$$

Fix x and n such that $\rho(\|x\|) \leq \eta n$ and use the above inequality with $N = 2n$ to obtain

$$\nu^{(4n)}(x) \geq \left(1 - (2C'\eta)^{1/2}\right)\nu^{(4n)}(e).$$

Hence, we can choose $\eta > 0$ such that

$$\nu^{(4n)}(x) \geq c\nu^{(4n)}(e).$$

Since $\nu(e) > 0$, this also holds for $4n + i$, $i = 1, 2, 3$, at the cost of changing the value of the positive constant c. □

4.2.4 Strong control

Definition 4.2.7. We say that $\|\cdot\|$ is well-connected if there exists $b \in (0,\infty)$ such that, for any $x \in G$ there exists $(x_0)_1^N \in G$ with $\|x_i\| \le r$, $\|x_i^{-1}x_{i+1}\| \le b$, $x_0 = e$ and $x_N = x$.

Lemma 4.2.8. *Assume that $\|\cdot\|$ is well-connected and V is doubling. Then for any fixed $\epsilon > 0$ there exists M_ϵ such that for any $r \ge 8b/\epsilon$ and any $\|x\| \le r$ we can find $(z_i)_0^M$, $z_0 = e$, $z_M = x$, $M \le M_\epsilon$, such that $\|z_i^{-1}z_{i+1}\| \le \epsilon r$.*

Proof. Let $\{y_i : 1 \le i \le M'\}$ be a maximal $\epsilon r/4$-separated set of points in $B(e, 2r) = \{\|g\| \le 2r\}$. The ball $B_i = \{y_i, \epsilon r/9\}$ are disjoints and have volume $V(\epsilon r/9)$ comparable to $V(2r)$. Hence $M' \le M'_\epsilon$ for some finite M'_ϵ independent of r. The union of the balls $B'_i = \{y_i, \epsilon r/4)$ covers $B(2r)$ (otherwise, $\{y_i : 1 \le i \le M'\}$ would not be maximal). In particular, these balls cover the sequence $(x_i)_0^N$ and we can extract a sequence $B_i^* = B'_{j_i}$, $1 \le i \le N \le M'_\epsilon$, such that $B_i^* \ni e$, $B_N^* \ni x$ and

$$\inf\{\|h^{-1}g\| : h \in B_i^*, g \in B_{i+1}^*\} \le b.$$

Set $x_0 = e$, $x_i = y_{j_i}$, $1 \le i \le N$, $x_{N+1} = z$. Then $\|x_i^{-1}x_i\| \le 2\epsilon r/4 + b \le \epsilon r$ as desired. □

Proposition 4.2.9. *Assume that the norm $\|\cdot\|$ is such that V is doubling and $\|\cdot\|$ is well-connected. Let r be an positive continuous increasing doubling function. Let ν be a symmetric probability measure that is $(\|\cdot\|, r)$-controlled and satisfies $\nu(e) > 0$ and a point-wise $(\|\cdot\|, r)$-pseudo-Poincaré inequality. Then ν is also strongly $(\|\cdot\|, r)$-controlled.*

Proof. First, we show that for any $\kappa > 0$ there exists $c_\kappa > 0$ such that $\|x\| \le \kappa r(n)$

implies

$$\nu^{(n)}(x) \geq cV(r(n))^{-1}.$$

By Theorem 4.2.6, there exists η such that $\nu^{(n)}(x) \geq c_1 V(r(n))^{-1}$ for all $\|x\| \leq \eta r(n)$. By Lemma 4.2.8, for any fixed κ there exists M_κ such that for any $\|x\| \leq \kappa r(n)$ we can find $(z_i)_0^M$, $z_0 = e$, $z_M = x$, $M \leq M_\kappa$, such that $\|z_i^{-1} z_{i+1}\| \leq \eta r(n)/4$. Write $B_i = \{\|z_i^{-1} g\| \leq \eta r(n)/4\}$ and

$$\begin{aligned} \nu^{(Mn)}(x) &\geq \sum_{(y_1,\dots,y_M)\in\otimes_1^M B_i} \nu^{(n)}(y_1)\cdots\nu^{(n)}(y_i^{-1}y_{i+1})\dots\nu^{(n)}(y_M^{-1}x) \\ &\geq c_1^{M+1} V(\eta r(n)/4)^M V(r(n))^{-M-1} \simeq c_1' V(r(n))^{-1}. \end{aligned}$$

Since $\nu(e) > 0$, this shows that $\|x\| \leq \kappa r(n)$ implies $\nu^{(n)}(x) \geq cV(r(n))^{-1}$ as stated. In particular, for any fixed κ, there exists $\epsilon > 0$ such that for any x, n with $\kappa r(n) \leq \tau$ and $\|x\| \leq \tau$,

$$\mathbf{P}_x(\|X_n\| \leq \tau) \geq \epsilon.$$

Now, fix $\gamma_1 \in (1, \infty)$. Let $\epsilon_0 > 0$ be such that, for any x, n, τ with

$$\|x\| \leq \tau \leq \gamma_1 r(2n),$$

we have $\mathbf{P}_x(\|X_n\| \leq \tau) \geq \epsilon_0$. Let $\gamma \geq 1$ be given by Definition 4.1.2 so that

$$\mathbf{P}_e\left(\sup_{k\leq n}\{\|X_k\|\} \geq \gamma r(n)\right) \geq \epsilon_0/2.$$

Set $\gamma_2 = \gamma/\gamma_1 + 1$ and, for any x, n, τ with $\|x\| \leq \tau$ and $\frac{1}{2}\rho(\tau/\gamma_1) \leq n \leq \rho(\tau/\gamma_1)$, write

$$\begin{aligned} &\mathbf{P}_x\left(\sup_{k\leq n}\{\|X_k\|\} \leq \gamma_2\tau, \|X_n\| \leq \tau\right) \\ &\quad = \mathbf{P}_x\left(\|X_t\| \leq \tau\right) - \mathbf{P}_x\left(\sup_{k\leq n}\|X_k\| \geq \gamma_2\tau, \|X_n\| \leq \tau\right) \\ &\quad \geq \epsilon_0 - \mathbf{P}_e\left(\sup_{k\leq n}\|X_k\| \geq \gamma\tau/\gamma_1\right) \geq \epsilon_0/2. \end{aligned}$$

This proves that ν is strongly $(\|\cdot\|, r)$-controlled. $\square$

As a simple illustration of these techniques, consider the case of of an arbitrary symmetric measure ν with generating support and finite second moment (with respect to the word-length $|\cdot|$) on a group with polynomial volume growth of degree $D(G)$. It follows from [30] that $\nu^{(n)}(e) \simeq n^{-D(G)/2}$ and satisfies a point-wise classical pseudo-Poincaré inequality. Proposition 4.2.9 yields the following result.

Theorem 4.2.10. *Let G be a finitely generated group with polynomial volume growth with word-length $|\cdot|$. Assume that ν is symmetric, satisfies $\mu(e) > 0$, has generating support and satisfies $\sum |g|^2\nu(g) < \infty$. Then ν is strongly $(|\cdot|, t \mapsto t^{1/2})$-controlled.*

4.3 Measures supported on powers of generators

4.3.1 The measure $\mu_{S,a}$

In this subsection we consider the special case when G is a nilpotent group equipped with a generating k-tuple $S = (s_1, \dots, s_k)$ and

$$J(x, y) = \mu_{S,a}(x^{-1}y), \quad a = (\alpha_1, \dots, \alpha_k) \in (0, \infty)^k \tag{4.1}$$

with $\mu_{S,a}$ degiven by (4.4). Our aim is to prove Theorem 4.1.6 stated in the introduction. The study of the random walks driven by this class of measure is a continuation of our work in Chapter 3 and we will refer to and use some of the main results there.

Following Definiton 3.1.3, let $\mathfrak{w}$ be the power weight system on the formal commutators on the alphabet S associated with setting $w_i = 1/\widetilde{\alpha}_i$, $\widetilde{\alpha}_i = \min\{\alpha_1, 2\}$. Namely, The weight of any commutator c using the sequence of letters $(s_{i_1}, \dots, s_{i_m})$

from S (or their formal inverse) is $w(c) = \sum_1^m w_{i_j}$. Recall that in Chapter 3, we proved the following result.

Theorem 4.3.1 (Theorem 3.1.8, Theorem 3.2.18)**.** *Referring to the above setting and notation, assume that the subgroup of G generated by $\{s_i : \alpha_i < 2\}$ is of finite index. Then there exists a real $D_{S,a} = D(S, \mathfrak{w})$ such that*

$$Q_{S,a}(r) \simeq r^{D_{S,a}}, \;\; \mu_{S,a}^{(n)}(e) \simeq n^{-D_{S,a}}.$$

The real $D_{S,a} = D(S, \mathfrak{w})$ is given by Definition 3.1.7 *. Further, there exists a k-tuple $b = (\beta_1, \cdots, \beta_k) \in (0,2)^k$ such that $\beta_i = \alpha_i$ is $\alpha_i < 2$, $D(S,a) = D(S,b)$, and*

$$\forall\, g \in G, \|g\|_{S,a}^{\alpha_*} \simeq \|g\|_{S,\beta}^{\beta_*}.$$

In addition, $\mu_{S,a}$ satisfies a point-wise $(\|\cdot\|_{S,a}, t \mapsto t^{1/\alpha_*})$-pseudo-Poincaré inequality.

The volume estimate $Q_{S,a}(r) \simeq Q_{S,b}(r) \simeq r^{D_{S,a}}$ shows, in particular, that $(G, \|\cdot\|_{S,b})$ has the volume doubling property. The upper bound $\mu_{S,a}^{(n)}(e) \le Cn^{-D_{S,a}}$ implies that the continuous time process with jump kernel J defined above satisfies

$$\forall\, t > 0, x \in G, \;\; p(t,x,x) \le m(t) = Ct^{-D_{S,a}}.$$

Note that $\|\cdot\|$ is clearly well-connected (Definition 4.2.7). In order to apply Propositions 4.2.2 and 4.2.9 to the present case and prove Theorem 4.1.6, it clearly suffices to prove the following lemma which provides estimates for δ_R and $\mathcal{G}(R)$.

Lemma 4.3.2. *Referring to the setting and hypotheses of* Theorem 4.3.1*, for J given by* (4.1)*, let $\|\cdot\| = \|\cdot\|_{S,b}$, $D = D_{S,b} = D_{S,a}$, we have*

$$\begin{aligned} V(r) &= \#\{g \in G : \|g\| \le r\} \simeq r^{D\beta_*}, \\ \delta_R &\simeq R^{-\beta_*}, \\ \mathcal{G}(R) &\simeq R^{2-\beta_*}. \end{aligned}$$

Proof. The volume estimate follows immediately from Theorem 4.3.1. Let $\mathfrak{v}$ be the power weight system associated with b (in particular, $v_i = 1/\beta_i > 1/2$). By Proposition 3.2.172.17, for each i there exists $0 < \beta_i' \le \beta_i \le \beta_* < 2$ such that

$$\|s_i^n\| \simeq |n|^{\beta_i'/\beta_*}.$$

In the notation of Chapter 2, $\beta_i = \overline{v}_{j_{\mathfrak{v}}(s_i)}$.

We have

$$\begin{aligned}\delta_R &= \sum_{\|x\|>R} \mu_{S,a}(x) = \sum_{i=1}^{k} \sum_{\|s_i^n\|>R} \frac{\kappa_i}{(1+|n|)^{1+\alpha_i}} \\ &\simeq \sum_{i=1}^{k} \sum_{n>R^{\beta_*/\beta_i'}} \frac{\kappa_i}{(1+|n|)^{1+\alpha_i}} \simeq \sum_{i=1}^{k} R^{-\beta_*\alpha_i/\beta_i'} \simeq R^{-\beta_*}.\end{aligned}$$

The last estimate use that fact that there must be some $i \in \{1, \dots, k\}$ such that $\alpha_i = \beta_i'$ and that, always, $\alpha_i \ge \beta_i'$.

Similarly, since $\alpha_i \ge \beta_i'$ and $\beta_* < 2$, we have $2\beta_i'/\beta_* - \alpha_i > 0$. This yields

$$\begin{aligned}\mathcal{G}(R) &= \sum_{\|x\|\le R} \|x\|^2 \mu_{S,a}(x) = \sum_{i=1}^{k} \sum_{\|s_i^n\|\le R} \frac{\kappa_i \|s_i^n\|^2}{(1+|n|)^{1+\alpha_i}} \\ &\simeq \sum_{i=1}^{k} \sum_{0\le n\le R^{\beta_*/\beta_i'}} \frac{\kappa_i |n|^{2\beta_i'/\beta_*}}{(1+|n|)^{1+\alpha_i}} \simeq \sum_{i=1}^{k} R^{2-\beta_*\alpha_i/\beta_i'} \simeq R^{2-\beta_*}.\end{aligned}$$

□

4.3.2 Some regular variation variants of $\mu_{S,a}$

Consider the class of measure μ of the form

$$\mu(g) = \frac{1}{k} \sum_{1}^{k} \sum_{m\in\mathbb{Z}} \frac{\kappa_i \ell_i(|n|)}{(1+|n|)^{1+\alpha_i}} \tag{4.2}$$

where each ℓ_i is a positive slowly varying function satisfying $\ell_i(t^b) \simeq \ell_i(t)$ for all $b > 0$ and $\alpha_i \in (0, 2)$. For each i, let F_i be the inverse function of $r \mapsto r^{\alpha_i}/\ell_i(r)$. Note that F_i is regularly varying of order $1/\alpha_i$ and that $F_i(r) \simeq [r\ell_i(r)]^{1/\alpha_i}$, $r \geq 1$, $i = 1, \dots, k$. We make the fundamental assumption that the functions F_i have the property that for any $1 \leq i, j \leq k$, either $F_i(r) \leq CF_j(r)$ of $F_j(r) \leq CF_i(r)$. For instance, this is clearly the case if all α_i are distinct. Set $a = (\alpha_1, \dots, \alpha_k) \in (0, 2)^k$ and consider also the power weight system $\mathfrak{v}$ generated by $v_i = 1/\alpha_i$, $1 \leq i \leq k$, as in Definition 3.1.3. Fix $\alpha_0 \in (0, 2)$ such that

$$\alpha_0 > \max\{\alpha_i : 1 \leq i \leq k\}$$

and $\alpha_0/\alpha_i \notin \mathbb{N}$, $i = 1, \dots, k$. Observe that there are convex functions $K_i \geq 0$, $i = 0, \dots, k$, such that $K_i(0) = 0$ and

$$\forall r \geq 1, \quad F_i(r^{\alpha_0}) \simeq K_i(r). \tag{4.3}$$

Indeed, $r \mapsto F_i(r^{\alpha_0})$ is regularly varying of index α_0/α_i with $1 < \alpha_0/\alpha_i \notin \mathbb{N}$. By [6, Theorems 1.8.2-1.8.3] there are smooth positive convex functions $\tilde{K}_i$ such that $\tilde{K}_i(r) \sim F_i(r^{\alpha_0})$. If $\tilde{K}_i(0) > 0$, it is easy to construct a convex function $K_i : [0, \infty) \to [0, \infty)$ such that $K_i \simeq \widetilde{K}_i$ on $[1, \infty)$ and $K_i(0) = 0$. Let use $\mathfrak{K}$ to denote the collection $(K_i)_1^r$.

Now, set

$$\|g\| = \|g\|_{\mathfrak{K}} = \min\left\{ r : g = \prod_{j=1}^{m} s_{i_j}^{\epsilon_j} : \epsilon_j = \pm 1, \ \ \#\{j : i_j = i\} \leq K_i(r) \right\}.$$

Because of the convexity property of the K_i, $\| \cdot \|$ is a norm. Note also that it is well-connected. The following theorem is proved in Chapter 3.

Theorem 4.3.3. *Referring to the above notation and hypothesis, there exists a real $D = D_{S,a} = D(S, \mathfrak{v})$ and a positive slowly varying function L (explicitly given in* [35, Theorem 5.15] *and which satisfies $L(t^a) \simeq L(t)$ for all $a > 0$) such that:*

- *For all $r \geq 1$, $V(r) = \#\{g : \|g\| \leq r\} \simeq r^{\alpha_0 D} L(r)$*
- *For a each $1 \leq i \leq k$, there exists a regularly varying function $\widetilde{F}_i$ such that $\|s_i^n\|^{\alpha_0} \leq C \widetilde{F}_i^{-1}(n)$ where $\widetilde{F}_i \geq F_i$ and with equality for some $1 \leq i \leq r$.*
- *For all $n \geq 1$, $\mu^{(2n)}(e) \leq C(n^D L(n))^{-1}$.*
- *The measure μ satisfies a point-wise $(\|\cdot\|, t \mapsto t^{1/\alpha_0})$-pseudo-Poincaré inequality.*

Here, we prove the following result.

Theorem 4.3.4. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1, \ldots, s_k)$. Assume that μ is a probability measure on G of the form (4.2). Let ℓ_i, F_i, L, $D = D_{S,a}$, $\alpha_0 \in (0, 2)$ and $\|\cdot\|$ be as described above. Then μ is strongly $(\|\cdot\|, t \mapsto t^{1/\alpha_0})$-controlled.*

Proof. It suffices to estimate the quantities δ_R and $\mathcal{G}(R)$ in the present context. For δ_R, we have

$$\delta_R \simeq \sum_1^k \sum_{n \geq \widetilde{F}_i(R^{\alpha_0})} \frac{1}{n F_i^{-1}(n)} \simeq \sum_1^k \frac{1}{F_i^{-1} \circ \widetilde{F}_i(R^{\alpha_0})} \simeq R^{-\alpha_0}.$$

A similar computation gives

$$\mathcal{G}(R) \simeq \sum_1^k \frac{R^2}{F_i^{-1} \circ \widetilde{F}_i(R^{\alpha_0})} \simeq R^{2-\alpha_0}.$$

□

4.3.3 The critical case when $\alpha_i = 2$, $1 \leq i \leq k$

When $a = \mathbf{2} = (2, \ldots, 2)$, that is, $\alpha_i = 2$ for all $1 \leq i \leq k$, we work with the usual word-length function $|g|$ associated with the generating set $\mathcal{S} = \{s_1^{\pm 1}, \ldots, s_k^{\pm 1}\}$. In this case, $V(r) = \#\{g : |g| \leq r\} \simeq r^{D(G)}$ where $D(G)$ is the classical degree of polynomial growth for the nilpotent group G. It is proved in Theorem 3.4.8 that $\mu_{S,\mathbf{2}}^{(n)}(e) \leq C(n \log n)^{-D/2}$ and that $\mu_{S,\mathbf{2}}$ satisfies a point-wise $(|\cdot|, t \mapsto (t \log t)^{1/2})$-pseudo-Poincaré inequality. Further, $|s_i^n| \simeq |n|^{1/\beta_i}$ with $\beta_i \geq 1$ and $\beta_i = 1$ for some i. From this it easily follows that

$$\delta_R \simeq R^{-2}, \;\; \mathcal{G}(R) \simeq \log R.$$

Applying Proposition 4.2.9 with $r(t) = (t \log t)^{1/2}$ yields the following theorem.

Theorem 4.3.5. *Let G be a finitely generated nilpotent group equipped with a generating k-tuple $(s_1, \ldots, s_k)$. Let $D(G)$ be the volume growth degree of G. Then $\mu_{S,\mathbf{2}}$ is strongly $(|\cdot|, t \mapsto (t \log t)^{1/2})$-controlled.*

4.4 Norm-radial measures

In this section we assume that G is a finitely generated group with polynomial volume growth of degree $D(G)$ and we consider norm-radial symmetric probability measures and kernels of the form

$$\nu_\alpha(x) \simeq \frac{1}{(1 + \|x\|)^\alpha V(\|x\|)}, \quad J(x,y) \simeq \nu_\alpha(x^{-1}y), \tag{4.1}$$

where $\alpha \in (0, 2]$, $\|\cdot\|$ is a norm on G and $V(r) = \#\{g : \|g\| \leq r\}$.

The case when $\alpha \in (0, 2)$ and $V(r) \simeq r^d$ for some d is treated in [2, 27] where global matching upper and lower bounds are obtained. We note that [2] is set in a more general context where the group structure play no role. Here we are primarily interested in the case $\alpha = 2$ as well as in the case when $V(r)$ is given by a regularly varying function.

We start with the following easy observation.

Lemma 4.4.1. *Referring the situation described above, assume that V satisfies $V(2r) \le C_{DV}V(r)$ for all $r > 0$. Then $\delta_R \simeq R^{-\alpha}$ and*

$$\mathcal{G}(R) \simeq \begin{cases} R^{2-\alpha} & \text{if } \alpha \in (0,2) \\ \log R & \text{if } \alpha = 2. \end{cases}$$

Proof. This follows by inspection. □

The next lemma follows by application of Proposition 4.2.2. However, in this lemma, we make a key hypothesis on $\nu_\alpha^{(n)}(e)$.

Lemma 4.4.2. *Set $r_\alpha(t) = t^{1/\alpha}$ if $\alpha \in (0,2)$, $r_2(t) = (t\log t)^{1/2}$. Referring the situation described above, assume that V is regularly varying of positive index and*

$$\nu_\alpha^{(n)}(e) \le CV(r_\alpha(n))^{-1}.$$

Then ν_α is $(\|\cdot\|, r_\alpha)$-controlled.

The next two theorems provide key examples when the hypothesis regarding ν_α can indeed be verified.

Theorem 4.4.3. *Referring the situation described above, assume that V is regularly varying of positive index and $\alpha \in (0,2)$. Then*

$$\nu_\alpha^{(n)}(e) \simeq V(n^{1/\alpha})^{-1}$$

and ν_α is $(\|\cdot\|, t \mapsto t^{1/\alpha})$ controlled.

Proof. It suffices to prove the upper bound $\nu_\alpha^{(n)}(e) \le CV(n^{1/\alpha})^{-1}$. Start by checking that

$$\nu_\alpha(x) \simeq \sum_0^\infty \frac{1}{(1+m)^{1+\alpha}} \frac{\mathbf{1}_{B(m)}(x)}{V(m)}.$$

where $B(m) = \{x \in G : \|x\| \le m\}$. Then apply the elementary technique of [5, Section 4.2] to derive the desired upper bound on $\nu_\alpha^{(n)}(e)$. □

Remark 4.4.4. In the context of Theorem 4.4.3, we do not know if $\|\cdot\|$ is well-connected and we also do not know if ν_α satisfies a *point-wise* $(\|\cdot\|, r_\alpha)$-pseudo-Poincaré inequality. Hence, the techniques used in this chapter do not suffice to obtain strong control. However, if $\|\cdot\|$ is well-connected and ν_α satisfies a point-wise $(\|\cdot\|, r_\alpha)$-pseudo-Poincaré inequality then the strong $(\|\cdot\|, r_\alpha)$-control follows by Proposition 4.2.9. This proves the second statement in Theorem 4.1.6.

The case $\alpha = 2$ is significantly more difficult and, indeed, we do not know how to treat this case in the generality described above. The following theorem treats the case when $\|\cdot\|$ is the usual word-length function $\|\cdot\| = |\cdot|$ on G.

Theorem 4.4.5. *Assume that G is a group of polynomial volume growth equipped with generating k-tuple $S = (s_1, \dots, s_k)$ and the associated word-length $|\cdot|$ and volume function V. Let $D(G)$ be the degree of polynomial volume growth of G. Let ν_2 be a symmetric probability measure such that $\nu_2(g) \simeq (|g|^2 V(|g|))^{-1}$. Then we have*

$$\nu_2^{(n)}(e) \simeq (n \log n)^{-D(G)/2}.$$

Further, ν_2 is strongly $(|\cdot|, t \mapsto (t \log t)^{1/2})$-controlled.

Proof. We apply Lemma 4.4.2 and Proposition 4.2.9. When G is nilpotent, the upper bound $\nu_2^{(n)}(e) \leq (n \log n)^{-D(G)/2}$ follows from Theorems 3.4.8 and Theorem 3.5.7. Namely, Theorem 3.5.7 shows that

$$\nu_2^{(n)}(e) \leq C \mu_{S,\mathbf{2}}^{(Kn)}(e)$$

and Theorem 3.4.8 gives $\mu_{S,\mathbf{2}}^{(n)}(e) \leq C(n \log n)^{-D(G)/2}$. Further, Theorem 3.5.7 shows that ν_2 satisfies a point-wise $(|\cdot|, t \mapsto (t \log t)^{1/2})$-pseudo-Poincaré inequality.

Since any group of polynomial volume growth of degree $D(G)$ contains a nilpotent subgroup of finite index (hence, with the same degree of polynomial volume growth) the upper bound $\nu_2^{(n)}(e) \le C(n \log n)^{-D(G)/2}$ follows from the comparison result Theorem 3.5.7. By direct inspection, the desired pseudo-Poincaré inequality also follows. $\square$

Proof of Theorem 4.1.5. The same technique of proof gives the much more complete and subtle result stated in the introduction as Theorem 4.1.5. Namely, let $\phi : [0,\infty) \to [1,\infty)$ be a continuous increasing regularly function of index 2 and let ν_ϕ be as in (4.3), that is, assume that ν_ϕ is symmetric and satisfies $\nu(g) \simeq [\phi(|g|)V(|g|)]^{-1}$. First, assume that G is nilpotent and let $\mu_{S,\phi}$ be the measure given by

$$\mu_{S,\phi}(g) = \frac{1}{k}\sum_1^k \frac{\kappa}{(1+|n|)\phi(n)} \mathbf{1}_{s_i^n}(g).$$

Let r be the inverse function of $t \mapsto t^2/\int_0^t \frac{tdt}{\phi(t)}$. By Lemma 3.4.4 the measure $\mu_{S,\phi}$ satisfies the point-wise $(|\cdot|, r)$-pseudo-Poincaré inequality. By [35, Theorem 4.1], it follows that $\mu_{S,\phi}^{(n)}(e) \le V(r(n))^{-1}$. By [35, Theorem 5.7], we have the Dirichlet form comparison $\mathcal{E}_{\mu_{S,\phi}} \le C\mathcal{E}_{\nu_\phi}$.

Now, if G has polynomial volume growth then it contains a nilpotent subgroup with finite index, G_0. By inspection, quasi-isometry and comparison of Dirichlet forms (see [30]), it is easy to transfer both the point-wise $(|\cdot|, r)$-pseudo-Poincaré inequality and the decay $\nu_\phi^{(n)}(e) \le V(r(n))^{-1}$ from G_0 to G. Further, one checks that the functions δ_R and $\mathcal{G}(R)$ satisfy $\delta_R \simeq 1/\phi(R)$ and $\mathcal{G}(R) \simeq \int_0^R \frac{tdt}{\phi(t)}$. Proposition 4.2.2 with $r(t) = R(t)$ equals to the inverse function of $s \mapsto s^2/\int_0^s \frac{tdt}{\phi(t)}$ shows that ν_ϕ is $(|\cdot|, r)$-controlled. By Proposition 4.2.9, ν_ϕ is strongly $(|\cdot|, r)$-controlled. $\square$

The approach presented here is applicable even in cases were we are not able to obtain sharp results and we illustrate this by an example. Le G be a nilpotent group equipped with a generating k-tuple $S = (s_1, \dots, s_k)$. Fix $a \in (0,2]^k$ and set

$\alpha_* = \max\{\alpha_i, 1 \leq i \leq k\} \leq 2$. Consider the norm $\|\cdot\|_{S,a}$ defined at (4.5). Let ν_* be any symmetric probability measure such that

$$\nu_*(g) \simeq \frac{1}{(1+\|g\|_{S,a})^2 V(\|g\|_{S,a})}, \quad V(r) = \#\{g : \|g\|_{S,a} \leq r\}.$$

Theorems 3.3.2, 3.4.8 and 3.5.7 gives the following information. There exists two reals $D = D_{S,a}$ and $d = d_{S,a}$ and a constant $C_1 \in (0, \infty)$ such that

$$\nu_*^{(n)}(e) \leq C_1 n^{-\alpha_* D/2} (\log n)^{-d} \tag{4.2}$$

$$V(r) \simeq r^{\alpha_* D}. \tag{4.3}$$

Theorem 4.4.6. *For the probability measure ν_* on a finitely generated nilpotent group as described above, we have*

$$c(\log\log n)^{-\alpha_* D}(n \log n)^{-\alpha_* D/2} \leq \nu_*^{(n)}(e) \leq C n^{-\alpha_* D/2}(\log n)^{-d}.$$

Proof. The volume estimate (4.3) and Lemma 4.4.1 gives $\delta_R \simeq R^{-2}$ and $\mathcal{G}(R) \simeq \log R$. In order to apply Proposition 4.2.2, we set $R(t) \simeq (t \log t)^{1/2}$. Further, we use (4.2)–(4.3) to verify that the choice $r(t) = 6AR(t) \log\log t$ with A large enough satisfies the condition of Proposition 4.2.2. Indeed, we have $m(t) \simeq t^{-\alpha_* D/2}(\log t)^{-d}$, $V(r) \simeq r^{\alpha_* D}$ so that

$$m(t)V(r(t))e^{-r(t)/6R(t)} \leq C(\log t)^{-d+\alpha_* D/2}(\log\log t)^{\alpha_* D} e^{-A \log\log t}.$$

Clearly, for A large enough, the right-hand side is bounded above by a constant as required by Proposition 4.2.2 which now gives the stated lower bound on $\nu_*^{(n)}(e)$.

□

4.4.1 Complementary off-diagonal upper bounds

In contrast with the case (4.4) of measures supported on powers of generators, for norm-radial kernels of type (4.1), we can use Meyer's construction to derive good off-diagonal bounds for $p(t,e,x)$.

Proposition 4.4.7. *Let G be a finitely generated group equipped with a norm $\|\cdot\|$. For $\alpha \in (0,2)$, let ν_α be a symmetric probability measure on G satisfying* (4.1). *Assume that there are positive slowly varying function ℓ_1, ℓ_2 and $D > 0$ such that:*

1. $\forall\, r > 1,\;\; V(r) \simeq r^D \ell_1(r)$;
2. $\forall\, t > 0,\; x \in G,\;\; p(t,x,x) \le m(t) \simeq [(1+t)^{D/\alpha}\ell_1((1+t)^{1/\alpha})]^{-1}$.

Then there exists C such that, for all $t > 1$ and $x \in G$, we have

$$p(t,e,x) \le Cm(t)\min\left\{\left(\frac{t}{\|x\|^\alpha}\right)^{1+D/\alpha}\frac{\ell_1(t^{1/\alpha})}{\ell_1(\|x\|)}, 1\right\}.$$

Proposition 4.4.8. *Let G be a finitely generated group equipped with a norm $\|\cdot\|$ with volume V. Let ν_2 be a symmetric probability measure on G satisfying* (4.1) *with $\alpha = 2$. Assume that:*

1. $\forall\, r > 1, V(r) \simeq r^D$,
2. $\forall t > 1,\; x \in G,\;\; p(t,x,x) \le m(t) \simeq ((1+t)\log(1+t))^{-D/2}$.

Then there exists C such that, for all $t > 1$ and $x \in G$, we have

$$p(t,e,x) \le Cm(t)\min\left\{\left(\frac{t\log\|x\|}{\|x\|^2}\right)^{1+D/2}, 1\right\}$$

Further, for any $\gamma \in (0,2)$, there exist C_γ such that if $1 \le t \le \|x\|^\gamma$ then

$$p(t,e,x) \le \frac{C_\gamma}{t^{D/2}}\left(\frac{t}{\|x\|^2}\right)^{1+D/2}.$$

Proof of Proposition 4.4.7. Under the stated hypothesis, we have $\delta_R \simeq R^{-\alpha}$ and $\mathcal{G}(R) \simeq R^{2-\alpha}$ and, for $t \le \eta R^\alpha$ (with η to be fixed later, small enough), Proposition 4.2.1 gives

$$p_R(t,e,x) \le Cm(t)\left(\frac{t}{\|x\|^\alpha}\right)^{\|x\|/3R}.$$

By Meyer's construction, we have

$$
\begin{aligned}
p(t,x,y) &\le p_R(t,x,y) + t\,\|\nu_R'\|_\infty \\
&\le C_1 \frac{1}{t^{D/\alpha}\ell_1(t^{1/\alpha})}\left(\frac{t}{R^\alpha}\right)^{\|x\|/3R} + \frac{t}{R^{\alpha(1+D/\alpha)}\ell_1(R)}.
\end{aligned}
$$

Choose $R = R(x,t)$ such that the two terms of the sum on the left-hand side are essentially equal, namely, set

$$
\left(\log\frac{R^\alpha}{t}\right)\frac{\|x\|}{3R} = \left(\log\frac{R^\alpha}{t}\right)\left(1+\frac{D}{\alpha}\right) + \log\frac{\ell_1(R)}{\ell_1(t^{1/\alpha})}.
$$

As long as η is small enough, this choice of R gives $\|x\| \simeq R$ and

$$
p(t,x,y) \le \frac{2t}{\|x\|^{\alpha(1+D/\alpha)}\ell_1(\|x\|)} \simeq \frac{1}{t^{D/\alpha}\ell_1(t^{1/\alpha})}\left(\frac{t}{\|x\|^\alpha}\right)^{1+D/\alpha}\frac{\ell_1(t^{1/\alpha})}{\ell_1(\|x\|)}.
$$

For any t (in particular, $t \ge \eta R^\alpha$) we can also use $m(t)$ for an easy upper bound. This gives

$$
p(t,e,x) \le Cm(t)\min\left\{\left(\frac{t}{\|x\|^\alpha}\right)^{1+D/\alpha}\frac{\ell_1(t^{1/\alpha})}{\ell_1(\|x\|)}, 1\right\}
$$

or, equivalently,

$$
p(t,e,x) \le C\min\left\{t\nu_\alpha(\|x\|), m(t)\right\}.
$$

□

Proof of Proposition 4.4.8. In the context of proposition 4.4.8, we have $\delta_R \simeq R^2$ and $\mathcal{G}(R) \simeq \log R$. For $1 \le t \le \eta R^2$, $\eta > 0$ small enough, Proposition 4.2.1 and Meyer's decomposition gives

$$
\begin{aligned}
p(t,x,y) &\le p_R(t,x,y) + t\,\|\nu_R'\|_\infty \\
&\le C(t\log t)^{-D/2}\left(\frac{t\log R}{R^2}\right)^{\|x\|/3R} + \frac{t}{R^{2+D}}.
\end{aligned}
$$

If $R^2/\log R \le t \le R^2$ then this bound is not better than the easy bound $p(t,x,y) \le m(t)$. By taking R such that $\|x\| = 3R(1+D/2)$, we obtain

$$
p(t,x,y) \le Cm(t)\min\left\{\left(\frac{t\log\|x\|}{\|x\|^2}\right)^{1+D/2}, 1\right\}.
$$

However, if $t \leq \eta\|x\|^\gamma$ with $\gamma \in (0,2)$ and η small enough, then we can choose $R \simeq \|x\|$ so that

$$(t\log t)^{-D/2}\left(\frac{t\log R}{R^2}\right)^{\|x\|/3R} = \frac{t}{R^{2+D}},$$

equivalently,

$$\left(\frac{R^2}{t\log R}\right)^{\|x\|/3R} = \left(\frac{R^2}{t\log R}\right)^{1+D/2}\frac{(\log R)^{1+D/2}}{(\log t)^{D/2}}.$$

In the region $t \leq \|x\|^\gamma$, this yields

$$p(t,e,x) \leq \frac{2t}{\|x\|^{2+D}} \simeq t^{-D/2}\left(\frac{t}{\|x\|^2}\right)^{1+D/2}.$$

$\square$

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