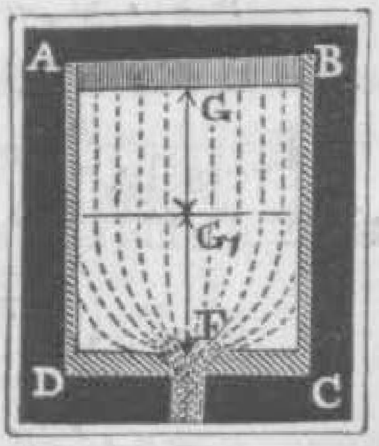
CHAPTER V.

ON THE EFFLUX OF WATER UNDER VARIABLE PRESSURE.

§ 345. *Prismatic Vessels.*—If a cistern from which water flows through an orifice at the side or bottom, has no influx to it from any other side, a gradual sinking of the surface of water will take place, and the cistern at last empty itself. If, further, the quantity of influx Q, be greater or less than the quantity of efflux $\mu F \sqrt{2 g h}$, the surface of water will then rise or fall until the head of water $h = \frac{1}{2g}\left(\frac{Q}{\mu F}\right)^2$, and after this the head of water and the velocity of efflux will remain unaltered. Our problem, then, is to find how the time, the rise and fall of the water, and the emptying of vessels of given form and dimensions, depend on each other.

The efflux from a prismatic vessel presents the most simple case when it takes place through an opening in the bottom, and when there is no efflux from above or below. If x is the variable head of water FG_1, F the area of the orifice, and G the transverse section of the vessel AC, Fig. 473, we have then the theoretical velocity of efflux $v = \sqrt{2 g x}$, the theoretical velocity of the falling surface of the water $= \frac{F}{G} v = \frac{F}{G} \sqrt{2 gx}$, and the effective velocity $v_1 = \frac{\mu F}{G} \sqrt{2 gx}$. At the commencement: $x = FG = h$, and at the end of the efflux $x = 0$, therefore, the initial velocity is:

Fig. 473.



$c = \frac{\mu F}{G} \sqrt{2 gh}$, and the final velocity $c_1 = 0$.

It is seen from the formula $v_1 = \sqrt{2\left(\frac{\mu F}{G}\right)^2 g\, x}$, that the motion of the surface is uniformly retarded, and the measure of the retardation $p = \left(\frac{\mu F}{G}\right)^2 g$, hence we also know (§ 14), that this velocity $= 0$, and the discharge ceases, when

$$t = \frac{c}{p} = \frac{\mu F}{G} \sqrt{2 g h} \div \left(\frac{\mu F}{G}\right)^2 g = \frac{G}{\mu F} \sqrt{\frac{2 g h}{g^2}}, \text{ i. e. } t = \frac{2G\sqrt{h}}{\mu F\sqrt{2g}}.$$

We may also put:

$$t = \frac{2 G h}{\mu F \sqrt{2gh}} = \frac{2 G h}{Q},$$

and, according to this, assume that double the time is required for the efflux of the discharge Gh through the orifice at the bottom F, under a head of water decreasing from h to 0, than under a uniform pressure.

As the co-efficient of efflux μ is not quite constant, but is greater for a diminution of pressure, we must, therefore, in calculations of this kind, substitute a mean value of this co-efficient.

Example. In what time will a rectangular cistern, of 14 square feet section, empty itself through a round orifice at the bottom, of 2 inches width, if the original head of water amount to 4 feet? The time of efflux would be theoretically:

$$t = \frac{2 \cdot 14 \sqrt{4}}{8{,}02 \cdot \frac{\pi}{4} (\frac{1}{6})^2} = \frac{2 \cdot 14 \cdot 144 \cdot 2}{8{,}02 \cdot \pi} = \frac{8064}{8{,}02 \cdot \pi} = 320'' = 5 \text{ min. } 20 \text{ sec.}$$

At the end of half the time of efflux, the head of water will be $= (\frac{1}{2})^2 \cdot h = \frac{1}{4} \cdot 4 = 1$ ft. Now the co-efficient of efflux, which corresponds to the head of water $= 1$ foot is 0,613, hence the effective time of discharge will be $= \frac{320''}{0{,}613} = 521'' = 8$ minutes, 41 seconds.

§ 346. *Vessels of Communication.*—Since for an initial head of water h_1, the time of efflux $t_1 = \frac{2\, G \sqrt{h_1}}{\mu\, F \sqrt{2g}}$, and for an initial head of water h_2 this time $t_2 = \frac{2\, G \sqrt{h_2}}{\mu F \cdot \sqrt{2g}}$, it then follows by subtraction, that the time within which the head of water passes from h_1 to h_2, and the surface of water sinks $h_1 - h_2$ is:

$$t = \frac{2\, G}{\mu\, F \cdot \sqrt{2\, g}} (\sqrt{h_1} - \sqrt{h_2}), \text{ or for the English foot measure:}$$

$$t = 0{,}249\, \frac{G}{\mu F} (\sqrt{h_1} - \sqrt{h_2}).$$

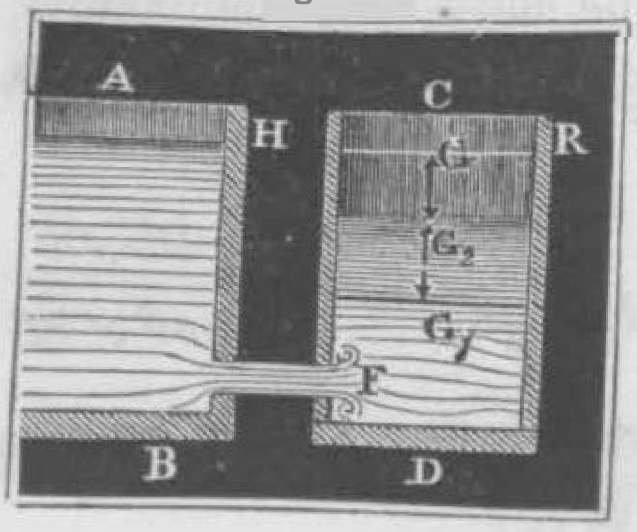
Inversely, the depression of the surface corresponding to a given time of efflux is $s = h_1 - h_2$, and is given by the formula:

$$h_2 = \left(\sqrt{h_1} - \frac{\mu \sqrt{2g} \cdot F}{2G}\, t\right)^2, \text{ or,}$$

$$s = \frac{\mu \sqrt{2g} \cdot F t}{G} \left(\sqrt{h_1} \mp \frac{\mu \sqrt{2g}}{4\, G} \cdot t\right).$$

The same formulæ are further applicable, when a vessel CD, Fig. 474, is filled by another AB in which the water maintains a uniform height. If the transverse section of the tube of communication, or of the orifice $= F$, the transverse section of the vessel to be filled $= G$, and the original level $G\, G_1$ of the two surfaces of water $= h$, we have then, since here the surface of water G_1 in the second vessel is uniformly retarded, the time of filling likewise, or the time within which the second surface of water comes to the level HR of the first:

Fig. 474.



$$t = \frac{2\,G\sqrt{h}}{\mu F.\sqrt{2g}},$$

and likewise the time in which the height of level h_1 passes into h_2, and, therefore, the surface of water ascends to:

$$GG_1 = s = h_1 - h_2.$$

$$t = \frac{2\,G}{\mu\,F.\sqrt{2g}}\,(\sqrt{h_1} - \sqrt{h_2}).$$

Examples. 1.—How much will the surface of water in the vessel of the last example sink in two minutes? $h_1 = 4$, $t = 2\,.\,60 = 120$, $\frac{F}{G} = \frac{\pi.}{14\,.\,144}$ and if we assume, further, $\mu = 0{,}605$, it follows then $h_2 = \left(\sqrt{h_1} - \mu\,.\sqrt{2\,g}\,.\,\frac{F\,t}{2\,G}\right)^2$ $= \left(2 - \frac{0{,}605\,.\,8{,}02\,.\,\pi\,.\,120}{2\,.\,14\,.\,144}\right)^2 = \left(2 - 0{,}605\,.\,8{,}02\,.\,\frac{5\,.\,\pi}{168}\right)^2 = 2{,}393$ feet, and the depression sought is $s = 4 - 2{,}393 = 1{,}607$ feet.

2. What time does the water in the 18 inch wide tube *CD*, Fig. 475, require to run over if it communicates with a vessel *AB* by a short $1\frac{1}{2}$ inch wide tube, and the rising surface of water *G* stands, at the beginning, 6 feet below the uniform surface of water *A*, and $4\frac{1}{2}$ feet below the head *C* of the tube. It is:

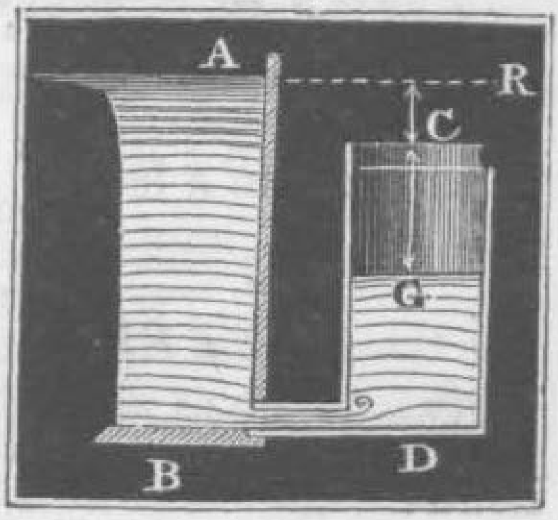
$$t = \frac{2\,G}{\mu\sqrt{2g}\,.\,F}\,(\sqrt{h_1} - \sqrt{h_2}),$$

$h_1 = 6$, $h_2 = 6 - 4{,}5 = 1{,}5$, $\frac{G}{F} = \left(\frac{18}{1{,}5}\right)^2 = 144$ and $\mu = 0.81$, whence it follows that:

$$t = \frac{2\,.\,144}{0{,}81\,.\,8{,}02}\,(\sqrt{6} - \sqrt{1{,}5}) = \frac{288\,.\,1{,}2248}{0{,}81\,.\,8{,}02}$$

$= 54{,}3$ sec.

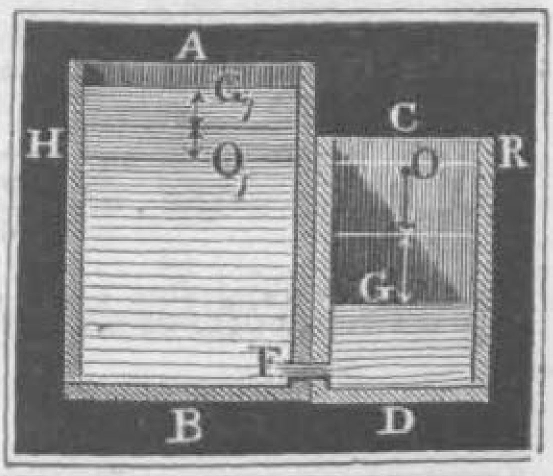
Fig. 475.



§ 347

If the first vessel *AB*, Fig. 476, from which the water runs into the other, has no influx, and its section G_1 also not to be considered as indefinitely great compared with the section *G* of the subsequent vessel *CD*, we have then to modify the condition. If the variable distance $G_1 O_1$ of the first surface of water from the level *HR* at which both surfaces stand at the end of the efflux $= x$, and the distance *GO* of the second surface of water from this same plane $= y$, we have then the variable head of water $= x + y$, and the corresponding velocity of efflux: $v = \sqrt{2g\,(x + y)}$, and the quantity of water:

Fig. 476.



$$G_1 x = Gy, v = \sqrt{2\,g\left(1 + \frac{G}{G_1}\right)y}.$$

The velocity with which the surface of water in the second vessel ascends is now:

$$v_1 = \frac{\mu\,F}{G}\,v = \frac{\mu\,F}{G}\sqrt{2\,g\left(1 + \frac{G}{G_1}\right)y},$$

consequently the retardation:

$$p = \left(\frac{\mu F}{G}\right)^2 \left(1 + \frac{G}{G_1}\right) g,$$

and the time of efflux:

$$t = \frac{\mu F}{G} \sqrt{2g\left(1 + \frac{G}{G_1}\right) y} \div \left(\frac{\mu F}{G}\right)^2 \left(1 + \frac{G}{G_1}\right) g$$

$$= \frac{2\, G \sqrt{y}}{\mu F \sqrt{2\, g\left(1 + \frac{G}{G_1}\right)}}.$$

Let us substitute for x and y, the initial height of level h, and therefore put:

$$x + y = h, \text{ or } \left(1 + \frac{G}{G_1}\right) y = h,$$

and we then obtain:

$y = \dfrac{h}{1 + \dfrac{G}{G_1}}$, and the time in which the two surfaces of water come to a level:

$$t = \frac{2\, G \sqrt{h}}{\mu F\left(1 + \frac{G}{G_1}\right)\sqrt{2g}} = \frac{2\, G G_1 \sqrt{h}}{\mu F (G + G_1) \sqrt{2g}}.$$

The time within which the level falls from h to h_1, is, on the other hand:

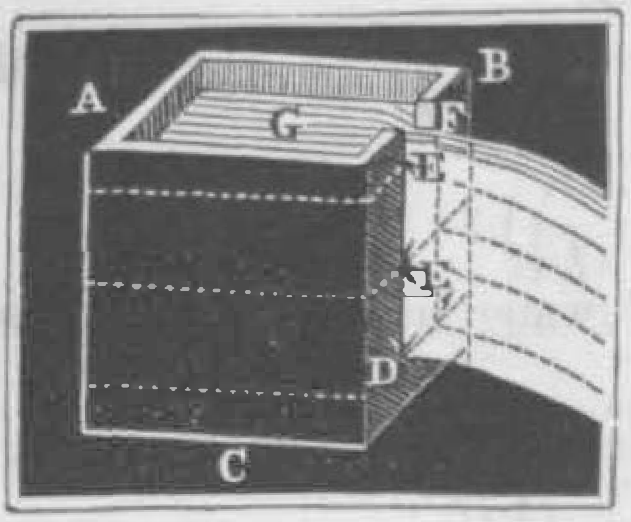
$$t = \frac{2\, G G_1 (\sqrt{h} - \sqrt{h_1})}{\mu \sqrt{2g}\, F (G + G_1)}.$$

Example. If the section of a cistern from which water flows is 10 square feet, and the section G of the recipient cistern 4 square feet; if, further, the initial level h of the two surfaces amounts to 3 feet, and the cylindrical tube of communication is 1 inch wide, then the time in which the water comes in both vessels to the same level is:

$$t = \frac{2 \,.\, 10 \,.\, 4 \,.\, \sqrt{3}}{0{,}82 \,.\, 8{,}02 \,.\, \frac{\pi}{4} \,.\, \frac{14}{144}} = \frac{320 \,.\, 72 \,.\, \sqrt{3}}{0{,}82 \,.\, 8{,}02 \,.\, 7\pi} = 275{,}9 \text{ sec.}$$

§ 348. *Notches in a Side.*—If water flows through the notch or cut DE of a prismatic cistern ABC, Fig. 477, to which there is no influx, the time of efflux may then be estimated in the following manner. Let us represent the transverse section of the cistern by G, the breadth EF of the notch by b, and the depth DE by h, and divide the whole aperture of efflux by horizontal lines into small slices, each of the breadth b and depth $\frac{h}{n}$. At a constant pressure the discharge per second will

Fig. 477.



be, $Q = \frac{2}{3}\mu b\sqrt{2gh^3}$, if we divide this into the area $\frac{Gh}{n}$ of a stratum of water, we shall then obtain the time of efflux $\tau = \frac{Gh}{\frac{2}{3}\mu nb\sqrt{2gh^3}}$, which we may write:

$$\frac{3\,Gh}{2\,\mu\,n\,b\sqrt{2g}}\,.\,h^{-\frac{3}{2}}.$$

Now, to obtain the time of efflux t for a quantity of water $G(h - h_1)$, or to determine the time in which the head of water above the line $DE = h$ sinks to $DE_1 = h_1$, let us make $h_1 = \frac{m}{n}h$, and therefore h_1 to consist of m parts, and let us now substitute for $h^{-\frac{3}{2}}$, successively:

$$\left(\frac{m}{n}h\right)^{-\frac{3}{2}},\ \left(\frac{m+1}{n}h\right)^{-\frac{3}{2}},\ \left(\frac{m+2}{n}h\right)^{-\frac{3}{2}}\ \ldots\left(\frac{nh}{n}\right)^{-\frac{3}{2}},$$

and finally add the results obtained. In this manner we shall obtain the time required:

$$t = \frac{3\,G\,h}{2\,\mu\,n\,b\sqrt{2\,g}}\left[\left(\frac{mh}{n}\right)^{-\frac{3}{2}} + \left(\frac{m+1}{n}h\right)^{-\frac{3}{2}} + \ldots + \left(\frac{nh}{n}\right)^{-\frac{3}{2}}\right]$$

$$= \frac{3\,G\,h}{2\,\mu\,n\,b\sqrt{2\,g}}\,.\,\frac{h^{-\frac{3}{2}}}{n^{-\frac{3}{2}}}\left(m^{-\frac{3}{2}} + (m+1)^{-\frac{3}{2}} + \ldots + n^{-\frac{3}{2}}\right)$$

$$= \frac{3\,Gh^{-\frac{1}{2}}}{2\,\mu\,n^{-\frac{1}{2}}\,b\sqrt{2g}}\left[\left(1^{-\frac{3}{2}} + 2^{-\frac{3}{2}} + 3^{-\frac{3}{2}} + \ldots + n^{-\frac{3}{2}}\right) - \left(1^{-\frac{3}{2}} + 2^{-\frac{3}{2}} + 3^{-\frac{3}{2}} + \ldots m^{-\frac{3}{2}}\right)\right],$$

or, from the "Ingenieur," Arithmetic, § 28:

$$t = \frac{3\,Gh^{-\frac{1}{2}}}{2\,\mu\,n^{-\frac{1}{2}}\,b\sqrt{2g}}\left(\frac{n^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} - \frac{m^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right)$$

$$= \frac{3\,Gh^{\frac{1}{2}}}{2\,\mu\,b\sqrt{2\,gh}}\,.\,2\left(m^{-\frac{1}{2}} - n^{-\frac{1}{2}}\right)$$

$$= \frac{3\,G}{\mu\,b\sqrt{2\,g\,h}}\left[\left(\frac{m}{n}\right)^{-\frac{1}{2}} - 1\right]$$

$$= \frac{3\,G}{\mu\,b\sqrt{2g}}\left[\left(\frac{m}{n}h\right)^{-\frac{1}{2}} - h^{-\frac{1}{2}}\right] = \frac{3\,G}{\mu b\sqrt{2g}}\left(\frac{1}{\sqrt{h_1}} - \frac{1}{\sqrt{h}}\right).$$

Let $h_1 = 0$, we have then $\frac{1}{\sqrt{h_1}}$, and therefore also $t = \infty$; an indefinite time, therefore, is required for the water to run down to the sill.

5). by Calculus. g_1 = section at dist. x from orifice

$t = \int \frac{g_1 dx}{\mu F \sqrt{2gx}}$ { $g_1 dx$ = volume discharged in an element of time; $\mu F \sqrt{2gx}$ = volume " " " unit " }

but $g : g_1 :: h : x$ or $H : g_1 :: h : x$

Example. If the water flows through a notch in a side, of 8 inches in breadth, from a reservoir 110 feet long and 40 feet broad, what time will it require to pass from a head of water of 15 inches to one of 6 inches?

$$t = \frac{3 \, . \, 110 \, . \, 40}{\mu \, . \, \frac{2}{3} \, . \, 8{,}02} \left(\frac{1}{\sqrt{0{,}5}} - \frac{1}{\sqrt{1{,}25}} \right) = \frac{19800}{\mu \, . \, 8{,}02} \left(\sqrt{2} - \sqrt{\tfrac{4}{5}} \right)$$

$$= \frac{19800}{8{,}02 \, \mu} (1{,}4142 - 0{,}8944) = \frac{19800 . \, 0{,}5198}{8{,}02 \, \mu} = \frac{1283}{\mu} \text{ sec.}$$

$g_1 = \frac{blx}{h}$

$t = \int \frac{blx \, dx}{\mu F h \sqrt{2gx}}$

$= \frac{2}{3} \frac{bl x^{3/2}}{\mu F h \sqrt{2g}} + C$

$\frac{2}{3} \frac{bl h^{1/2}}{\mu F \sqrt{2g}}$

If we assume the co-efficient $\mu = 0{,}60$, the effective time of efflux will be

$$t = \frac{1283}{0{,}6} = 2138 \text{ sec.} = 35 \text{ min. } 38 \text{ sec.}$$

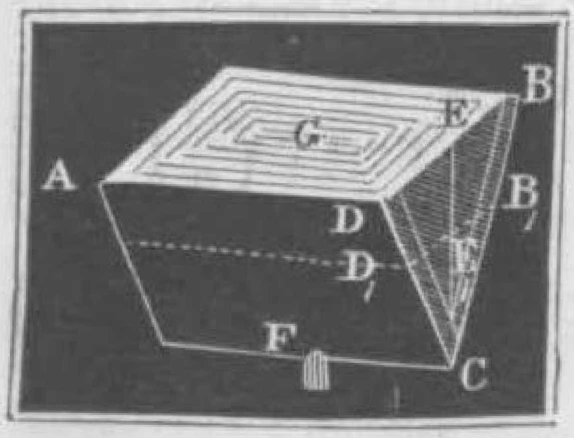
Remark. We may put for a rectangular lateral opening, approximatively:

$$t = \frac{2 \, G_1}{\mu \, F \sqrt{2g}} \left[\left(\sqrt{h_1} - \sqrt{h_2} \right) - \frac{a^2}{288} \left(\sqrt{h_1^{-3}} - \sqrt{h_2^{-3}} \right) \right],$$

and F and G represent the transverse sections of the opening and of the vessel, a the depth of the opening, h_1 the head of water at the commencement, h_2 that at the end of the efflux. If $h_1 = \frac{a}{2}$, the opening becomes a notch, and we must then apply the proper formula.

§ 349. *Wedge and Pyramidal-shaped Vessels.*—If the cistern of discharge ABF, Fig. 478, forms a horizontal triangular prism, the time of efflux may be found in the following manner. Let us divide the height $CE = h$ into n equal parts, and carry horizontal planes through the points of division; let us then decompose the whole quantity of water into equally thick strata of equal length $AD = l$, and of breadths diminishing downwards. If the breadth of the upper stratum $BD = b$, we have then the breadth of another stratum D_1B_1, which stands about $CE_1 = x$ above the orifice F, lying at the lower edge, $b_1 = \frac{x}{h} b$, and its volume $= b_1 \, l \, . \, \frac{h}{n} = \frac{b \, l \, x}{n}$. But now the discharge referred to a unit of time is: $Q = \mu \, F \sqrt{2 \, g \, x}$, hence then the small time in which the surface of water sinks about $\frac{h}{n}$ is $\tau = \frac{bl}{n} x \div \mu \, F \sqrt{2 \, g \, x} = \frac{b \, l}{n \, \mu \, F \sqrt{2g}} . \sqrt{x}$. Finally, since the sum of all the $x^{\frac{1}{2}}$ from $x = \frac{h}{n}$ to $x = \frac{nh}{n}$ are $= \left(\frac{h}{n} \right)^{\frac{1}{2}} . \frac{n^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3} \, n \, h^{\frac{1}{2}}$, we have the time for the discharge of the entire prism of water:

Fig. 478.

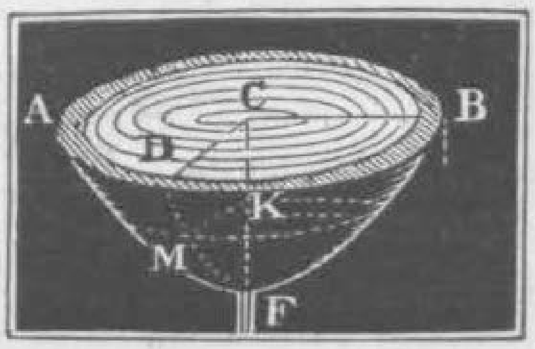


$$t = \frac{b \, l}{n \, \mu \, F \sqrt{2g}} \, . \, \tfrac{2}{3} \, n \, h^{\frac{1}{2}} = \tfrac{2}{3} \frac{b \, l}{\mu \, F \sqrt{2 \, g}} \, . \, h^{\frac{1}{2}} = \tfrac{4}{3} \frac{\frac{1}{2} \, b \, l \, h}{\mu \, F \sqrt{2 \, g \, h}}$$

$= \frac{4}{3} \, . \, \frac{V}{\mu \, F \, c}$, if V represents the whole quantity of water and c the initial velocity of efflux. Here the water, therefore, requires $\frac{1}{3}$ more time than if the velocity of efflux c were uniform.

If the vessel ABF, Fig. 479, forms an erect paraboloid, we then have for the ratio of the radii $KM = y$ and $CD = b$: $\frac{y}{b} = \frac{\sqrt{x}}{\sqrt{h}}$, and hence the ratio of the principal sections $\frac{G_1}{G} = \frac{y^2}{b^2} = \frac{x}{h}$, consequently $G_1 = \frac{Gx}{h}$ and the contents of a stratum of water $= G_1 \cdot \frac{h}{n} = \frac{Gx}{n}$. The perfect accordance of this expression with that found for the triangular prism, admits of our here putting $t = \frac{4}{3} \cdot \frac{\frac{1}{2} Gh}{\mu F \sqrt{2gh}}$, or, as $V = \frac{1}{2} Gh$ (§ 118), also $t = \frac{4}{3} \cdot \frac{V}{\mu F c}$.

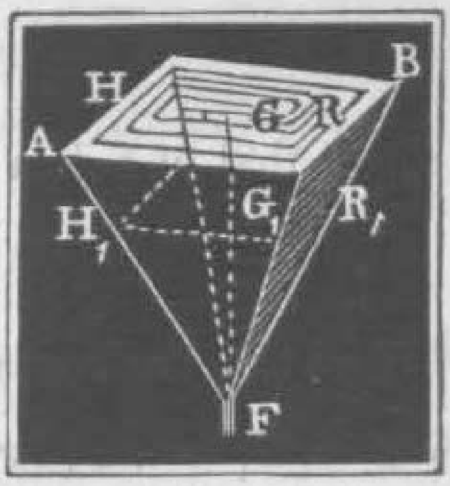
Fig. 479.



The formula may be used in many other cases for the approximative determination of the time of efflux, especially for that of the emptying of reservoirs. It is especially true in all cases where the horizontal sections increase as the distances from the bottom.

If, lastly, a vessel ABF be pyramidal, Fig. 480, then $G_1 : G = x^2 : h^2$, and hence $G_1 = \frac{Gx^2}{h^2}$ further the contents of the stratum $H_1 R_1$: $\frac{G_1 h}{n} = \frac{Gx^2}{nh}$, and the time for its discharge:

Fig. 480.



$$\tau = \frac{Gx^2}{nh} : \mu F \sqrt{2gx} = \frac{G}{n\mu Fh\sqrt{2g}} \cdot x^{\frac{3}{2}}.$$

But as the sum of all the $x^{\frac{3}{2}}$ taken from $x = \frac{h}{n}$ to $x = \frac{nh}{n} = \left(\frac{h}{n}\right)^{\frac{3}{2}} \cdot \frac{n^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2}{5}\, n\, h^{\frac{3}{2}}$,

it follows that the time for the emptying of the whole pyramid is:

$$t = \frac{G}{n\mu Fh\sqrt{2g}} \cdot \frac{2}{5}\, nh^{\frac{3}{2}} = \frac{2}{5} \cdot \frac{Gh^{\frac{1}{2}}}{\mu F\sqrt{2g}} = \frac{6}{5} \cdot \frac{\frac{1}{3} Gh}{\mu F\sqrt{2gh}},$$

or if $\frac{1}{3} Gh$ be put $= V$, then will $t = \frac{6}{5} \cdot \frac{V}{\mu Fc}$.

As in this efflux the initial velocity of flow decreases gradually from c to zero, the time of efflux is then $\frac{1}{5}$th greater than if the velocity c remained uniform.

Example. In what time will a pond, whose surface has an area of 765000 square feet, empty itself, if there be a conduit 15 feet below the surface, and at the deepest place, which forms a channel 15 inches wide and 50 feet long? Theoretically, the time of efflux is $t = \frac{4}{3} \cdot \frac{V}{F\sqrt{2gh}} = \frac{4}{3} \cdot \frac{1}{2} \cdot \frac{765000 \,.\, 15}{\frac{\pi}{4} \cdot \left(\frac{5}{4}\right)^2 . \, 8{,}02\sqrt{15}}$

$$= \frac{19584000}{\pi \,.\, 8{,}02\sqrt{15}} = 200848 \text{ sec.}$$

But now the co-efficient of resistance for entrance into the channel, inclined about 45°, is, $\zeta = 0{,}505 + 0{,}327$ (*see* § 323) $= 0{,}832$, and the resistance of the conduit due to friction $5 = 0{,}025 \, \frac{l}{d} \cdot \frac{v^2}{2g} = 0{,}025 \, . \, \frac{50}{\frac{5}{4}} \cdot \frac{v^2}{2g} = \frac{v^2}{2g}$; hence, the complete co-efficient of efflux for the channel is:

$\mu = \frac{1}{\sqrt{1 + 0{,}832 + 1}} = \frac{1}{\sqrt{2{,}832}} = 0{,}594$, and the time of efflux demanded:

$t = 200848 \div 0{,}594 = 338128 = 93$ hours, 55 minutes, 28 seconds.

§ 350. *Spherical and Obelisk-shaped Vessels.*—By means of the formula of the last paragraph, we may now find the times of efflux for many other vessels, such as spherical, pontoon-shaped, pyramidal, &c. For the emptying of a spherical segment AB, Fig. 481, we obtain:

Fig. 481.

$$t = \frac{4}{3} \frac{\pi r h^2}{\mu F \sqrt{2gh}} - \frac{2}{5} \cdot \frac{\pi h^3}{\mu F \sqrt{2gh}}$$

$$= \frac{2}{15} \pi \frac{(10r - 3h)\, h^{\frac{3}{2}}}{\mu F \sqrt{2g}},$$

therefore, for the emptying of a full sphere, where $h = 2r$,

$$t = \frac{16 \pi r^2 \sqrt{2r}}{15 \mu F \sqrt{2g}},$$

and for that of half a sphere where:

$$h = r, t_1 = \frac{14 \pi r^2 \sqrt{r}}{15 \mu F \sqrt{2g}}.$$

Here the horizontal stratum H_1R_1 corresponding to the depth FG_1 $= x = G_1 = \pi x (2r - x) . \frac{h}{n} = \frac{2\pi r h x}{n} - \frac{\pi h x^2}{n}$, therefore:

$$\tau = \frac{2 \pi r h}{n \mu F \sqrt{2g}} . x^{\frac{1}{2}} - \frac{\pi h}{n \mu F \sqrt{2g}} . x^{\frac{3}{2}};$$

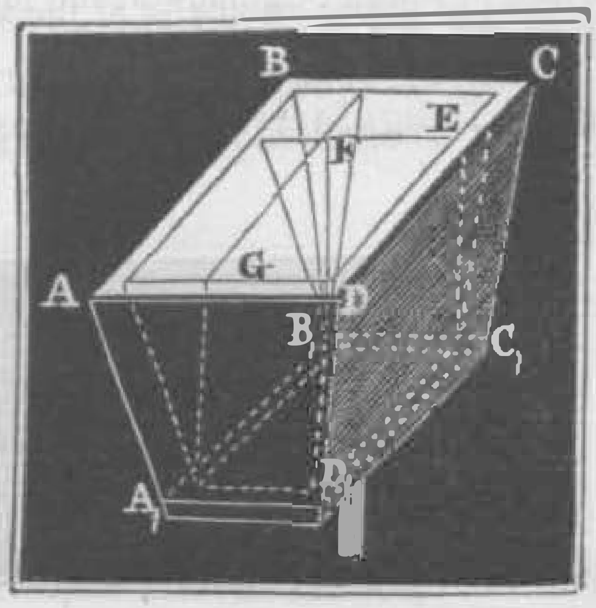
as the first part of this expression agrees with the formula for the emptying of a prismatic, and the second part for the emptying of a pyramidal vessel, if we put first $2\pi r h$ in place of bl, and secondly πh^2 in place of G, we shall obtain by means of the difference of the times of emptying of a prismatic and pyramidal vessel, found in the former paragraph:

$$t = \frac{2}{3} \cdot \frac{b l h}{\mu F \sqrt{2gh}}, \text{ and } t = \frac{2}{5} \cdot \frac{G h}{\mu F \sqrt{2gh}},$$

the time also of the emptying of a spherical segment.

The above formula may be likewise applied to the case of an obelisk or pontoon-shaped vessel ACD_1, Fig. 482, since this is composed of a parallelopiped, two prisms, and a pyramid. Let b be the

Fig. 482.



breadth at top AD, and b_1 the breadth $A_1 D_1$ at bottom, l the length at top AB, and l_1 the length at bottom A_1B_1, and lastly, h the height of the vessel, we have then for the area of the surface AC: $bl = b_1l_1 + b_1(l - l_1) + l_1(b - b_1) + (l - l_1)(b - b_1)$, of which b_1l_1 belongs to the parallelopiped A_1C_1EG, $b_1(l - l_1) + l_1(b - b_1)$ to the two prisms CFB_1C_1, and AFB_1A_1 and $(l - l_1)(b - b_1)$ to the pyramid BFB_1. But now the time of efflux for the parallelopiped, whose base is b_1l_1, is $t_1 = \frac{2\, b_1 l_1 \sqrt{h}}{\mu F \sqrt{2g}}$; further, that for the two triangular prisms

$$t_2 = \tfrac{2}{3}\, \frac{[b_1(l - l_1) + l_1(b - b_1)]\sqrt{h}}{\mu F\sqrt{2g}}$$

and, lastly, for the pyramid:

$$t_3 = \tfrac{2}{5}\, \frac{(l - l_1)(b - b_1)\sqrt{h}}{\mu F\sqrt{2g}};$$

hence the time of discharge for the whole vessel is:

$$t = t_1 + t_2 + t_3$$

$$= [30\, b_1l_1 + 10\, b_1(l - l_1) + 10\, l_1(b - b_1) + 6\,(l - l_1)(b - b_1)]\, \frac{\sqrt{h}}{15\mu F\sqrt{2g}}$$

$$= [3\, bl + 8\, b_1l_1 + 2(bl_1 + b_1l)]\, \frac{2\sqrt{h}}{15\mu F\sqrt{2g}}.$$

If $\frac{b_1}{l_1} = \frac{b}{l}$, we have then a truncated pyramid to consider. Let the one base $bl = G$, and the other $b_1l_1 = G_1$, we then obtain:

$$t = (3\, G + 8\, G_1 + 4\sqrt{G G_1})\, \frac{2\sqrt{h}}{15\mu F\sqrt{2g}}.$$

It would be easy to show that this formula holds true also for every trilateral or multilateral pyramid.

Example. An obelisk-shaped water-cask is 5 feet long, and 3 feet broad at top, and at the depth of 4 feet, that is, at the level of a short horizontal discharge-tube, 1 inch in width, and 3 inches in length, it is 4 feet long and 2 feet broad, what time will be required for the water in the full cask to sink $2\frac{1}{2}$ feet? The time for emptying is, μ being taken $= 0{,}815$:

$$t = [8 \,.\, 4 \,.\, 2 + 3 \,.\, 5 \,.\, 3 + 2\,(3 \,.\, 4 + 5 \,.\, 2)]\, \frac{2\sqrt{4}}{15 \,.\, 0{,}815 \,.\, \frac{\pi}{4} \,.\, \left(\frac{1}{12}\right)^2 \,.\, 8{,}02}$$

$$= \frac{153 \,.\, 4 \,.\, 4 \,.\, 144}{15 \,.\, 0{,}815 \,.\, 8{,}02\, \pi} = 153 \,.\, \frac{2304}{12{,}225 \,.\, 8{,}02\, \pi} = 153 \,.\, 7{,}481 = 1145 \text{ sec.}$$

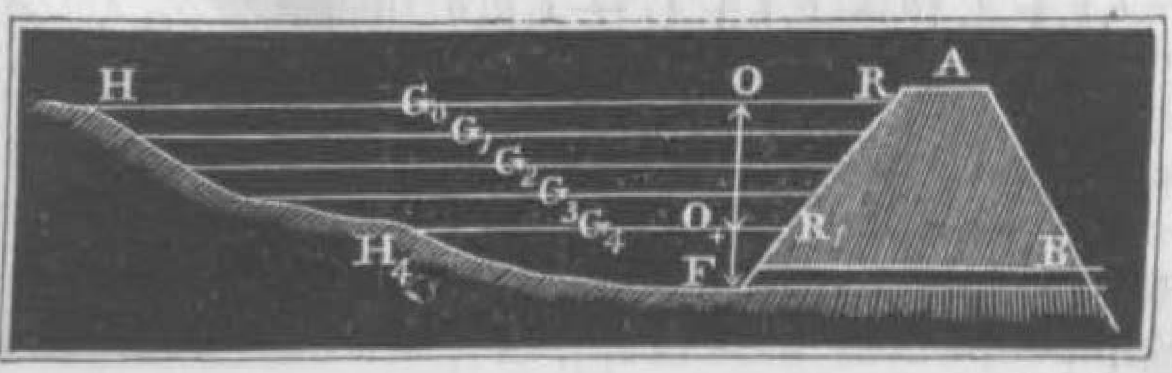
As the level $4 - 2\frac{1}{2} = 1\frac{1}{2}$ feet above the tube $l = l_1 + \frac{3}{8} = 4\frac{3}{8}$ and $b = b_1 + \frac{3}{8} = 2\frac{3}{8}$ feet, hence the time for emptying if the vessel be filled only up to this level, is:

$$t_1 = [8 \,.\, 4 \,.\, 2 + 3 \,.\, \tfrac{35}{8} \,.\, \tfrac{19}{8} + 2\,(2 \,.\, \tfrac{35}{8} + 4 \,.\, \tfrac{19}{8})] \,.\, \frac{1152\sqrt{1{,}5}}{15 \,.\, 0{,}815 \,.\, 8{,}02\, \pi} = 131{,}672 \,.$$

$4{,}5779 = 602{,}76$ sec. The difference of the times found gives the time in which the surface of water originally at the top of the vessel sinks $2\frac{1}{2}$ feet.

§ 351. *Irregular Vessels.*—When we have to find the time of efflux for an irregularly formed vessel HFR, Fig. 483, we must apply Simp-

Fig. 483.



son's rule as a method of approximation. If we divide the whole mass of water into four equally thick strata, and the heads of water G_0, G_1, G_2, G_3, G_4, corresponding to the horizontal slices, represented by h_0, h_1, h_2, h_3, h_4, the time of efflux will be given by Simpson's rule

$$t = \frac{h_0 - h_4}{12\,\mu\,F\,\sqrt{2g}}\left(\frac{G_0}{\sqrt{h_0}} + \frac{4\,G_1}{\sqrt{h_1}} + \frac{2\,G_2}{\sqrt{h_2}} + \frac{4\,G_3}{\sqrt{h_3}} + \frac{G_4}{\sqrt{h_4}}\right).$$

In assuming six strata:

$$t = \frac{h_0 - h_6}{18\mu F\sqrt{2g}}\left(\frac{G_0}{\sqrt{h_0}} + \frac{4G_1}{\sqrt{h_1}} + \frac{2G_2}{\sqrt{h_2}} + \frac{4G_3}{\sqrt{h_3}} + \frac{2G_4}{\sqrt{h_4}} + \frac{4G_5}{\sqrt{h_5}} + \frac{G_6}{\sqrt{h_6}}\right).$$

The discharge in the first case is:

$$Q = \frac{h_0 - h_4}{12}(G_0 + 4\,G_1 + 2\,G_2 + 4\,G_3 + G_4),$$ in the second:

$$Q = \frac{h_0 - h_6}{18}(G_0 + 4\,G_1 + 2\,G_2 + 4\,G_3 + 2\,G_4 + 4\,G_5 + G_6).$$

When the form and dimensions of the vessel of efflux are not known, we may then calculate very nearly the discharge by the heads of water noted in equal intervals of time. Let t be one such interval, we have then for apertures at the bottom and sides:

$$Q = \frac{\mu F t\sqrt{2g}}{3}(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}),$$

and for divisions or notches in a side:

$$Q = \tfrac{2}{3}\,\mu\,b\,t\sqrt{2g}\,(\sqrt{h^3_0} + 4\sqrt{h^3_1} + 2\sqrt{h^3_2} + 4\sqrt{h^3_3} + \sqrt{h^3_4}).$$

Example. In what time will the surface of water in a pond sink 6 feet, if the sluice forms a half cylinder, 18 inches wide, 9 inches deep, and 60 feet long, and the surfaces of water have the following areas?

G_0, at 20	feet head of water,	$= 600000$	square feet.
G_1, " 18,5	" "	$= 495000$	"
G_2, " 17,0	" "	$= 410000$	"
G_3, " 15,5	" "	$= 325000$	"
G_4, " 14,0	" "	$= 265000$	"

$F = \frac{\pi}{8} \cdot \left(\frac{3}{2}\right)^2 = \frac{9\pi}{32} = 0{,}8836$ square feet. Let the coefficient of resistance for the entrance $= 0{,}832$, and that for the friction:

$= 0{,}025 \cdot \frac{l}{d} = 0{,}025 \cdot 60 : 1{,}091 = 1{,}6356$, then is the coefficient of efflux

$$\mu = \frac{1}{\sqrt{1 + 0{,}832 + 1{,}6365}} = \frac{1}{\sqrt{3{,}4685}} = 0{,}537,$$

and $\mu F\sqrt{2g} = 0{,}537 \,.\, 0{,}8836 \,.\, 8{,}02 = 3{,}8054$. Now

$$\frac{G_0}{\sqrt{h_0}} = \frac{600000}{\sqrt{20}} = 134170, \quad \frac{G_1}{\sqrt{h_1}} = \frac{495000}{\sqrt{18{,}5}} = 115090,$$

$$\frac{G_2}{\sqrt{h_2}} = \frac{410000}{\sqrt{17}} = 99440, \quad \frac{G_3}{\sqrt{h_3}} = \frac{325000}{\sqrt{15{,}5}} = 82550,$$

$$\frac{G_4}{\sqrt{h_4}} = \frac{265000}{\sqrt{14}} = 70830;$$ hence, then, the time of efflux follows:

$$t = \frac{6}{12 \,.\, 3{,}8054}(134170 + 4 \,.\, 115090 + 2 \,.\, 99440 + 4 \,.\, 82550 + 70830)$$

$$= \frac{1194440}{7{,}6108} = 156940 \text{ sec.}, = 43 \text{ hours, } 35 \text{ min. } 40 \text{ sec.}$$

The discharge is:

$$Q = \tfrac{6}{12}(600000 + 4 \,.\, 495000 + 2 \,.\, 410000 + 4 \,.\, 325000 + 265000)$$

$$= \frac{4965000}{2} = 2482500 \text{ cubic feet.}$$

§ 352. *Influx and Efflux.*—If the vessel during the efflux from below has an influx to it from above, the determination of the time in which the surface of water rises or falls a certain height becomes more complicated, so that we must be satisfied generally with but an approximate determination. If the discharge per second Q_1 is $> \mu F\sqrt{2gh}$, then there is a rise, and if $Q_1 < \mu F\sqrt{2gh}$, a fall of the surface. Moreover, a state of permanency occurs whenever the head of water is increased or decreased by $k = \frac{1}{2g}\left(\frac{Q_1}{\mu F}\right)^2$. The time τ, in which the variable head of water x increases by the small amount ξ, is given by the equation

$$G_1 \xi = Q_1 \tau - \mu F\sqrt{2gx} \,.\, \tau,$$

and, on the other hand, the time in which it sinks the height k, by

$$G_1 \xi = \mu F\sqrt{2gx} \,.\, \tau - Q_1 \tau.$$

Hence we have in the first case $\tau = \frac{G_1\xi}{Q_1 - \mu F\sqrt{2gx}}$, and in the second $\tau = \frac{G_1\xi}{\mu F\sqrt{2gx} - Q_1}$. By the application of Simpson's rule we then obtain the time of efflux, during which the lowering surface passes from G_0 to G_1, G_2 . . ., and the head of water from h_0 to h_1, h_2 . . .

$$t = \frac{h_0 - h_4}{12}\left[\frac{G_0}{\mu F\sqrt{2gh_0} - Q_1} + \frac{4\,G_1}{\mu F\sqrt{2gh_1} - Q_1} + \frac{2\,G_2}{\mu F\sqrt{2gh_2} - Q_1} + \frac{4\,G_3}{\mu F\sqrt{2gh_3} - Q_1} + \frac{G_4}{\mu F\sqrt{2gh_4} - Q_1}\right],$$

or, more simply, if we represent $\frac{Q_1}{\mu F\sqrt{2g}}$ by $\sqrt{k}$,

$$= \frac{h_0 - h_4}{12\mu F\sqrt{2g}}\left[\frac{G_0}{\sqrt{h_0} - \sqrt{k}} + \frac{4\,G_1}{\sqrt{h_1} - \sqrt{k}} + \frac{2\,G_2}{\sqrt{h_2} - \sqrt{k}} + \frac{4\,G_3}{\sqrt{h_3} - \sqrt{k}} + \frac{G_4}{\sqrt{h_4} - \sqrt{k}}\right].$$

By Calculus. $dt = \frac{G\,dx}{\mu F \sqrt{2g}\sqrt{x} - Q} = \frac{G}{\mu F \sqrt{2g}} \cdot \frac{dx}{\sqrt{x}-\sqrt{k}}$; But $\int_{h_1} \frac{dx}{\sqrt{x}-\sqrt{k}} = \int_{h_1} \frac{dx}{\sqrt{x}} + \frac{\sqrt{k}\frac{dx}{\sqrt{x}}}{\sqrt{x}-\sqrt{k}} = 2\left(\sqrt{x} + \sqrt{k}\,\log(\sqrt{x}-\sqrt{k})\right)_{h_1}^{h}$; $\therefore t = \frac{2G}{\mu F \sqrt{2g}}\left(\sqrt{h} - \sqrt{h_1} + \sqrt{k}\,\log \frac{\sqrt{h}-\sqrt{k}}{\sqrt{h_1}-\sqrt{k}}\right)$

If the vessel is prismatic, and has a uniform transverse section G, we then have:

$$t = \frac{2\,G}{\mu\,F\,\sqrt{2g}}\left(\sqrt{h} - \sqrt{h_1} + \sqrt{k}\,.\ hyp.\ log.\left(\frac{\sqrt{h}-\sqrt{k}}{\sqrt{h_1}-\sqrt{k}}\right)\right),$$

the time in which the head of water passes from h to h_1. Since for:

$$h_1 = k,\ \frac{\sqrt{h}-\sqrt{k}}{\sqrt{h_1}-\sqrt{k}} = \frac{\sqrt{h}-\sqrt{k}}{0} = \infty,$$

it follows that the condition of permanency takes place indefinitely late.

The following formula is the result of investigation for a wier or notch in a side.

$$t = \frac{G\,k}{3\,Q_1}\Big[hyp.\ log.\ \frac{(\sqrt{h}-\sqrt{k})^2\,(h_1+\sqrt{h_1 k}+k)}{(\sqrt{h_1}-\sqrt{k})^2\,(h+\sqrt{hk}+k)}$$
$$+ \sqrt{12}\ .\ arc\ \left(tang. = \frac{(\sqrt{h}-\sqrt{h_1})\sqrt{12k}}{3\,k + (2\sqrt{h}+\sqrt{k})\,(2\sqrt{h_1}+\sqrt{k})}\right)\Big],$$

where $k = \left(\frac{Q_1}{\frac{2}{3}\,\mu\,b\,\sqrt{2\,g}}\right)^{\frac{2}{3}}$, *hyp. log.* represents the hyperbolic logarithm, and *arc* $(tang. = y)$ the arc whose tangent $= y$.

According as k is $\lessgtr h$, and the inflowing quantity of water

$Q_1 \gtrless \frac{2}{3}\,\mu\,b\,\sqrt{2\,g\,h^3}$, there is a rise or fall of the fluid surface. The condition of permanency occurs, when $h_1 = k$, and the time corresponding becomes ∞.

Example. In what time will the water in a rectangular tank 12 feet long and 6 feet broad rise from 0 to 2 feet above the edge of a notch $\frac{1}{2}$ foot broad, if 5 cubic feet of water flow in per second? We have here $h = 0$; hence, more simply:

$$t = \frac{G\,k}{3\,Q_1}\Big[\ hyp.\ log.\ \frac{h_1+\sqrt{h_1\,k}+k}{(\sqrt{h_1}-\sqrt{k})^2} + \sqrt{12}\ \ arc\ \left(tang. = \frac{-\sqrt{3\,h_1}}{2\,\sqrt{k}+\sqrt{h_1}}\right)\Big].$$

Now $G = 12\,.\,6 = 72$, $Q_1 = 5$, $h_1 = 2$, $b = \frac{1}{2}$, and $\mu = 0{,}6$,

$k = \left(\frac{5}{\frac{2}{3}\,.\,0{,}6\,.\,\frac{1}{2}\,.\,8{,}02}\right)^{\frac{2}{3}} = 2{,}1338$, and the time sought is:

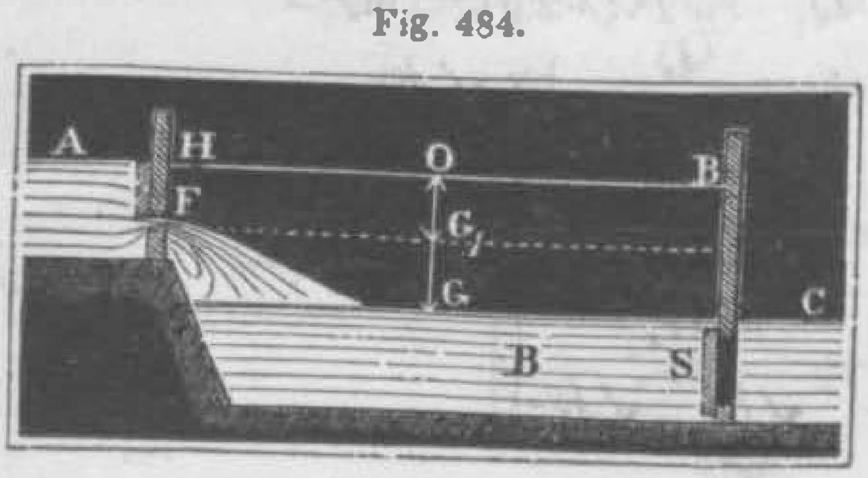
$$t = \frac{72\,.\,2{,}1338}{3\,.\,5}\Big[\ hyp.\ log.\ \frac{4{,}1338+\sqrt{4{,}2676}}{(1{,}4142-1{,}4607)^2} - \sqrt{12}\,.\ arc\ \left(tang. = \frac{\sqrt{6}}{1{,}4142+2{,}9214}\right)\Big]$$

$$= 10{,}242\,\Big[\ hyp.\ log.\ \frac{6{,}1996}{0{,}002162} - \sqrt{12}\,.\ arc\ \left(tang. = \frac{\sqrt{6}}{4{,}3356}\right)\Big]$$

$$= 10{,}242\ (7{,}961 - 1{,}781) = 10{,}242\,.\,6{,}18 = 63{,}29\ sec.$$

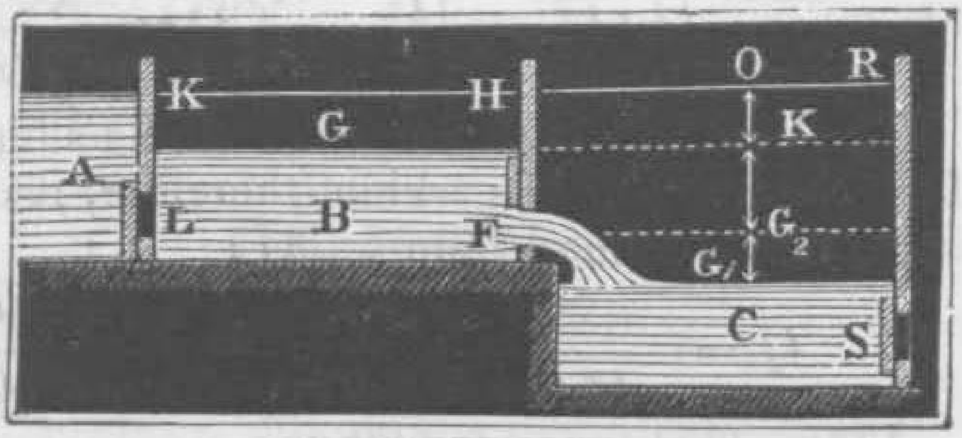
§ 353. *Locks.*—A very useful application of the doctrines hitherto treated of may be made to the filling and emptying of canal locks. We distinguish two kinds of navigation locks, single and double. The single lock, Fig. 484, consists of a chamber B, which is separated by the upper gate HF from the upper reach A, and by the

Fig. 484.



lower gate RS from the lower reach C. The double lock, Fig. 485, on the other hand, consists of two chambers, with the upper gate KL, the middle one HF, and the lower one RS.

Fig. 485.



Let the mean horizontal transverse section of a simple lock chamber $= G$, the distance of the middle of the sluice in the upper gate from the upper surface HR of the upper reach $= h_1$, and from that of the lower reach $= h_2$, and, lastly, the area of the aperture or sluice opening $= F$, we then obtain the time of filling up to the middle of the aperture $t_1 = \frac{G h_2}{\mu F \sqrt{2 g h_1}}$, and the time for filling the remaining space, where a gradual diminution of the head of water takes place, $t_2 = \frac{2\ G h_1}{\mu F \sqrt{2 g h_1}}$; consequently, the time for filling the single sluice is;

$$t = t_1 + t_2 = \frac{(h_2 + 2\,h_1)\,G}{\mu\ F \sqrt{2 g h_1}}.$$

If the aperture in the lower gate is entirely under water, then while emptying, the head of water gradually decreases from $h_1 + h_2$ to zero, hence the time for emptying or running off is:

$$t = \frac{2\ G \sqrt{h_1 + h_2}}{\mu\ F \sqrt{2 g}}.$$

If, on the other hand, a part of the aperture stands above the lower water, we then have two discharges to take into account; the one flowing above and the other below the water. Let the height of the part of the aperture above the water $= a_1$, and that under the water $= a_2$, the breadth of the aperture $= b$, we then obtain the time of efflux from the expressione

$$t = \frac{2\ G\,(h_1 + h_2)}{\mu\ b \sqrt{2 g}\left(a_1 \sqrt{h_1 + h_2 - \frac{a_1}{2}} + a_2 \sqrt{h_1 + h_2}\right)}.$$

In double locks, the head of water gradually decreases in the chamber which is closed from the upper reach, during the discharge into the second chamber. If G is the horizontal transverse section of the first chamber, and the original head of water h_1 in this chamber sinks to x, whilst the water in the second chamber rises to the middle of the aperture of the sluice, we have then the corresponding time $t_1 = \frac{2\ G}{\mu\ F \sqrt{2 g}} (\sqrt{h_1} - \sqrt{x})$. Now the quantity of water

$G(h_1 - x) = G_1 h_2$, hence $x = h_1 - \frac{G_1}{G} h_2$, and

$$t_1 = \frac{2\,G}{\mu F \sqrt{2g}}\left(\sqrt{h_1} - \sqrt{h_1 - \frac{G_1 h_2}{G}}\right) = \frac{2\sqrt{G}}{\mu F \sqrt{2g}}\left(\sqrt{Gh_1} - \sqrt{Gh_1 - G_1 h_2}\right).$$

The time in which the water rises as high in the second as in the first chamber, and in which, therefore, it comes to the same level in both, may be found from § 347:

$$t_2 = \frac{2\,GG_1\sqrt{x_0}}{\mu F(G+G_1)\sqrt{2g}} = \frac{2\,G_1\sqrt{G}\sqrt{Gh_1 - G_1 h_2}}{\mu F(G+G_1)\sqrt{2g}},$$

and the whole time for filling:

$$t = t_1 + t_2 = \frac{2\sqrt{G}}{\mu F\sqrt{2g}}\left(\sqrt{Gh_1} - \frac{G}{G+G_1}\sqrt{Gh_1 - G_1 h_2}\right).$$

Example. What time is required for the filling and running off of the following single lock chamber? The mean length of the lock = 200 feet, mean breadth = 24 feet, therefore $G = 200 \cdot 24 = 4800$ square feet, distance of the centre of the aperture of the sluice in the upper gate from the two surfaces of water 5 feet, breadth of both apertures $2\frac{1}{2}$ feet, height of the aperture in the upper gate 4 feet, and of that in the lower gate (entirely under water) 5 feet. Let

$t = \frac{(2\,h_1 + h_2)\,G}{\mu\,F\sqrt{2\,g\,h_1}}$, $h_1 = 5$, $h_2 = 5$, $G = 4800$, $\mu = 0{,}615$, $F = 4 \cdot 2\frac{1}{2} = 10$, $\sqrt{2g} = 8{,}02$, we then obtain the time of filling:

$$t = \frac{3 \cdot 5 \cdot 4800}{6{,}15 \cdot 8{,}02\sqrt{5}} = \frac{14400}{1{,}23 \cdot 8{,}02\sqrt{5}} = 652{,}85 \text{ seconds.}$$ If we substitute in the formula $t = \frac{2\,G\sqrt{h_1 + h_2}}{\mu\,F\sqrt{2g}}$, $G = 4800$, $h_1 + h_2 = 10$, $F = 5 \cdot 2\frac{1}{2} = 12{,}5$, we then obtain the time for emptying of the sluice:

$$t = \frac{2 \cdot 4800\sqrt{10}}{0{,}615 \cdot 12{,}5 \cdot 8{,}02} = 491{,}78 \text{ sec.} = 8 \text{ min. } 21{,}78 \text{ sec.}$$

CHAPTER VI.

ON THE EFFLUX OF AIR FROM VESSELS AND TUBES.

§ 354. *Efflux of Still Air.*—Condensed air does not flow from vessels quite in accordance with the law which regulates the flow of water, because an expansion takes place during its discharge, which is not the case with water. But in order to discover a similar law for air and other gases, let us make the mechanical effect $Q\gamma \frac{v^2}{2g}$, which a quantity of air Q of the density γ requires to pass from a state of rest into that of the velocity v, equal to the mechanical effect $Q\,p$ *hyp. log.* $\left(\frac{p_1}{p}\right)$ found in § 298, which the same quantity of air produces when it passes from a greater pressure p_1 to a less p. If, therefore, p_1 be the elastic force of air enclosed in a vessel, v its velocity of efflux for the tension of the external air, and γ its density, then $Q\gamma \cdot \frac{v^2}{2g} = Qp \text{ hyp. log. } \left(\frac{p_1}{p}\right)$, therefore, *the height due to the velocity:*

$$\frac{v^2}{2g} = \frac{p}{\gamma} \text{ hyp. log. } \left(\frac{p_1}{p}\right) = 2{,}3026 \frac{p}{\gamma} \log. \left(\frac{p_1}{p}\right);$$

and the velocity itself:

$$v = \sqrt{2g \frac{p}{\gamma} \text{ hyp. log. } \left(\frac{p_1}{p}\right)}.$$

When the tensions p and p_1 differ little from each other, when $p_1 - p$ is $< \frac{1}{10} p$, then we may put:

$$\text{hyp. log. } \frac{p_1}{p} = \text{hyp. log. } \left(1 + \frac{p_1 - p}{p}\right) = \frac{p_1 - p}{p}, \text{ and hence}$$

$$v = \sqrt{2g\left(\frac{p_1 - p}{\gamma}\right)}.$$

But the height of an external column of air which is in equilibrium by its weight with the pressure $p_1 - p$ (§ 294), is $h = \frac{p_1 - p}{\gamma}$; hence we may put the velocity of efflux $v = \sqrt{2gh}$, and a perfect analogy with the efflux of water will hereby subsist. For high pressure this formula is not of course sufficient, for in this case:

$$\text{hyp. log. } \left(\frac{p_1}{p}\right) = \frac{p_1 - p}{p} - \frac{1}{2}\left(\frac{p_1 - p}{p}\right)^2 \text{ at least.}$$

Hence, then, more accurately

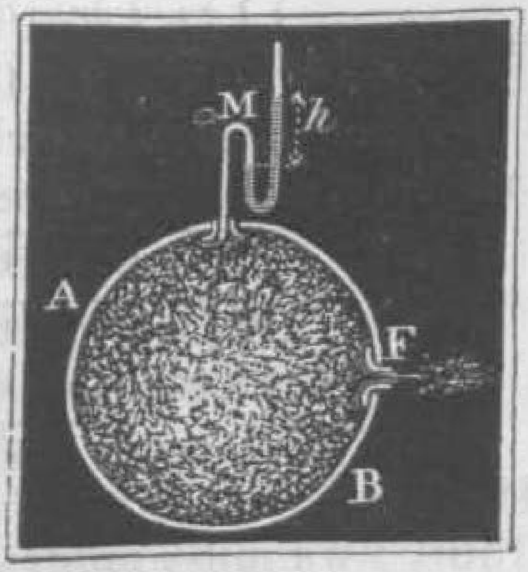
$$v = \sqrt{2g\left(\frac{p_1 - p}{\gamma} - \frac{1}{2}\frac{(p_1 - p)^2}{p\gamma}\right)} = \sqrt{2g\left(1 - \frac{p_1 - p}{2p}\right)h},$$

or if we represent the height of the barometer by b, $p = b\gamma$, and

$$v = \sqrt{2g\left(1 - \frac{h}{2b}\right)h} = \left(1 - \frac{h}{4b}\right)\sqrt{2gh}.$$

If the discharging orifice F of the vessel AB, Fig. 486, is accurately and smoothly rounded, the particles of air then flow in parallel lines, and hence the quantity of air flowing through the orifice in each second, and measured by the height of the external barometer, is:

Fig. 486.



$$Q = Fv = F\left(1 - \frac{h}{4b}\right)\sqrt{2gh},$$

or more accurately:

$$= F\sqrt{2gb \text{ hyp. log. } \left(\frac{b + h}{b}\right)}.$$

§ 355. The above formulæ do not admit of direct application, because we cannot measure the internal or the external pressure by the length $b+h$, and b of the columns of air. These pressures are generally measured by columns of water or mercury. As regards the quotient $\frac{p_1}{p} = \frac{b+h}{b}$, it is immaterial whether b and h be expressed in columns of air, water, or mercury, because each reduction of b and h leaves the fraction $\frac{b+h}{b}$ constant, except that the quotient $\frac{p}{\gamma} = b$, is still dependent on the temperature of the effluent air, and varies for different kinds of gas. For atmospheric air (§ 301), if p represent the pressure of air on one square centimetre, and γ the weight of a cubic metre of air, and t the temperature in degrees centigrade, we have

$$\frac{p}{\gamma} = \frac{1+0{,}00367 \,.\, t}{1{,}2572}, \text{ on the other hand, for steam,}$$

$$\frac{p}{\gamma} = \frac{1+0{,}00367\, t}{0{,}7857}.$$

If we substitute these values in the general formula for v, we shall obtain for atmospheric air:

$$v = 395 \sqrt{(1+0{,}00367a\; t)\; hyp.\, log. \left(\frac{b+h}{b}\right)} \text{ metres,}$$

or $\frac{h}{b}$ being small:

$$v = 395 \sqrt{(1+0{,}00367a\; t)\frac{h}{b}} \text{ metres, and for steam}$$

$$v = 500{,}6 \sqrt{(1+0{,}00367 \,.\, t)\; hyp.\, log. \left(\frac{b+h}{b}\right)} \text{ metres.}$$

The theoretical discharge as estimated under the external pressure is $Q = Fv$, but if this is to be estimated at the internal pressure, we must then make $Q_1 p_1 = Q p$, hence $Q_1 = \frac{p}{p_1} Q = \frac{b\,Q}{b+h}$. Reduced to the temperature of zero, the quantity discharged is:

$$Q_2 = \frac{Q}{1+0{,}00367 \,.\, t}, \text{ therefore, for atmospheric air}$$

$$= 395\, F \sqrt{\frac{hyp.\, log.\, (b+h) - hyp.\, log.\, b}{1+0{,}00367 \,.\, t}} \text{ cubic metres.}$$

If equal masses of air of different temperatures issue from different orifices F and F_1 at the same tension, we then have:

$$\frac{F_1}{F} = \sqrt{\frac{1+0{,}00367\, t_1}{1+0{,}00367\, t}}.$$

If, for example, $t = 0$ and $t_1 = 150°$ C., we then have:

$$F_1 = \sqrt{1{,}5505a}\; F = 1{,}245\, F.$$

If, therefore, a blast furnace is to be supplied with heated air of 150°, we mustoapply nozzle pipes, which have a one-fourth greater transverse section at the discharging orifice than if cold air were to be used.

For Prussian measure, and centigrade scale of temperature:

$$v = 1258 \,.\, \sqrt{(1 + 0{,}00367\, t)\, hyp.\, log. \left(\frac{b+h}{b}\right)}, \text{ and for steam}$$

$$v = 1595 \,.\, \sqrt{(1 + 0{,}00367\, t)\, hyp.\, log. \left(\frac{b+h}{b}\right)}.$$

For English measure, and Fahrenheit's scale of temperature:

$$v = 1295 \,.\, \sqrt{(1 + 0{,}00204\, t)\, hyp.\, log. \left(\frac{b+h}{b}\right)}, \text{ and for steam}$$

$$v = 1642 \,.\, \sqrt{(1 + 0{,}00204\, t)\, hyp.\, log. \left(\frac{b+h}{b}\right)}.$$

Example. In a large reservoir, air at 120° C., temperature is enclosed, which corresponds to the height of a mercurial manometer of 5 inches, whilst the external barometer stands at 27,2 inches; what quantity of air will flow from this through a round aperture $1\frac{1}{2}$ inch wide? It is:

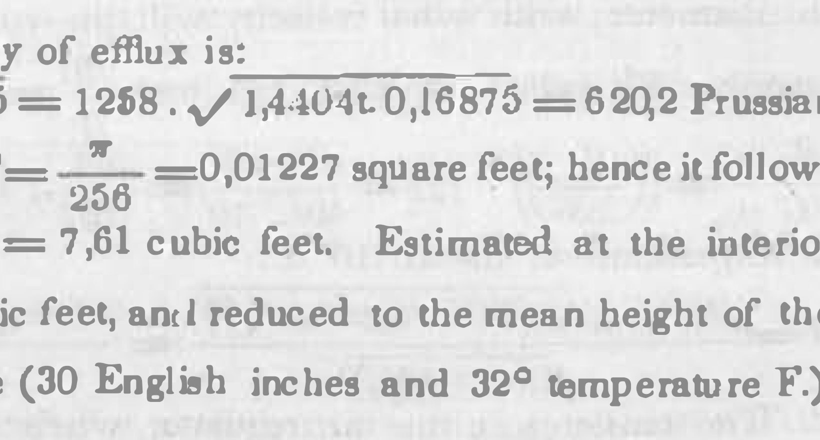
$hyp.\ log. \left(\frac{b+h}{b}\right) = hyp.\ log. \left(\frac{32{,}2}{27{,}2}\right) = hyp.\ log.\ 32{,}2 - hyp.\ log.\ 27{,}2 = 5{,}77455 - 5{,}60580 = 0{,}16875$, hence the velocity of efflux is:

$v = 1258 \,.\, \sqrt{(1 + 0{,}00367 \,.\, 120)\, 0{,}16875} = 1258 \,.\, \sqrt{1{,}4404 \,.\, 0{,}16875} = 620{,}2$ Prussian feet. Now the area of the orifice $= \frac{\pi}{4} \,.\, (\frac{1}{8})^2 = \frac{\pi}{256} = 0{,}01227$ square feet; hence it follows that the discharge $Q = 0{,}01227 \,.\, 620{,}2 = 7{,}61$ cubic feet. Estimated at the interior pressure, it is $= \frac{272}{322} \,.\, 7{,}61 = 6{,}43$ cubic feet, and reduced to the mean height of the barometer, 28 inches and 0° temperature (30 English inches and 32° temperature F.), the quantity discharged is:

$= 7{,}61 \,.\, \frac{272}{280} \,.\, \frac{1}{1{,}4404} = 5{,}13$ cubic feet.

§ 356. *Efflux of Air in Motion.*—The formula of efflux given: suppose the pressure p_1 or the height of the manometer h to be measured at a place where the air is at rest, or has a very slight motion, but if p_1 or h_1 is measured at a place where the air is in motion, if, for instance, the manometer M_1 communicates with the air in a conducting tube CF, Fig. 487, we shall then have to take into account the *vis viva* of the arriving air. If now c be the velocity of the air passing the orifice of the manometer we shall accordingly have to make:

Fig. 487.



$$Q\gamma \frac{v^2}{2g} = Q\gamma \,.\, \frac{c^2}{2g} + Q\, p\, hyp.\, log. \left(\frac{p_1}{p}\right),$$

or if F be the transverse section of the orifice, and G that of the tube, or of the air passing the orifice of the manometer, according to the law of Mariotte, $\frac{Gc}{Fv} = \frac{p}{p_1}$, or $Gcp_1 = Fvp$, therefore,

$$c = \frac{F}{G} \,.\, \frac{p}{p_1}\, v,\ Q\gamma \left[1 - \left(\frac{F}{G}\right)^2 \left(\frac{p}{p_1}\right)^2\right] \frac{v^2}{2g} = Qp\, hyp.\, log. \left(\frac{p_1}{p}\right),$$

and the velocity of efflux in question:

$$v = \frac{\sqrt{2g\frac{p}{\gamma}\, hyp.\, log.\left(\frac{p_1}{p}\right)}}{\sqrt{1-\left(\frac{Fp}{Gp_1}\right)^2}}.$$

The velocity of efflux is, therefore, here exactly like that of water from vessels, the velocity is greater, the greater the ratio $\frac{F}{G}$ of the transverse section of the orifice to that of the tube or the arriving current of air. From this it is evident, that under otherwise similar circumstances, the height of the manometer p_1 is so much the less the narrower the conducting tube is, or the greater the velocity of the air issuing from it.

Examples.—1. A mercurial manometer, placed upon an air tube $3\frac{1}{2}$ inches wide, stands at $2\frac{1}{2}$ inches, while the air flows from its conical extremity through a round orifice 2 inches in diameter; with what velocity will the current move? If the external barometer stand at $27\frac{1}{2}$ inches, we shall then have $\frac{p_1}{p} = \frac{27\frac{1}{2}+2\frac{1}{2}}{27\frac{1}{2}} = \frac{30}{27,5} = \frac{12}{11}$ and $\frac{F}{G}\frac{p}{p_1} = \left(\frac{2}{3,5}\right)^2 \cdot \frac{11}{12} = \frac{16\,.\,11}{49\,.\,12} = \frac{44}{147}$; hence the theoretical velocity of efflux at a temperature of the air 10° C.:

$$v = \frac{1258\,.\sqrt{1,0367\cdot hyp.\, log.\left(\frac{12}{11}\right)}}{\sqrt{1-\left(\frac{44}{147}\right)^2}} = \frac{1258\cdot\sqrt{1,0367\cdot 0,087}}{\sqrt{0,9104}} = 396 \text{ Prussian feet.}$$—

2. The tension p_2 in the air regulator, where the air is without motion, is given by the formula,

$$hyp.\, log.\left(\frac{p_2}{p}\right) = \frac{v^2}{2g}\cdot\frac{\gamma}{p}, \text{ or } hyp.\, log.\, p_2 = hyp.\, log.\, p + \frac{hyp.\, log.\left(\frac{p_1}{p}\right)}{1-\left(\frac{Fp}{Gp_1}\right)^2}$$

therefore, in the present case, $= hyp.\, log.\, 27,5 + \frac{0,087}{0,9104} = 3,3142 + 0,0965 = 3,4107.$ Hence it follows that $p_2 = 30,3$ inches.

§ 357. *Efflux under Decreasing Pressure.*—If an air reservoir has no influx, whilst an uninterrupted efflux goes on, the density and tension gradually diminish, and hence the velocity of efflux becomes less and less. We may determine in the following manner in what ratio this diminution is to the time and to its discharge.

Let V be the volume of the reservoir, h_0 the initial height of the manometer, and h_n the height of the manometer at the end of a certain time t, b the height of the external barometer. Then the quantity of air in the reservoir at the commencement reduced to the external pressure $= \frac{V(b+h_0)}{b}$, and at the end of the time t, $= \frac{V(b+h_n)}{b}$, and, consequently, the quantity discharged in the time t, and at the external pressure is:

$$V_n = \frac{V(b+h_0)}{b} - \frac{V(b+h_n)}{b} = \frac{V(h_0-h_n)}{b};$$

and, inversely, the height of the manometer corresponding to the discharge V_n is:

$$h_n = h_0 - \frac{V_n}{V} \cdot b.$$

If we take four intervals, and the initial height of the manometer h_0, and at the end of the time $t = h_4$, and

$$h_1 = h_0 - \frac{h_0 - h_4}{4},\ h_2 = h_0 - \tfrac{2}{4}(h_0 - h_4), \text{ and}$$

$h_3 = h_0 - \frac{3}{4}(h_0 - h_4)$, we shall then obtain by Simpson's rule the time

$$t = \frac{V(h_0 - h_4)}{12\, Fb\sqrt{2g\frac{p}{\gamma}}}\left(\frac{1}{\sqrt{hyp.\,log.\left(\frac{b+h_0}{b}\right)}} + \frac{4}{\sqrt{hyp.\,log.\left(\frac{b+h_1}{b}\right)}}\right.$$

$$+ \frac{2}{\sqrt{hyp.\,log.\left(\frac{b+h_2}{b}\right)}} + \frac{4}{\sqrt{hyp.\,log.\left(\frac{b+h_3}{b}\right)}}$$

$$\left. + \frac{1}{\sqrt{hyp.\,log.\left(\frac{b+h_4}{b}\right)}}\right).$$

For moderate pressures or heights of the manometer:

$$hyp.\,log.\left(\frac{b+h}{b}\right) = \frac{h}{b}\left(1 - \frac{h}{2b}\right),$$

consequently $\sqrt{hyp.\,log.\left(\frac{b+h}{b}\right)} = \left(1 - \frac{h}{4b}\right)\sqrt{\frac{h}{b}}$ and

$$\frac{1}{\sqrt{hyp.\,log.\left(\frac{b+h}{b}\right)}} = \left(1 + \frac{h}{4b}\right)\sqrt{\frac{h}{b}}.$$

If we now take n intervals, and therefore the discharge for one interval: $\frac{V_1}{n} = \frac{V(h_0 - h_n)}{nb}$, we shall then obtain the corresponding element of time:

$$\tau = \frac{V(h_0 - h_n)}{nb} \div \mu . F\sqrt{2g\frac{p}{\gamma}\,hyp.\,log.\left(\frac{b+h}{b}\right)}$$

$$= \frac{V(h_0 - h_n)}{nb}\,\frac{\left(1 + \frac{h}{4b}\right)\sqrt{\frac{b}{h}}}{F\sqrt{2g\frac{p}{\gamma}}}$$

$$= \frac{V(h_0 - h_n)\left(h^{-\frac{1}{2}} + \frac{h^{\frac{1}{2}}}{4b}\right)}{n.F\sqrt{2gb\frac{p}{\gamma}}}.$$

Now if we substitute for h; $h_0, h_1, h_2, \ldots h_n$, we shall then obtain the sum of all the

$$\left(\frac{h_0-h_n}{n}\right) h^{-\frac{1}{2}} = 2(h_0^{\frac{1}{2}}-h_n^{\frac{1}{2}}) = 2(\sqrt{h_0}-\sqrt{h_n}).$$

and the sum of all the

$$\left(\frac{h_0-h_n}{n}\right) h^{\frac{1}{2}} = \tfrac{2}{3}\left(h_0^{\frac{3}{2}}-h_n^{\frac{3}{2}}\right) = \tfrac{2}{3}(\sqrt{h_0^3}-\sqrt{h_n^3}),$$

whence the sum of all the small intervals of time, or the whole time in which h_n passes into h_0, and the quantity of air

$V_n = \frac{V(h_0-h_n)}{b}$ which flows out, is:

$$t = \frac{2V}{F\sqrt{2gb\frac{p}{\gamma}}}\left[(\sqrt{h_0}-\sqrt{h_n}) + \frac{1}{12b}(\sqrt{h_0^3}-\sqrt{h_n^3})\right], \text{ or}$$

$$= \frac{2V}{F\sqrt{2gb\frac{p}{\gamma}}}(\sqrt{h_0}-\sqrt{h_n})\left(1+\frac{h_0+\sqrt{h_0h_n}+h_n}{12b}\right),$$

approximately:

$$= \frac{2V}{F\sqrt{2gb\frac{p}{\gamma}}}(\sqrt{h_0}-\sqrt{h_n})\left(1+\frac{h_0+h_n}{8b}\right).$$

Example. A 50 feet long and 5 feet wide cylindrical air-regulator of a blowing machine is filled with air; the height of its manometer $h = 10$ inches, and the thermometer stands at 6° C. If now a flow of air takes place in a space where the height of the barometer is 27 inches, through a 1-inch wide circular orifice, then the question arises, in what time will the height of the manometer fall to 7 inches, and what will be the corresponding discharge? The volume of the chamber is for Prussian measures,—

$= \frac{\pi}{4} \cdot 5^2 \cdot 50 = 1250 \cdot \frac{\pi}{4} = 981{,}75$ cubic feet, hence the discharge, measured at the external pressure, is $V_1 = \left(\frac{h_0-h_n}{b}\right) V = \left(\frac{10-7}{27}\right) \cdot 981{,}75 = 109{,}08$ cubic feet.

Now $\sqrt{2g\frac{p}{\gamma}} = 1258\sqrt{1+0{,}00367 \cdot t} = 1258\sqrt{1{,}02202} = 1272$, and

$F = \frac{\pi}{4}\left(\frac{1}{12}\right)^2 = \frac{\pi}{576} = 0{,}005454$ square feet, hence the time of efflux in question is

$$t = \frac{2 \cdot 981{,}75}{0{,}005454 \cdot 1272}\left(\sqrt{\frac{10}{27}}-\sqrt{\frac{7}{27}}\right)\left(1+\frac{10+7}{8 \cdot 27}\right)$$

$$= \frac{1963{,}5}{5{,}454 \cdot 1{,}272} \cdot 0{,}0994 \cdot 1{,}079 = 30{,}3 \text{ seconds.}$$

§ 358. *Co-efficients of Efflux.*—The phenomena of contraction, which we have considered in the efflux of water from vessels, occur also in the efflux of air. If the orifice of efflux be cut in a thin plate, the air passing through it has a smaller transverse section than the orifice, and on this account the discharge is less than the product Fv of the transverse section F of the orifice and the theoretical velocity v. Let $\frac{F_1}{F}$ be the ratio of the transverse section F_1 of the blast to that of the orifice F, $= \mu$, we then have the effective discharge as for water:

$$Q_1 = \mu\, Q = F_1 v = \mu\, Fv = \mu\, F \sqrt{2g \frac{p}{\gamma}\, hyp.\, log. \left(\frac{p_1}{p}\right)}.$$

From the author's reduction of Koch's experiments at pressures of the manometer of from $\frac{1}{200}$ to $\frac{1}{5}$ of an atmosphere, we may take the mean of $\mu = 0,58$.

The effective discharge in the issuing of air through short cylindrical adjutages, is likewise less than that determined theoretically; we have, therefore, to multiply this latter by a number deduced from experiment, the co-efficient of efflux, μ in order to obtain the former; only here μ is not the ratio of the transverse section $\frac{F_1}{F}$, but the ratio $\frac{v_1}{v}$ of the effective velocity of efflux v_1 to the theoretical v. Koch's experiments give for the above pressures, in the flow of air through cylindrical adjutages, which were nearly all six times as long as wide, as a mean $\mu = 0,74$.

Conically convergent adjutages, similar to the nozzles of bellows, give a still greater co-efficient of efflux; a tube of 6° lateral convergence in the experiments of Koch, gave when five times as long as wide, the mean co-efficient $\mu = 0,85$.

From this, therefore, the effective discharge for the flow of air through orifices in a thin plate, measured at the external pressure, is

$$Q_1 = 751,1\, F \left(1 - \frac{h}{4b}\right) \sqrt{(1 + 0,00367 t) \frac{b}{h}} \text{ cubic feet (Eng.),}$$

for efflux through short cylindrical adjutages:

$$Q_1 = 958,3\, F \left(1 - \frac{h}{4b}\right) \sqrt{(1 + 0,00367\, t) \frac{h}{b}} \text{ cubic feet,}$$

and through conical adjutages of 6° convergence.

$$Q_1 = 1090,7\, F \left(1 - \frac{h}{4b}\right) \sqrt{(1 + 0,00367\, t) \frac{h}{b}} \text{ cubic feet.*}$$

* Experiments on the efflux of air have been undertaken by Young, Schmidt, Lagerhjelm, Koch, d'Aubuisson, Buff, and in later time, by Pecqueur, Saint-Venant, and Wantzel. For an account of the experiments of Young and Schmidt, we may refer to Gilbert's "Annalen," vol. 22, 1801, and vol. 6, 1820, and to Poggendorff's "Annalen," vol. 2, 1824; for those of Koch and Buff, to the "Studien des göttingschen Vereines bergmännischer Freunde," vol. 1, 1824; vol. 3, 1833; vol. 4, 1837, and vol. 5, 1838; also in Poggendorff's "Annalen," vol. 27, 1836, and vol. 40, 1837. The experiments of Lagerhjelm are described in the Swedish work, "Hydrauliska Försök of Lagerhjelm, Forselles och Kallstenius," 1 vol. Stockholm, 1818. D'Aubuisson's experiments are to be found in the "Annales des Mines," vol. 11, 1825; vol. 13, 1826; vol. 14, 1827, and likewise in his "Traité d' Hydraulique." The latest experiments instituted in France are reported in the "Polytechnischen Centralblatt," vol. 6, 1845. Most of these experiments were made with very narrow orifices, and, therefore, scarcely answer the purpose in practice. The experiments of d'Aubuisson and Koch deserve most consideration; and next to them, perhaps, those of Pecqueur; but the most extensive are those of Koch. The wished-for accordance is hardly to be met with in the results of all these experiments; the co-efficients of efflux found by d'Aubuisson vary considerably from those calculated by Koch. The grounds for my placing the most confidence in the co-efficients of Koch, are given in the "Allgemeinen Maschinenencyclopädie," under the article "Ausfluz," and in a Memoir of mine in Poggendorff's "Annalen," vol. 51, 1840. [For calculations of the above, and all similar cases, the co-efficient of t for the Fahrenheit's thermometer is 0,002039 instead of 0,00367; (see above, p. 346;) but the degrees computed are actually $t - 32$ on that scale.]—Am. Ed.

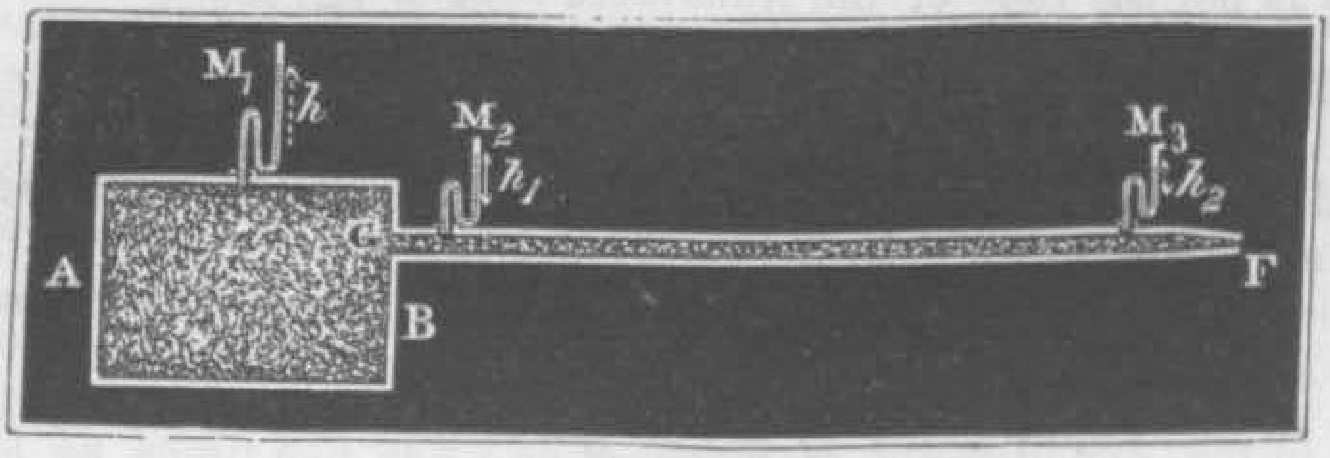
Example. If the two orifices of a bellows together possess an area of 3 square inches, if, further, the pressure of the manometer is 3 inches, the external barometer 27½ inches, and the temperature of the air 15°, then is the discharge:

$$Q = 1069 \cdot \frac{3}{144} \left(1 - \frac{3}{4 \cdot 27,5}\right) \sqrt{(1 + 0,00367 \cdot 15) \frac{3}{27,5}}$$

$$= 22,27 \cdot \frac{107}{110} \sqrt{1,055 \cdot \frac{6}{55}} = 21,66 \sqrt{0,1151} = 7,34 \text{ cubic feet.}$$

§ 359. *Flow through Tubes.*—If the air issues through a long tube GF, Fig. 488, it has then the resistance of friction to overcome in

Fig. 488.



the same manner as water; this resistance may also be measured by the height of a column of air, which has for expression

$h_n = \zeta \cdot \frac{l}{d} \cdot \frac{v^2}{2g}$, where, as in the conducting of water, v represents the velocity, l the length, d the width of the tube, and ζ a co-efficient of resistance to be determined by experiment.

Numerous experiments of Girard, d'Aubuisson, Buff and Pecqueur, lead to the mean value $\zeta = 0,024$. From this, therefore, the resistance generated by the friction of air in tubes may be measured by the height $h_n = 0,024 \frac{l}{d} \cdot \frac{v^2}{2g}$ of a column of air, or by the height

$h_n = 0,0000023 \frac{l}{d} \cdot \frac{v^2}{2g}$ of a column of quicksilver, and the manometer will stand at this much less height at the end of the conducting tube than at the beginning.

If at the end of a conducting tube of the width d, the manometer stands at h_2, whilst the air flows through an orifice of the width d_1 then from what precedes, the velocity of discharge will be:

$$v = \frac{\sqrt{2\, g \frac{p}{\gamma}\ hyp.\ log. \left(\frac{b + h_2}{b}\right)}}{\sqrt{1 - \left(\frac{b}{b + h_2}\right)^2 \left(\frac{d_1}{d}\right)^4}};$$

but if h_1 be the height of the manometer at the beginning of the conduit, we shall then have:

$$\frac{p}{\gamma}\ hyp.\ log. \left(\frac{b - h_1}{b}\right) = \left[1 + \left(\frac{b}{b + h_1}\right)^2 \left(\frac{d_1}{d}\right)^4 + 0,024 \frac{l}{d} \left(\frac{d_1}{d}\right)^4\right] \frac{v^2}{2g},$$

because the velocity in the tube $= \frac{d_1^2}{d^2} v$; hence in this case

$$v = \frac{\sqrt{2\,g\,\frac{p}{\gamma}\;hyp.\,log.\left(\frac{b+h_1}{b}\right)}}{\sqrt{1+\left[0{,}024\,\frac{l}{d}+\left(\frac{b}{b+h_1}\right)^2\right]\left(\frac{d_1}{d}\right)^4}}.$$

If, lastly, the height of the manometer h is measured in the reservoir at the beginning of the conduit, where the air may be regarded as at rest, we then have:

$$v = \frac{\sqrt{2\,g\,\frac{p}{\gamma}\;hyp.\,log.\left(\frac{b+h}{b}\right)}}{\sqrt{1+0{,}024\,\frac{l\,d_1^4}{d^5}}}.$$

If, further, we put the co-efficient of resistance ζ for entrance into the tube, which when $\mu_1 = 0{,}74$ amounts to 0,826, and, further, join to it the co-efficient of efflux μ for the outer adjutage, we then obtain for the velocity:

$$v = \frac{\mu\sqrt{2\,g\,\frac{p}{\gamma}\;hyp.\,log.\left(\frac{b+h}{b}\right)}}{\sqrt{1+\zeta+0{,}024\,\frac{l\,d_1^4}{d^5}}}$$

$$\text{or} = \frac{1294\,\mu\sqrt{(1+0{,}00367\,t)\;hyp.\,log.\left(\frac{b+h}{b}\right)}}{\sqrt{1+\zeta+0{,}024\,\frac{l\,d_1^4}{d^5}}}\ \text{feet (Pruss.)}$$

According as the point of the interior orifice lies s lower or higher than the point of the exterior orifice, we have to add $\pm s$ to the quantity under the radical in the denominator. Moreover, other hindrances may present themselves in the tube, such as curvatures, contractions, and widenings, &c. Satisfactory experiments on these obstacles do not exist, but we may assume with great probability that these resistances are not much different from what takes place in the case of water, because the co-efficients of efflux, and the co-efficient of friction are nearly the same for air as for water.

As long, therefore, as no further experiments are made on this subject, we may avail ourselves with tolerable safety of the co-efficient of resistance found for water in investigations on the motion and flow of air.

Example. In the regulator at the head of a 320 feet long and 4 inch wide air-conductor, the mercurial manometer stands at 3,1 inch, whilst the external barometer is at 27,2 inch; further, the width of the orifice of the conically contracted extremity of the conductor is 2 inches, and the temperature of the air 20° C., what quantity of air will this conductor deliver? It will be:

$1+\zeta+0{,}024\,\frac{l d_1^4}{d^5} = 1{,}826 + 0{,}0245\,\frac{320}{\frac{1}{3}}\cdot\left(\frac{1}{2}\right)^4 = 1{,}826 + 0{,}024\,.\,\frac{320\,.\,3}{16} = 1{,}826$ $+\,1{,}44 = 3{,}266$; further, $(1+0{,}00367\,t)\;hyp.\,log.\left(\frac{b+h}{b}\right)$

$= (1 + 0{,}00367 \,.\, 20)\, hyp.log. \left(\frac{30{,}3}{27{,}2}\right) = 1{,}0734 \,.\, (5{,}7137 - 5{,}6058)$
$= 1{,}0734 \,.\, 0{,}1079 = 0{,}1158$; if now, further, we introduce the co-efficient of efflux, $\mu = 0{,}85$, we shall then obtain the velocity of flow:

$$v = \frac{1258 \,.\, 0{,}85 \sqrt{0{,}1158}}{\sqrt{3{,}266}} = 201{,}3 \text{ feet; and lastly, the discharge:}$$

$$Q = \frac{\pi d_1^2}{4} \,.\, v = \frac{\pi}{4} \,.\, \frac{201{,}3}{36} = 4{,}39 \text{ cubic feet (Prussian).}$$

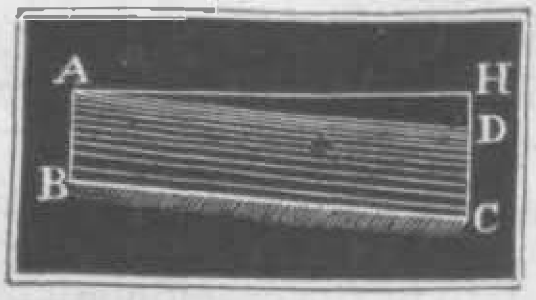
CHAPTER VII.

ON THE MOTION OF WATER IN CANALS AND RIVERS.

§ 360. *Running Water.*—The doctrine of the motion of water in canals and rivers, forms the second main division of hydraulics. Water flows either in a natural or in an artificial bed. In the first case, it forms streams, rivers, brooks; in the second, canals, cuts, drains, &c. In the theory of the motion of flowing water, this distinction is of little moment.

The bed of a river consists of the *bottom* and the two *banks* or *shores*. The *transverse section* is obtained by a plane at right angles to the direction of motion of the flowing water. Its *perimeter* is that of the transverse section, which again consists of the *air* and the *water section*. A vertical plane in the direction of the flowing water gives the *longitudinal section* or *profile*. By the *slope* or *declivity* of flowing water is understood the angle of inclination of its surface to the horizon. The *fall*, which is the vertical distance of the two extreme points of a definite length of the fluid surface, serves to assign the angle for a definite length of the flowing stream. For the length of course, $AD = l$, Fig. 489, BC is the bottom of the channel, $DH = h$ the fall, and the angle $DAH = \delta$, the slope $\sin. \delta = \frac{h}{l} =$ absolute fall per unit of length.*

Fig. 489.



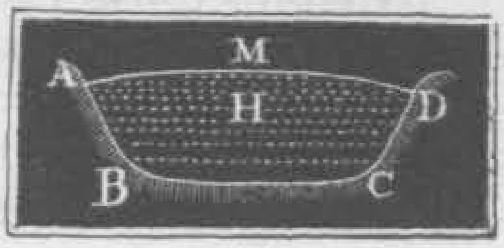
§ 361. *Different Velocities in the Transverse Section.*—The velocity of water in one and the same transverse section is very different at

* The fall of brooks and rivers is very various. The Elbe, for example, for the extent of a German mile from the Upper Elbe to Podiebrad, has a fall of 57 feet, from thence to Leitmeritz 9 feet, from there to Mühlberg a mean of 5,8, and from thence to Magdeburg 2,5 feet. Mountain brooks have a fall of from 40 to 400 feet per German mile. For further particulars, see "Vergleichende hydrographische Tabellen," &c., von Stranz. Canals and other artificial water conduits have much smaller falls. Here the absolute fall, at most, is 0,001, often 0,0001, and even less. More on this subject will be given in the Second Part.

different points. The adhesion of the water to its bed, and the cohesion of the particles to each other, cause those lying nearer to the sides of the bed to suffer some constraint in their motion, and hence, to flow more slowly than the more remote. For this reason the velocity diminishes from the surface downwards to the bed, and is least near the side or at the bottom. The greatest velocity is found for straight rivers, generally in the middle, or at that part of the free surface of the water where there is the greatest depth. The place where the water attains its maximum velocity is called the ***line of current***, and the deepest part of the bed; the ***mid-channel.***

The upper surface does not form an exact horizontal line, because the elements lying on the surface of water, flow on with different velocities with respect to each other, they therefore exert on each other different pressures; the quicker ones a less, and the slower a greater pressure, and thus for the maintenance of relative equilibrium, the quicker elements superpose themselves on the slower. If v and v_1 are the velocities of two elements M and A, Fig. 490, then according to the doctrine of hydraulic pressure (§ 307) the difference of level of the two elements ise

Fig. 490.



$$MH = h = \frac{v^2}{2g} - \frac{v_1^2}{2g} = \frac{v^2 - v_1^2}{2g}.$$

This difference of level is always very small. If, for example, $v_1 = 0{,}9\, v$, and $v = 5$ feet, we then have this

$$= (1 - 0{,}81) \frac{v^2}{2g} = 0{,}19 \,.\, 0{,}0155 \,.\, 25 = 0{,}0736 \text{ feet} = 0{,}88 \text{ inches}$$

(Eng.). For this reason the water stands highest in the current, and lowest at the banks.

In bends, the current is generally near the concave banke

§ 362. *Permanent Motion of Water.*—The mean velocity of water in a transverse section is, according to § 308:

$$c = \frac{Q}{F} = \frac{\text{quantity of water per second}}{\text{area of section}}.$$

The mean velocity besides may be further calculated from the velocities c_1, c_2, c_3, &c., of the separate portions of the section, and from the areas F_1, F_2, F_3, &c. It is namely:

$$Q = F_1 c_1 + F_2 c_2 + F_3 c_3 + \ldots,$$

and hence also:

$$c = \frac{F_1 c_1 + F_2 c_2 + \ldots}{F_1 + F_2 + \ldots}.$$

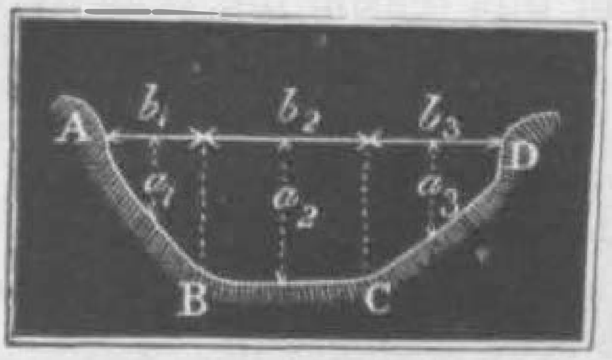
Besides the mean velocity, the mean depth of water has to be introduced, that is, the depth a which a section must have at all points that it may have the same area as it actually has with the variable depths a_1, a_2, a_3, &c. Hence, therefore,

$$a = \frac{F}{b} = \frac{\text{area of section}}{\text{breadth of section}}.$$

If the separate parts of the breadth b_1, b_2, b_3, have the corresponding mean depths a_1, a_2, a_3, &c., Fig. 491, we then have:

$$F = a_1 b_1 + a_2 b_2 + \ldots,$$

Fig. 491.



and hence also:

$$a = \frac{a_1 b_1 + a_2 b_2 + \ldots}{b_1 + b_2 + \ldots}$$

Lastly:

$$c = \frac{a_1 b_1 c_1 + a_2 b_2 c_2 + \ldots}{a_1 b_1 + a_2 b_2 + \ldots},$$

and if the portions b_1, b_2, &c., be of equal size,

$$c = \frac{a_1 c_1 + a_2 c_2 + \ldots}{a_1 + a_2 + \ldots}.$$

A river or brook is in a state of *permanency* when an equal quantity of water flows through each of its transverse sections in an equal time; when, therefore, Q or the product Fc of the area of the section and the mean velocity throughout the whole extent of the stream is a constant number. Hence this simple law comes out: ***in the permanent motion of water, the mean velocities in two transverse sections are to each other inversely as the areas of these sections.***

Examples.—1. At the section of a canal, $ABCD$, Fig. 491, it was found that the

Portions of the breadth	$b_1 = 3{,}1$ feet,	$b_2 = 5{,}4$ feet,	$b_3 = 4{,}3$ feet
Mean depth	$a_1 = 2{,}5$ "	$a_2 = 4{,}5$ "	$a_3 = 3{,}0$ "
Corresponding mean velocities	$c_1 = 2{,}9$ "	$c_2 = 3{,}7$ "	$c_3 = 3{,}2$ "

Hence the area of these profiles $F = 3{,}1 \,.\, 2{,}5 + 5{,}4 \,.\, 4{,}5 + 4{,}3 \,.\, 3{,}0 = 44{,}95$ square feet, and the discharge

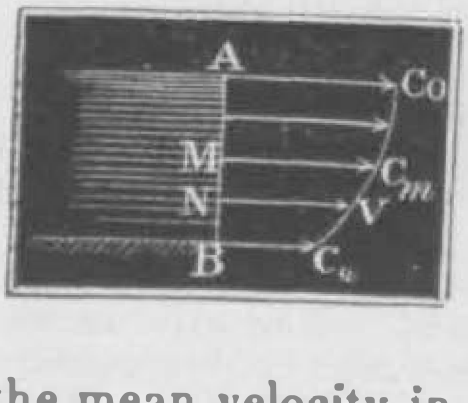
$Q = 3{,}1 \,.\, 2{,}5 \,.\, 2{,}9 + 5{,}4 \,.\, 4{,}5 \,.\, 3{,}7 + 4{,}3 \,.\, 3{,}0 \,.\, 3{,}2 = 153{,}665$ cubic feet, and the mean velocity $c = \frac{Q}{F} = \frac{153{,}665}{44{,}95} = 3{,}419$ feet.

2. When a cut is to conduct 4,5 cubic feet of water with a mean velocity c of 2 feet, we must then give to it a transverse section of $\frac{4{,}5}{2} = 2{,}25$ square foot area.—3. If one and the same stream has a mean velocity of $2\frac{1}{4}$ feet at a place 560 feet broad and 9 feet mean depth, it will then have, at a place 320 feet broad and 7,5 feet mean depth, the mean velocity

$$c = \frac{560 \,.\, 9}{320 \,.\, 7{,}5} \,.\, 2{,}25 = \frac{567}{120} = 4{,}725 \text{ feet.}$$

§ 363. *Mean Velocity.*—If we divide the depths of water at any point of a flowing stream into equal parts, and raise ordinates upon them corresponding to the velocities, we shall then obtain a scale of the velocity of the current AB, Fig. 492. Although it may be granted that the law of this scale, or of the difference of velocity is expressed by some curve, as according to Gerstner by an ellipse, yet it is allowable, without fear of any great error, to substitute for this a straight line, or assume that the velocity diminishes uniformly with the depth, because the diminution of velocity downwards is always very small. From the experiments of Ximenes, Brünnings, and Funk, the mean velocity in a perpendicular $c_m = 0{,}915\, c_0$, where c_0 repre-

Fig. 492.



sents the velocity at the surface, or the maximum velocity. The velocity, therefore, diminishes from the surface to the middle M

$$\text{by } c_0 - c_m = (1 - 0{,}915)\, c_0 = 0{,}085\, c_0,$$

and, consequently, the velocity below or at the foot of the perpendicular may be put

$$c_u = c_0 - 2 \,.\, 0{,}085\, c_0 = (1 - 0{,}170)\, c_0 = 0{,}83\, c_0.$$

If, now, the whole depth $= a$, we then have, by assuming a straight line for the scale of the velocities, the corresponding velocity for a depth $AN = x$, below the water

$$v = c_0 - (c_0 - c_u)\frac{x}{a} = \left(1 - 0{,}17\frac{x}{a}\right) c_0.$$

Further, let $c_0, c_1, c_2, \ldots$ be the superficial velocities of a whole transverse profile of not very variable depth, we have then the corresponding velocities at a mean depth: $0{,}915\, c_0$, $0{,}915\, c_1$, $0{,}915\, c_2$, and hence the mean velocity in the whole profile:

$$c = 0{,}915\, \frac{(c_0 + c_1 + c_2 + \ldots c_n)}{n}.$$

Lastly, if we assume that the velocity diminishes from the line of current towards the banks, as it does according to the depth, we may then again put the mean superficial velocity

$$\frac{(c_0 + c_1 + \ldots + c_n)}{n} = 0{,}915\, c_0,$$

and so obtain the mean velocity in the whole profile:

$$c = 0{,}915 \,.\, 0{,}915 \,.\, c_0 = 0{,}837 \,.\, c_0,$$

i. e. from 83 to 84 per cent. of the maximum velocity, or of that of the line of current.

Prony deduced from Du Buat's experiments conducted with very small channels, and for these cases perhaps more correctly:

$$c_m = \left(\frac{2{,}372 + c_0}{3{,}153 + c_0}\right) c_0 \text{ metre} = \left(\frac{7{,}71 + c_0}{10{,}25 + c_0}\right) c_0 \text{ feet English.}$$

For medium velocities of 3 feet it hence follows that $c_m = 0{,}81\, c_0$.

Example. In the line of current of a brook the velocity of the water is 4 feet, and the depth 6 feet, we have then the mean velocity at a corresponding perpendicular $c_m = 0{,}915 . 4 = 3{,}66$ feet, and that at the bottom $= 0{,}83 . 4 = 3{,}32$ feet; further, the velocity 2 feet below the surface is $v = (1 - 0{,}17 \,.\, \frac{2}{6})\, 4 = (1 - 0{,}057)\, 4 = 3{,}772$ feet; lastly, the mean velocity throughout the profile is, $c = 0{,}837 \,.\, 4 = 3{,}348$ feet, and according to Prony, $c = \frac{11{,}50}{13{,}97} \,.\, 4 = \frac{46}{13{,}97} = 3{,}29$ feet.*

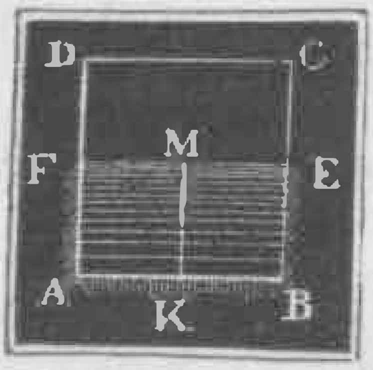
§ 364. *The Best Form of Transverse Sections.*—The resistance which the bed opposes to the motion of the water in virtue of its adhesion, viscosity, or friction, increases with the surface of contact between the bed and the water, and therefore with the perimeter p of the water profile, or of the portion of the transverse section which comprises the bed. But as more filaments of water pass through a

* This and the following subjects have been fully treated of under the article "Bewegung des Wassers," in the "Allgemeinen Maschinenencyclopädie." New experiments and new views may be found in the following writings: Lahmeyer's "Erfahrungsresultate über die Bewegung des Wassers in Fluszbetten und Kanälen." Brunswick, 1845.

profile, the greater its area is, so this resistance of a filament increases also inversely as the area, and hence on the whole as the quotient $\frac{p}{F}$ of the perimeter of the water profile, and the area of the whole transverse profile.

That the resistance of friction of a running stream or river may be the smallest possible, we must give to its transverse section that form for which the perimeter p for a given area is a minimum, or the area for a given perimeter a maximum. In enclosed conduits, as, for example, pipes, p is the entire perimeter of the figure formed by the transverse profile. Now of all figures having an equal number of sides, the regular figure, and again, of all regular figures that which has the greater number of sides, has for the same area the least perimeter; hence for enclosed conduits, the co-efficient of friction comes out the less, the nearer its transverse profile approaches to a regular figure, and the greater its number of sides; and the circle, which is a regular figure of an infinite number of sides, is in this case the profile which corresponds to the minimum of friction. We must, therefore, in estimating this resistance of friction, leave out of our consideration in the quotient $\frac{p}{F}$ the upper side or surface in contact with the air.

Fig. 493.



The rectangular and trapezoidal sections are those generally applied to canals, cuts, water-courses, &c. A horizontal line EF, Fig. 493, passing through the centre M of the square AC, divides as well the area as also the perimeter into two equal parts, hence it follows that what is true for the square is also correct for these halves, and, accordingly, of all rectangular transverse profiles, the half square AE, or that which is twice as broad as it is deep, corresponds to the least resistance of friction. The regular hexagon ACE, Fig. 494, may be likewise divided by a horizontal line CF into two equal trapeziums, each of which, like the entire hexagon, has the greatest relative area, and, consequently, of all trapezoidal profiles, half the regular hexagon or the trapezium $ABCF$ with the angle of slope $AFM = BCM$ of 60° is that

Fig. 494.

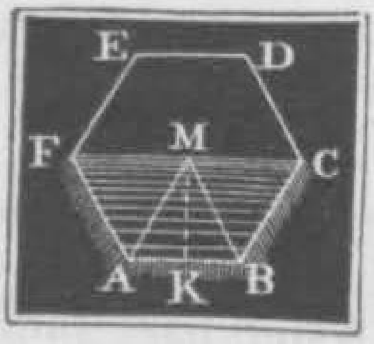


Fig. 495.

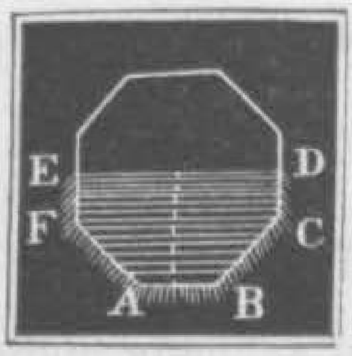
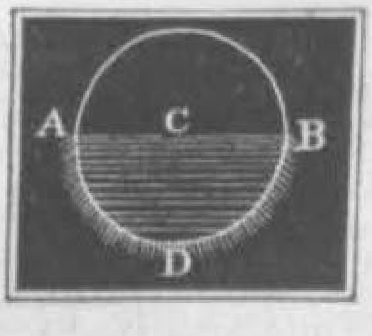


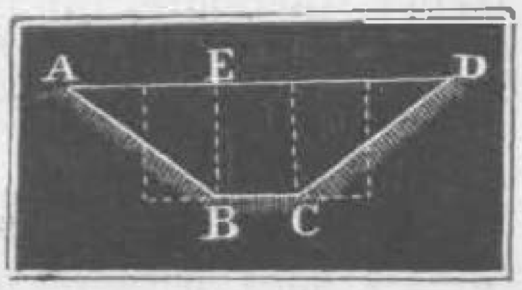
Fig. 496.



which, when applied, gives the least resistance of friction. Half the regular octagon ADE, Fig. 495, half the regular decagon, and, lastly, the semi-circle ADB, Fig. 496, afford under given circumstances the most advantageous transverse profiles for canals. The trapezoidal, or half the regular hexagon, gives a still less resistance than half the square or rectangle, the ratio of whose sides is 1 to 2, because the hexagon has a less relative perimeter than the square. Half the regular decagon gives a still less friction, and, in general, the minimum of friction corresponds to the semi-circle. The profiles of channels of wood, stone or iron only, are made semi-circular and rectangular; the profiles of canals, on the other hand, which are cut and bricked, are constructed of the trapezoidal figure. Other figures, in consequence of difficulties in the execution, are not easily applicable.

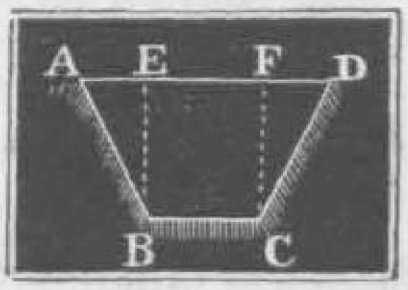
§ 365. In the case where a canal is not walled up, but dug out of loose earth or sand, the angle of 60° slope is too great, and the relative slope *cotg.* 60° $= 0{,}57735$ too small, because the banks would not have a sufficient stability; we are, therefore, under the necessity of applying the trapezoidal profile, for which the inclination of the sides to the base must be still less than 60°, perhaps scarcely 45°, or even less. For a trapezoidal profile $ABCD$, Fig. 497, which has a perimeter and area equal to that of half the square, the relative slope $= \frac{4}{3}$, and the angle of slope hardly 36° 52′. If the height BE be divided into three equal parts, the base BC will then have two of them, the parallel line AD ten, and each of the sides $AB = CD$ = five parts. In many cases the slope is made $= 2$, to which belongs an angle of 26° 34′, and sometimes it is even made still greater.

Fig. 497.



In every case the angle of slope $BAE = \Theta$, Fig. 498, or the slope $n = \frac{AE}{BE} = cotang. \Theta$ may be regarded as a given quantity dependent on the nature of the ground in which the canal is dug, and hence the dimensions of the profile which offers the least resistance have only further to be determined. Let the lower breadth $BC = b$, the depth $BE = a$, and the slope $= n$, we then obtain for the perimeter:

Fig. 498.



$$AB + BC + CD = p = b + 2\sqrt{a^2 + n^2a^2} = b + 2a\sqrt{1 + n^2},$$

for the area:

$$F = ab + naa = a(b + na),$$

and hence, inversely, $b = \frac{F}{a} - na$, and the ratio:

$$\frac{p}{F} = \frac{1}{a} + \frac{a}{F}(2\sqrt{n^2 + 1} - n).$$

If we substitute for a, $a + x$, where x is a small number, we may then put:

$$\frac{p}{F} = \frac{1}{a + x} + \frac{(a + x)}{F}(2\sqrt{n^2 + 1} - n)$$

$$= \frac{1}{a}\left(1 - \frac{x}{a} + \frac{x^2}{a^2}\right) + \frac{a+x}{F}\left(2\sqrt{n^2+1} - n\right)$$

$$= \frac{1}{a} + \frac{a}{F}\left(2\sqrt{n^2+1} - n\right) + \left(\frac{2\sqrt{n^2+1} - n}{F} - \frac{1}{a^2}\right)x + \frac{x^2}{a^3}.$$

Now that this value may be greater not only for a positive, but also for a negative value of x, than the first

$$\frac{1}{a} + \frac{a}{F}\left(2\sqrt{n^2+1} - n\right),$$

it is necessary that the member with the factor x should vanish, and therefore that $\frac{p}{F}$ may become a minimum, we must have

$$\frac{2\sqrt{n^2+1} - n}{F} - \frac{1}{a^2} = 0, \text{ i. e. } a^2 = \frac{F}{2\sqrt{n^2+1} - n},$$

or since:

$$n = cotang.\,\Theta \text{ and } \sqrt{n^2+1} = \frac{1}{sin.\,\Theta},\ a^2 = \frac{F\,sin.\,\Theta}{2 - cos.\,\Theta}.$$

Hence, therefore, the most appropriate form of profile corresponding to a given angle of slope Θ, and a given area is determined by

$$a = \sqrt{\frac{F\,sin.\,\Theta}{2 - cos.\,\Theta}} \text{ and } b = \frac{F}{a} - a\,cotang.\,\Theta.$$

Example. What dimensions must be given to the transverse profile of a canal, whose banks are to have 40° slope, and which is to conduct a quantity of water Q of 75 cubic feet, with a mean velocity of 3 feet? $F = \frac{Q}{c} = \frac{75}{3} = 25$ square feet, hence the depth $a = \sqrt{\frac{25\,sin.\,40^\circ}{2 - cos.\,40^\circ}} = 5\sqrt{\frac{0,64279}{1,23396}} = 3,609$ feet, the lower breadth $b = \frac{25}{3,609} - 3,609\,cotang.\,40^\circ = 6,927 - 4,301 = 2,626$ feet the slope or cut of the banks $= 3,609\,.\,cotang.\,40^\circ = 4,301$, the upper breadth $= 6,927 + 4,301 = 11,228$ feet, the perimeter $p = b + \frac{2a}{sin.\,\Theta} = 2,626 + \frac{7,218}{sin.\,40^\circ} = 13,855$ feet, and the ratio determining the friction $\frac{p}{F} = \frac{13,855}{25} = 0,5542$.

§ 366. The dimensions of the most suitable profiles which correspond to different angles of slope and to a given profile are to be found in the following table.

Angle of slope. θ	Relative slope.	Dimensions of transverse profile.				Quotient $\frac{p}{F}$
		Depth a.	Lower breadth b.	Absolute slope na.	Upper breadth $b + 2na$.	
90°	0	$0,707 \sqrt{F}$	$1,414 \sqrt{F}$	0	$1,414 \sqrt{F}$	$\frac{2,828}{\sqrt{F}}$
60°	0,577	$0,760 \sqrt{F}$	$0,877 \sqrt{F}$	$0,439 \sqrt{F}$	$1,755 \sqrt{F}$	$\frac{2,632}{\sqrt{F}}$
45°	1,000	$0,740 \sqrt{F}$	$0,613 \sqrt{F}$	$0,740 \sqrt{F}$	$2,092 \sqrt{F}$	$\frac{2,704}{\sqrt{F}}$
40°	1,192	$0,722 \sqrt{F}$	$0,525 \sqrt{F}$	$0,860 \sqrt{F}$	$2,246 \sqrt{F}$	$\frac{2,771}{\sqrt{F}}$
36° 52′	1,333	$0,707 \sqrt{F}$	$0,471 \sqrt{F}$	$0,943 \sqrt{F}$	$2,357 \sqrt{F}$	$\frac{2,828}{\sqrt{F}}$
35°	1,402	$0,697 \sqrt{F}$	$0,439 \sqrt{F}$	$0,995 \sqrt{F}$	$2,430 \sqrt{F}$	$\frac{2,870}{\sqrt{F}}$
30°	1,732	$0,664 \sqrt{F}$	$0,356 \sqrt{F}$	$1,150 \sqrt{F}$	$2,656 \sqrt{F}$	$\frac{3,012}{\sqrt{F}}$
26° 34′	2,000	$0,636 \sqrt{F}$	$0,300 \sqrt{F}$	$1,272 \sqrt{F}$	$2,844 \sqrt{F}$	$\frac{3,144}{\sqrt{F}}$
Semicircle		$0,798 \sqrt{F}$			$1,596 \sqrt{F}$	$\frac{2,507}{\sqrt{F}}$

We see from this table that the quotient $\frac{p}{F}$ is least for the semicircle, namely, $= \frac{2,507}{\sqrt{F}}$; greater for the semi-hexagon, and greater still for the half square, and the trapezium of 36° 52′, &c.

Example. What dimensions must be given to a profile, which has for an area of 40 square feet, a slope of its banks of 35°? From the preceding table, the depth $a = 0,697 \sqrt{40} = 4,408$, the lower breadth $= 0,439 \sqrt{40} = 2,777$ feet, the absolute slope $= 0,995 \sqrt{40} = 6,293$ feet, the upper breadth $= 15,363$, and the quotient

$$\frac{p}{F} = \frac{2,870}{\sqrt{40}} = 0,4538.$$

§ 367. *Uniform Motion.*—The motion of water in beds is for a certain tract either *uniform* or *variable;* it is uniform when the mean velocity at all transverse sections of this length remains the same, and therefore, also, the areas of the sections equal; and variable, on the other hand, when the mean velocities, and therefore, also, the areas of the sections vary. We shall treat first of uniform motion.

In the uniform motion of water along the distance $AD = l$, Fig. 489, the whole fall $HD = h$ is expended in overcoming the friction of the water in the bed, because the water flows on with the same velocity with which it arrives, therefore, a height due to a velocity is neither taken up nor set free. If we measure this friction by the height of this column of water, we may then make the fall equal to this height. But the height due to the resistance of friction increases with the quotient $\frac{p}{F}$, with l and with the square of the mean velocity c (§ 329); hence then the formula holds good:

$$1.\ h = \zeta \cdot \frac{lp}{F} \cdot \frac{c^2}{2g},$$

in which ζ expresses a number deduced from experiment which may be called the *co-efficient of the resistance of friction.*

By inversion it follows:

$$2.\ c = \sqrt{\frac{F}{\zeta \cdot lp} \cdot 2gh}.$$

In determining the fall, therefore, when the length, the cross section and the velocity are given, and inversely, in deducing the velocity from the fall, the length and the cross section, we must know the co-efficient of friction ζ. According to Eytelwein's reduction of the ninety-one observations of Du Buat, Brünings, Funk and Waltmann, $\zeta = 0{,}007565$, and hence

$$h = 0{,}007565 \cdot \frac{lp}{F} \cdot \frac{c^2}{2g}.$$

If we put $g = 9{,}809$ metres or $31{,}25$ feet ($32{,}2$ feet English), we have for the metrical measure

$$h = 0{,}0003856 \, \frac{lp}{F} \cdot c^2 \text{ and } c = 5{,}09 \sqrt{\frac{Fh}{pl}},$$

and for the foot measure:

$$h = 0{,}00011726 \, \frac{lp}{F} \cdot c^2 \text{ and } c = 92{,}35 \sqrt{\frac{Fh}{pl}} \text{ English measure.}$$

For conduit pipes $\frac{lp}{F} = \frac{\pi l d}{\frac{1}{4} \pi d^2} = \frac{4l}{d}$, hence this formula gives for pipes $h = 0{,}03026 \, \frac{l}{d} \cdot \frac{v^2}{2g}$, whilst we have found more correctly for these (§ 331) for mean velocities

$$h = 0{,}025 \, \frac{l}{d} \cdot \frac{v^2}{2g}.$$

The friction, therefore, as might be expected, is greater in the beds of rivers than in metallic conducting pipes.

Examples. 1. What fall must be given to a canal of the length $l = 2600$ feet, lower breadth $b = 3$ feet, upper breadth $b_1 = 7$ feet, and depth $a = 3$ feet, if it is to conduct a quantity of water of 40 cubic feet per second? It is:

$p = 3 + 2\sqrt{2^2 + 3^2} = 10{,}211$, $F = \frac{(7+3)\,3}{2} = 15$ and $c = \frac{40}{15} = \frac{8}{3}$, hence the fall sought, $h = 0{,}00011726 \, \frac{2600 \cdot 10{,}211}{15} \cdot (\frac{8}{3})^2 = \frac{0{,}305 \cdot 10{,}211 \cdot 64}{15 \cdot 9} = 1{,}476$ feet.—2. What quantity of water does a canal 5800 feet long, having a 3 feet fall, 5 feet deep, 4 feet lower and 12 feet upper breadth? Here:

$$\frac{p}{F} = \frac{4 + 2\sqrt{5^2 + 4^2}}{5 \cdot 8} = \frac{16{,}806}{40} = 0{,}42015;$$

hence the velocity

$$c = 92{,}35 \sqrt{\frac{3}{0{,}42015 \cdot 5800}} = \frac{92{,}35}{\sqrt{0{,}14005 \cdot 5800}} = \frac{92{,}35}{\sqrt{812{,}29}} = \frac{92.35}{28{,}5} = 3{,}24 \text{ feet,}$$

and the quantity of water $Q = Fc = 40 \cdot 3{,}24 = 129{,}6$ cubic feet, English measure.

§ 368. *Co-efficients of Friction.*—The co-efficient of friction for rivers, brooks, &c., the mean value of which, in the foregoing paragraphs, we have taken at 0,007565, is not constant, but, as in pipes, increases somewhat for small and diminishes for great velocities. We have, therefore, to put:

$$\zeta = \zeta_1 \left(1 + \frac{a}{c}\right) \text{ or } \zeta_1 \left(1 + \frac{a}{\sqrt{c}}\right).$$

The author of the work alluded to in § 363, finds from 255 experiments, the greater part of them undertaken by himself, for the Prussian measure $\zeta = 0,007409 \left(1 + \frac{0,0299}{c}\right)$, and hence it follows for the metre $\zeta = 0,007409 \left(1 + \frac{0,00939}{c}\right)$,

and for English measure $007409 \left(1 + \frac{0,0308}{c}\right)$.

It is manifest that these formulæ, for a velocity $c = 1\frac{1}{2}$ feet, give again the above assigned mean co-efficient of resistance $\zeta = 0,007565$. The following useful table of the co-efficients of resistance in the metrical measure serves for facilitating calculation.

Velocity c.	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	Meter.
Co-efficient of resistance $\zeta = 0,00$	811	776	764	758	755	753	751	750	749	

Velocity c.	1	1,2	1,5	2	3	Meter.
Co-efficient of resistance $\zeta = 0,00$	748	747	746	744	743	

The following table serves for the Prussian or English measure:

Velocity c.	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1	$1\frac{1}{2}$	2	3	5	10 ft.
Co-efficient of resistance $\zeta = 0,00$	815	797	785	778	773	769	766	763	759	752	749	745	743

These tables find their direct application in all cases where the velocity c is given and the fall to be found, and where the formula No. 1 of the former paragraph is applicable. But if the velocity c is unknown, and its amount to be determined, these tables will then only admit of a direct application, when we have already an approximate value of c. We may set to work in the simplest manner by determining c approximately by the formula $c = 50,9 \sqrt{\frac{Fh}{pl}}$, and from this a value of ζ, taken from the table, and the value so obtained put into the formula

$$\frac{c^2}{2g} = \frac{h}{\zeta} \cdot \frac{F}{lp} \text{ or } c = \sqrt{\frac{F}{\zeta\, l p} \cdot 2gh}.$$

From the velocity c, the quantity of water is then given by the formula $Q = Fc$.

If, lastly, the quantity and the fall are given, and, as is often requisite in the construction of canals, it be required to determine the transverse section, we may put $\frac{p}{F} = \frac{m}{\sqrt{F}}$ (*see* Table, § 366) and $c = \frac{Q}{F}$ into the formula $h = 0{,}007565 \frac{lp}{F} \cdot \frac{c^2}{2g}$, and write, therefore, $h = 0{,}007565 \frac{m l Q^2}{2gF^{\frac{5}{2}}}$, and accordingly determine:

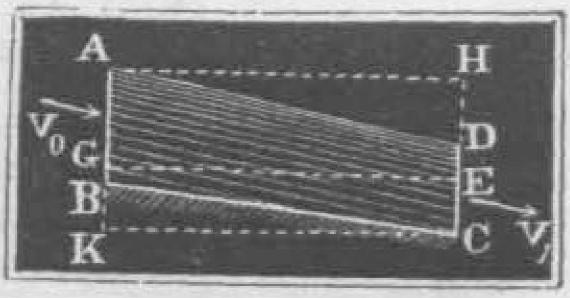
$F = \left(0{,}007565 \frac{m\, l\, Q^2}{2\, gh}\right)^{\frac{2}{5}}$, *i. e.*, for the metre $F = 0{,}0431 \left(\frac{m\, l\, Q^2}{h}\right)^{\frac{2}{5}}$ or the English foot measure $F = 0{,}0268 \left(\frac{m\, l\, Q^2}{h}\right)^{\frac{2}{5}}$. Hence it follows, approximately, that $c = \frac{Q}{F}$; if we take a correspondent value of ζ from one of the tables, more accurately $F = \left(\zeta \cdot \frac{m\, l\, Q^2}{2\, gh}\right)^{\frac{2}{5}}$; and hence, more exact values for $c = \frac{Q}{F}$, $p = m \sqrt{F}$, as also for a, b, &c.

Examples.—1. What fall does a canal 1500 feet long, 2 feet lower and 8 feet upper breadth, and 4 feet depth require to give a discharge of 70 cubic feet per second? It is $p = 2 + 2\sqrt{4^2 + 3^2} = 12$, $F = 5 \cdot 4 = 20$, $c = \frac{70}{20} = 3{,}5$, hence $\zeta = 0{,}00748$, and $h = 0{,}00748 \cdot \frac{1500 \cdot 12}{20} \cdot \frac{3{,}5^2}{2g} = 6{,}732 \cdot 0{,}1902 = 1{,}28$ ft. (Eng.)—2. What discharge does a brook 40 feet broad, $4\frac{1}{2}$ feet mean depth, and 46 feet water profile, if it has a fall of 10 inches for a length of 750 feet? It is about $c = 92{,}35 \cdot \sqrt{\frac{40 \cdot 4{,}5 \cdot 10}{46 \cdot 750 \cdot 12}} = \frac{92{,}35}{\sqrt{230}} = 6{,}089$ feet, and hence $\zeta = 0{,}00745$. Hence we obtain, more correctly: $\frac{c^2}{2g} = \frac{Fh}{\zeta lp} = \frac{4{,}5 \cdot 40 \cdot 10}{0{,}00745 \cdot 46 \cdot 750 \cdot 12} = \frac{1}{1{,}7112} = 0{,}5844$, and $c = 6{,}119$ feet. Lastly, the corresponding discharge is $Q = 4{,}5 \cdot 40 \cdot 6{,}119 = 1101$ cubic feet, (Eng.)—3. A trench 3650 feet long is to be cut, which for a total fall of 1 foot is to carry off a discharge of 12 cubic feet per second, what dimensions are to be given to the transverse profile, if it is to preserve a regular semi-hexagonal figure? Here $m = 2{,}632$ (*see* Table, § 366), hence, approximately, $F = 0{,}0268\ (2{,}632 \cdot 3650 \cdot 144)^{\frac{2}{5}} = 7{,}665$ square feet, and $c = \frac{12}{7{,}665} = 1{,}539$ feet. Hence ζ is to be taken $= 0{,}00758$, and $F = \left(0{,}00758 \cdot 2{,}632 \cdot \frac{3650 \cdot 144}{6{,}44}\right)^{\frac{2}{5}} = 7{,}67$ square feet. Therefore the depth must be made: $a = 0{,}760 \sqrt{F} = 2{,}104$ feet, the lower breadth $= 0{,}877 \sqrt{F} = 2{,}428$, and the upper breadth $= 2 \cdot 2{,}428 = 4{,}846$ English feet.

§ 369. *Variable Motion.*—The theory of the variable motion of water in beds of rivers may be reduced to the theory of uniform motion, provided the resistance of friction for a short length of the river may be considered as constant, and the corresponding height, in like manner, as $= \zeta \cdot \frac{lp}{F} \cdot \frac{v^2}{2g} B$ But, besides this, regard must be had to the *vis viva* of the water, which corresponds to a change of velocity.

Let $ABCD$, Fig. 499, be a short extent of river, of the length $ADB = l$, the fall $DH = h$, and let v_0 be the velocity of the arriving, and v_1 that of the departing water. If we apply the rules of efflux to an element D of the surface, we shall obtain for its velocity v_1,

Fig. 499.



$$\frac{v_1^2}{2g} = h + \frac{v_0^2}{2g};$$

as regards an element E below the surface, it is true that on the one side it has a greater height of pressure $AG = EH$; but as the downstream water reacts with a pressure DE, there remains for it only the fall $DH = EH - ED$, as pressure inducing motion, and so, for this or any other element, the formula:

$$h = \frac{v_1^2 - v_0^2}{2g} \text{ answers;}$$

and if, further, the resistance due to friction be added, we then obtain:

$$h = \frac{v_1^2 - v_0^2}{2g} + \zeta \cdot \frac{lp}{F} \cdot \frac{v^2}{2g},$$

where p, F and v are the mean values of the wetted perimeter, transverse section, and velocity. If F_0 is the area of the upper, and F_1 that of the lower section, we may then put:

$$F = \frac{F_0 + F_1}{2}, \text{ and } Q = F_0 v_0 = F_1 v_1,$$

whence it follows that:

$$\frac{v_1^2 - v_0^2}{2g} = \frac{1}{2g}\left[\left(\frac{Q}{F_1}\right)^2 - \left(\frac{Q}{F_0}\right)^2\right] = \left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g}, \text{ and}$$

$$\frac{v^2}{F} = \frac{v_0^2 + v_1^2}{F_0 + F_1} = \left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{F_0 + F_1}, \text{ we obtain:}$$

$$1.\ h = \left[\frac{1}{F_1^2} - \frac{1}{F_0^2} + \zeta \frac{lp}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\right]\frac{Q^2}{2g}, \text{ as also}$$

$$2.\ Q = \frac{\sqrt{2gh}}{\sqrt{\frac{1}{F_1^2} - \frac{1}{F_0^2} + \zeta \frac{lp}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)}}.$$

The corresponding fall h may be calculated by means of the formula 1, from the quantity of water, the length and transverse section of a river or canal; and, inversely, the quantity of water from the fall, the length and the transverse section, by formula 2. To obtain greater accuracy, we may make the calculation for several short portions of the river, and take the arithmetical mean. If the total fall only is known, we must substitute this at once for h in the last formula, and put

$$\frac{1}{F_1^2} - \frac{1}{F_0^2} = \frac{1}{F_n^2} - \frac{1}{F_0^2},$$

where F_n denotes the area of the last section, and in place of

$$\zeta \cdot \frac{l\,p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)$$

the sum of all similar values of the separate lengths of the river.

Example. A brook has for a distance of 300 feet a fall of 9,6 inches, the mean perimeter of its water profile is 40 feet, the area of the upper transverse profile 70, that of the lower 60 square feet; what quantity of water does this brook discharge? It is

$$Q = \frac{8{,}02\sqrt{0{,}8}}{\sqrt{\frac{1}{60^2} - \frac{1}{70^2} + 0{,}007565 \cdot \frac{300 \cdot 40}{130}\left(\frac{1}{60^2} + \frac{1}{70^2}\right)}} =$$

$$= \frac{7{,}173}{\sqrt{0{,}0000731 + 0{,}0003365}} = \frac{7{,}173}{\sqrt{0{,}0004096}} = 354{,}43 \text{ cubic feet.}$$ The mean velocity is $\frac{2\,Q}{F_0 + F} = \frac{708{,}8}{130} = 5{,}452$ feet; hence, more accurately, ζ must be taken $= 0{,}00745$ in place of $0{,}007565$, and therefore more nearly:

$$Q = \frac{7{,}173}{\sqrt{0{,}0000731 + 0{,}0003314}} = 357{,}5 \text{ cubic feet.}$$ If the same brook, with the same head of water, had for a length of 450 feet, a fall of 11 inches, and if its upper transverse profile had an area of 50 and its lower of 60 square feet, and the mean perimeter of the profile measured 36 feet, we should then have:

$$Q = \frac{8{,}02\sqrt{0{,}9167}}{\sqrt{\frac{1}{60^2} - \frac{1}{50^2} + 0{,}00745 \cdot \frac{450 \cdot 36}{110}\left(\frac{1}{60^2} + \frac{1}{50^2}\right)}},$$

$$= 8{,}02\sqrt{\frac{0{,}9167}{-0{,}0001222 + 0{,}0007436}} = 308 \text{ cubic feet.}$$

The mean of these two values is $Q = \frac{357{,}5 + 308}{2} = 332{,}75$ cubic feet.

§ 370. In order to obtain a formula for the depth of water, let the upper depth $= a_0$ and the lower $= a_1$, the slope of the bed $= \alpha$, consequently the fall of the bed $= l \sin. \alpha$. We then obtain the fall of the water $h = a_0 - a_1 + l \sin. \alpha$, and there results the equation:

$$a_0 - a_1 - \left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g} = \left[\zeta\,\frac{p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g} - \sin. \alpha\right] l,$$

hence
$$l = \frac{a_0 - a_1 - \left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g}}{\zeta\,\frac{p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g} - \sin. \alpha}.$$

The length l which corresponds to a difference $a_0 - a_1$ of the depth of water, may be determined by this formula. But if the reverse problem is to be solved, we must do it by the method of approximation, and first determine the distances l_1 and l_2 corresponding to the assumed depressions $a_0 - a_1$, and $a_1 - a_2$, and from these calculate by a proportion, the depression corresponding to a given distance l.*

* See "Ingenieur," Arithmetik, § 16, v.

The formula is further capable of simplification when the breadth b of the running water is constant, or may be considered as such. In this case we put:

$$\left(\frac{1}{F_1^2}-\frac{1}{F_0^2}\right)\frac{Q^2}{2g0}=\frac{F_0^2-F_1^2}{F_0^2F_1^2}\cdot\frac{Q^2}{2g0}=\frac{(F_0-F_1)(F_0+F_1)}{F_1^2}\cdot\frac{v_0^2}{2g}$$

$$=\frac{(a_0-a_1)(a_0+a_1)}{a_1^2}\cdot\frac{v_0^2}{2g}\text{ approximately }=2\,\frac{(a_0-a_1)}{a_0}\cdot\frac{v_0^2}{2g},$$

and likewise:

$$\frac{p}{F_0+F_1}\left(\frac{1}{F_0^2}+\frac{1}{F_1^2}\right)\frac{Q^2}{2g}=\frac{p\,(F_0^2+F_1^2)}{(F_0+F_1)F_1^2 0}\cdot\frac{v_0^2}{2g}\text{ approximately}$$

$$=\frac{p}{a_0 b}\cdot\frac{v_0^2}{2g},\text{ hence } l=\frac{(a_0-a_1)\left(1-\frac{2}{a_0}\cdot\frac{v_0^2}{2g}\right)}{\zeta\cdot\frac{p}{a_0 b}\cdot\frac{v_0^2}{2g}-sin.\,\alpha},\text{ and hence}$$

$$\frac{a_0-a_1}{l}=\frac{\zeta\cdot\frac{p}{a_0 b}\cdot\frac{v_0^2}{2g}-sin.\,\alpha.}{1-\frac{2}{a_0}\cdot\frac{v_0^2}{2g}}.$$

The difference (a_0-a_1) of the depth corresponding to a given extent l may be calculated directly by this formula.

Example. In a horizontal trench, 5 feet broad and 800 feet long, it is desired to carry off a 20 cubic feet discharge, and to let it flow in at a depth of 2 feet, what depth will the water at the end of the canal have? Let us divide the whole length into two equal portions, and determine from the last formula the fall for each of them.

Here the $sin.\,\alpha=0$, $l=\frac{800}{2}=400$, and $b=5$; for the first portion $v=\frac{20}{2\,.\,5}=2$, hence $\zeta=0{,}00752$, also $a_0=2$; since $p=8\frac{1}{2}$, it follows that $a_0-a_1=$

$$\left(\frac{0{,}00752\,.\,\frac{8{,}5}{10}\,.\,\frac{4}{2g}}{1-\frac{2}{2}\,.\,\frac{4}{2g}}\right).\,400=\frac{0{,}15877}{0{,}9379}=0{,}1692\text{ feet. Now, for the second half, } a_1=$$

$2-0{,}1692=1{,}8308$, and $p_1=8{,}2$, $v_1=\frac{20}{9{,}154}=2{,}1848$, and the depression of the second portion:

$$a_1-a_2=\left(\frac{0{,}00752\,.\,\frac{8{,}2}{9{,}154}\,.\,\frac{2{,}1848^2}{2g}}{1-\frac{2}{1{,}8308}\,.\,\frac{2{,}1848^2}{2g}}\right).\,400=\frac{0{,}1997}{0{,}919}=0{,}2173\text{ feet, hence the}$$

whole depression $=0{,}1692+0{,}2173=0{,}3865$, and the depth of water at the lower end $=2-0{,}3865=1{,}6135$.

§ 371. *Floods.*—When the depth of water in rivers and canals varies, variations in the velocity and discharge take place likewise. A greater depth of water not only involves a greater section, but also a greater velocity, and hence, for two reasons, a greater quantity of water, and likewise a diminution of the depth of water, gives a diminution of the section and the velocity, and hence also a decrease of the discharge. If the original depth $=a$, and any increased depth

$= a_1$, the upper breadth of the canal $= b$, then the augmentation of the section may be put $= b\,(a_1 - a)$, and hence the section afterwards $a_1 - a$, $F_1 = F + b\,(a_1 - a)$, it also follows from this that

$$\frac{F_1}{F} = 1 + \frac{b\,(a_1 - a)}{F}, \text{ and}$$

$$\sqrt{\frac{F_1}{F}} \text{ approximately} = 1 + \frac{b\,(a_1 - a)}{2\,F}.$$

If further p be the original, p_1 the increased perimeter of the water profile, and Θ the angle of slope of the banks, then

$$p_1 = p + \frac{2\,(a_1 - a)}{p\ sin.\ \Theta}, \text{ hence } \frac{p_1}{p} = 1 + \frac{2\,(a_1 - a)}{p\ sin.\ \Theta}, \text{ and}$$

$$\sqrt{\frac{p_1}{p}} = 1 + \frac{a_1 - a}{sin.\ \Theta}, \text{ as also } \sqrt{\frac{p}{p_1}} = 1 - \frac{a_1 - a}{p\ sin.\ \Theta}.$$

Now the velocity with the first depth of water is

$$c = 92{,}35\sqrt{\frac{Fh}{pl}}, \text{ and with the second } c_1 = 92{,}35\sqrt{\frac{F_1}{p_1} \cdot \frac{h}{l}},$$

hence we may put:

$$\frac{c_1}{c} = \sqrt{\frac{F_1}{F}} \cdot \sqrt{\frac{p}{p_1}} = \left(1 + \frac{b\,(a_1 - a)}{2\,F}\right)\left(1 - \frac{a_1 - a}{p\ sin.\ \Theta}\right)$$

$$= 1 + (a_1 - a)\left(\frac{b}{2F} - \frac{1}{p\ sin.\ \Theta}\right),$$

therefore the relative change of velocity:

$$1.\quad \frac{c_1 - c}{c} = (a_1 - a)\left(\frac{b}{2\,F} - \frac{1}{p\ sin.\ \Theta}\right).$$

On the other hand, the ratio of the discharge is:

$$\frac{Q_1}{Q} = \frac{F_1 c_1}{Fc} = \left(1 + \frac{b\,(a_1 - a)}{F}\right)\left[1 + (a_1 - a)\left(\frac{b}{2\,F} - \frac{1}{p\ sin.\ \Theta}\right)\right]$$

$$= 1 + (a_1 - a)\left(\frac{3\,b}{2F} - \frac{1}{p\ sin.\ \Theta}\right),$$

and the relative increase:

$$2.\quad \frac{Q_1 - Q}{Q} = (a_1 - a)\left(\frac{3\,b}{2F} - \frac{1}{p\ sin.\ \Theta}\right).$$

Less accurately, but in many cases, especially in broad canals with little slope, we may put $F = ab$, and neglect $\frac{1}{p\ sin.\ \Theta}$, whence it follows more simply that:

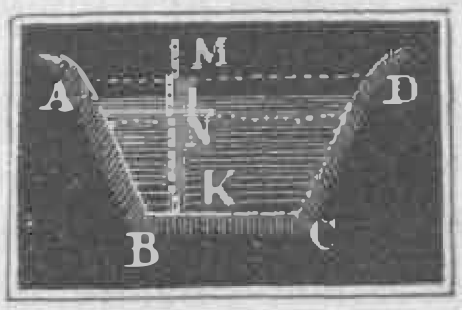
$$\frac{c_1 - c}{c} = \tfrac{1}{2}\,\frac{a_1 - a}{a}, \text{ and}$$

$$\frac{Q_1 - Q}{Q} = \tfrac{3}{2} \cdot \frac{a_1 - a}{a}.$$

From this, therefore, the relative change of velocity is $\tfrac{1}{2}$, and the relative change in the quantity of water $\tfrac{3}{2}$, that of the relative change in the depth of water.

Examples.—1. When the head of water increases $\tfrac{1}{10}$ of its original amount, the velocity is then $\frac{1}{20}$, and the quantity $\frac{3}{20}$ greater than its original value.—2.—When the depth diminishes 8 per cent., the velocity then diminishes 4, and the quantity 12 per cent.—3. From the more correct formula:

Fig. 500.



$$\frac{Q_1 - Q}{Q} = (a_1 - a)\left(\frac{3\,b}{2F} - \frac{1}{p\ sin.\ \Theta}\right)$$

a scale of the depth of water KM, Fig. 500, may be constructed, on which the discharge of a canal corresponding to any depth KL, may be read off, when the quantity of water for a certain mean depth is once known. If $b = 9$ feet, $b_1 = 3$, $a = 3$, and $\Theta = 45^{\circ}$, we then have $F = \frac{(9 + 3)\,3}{2} = 18$ square ft., $p = 3 + 2 \cdot 3\sqrt{2} = 11{,}485$ and $sin.\ \Theta = \sqrt{\tfrac{1}{2}} = 0{,}707$, hence:

$$\frac{Q_1 - Q}{Q} = \left(\frac{3 \cdot 9}{2 \cdot 18} - \frac{1}{11{,}485 \cdot 0{,}707}\right)(a_1 - a) = (0{,}750 - 0{,}123)\,(a_1 - a) = 0{,}627$$

$(a_1 - a)$. If the quantity corresponding to a mean head of water $Q = 40$ cubic feet, we then have $Q_1 = 40 + 40 \cdot 0{,}627\,(a_1 - a) = 40 + \frac{a_1 - a}{0{,}04}$. If $a_1 - a = 0{,}04$ feet $= 5{,}76$ lines, it follows that $Q_1 = 41$; $a_1 - a = 0{,}08$ feet $= 11{,}52$ lines, we then have $Q_1 = 42$ cubic feet; if, further, $a_1 - a = -0{,}04$, then is $Q_1 = 39$ cubic feet, &c. A scale, therefore, whose intervals are $LM = LN = 5{,}76$ lines, gives the discharge accurately to a cubic foot. Of course the accuracy is the less, the more the head of water differs from a mean value.

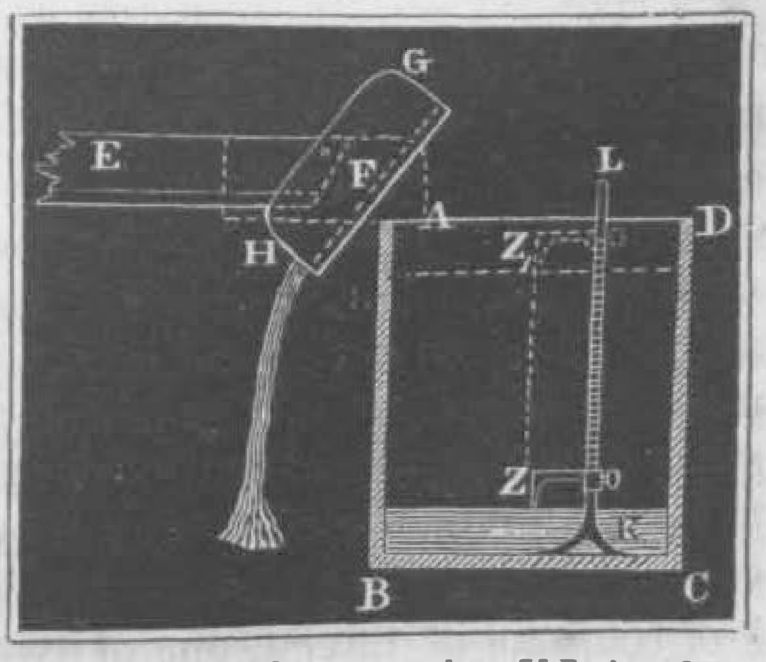
Remark. The conducting and carrying off of water in canals, as well as the subject of weirs and dams, will be fully treated of in the Second Part.

CHAPTER VIII.

HYDROMETRY, OR THE DOCTRINE OF THE MEASUREMENT OF WATER.

§ 372. *Gauges.*—The quantity, which a stream discharges in a certain time, is determined either by a gauge, by an apparatus of efflux, or by an hydrometer. The most simple way of measuring water is by the guage, *i. e.* by the use of a graduated vessel, but this method is only applicable to small discharges, carried off by pipes or small brooks, or drains. The gauge vessel is generally made of wood, and of a rectangular form, and to increase its strength is bound round with iron-hooping. The water is conducted into it by a trough EF, Fig. 501, at whose extremity there is a double valve GH, by which the water may be made to flow at will into the vessel AC, or by the side of it. To obtain the exact depth of the body of water in the vessel, a scale KL is fur-

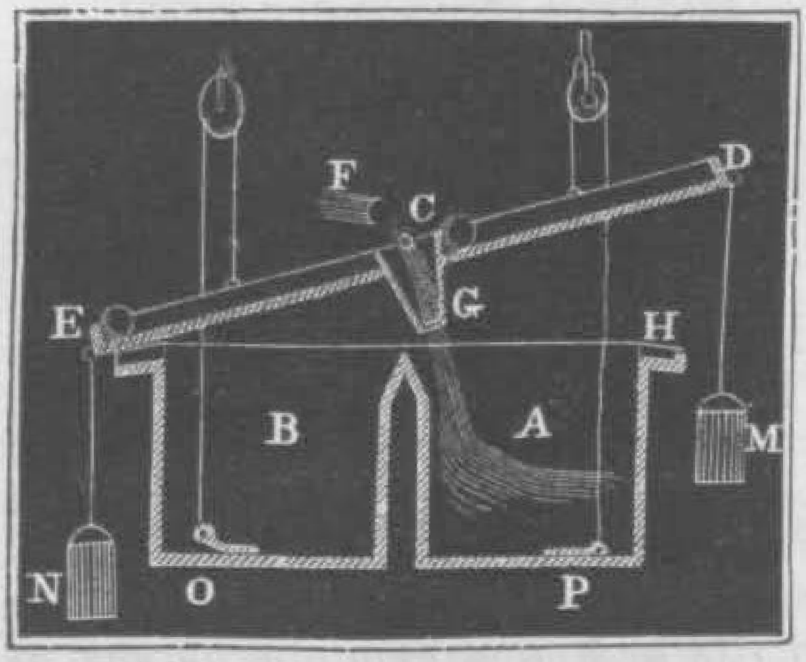
Fig. 501.



ther applied. If before measurement, the index Z be moved down to the surface of the water, already in the vessel, and merely covering the bottom, and the head of water read off from the scale, we shall obtain the height ZZ_1 of the gauged water by subtraction of this from the head of water which the scale indicates when the index hand Z_1 is brought into contact with the surface of water at the end of the observation. Before measurement, the valve must be so placed that the water may flow off outside the vessel. When we are convinced that the efflux in the trough is in a state of permanency, and, watch in hand, have noted a certain moment, the valve must then be turned, so that the water may run into the gauge vessel, and after it is either partly or entirely filled, a second interval is noted by the watch, and the valve again brought into its first position. From the mean section F of the vessel, and the depth $ZZ_1 = a$ of the body of water, the whole quantity $= Fa$ may be estimated, and again from the time of filling t, given by the difference of the times observed, the quantity of water per second $Q = \frac{Fa}{t}$.

Remark. To determine a variable quantity of efflux at each period of the day, we may make use of the apparatus represented in Fig. 502, as applicable especially in salt works. There are here two gauge vessels, A and B, which alternately fill and empty themselves, and the water which is conducted by the pipe F passes through a short pipe CG, which is rigidly connected with a lever DE, revolving about C. When one vessel A becomes filled, the water then flows through a short tube H into the little vessel M, this draws the lever down again on one side, and the pipe CG comes into such a position that the water is conducted into B. The drawing up of the valves O and P takes place by means of strings passing over pulleys, whose extremities are connected with the lever, and sustained by iron balls, which impart a final impulse to the descent of the lever. The vessels M and N have small efflux orifices, by which they empty themselves after each reversion of the lever. An apparatus is besides applied, by which the number of strokes may be read off at any time.

Fig. 502.



§ 373. *Efflux Regulators.*—Small and medium discharges are very frequently determined by means of their flow through a definite orifice, and under a known head. From the area F of the orifice, the head of water h, and the efflux co-efficient μ, the discharge per second $Q = \mu F \sqrt{2gh}$ is given. The Poncelet orifices are those best adapted for this purpose, because the co-efficients of efflux of these under different heads of water are known with great accuracy (§ 316), still they are applicable only to certain medium discharges. The author availed himself of four such orifices for his measurements, one of five, one of ten, one of fifteen, one of twenty centimetres depth, but all of twenty centimetres width. These orifices are cut out of brass plate, and fixed to a wooden frame AC, Fig. 503, which is fastened by four strong iron screws to each wall. In many cases, indeed, greater orifices, the co-efficients of efflux for which are not so accurately determined, and sometimes wiers must be used, which admit generally of a still less accuracy. In all cases, however, the rule holds good, that we must endeavor to get as complete and perfect a contraction as possible, and for this reason must give to the orifice, if it is in a thick plate, a slope outward. The corrections which must be applied for incomplete and partial contraction, have been sufficiently distinguished in paragraphs 319, 320, &c. To measure the water of a brook we must set the frame with its orifice, and wait for the moment when the head of water is permanent. For the measurement of the head of water we must avail ourselves of the index scale, Fig. 500, or of the movable scale EF, Fig. 505. If we would note the efflux directly from the apertures of sluices, it is better to fix before hand a pair of brass scales BC and DE, Fig. 504, with

Fig. 503.

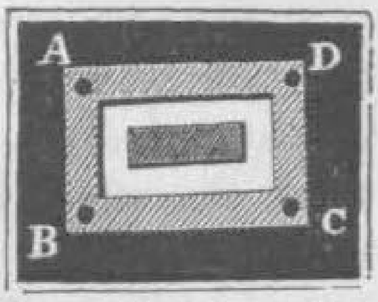


Fig. 504.

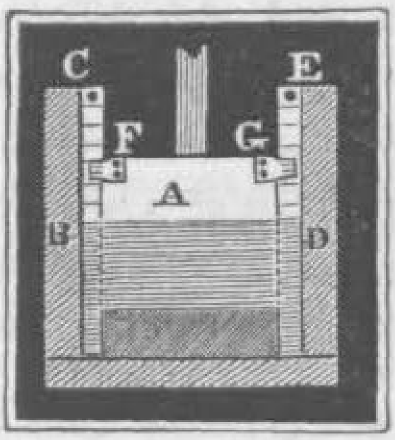
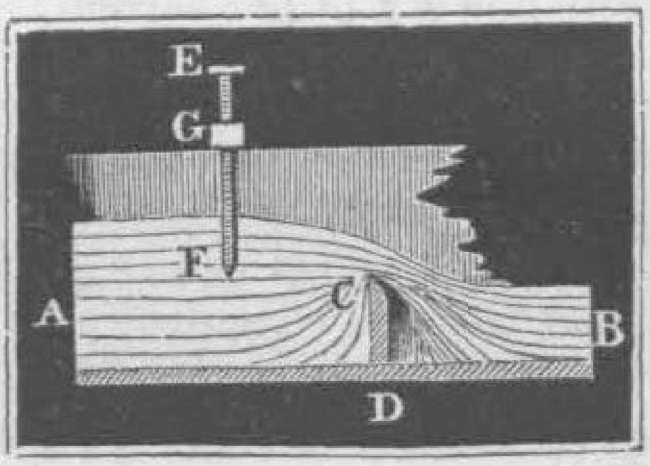


Fig. 505.



their indices F and G to the slide, and to the sluice-board A, in order to read off more safely the height of the aperture. It is generally better for the purpose of measuring water, to put on a new sluice-board with its guide, and with the requisite slope outwards. The simplest means of measuring water in a channel, consists in putting in a board CD, with its upper edge sloped off, Fig. 505, and measuring the fall produced by it. If the channel is long, and there is little rise, it is generally some time before the condition of permanency takes place, and it is for this reason good, before measurement, to put on a second board, so as to impede the efflux of water for a long time, in order to accelerate the rise to a height corresponding to a state of permanency.

To measure the quantity of water of a brook, we may dam it up with posts and boards as in Fig. 506, and let the water C run off through an aperture, or we may avail ourselves of a simple overfall or wier, but of this we shall treat in the second part.

Fig. 506.

§ 374. But as it is often long before a state of permanency occurs in water dammed up by this construction, we may adopt with advantage the following method, first proposed by Prony. We may first close entirely the aperture by a sluice-board, and let the water rise to some height, or as high as circumstances will admit, then draw it so far up that more water may flow in than out, and measure the heads of water at equal and very short intervals; lastly, the aperture of the sluice must be again perfectly closed, and the time t in which the water rises to the first height, further noted. In each case, then, during the whole time of observation $t+t_1$, as much water flows in as out, and hence the quantity flowing in, in the time $t+t_1$, may be expressed by the quantity flowing out in the time t_1. If the heads of water during the depressions are h_0, h_1, h_2, h_3, and h_4, we have then the mean velocity of efflux:

$$v = \frac{\sqrt{2g}}{12}\left(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}\right) \text{ (see § 351)},$$

and if the area of the aperture $= F$, we have then the quantity of efflux in the time t:

$$V = \frac{\mu F t \sqrt{2g}}{12}\left(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}\right), \text{ and the}$$

quantity flowing in per seconds

$$Q = \frac{V}{t+t_1} = \frac{\mu F t \sqrt{2g}}{12(t+t_1)}\left(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}\right).$$

Example. To measure the water of a brook used for the driving of a water-wheel, which has been dammed up by a sluice, Fig. 506, after opening the rectangular aperture, the following is observed: the original head of water is 2 feet, after 30″ 1,8 feet, after 60″ 1,55 feet, after 90″ 1,3 feet, after 120″ 1,15 feet, after 150″ 1,05 feet, and after 180″ 0,9 feet, breadth of the aperture 2 feet, depth $\frac{1}{2}$ foot, time of rising to the first height with closed aperture $=$ 110″. The mean velocity of efflux is:

$$v = \frac{8,02}{18}\left(\sqrt{2} + 4\sqrt{1,8} + 2\sqrt{1,55} + 4\sqrt{1,3} + 2\sqrt{1,15} + 4\sqrt{1,05} + \sqrt{0,9}\right) =$$
$$0,440\,(1,414 + 5,364 + 2,490 + 4,561 + 2,145 + 4,099 + 0,949) = 0,440 \,.\, 21,022 =$$

9,248 feet; but now $F = 2 \,.\, \frac{1}{2} = 1$ square foot, hence it follows that the theoretical discharge is $=$ 9,248 cubic feet. If the co-efficient of efflux is taken $=$ 0,61, we finally obtain the quantity of water sought:

$$Q = \frac{0,61 \,.\, 180}{180 + 110}\, 9,248 = 3,5015 \text{ cubic feet, (English.)}$$

§ 375. *The "Pouce d'Eau," or Water-Inch.*—To measure small discharges, we avail ourselves of the flow through round 1 inch wide orifices, in a thin plate, under a given pressure. The discharge given through such an aperture under the least pressure, or when the surface is only a line above the uppermost position of the orifice, is called an *inch of water*. The French assume for the water-inch (old Paris measure) 15 pints, or 19,1953 cubic metres of water in the 24 hours; therefore in 1 hour 0,7998, and in 1 minute 0,0133 cubic metres; yet older data, by Mariotte, Couplet, and Bossut, vary not a little from the above. According to Hagen, an inch of water (Prussian measure) delivers in 24 hours 520 cubic feet, therefore, in a minute, 0,3611 cubic feet. The double water modulus of Prony, which corresponds to an orifice of 2 centimetres diameter, with a pressure of 5 centi-

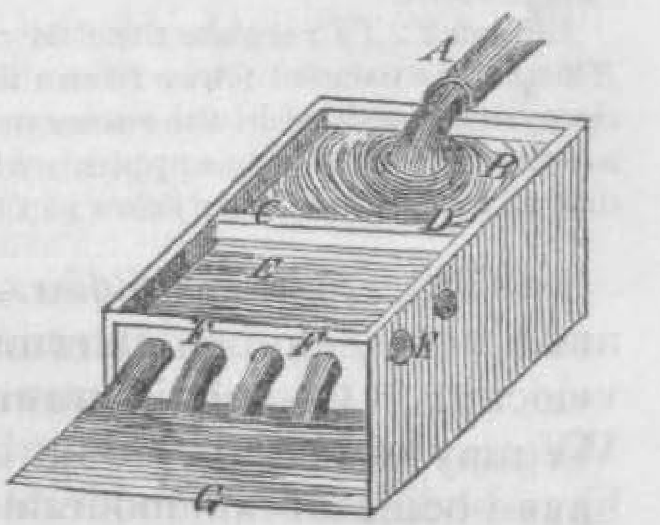


Fig. 507.

metres, and discharges 20 cubic metres of water in 24 hours, has not been adopted. The apparatus by which water is measured by the inch is represented in Fig. 507. The water to be measured flows through the tube *A* into a box; from this it passes through holes below in the partition *CD* into the box *E*, and from this through a horizontal row of round orifices *F*, of exactly 1 inch width, and cut in tin plate, into the reservoir *G*. That the surface of water may stand a line above the heights of these orifices, it is necessary that there be a sufficient number of them, and that a part of them be closed by stoppers. For great discharges the whole water is divided, and in this way a part, only one-tenth, is measured. This division may be accomplished easily, by first conducting the water into a reservoir, with a certain number of orifices at the same level, and only to receive the quantity delivered by one orifice in the apparatus represented above.

Remark 1. We may apply also cocks and other regulating apparatus to the measurement of water, if we know the coefficient of resistance for each position. If h is the head of water, F the transverse section of the pipe, and μ the co-efficient of efflux, for a cock quite opened, we then have the discharge $Q = \mu F \sqrt{2 g h}$, as inversely, $\mu = \frac{Q}{F \sqrt{2 g h}}$ and $\frac{1}{\mu^2} = \left(\frac{F}{Q}\right)^2 . 2 g h$. If now we put the co-efficient of resistance corresponding to a position of the cock, and taken from the tables already given $= \zeta$, we then have the corresponding discharge:

$$Q_1 = F \sqrt{\frac{2 g h}{\frac{1}{\mu^2} + \zeta}} = \frac{\mu F \sqrt{2 g h}}{\sqrt{1 + \mu^2 \zeta}} = \frac{Q}{\sqrt{1 + \zeta \left(\frac{Q}{F}\right)^2 . \frac{1}{2 g h}}}.$$

For convenience sake, we may construct for ourselves a table, so that we can find at a glance the discharge corresponding to a position of the cock, or the position of the cock corresponding to a given discharge. If, for example, $\mu = 0{,}7$ and $F = 5$ square inches, we have then:

$$Q_1 = \frac{0{,}7 . 4 . 12 . 8{,}02 \sqrt{h}}{\sqrt{1 + 0{,}49 . \zeta}} = 269{,}5 \sqrt{\frac{h}{1 + 0{,}49 \zeta}} \text{ cubic inches,}$$

or if h is constantly 1 foot, $Q_1 = \frac{269{,}5}{\sqrt{1 + 0{,}49 \zeta}}$. If now the positions of the cock are at 5°, 10°, 15°, 20°, 25°, &c., the co-efficients of resistance, 0,057; 0,293; 0,758; 1,559;

Fig. 508.

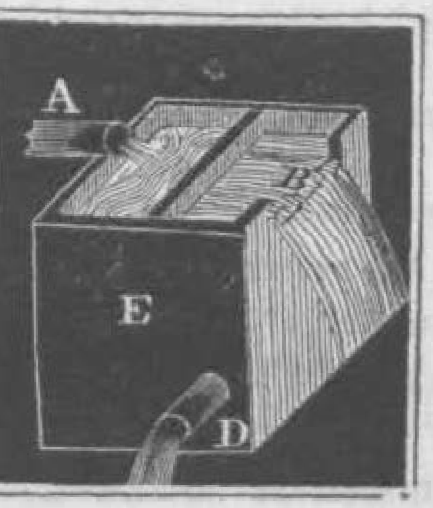
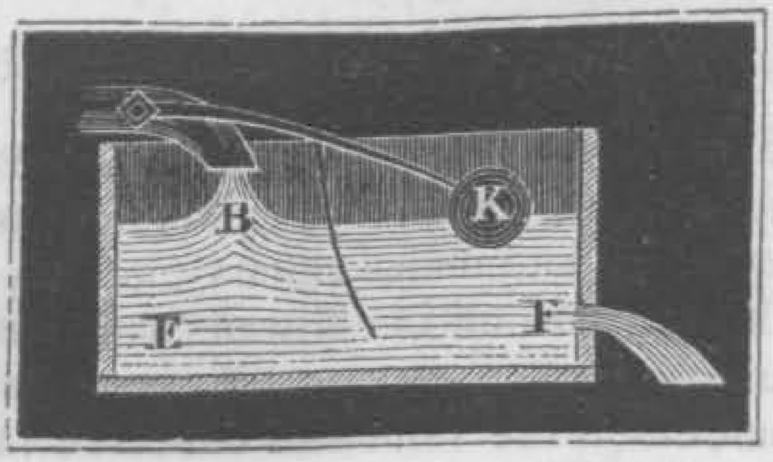


Fig. 509.



3,095, the discharges corresponding to these are: 265,8; 252; 230,1; 202,8; 169,9 cubic inches.

Remark 2. To regulate the flow through an orifice D, Fig. 508, we may apply a weir B that the excess of water from the pipe A may flow over, and that a constant pressure may be maintained in the reservoir DE. That there may be no loss of water, a cock or a valve A, Fig. 509, is applied, which is regulated by a float K acting upon a lever, so that as much water only flows in through B as flows out through F.

§ 376. *Floating Bodies.*—The discharge of large brooks, canals, and rivers, can be determined only by an hydrometer indicating the velocity. Of such instruments floating bodies are the most simple. We may use any floating body for this purpose, but it is better to have bodies of a moderate size, which are only a little specifically lighter than water. Substances of about $\frac{1}{10}$ of a cubic foot content are large enough. Very large ones do not easily assume the velocity of water, and very small ones again, especially when much above the water, are easily disturbed in their motion by accidental circumstances, sometimes by the air on the surface of the fluid. Often, plain pieces of wood are sufficient; it is better, however, if they have a coating of some bright varnish, and better still if the floats are hollow, such as glass flasks, tin balls, &c., because these may be filled at will with water. Swimming balls are most frequently used. They are from 4 to 12 inches diameter, and made of brass, and painted over with some light oil-color, to make them more visible to the eye, and have an opening with a neck, that they may be filled with water and stopped. A floating ball, such as A, Fig. 510, gives only the velocity at the surface, and often only that of the main current; but by suspending two balls one to the other, A and B, Fig. 511, we may determine the velocity at different depths. In this case, the one ball B, which swims under water, is quite filled with the fluid; the other, however, which swims on the surface, is only filled just enough to make it float a little above the surface. Both balls are connected with each other by a string or wire, or by a light wire chain. The velocity c_0 of the surface is first determined by the single ball, and then the mean velocity of the two observed by the connection of balls.

Fig. 510.

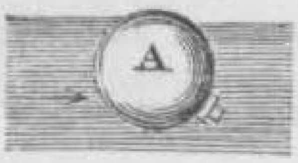
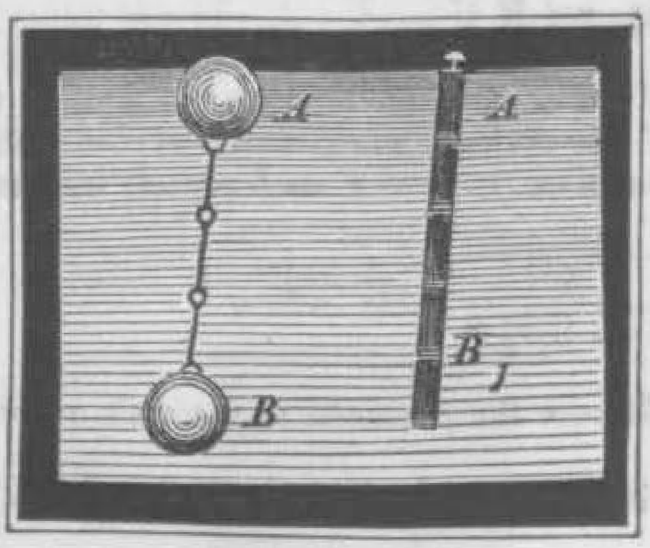


Fig. 511.



If, now, the velocity at the depth of the second ball be denoted by c_1, we may then put $c = \frac{c_0 + c_1}{2}$, and hence, inversely, $c_1 = 2\,c - c_0$. Whilst now both balls are connected with one another by longer and longer wires, we may, in this manner, find the velocities at greater depths. The mean velocity c of a perpendicular is otherwise given if the second ball is allowed to swim a little above the bottom, and c_1 is made $= 2\,c_2 - c_0$; still more accurately, however, if for c_1 the mean of all the velocities observed in a perpendicular be taken.

To find the mean velocity in a perpendicular, the floating staff $A_1 B_1$, represented in Fig. 512, is used. This is particularly convenient for measurements in canals and cuts when it is composed of short pieces screwed together. The floating staff which the author uses is composed of 15 hollow portions, each 1 decimetre in length. That this may swim pretty nearly upright, the lowermost piece is loaded with shot, so that the top just rises above the water. The number of pieces composing the staff, depends, of course, on the depth of the canal.

Both with the floating staff as well as the connection of balls, it may be observed that the velocity at the surface, when the motion of the water in beds is unimpeded, is greater than at the bottom, because the top of the staff swims in advance of the bottom, and the upper ball in advance of the lower. In contraction only, for example, when the water is dammed up by piles, &c., does the contrary take place.

Remark. As a rule, especially with large and floating bodies, as ships, &c., the velocity of the swimming body is somewhat greater than that of the water; not so much because these bodies in swimming float down an inclined plane formed by the surface of the water, but because they take none, or scarcely any, part in the irregular intimate motion of the water; still, the variation for small floating bodies is so slight that it may be neglected.

§ 377. The velocity of a floating ball is found by noting the time t with a good seconds watch, or a half-second pendulum (§ 247), which it takes while floating on the water to describe a measured distance s, marked out on the banks. Then the required velocity of the ball is $c = \frac{s}{t}$. That the time t corresponding to the space described along the bank may be accurately found, it is necessary, with the assistance of a cross line or lines, to erect at the opposite bank two signal staves C and D, perpendicular at A and B, Fig. 512. If we place ourselves behind A, we may then note the moment when the float K, dropped in a little above A, comes into the line AC, and if behind B, we may then also observe the time by a watch held in the hand, when

Fig. 512.

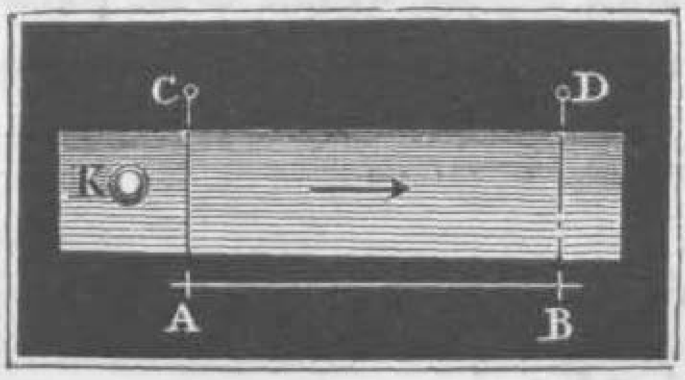
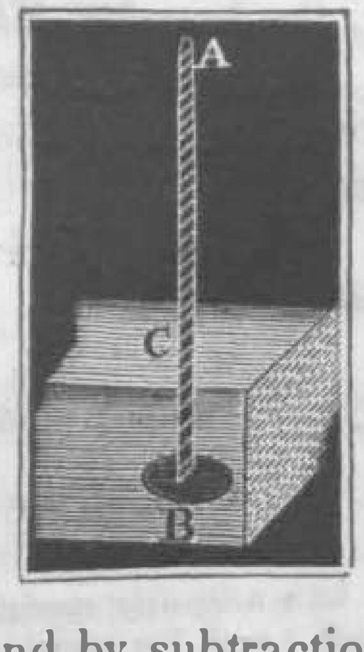


Fig. 513.



the float reaches the line BD, and we then find by subtraction of the times of observation, the required time t corresponding to the describing of the space s. Besides the mean velocity c of the water, the area

F of the transverse profile is further required for determining the quantity of water $Q = Fc$. To find this area, it is necessary to know the breadth and the mean depth of the water. The depths are measured by a sounding rod AB, Fig. 513, having a rhomboidal section, and a board B at the foot; for greater depths we may also use a sounding chain, at whose extremity there is an iron plate, which, in sinking, rests on the bottom. The breadth and the abscissæ corresponding to the measured depths, or the distances from the banks in canals and small brooks EFG, Fig. 514, are found by stretching across a mea-

Fig. 514.

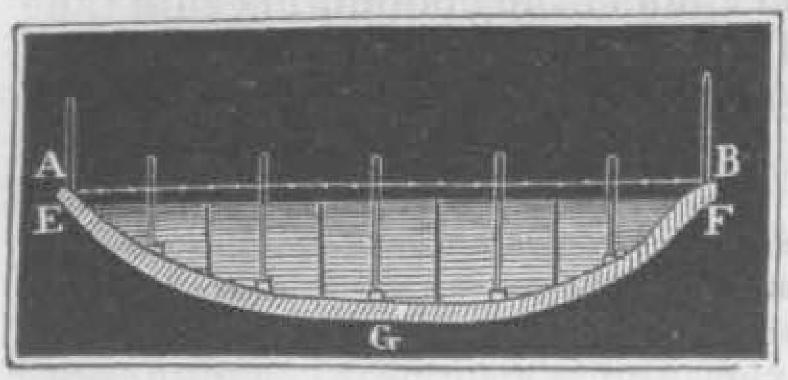
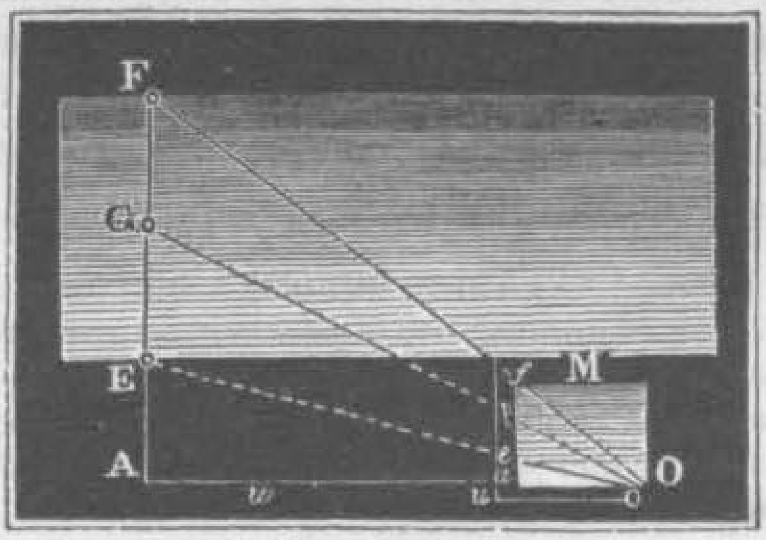


Fig. 515.



suring chain AB, or the placing of a rod right across the running water. For broad rivers this is determined by a measure table M, which is placed at a proper distance AO, from the section EF, Fig. 515, which is to be measured. If ao is the distance AO between A and O, reduced to the table, and if ao is placed in the direction of AO, and thereby also the direction of the breadth af made parallel to the line of breadth AF marked out, then each line of vision will intersect in the direction of the points E, F, G, &c., in the profile, the corresponding points e, f, g on the table, and ae, af, ag, &c., are the distances AE, AF, AG, &c., in the reduced measure. It is not, therefore, necessary on putting in the sounding rod, and measuring the depths by it, to measure the distances of the corresponding points of the banks, if the engineer standing by the measure table looks at the sound on its being put in, in the line EF.

If, now, the breadth EF, Fig. 514, of a transverse profile, consist of parts b_1, b_2, b_3, &c., and the mean depths within those parts a_1, a_2, a_3, and the mean velocities c_1, c_2, c_3, &c., we have then the area of the profile:

$$F = a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots,$$

the discharge:

$$Q = a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3 + \ldots,$$

and, lastly, the mean velocity:

$$c = \frac{Q}{F} = \frac{a_1 b_1 c_1 + a_2 b_2 c_2 + \ldots}{a_1 b_1 + a_2 b_2 + \ldots}.$$

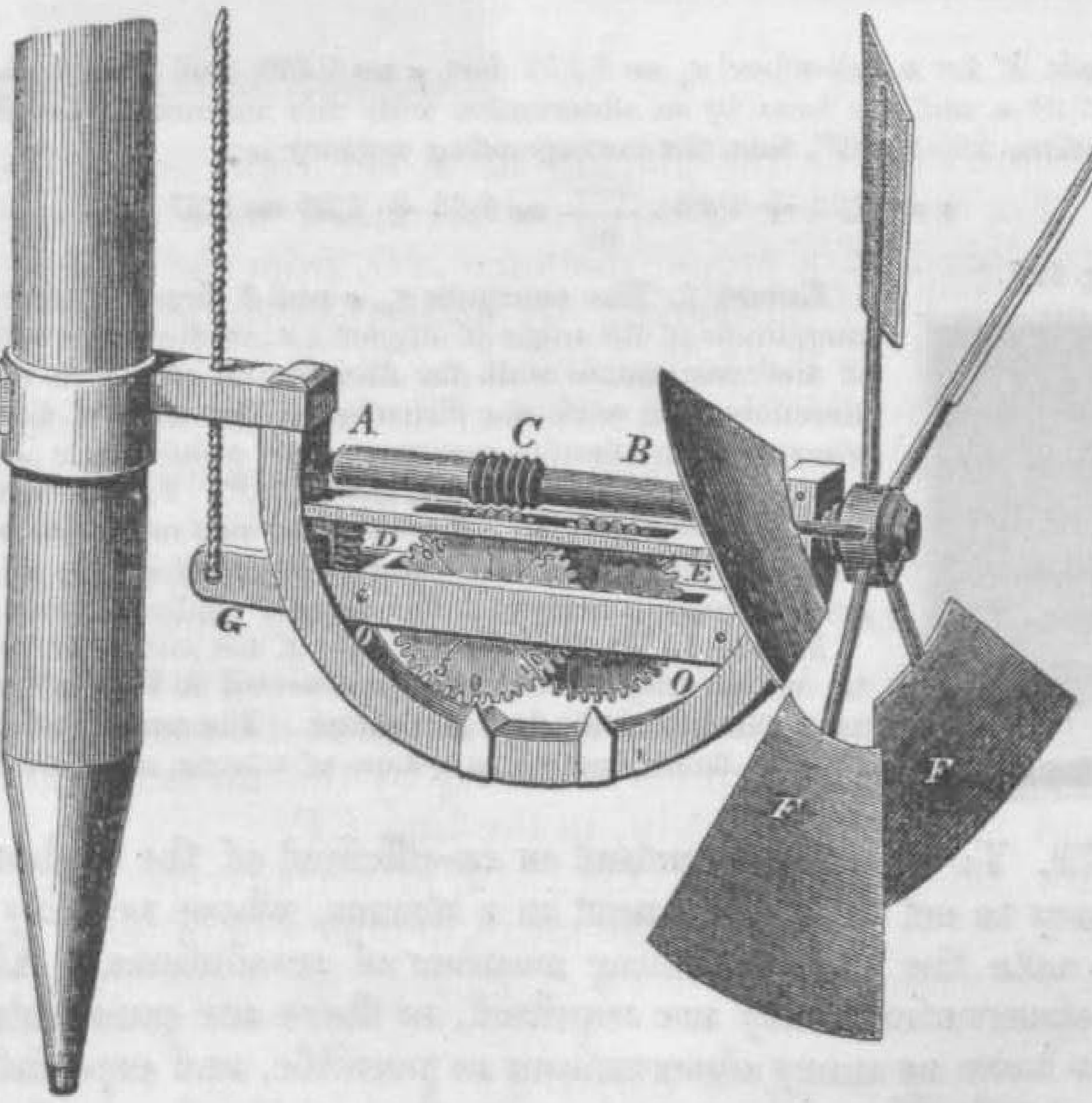
Example. In a tolerably straight and uniform extent of river, we have at the middle points of portions of the breadth:

	5 feet,	12 feet.	20 feet,	15 feet,	7 feet,
The depths	3 "	6 "	11 "	8 "	4 "
The mean velocities . .	1,9 "	2,3 "	2,8 "	2,4 "	2,1

Hence we may put:
The area of the profile $F = 5 \,.\, 3 + 12 \,.\, 6 + 20 \,.\, 11 + 15 \,.\, 8 + 7 \,.\, 4 = 455$ square feet.
The quantity of water $Q = 15 \,.\, 1{,}9 + 72 \,.\, 2{,}3 + 220 \,.\, 2{,}8 + 120 \,.\, 2{,}4 + 28 \,.\, 2{,}1 = 1156{,}9$ cubic feet. The mean velocity $c = \frac{1156{,}9}{455} = 2{,}54$ feet.

§ 378. *The Tachometer.*—The most eligible hydrometer is the tachometer of Woltmann, Fig. 516. It consists of a horizontal axle *AB*, with from two to five vanes *F*, placed at an inclination to the direction of the axis, and gives, when immersed in the water and held at right angles to the direction of motion, by the number of its revolutions in a certain time, the velocity of the running water. To read off the number of these revolutions, the axle has a few turns of a screw *C*, and these work into the teeth of a wheel *D*, upon whose lateral surfaces numbers are engraved, which give, by means of an index, the number of revolutions of the wheel. But to be able to register a great number of revolutions upon the axle of this toothed

Fig. 516.



wheel, there is a pinion which works into the teeth of the wheel *E*, by which, like the hands of a watch, several multiple revolutions may be read off. If, for example, each of the two toothed wheels has fifty teeth, and the trundle ten, then the second wheel revolves one tooth whilst the first advances five teeth, or the vanes make five revolutions, if the index of the first wheel points to $27 = 25 + 2$, and that of the second to 32, the corresponding number of revolutions of the vanes is accordingly: $= 32 \,.\, 5 + 2 = 162$. The entire instrument is screwed to a staff having a tin vane attached, to admit of easy immersion in the water, and of being kept opposed to the current. But that the wheelwork may only revolve during the time of observation, the axis is connected with a lever GO, which is pressed down by a spring, so that the teeth of the first wheel are thrown into gear with the screw only when the lever is drawn up by a string.

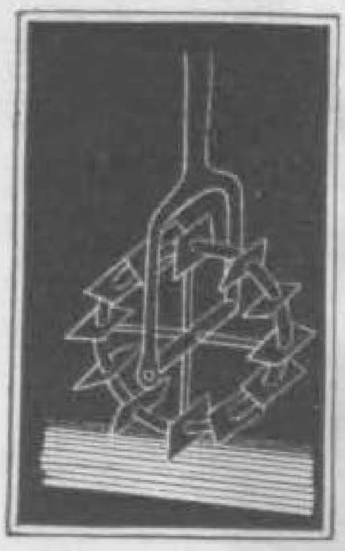
The number of revolutions of a wheel in a certain time, for example, in a second, is not exactly proportional to the velocity of the water, hence we cannot put $v = \alpha u$, where u is the number of revolutions, v the velocity, and α a number deduced from experiments, but rather $v = v_0 + \alpha u$, or more correctly $v = v_0 + \alpha u + \beta u^2 \ldots$, or still more correctly:

$v = \alpha u + \sqrt{v_0^2 + \beta u^2}$, where v_0 is the velocity, at which the water is no longer able to turn the wheel, and α and β are co-efficients from experiment. The constants v_0, α and β, are to be determined for each instrument in particular. With their assistance the velocity is known from a single observation, nevertheless it is always safer to make at least two, and to substitute the mean value as the correct one.

Example. If for a sail-wheel $v_0 = 0{,}110$ feet, $\alpha = 0{,}480$, and $\beta = 0$, therefore $v = 0{,}11 + 0{,}48\,u$, and we have by an observation with this instrument found the number of revolutions 210 in 80″, then the corresponding velocity is:

$$v = 0{,}11 + 0{,}48 \cdot \frac{210}{80} = 0{,}11 + 1{,}26 = 1{,}37 \text{ feet.}$$

Fig. 517.



Remark 1. The constants v_0, α and β depend principally on the magnitude of the angle of impact, *i.e.*, on the angle which the plane of the vane makes with the direction of motion of the water, and therefore, also, with the direction of the axis of the wheel. To observe with tolerable accuracy small velocities, it is well to have a large angle of impulse, *i. e.*, one of 70°. For the rest, it is desirable to have vanes of different sizes and with different angles of impulse, and to use the vane with small angles of impulse for great velocities, and a smaller one for shallow water.

Remark 2. To find the velocity of the surface of water, a small tin wheel may be used, as represented in Fig. 517, and its under part allowed to dip into the water. The number of its revolutions may be determined by a system of wheels, as in the tachometer.

§ 379. To find the constant or co-efficient of the tachometer, it is necessary to set this instrument in a stream, whose velocity is known, and to note the corresponding number of revolutions. Although as many observations only are required, as there are constants, it is still safer to have as many observations as possible, and especially for very different velocities, because we may then apply the method of least squares, and thereby eliminate the effect of accidental errors of observation. The velocity of the water may be found by the floating ball, or by receiving the water in a gauge vessel, and dividing the measured discharge by the transverse section. In using floating balls, the air should be still, and the tract of water straight and uniform. The tachometer is to be held at several places of the space described by the floating ball, and it is also requisite for accuracy, that the diameter of the ball should be equal to that of the tachometer.

The second method of determination has several advantages when the water in which the instrument is immersed is received into a gauge vessel. For this purpose, and especially for adjusting the hydrometer, it is well if the engineer can erect a proper hydraulic observatory, consisting of a vessel of efflux, a gauge reservoir, and a channel of communication between the two. With such an arrangement, we may impart to the water any arbitrary velocity, because we can not only regulate the entrance into the channel, but also the motion by means of boards placed in at pleasure. During observations we must keep the tachometer at different parts of the transverse section of the channel, measure the depth of this section by a scale, and, lastly, gauge the water running through in a definite time, in the lower reservoir (§ 372). We obtain the area F of the transverse profile by multiplication of the mean depth with the mean breadth, and the quantity of water Q is found from the mean transverse section G of the gauge measure, and the height (s) of the quantity which has flowed in during the time by the formula $Q = \frac{Gs}{t}$; but the mean velocity of the water: $v = \frac{Q}{F} = \frac{Gs}{Fl}$ follows from Q and F.

The corresponding number of revolutions u of the wheel is the mean of all the revolutions which are obtained when the instrument is immersed at different breadths and depths of the measured profile.

If from a series of experiments we have found the mean velocities v_1, v_2, v_3, &c., and the corresponding number of revolutions, we then obtain by substitution in the formula $v = v_0 + \alpha u$, or in the more correct one: $v = \alpha u + \sqrt{v_0^2 + \beta u^2}$ as many equations of condition for the constants v_0, α, β, as there have been observations made, and we may from these find the constants, if these equations are divided into as many groups as there are unknown constants, and these added together for as many equations of condition as are requisite for determining v_0, α, and also β when required.

Remark. If we adopt the more simple formula with 2 constants, we may then, after the method of least squares, put:

$$v_0 = \frac{\Sigma(y)^2\,\Sigma(x) - \Sigma(xy)\,\Sigma(y)}{\Sigma(x^2)\,\Sigma(y^2) - [\Sigma(xy)]^2} \text{ and } \alpha = \frac{\Sigma(x^2)\,\Sigma(y) - \Sigma(xy)\,\Sigma(x)}{\Sigma(x^2)\,\Sigma(y^2) - [\Sigma(xy)]^2},$$

where $x = \frac{1}{v}$ and $y = \frac{u}{v}$, and the sign Σ represents the sum of all successive similar values, for example, $\Sigma(x) = \frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} + \ldots$,

$$\Sigma(xy) = \frac{1}{v_1} \cdot \frac{u_1}{v_1} + \frac{1}{v_2} \cdot \frac{u_2}{v_2} + \frac{1}{v_3} \cdot \frac{u_3}{v_3} + \ldots$$

Example. For a small tachometer the velocities are: 0,163; 0,205; 0,298; 0,36[illegible] 0,610 metres, the number of revolutions per second: 0,600; 0,835; 1,467; 1,805; 3,142 required to determine the constants corresponding to this wheel. From the formula given in the remark it follows, that:

$$\Sigma(x) = \frac{1}{0,163} + \frac{1}{0,205} + \ldots = 18,740,\ \Sigma(y) = \frac{0,600}{0,163} + \ldots = 22,759$$

$$\Sigma(x^2) = \left(\frac{1}{0,163}\right)^2 + \left(\frac{1}{0,205}\right)^2 + \ldots = 82,846,\ \Sigma(y^2) = 105,223, \text{ and}$$

$$\Sigma(xy) = \frac{0,600}{(0,163)^2} + \frac{0,835}{(0,205)^2} + \ldots = 80,961,$$

$$v_0 = \frac{105{,}223 \,.\, 18.740 - 80{,}961 \,.\, 22{,}759}{82{,}846 \,.\, 105{,}223 - (80{,}961)^2} = \frac{129{,}5}{2162} = 0{,}060 \text{ and}$$

$a = \frac{368.3}{2162} = 0{,}1703$, hence for this instrument the formula $v = 0{,}060 + 0{,}1703\, u$.

If in this we put $u = 0{,}6$, we then obtain

$v = 0{,}060 + 0{,}102 = 0{,}162$; further, $u = 0{,}835$,
$v = 0{,}060 + 0{,}142 = 0{,}202$; further, $u = 1{,}467$,
$v = 0{,}060 + 0{,}249 = 0{,}309$. $u = 1{,}805$,
$v = 0{,}060 + 0{,}307 = 0{,}367$; lastly $u = 3{,}142$,
$v = 0{,}060 + 0{,}535 = 0{,}595$;

therefore, the calculated values agree very well with the observed.

§ 380. *Pitot's Tube.*—Other hydrometers are not so satisfactory as the tachometer, for they either admit of less accuracy, or they are more complicated in their use. The most simple instrument of this kind is *Pitot's tube.* In its simplest form it consists of a bent glass tube ABC, Fig. 518, which is held in the water in such a manner that its lower part stands horizontally, and is opposed to the water. By the percussion of the water, a column of water is sustained in this tube, which stands above the level of the exterior fluid surface, and the elevation DE of this column is greater, the greater the percussion or the velocity of the water generating it; this elevation or difference of level may hence serve inversely for a measure of the velocity of the water. Let this elevation DE above the external surface of water $= h$, and the velocity $= v$, then $h = \frac{v^2}{2g\mu^2}$, where μ is a number derived from experiment, and we have inversely, $v = \mu \sqrt{2gh}$, or more simply: $v = \psi \sqrt{h}$. To find the constant ψ, the instrument is immersed at a place in the water where the velocity v_1 is known; if the elevation is here $= h_1$, we then have the constant $\psi = \frac{v_1}{\sqrt{h_1}}$, which is to be applied in other cases, where the velocity is to be determined with this instrument.

Fig. 518.

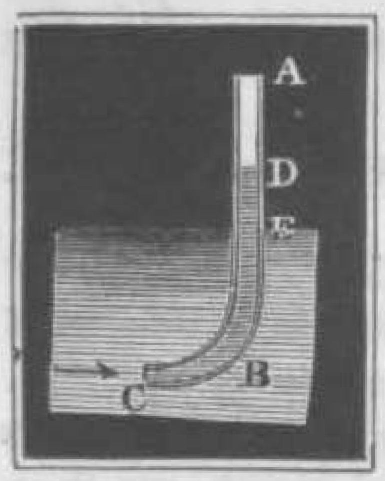
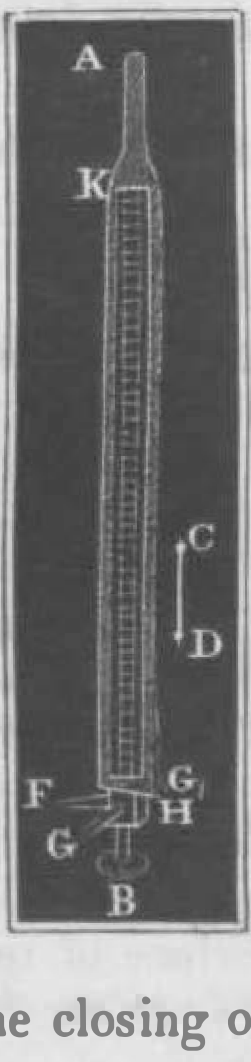


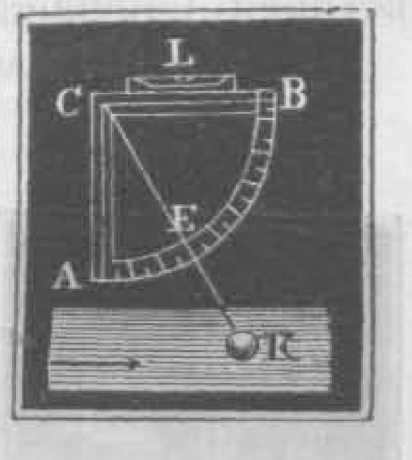
Fig. 519.



To facilitate the reading off of the height h, the instrument consists of two tubes, as shown in Fig. 519, and from the one a small tube F is directed against the stream, from the other two small tubes G and G_1 at right angles to the direction of the stream, both tubes are connected with a single cock H, by which the water can be retained in them. When the instrument is drawn out of the water, we may conveniently read off on a scale attached to both the tubes, the difference $CD = h$ of the two columns of water. That the water in the tube may not oscillate much, it is necessary to make the exterior orifices of the tubes narrow, and that the closing of them may take place quickly and safely; the cock is provided with an arm and an even rod HK, which terminates above, near the handle of the instrument.

§ 381. *Hydrometric Pendulum.*—The hydrometric pendulum has been used in preference by Ximenes, Michelotti, Gerstner and Eytelwein for the measurement of the velocity of running water. This instrument consists of a quadrant AB, Fig. 520, divided into degrees and smaller parts, and a metallic or ivory ball K of from two to three inches diameter, suspended by a thread from the centre C, the velocity of the water is given by the angle ACE, at which the thread when stretched by the ball deviates from the vertical, when the plane of the instrument is brought into the direction of the stream, and the ball submerged in the water. As the angle rarely amounts to forty or more degrees, this instrument has often the form of a right angled triangle given to it, and the divisions made on its horizontal arm. For the placing of the index or zero line in the vertical, it is best to use a spirit level on the horizontal arm of the instrument, or the ball itself may serve for this purpose, by letting it be suspended out of the water, and the instrument revolve until the thread coincides with the zero line of the division.

Fig. 520.



For velocities under four feet we may use the ivory ball, but for greater velocities the hollow metal ball. On account of the constant undulations of the ball in the direction of the motion of the water, as also at right angles to the direction of the current, the reading off is somewhat difficult, and leaves a good deal of uncertainty, for which reason this instrument cannot be relied upon for the more exact numbers.

The dependence between the angle of deviation and the velocity of the water may be determined in the following manner when the ball is not very deeply immersed. From the weight G of the ball and from the impulse of the water $P = \mu F v^2$, increasing simultaneously with the square of the velocity v and the section of the ball F, the resultant R, whose direction the thread assumes, follows, and is determined by the angle of deviation β, for which the $tang.\ \beta = \frac{P}{G} = \frac{\mu F v^2}{G}$, hence also inversely:

$$v^2 = \frac{G\ tang.\ \beta}{\mu F}, \text{ and } v = \sqrt{\frac{G}{\mu F}} \cdot \sqrt{tang.\ \beta}, \text{ i. e. } v = \psi \sqrt{tang.\ \beta},$$

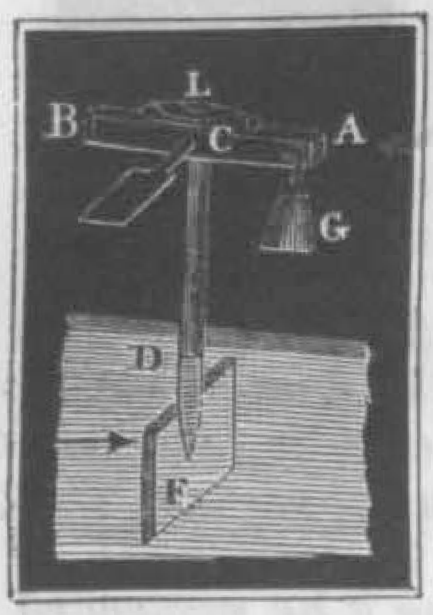
if ψ represents a co-efficient derived from experiment, which must be obtained before use, according to the above-mentioned instructions.

§ 382. *Rheometer.*—The remaining hydrometers, such as Lorgna's water lever, Ximenes's water vane, Michelotti's hydraulic balance, Brünning's tachometer, Poletti's rheometer, are more complicated in their use, and not altogether to be relied on. The principle of all these instruments is the same, they are composed of a surface of impulse and a balance, and the last serves for the purpose of giving the percussion P of the water against the former, but since this $= \mu\, F v^2$, we then have inversely:

$= \sqrt{\frac{P}{\mu F}} = \psi \sqrt{P}$, where ψ denotes a constant deduced from experiment dependent on the magnitude of the surface of impact F.

The *rheometer*, which was lately proposed by Poletti, and does not materially differ from the hydrometric balance of Michelotti, consists of a lever AB, Fig. 521, turning about a fixed axis C, and an arm CD to which the surface of impulse, or according to Poletti, a mere impulse-staff is screwed. To maintain equilibrium with the percussion of the water against the surface, the boxes suspended at the extremity A of the lever are loaded with weight or shot, and to put the empty balance in equilibrium in still water, a counterpoise is placed at B, which makes up the outermost end of the arm CB. From the weight put on G, the impulse P is found by means of the arm $CA = a$ and $CF = b$ from the formula $Pb = Ga$, whence, therefore,

Fig. 521.



$$P = \frac{a}{b} G, \text{ and } v = \sqrt{\frac{P}{\mu F}} = \sqrt{\frac{a\,G}{\mu\, b\, F}} = \psi \sqrt{G},$$

where ψ is a constant derived from experiment.

Remark. With respect to the last hydrometer, ample details will be found in Eytelwein's "Handbuch der Mechanik fester Korper und der Hydraulik;" further, in Gerstner's "Handbuch der Mechanik," vol. 2; in Brünning's "Treatise on the velocity of running water;" in Venturoli's "Elementi di Meccanica e d'Idraulica," vol. 2. Concerning Poletti's hydrometer, we must refer to Dingler's "Polytechn. Journal," vol. 20, 1826. The hydrometer described in Stevenson's treatise on Marine Surveying and Hydrometry is the tachometer of Woltmann, *see* Dingler's "Journal," vol. 65, 1842.

CHAPTER IX.

ON THE IMPULSE AND RESISTANCE OF FLUIDS.

§ 383. *Impulse and Resistance of Water.*—Water or any other fluid imparts a shock to a rigid body, when it meets it in such a manner that its condition of motion is thereby altered. The resistance which water opposes to the motion of a body, does not essentially differ from impulse. The investigation of these two forms the third principal division of hydraulics. We distinguish from each other:

1. The impulse of an isolated stream.
2. The impulse of a limited stream.
3. The impulse of an unlimited stream.

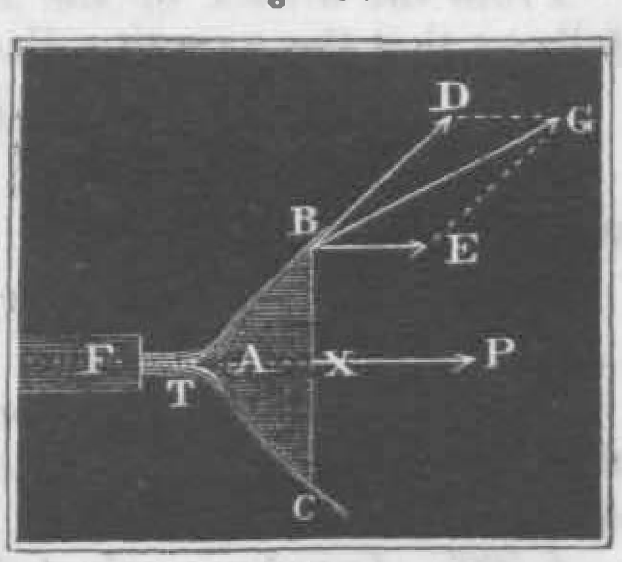
An impulse of the first kind takes place when a body, for instance, the float board of an over-shot water-wheel, is opposed to a stream of water issuing from a reservoir; an impulse of the second kind occurs where water, in a canal or in a water-course, impinges against a body which entirely fills up its transverse section, as for instance, against the float board of an under-shot wheel; the third kind, lastly, presents itself, when running water strikes against a body immersed in it, whose transverse section is only a very small part of that of the current of water, as, for instance, against the float boards of a floating mill-wheel.

We must distinguish the impulse of water against a body at rest and against a body in motion, and further, the impulse against a curved and against a plane surface, and in this last again, between the perpendicular and the oblique impulse.

Let us consider at once the general case, namely, the impulse of an isolated stream against a surface of rotation which moves in its proper axis, and in the direction of motion of the stream.

§ 384. *Impact of Isolated Streams.*—Let BAC, Fig. 522, be a surface of rotation, AX its axis, and FA a fluid stream meeting it in this direction. Let the velocity of the water $= c$, that of the surface $= v$, and the angle BTX, which the tangent DT at the extremity B of the generating curve or of each of the filaments of water BT leaving the surface, includes with the direction of the axis $BE = \alpha$; lastly, let us further assume that the water in running off from the surface loses nothing in *vis viva* by friction. The water strikes against the surface with the relative velocity $c - v$, and hence leaves the surface with this, and therefore quits it in the tangential directions TB, TC, &c. From the tangential velocity $BD = c - v$, and the velocity of the axis $BE = v$, the absolute velocity $BG = c_1$ of the water after impinging against the surface is found by the known formula:

Fig. 522.



$$c_1 = \sqrt{(c-v)^2 + 2(c-v)\, v \cos. \alpha + v^2}.$$

But now a quantity of water Q is able to produce by virtue of its *vis viva* the mechanical effect $\frac{c^2}{2g} \cdot Q\gamma$, if its velocity c is fully imparted; accordingly the residuary effect of the water:

$= \frac{c_1^2}{2g} \cdot Q\gamma$; consequently the mechanical effect distributed over the surface is:

$$Pv = \frac{c^2}{2g}\, Q\gamma - \frac{c_1^2}{2g}\, Q\gamma = \frac{c^2 - c_1^2}{2g} \cdot Q\gamma.$$

$$= \frac{[c^2 - (c-v)^2 - 2(c-v)\, v \cdot \cos. \alpha - v^2]}{2g} \cdot Q\gamma$$

$$= \frac{2\,c\,v - 2\,v^2 - 2\,(c-v)\,v\cos.\alpha}{2g}\,Q\gamma,\ i.e.$$

$$Pv = (1 - \cos.\alpha)\frac{(c-v)\,v}{g}\,Q\gamma,$$

and the force or the impulse of the water in the direction of its axis is:

$$P = (1 - \cos.\alpha)\frac{(c-v)}{g}\,Q\gamma.$$

If the surface meets the water with the velocity v, we then have:

$$P = (1 - \cos.\alpha).\frac{(c+v)}{g}\,Q\gamma,$$

and if this is without motion, therefore, $v = 0$, the impulse or hydraulic pressure of the axis comes out:

$$P = (1 - \cos.\alpha)\frac{c}{g}.\,Q\gamma.$$

It follows from this, that the impulse of one and the same mass of water under otherwise similar circumstances is proportional to the relative velocity $c \mp v$ of the water.

From the area F of the transverse section of the fluid stream, it follows that the quantity discharged is $Q = Fc$; hence

$$P = (1 - \cos.\alpha)\frac{(c \mp v)\,c}{g}\,F\gamma;$$

and for $v = 0$:

$$P = (1 - \cos.\alpha)\frac{c^2}{g}\,F\gamma.$$

For an equal transverse section of the stream, the impulse against a surface at rest increases therefore as the square of the velocity of the water.

§ 385. *Impulse against Plane Surfaces.*—The impulse of one and the same fluid stream depends principally on the angle α, under which the water, after the impulse, leaves the axis; it is nothing if this angle $= 0$; and, on the other hand, a maximum, namely, $= 2\,\frac{(c \mp v)}{g}\,Q\,\gamma$, if this angle is 180°, therefore its cosine $= -1$, where the water, as represented in Fig. 523, leaves the surface in a

Fig. 523.

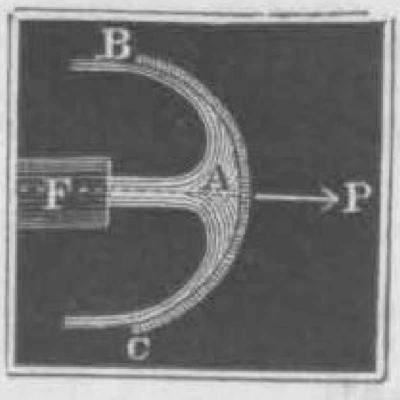
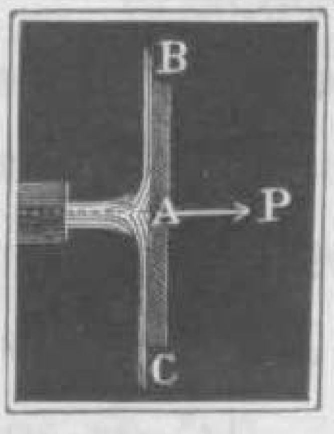


Fig. 524.



direction opposite to that in which it impinges. This is generally greater for concave surfaces than for convex, because the angle is there oblique, therefore the cosine negative and $1 - \cos.\, \alpha$ becomes $1 + \cos.\, \alpha$.

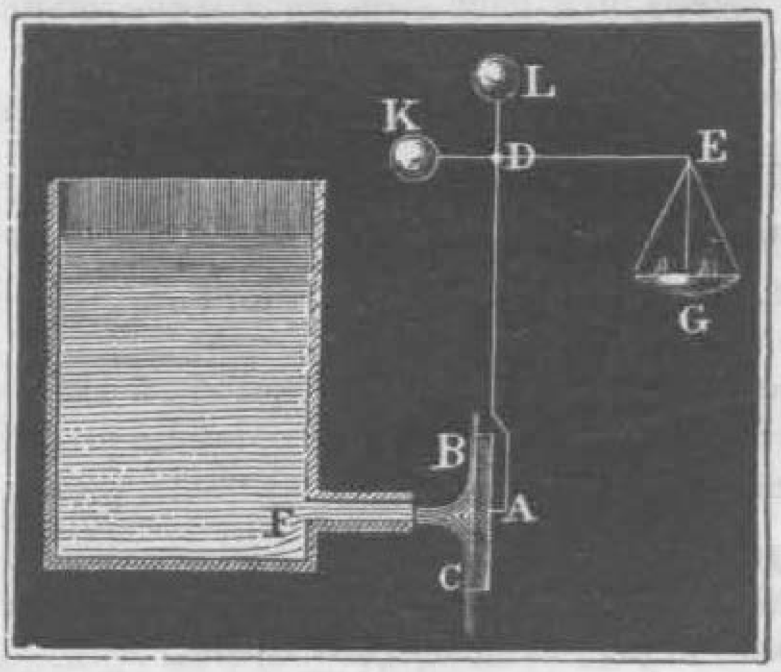
Most frequently the surface, as represented in Fig. 524, is plane, and hence $\alpha = 90^0$, therefore $\cos.\, \alpha = 0$, and the impulse $P = \frac{(c \mp v)}{g} \,.\, Q\gamma$; for a surface at rest:

$$P = \frac{c}{g}\, Q\gamma = \frac{c^2}{g} F\gamma = 2\,.\,\frac{c^2}{2g} F\gamma = 2\, F h \gamma.$$

The normal impulse of water against a plane surface is therefore equivalent to the weight of a column of water which has for base the transverse section F of the stream, and for altitude, twice the height due to the velocity $2\, h = 2\,.\,\frac{c^2}{2g}$.

The experiments made on this subject by Michelotti, Vince, Langsdorf., Bossut, Morosi, and Bidone, have nearly led to the same results when the transverse section of the impinged surface was at least six times as great as that of the stream, and when this surface was twice as far from the plane of the orifice as the thickness of the stream. The apparatus which was used for this purpose consisted of a lever, similar to that of Poletti's rheometer, which received upon one side the impulse of the water, and whilst its other side was kept in equilibrium by weights. The instrument which Bidone made use of is represented in Fig. 525. *BC* is the surface impinged on by the stream *FA*, *G* is the scale-pan for the reception of the weights, *D* the axis of rotation, *KL* counter-weights.*

Fig. 525.



§ 386. *Maximum Effect of Impulse.*—The mechanical effect of impulse:

$$Pv = (1 - \cos.\, \alpha)\, \frac{(c - v)\, v}{g}\, Q\gamma$$

depends principally on the velocity v of the impinged surface; it is,

* The latest and most extensive experiments on the percussion of water are those of Bidone. See "Memorie de la Reale Accademia delle Scienze di Torino," vol. 40, 1838. They were performed with a velocity of at least 27 feet, and on brass plates of from 2 to 9 inches diameter. In general, Bidone found that the normal impulse against a plane surface was somewhat greater than $2\, F h\gamma$, yet this variation is perhaps to be attributed to an augmentation of the leverage which is produced by the falling back of the water. See Duchemin's "Recherches expérim. sur les lois de la résistance des fluides." When the impinged surface was quite near the orifice, Bidone found that P was only $1{,}5\, F h\gamma$; when, further, the surface had a transverse section equal to that of the stream, in which case the water only deviated by an acute angle α, then, after Du Buat and Langsdorf, P was only $= F h\gamma$. Lastly, it has been deduced by Bidone and others that the impulse is in the first moment nearly as great again as the permanent impulse.

for example, nothing, not only for $v = c$, but also for $v = 0$; hence there is a velocity for which the effect of the impulse is a maximum. It is manifest that it only depends on $(c - v)\, v$ becoming a maximum. If we consider c as half the perimeter of a rectangle, and v as its base, we have then its height $= c - v$ and its area $= (c - v)\, v$. But of all rectangles the square is that which has for a given perimeter $2\,c$ the greatest area, hence also $(c - v)\, v$ is a maximum, when $c - v = v$, i. e. $v = \frac{c}{2}$, *and we therefore obtain the maximum value of the mechanical effect of the impulse when the surface moves from it with half the velocity of the water, and indeed*

$$Pv = (1 - cos.\, \alpha) \,.\, \tfrac{1}{2} \,.\, \frac{c^2}{2g} \,.\, Q\gamma = (1 - cos.\, \alpha) \,.\, \tfrac{1}{2}\, Qh\gamma.$$

If now $\alpha = 180°$, and if, therefore, the motion of the water be reversed by the impulse, we then have the effect equal to $2 \,.\, \frac{1}{2}\, Qh\gamma = Qh\gamma$. But if $\alpha = 90°$, *i. e.* if it impinges against a plane surface, this effect is then onlye$\frac{1}{2}\, Qh\gamma$, therefore, in the last case, the half only of the whole disposable effect, or that which corresponds to the *vis viva* of the water, is gained or brought to bear upon the surface.

Examples.—1. If a stream of water, of 40 square inches transverse section, delivers a quantity of 5 cubic feet per second, and strikes normally against a plane surface, and escapes with a 12 feet velocity, the effect of impulse is then:

$$P = \frac{(c - v)}{g}\, Q\gamma = \left(\frac{5 \,.\, 144}{40} - 12\right) . \, 0{,}031 \,.\, 5 \,.\, 62.5 = 6 \,.\, 0{,}031 \,.\, 312{,}5 = 58{,}12 \text{ lbs.},$$

and the mechanical effect brought to bear upon the surface $Pv = 58{,}12 \times 12 = 697{,}44$ ft. lbs. The greatest effect is for $v = \frac{c}{2} = \frac{1}{2} \,.\, \frac{5 \,.\, 144}{40} = 9$ feet, and indeed:

$$= \tfrac{1}{2} \,.\, \frac{c^2}{2g} \,.\, Q\gamma = \tfrac{1}{2} \,.\, 18^2 \,.\, 0{,}0155 \,.\, 5 \,.\, 62{,}5 = 81 \,.\, 0{,}0155 \,.\, 62{,}5 = 784{,}68 \text{ ft. lbs.};$$

the corresponding impulse, or hydraulic pressuret$= \frac{784{,}68}{9} = 87{,}18$ lbs.—2. If a stream FA, Fig. 526, of 64 square inches section, strikes with a 40 feet velocity against an immovable cone, having an angle of convergence $BAC = 100°$, then is the hydraulic pressure in the direction of the stream:

$$P = (1 - cos.\, \alpha)\, \frac{c}{g}\, Q\gamma = (1 - cos.\, 50°)\, 400\; 0{,}0310\, \frac{64}{144} \,.\, 400\; 62{,}5$$

$$= (1 - 0{,}64279) \,.\, 1 \,.\, 24 \,.\, \frac{10000}{9} = 0{,}35721 \,.\, 1377{,}7 = 492 \,.\, 13 \text{ lbs.}$$

Fig. 526.

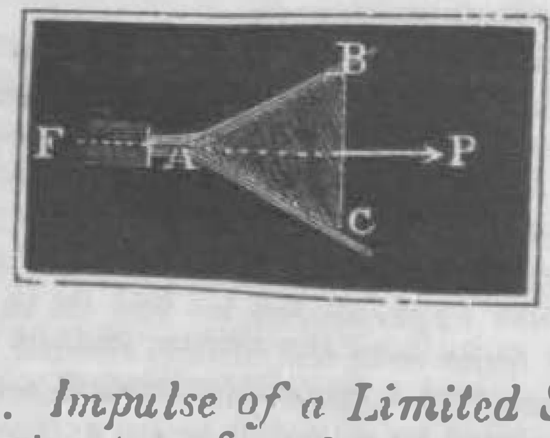
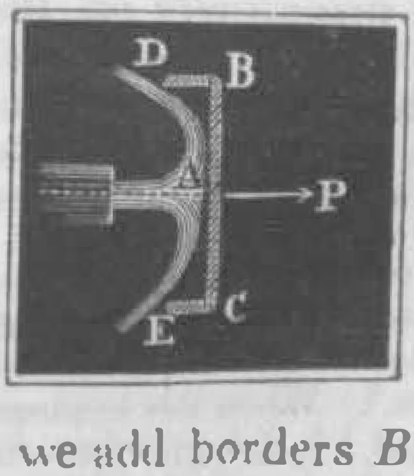


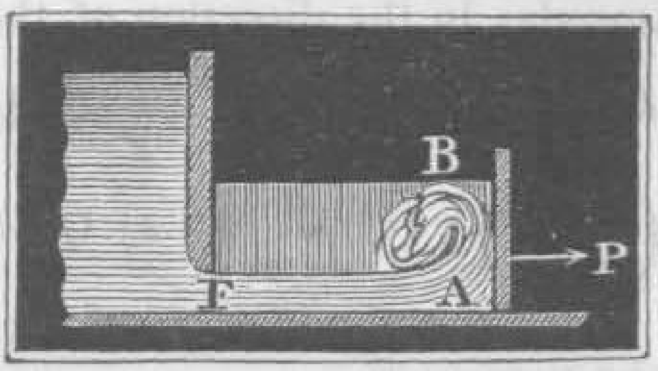
Fig. 527.



§ 387. *Impulse of a Limited Stream.*—If we add borders BD, CE, to the perimeter of a plane surface BE, Fig. 527, which project from the side impinged upon by the water, then will the water deviate from its direction at an obtuse angle, in a similar manner as from concave surfaces, and hence the impulse will be greater than for plane surfaces. The effect of this impulse depends principally on the height of the border and the ratio of the transverse section between the stream and the part confined. In an experiment, where the stream was 1 inch thick, the cylindrical enclosure 3 inches wide and $3\frac{1}{2}$ lines deep, the water ran off almost in a reversed direction, and the impulse amounted to $3{,}93\,\frac{c^2}{2g}\,F\gamma$; in every other case this force was less. In consequence of the friction of the water at the surface and the sides, the theoretical maximum value never reaches $4\,\frac{c^2}{2g}\,F\gamma$.

In the impulse of a limited stream FAB, Fig. 528, a rising at the edges takes place; this rising occupies only a portion of the perimeter, and extends itself, on the other hand, simultaneously to the impinged surface and the fluid stream. The impinging water takes the direction of the unbordered portion of the perimeter, and here, therefore, becomes deflected 90 degrees, whence the formula above found for the isolated stream $P = \frac{(c-v)}{g}\,Q\gamma$ holds good; yet this may also be deduced in the following manner. If we assume that the velocity c of the arriving water by the impulse against its surface is changed into the velocity v of the surface, we may then also assume that a loss of mechanical effect $\frac{(c-v)^2}{2g}\,Q\gamma$ (similar to that in § 337), expended in the division of the water, is connected with it. But now the effect due to the *vis viva* of the arriving watere $= \frac{c^2}{2g}\,Q\gamma$ and to that of the water going on $= \frac{v^2}{2g}\,Q\gamma$, hence it follows that the mechanical effiect imparted to the surface is:

Fig. 528.



$$Pv = [c^2-(c-v)^2-v^2]\,\frac{1}{2g}\,Q\gamma = \frac{(c-v)\,v}{g}\,Q\gamma.^*$$

§ 388. *Oblique Impulse.*—In oblique impulse against a plane surface, we must distinguish whether the water flows away in one, two, or in all directions in the plane. If, as in the impact of limited water, the surface AB, Fig. 529, is confined at three sides, so that the water can run off only in one direction, we have then the hydraulic pressure against the surface in the direction of the stream $P = (1\text{-}cos.\,\alpha)\,\frac{(c-v)}{g}\,Q\gamma$.

* This formula will be found applicable hereafter, when we come to the theory of water-wheels.

Fig. 529.

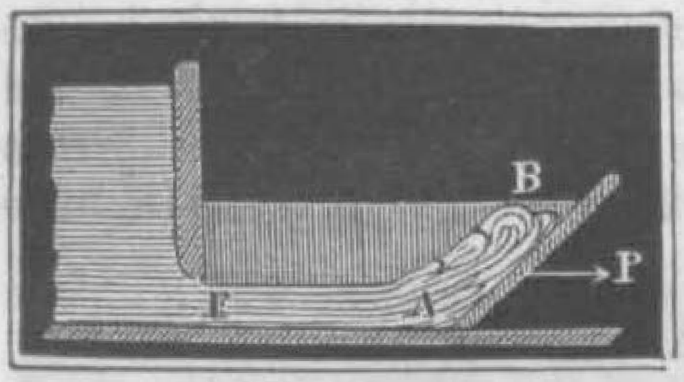
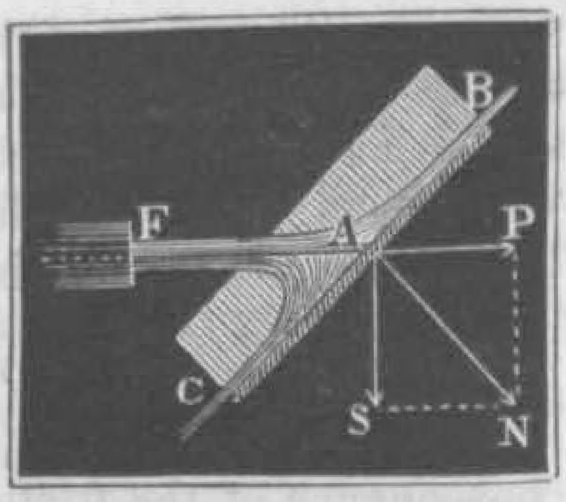


Fig. 530.



But if the impinged plane BC, Fig. 530, is only bordered on two oppositely situated sides, the stream then divides itself into two unequal portions; the greater portion Q_1 takes the small deflexion α, and the lesser Q_2, the greater deflexion $180 - \alpha$; hence, the whole impulse in the direction of the stream ise

$$P = (1 - \cos. \alpha) \cdot \frac{c - v}{g} Q_1 \gamma + (1 + \cos. \alpha) \cdot \frac{c - v}{g} Q_2 \gamma = \left(\frac{c - v}{g}\right) \gamma \left[(1 - \cos. \alpha) Q_1 + (1 + \cos. \alpha) Q_2\right].$$

Now the equilibrium of the two portions of the stream requires that the pressures

$$\frac{(c - v)}{g} \gamma (1 - \cos. \alpha) Q_1 \text{ and } \frac{(c - v)}{g} \gamma (1 + \cos. \alpha) Q_2$$

between them should be equal; hence, also:

$$(1 - \cos. \alpha) Q_1 = (1 + \cos. \alpha) Q_2, \text{ or since } Q_1 + Q_2 = Q,$$

$$(1 - \cos. \alpha) Q_1 = (1 + \cos. \alpha)(Q - Q_1), \text{ i. e.,}$$

$$Q_1 = \left(\frac{1 + \cos. \alpha}{2}\right) Q, \text{ and } Q_2 = \left(\frac{1 - \cos. \alpha}{2}\right) Q,$$

so that the whole impulse in the direction of the stream is finally:

$$P = \frac{(c - v)}{g} \gamma \cdot 2 (1 - \cos. \alpha) \frac{(1 + \cos. \alpha) Q}{2} = \frac{(c - v)\gamma}{g} (1 - \cos. \alpha^2) Q,$$

$$\text{i. e., } P = \frac{c - v}{g} \sin. \alpha^2 \cdot Q \gamma.$$

Besides the *parallel impulse* P, acting in the direction of the stream, we distinguish, further, the *lateral impulse* S, acting at right angles to the direction of the stream, and the *normal impulse* N, composed of these two, and at right angles to the surface. In every case $P = N \sin. \alpha$, and $S = N \cos. \alpha$; hence, inversely,

$$N = \frac{P}{\sin. \alpha} = \frac{c - v}{g} \sin. \alpha \; Q\gamma \text{ and } S = \frac{c - v}{2g} \sin. 2\alpha \cdot Q\gamma.$$

The normal impulse, therefore, increases as the sine, the parallel impulse as the square of the sine of the angle of incidence, and the lateral impulse as double the same angle. Lastly, if the inclined surface impinged on is not bordered, then the water can spread over it in all directions; the impulse is then greater, because of all the angles by which the filaments of water are deflected, α is the least; and hence, each filament which does not move in the normal plane, exerts a greater pressure than the filament in this plane. Let us assume tha a portion Q_1 corresponding to the sectors AOB and DOE, Fig. 531, is deflected by the angles α and $180^\circ - \alpha$, and another Q_2, corresponding to the sectors AOD and BOE, by 90°, and that both portions exert a parallel impulse, we may then put:

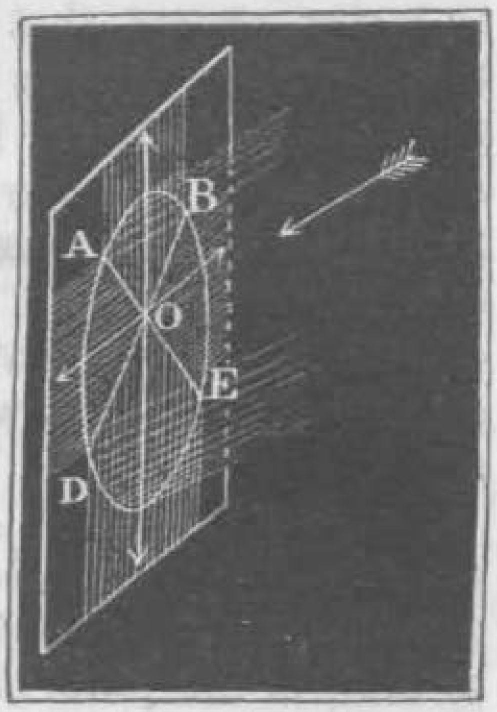
$$P = \frac{c - v}{g} Q_1 \gamma \sin. \alpha^2 + \frac{c - v}{g} Q_2 \gamma, \; Q_1 \sin. \alpha^2$$

$= Q_2$, and $Q_1 + Q_2 = Q$; hence it follows, that $Q_1 (1 + \sin. \alpha^2) = Q$, and the whole parallel impulse $P =$

$$\left(\frac{c - v}{g}\right) \frac{2 Q \gamma \sin. \alpha^2}{1 + \sin. \alpha^2} = \frac{2 \sin. \alpha^2}{1 + \sin. \alpha^2} \cdot \frac{c - v}{g} \cdot Q \gamma.$$

Although this hypothesis is only approximately correct, it tolerably well agrees, nevertheless, with the latest experiments of Bidone.

Fig. 531.



§ 389. *Action of an Unlimited Stream.*—If a body moves progressively in an unlimited fluid, or if a body is put into a fluid which is in motion, it then suffers a pressure which is dependent on the form and dimensions of this body, as well as on the density and on the velocity of the one or the other mass, and in the one case is called the *resistance*, and in the other the *impulse* of the fluid. This hydraulic pressure arises principally from the inertia of the water, whose condition of motion is altered by striking against the solid body, and also, further, from the force of cohesion of the particles of water, which are hereby partially separated from one another, or pushed aside. If a body AC moves against running water, Fig. 532, it pushes away

Fig. 532.

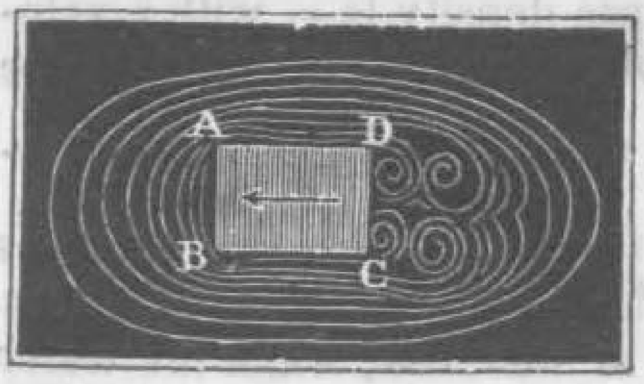
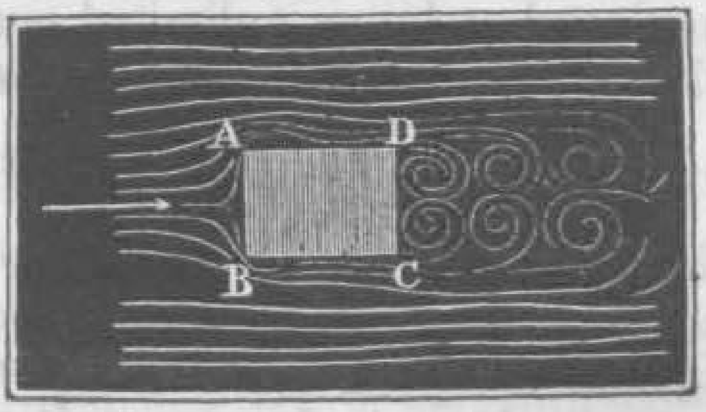


Fig. 533.



before it a certain quantity with an augmented pressure. Whilst this mass of water, by the further advance of the body, always increases on the one side, on the other a constant flowing away takes place, while the particles lying near the anterior surface assume a motion in the direction of this surface. If the moving mass of water strikes against a body at rest, Fig. 533, then is there likewise an increased pressure produced in front of it, which causes the particles before the body to deviate from their original direction, and to run off at the surface AB. When these particles have reached the limits of the surface, they then turn and flow away by the lateral surfaces until they come to the back, when they then again immediately unite, but assume an eddying motion. It is manifest that the general circumstances of motion of the particles surrounding the body are the same in the impact of moving water as in the resistance of a body moving in water, except that in the eddies a difference so far takes place, that with short bodies the eddy in the latter case occupies a less space than in the former. In both cases the velocity of the particles increases more and more from the middle of the anterior surface to its limits, attains its maximum at the commencement of the lateral surfaces, where, for the most part, a contraction takes place, gradually diminishes in the water which passes away at the sides, and lastly, attains its minimum when the water reaches the back and passes into a whirling motion.

§ 390. *Theory of Impulse and Resistance.*—The normal pressure varies at different points of the body; it is greatest at the middle of the anterior, and least at the middle of the posterior surface, and, next to that, at the parts of the sides nearest this; because, in respect to the body, there is at the one place rather a flow to, and at the other a flow from these surfaces. If the body be symmetrical, as we shall suppose it to be, with respect to the direction of motion, then the aggregate pressures in this direction counteract each other, and hence only the pressures in the direction of motion are to be taken into account. But now the pressures on the posterior surface are opposed to those on the anterior; hence the *resultant impulse or resistance of the water may be equated to the difference of pressure of the anterior and posterior surfaces.*

If we cannot assign the amount of these pressures *à priori*, we may, nevertheless, from the great similarity of the circumstances to the impulse of isolated streams, assume that at least the general law for the impulse of unlimited water does not differ from that of the impulse of isolated streams. If, therefore, F is the area of a surface, which is impinged on by an unlimited current whose density is γ, with a velocity v, then the corresponding impulse or hydraulic pressure may be put $P = \zeta \frac{v^2}{2g} F\gamma$, where ζ represents a number deduced from experiment, dependent on the form of the surface. But this expression is not only applicable to action against the anterior, but also to that against the posterior surface, only that in this last, when the water has a tendency to flow away, it consists of a draught or negative pressure. If now $Fh\gamma$ is the hydrostatic pressure (§ 276) against the front and back surface of a body, the whole pressure against the front is: $P_1 = Fh\gamma + \zeta_1 . \frac{v^2}{2g} F\gamma$, and that against the back: $P_2 = Fh\gamma - \zeta_2 . \frac{v^2}{2g} F\gamma$, and the resultant impulse or resistance of the water is then found:

$P = P_1 - P_2 = (\zeta_1 + \zeta_2) . \frac{v^2}{2g} F\gamma = \zeta . \frac{v^2}{2g} F\gamma$, if $\zeta_1 + \zeta_2 = \zeta$. This general formula for the impulse of unlimited water is applicable to the percussion of the wind or to the resistance of the air. Besides the difference of aërodynamic pressure at the front and back, there is further a difference of aërostatic pressure, because the air in front, in consequence of its greater elasticity, has a greater density (γ) than that at the back. For this reason, in high velocities, as those of cannon-balls, the co-efficient of the resistance of air is greater than that of water.

Remark.—The adhesion of a certain quantity of air or water to the body, is a peculiar phenomenon of the impulse or resistance of an unlimited medium (water or air), whose influence is particularly remarkable in the variable motion of bodies, as, for example, in the oscillations of the pendulum. For a ball, the air or water adhering to the moving body is equal to 0,6 of the volume of the ball. For a prismatic body moved in the direction of its axis, the ratio of this volume $= 0{,}13 + 0{,}705 \dfrac{\sqrt{F}}{l}$, where l is the length, and F the transverse section of the body. These relations, discovered by Du Buat, have been fully confirmed by the later observations of Bessel, Sabine, and Baily.

§ 391. *Impulse and Resistance against Surfaces.*—The co-efficient of resistance ζ, or the number with which the height due to the velocity is to be multiplied to obtain the height of a column of water measuring this hydraulic pressure, varies for bodies of different figures, and only for plates which are at right angles to the direction of motion is it nearly a definite quantity. According to the experiments of Du Buat and those of Thibault, we may put $\zeta = 1{,}85$ for the impulse of air or water against a plane surface at rest, and, on the other hand, assume, but with less accuracy, for the resistance of air or water against a surface in motion $\zeta = 1{,}40$. In both cases, about two-thirds of the whole effect are expended on the front, and one-third on the back. The resistance which the air opposes to a surface revolving in a circle, has been found by Borda, Hutton, and Thibault to vary a good deal, but may be expressed by a mean of $\zeta = 1{,}5$. If the surface does not stand at right angles to the direction of the motion, but makes with it an acute angle α, we may then, with Duchemin, substitute for ζ, $\dfrac{2\,\zeta\, \sin.\alpha^2}{1 + \sin.\alpha^2}$ with tolerable correctness.

The impulse and resistance of unlimited media are also augmented when the surfaces are hollowed out or have projecting edges at their perimeters, but we have arrived at no general results on this subject.

Example. If the wind impinges with a 20 feet velocity against a firmly fixed wind-mill wheel, which consists of four wings, of which each has an area of 200 square feet and 75° inclination to the direction of the wind, then is the impinging force of the wind in its direction, or in that of the axis of the wheel:

$$P = 1{,}85 \, . \, \frac{2\,(\sin.75)^2}{1 + (\sin.75)^2} \, . \, \frac{20^2}{2g} \, . \, 4 \, . \, 200 \, . \, 0{,}081 = 1{,}85 \, . \, 0{,}965 \, . \, 6{,}21 \, . \, 800 \, . \, 0{,}081$$

$= 718{,}4$ ft. lbs., when the density of the wind is (from § 301) taken at 0,081 lbs.

Remark. Views, with respect to the impulse and resistance of unlimited fluids, entirely at variance with these, are put forward in the above-mentioned work of Duchemin. It is there maintained, for instance, that the impulse and resistance against the front surface of a thin plate amounts to $2 \, . \, \dfrac{v^2}{2g} F h$, and is not negative at the back, that the impulse $= 0{,}136 \dfrac{v^2}{2g} F\gamma$, and the resistance $= 0{,}746 \dfrac{v^2}{2g} F\gamma$. It would be too circumstantial here to give a detail of the reasons why the author cannot agree with the views of Duchemin, but more with reference to this will be found in Poncelet's "Introduction à la mécanique industrielle," 2d edition, 1841.

§ 392. *Impulse and Resistance to Bodies.*—The impulse and resistance of water to prismatic bodies, whose axis coincides with the direction of motion, diminishes when the length of the body is considerable. From the experiments of Du Buat and Duchemin, the impulse of the front surface is invariable, and only the effect against the back surface variable. To this corresponds the co-efficient $\zeta_1 = 1,186$, for the total effect, however, with the relative lengths

$$\frac{l}{\sqrt{F}} = 0, \quad 1, \quad 2, \quad 3,$$

$$\zeta = 1,86; \ 1,47; \ 1,35; \ 1,33.$$

For still greater ratios between the length l and the mean breadth $\sqrt{F}$ of the body ζ diminishes, owing to the friction of the water at the lateral surfaces of the body. From the resistance of the water, reverse relations take place. Here, from Du Buat, for the effect on the front surface, $\zeta_1 = 1$ invariably; for the total effect, however, with

$$\frac{l}{\sqrt{F}} = 0, \quad 1, \quad 2, \quad 3,$$

$\zeta = 1,25;\ 1,28;\ 1,31;\ 1,33$, so that, for a prism which is 3 times as long as broad, the impulse is the same as the resistance.

The experiments undertaken by Borda, Hutton, Vince, Desaguilliers, Newton, and others, with angular and with round bodies, leave still much uncertainty. In what relates to spheres, it appears that for moderate velocities the mean co-efficient for motion in air or water $= 0,6$. For a greater velocity and for motion in air, according to Robins and Hutton, for the velocities

$v = 1, \quad 5, \quad 25, \quad 100, \quad 200, \quad 300, \quad 400, \quad 500, \quad 600$ metr.
$\zeta = 0,59;\ 0,63;\ 0,67;\ 0,71;\ 0,77;\ 0,88;\ 0,99;\ 1,04;\ 1,10.$

Duchemin and Piobert have given particular formulæ for the rate of increase of these co-efficients.

For the impulse of water against a sphere, Eytelwein found $\zeta = 0,7886$.*

Example. If, according to Borda, we put the resistance and impact at right angles to the axis of a cylinder at half as great as that against a parallelopiped which has the same dimensions, we then obtain for the resistance $\zeta = \frac{1}{2} \,.\, 1,28 = 0,64$ and the impact $= \frac{1}{2} \,.\, 1,47 = 0,735$. If we apply these values to the human body, whose section has an area of some 7 square feet, we then find for the resistance and impulse of air against it, the values:

$P = 0,64 \,.\, 0,0155 \,.\, 7 \,.\, 0,081\, v^2 = 0,00562\, v^2$, and

$P = 0,735 \,.\, 0,0155 \,.\, 7 \,.\, 0,081\, v^2 = 0,00646\, v^2$. Hence the resistance of air for a velocity of 5 feet is only $0,00562 \,.\, 25 = 0,1405$ lbs.; and the corresponding mechanical effect per second $= 5 \,.\, 0,1405 = 0,70$ ft. lbs.; for a velocity of 10 feet this resistance is four times, and the effect expended eight times as great, and for a velocity of 15 feet, the resistance is 9 times and the effect 27 times as great as for a 5 feet velocity. If a man, with a 5 feet velocity, moves against wind having a 50 feet velocity, he has then a resistance $0,00646 \,.\, 55^2 = 19,54$ lbs. to overcome, corresponding to the relative velocity $50 + 5 = 55$ feet, and thereby to produce the mechanical effect of $19,54 \,.\, 5 = 97,7$ ft. lbs. (English.)

* Poncelet, in his work above cited, and Duchemin and Thibault in their "Recherches expérimentales," have treated very fully of these circumstances. In the Second Part we shall treat of the resistance to floating bodies, especially to ships, &c., as also the impact of the wind on wheels, &c.

§ 393. *Motion in Resisting Media.*—The laws of the motion of a body in a resisting medium are rather complex, because we have here to deal with a variable force, *i. e.*, one increasing with the square of the velocity. From the force P_1 which urges the body forward, and from the resistance $P_2 = \zeta \cdot \frac{v^2}{2g} F\gamma$, which the medium opposes to the motion, the motive force is:

$$P = P_1 - P_2 = P_1 - \zeta \cdot \frac{v^2}{2g} F\gamma,$$

but since the mass of the body $= M = \frac{G}{g}$, the accelerating force is:

$$p = \frac{P}{M} = \left(P_1 - \zeta \frac{v^2}{2g} F\gamma\right) \div M = \left(\frac{P_1 - \zeta \frac{v^2}{2g} F\gamma}{G}\right) \cdot g,$$

or if we represent $\frac{F\gamma}{2gP_1}$ by $\frac{1}{w^2}$.

$p = \left[1 - \zeta \left(\frac{v}{w}\right)^2\right] \frac{P_1}{G} g$. But the velocity v is accelerated in the instant of time τ by $\varkappa = p\,\tau$, hence:

$\varkappa = \left[1 - \zeta \left(\frac{v}{w}\right)^2\right] \frac{P_1}{G} g\,\tau$, and inversely:

$$\tau = \frac{G}{P_1} \cdot \frac{\varkappa}{g\left[1 - \zeta \left(\frac{v}{w}\right)^2\right]}.$$

Now to find the time corresponding to a given change of velocity, let us divide the difference $v_n - v_0$, of the final and initial velocity into n parts, let any such part $\frac{v_n - v_0}{n} = \varkappa$, and let us calculate the velocities:

$$v_1 = v_0 + \varkappa,\ v_2 = v_0 + 2\,\varkappa,\ v_3 = v_0 + 3\,\varkappa, \text{ \&c.},$$

and substitute these values in the formula of Simpson. In this manner, by taking four parts we shall obtain the time sought

$$1.\ t = \frac{G}{P_1} \cdot \frac{v_n - v_0}{12g} \left(\frac{1}{1 - \zeta\left(\frac{v_0}{w}\right)^2} + \frac{4}{1 - \zeta\left(\frac{v_1}{w}\right)^2} + \frac{2}{1 - \zeta\left(\frac{v_2}{w}\right)^2} + \frac{4}{1 - \zeta\left(\frac{v_3}{w}\right)^2} + \frac{1}{1 - \zeta\left(\frac{v_4}{w}\right)^2}\right).$$

Further, the small space described in any instant τ (§ 19), is $\sigma = v\,\tau$, or since $\tau = \frac{\varkappa}{p}$, $\sigma = \frac{v\varkappa}{p}$, therefore,

$\sigma = \frac{v\varkappa}{1 - \zeta\left(\frac{v}{w}\right)^2} \cdot \frac{G}{P_1 g}$. By the application of Simpson's rule, we shall now find the space which is described while the velocity v passes into that of v_n.

$$2.\quad s = \frac{G}{P_1} \mp \frac{v_n - v_0}{12\,g}\left(\frac{v_0}{1-\zeta\left(\frac{v_0}{w}\right)^2} + \frac{4v_1}{1-\zeta\left(\frac{v_1}{w}\right)^2} + \frac{2v_2}{1-\zeta\left(\frac{v_2}{w}\right)^2} + \frac{4v_3}{1-\zeta\left(\frac{v_3}{w}\right)^2} + \frac{v_4}{1-\zeta\left(\frac{v_4}{w}\right)^2}\right).$$

Of course the accuracy is greater, when we take six, eight, or more parts. This formula takes into account the variability of the co-efficients of resistance, which in considerable velocities is necessary. For the free descent of bodies in air or water $P_1 = G$, and for motion on a horizontal plane $P_1 = 0$, is more correctly equal to the friction $f\,G$. Since this is a resistance, we have then to introduce it as negative into the calculation, whence

$$P = -(P_1 + P_2),\ \text{and}\ p = -\left[1+\zeta\left(\frac{v}{w}\right)^2\right]\frac{P_1}{G}g.$$

As it cannot be a question here of an increase, but only of a diminution of velocity, we have then to substitute in the above formula $v_0 - v_n$ for $v_n - v_0$.

In the case, where the body is urged by a force, by its weight for instance, the motion approximates more and more to a uniform one, so that after the lapse of a certain time, it may be considered as such, although not so in reality. The accelerating force $p = 0$, when

$$\zeta \cdot \frac{v^2}{2g} F\gamma = P_1,\ \text{when, therefore,}\ v = \sqrt{\frac{2g\,P_1}{\zeta\,F\gamma}}$$

The velocity of a falling body approximates, therefore, to this limit more and more, without ever actually attaining it.

Example. Piobert, Morin, and Didion found, for a parachute whose depth was 0,31 that of the diameter of its opening $\zeta = 1{,}94 \cdot 1{,}37 = 2{,}66$. Hence, from what height in Prussian feet will a man, of 150 lbs. weight, be able to descend with a similar parachute, of 10 lbs. weight and 60 square feet transverse section, without acquiring a greater velocity than that which he would have acquired by jumping from a 10 feet height, without a parachute? The last velocity is $v = 7{,}906\sqrt{10} = 25$ feet, the force is $P_1 = G = 150 + 10 = 160$ lbs., the surface $F = 60$ square feet, the density $\gamma = 0{,}0859$, and the co-efficient of resistance $\zeta = 2{,}66$, hence:

$\frac{1}{w^2} = \frac{60 \cdot 0{,}0859}{63{,}3 \cdot 160} = 0{,}000515$, and $\zeta \cdot \frac{v^2}{w^2} = 2{,}66 \cdot 0{,}000515 \cdot 25^2 = 0{,}85625$. If, therefore, we take 6 parts, we then obtain for these:

$1 - \zeta \cdot \frac{v^2}{w^2} = 0{,}97621;\ 0{,}90486;\ 0{,}78593;\ 0{,}61944;\ 0{,}40537;\ 0{,}14375$, and for

$\frac{v}{1-\zeta\frac{v^2}{w^2}} = 0;\ 4{,}268;\ 9{,}210;\ 15{,}905;\ 26{,}910;\ 51{,}393$, and $173{,}913$; from Simpson's rule the mean value is:

$= (1 \cdot 0 + 4 \cdot 4{,}268 + 2 \cdot 9{,}210 + 4 \cdot 15{,}905 + 2 \cdot 26{,}910 + 4 \cdot 51{,}393 + 1 \cdot 173{,}913) \div 3 \cdot 6$

$= \frac{532{,}42}{18} = 29{,}58$; and from this the space of descent sought:

$$s = \frac{v_n - v_0}{g}\ \text{times the mean value of}\ \frac{v}{1-\zeta \cdot \frac{v^2}{w^2}} = \frac{25 - 0}{31{,}25} \cdot 29{,}58 = 23{,}6\ \text{feet}.$$

The corresponding time of descent is since the mean value of $\frac{1}{1-\zeta\frac{v^2}{w^2}}$

$= (1.0 + 4 . 1{,}024 + 2 . 1{,}105 + 4 . 1{,}272 + 2 . 1{,}614 + 4 . 2{,}467 + 1 . 6{,}957) \div 18$
$= 1{,}747$, $t = \frac{25}{31.25} . 1{,}747 = 1{,}4$ sec.

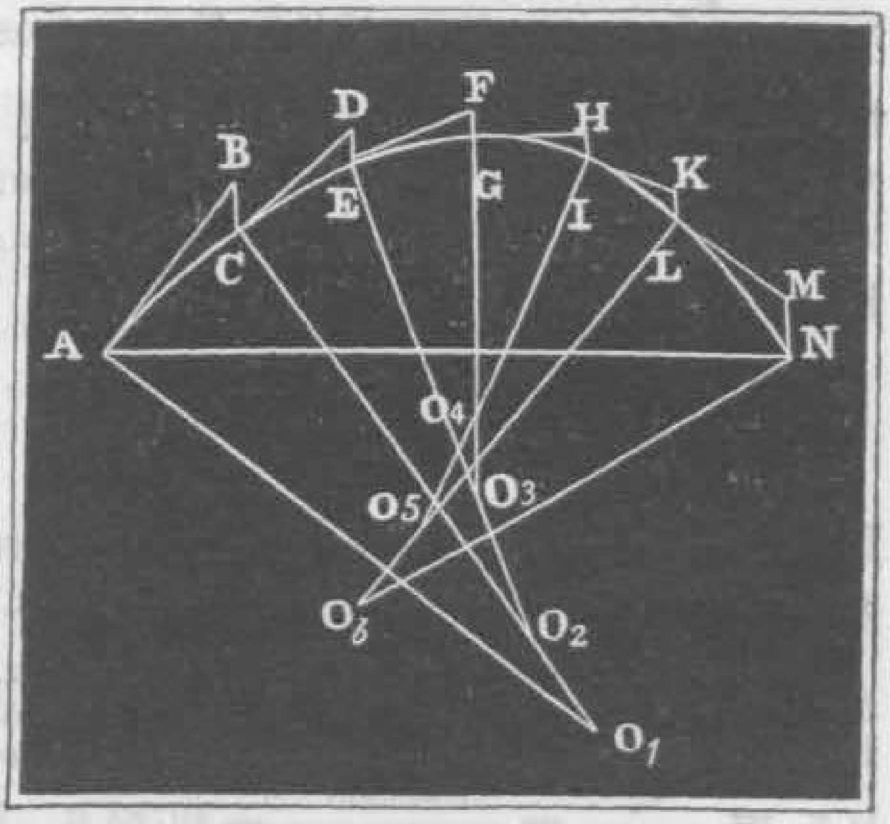
Remark. For a constant co-efficient of resistance, the higher calculus gives us:

$$v = \left(\frac{e^{\mu t}-1}{e^{\mu t}+1}\right)\sqrt{2g\frac{P}{\zeta F\gamma}} \text{ and } s = \frac{G}{\zeta F\gamma} L n \div \left(\frac{(e^{\mu t}+1)^2}{4e^{\mu t}}\right),$$

where $\mu = \sqrt{2\,g\,\zeta\frac{P F\gamma}{G^2}}$, e being the base of the hyperbolic system of powers, and Ln the hyperbolic logarithm.

§ 394. *Projectiles.*—We have already investigated the motion of projectiles in vacuo (§ 38), and found this motion to be parabolic; we may now obtain a more exact knowledge of motion in a resisting medium, and consider that, for instance, of a shot. In no case is the path AGN, Fig. 534, of a body passing through the air a symmetric curve; the portion GN in which the body descends is rather shorter, and, therefore, less inclined than the portion AG in which the body ascends, because the resistance of the air operating in the direction of motion tends always to shorten the portions of its path AC, CE, EG, &c., more and more; if, therefore, the first portion of the path AC, for motion in the air is only a little shorter than it would be in vacuo, the last portion LN is considerably shorter in the first motion than it is in the last. The construction of the path in a resisting medium by means of circles of curvature may be accomplised in the following manner.

Fig. 534.



From the initial velocity v_1, and the angle of elevation $BAN = \alpha_1$ it follows that the $\angle ABC = 90 - \alpha_1$, and $sin.\ ABC = cos.\ \alpha_1$, from § 40 the radius of curvature

$$O_1A = O_1C = r_1 = \frac{v_1^2}{g\cos.\alpha_1},$$

hence with this we may approximately describe the portion of arc AC. If now we assume the angle subtended at the centre $AO_1C = \phi^0_1$, therefore $AC = s_1 = r_1\,\phi_1$, we then obtain for the succeeding particle of space CE the angle of inclination $\alpha^0_2 = \alpha^0_1 - \phi^0_1$. Let further, the height of fall $BC = h_1$, and the measure of the retardation due to the air's resistance $\zeta . \frac{v_1^2}{2g} F\gamma$ being

$$\zeta \cdot \frac{v_1^2}{2g} \cdot \frac{F\gamma}{G} = \mu\, v_1^2, \text{ therefore } \zeta \cdot \frac{F\gamma}{2\,G} = \mu,$$

from the principle of *vires vivæ*, we then obtain for the velocity v_2 at the initial point of the second portion of arc:

$$\frac{v_2^2}{2g} = \frac{v_1^2}{2g} - h_1 - \mu\left(\frac{v_1^2+v_2^2}{2g}\right) s_1, \text{ or } (1+\mu s_1)\frac{v_2^2}{2g}$$

$$=(1-\mu s_1)\frac{v_1^2}{2g} - h_1, \text{ and hence } v_2 = \sqrt{\frac{(1-\mu s_1)\, v_1^2 - 2\,g\,h_1}{1+\mu s_1}}.$$

Since now the height of fall $h_1 = \frac{1}{2} g t^2 = \frac{1}{2} g \cdot \left(\frac{s_1}{s_1}\right)^2$, it follows that:

$$v_2 = \sqrt{\frac{(1-\mu s_1)\, v_1^2 - \left(\frac{g s_1}{v_1}\right)^2}{1+\mu s_1}} = v_1 \sqrt{\frac{1-\mu r_1 \phi_1 - \frac{\phi_1^2}{\cos.\, a_1^2}}{1+\mu r_1 \phi_1}}.$$

If we substitute these values of a_2 and v_2 in the equation:

$r_2 = \frac{v_2^2}{g \cos.\, a_2}$, we then obtain the radius of curvature $O_2C = O_2E$ of the succeeding portion of arc CE, and if we assume an angle of revolution $CO_2E = \phi_2$, it again follows from this that the angle of inclination in the vicinity of E: $a_3 = a_2\ \phi_2$, and the velocity at this point

$$v_3 = v_2 \sqrt{\frac{1-\mu r_2 \phi_2 - \frac{\phi_2^2}{\cos.\, a_2^2}}{1+\mu r_2 \phi_2}}.$$

It is therefore easy to see how the entire path of the projectile may be successively composed of circular arcs.

Example. A cast-iron ball, of 4 inches diameter, is shot off at an angle of elevation of $50°$ with a velocity of 1000 feet, required its path, if only approximately, according to Prussian weights and measures. The radius of curvature of the first portion of arc is $r_1 = \frac{v_1^2}{g \cos.\, a} = \frac{1000000}{31{,}25 \cos.\, 50°} = 49783$ ft. As the density of the air $= 0{,}0859$, and that of cast-iron $= 470$ lbs., we have then $\mu = \zeta \cdot \frac{F\gamma}{2G} = \zeta \cdot \frac{3.3.0{,}0859}{4.470} = 0{,}00041122$ $. \zeta$; now for $v = 1000$, $\zeta = 0{,}90$, hence $\mu = 0{,}0003701$. If we take an arc of $1°$ only, we then obtain the velocity at the end of it:

$$v_2 = 1000 \sqrt{\frac{1 - 0{,}0003701 . 49783 . 0{,}017453 - (0{,}017453 \div \cos.\, 50°)^2}{1 + 0{,}0003701 . 49783 . 0{,}017453}}$$

$= 7697$ feet.

and the radius of curvature for a second portion of arc:

$$r_2 = \frac{(769{,}7)^2}{31{,}25 \cos.\, 49°} = 28897 \text{ feet.}$$

For $v_2 = 769{,}7$ feet, $\zeta = 0{,}81$, therefore $\mu = 0{,}0003331$. If, therefore, we describe with the last radius, an arc $\phi_2 = 2°$, the velocity at its ending point will be

$$v_3 = 769{,}7 \sqrt{\frac{1 - 0{,}33598 - 0{,}002831}{1{,}33598}} = 541{,}47 \text{ feet.}$$

For a third arc Q_3, the radius of curvature $r_3 = 13757$ feet, and if, therefore, we assume $\zeta = 0{,}75$, we shall then obtain at the end of a length of are of $4°$, the velocity $v_4 = 398{,}85$ feet. The radius of curvature for a fourth are may be likewise found $r_4 = 6960{,}5$ by assuming $\zeta = 0{,}72$, and we shall then obtain the velocity $v_5 = 288{,}85$ feet, at the end of an arc of $8°$, from which a fifth radius of curvature $r_5 = 3259$ feet may be calculated. Proceeding in this manner, we shall obtain, by degrees, the collective elements for the construction of the line of projection in question.