THE FUNDAMENTAL GROUP, HOMOLOGY, AND COHOMOLOGY OF TORIC ORIGAMI 4-MANIFOLDS

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THE FUNDAMENTAL GROUP, HOMOLOGY, AND COHOMOLOGY OF TORIC ORIGAMI 4-MANIFOLDS

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This is a collection of algebraic topological results for toric origami manifolds, mostly in dimension 4. Using a known formula for the fundamental group of a compact orientable toric origami manifold, a list of all groups obtainable as the fundamental group of a compact orientable toric origami 4-manifold is given, along with example manifolds that realize them. The known fundamental group formula is generalized to compact non-orientable toric origami manifolds of all dimensions. The homology and cohomology groups of toric origami 4-manifolds are explicitly constructed with generators realized as embedded submanifolds, and the intersection form and cohomology ring are calculated.

BIOGRAPHICAL SKETCH

Ian Pendleton was born in Santa Monica, CA in 1991. His family settled in the San Francisco Bay Area in Pleasanton, CA in 1996. He attended UC Berkeley from 2009 - 2013 and earned a Bachelor's degree in Mathematics. He then moved across the country to attend Cornell University from 2013 - 2019. He earned a Master's degree in Mathematics in 2016.

For my grandmother Ruth, who I miss more than anything.

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CHAPTER 1

INTRODUCTION

1.1 Thesis Overview

This thesis is a collection of results concerning the topology of toric origami manifolds, mostly in dimension four. Toric origami manifolds are a generalization of toric symplectic manifolds. Symplectic geometers often use toric symplectic manifolds as a source of inspiration and examples because $2n$-dimensional toric symplectic manifolds are in one-to-one correspondence with Delzant polytopes in $\mathbb{R}^n$. In other words, all of these manifolds' interesting symplectic geometry is captured by the combinatorics of the half-dimensional polytopes that represent them.

There is a similar correspondence for toric origami manifolds, which is what makes them so interesting to study. Each $2n$-dimensional toric origami manifold can be represented by a collection of n-dimensional Delzant polytopes together with "folding data" that records the gluing of specific facets of the polytopes to one another. These polytopes and folding data together are called an origami template for the toric origami manifold.

In [3] and [4], Ana Cannes da Silva and her collaborators start from the symplectic geometry, build up the definitions for folded symplectic forms and origami forms, and then explain how to prove the one-to-one correspondence of toric origami manifolds to origami templates. This viewpoint is the best when the goal is to study and understand the origami forms on the toric origami manifolds.

However, this thesis will largely ignore the symplectic and origami geometry. The results we prove will be about the underlying topology forced on the manifolds by the symplectic and origami structure, so we will take a slightly different approach. We will begin by defining an origami template. We will then explain how to construct a topological manifold from an origami template. Finally, we will define and explain how these constructed manifolds are endowed with an origami form, but this will be incidental to the topological results presented.

1.2 How Does a Polytope Become a Manifold?

A polytope in $\mathbb{R}^n$ is the convex hull of a finite number of points in $\mathbb{R}^n$. A facet of a polytope is a top-dimensional (i.e. $n-1$) face.

Definition. A **Delzant polytope** in $\mathbb{R}^n$ is a polytope that is:

- **simple**: there are n facets adjacent to each vertex
- **rational**: the primitive normal vector to each facet lives in the $\mathbb{Z}^n$ lattice
- **smooth**: at each vertex, the primitive normal vectors to the n facets adjacent to the vertex form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

In order to construct a manifold from a Delzant polytope, we are going to rely heavily on a chosen isomorphism $\mathfrak{t}^n \cong \mathbb{R}^n$ where $\mathfrak{t}^n$ is the Lie algebra associated to the n-dimensional torus T^n. Our isomorphism sends the generator of the i-th copy of $\mathfrak{t}$ to the basis element $x_i \in \mathbb{R}^n$. Thus the i-th S^1 factor in T^n can now be identified with the i-th basis element of $\mathbb{R}^n$. This induces an isomorphism between $\mathbb{Z}^n$ and $\pi_1(T^n)$ which takes x_i to $[S^1_i]$, the loop that goes once around the i-th circle factor of T^n. For example, the vector $(2, -3) \in \mathbb{Z}^2$ corresponds to the loop in $\pi_1(T^2)$ that goes 2 times around the first circle factor (in the positive direction) and 3 times around the second circle factor (in the negative direction).

Let P be an n-dimensional Delzant polytope embedded in $\mathbb{R}^n$, with facets $F_1, \ldots, F_m$. Let $\nu_i \in \mathbb{Z}^n$ be the primitive outward normal vector to the facet F_i. To construct the $2n$-dimensional manifold M_P that corresponds to P, do the following:

$$M_P := (P \times T^n)/\sim,$$

where $(p_1, t_1) \sim (p_2, t_2)$ if and only if $p_1 = p_2 \in F_i$ for some facet F_i and $t_2 - t_1 = r\nu_i$ for some $r \in \mathbb{R}$. In each fiber over the facet F_i, the relation $\sim$ collapses the circle of T^n spanned by the normal vector ν_i, leaving a fiber homeomorphic to T^{n-1}. This is not a fiber bundle because the dimension of the fibers drops over the facets, but it helps to think of M_P as having base space P with T^n fibers and weirdness over the facets.

Let $q : M_P \to P$ be the projection $q(p, t) = p$. The preimage $q^{-1}(F_i)$ of a facet F_i is a codimension-2 submanifold of M. As an example, consider P as the standard 2-simplex embedded in $\mathbb{R}^2$ (Exercise: quickly check that P is Delzant). See Figure 1.1.

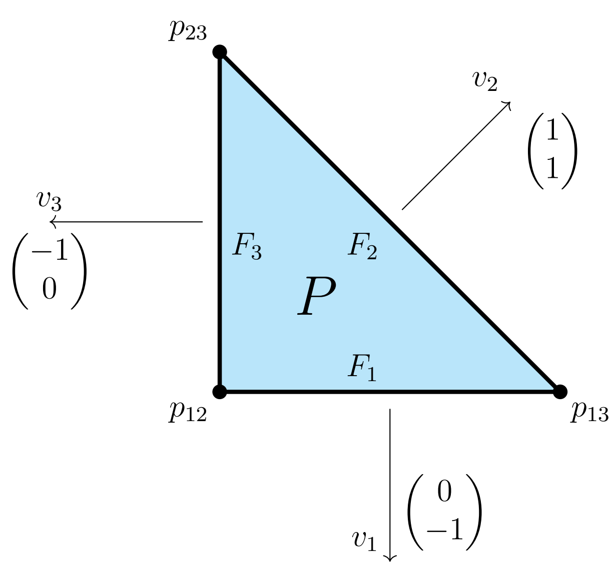


Figure 1.1: The Delzant polytope P corresponding to $M_P \cong \mathbb{C}P^2$.

The corresponding manifold M_P has facets F_1, F_2, and F_3. The preimage of each facet $q^{-1}(F_i)$ is homeomorphic to a 2-sphere S^2. Notice that the preimage of the vertex p_{12} in the intersection of F_1 and F_2 has the circle fibers corresponding to both v_1 and v_2 collapsed. The smooth condition on P means that v_1 and v_2 span all of $\mathbb{Z}^2$, and thus the entire T^2 fiber over p_{12} is collapsed. In particular, $q^{-1}(p_{12})$ is a single point.

Although it's neither obvious nor trivial, the 4-dimensional manifold M_P corresponding to the 2-simplex P in Figure 1.1 is homeomorphic to $\mathbb{C}P^2$. The preimages of all three facets are homologous, and they are all representatives of the homology class of the non-trivial $\mathbb{C}P^1$ inside $\mathbb{C}P^2$.

1.3 Toric Symplectic Definitions

At this point we have shown how to construct a manifold from a Delzant polytope. However, we promised a toric symplectic manifold. This requires some definitions. Again, for the results in this thesis the toric and symplectic structures are mostly unnecessary. We include their definitions and constructions because we expect anyone who cares about the topology of these manifolds will also care about how the toric and symplectic structures interact with the topology. A good introduction to symplectic manifolds can be found in [2].

A **symplectic manifold** (M, ω) is a manifold M along with a 2-form $\omega \in \Omega^2(M)$ called the **symplectic form** which is closed (i.e. $d\omega = 0$) and non-degenerate (i.e. M is even dimensional and the top wedge-power of ω is a volume form on M).

Suppose that a compact connected abelian Lie group $T^n = (S^1)^n$ acts on M preserving ω. The action is **weakly Hamiltonian** if for every vector $\xi \in \mathfrak{t}$ in the Lie algebra $\mathfrak{t}^n$ of T^n, the vector field

$$V_\xi(x) = \frac{d}{dt}\bigg|_{t=0} \exp(t\xi) \cdot x$$

is a **Hamiltonian vector field**. That is, we require $\omega(V_\xi, \cdot)$ to be an exact one-form

$$\omega(V_\xi, \cdot) = d\mu^\xi.$$

Each μ^ξ is a smooth function on M defined up to a constant by the differential equation above. Taking them together we may define a **moment map**

$$\mu : M \to (\mathfrak{t}^n)^*, \quad x \mapsto \Big(\xi \mapsto \mu^\xi(x)\Big).$$

The action is **Hamiltonian** if the moment map μ can be chosen to be T^n-invariant with respect to the coadjoint action of $(\mathfrak{t}^n)^*$. Atiyah [1] and Guillemin and Sternberg [9] have shown that when M is a compact Hamiltonian T^n-manifold, the image $\mu(M)$ is a convex polytope, and is the convex hull of the images of the fixed points of the T^n action.

If the Hamiltonian action is **effective** (i.e. no non-trivial subgroup of T^n acts trivially on M), then $\dim(T^n) \leq \frac{1}{2}\dim(M)$. The action is called **toric** if it is effective and this inequality is an equality, so $\dim(M) = 2n$. A symplectic manifold M^{2n} with a toric Hamiltonian T^n action is called a **toric symplectic manifold**.

Fun fact: if we start with a Delzant polytope P, the manifold M_P we construct has a symplectic form ω which can be described locally in coordinates. Let $x_1 \ldots x_n$ be coordinate functions for $\mathbb{R}^n$ around $p \in P$. Let $t_1, \ldots t_n$ be the corresponding coordinate functions for the torus fiber T^n above p. Then $\omega_p = \sum_{i=1}^n dx_i \wedge dt_i$.

In addition, M_P inherits the structure of a toric symplectic manifold from the action of T^n on each T^n fiber. The moment map $\mu : M_P \to (\mathfrak{t}^n)^* \cong \mathbb{R}^n$ is exactly the projection map q which is also the quotient map for the T^n action. In particular, the orbit space for the T^n action on M_P is P. None of this is trivial, and is in fact the meat of Delzant's theorem which is nicely presented in [2]. As mentioned earlier, toric symplectic manifolds are well-understood from many different perspectives and provide a great class of examples for symplectic geometers. This thesis will attempt to take the intuition from toric symplectic manifolds and expand it to the more general situation of toric origami manifolds.

1.4 How Does an Origami Template Become a Manifold?

Let $\mathcal{D}_n$ be the set of all Delzant polytopes in $\mathbb{R}^n$ and let $\mathcal{E}_n$ be the set of all subsets of $\mathbb{R}^n$ which are facets of elements of $\mathcal{D}_n$.

Definition. An n-dimensional **origami template** is a graph $G = (V, E)$, called the **template graph**, along with a pair of maps $\Psi_V : V \to \mathcal{D}_n$ and $\Psi_E : E \to \mathcal{E}_n$ such that:

1. If e is an edge between v_1 and v_2, then $\Psi_E(e)$ is a facet of each of the polytopes $\Psi_V(v_1)$ and $\Psi_V(v_2)$, and these polytopes agree on a neighborhood of $\Psi_E(e)$.
2. If $e_1, e_2 \in E$ are two edges of G adjacent to $v \in V$, then $\Psi_E(e_1)$ and $\Psi_E(e_2)$ are disjoint facets of $\Psi_V(v)$.

The manifold M we will construct from the origami template will again have a T^n action on it, so we start by constructing the **template-space** X which will end up being M/T. We define X as

$$X = \left(\bigsqcup_{v \in V} \Psi_V(v) \right) \Big/ \sim$$

where $x \in \Psi_V(u)$ is equivalent to $y \in \Psi_V(v)$ if and only if there exists an edge $e \in E$ with endpoints u and v so that $x = y \in \Psi_E(e)$. Thus X is the disjoint union of the Delzant polytopes corresponding to the vertices of G where the facets corresponding to each edge of G have been glued via the identity map. In two dimensions, X will resemble the folded paper art from which it gets its name. See Figure 1.2 for an example. In Figure 1.2, the Delzant polytopes $P_1, \dots, P_4$ are glued along the red facets $F_1, \dots, F_4$.

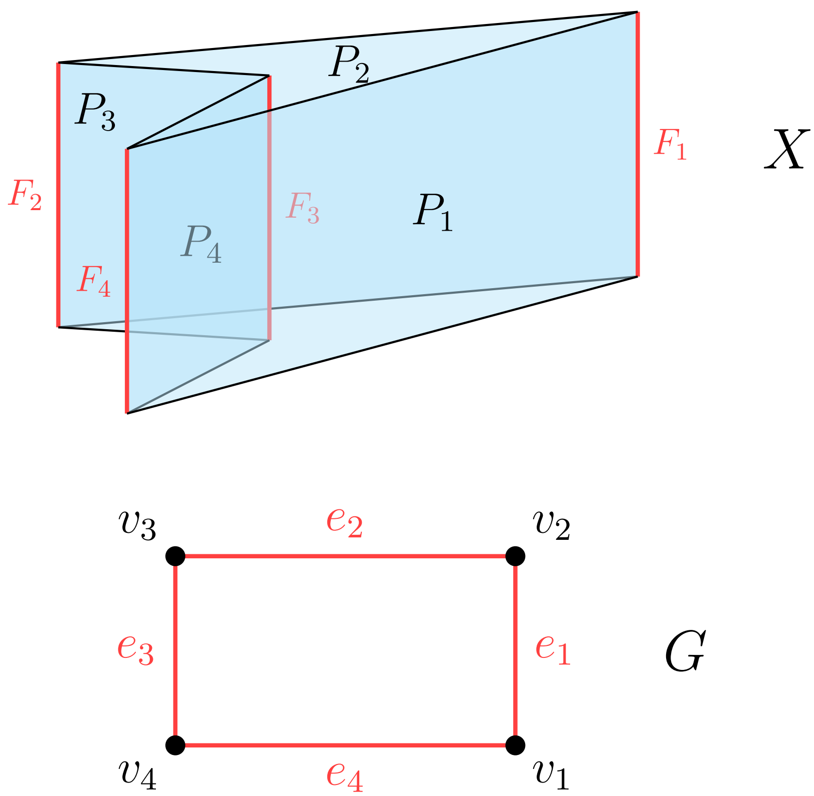


Figure 1.2: An example template-space X and corresponding template graph G.

A facet is called **folded** if it is $\Psi_E(e)$ for some $e \in E$ (i.e. it folds or glues two Delzant polytopes in X together via $\sim$). All other facets are called **unfolded**. A folded facet whose corresponding edge connects two distinct vertices of G is called a **coörientable**. A folded facet whose corresponding edge is a loop adjacent to a single vertex is called **non-coörientable**. In the template graph X, the coörientable facets glue distinct Delzant polytopes together. The non-coörientable facets are glued to themselves via the identity map, and therefore are indistinguishable from the unfolded facets. However, we remember they are folded for the construction of M.

To construct M, we mimic the toric symplectic case, but we will only be collapsing torus fibers above unfolded facets and non-coörientable facets. That is, given an n-dimensional origami template with template-space X define then $2n$-dimensional manifold M_X as

$$M_X := (X \times T^n)/\sim,$$

where $(x, t_1) \sim (x, t_2)$ if $x \in F$ for any *unfolded* facet F of X and $t_2 - t_1 = rv$ for some $r \in \mathbb{R}$ and v the normal vector to the facet F. Additionally, $(x, t_1) \sim (x, t_2)$ if $x \in F$ for any *folded, non-coörientable* facet F of X where t_1 and t_2 are antipodal points of the S^1 factor of T^n defined by v the normal vector to the facet F (in particular, $2(t_2 - t_1) = kv$ for some $k \in \mathbb{Z}$).

That is, in each T^n fiber above a point in an unfolded facet we collapse the circle corresponding to the facet's normal vector. Thus the fiber above a point in an unfolded facet will be homeomorphic to T^{n-1}. In each T^n fiber above a point in a non-coörientable folded facet we quotient the circle corresponding to the facet's normal vector by the antipodal map. This leaves fibers still homeomorphic to T^n.

Note that the manifold M_X has a T^n action given by the action of T^n on each T^n fiber. The orbit space M/T of this action of T^n on M_X is exactly X. We let $q : M \to M/T$ be the quotient map by the T-action. The antipodal map quotient makes a $q^{-1}(F)$ a one-sided codimension 1 submanifold of M for each non-coörientable facet F. The existence of non-coörientable folded facets will imply that M is non-orientable. In Chapters 2 and 4 we will be assuming M to be orientable, and thus all folded facets will be coörientable. Chapter 3 will deal with the non-orientable situation.

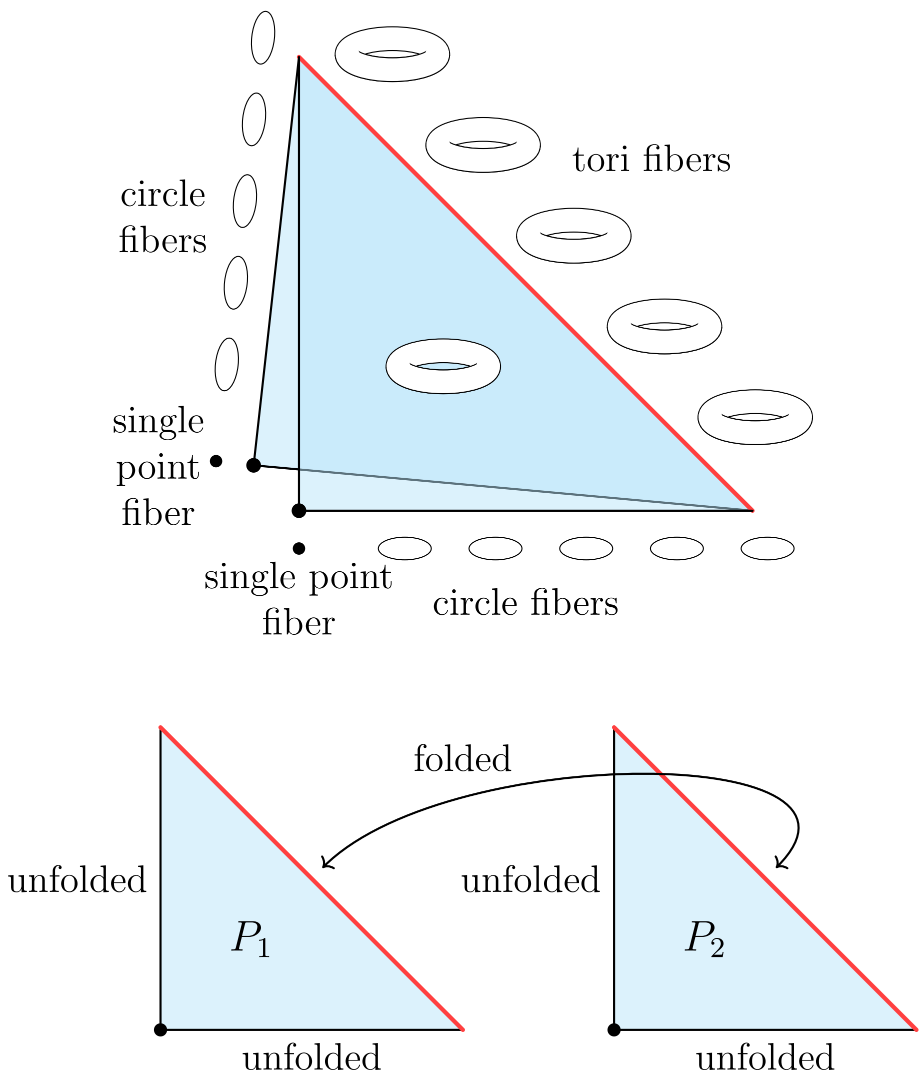


Figure 1.3: A template-space X for two polytopes folded along a single facet. The q^{-1} fibers above some points are illustrated. The constructed manifold is S^4.

See Figure 1.3 for an example topological construction. The template-space X consists of two identical polytopes $P_1 = \Psi_V(v_1)$ and $P_2 = \Psi_V(v_2)$ folded together along a single facet. The fibers $q^{-1}(p)$ above points p in the interior of the P_i are the full T^2 tori. The fibers above points contained in exactly one of the unfolded facets are $T^2/S^1 \cong S^1$, where the collapsed S^1 corresponds to the unfolded facet's normal vector. The fibers above the points contained in two unfolded facets (i.e. the bottom-left corner of either P_i) are single points, which are fixed points of the T^2 action. Fibers above the folded facet act like fibers above the polytope interior: they are full torus orbits above the interior of the facet, and circle orbits above the ends where the folded facet meets an unfolded facet. The manifold M constructed from this particular origami template is S^4, where $q^{-1}(P_1)$ and $q^{-1}(P_2)$ are both homeomorphic to D^4, and $q^{-1}(F)$ for the folded facet F is the boundary S^3 of D^4.

1.5 Toric Origami Definitions

We define a **folded symplectic form** on a $2n$-dimensional manifold M to be a 2-form $\omega \in \Omega^2(M)$ that is closed ($d\omega = 0$), whose top power ω^n intersects the zero section transversely on a subset Z of M, and whose restriction to points in Z has maximal rank. The transversality condition forces Z to be a codimension-1 embedded submanifold of M. We call Z the **folding hypersurface** or **fold**. Let $i : Z \hookrightarrow M$ be the inclusion of Z into M. The maximal rank assumption implies that $i^*\omega$ has a 1-dimensional kernel on Z. This line field is called the **null foliation** on Z. An **origami manifold** is a folded symplectic manifold (M, ω) whose null foliation is fibrating: $\pi : Z \to B$ is a fiber bundle with orientable circle fibers over a compact base B. The form ω is called an **origami form** and the bundle π is called the **null fibration**. A diffeomorphism ϕ between two origami manifolds (M_1, ω_1) and (M_2, ω_2) is called an **origami-symplectomorphism** if $\phi^*\omega_2 = \omega_1$.

Since the definition of a Hamiltonian action depends only on ω being closed, we may define moment maps and toric actions on origami manifolds exactly as we did on symplectic manifolds in Section 1.3.

The big picture is that an oriented origami manifold (M, ω) with folding hypersurface Z can be **unfolded** into a symplectic manifold. To do this, we first take the closures of the connected components of $M \setminus Z$. The result is a manifold with boundary, whose boundary is two copies of Z. We then use the origami structure to collapse the circle fibers of the null fibration: $p_1 \sim p_2$ if p_1 and p_2 are boundary points in the same fiber of the null fibration. The result is a disconnected smooth manifold $M_0 := (M \setminus Z) \cup B_1 \cup B_2$, where B_1 and B_2 are copies of the compact base B of the null fibration. M_0 has one connected component for each Delzant polytope in the origami template for M (i.e. for each vertex in M's template graph), and is exactly the disjoint union of the corresponding toric symplectic manifolds. Therefore M_0 inherits a symplectic form which on $M_0 \setminus (B_1 \cup B_2)$ coincides with the origami form on $M \setminus Z$. Because this process can be achieved using symplectic cutting techniques, the manifold M_0 is called the **symplectic cut space**, the connected components are called **symplectic cut pieces**, and the whole process is called **cutting**. The symplectic cut space of a nonorientable origami manifold is the $\mathbb{Z}_2$-quotient of the symplectic cut space of its orientable double cover.

Since M_0 is a symplectic manifold, it inherits an orientation from ω^n, the top wedge power of the symplectic form. If M is an orientable manifold with a chosen volume form vol_{2n}, then on each cut piece of M_0, the orientation from ω^n either agrees or disagrees with the orientation from vol_{2n}. Thus $M_0 = M_+ \sqcup M_-$ where M_+ is the disjoint union of the components of M_0 on which the orientations agree, and M_- is the disjoint union of the components of M_0 on which the orientations disagree. The transversality condition on the fold Z demands that if a cut piece $M_i \subseteq M_+$ is folded to a cut piece M_j, then $M_j \subseteq M_-$ (and vice versa). In particular, if $\omega^n > 0$ on one side of the folding hypersurface, then $\omega^n < 0$ on the other side.

1.6 The 1-Skeleton of M

It will be very helpful to have a way to reference the collections of unfolded facets of a template-space X that have been glued together as single entities. As an example, let P_i and P_j be Delzant polytopes within X, folded together by the facet F. If U_i is an unfolded facet in P_i adjacent to F, then there is a corresponding unfolded facet U_j in P_j adjacent to F with the same normal vector as U_i. The unfolded facets U_i and U_j are glued together at the edge of the folded facet F. When X is 2-dimensional (so M_X is 4-dimensional), $q^{-1}(U_i)$ and $q^{-1}(U_j)$ are hemispheres that glue together along their "equators" in F to form a 2-sphere $q^{-1}(U_i \cup U_j)$. We will refer to $q^{-1}(U_i \cup U_j)$ as a component of the 1-skeleton of M.

Formally, we define the 1-skeleton of M to be the union of all the 0 and 1-dimensional T orbits. Thus, if M is 4-dimensional then the 1-skeleton is q^{-1} of the collection of unfolded facets (i.e. edges) in M/T.

If a connected component of the 1-skeleton contains no T-fixed points, then it must be a circles' worth of circle orbits and therefore homeomorphic to $S^1 \times S^1 \cong T^2$. If a connected component of the 1-skeleton has j fixed points, then it is homeomorphic to $j-1$ spheres cyclically connected with the north pole of the i-th sphere glued to the south pole of the $(i+1)$-st sphere (think a torus with j pinched meridian circles).

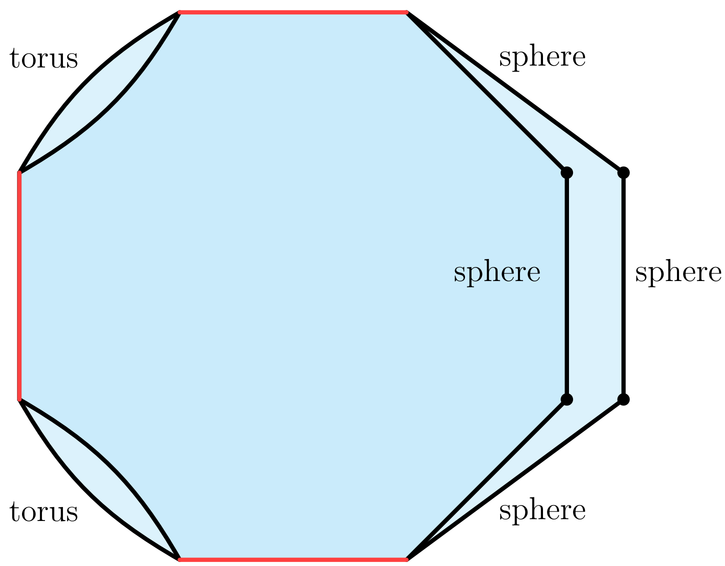


Figure 1.4: M/T with its 1-skeleton components labeled.

The notation here is messy because we want to distinguish between “components” and “connected components” of the 1-skeleton. We will use “connected component” as it is usually defined in topology. However, we will define a component of the 1-skeleton to mean a single torus or sphere in the 1-skeleton. In particular, if a connected component of the 1-skeleton has n fixed points, then it is comprised of n spheres and therefore n components. If a connected component has no fixed points, then it is a torus and is also a component.

CHAPTER 2

ORIENTABLE FUNDAMENTAL GROUPS

2.1 Orientable Fundamental Group Introduction

The goal of this chapter will be to prove a result about the fundamental group of toric origami manifolds. In [11], Holm and Pires prove that for a toric origami manifold $(M^{2n}, \omega, T^n, \mu)$ the fundamental group is given by $\pi_1(M) \cong N/N_X \times \pi_1(G)$, where N/N_X is a lattice quotient and G is the template graph for M. In particular, $N \cong H_1(T^n) \cong \mathbb{Z}^n$ and N_X is the span of the loops in $H_1(T^n)$ corresponding to the normal vectors to the unfolded facets of $X = M/T$. See Section 1.4 for a detailed explanation of the space X. Since adjacent facets cannot be folded, the Delzant smoothness condition will ensure that N_X always contains at least $n-1$ linearly independent vectors in $\mathbb{Z}^n$. Therefore N/N_X will always be isomorphic to $\mathbb{Z}/k\mathbb{Z}$ for some non-negative integer k, where $k = 0$ implies $N/N_X \cong \mathbb{Z}$, and $k = 1$ implies $N/N_X \cong 1$. Since G is a graph, $\pi_1(G)$ will always be a free group F_ℓ for some non-negative integer ℓ.

This chapter addresses the following question: For which combinations of non-negative integers k and ℓ does there exist a compact, orientable, 4-dimensional toric origami manifold M such that $\pi_1(M) \cong \mathbb{Z}/k\mathbb{Z} \times F_\ell$? For the combinations where such an M exists, we will produce an example. In all other cases, we will prove that no such M can exist.

2.2 Orientable Fundamental Group Theorem

Theorem 2.2.1 (Main Theorem)**.** *The following combinations of k and ℓ are the only ones for which there exists a compact, orientable, toric origami 4-manifold M with $\pi_1(M) \cong N/N_X \times \pi_1(G)$:*

N/N_X	$\pi_1(G)$	Proof
1	F_ℓ for any $\ell \geq 0$	Theorem 2.3.1
$\mathbb{Z}$	F_1	Theorem 2.3.2
$\mathbb{Z}/2\mathbb{Z}$	F_ℓ for any $\ell \geq 1$	Theorem 2.4.1
$\mathbb{Z}/3\mathbb{Z}$	F_ℓ for odd $\ell \geq 1$	Theorem 2.5.4
$\mathbb{Z}/k\mathbb{Z}$ for $k \geq 4$	F_1	Theorem 2.6.3

Note that $F_1 \cong \mathbb{Z}$ and that we let $F_0 := 1$.

We begin by outlining some preliminary tools and ideas that will be crucial to the proof of the theorem. First, note that if $\pi_1(G) \cong 1$ then $N/N_X \cong 1$ as well. This is Corollary 2-16 in [11], which states that $\pi_1(M) \cong 1$ if and only if $\pi_1(G) \cong 1$. The proof is that if $\pi_1(G) \cong 1$, then G has a leaf node. The polytope represented by the leaf node must have an unfolded vertex, and hence the Delzant smoothness condition at that vertex will force $N/N_X \cong 1$. Conversely, if N/N_X is non-trivial then it must be that $\pi_1(G)$ is non-trivial as well.

The second important tool is the correspondence between polygon corner chops and symplectic blow-ups. Suppose M is a toric symplectic manifold with Delzant polygon P. If $\tilde{P}$ is a Delzant polygon obtained from P by chopping off a corner, then the toric symplectic manifold $\tilde{M}$ represented by $\tilde{P}$ is a symplectic blow-up of M. When chopping off a corner, there is exactly one choice of normal vector to the new facet which will preserve the smoothness (and therefore Delzant-ness) of the new polygon. To chop off a corner between facets with normal vectors v and w, the new facet must have normal $v + w$. Corner chopping is important because:

Proposition 2.2.2. *Every compact toric symplectic 4-manifold can be obtained from either $\mathbb{C}P^2$ or a Hirzebruch surface by a succession of symplectic blow-ups at fixed points of the T-action.*

This translates into the discrete geometry of polygons as: We can obtain any 2-dimensional Delzant polygon by a sequence of corner chops, starting with either the standard triangle representing $\mathbb{C}P^2$ or one of the trapezoids representing a Hirzebruch surface. Hirzebruch surfaces are a family of symplectic manifolds whose Delzant polytopes are trapezoids with normal vectors $(0,-1)$, $(-1,0)$, $(0,1)$, and $(1,k)$ for $k \in \mathbb{Z}_{\geq 0}$. If k is even, then the Hirzebruch surface is homeomorphic to $S^2 \times S^2$. If k is odd then the Hirzebruch surface is homeomorphic to the twisted S^2 bundle over S^2. See [13, Lemma 2.15] for a detailed statement of the proposition, and see [8, Section 2.5] for a proof.

Finally, it should be noted here that if the goal is to obtain a toric origami manifold with $N/N_X \cong \mathbb{Z}/k\mathbb{Z}$ for $k \geq 1$, then no vertex can be left unfolded in the origami template. By the Delzant smoothness condition, if any vertex is unfolded then N/N_X will be trivial. However, the definition of an origami template also requires that adjacent facets not both be folded. Therefore every polygon in the origami template must have an even number of facets, every other one of which is folded. In the case where the origami template is two identical copies of some polygon P, it follows that to increase the number of generators in $\pi_1(G)$ by 1 we must create two additional facets in each copy of P, one pair of which can be folded together to create a new edge in G. We will use this strategy in all of the example manifolds we create.

2.3 Proof in Base Cases

Theorem 2.3.1. *For any $\ell \geq 0$ there exists a compact, orientable, toric origami manifold M so that $\pi_1(M) \cong 1 \times F_\ell$.*

Proof of Theorem 2.3.1. Any toric symplectic manifold M (take $\mathbb{C}P^2$ for example) satisfies $\pi_1(M) \cong 1$, which produces the $\ell = 0$ case. For the $\ell > 0$ case, start with the Delzant triangle for $\mathbb{C}P^2$. Do any sequence of 2ℓ corner chops to get a new Delzant polygon P which has $2\ell + 3$ facets. Take two identical copies of P and create an origami template by folding the two copies of P together via the identity map on any set of $\ell + 1$ non-adjacent facets. The resulting origami template will have an unfolded vertex, so $N/N_x \cong 1$. The template graph G will have 2 vertices and $\ell + 1$ edges, so $\pi_1(G) \cong F_\ell$ as desired. $\square$

Theorem 2.3.2. *If M is a compact, orientable, toric origami 4-manifold with $N/N_X \cong \mathbb{Z}$, then it must be that $\pi_1(G) \cong F_1$. There exists such an M with $\pi_1(M) \cong \mathbb{Z} \times \mathbb{Z}$.*

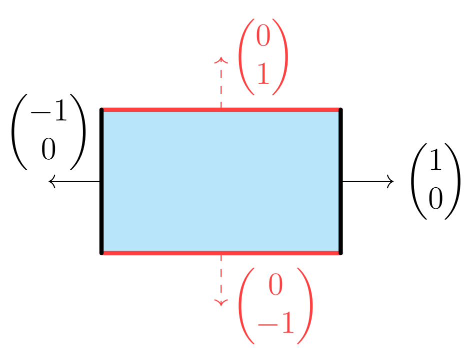


Figure 2.1: Delzant polygon P.

Proof of Theorem 2.3.2. This is the "prismatic" case from Definition 2-12 and Corollary 2-13 of [11]. Corollary 2-13 states that if a compact orientable toric origami manifold M has $N/N_X \cong \mathbb{Z}$, then $\pi_1(G) \cong \mathbb{Z} \cong F_1$ as well.

To get such an M, start with the Delzant polygon P representing $S^2 \times S^2$ which is a square or rectangle whose facets have outward pointing normal vectors:

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

To create an origami template for M, start with two copies of the polygon P. Fold together the two copies of the facet with normal vector $(0, -1)$ via the identity map. Do the same for the two copies of the facet with normal vector $(0, 1)$. This leaves only the facets with normal vectors $(-1, 0)$ and $(1, 0)$ unfolded. See Figure 2.1, where the black facets with solid normal vectors are unfolded, and the red facets with dashed normal vectors are folded.

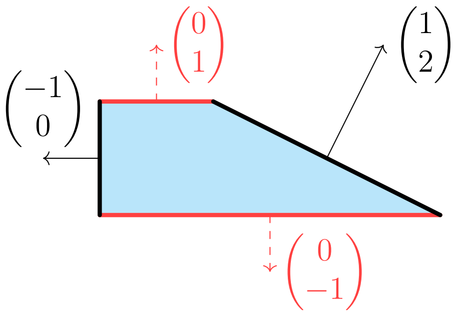
Recall that $N \cong \mathbb{Z}^2$ for all 4-dimensional M, and that the unfolded normal vectors generate N_X. Therefore

$$N/N_X \cong \mathbb{Z}^2/\left\langle(-1, 0), (1, 0)\right\rangle \cong \mathbb{Z}.$$

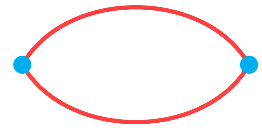
Since the template graph G has 2 vertices and 2 edges, we get that $\pi_1(G) \cong F_1 \cong \mathbb{Z}$, as desired. $\square$

2.4 Proof in $N/N_X \cong \mathbb{Z}/2\mathbb{Z}$ Case

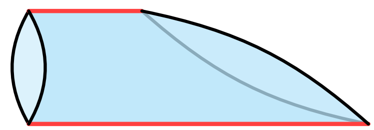
Theorem 2.4.1. *For any $\ell \geq 1$ there exists a compact, orientable, toric origami 4-manifold M such that $\pi_1(M) \cong \mathbb{Z}/2\mathbb{Z} \times F_\ell$.*



(a) Moment image of M_1.



(b) M_1 template graph G.



(c) Orbit space for M_1.

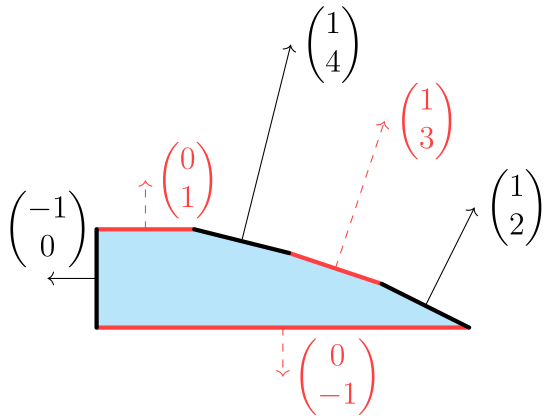
Figure 2.2: Origami template information for M_1.

Proof of Theorem 2.4.1. Let P_1 be the quadrilateral representing a Hirzebruch surface whose facets have the following outward primitive normal vectors:

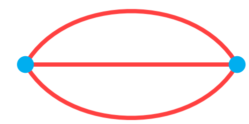
$$\begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}.$$

In the $\ell = 1$ case, begin with two identical copies of P_1. Create an origami template by first folding the two facets with outward normal vectors $(0, -1)$ together via the identity map, and then folding the two facets with outward normal vectors $(0, 1)$ together via the identity map. Let M_1 be the toric origami four manifold represented by this origami template. Figure 2.2a shows the moment map image of M_1. The red facets with dashed normal vectors are the folded facets. The black facets with solid normal vectors are the unfolded facets.

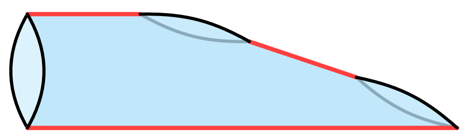
Recall that N_X is generated by the primitive normal vectors to the unfolded facets. Therefore,



(a) Moment image of M_2.



(b) M_2 template graph G.



(c) Orbit space for M_2.

Figure 2.3: Origami template information for M_2.

$$N/N_X \cong \mathbb{Z}^2 / \langle(-1,0),(1,2)\rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

The template graph G pictured in Figure 2.2b has two vertices connected by two edges. Each vertex represents one of the polygons seen in 2.2a, and each edge represents one of the red facets folding the polygons together. Therefore $\pi_1(G) \cong F_1$. Therefore $\pi_1(M_1) \cong \mathbb{Z}/2\mathbb{Z} \times F_1$.

Now consider the $\ell = 2$ case. By performing a corner chop between the $(1,2)$ facet and the $(0,1)$ facet, we get a new facet with normal vector $(1,3)$. Chopping again between $(1,3)$ and $(0,1)$ gives a polygon P_2 with normal vectors:

$$\begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}.$$

Create an origami template by taking two copies of P_2 and folding together the pairs of facets with normal vectors $(0,-1)$, $(1,3)$, and $(0,1)$.

The corresponding manifold M_2 will have

$$N/N_X \cong \mathbb{Z}^2/\langle(-1,0),(1,2),(1,4)\rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

Further, the template graph G pictured in Figure 2.3b has 2 vertices connected by 3 edges, so $\pi_1(G) \cong F_2$. This finishes the $\ell = 2$ case.

By continuing this corner chop pattern, we can produce examples of manifolds M with $\pi_1(M) \cong \mathbb{Z}/2\mathbb{Z} \times F_\ell$ for any $\ell \geq 1$. To see this, fix $\ell \geq 1$. Begin with P_1 from above, and perform pairs of corner chops until the outward primitive normal vectors are

$$(-1,0),(0,-1),(1,2),(1,3),(1,4),\ldots,(1,2\ell-1),(1,2\ell),(0,1).$$

Call this new polygon P_ℓ. Create an origami template by taking two copies of P_ℓ and folding together the pairs of facets with normal vectors $(0,-1)$, $(1,3)$, $(1,5)$, $\ldots$, $(1,2\ell-1)$, $(0,1)$. Call the manifold represented by this origami template M_ℓ. The origami template information for M_3 can be seen in Figure 2.4.

The sublattice N_X of M_ℓ will be generated by the unfolded facets, and so

$$N/N_X \cong \mathbb{Z}^2/\langle(-1,0),(1,2),(1,4),\ldots,(1,2\ell-2),(1,2\ell)\rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

The corresponding template graph G will have 2 vertices and $(\ell+1)$ edges, and so $\pi_1(G) \cong F_\ell$. Since $\pi_1(M_\ell) \cong \mathbb{Z}/2\mathbb{Z} \times F_\ell$, we have proved the theorem. □

2.5 Proof in $N/N_X \cong \mathbb{Z}/3\mathbb{Z}$ Case

In Theorem 2.5.4 we will prove that if M is a compact, orientable, toric origami 4-manifold with $N/N_X \cong \mathbb{Z}/3\mathbb{Z}$, then $\pi_1(M) \cong \mathbb{Z}/3\mathbb{Z} \times F_\ell$ for some odd integer

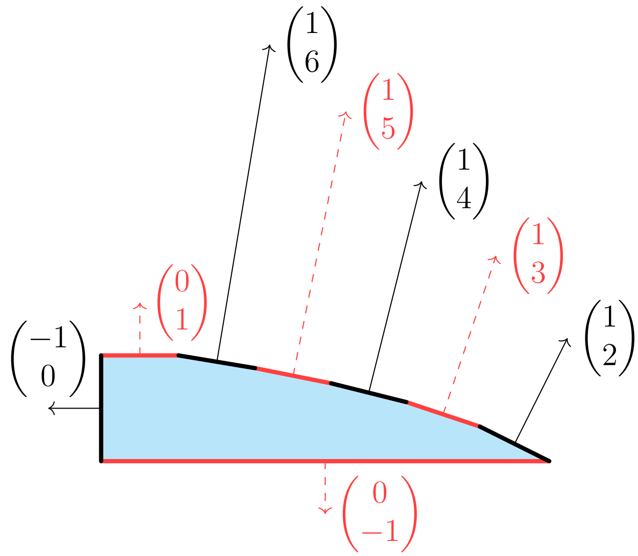


Figure 2.4: Moment image for M_3.

$\ell \geq 1$. We begin with a few lemmas and definitions we will use in the proof.

Lemma 2.5.1. *If (M, ω) is a compact, orientable, toric origami 4-manifold with origami template graph $G = (V, E)$, and if N/N_X is non-trivial, then it must be that $|V|$ is even.*

Proof of Lemma 2.5.1. Define Q be the set of all vectors $w \in \mathbb{R}^2$ for which there exists $v \in V$ such that w is a primitive outward normal vector to an unfolded facet of $\Psi_V(v)$. Choose any element of Q and label it q_1. Consider all the vectors of Q to be anchored at the origin, and label the vector closest to q_1 in the counter-clockwise direction as q_2. Continue in a counter-clockwise direction from q_2 to label the remaining elements of Q as $q_3, \ldots, q_n$. Order the elements of Q by $q_i < q_j$ if and only if $i < j$.

Define $q : V \to Q$ to be the function which for $v \in V$ returns the minimal element q_i of Q for which there exists an unfolded facet in $\Psi_V(v)$ with primitive outward normal vector q_i. For $1 \leq i \leq n$, define

$$V_i := \{v \in V \mid q(v) = q_i\}.$$

Notice that since each polygon $\Psi_V(v)$ will have a unique minimal element $q(v)$, it must be that the sets V_i are disjoint, and that $V = \bigsqcup_i V_i$. We will now show that each set V_i contains an even number of elements, and therefore V will also contain an even number of elements, as desired.

Fix any $1 \leq i \leq n$. We will show that $|V_i|$ is even. Suppose that $|V_i|$ is non-zero. Let $v \in V_i$. The polygon $P = \Psi_V(v)$ has adjacent facets F_1, F_2, F_3 labeled in counter-clockwise order, where F_3 is the unfolded facet with primitive outward normal vector q_i. Since N/N_X is non-trivial, F_2 must be a folded facet and F_1 must be an unfolded facet. Suppose primitive outward normal vector to F_1 is q_j for some $j \neq i$. Since $q(v) = q_i$, it must be that $j > i$.

Since F_2 is folded, there exists $w \in V$, $w \neq v$, with $\Psi_V(w)$ being identical to $\Psi_V(v)$ in a neighborhood of F_2. Therefore there are counter-clockwise labeled adjacent facets F_1', F_2', F_3' of $\Psi_V(w)$ with primitive outward normal vectors identical to F_1, F_2, and F_3 respectively. Since the normal to F_1' is q_j and the normal to F_3' is q_i, it must be that $q(w) = q_i$ and therefore $w \in V_i$.

Remove v and w from V_i. Either V_i is empty, or the process can be repeated. Since elements of V_i always come in pairs, it must be that $|V_i|$ is even, and therefore $|V|$ is even as well. $\square$

Definition. A cyclic sequence of integers $a_1, \ldots, a_{2m}$ is called **k-foldable** if it is even length, and if either

(1) $a_{2i} \equiv 0 \mod k$ for all i, or

(2) $a_{2i+1} \equiv 0 \mod k$ for all i.

By "cyclic" we mean that there is no true start or end to the sequence: we consider a_1 to follow a_{2m} in the sequence.

Definition. A cyclic sequence of integers $b_1, \ldots, b_{m+1}$ is the **blow-up** of a cyclic sequence of integers $a_1, \ldots a_m$ if there exists some i such that

$$\begin{aligned}
b_1 &= a_1 \\
&\vdots \\
b_i &= a_i \\
b_{i+1} &= a_i + a_{i+1} \\
b_{i+2} &= a_{i+1} \\
&\vdots \\
b_{m+1} &= a_m.
\end{aligned}$$

These definitions are useful because once we put an origami template in standard position, either $(0,-1)$ or $(-1,0)$ will be unfolded. If $(0,-1)$ is unfolded, then the problem of determining the total span of the unfolded facets reduces to determining the span of the remaining unfolded facets' x-coordinates. The unfolded x-coordinates will form a cyclic sequence of integers, and $N/N_X \cong \mathbb{Z}/k\mathbb{Z}$ if and only if this sequence is k-foldable. Similarly, if $(-1,0)$ is unfolded, then $N/N_X \cong \mathbb{Z}/k\mathbb{Z}$ if and only if the cyclic sequence of unfolded y-coordinates is k-foldable. Performing a corner chop on a polygon P is equivalent to performing a blow-up of the corresponding cyclic sequence of integers.

Lemma 2.5.2. *Let M be a toric origami manifold with template graph $G = (V, E)$.*

Let $v \in V$, and let $P = \Psi_V(v)$ in standard position. Let

$$\begin{pmatrix} x_1 = -1 \\ \\ y_1 = 0 \end{pmatrix}, \begin{pmatrix} x_2 = 0 \\ \\ y_2 = -1 \end{pmatrix}, \begin{pmatrix} x_3 \\ \\ y_3 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ \\ y_m \end{pmatrix}$$

be the outward primitive normal vectors to the facets of P. Let S_P be the span of the primitive normal vectors to the unfolded facets of P. If $N/S_P \cong \mathbb{Z}/k\mathbb{Z}$, then either $x_1, \ldots, x_m$ is k-foldable or $y_1, \ldots, y_m$ is k-foldable.

Proof of Lemma 2.5.2. Since N/S_P is not trivial, it must be that m is even. Suppose the facet with normal vector (x_1, y_1) is unfolded. Then all the odd index facets are unfolded and the even index facets are folded. Therefore

$$N/S_P \cong (\mathbb{Z} \times \mathbb{Z})/\operatorname{span}\langle (x_1, y_1), (x_3, y_3), \ldots (x_{m-1}, y_{m-1})\rangle.$$

Since $x_1 = -1$, it follows that

$$N/S_P \cong \mathbb{Z}/\gcd(y_1, y_3, \ldots, y_{m-1}).$$

Since $N/S_P \cong \mathbb{Z}/k\mathbb{Z}$, it is necessary that $y_{2i+1} \equiv 0 \mod k$ for all i. By definition, $y_1, \ldots, y_m$ is k-foldable.

On the other hand, if we suppose (x_1, y_1) is folded, then the odd facets are folded and the even facets are unfolded. Since $y_2 = -1$, we get that

$$N/S_P \cong \mathbb{Z}/\gcd(x_2, x_4, \ldots, x_m),$$

and so $x_{2i} \equiv 0 \mod k$ for all i. Therefore $x_1, \ldots, x_m$ is k-foldable, as desired. □

Lemma 2.5.3. *Let (M, ω) be a compact, orientable, toric origami 4-manifold with origami template graph $G = (V, E)$, and suppose $N/N_X \cong \mathbb{Z}/3\mathbb{Z}$. If $v \in V$, then $\Psi_V(v)$ is a polygon with a multiple of four facets, and therefore an even number of folded facets. This means that all vertices in the template graph have even degree.*

Proof of Lemma 2.5.3. Let $P = \Psi_V(v)$ for some $v \in V$. Put P in standard position and let $\{(x_i, y_i)\}_i$ be the ordered set of outward pointing normal vectors to facets of P. Since $N/N_X \cong \mathbb{Z}/3\mathbb{Z}$, either the sequence of x_i's or the sequence of y_i's must be 3-foldable by Lemma 2.5.2. By Proposition 2.2.2, we know that any Delzant polygon P with more than 4 facets can be obtained from some Hirzebruch surface by a sequence of corner chops. Let H_j be the Hirzebruch surface in standard position with outward pointing normal vectors

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ j \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

for some $j \in \mathbb{Z}_+$ so that after some number of corner chops, H_j becomes P.

This implies that after some number of blow-ups, either -1, 0, 1, 0, or 0, -1, j, 1 becomes a 3-foldable cyclic sequence. Since the case $j \equiv 0 \pmod 3$ is identical to the first sequence, suppose without loss of generality that the sequence 0, -1, j, 1 will blow-up to become 3-foldable. Consider the first corner chop in the sequence that eventually produces P. There are two main cases.

Case 1: Suppose $j \equiv 0 \pmod 3$. Then the first blow-up occurs between a 0 and a $\pm 1 \pmod 3$, since there are no other options. Any blow-up between a 0 and a ± 1 forces 3 additional unique blow-ups to occur before the subsequence between the original 0 and ± 1 is 3-foldable. Here is the sequence of blow-ups that must occur:

$$\begin{array}{cccccccc}
0 & & & & & \pm 1 & \pmod 3 \\
0 & \pm 1 & & & & \pm 1 & \pmod 3 \\
0 & \pm 1 & & \pm 2 & & \pm 1 & \pmod 3 \\
0 & \pm 1 & 0 & \pm 2 & & \pm 1 & \pmod 3 \\
0 & \pm 1 & 0 & \pm 2 & 0 & \pm 1 & \pmod 3
\end{array}$$

After performing these 3 necessary blow-ups (for a total of 4 blow-ups), we will have a sequence that is 3-foldable: every other integer is 0 mod 3. Since $2 \equiv -1 \pmod 3$, any additional unforced blow-ups will repeat the above pattern and must come in sets of four. Since H_j has four facets and corner-chops must come in sets of four, we get that P has a multiple of four facets.

Case 2: Suppose $j \equiv \pm 1 \pmod 3$. This means that the subsequence -1, ± 1, 1 must become 3-foldable. In both cases ($j \equiv 1$ and $j \equiv -1$), it takes exactly four corner chops to make this subsequence 3-foldable:

$$
\begin{array}{ccccccccccccccc}
-1 & & 1 & & & & 1 & & -1 & & & & -1 & & 1 \\
-1 & 0 & 1 & & & & 1 & & -1 & & -2 & & -1 & & 1 \\
-1 & 0 & 1 & & 2 & & 1 & \text{or} & -1 & 0 & -2 & & -1 & & 1 \\
-1 & 0 & 1 & 0 & 2 & & 1 & & -1 & 0 & -2 & 0 & -1 & & 1 \\
-1 & 0 & 1 & 0 & 2 & 0 & 1 & & -1 & 0 & -2 & 0 & -1 & 0 & 1
\end{array}
$$

After performing these four required blow-ups (and remembering that $2 \equiv -1 \pmod 3$), our sequence becomes 3-foldable and any additional blow-ups will occur between a 0 and a ± 1 mod 3. This puts us back into case 1, and means that additional blow-ups will happen in sets of four. Therefore P must have a multiple of four facets. $\square$

Theorem 2.5.4. *There exists a compact, orientable, toric origami 4-manifold M with $\pi_1(M) \cong \mathbb{Z}/3\mathbb{Z} \times F_\ell$ if and only if $\ell \geq 1$ is odd.*

Proof of Theorem 2.5.4. Suppose M is an orientable toric origami manifold with origami template graph $G = (V, E)$, and suppose that $N/N_X \cong \mathbb{Z}/3\mathbb{Z}$. We want to show that $\pi_1(G) \cong F_\ell$, where ℓ must be odd. We know that $\ell = |E| -$

$(|V|-1)$. By Lemma 2.5.1 we know that $|V|$ is even. We will show here that $|E|$ is also even, and therefore ℓ is odd.

By Lemma 2.5.3, we know that each vertex represents a polygon with an even number of folded facets. Since folded facets are represented by edges, this implies that each vertex in G has even degree. Veblen's Theorem states that the set of edges of a finite graph can be written as a union of disjoint simple cycles if and only if every vertex in the graph has even degree [15]. Since we assumed M to be an orientable manifold, we also know that there cannot be any odd cycles in the template graph. Since E can be written as the disjoint union of even cycles, it must be that $|E|$ is even, and hence ℓ must be odd.

Finally, we construct examples of orientable toric origami 4-manifolds M for which $\pi_1(M) \cong \mathbb{Z}/3\mathbb{Z} \times F_\ell$ for odd $\ell \geq 1$. In all cases, the origami template for M will be two identical polygons with folded facets glued via the identity map. Begin with the Hirzebruch surface with outward pointing normal vectors

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Folding second and fourth facets via the identity map will result in a manifold M with $\pi_1(M) \cong \mathbb{Z}/3\mathbb{Z} \times F_1$. Choose any two adjacent vectors, and perform the sequence of four corner chops outlined in the proof of Lemma 2.5.3 that preserves the $\mathbb{Z}/3\mathbb{Z}$ lattice. This will introduce two new folded facets, and so the new manifold will have fundamental group $\mathbb{Z}/3\mathbb{Z} \times F_3$. Repeat to get $\mathbb{Z}/3\mathbb{Z} \times F_5$, etc. This process creates orientable toric origami 4-manifolds M_ℓ with $\pi_1(M_\ell) \cong \mathbb{Z}/3\mathbb{Z} \times F_\ell$ for all odd $\ell \geq 1$. $\square$

2.6 Proof in $N/N_X \cong \mathbb{Z}/k\mathbb{Z}$, $k \geq 4$ Case

In Theorem 2.6.3 of this section we will prove that if M is an orientable toric origami 4-manifold with $N/N_X \cong \mathbb{Z}/k\mathbb{Z}$ for $k \geq 4$, then $\pi_1(M) \cong \mathbb{Z}/k\mathbb{Z} \times F_1$.

Lemma 2.6.1. *If any two adjacent elements of a cyclic sequence of integers are equivalent to 1 and 2 (mod k), respectively, or to -1 and -2 (mod k), respectively, for $k \geq 4$, then no finite number of blow-ups will result in a k-foldable cyclic sequence of integers.*

Proof of Lemma 2.6.1. Suppose that $a_1, \ldots a_m$ is a cyclic sequence of integers.

Case 1: Suppose some pair of adjacent elements are 1 and 2 mod k. The definition of k-foldable is that either every even index element must be equivalent to 0 mod k, or every odd index element must be equivalent to 0 mod k. Therefore $a_1, \ldots, a_m$ is not k-foldable. We will show that any attempt to create a sequence which is k-foldable by performing blow-ups will fail.

Since neither 1 nor 2 is equivalent to 0 mod k, a blow-up must be performed between them for the sequence to have a chance to become k-foldable. This creates the sub-sequence 1, 3, 2, which means we now have an adjacent 1 and 3 mod k. Again neither is 0 mod k, so a blow-up must be performed between them. This creates an adjacent 1 and 4 mod k. We continue making the necessary blow-ups until we get an adjacent 1 and $(k-2)$ mod k. The next blow-up creates an adjacent $(k-1)$ and $(k-2)$ mod k. This is equivalent to an adjacent -1 and -2 mod k.

In order to get an element which is 0 mod k between the 1 and 2, we had to introduce a new set of adjacent elements which are equivalent to -1 and -2 mod k. This puts us into Case 2.

Case 2: Suppose some pair of adjacent elements are -1 and -2 mod k. In order to obtain a sequence which is k-foldable, a blow-up will have to be performed between them. By the same logic as Case 1, this forces blow-ups until we have an adjacent $-(k-1)$ and $-(k-2)$ mod k. These are equivalent to 1 and 2 mod k, so we are back in Case 1.

Since the adjacent elements in Case 1 force at least two blow-ups and introduce a new pair of elements satisfying Case 2 in the process, and vice versa, no finite number of blow-ups of $a_1, \ldots, a_m$ will result in a k-foldable cyclic sequence. $\square$

Lemma 2.6.2. *Let H be a Delzant polygon representing a Hirzebruch surface, and let C be the Delzant polygon representing $\mathbb{C}P^2$. Suppose P is a Delzant polygon with at least 6 facets which is obtained from either H or C by a finite number of corner chops. If $P = \Psi_V(v)$ for some vertex v of an origami template graph G, then N/N_X is not isomorphic to $\mathbb{Z}/k\mathbb{Z}$ for any $k \geq 4$.*

Proof of Lemma 2.6.2. Suppose for a contradiction that $N/N_X \cong \mathbb{Z}/k\mathbb{Z}$ for some $k \geq 4$. Let S_P be the span of the outward primitive normal vectors to the unfolded facets of P. Since $S_P \leq N_X$, it follows that

$$\mathbb{Z}/k\mathbb{Z} \cong N/N_X \leq N/S_P.$$

Thus $\mathbb{Z}/k\mathbb{Z}$ is a subgroup of N/S_P, so $N/S_P \cong \mathbb{Z}/rk\mathbb{Z}$ for some integer $r \geq 1$. Let $s = rk$. Thus $N/N_X \cong \mathbb{Z}/k\mathbb{Z}$ implies that $N/S_P \cong \mathbb{Z}/s\mathbb{Z}$.

Put H and C into standard position by actions of $SL_2(\mathbb{Z})$, so that the primitive outward normal vectors to their facets are

$$H : \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ j \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

for some $j \in \mathbb{Z}$, and

$$C : \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let the normal vectors to the facets of P be

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}.$$

It is important to note that if $\{v_i\}$ are the normal vectors to a Delzant polytope, with $v_i = (a_i, b_i)$, then a corner chop between v_i and v_{i+1} results in a blow-up of the cyclic sequences $\{a_i\}$ and $\{b_i\}$ between the a_i and a_{i+1} elements and between the b_i and b_{i+1}. Therefore if P is obtained from H by some sequence of corner chops, it follows that $x_1 \ldots x_m$ is obtained from -1, 0, 1, 0 and $y_1, \ldots, y_m$ is obtained from 0, -1, j, 1 by the same sequence of blow-ups.

Since $(x_1, y_1) = (-1, 0)$ begins next to $(0, -1)$ in both H and C, it follows that any sequence of corner chops between these two vectors will result in $y_2 = -1$. Thus by Lemma 2.5.2, since $N/S_P \cong \mathbb{Z}/s\mathbb{Z}$ for $s \geq 4$, and $x_1 = -1$, and $y_2 = -1$, it must be that either $x_1, \ldots, x_m$ or $y_1, \ldots, y_m$ is s-foldable. We will show that neither of these cases can be true.

Case 1: Suppose $x_1, \ldots, x_m$ is s-foldable. If P is obtained from H by corner chopping, then some finite number of blow-ups of the cyclic sequence -1, 0, 1, 0 is s-foldable. However any blow-up to this sequence will result in either adjacent 1's or adjacent -1's. Suppose without loss of generality that we have adjacent 1's. To be s-foldable, every even index (or odd index) element of the sequence must be 0 mod s. Therefore another blow-up must be done between the adjacent 1's. This gives us the subsequence 1, 2, 1. It then follows from Lemma 2.6.1 that no finite sequence of blow-ups will result in s-foldability.

If P is obtained from C, then some finite number of blow-ups of the cyclic sequence -1, 0, 1 is s-foldable. Any blow-up to this sequence creates adjacent 1's or -1's. It follows from the same argument as above that no finite sequence will result in s-foldability. Therefore $x_1, \ldots, x_m$ cannot be s foldable.

Case 2: Suppose $y_1, \ldots, y_m$ is s-foldable. If P is obtained from C, then some finite number of blow-ups of 0, -1, 1 is s-foldable. After a single blow-up, either there are adjacent 1's or -1's, or the sequence looks like 0, -1, 0, 1. Since P has at least 6 facets, at least 2 more blow-ups must be done to obtain $y_1, \ldots, y_m$. The first of these blow-ups will result in adjacent 1's or -1's. By the same argument as in case 1, there is no sequence of additional blow-ups which will result in s-foldability. Thus P is not obtained from C.

Suppose P is obtained from H. Then after some finite number of blow-ups of 0, -1, j, 1 is s-foldable. Since s-foldability is defined mod s, we may assume that $0 \leq j < s$. If $j = 0$, then any additional blow-up will result in adjacent 1's or -1's and a contradiction. If $j = 1$, then we again have a contradiction. Suppose $2 \leq j < s$. Since neither -1 nor j is equivalent to 0 mod s, a blow-up must be done between them. This results in the subsequence -1, $(j-1)$, j. If $(j-1) = 1$, then $(j-1)$ and j are an adjacent 1 and 2. If not, a blow up must be done between -1 and $(j-1)$. This repeats until eventually there must be an adjacent 1 and 2. Then it follows from Lemma 2.5.2 that $y_1, \ldots y_m$ cannot be s-foldable.

Since neither $x_1, \ldots x_m$ nor $y_1, \ldots, y_m$ is s-foldable, Lemma 2.5.2 is in contradiction with our assumption and so it must be that N/S_P is not isomorphic to $\mathbb{Z}/s\mathbb{Z}$. This further implies that N/N_X is not isomorphic to $\mathbb{Z}/k\mathbb{Z}$, as desired. $\square$

Now finally we put this all together to prove Theorem 2.6.3.

Theorem 2.6.3. *If M is a toric origami 4-manifold with $N/N_X \cong \mathbb{Z}/k\mathbb{Z}$ for $k \geq 4$, then it must be that $\pi_1(G) \cong F_1$. For every $k \geq 4$ such a manifold exists.*

Proof of Theorem 2.6.3. Suppose that M is a toric origami 4-manifold with $N/N_X \cong \mathbb{Z}/k\mathbb{Z}$ for some $k \geq 4$. Let $G = (V, E)$ be the corresponding template graph. Suppose for the sake of a contradiction that $\pi_1(G)$ is not isomorphic to F_1. Since $\pi_1(G)$ is neither trivial nor isomorphic to F_1, it follows that some vertex $v \in V$ has at least 3 edges incident to it. The corresponding Delzant polygon $P = \Psi_V(v)$ must have at least 3 folded facets, and therefore at least 6 total facets.

By Proposition 2.2.2, P is obtained from a Delzant polygon H or C representing a Hirzebruch surface or $\mathbb{C}P^2$, respectively, by some finite number of corner chops. This contradicts Lemma 2.6.2. Therefore it must be that $\pi_1(G) \cong F_1$.

Examples of such manifolds M with $\pi_1(M) \cong \mathbb{Z}/k\mathbb{Z} \times F_1$ are provided in [11]. Given $k \geq 4$, they result as the manifolds represented by the origami template with 2 copies of the polygon H_k, where the primitive normal vectors to the facets of H_k are:

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ k \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

folded together along the facets with normal vectors $(0, -1)$ and $(0, 1)$. $\square$

CHAPTER 3

NON-ORIENTABLE FUNDAMENTAL GROUPS

3.1 Non-Orientable Fundamental Group Introduction

In this chapter we will investigate the fundamental groups of non-coörientable toric origami manifolds. This chapter is unique in that it applies to toric origami manifolds of all dimensions rather than specializing to dimension four.

Recall that a facet F of X is a fold facet if $F = \Psi_E(e)$ for some edge e in G. If e is an edge between distinct vertices u and v, then we call F **two-sided** because it will fold $\Psi_V(u)$ to $\Psi_V(v)$ and the tubular neighborhood of $q^{-1}(F)$ will be two-sided. If, on the other hand, e is a loop edge adjacent to a single vertex v, then $q^{-1}(F)$ is folded to itself with a one-sided tubular neighborhood and we call F **one-sided**.

Two-sided fold facets are also called **coörientable**, and one-sided fold facets are called **non-coörientable**. If every fold facet in M/T is coörientable, then we call M itself coörientable. Note that if M is orientable, then it must also be coörientable. The converse is not true however. If a coörientable M has an odd length cycle in its template graph, then M will be non-orientable.

Holm and Pires have a formula for the fundamental group of an orientable toric origami manifold $(M^{2n}, \omega, T^n, \mu)$ in terms of the combinatorics of its template graph [11]. In particular, $\pi_1(M) \cong N/N_X \times \pi_1(X)$ where $N \cong H_1(T^n) \cong \mathbb{Z}^n$ and N_X is the span of the loops in $H_1(T^n)$ corresponding to the normal vectors to the unfolded facets of M/T. Their proof of this isomorphism goes through word for word in the case where M is coörientable but not orientable. In this chapter we will explore the case where M is non-coörientable.

3.2 Non-Coörientable Fundamental Group Theorem

Theorem 3.2.1. *Let $(M^{2n}, \omega, T^n, \mu)$ be a compact toric origami manifold, with template graph G and template-space (i.e. orbit space) X. Note that we allow M to be non-coörientable and therefore non-orientable. Let $F_1, \ldots F_m$ be the one-sided facets in M, with corresponding primitive outward normal vectors $\tau_1, \ldots, \tau_m$. Let $U_1^i, \ldots, U_{k_i}^i$ be the unfolded facets adjacent to F_i, with corresponding primitive outward normal vectors $\eta_1^i, \ldots, \eta_{k_i}^i$. Then*

$$\pi_1(M) \cong \pi_1(X) \times \left\langle \begin{array}{c} \alpha_1, \ldots, \alpha_m \\ x_1, \ldots, x_n \end{array} \middle| \begin{array}{c} \alpha_i^2 - \tau_i \cdot \mathbf{x}, \ \textit{for all } 1 \leq i \leq m, \\ \eta_1^i \cdot \mathbf{x}, \ldots, \eta_{k_i}^i \cdot \mathbf{x}, \ \textit{for all } 1 \leq i \leq m, \\ [x_i, x_j], \ \textit{for all } 1 \leq i, j \leq n \end{array} \right\rangle.$$

Proof. (Proof of Theorem 3.2.1) We will be using the Seifert–van Kampen theorem to prove Theorem 3.2.1. Choose a basepoint $p \in M$ so that $q(p)$ is in the interior of M/T (i.e. not in any facet). We will begin by carefully choosing open sets that cover M, and calculating their fundamental groups.

Let V_0 be the open set which is M with all the non-orientable folded facets cut out, given by $V_0 = q^{-1}\big(X - \bigsqcup_{i=1}^m F_i\big) = M - q^{-1}\big(\bigsqcup_{i=1}^m F_i\big)$. Then by [11], we have that $\pi_1(V_0) \cong \pi_1(G) \times N/N_X$. Since G is a graph, it follows that $\pi_1(G)$ is always a free group.

N is the lattice generated by $x_1, \ldots, x_n$ inside $\mathfrak{t}^n$. Let $U_1, \ldots, U_r$ be the unfolded facets in X, with corresponding primitive outward normal vectors $\nu_1, \ldots, \nu_r$. Then since any loop in the torus factor of $M \cong \big(X \times T\big)/\sim$ is homotopic to the same loop moved into the facet U_i where the subtorus $\nu_i \cdot \mathbf{x}$ is collapsed, we get that

$$N/N_X \cong \left\langle \, x_1, \ldots, x_n \;\middle|\; \begin{array}{c} \nu_1 \cdot \mathbf{x}, \ldots, \nu_r \cdot \mathbf{x}, \\ [x_i, x_j] \text{ for all } 1 \leq i, j \leq n \end{array} \right\rangle .$$

If ν_i is the vector $(\nu_{i,1}, \ldots, \nu_{i,n})$ in $\mathbb{Z}^n \cong N$, we use the notation $\nu_i \cdot \mathbf{x}$ to represent the linear combination $\nu_{i,1}x_1 + \cdots + \nu_{i,n}x_n \in N$. In this notation, $N_X = \langle \nu_1 \cdot \mathbf{x}, \ldots, \nu_r \cdot \mathbf{x} \rangle$.

Therefore,

$$\pi_1(V_0) \cong \pi_1(X) \times \left\langle \, x_1, \ldots, x_n \;\middle|\; \begin{array}{c} \nu_1 \cdot \mathbf{x}, \ldots, \nu_r \cdot \mathbf{x}, \\ [x_i, x_j] \text{ for all } 1 \leq i, j \leq n \end{array} \right\rangle .$$

Next, we define V_i for $1 \leq i \leq m$ to be the one-sided open tubular neighborhood of $q^{-1}(F_i)$, together with an open tube or "tongue" in M connecting to p. Let $U^i_1, \ldots, U^i_{k_i}$ be the unfolded facets of X adjacent to F_i, and with corresponding outward primitive normal vectors $\eta^i_1, \ldots, \eta^i_{k_i}$. Let τ_i be the primitive outward normal vector to F_i.

Since $q^{-1}(F_i)$ is a one-sided hypersurface in M, we understand it as the natural $\mathbb{Z}_2$-quotient of its orientation double cover. Let $d : \hat{M} \to M$ be the orientation double cover of M. In the orientation double cover, $d^{-1}(V_i)$ is a two-sided neighborhood of the coörientable fold $d^{-1}(q^{-1}(F_i))$. Therefore we have that

$$\pi_1\left(d^{-1}\left(V_i\right)\right) = \left\langle \, x_1, \ldots, x_n \;\middle|\; \begin{array}{c} \eta^i_1 \cdot \mathbf{x}, \ldots, \eta^i_{k_i} \cdot \mathbf{x}, \\ [x_i, x_j] \text{ for all } 1 \leq i, j \leq n \end{array} \right\rangle .$$

At a point in $d^{-1}(q^{-1}(F_i))$, the direction of the null fibration is given by the tangent vector τ_i. Since $d^{-1}(F_i)$ is toric origami, the null fibration integrates into S^1 fibers. In particular, the S^1 fibers are the loops generated by τ_i in T^n. The map d acts by collapsing the null fibration via the antipodal map. Topologically, this means that there is a generator in $\pi_1(V_i)$ which squares to be the loop $\tau_i \cdot \mathbf{x}$. Call this generator α_i. Then $\alpha_i^2 = \tau_i \cdot \mathbf{x}$ and thus

$$\pi_1(V_i) \cong \left\langle \ \alpha_i, x_1, \ldots, x_n \ \middle| \ \begin{array}{c} \alpha_i^2 - \tau_i \cdot \mathbf{x}, \\ \eta_1^i \cdot \mathbf{x}, \ldots, \eta_{k_i}^i \cdot \mathbf{x}, \\ [x_i, x_j] \text{ for all } 1 \le i, j \le n \end{array} \ \right\rangle.$$

The theorem result is now a direct application of the Seifert–van Kampen theorem. Let $\iota_i : V_i \hookrightarrow M$ be the inclusions of the open sets V_i into M for $0 \le i \le m$. We notice that for $1 \le i \le m$, we have that $\iota_0(x_j) = \iota_i(x_j)$ for all $1 \le j \le n$. In particular, the generators for the loops in the torus fibers have been chosen in a consistent manner for each open set V_i. Since the generators $x_1, \ldots, x_n$ are the only generators in each $V_0 \cap V_i$, Seifert–van Kampen implies that $\pi_1(M)$ is essentially just combining the relations from $\pi_1(V_0)$ with the relations for $\pi_1(V_i)$ for each i. This gives exactly the group presentation in Theorem 3.2.1. $\square$

CHAPTER 4

HOMOLOGY AND COHOMOLOGY

4.1 Homology Groups of Toric Symplectic Manifolds

This chapter will focus on explicitly describing minimal generating sets for the homology groups of all ranks of any compact, orientable, toric origami 4-manifold. After finding embedded submanifolds representing each homology class, we will then use Poincaré duality to understand the corresponding cohomology groups and ring structure. Before getting to the homology and cohomology of toric origami manifolds, however, it is instructive to understand the homology and cohomology of toric symplectic manifolds. As a note, all cohomology groups in this chapter will be assumed to have $\mathbb{Z}$ coefficients unless otherwise specified.

Let $(M^{2n}, \omega, T^n, \mu)$ be a toric symplectic manifold with Delzant polytope $P = M/T$. Suppose P has facets $F_0, \ldots, F_{m-1}$ with normal vectors $\nu_0, \ldots \nu_{m-1}$ such that F_0 and F_1 are in standard position (so $\nu_0 = -x_0$ and $\nu_1 = -x_1$). Then $q^{-1}(F_i)$ is an embedded co-dimension 2 submanifold of M. If M is a 4-manifold, then $q^{-1}(F_i)$ is an embedded 2-sphere for each i. The Danilov-Jurkiewicz theorem, originally from [6] and [12] and carefully described in Theorem 12.4.4 of [5], states that the cohomology ring $H^*(M; \mathbb{Z})$ is generated in degree 2 by the classes $[F_i]$ dual to $q^{-1}(F_i)$ for $2 \leq i \leq m-1$. When $\dim(M) = 4$,

$$\begin{aligned}
H_0(M) &\cong \mathbb{Z} & H^0(M) &\cong \mathbb{Z} \\
H_1(M) &\cong 0 & H^1(M) &\cong 0 \\
H_2(M) &\cong \mathbb{Z}^{m-2} & H^2(M) &\cong \mathbb{Z}^{m-2} \\
H_3(M) &\cong 0 & H^3(M) &\cong 0 \\
H_4(M) &\cong \mathbb{Z} & H^4(M) &\cong \mathbb{Z}.
\end{aligned}$$

Explicit details are given in [7] for this case, including a calculation of the intersection form. In particular, the spheres $q^{-1}(F_i)$ intersect their neighbors transversely so that $[F_i] \cdot [F_{i+1}] = +1$. Non-adjacent spheres have intersection number 0. The self-intersection number of $[F_i]$ is given by $[F_i] \cdot [F_i] = \det(\nu_{i-1}, \nu_{i+1})$, where $\nu_0 = \nu_m$. Since $H^1(M)$ and $H^3(M)$ are 0, this completely describes the cohomology ring of a 4-dimensional toric symplectic manifold.

4.2 Homology Groups of Toric Origami Manifolds

Let (M, ω, T, μ) be a compact, orientable, toric origami 4-manifold with template graph $G = (G_V, G_E)$ and graph maps $\Psi_V : G_V \to \mathcal{D}_2$ and $\Psi_E : G_E \to \mathcal{E}_2$, as in Section 1.2. Suppose $|G_V| = n$, so there are n polygons folded together to create M/T. Let $P_1, \ldots, P_n$ be these polygons. In particular, $P_i := \Psi_V(v_i)$ for $v_i \in G_V$. We will also assume that M (and therefore G) is path-connected.

The calculations by Holm and Pires in [11] for the fundamental group and Betti numbers of a toric origami 4-manifold M give that $\pi_1(M) \cong \pi_1(G) \times N/N_X$ where $\pi_1(G)$ is a free group on ℓ generators, and $N/N_X \cong \mathbb{Z}/k\mathbb{Z}$ for some $k \in \mathbb{Z}_{\geq 0}$. The case that $N/N_X \cong \mathbb{Z}$ is called the "primsatic" case, and they show that if M is prismatic then M is homeomorphic to $S^2 \times T^2$. For the rest of the chapter, we will assume M is not prismatic, so N/N_X is 0 or finite cyclic. The Betti number calculations in [11] for non-prismatic M give

$$b_i(M) = \begin{cases} 1, & i = 0, 4 \\ \ell, & i = 1, 3 \\ 2\ell + \#M^T - 2, & i = 2, \end{cases}$$

where $\#M^T$ is the number of fixed points of the torus action, or equivalently the number of components of the 1-skeleton which are spheres (see Section 1.6). Combining these Betti number calculations calculations with the Universal Coefficients Theorem and Poincaré Duality gives the following list of homology and cohomology groups:

$$
\begin{array}{ll}
H_0(M) \cong \mathbb{Z} & H^0(M) \cong \mathbb{Z} \\
H_1(M) \cong \mathbb{Z}^\ell \times \mathbb{Z}/k\mathbb{Z} & H^1(M) \cong \mathbb{Z}^\ell \\
H_2(M) \cong \mathbb{Z}^{2\ell+\#M^T-2} \times \mathbb{Z}/k\mathbb{Z} & H^2(M) \cong \mathbb{Z}^{2\ell+\#M^T-2} \times \mathbb{Z}/k\mathbb{Z} \\
H_3(M) \cong \mathbb{Z}^\ell & H^3(M) \cong \mathbb{Z}^\ell \times \mathbb{Z}/k\mathbb{Z} \\
H_4(M) \cong \mathbb{Z} & H^4(M) \cong \mathbb{Z}
\end{array}
$$

Our goal is to find explicit embedded submanifolds representing generating sets for each $H_k(M)$, which will then be dual to generating classes for $H^{4-k}(M)$. We will start with $H_1(M)$ by choosing explicit generators for $\pi_1(M) \cong \pi_1(G) \times N/N_X$. To do this, choose a fixed spanning tree within the template graph G. Since G is connected with n vertices, a spanning tree is comprised of $n-1$ edges. Let $e_1^*, \dots e_{n-1}^*$ be a spanning set of edges in G_E. Label the remaining edges in G_E as $e_1, \dots, e_\ell$, where $\ell = |G_E| - (n-1)$. In the acyclic case where G is a tree, $\ell = 0$. The edges $e_1^*, \dots e_{n-1}^*$ of the fixed spanning tree in G correspond via Ψ_E to a fixed collection of folded facets $E_1^*, \dots, E_{n-1}^*$ in M/T which glue the polygons $P_1, \dots, P_n$ into a tree-like structure. The remaining edges $e_1, \dots e_\ell$ correspond to the remaining folded facets $E_1, \dots E_\ell$ and create cycles in the graph. The fundamental group of a graph will always be a free group whose rank can be determined by collapsing a spanning tree (creating a bouquet of circles) and counting how many edge loops remain. Each e_i for $1 \le i \le \ell$ corresponds to one of the generators of $\pi_1(G)$ by taking a loop that is entirely contained within the (acyclic) spanning tree, except for passing exactly once along e_i.

Let α_i for $1 \le i \le \ell$ be the inclusion of the generator of $\pi_1(G)$ corresponding to $e_i \in G_E$ into $\pi_1(M)$ using the isomorphism $\pi_1(M) \cong \pi_1(G) \times N/N_X$. Thus, α_i is a lift of the generator loop in $\pi_1(G)$ that passes exactly once along the edge e_i. In particular, α_i is a loop in M that passes exactly once through the folded facet preimage $q^{-1}(E_i)$ and otherwise lives in the spanning tree-like structure of the $\bigcup q^{-1}(P_i)$ polytopes and $\bigcup q^{-1}(E_i^*)$ folded facets. Since $\alpha_1, \ldots, \alpha_\ell$ generate the free part of $\pi_1(M)$ (i.e. they represent $\pi_1(G)$), they will also generate the free abelian part of $H_1(M)$. In the case that $N/N_X \cong \mathbb{Z}/k\mathbb{Z}$ with $k \geq 2$, let $\beta \in H_1(M)$ be the class of a loop that generates N/N_X in $\pi_1(M)$. Then β will be a generator for the k-torsion subgroup of $H_1(M)$.

The facets $E_1, \ldots, E_\ell$ are the folded facets of X which do not correspond to the spanning tree of G. In particular, α_i is a loop in M that passes through $q^{-1}(E_i)$ exactly once, with intersection number $+1$. Let $L_i := q^{-1}(E_i)$ for $1 \leq i \leq \ell$. Taking the intersection number of a class in $H_1(M)$ with L_i defines an element $[L_i] \in \mathrm{Hom}_{\mathbb{Z}}(H_1(M), \mathbb{Z})$ dual to L_i which sends α_i to 1 and all other α_j to 0. Since $H^1(M)$ is torsion-free by the Universal Coefficients Theorem, it follows that $\mathrm{Hom}_{\mathbb{Z}}(H_1(M), \mathbb{Z}) \cong H^1(M; \mathbb{Z})$ and thus $[L_1], \ldots, [L_\ell]$ form a free $\mathbb{Z}$-module basis for $H^1(M; \mathbb{Z})$. In addition, by Poincaré duality $H_3(M) \cong \mathbb{Z}^\ell$ is generated by the facet preimages $L_1, \ldots L_\ell$ (which are 3-dimensional lens spaces).

The remainder of this chapter will be devoted to understanding $H_2(M)$ and $H^2(M; \mathbb{Z})$, and then finally the ring structure on $H^*(M, \mathbb{Z})$. The ring structure results are collected in Theorem 4.5.1.

We will be understanding $H_2(M)$ using a generalized Mayer-Vietoris argument with one open set for each Delzant polytope P_i in X. We define A_i to be an open neighborhood of $M_i = q^{-1}(P_i)$ in M that deformation retracts onto M_i. We choose the A_i so that if P_i is folded to P_j along a facet $F = \Psi_E(e)$, then the corresponding connected component of the intersection $A_i \cap A_j$ is an open set in M that deformation retracts to $\mu^{-1}(F)$.

Notice that $\{A_i\}_{i=1}^n$ is an ordered collection of open sets covering M, and that all triple intsersections amongs the A_i are empty. Mirroring the argument in Section 2.2 of Hatcher's Algebraic Topology [10] for the Mayer-Vietoris sequence for two open sets, there are short exact sequences of chain complexes

$$0 \longrightarrow C_k\left(\bigsqcup_{1\le i<j\le n} A_i \cap A_j\right) \longrightarrow \bigoplus_{i=1}^{n} C_k(A_i) \longrightarrow C_k(M) \longrightarrow 0$$

where the inclusion $C_k(A_i \cap A_j) \hookrightarrow C_k(A_k)$ for $i < j$ is positive if $k = i$ and negative if $k = j$ to create exactness. These short exact sequences combine to form a generalized Mayer-Vietoris long exact sequence

$$\cdots \longrightarrow H_{k+1}(M) \longrightarrow \bigoplus_{1\le i<j\le n} H_k(A_i \cap A_j) \longrightarrow \bigoplus_{i=1}^{n} H_k(A_i) \longrightarrow H_k(M) \longrightarrow \cdots,$$

which we will use to understand the homology of M. Since each component of $A_i \cap A_j$ deformation retracts to a lens space, we have that $H_2(A_i \cap A_j) \cong 0$. Therefore we will be using the exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{n} H_2(A_i) \xrightarrow{f} H_2(M) \xrightarrow{g} \bigoplus_{1\le i<j\le n} H_1(A_i \cap A_j) \xrightarrow{h} \bigoplus_{i=1}^{n} H_1(A_i)$$

to understand $H_2(M)$. In what follows, we will describe an embedded collection of submanifolds of M that represent a generating set of homology classes in $H_2(M)$. In the case where $H_2(M)$ is a free $\mathbb{Z}$-module, these classes will form a basis for $H_2(M)$.

4.3 A Dual Basis for Toric Symplectic $H_2(M)$

To understand $H_2(M)$ for toric origami M via the Mayer-Vietoris sequence in Section 4.2, we first need to understand $H_2(A_i) \cong H_2(M_i)$ for the toric symplectic manifolds with boundary M_i that cover M. To do this, we need a better understanding of $H_2(M)$ when M is a toric symplectic manifold.

As described in Section 4.1 with results from [7], if M is a toric symplectic 4-manifold M with Delzant polygon P with m facets, $H_2(M) \cong \mathbb{Z}^{m-2}$, with the sphere preimages of $m-2$ of the facets generating $H_2(M)$ and dual to generators for the entire ring $H_*(M)$. It is shown by Holm and Pires in [11] that in an open toric symplectic 4-manifold M_i with Delzant polygon P_i with m_i facets, k_i of which are folded, $b_2(M_i) = m_i - 2 - k_i$.

If P_i is P with k_i facets designated as folded, then the topological difference between M_i and M is that M_i is M with an open neighborhood of q^{-1} of each folded facet removed. This is topologically the same as deleting a neighborhood of the sphere corresponding to each of the k_i folded facets from M. This formula then makes some sense, because it is saying that deleting k_i of the generators for $H_2(M)$ from M exactly decreases the rank of $H_2(M)$ by k_i.

However, since the removed spheres are preimages of folded facets which intersect adjacent unfolded facets, removing the folded facets also rips a point out of the spheres representing the unfolded facets. Our goal is to find a different basis (which will end up being dual to the usual sphere basis) for $H_2(M)$ when M is toric symplectic that translates very nicely to a basis for $H_2(M_i)$ when M_i is toric symplectic with boundary.

4.3.1 Defining Spider Homology Classes

We begin constructing our new basis by setting out some notation. Let M be a toric symplectic manifold with Delzant polytope P whose facets $F_0, \ldots, F_{m-1}$ have normal vectors $\nu_0, \ldots, \nu_{m-1}$. Suppose F_0 and F_1 are in standard position so that $\nu_0 = (0, -1)$ and $\nu_1 = (-1, 0)$. Let $[\nu_i] \in H_1(T^2)$ be the homology class of the loop in the torus fiber corresponding to the normal vector ν_i. In particular, if $\nu_i = (a, b)$ then $[\nu_i]$ is the class representing the curve which goes a times around the first S^1 factor of T^2 and b times around the second S^1 factor of T^2. Then since F_0 and F_1 are in standard position, $[\nu_0]$ and $[\nu_1]$ form a basis for $H_1(T^2)$. Therefore for each $2 \leq i \leq m-1$, there is a linear relation $c_0^i[\nu_0] + c_1^i[\nu_1] + [\nu_i] = 0$ in $H_1(T^2)$. By definition, this means there is a 2-chain K_i in $C_2(T^2)$ with boundary $\partial K_i = c_0^i[\nu_0] + c_1^i[\nu_1] + [\nu_i]$.

By using disks to cap off each component of ∂K_i, we will create a 2-cycle representing an element of $H_2(M)$. Let p be a point in the interior of P, and let K_i live in the T^2 fiber over p. This fiber is the core, or "body" of the "spider" class we are building. Let γ_i be a path in P from p to a point on the unfolded facet with normal vector ν_i. See Figure 4.1. Then $\gamma_i \times [\nu_i]$ is a 2-disk with boundary $[\nu_i]$. This is because γ_i is an interval, and $[\nu_i]$ is a circle fiber above all points of γ_i (creating a cylinder), except above the endpoint of γ_i in the unfolded facet where $[\nu_i]$ is collapsed to a point. We call $\gamma_i \times [\nu_i]$ a "leg" of the spider.

By taking c_0^i copies of the leg $\gamma_0 \times [\nu_0]$, and c_1^i copies of $\gamma_1 \times [\nu_1]$, and 1 copy of the leg $\gamma_i \times [\nu_i]$, we get a collection of disks that will cap off all the boundary components of K_i. To get a leg with a negative coefficient, simply reverse the orientation of γ_i so that it begins at the facet and ends at the spider body. Once the legs are glued to the body K_i, this spider represents a class in $H_2(M)$ that

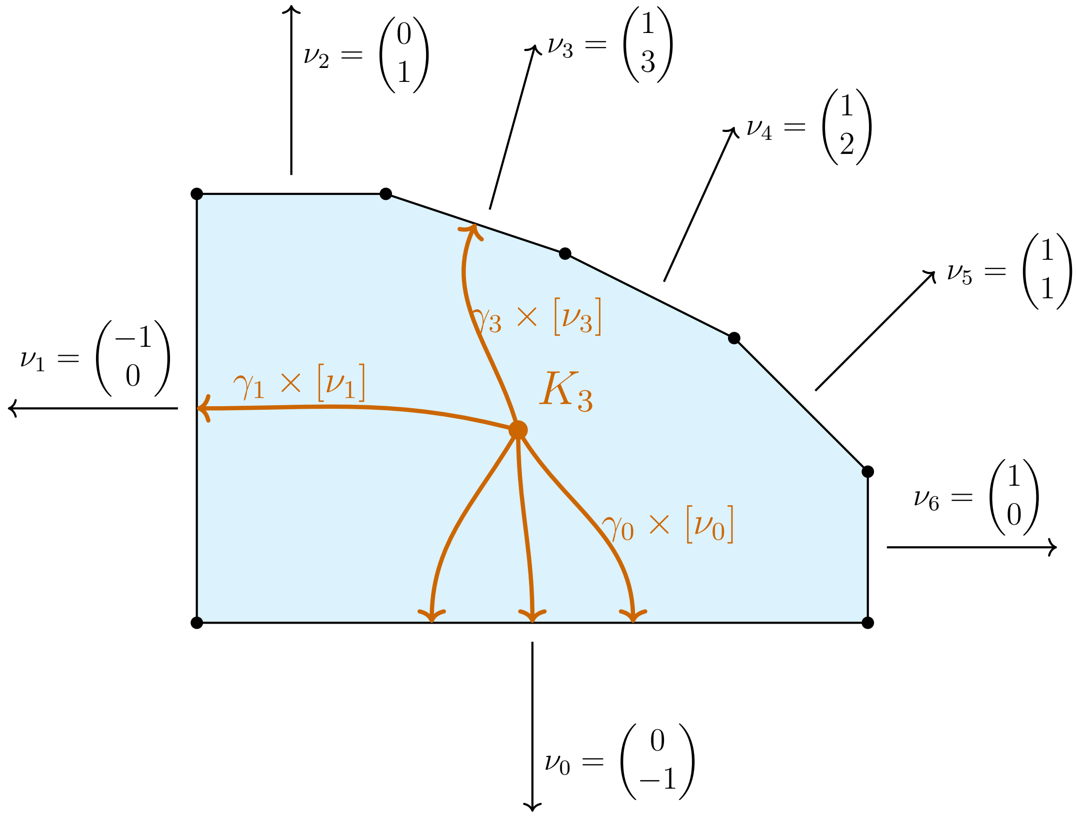


Figure 4.1: The image of a spider in the polytope P.

never interacts with any of the folded facets of P.

Although it is particularly nice when F_0 and F_1 are in standard position, it is not actually necessary. This construction will create a spider homology class representing any relation of the form $\sum_{i=0}^{m-1} c_i[\nu_i] = 0$ in $H_1(T^2)$, with a single body fiber and c_i legs to the ν_i facet.

4.3.2 Embedding Spiders

These spiders are nice classes in singular homology. Each leg can be realized by a map of a single 2-simplex into M. The body of the spider is b of copies of the torus 2-cell, for some $b \in \mathbb{Z}$. Since the torus has a nice simplicial complex structure with two 2-simplices, we can realize the body by $2b$ maps of the 2-simplex into M. These maps form a chain with trivial boundary, since we choose the legs exactly to cancel the boundary of the body maps. The problem that remains is to find an embedded submanifold in the same homology class as our spider cycle. This is important because we want to compute the cup product structure by interpreting the cup product as the intersection of embedded submanifolds in the dual homology classes.

We will first show that the spiders are a basis for $H_2(M)$ when M is a 4-dimensional toric symplectic manifold represented by Delzant polygon P. Let $[s_i]$ be the spider class that connects the facet F_i to the facets F_0 and F_1, as defined in Section 4.3.1. Thus, its generating relation is $c_0^i[\nu_0] + c_1^i[\nu_1] + [\nu_i] = 0$ in $H_1(T^2)$. Our first goal is to find an embedded submanifold in the equivalence class $[s_i]$.

We begin by considering the collection of maps $f_j^i : \Delta_j^2 \to M$ that realize $[s_i]$ as a cycle in singular homology. The problem with the maps f_j^i is that their image as a chain is not generally an embedded submanifold. Since we want to compute the cup product structure on $H^2(M)$ as the intersection form on dual embedded submanifolds, we need to find an embedded submanifold in the same homology class as $[s_i]$. By identifying the boundaries of some of the 2-simplices Δ^2, we can get a map of a Δ-complex D into M given by $f : D \to M$ which will exactly agree with the maps f_i on each 2-simplex. We will then show that D is homeomorphic to S^2. Once we have a map $f : S^2 \to M$ with image $[s_i]$, we will approximate f by an immersion whose only failure to be embedded will be a finite number of transverse double points. We will then do local surgeries on these double points to remove the intersections. These surgeries may increase the genus of the representing submanifold, but they will maintain the homology class.

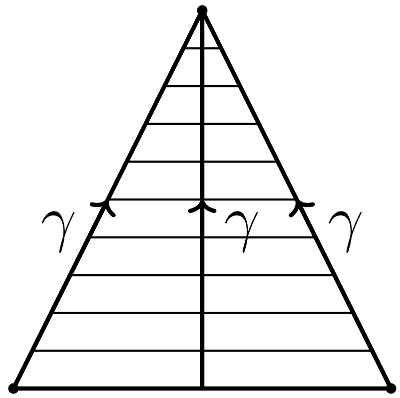


Figure 4.2: The map from Δ^2 to the leg $\gamma \times \nu$. Each horizontal line maps to the circle ν above a different point on the path γ.

We start by defining the map $f : D \to M$. As an example, consider the spider connecting the facets $\nu_0 = (-1, 0)$, $\nu_1 = (0, -1)$, and $\nu_i = (3, 2)$. Note that we can always use a linear transformation to put ν_0 and ν_1 into this standard position. Thus the generating relation is $3[\nu_0]+2[\nu_1]+[\nu_i] = 0$. This spider has six legs total: three to the ν_0 facet, two to the ν_1 facet, and one to the ν_i facet. Each of these legs is an embedded disk in M, and is the image of a single 2-simplex. To define the map from Δ^2 to an arbitrary leg $\gamma \times \nu$ with path γ to the facet that collapses the circle ν, consider the 2-simplex Δ^2 in Figure 4.2. Each horizontal line in Δ^2 will be mapped to the circle $\{p\} \times \nu$ for some point $p \in M/T$ on γ. Straight lines from each point on the bottom edge to the top vertex will be mapped to $\gamma \times \{c\}$ for some point $c \in \nu$. The bottom edge will be mapped to ν on the body fiber. The top vertex will be mapped to the facet where ν is collapsed to a point. Since there are six legs in our example, we will need six such maps. We label the maps f_i with domain Δ_i^2 for $i = 0, \ldots, 5$.

The body requires six copies of the generator of $H_2(T^2)$, each of which will be the image of two 2-simplices. The two maps from the two 2-simplices to the torus fiber are given by putting a simple Δ-complex structure on T^2, as shown in

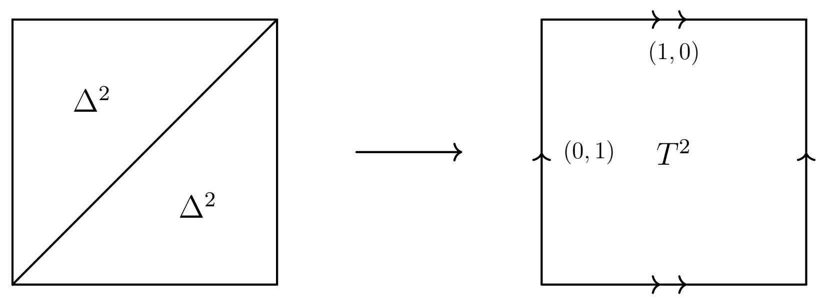


Figure 4.3: The pair of maps from two copies of Δ^2 to the torus fiber above a single point in M/T. The body of the spider consists of copies of these maps.

Figure 4.3. To create D in our example, we need six copies of this pair of maps together with the six leg maps. Label the body maps $f_i : \Delta_i^2 \to M$ for $i = 6, \dots 17$. Taken together, the twelve 2-simplices representing the six torus cells and the six 2-simplices representing the six legs glue together into a rectangular disk with triangle protrusions as shown in Figure 4.4. This gluing can be done because the maps f_i into M agree on the overlapping boundaries.

To turn this disk into a sphere, we glue each of the 2-simplices that will get mapped to a leg into a cone. In Figure 4.4, this means making six edge identifications: glue the two edges labeled e_0 together, glue the two edges labeled e_1 together, etc. We call this space D. To see that D is homeomorphic to a sphere, note that these leg edge gluings affect the rectangular disk representing the body of the spider: it becomes a sphere with boundary a wedge of six circles. The six leg cones exactly cap off these boundary circles with disks, making D homeomorphic to S^2. We call the union of the maps f_i with domains identified to become $D \cong S^2$ the map $f : S^2 \to M$. Note that the image of f is exactly the cycle $[s_i]$.

Now we want to find an embedded submanifold in the same homology class as $[s_i]$. To do so, we invoke a result of Whitney. Theorem 2 in [16] states that if $f : N \to M$ is a continuous map of differentiable manifolds with $2\dim(N) \leq \dim(M)$, then there is a differentiable immersion $F : N \to M$ approximating f arbitrarily

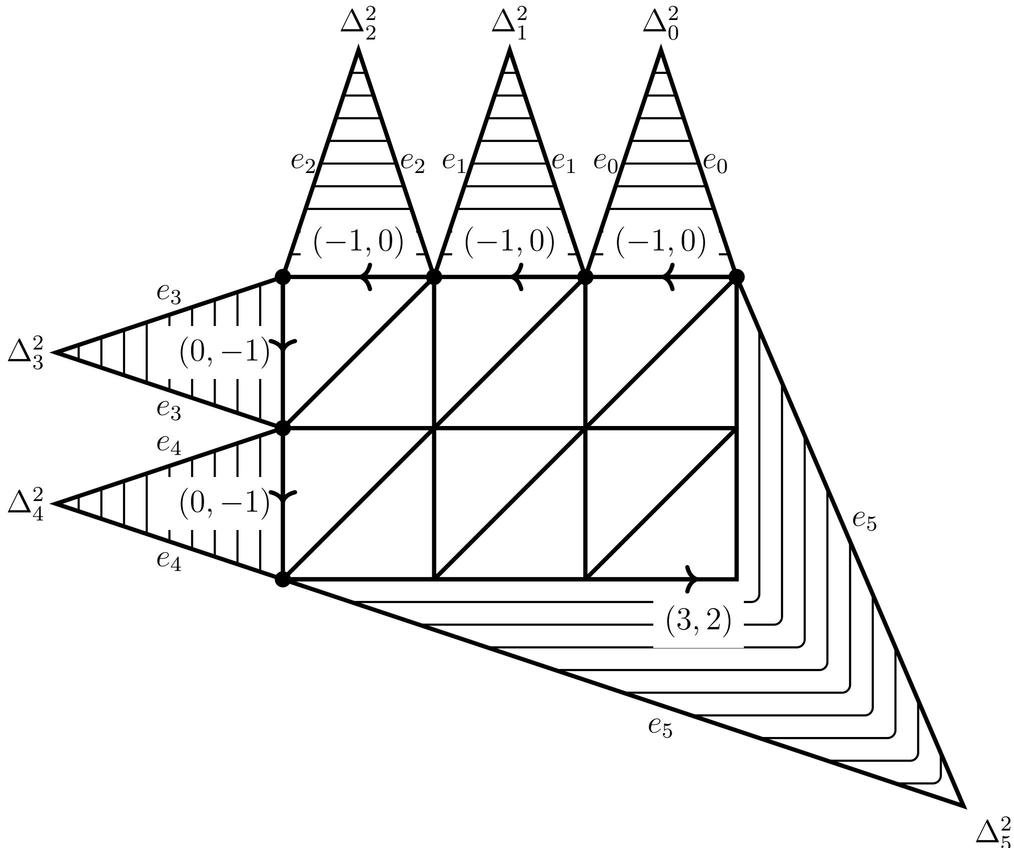


Figure 4.4: The disk of 2-simplices which will become the sphere D.

closely in the uniform topology whose only failure to being an embedding is a finite set of transverse double-points. Since $f : S^2 \to M$ is a continuous map with $2\dim(S^2) \leq \dim(M)$, there is such an approximating map F.

We can turn the map F into an embedding by following the argument in Section 3.1 of [14]. Suppose we have local coordinates around a double-point so the image is two planes meeting transversely at the origin in $\mathbb{R}^4$. Their intersection with the 3-sphere S^3 is two circles, linking once. We remove the portions of the planes in the open 4-ball bounded by S^3, and instead connect the two circles by an annulus. This will increase the genus of the image of F, but maintain the homology class and remove the double-point. Repeating this for each double-point will yield an embedded submanifold in the same homology class as $[s_i]$. Equally importantly, all of the changes we made can be contained to a neighborhood of the body of $[s_i]$, leaving the legs (which were already embedded away from the body) alone. This means we will be able to compute the intersection number of two spider classes by placing their bodies in (or near) disjoint fibers, and then looking only at the intersections of their legs.

4.3.3 Spiders Form a Dual Basis

From this point forward, we will conflate the original singular spider class $[s_i]$ with its homologous embedded submanifold. Our goal in this section is to prove that the $m-2$ spider classes $[s_2], \ldots, [s_{m-1}]$ defined in Section 4.3.2 form a basis for $H_2(M)$ when M is a toric symplectic manifold. To do so we will use Poincaré duality, and the fact that the cup product on H^2 can be computed using the intersection number of dual embedded submanifolds. In order to do this, we need to be very clear about the orientations placed on the spiders, and on M.

Let x_0 and x_1 be the coordinate functions corresponding to e_0 and e_1 in the base space M/T. Note that these are well-defined on each P_i, but don't form global functions on M/T. Let t_0 and t_1 be the corresponding coordinate functions on the fiber T^2, so that the standard volume form providing an orientation on each toric symplectic piece M_i is given by $dx_0 \wedge dt_0 \wedge dx_1 \wedge dt_1$. However, recall from Section 1.5 that the orientations of the M_i only agree with the global orientation of M up to a sign, and the sign swaps when passing through a folded facet to an adjacent M_j. Therefore the oriented basis to the tangent space of M_i is $dx_0 \wedge dt_0 \wedge dx_1 \wedge dt_1$ if $M_i \subseteq M_+$ and is $-dx_0 \wedge dt_0 \wedge dx_1 \wedge dt_1$ if $M_i \subseteq M_-$.

To compute the orientation on a leg $\gamma \times \nu$, assume ν is the circle (a, b) in

$H_1(T^2)$ and γ is a path starting at some point in M/T and terminating in the facet corresponding to ν. Then in local coordinates a basis for the tangent space to γ is $a\,dx_0 + b\,dx_1$, and a basis for the tangent space to ν is $a\,dt_0 + b\,dt_1$. Therefore a basis for the tangent space to the leg $\gamma \times \nu$ is given by $\alpha(a, b) := (a\,dx_0 + b\,dx_1) \wedge (a\,dt_0 + b\,dt_1)$. This is equal to $\alpha(a, b) = (a^2\,dx_0 dt_0 + ab\,dx_0 dt_1 + ab\,dx_1 dt_0 + b^2\,dx_1 dt_1)$.

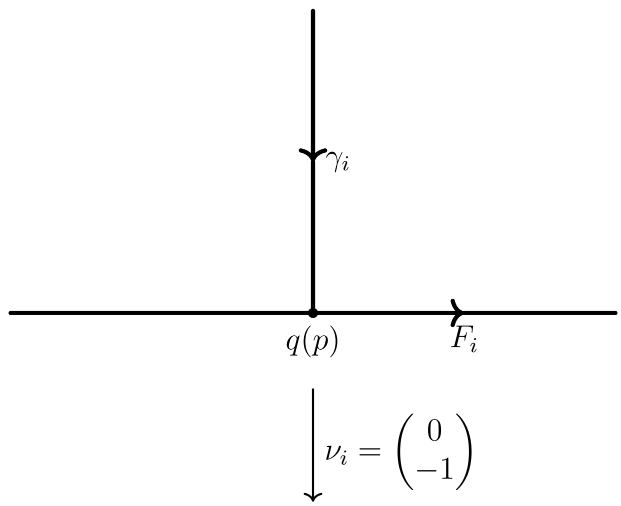


Figure 4.5: The intersection of $\gamma_i \times \nu_i$ with $q^{-1}(F_i)$ projected into M/T.

The first calculation we want to make is the oriented intersection number of a leg $\gamma_i \times \nu_i$ with the 2-sphere $q^{-1}(F_i)$ where F_i is the facet with primitive normal vector ν_i. By using a linear transformation, we can always put the point of intersection into a standard form. We let ν_i be the vector $(0, -1)$, and we let the facet F_i lie along the x_0-axis of $\mathbb{R}^2$. We let γ_i be a vertical path in the $-x_1$ direction of $\mathbb{R}^2$, terminating on F_i. Let $p \in M$ be the point of intersection between $q^{-1}(F_i)$ and $\gamma_i \times \nu_i$. See Figure 4.5. We assume that the orientations of the spheres $q^{-1}(F_i)$ are consistently counter-clockwise around P_i for $M_i \subseteq M_+$ and consistently clockwise around P_i for $M_i \subseteq M_-$. Then an oriented basis for the tangent space to $\gamma_i \times \nu_i$ is given by $-dx_1 \wedge -dt_1 = dx_1 dt_1$, and an oriented basis for the tangent space to $q^{-1}(F_i)$ in the M_+ case is given by $dx_0 \wedge dt_0 = dx_0 dt_0$. The direct sum of these oriented bases is $dx_1 dt_1 \wedge dx_0 dt_0 = dx_0 dt_0 dx_1 dt_1$. Since this is exactly the chosen oriented basis for M_+, the intersection $[s_i] \cdot [q^{-1}(F_i)] = +1$. A similar calculation yields the same result in the M_- case.

Since the spider $[s_i]$ need not intersect the sphere $q^{-1}(F_j)$ whenever $i \neq j$, we have that $[s_i] \cdot [q^{-1}(F_j)] = 0$ for $i \neq j$. From these two facts and the unimodularity of the intersection form it follows that the spiders $[s_2], \ldots, [s_{m-1}]$ are a basis for $H_2(M)$ because they are dual to standard basis of spheres $[q^{-1}(F_2)], \ldots, [q^{-1}(F_{m-1})]$. See Section 3.2 of [14] for more details on dual bases and intersection forms.

4.4 Generators for Toric Origami $H_2(M)$

4.4.1 Spiders in the Open Submanifolds A_i

To understand $H_2(M)$ for toric origami M, we will use Mayer-Vietoris as outlined in Section 4.2. This will require us to understand $H_2(A_i)$ for the open sets A_i which cover M.

Each A_i is an open set in M which deformation retracts onto $M_i = q^{-1}(P_i)$ the toric symplectic manifold with boundary homeomorphic to the following construction: Start with the Delzant polytope P_i which has m_i facets, and let all facets of P_i corresponding to an edge of the template graph G be called folded, and the rest be unfolded. Let $\nu_0, \ldots, \nu_{k_i-1}$ be the set of normal vectors to the k_i *unfolded* facets $F_0, \ldots F_{k_i-1}$. Let T_j be the circle subgroup of T^2 generated by ν_j. Then

$$M_i \cong \left(P_i \times T^2\right) / \sim$$

where $(p_1, t_1) \sim (p_2, t_2)$ if and only if $p_1 = p_2$ are both in some *unfolded* facet F_j and

$t_1 - t_2 \in T_j$. This is the almost the topological construction of a toric symplectic manifold as given in Section 1.2, but the quotient by the normal direction circle only happens in fibers above points in the unfolded facets. The preimage of each folded facet is a 3-dimensional lens space and a component of ∂M_i. There are $m_i - k_i$ such folded facets and thus ∂M_i has $m_i - k_i$ connected components.

Holm and Pires show in [11] that $H^1(M_i) \cong 0$, and that $H^2(M_i) \cong \mathbb{Z}^{k_i-2}$. Therefore by Poincaré duality for manifolds with boundary $H_3(M_i, \partial M_i) \cong 0$, and $H_2(M_i, \partial M_i) \cong \mathbb{Z}^{k_i-2}$.

The long exact sequence for relative homology

$$\cdots \to H_3(M_i, \partial M_i) \to H_2(\partial M_i) \to H_2(M_i) \to H_2(M_i, \partial M_i) \to H_1(\partial M_i) \to \cdots$$

becomes

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}^s \longrightarrow H_2(M_i) \longrightarrow \mathbb{Z}^{k_i-2} \longrightarrow \mathbb{Z}^s \oplus \left(\bigoplus_{j=1}^{m_i-k_i-s} \mathbb{Z}/p_j\mathbb{Z} \right) \longrightarrow \cdots$$

where s is the number of components of ∂M_i homeomorphic to $S^1 \times S^2$ rather than "normal" lens spaces and the $\mathbb{Z}/p_j\mathbb{Z}$ comes from H_1 of the "normal" lens spaces. It follows from this piece of the long exact sequence that $H_2(M_i) \cong \mathbb{Z}^{k_i-2}$.

Let N_X be the $\mathbb{Z}$-span of the normal vectors $\nu_0, \ldots, \nu_{k_i-1}$ in $\mathbb{Z}^2 \cong N$. If $N/N_X \cong \mathbb{Z}$, then M_i is prismatic which forces the entire toric origami manifold M to be prismatic and homeomorphic to $S^{\times}T^2$ so we ignore this case. Thus we assume N/N_X is zero or finite cyclic. In either case, we have k_i vectors $\mathbb{Z}$-spanning a 2-dimensional subspace of N. There must therefore be $k_i - 2$ linearly independent linear relations amongst the ν_i. In particular, no non-trivial linear combination of the linear relations can create the trivial linear relation. Suppose the linear relations are given for $1 \leq j \leq k_i - 2$ by:

$$\sum_{r=0}^{k_i-1} c_r^j \nu_r = 0.$$

where $c_r^j \in \mathbb{Z}$ not all zero for fixed j. Each of these $k_i - 2$ linear relations specifies a spider class in $H_2(M_i)$. Let $[s_j] \in H_2(M_i)$ be the spider with c_r^j legs to the facet with normal vector ν_r.

We claim that the spiders $[s_j]$ are linearly independent in $H_2(M_i)$ and therefore form a basis for $H_2(M_i)$. Suppose for a contradiction that there were a linear dependence $\sum_{j=1}^{k_i-2} a_j[s_j] = 0$ with not all $a_j = 0$. Consider the map $\hat{q} : M_i \to \hat{M}_i$ where $\hat{M}_i$ is the toric symplectic manifold corresponding to P_i. The map $\hat{q}$ is a quotient map which collapses the circle fibers above the folded facets of M_i, finishing the topological construction for a toric symplectic manifold.

Then $H_2(\hat{M}_i) \cong \mathbb{Z}^{m_i-2}$ as described in Sections 4.3.1 and 4.3.3. We label the normal vectors $\hat{\nu}_0, \ldots, \hat{\nu}_{m_i-1}$ in $\hat{M}_i$ so that $\hat{\nu}_j = \nu_j$ for $0 \leq j \leq k_i - 1$. Further, we can assume that we chose labels such that the 2 basis normal vectors in M_i are $\hat{\nu}_0$ and $\hat{\nu}_{m_i-1}$. Therefore the spider basis classes in $H_2(\hat{M}_i)$ are given by $[\hat{s}_1], \ldots, [\hat{s}_{m_i-2}]$. Then map $\hat{q}_* : H_2(M_i) \to H_2(\hat{M}_i)$ is a homomorphism such that

$$q_*([s_j]) = \sum_{r=1}^{k_i-1} c_r^j [\hat{s}_r].$$

Therefore if $\sum_{j=1}^{k_i-2} a_j[s_j] = 0$ then

$$0 = q_* \left(\sum_{j=1}^{k_i-2} a_j[s_j] \right) = \sum_{j=1}^{k_i-2} \left(a_j \sum_{r=1}^{k_i-1} c_r^j [\hat{s}_r] \right) = \sum_{r=1}^{k_i-1} \left(\sum_{j=1}^{k_i-2} a_j c_r^j \right) [\hat{s}_r].$$

The linear independence of the $[\hat{s}_r]$ implies that $\sum_{j=1}^{k_i-2} a_j c_r^j = 0$ for all $1 \leq r \leq k_i - 1$. However, this exactly contradicts the assumption that the linear relations

$\sum_{r=0}^{k_i-1} c_r^j \nu_r = 0$ are linearly independent: the a_j (with $a_0 = 0$) are a non-trivial linear dependence between the relations. Therefore it must be the case that no such a_j exist and the $[s_j]$ are linearly independent in $H_2(M_i)$. Since $H_2(M_i) \cong \mathbb{Z}^{k_i-2}$, the $[s_j]$ in fact form a basis.

4.4.2 Origami Spiders

Let M be a toric origami 4-manifold covered by the open sets $A_1, \ldots, A_n$ which deformation retract onto the toric symplectic manifolds with boundary $M_1, \ldots, M_n$. Each component of the 1-skeleton of M as defined in Section 1.6 is made up of unfolded facets from the M_i. Let $C_0, \ldots, C_{m-1}$ be the 1-skeleton components of M with corresponding normal vectors $\nu_0, \ldots, \nu_{m-1}$. Exactly as in the case for the individual M_i in Section 4.4.1, we can define spider classes in $H_2(M)$ by taking $m-2$ linearly independent linear relations amongst the vectors $\nu_0, \ldots, \nu_{m-1}$ in $\mathbb{Z}^2$. In particular, let

$$\sum_{r=0}^{m-2} c_r^j \nu_r = 0$$

for $1 \leq j \leq m-2$ be linearly independent relations. Then for each j there is a spider class $[s_j] \in H_2(M)$ with a single body fiber in above a point in the interior of M/T and c_r^j legs to the 1-skeleton component with normal vector ν_r. Recall that the image of a leg under the map $q : M \to M/T$ is a path γ starting at the point corresponding to the body fiber, and ending in an unfolded facet. The difference between spiders in the M_i and spiders in M is that $\pi_1(M_i/T) \cong 0$, but $\pi_1(M/T)$ may be non-trivial. Therefore the path γ may be somewhat complicated and the $[s_j]$ are not well-defined.

To fix a choice of leg-path for each leg of each $[s_j]$, we recall from the notation set out in Section 4.2 that we have chosen a collection $E_1^*, \ldots, E_{m-1}^*$ of folded facets in M/T that correspond to a spanning tree in the template graph G. The remaining folded facets are labeled $E_1, \ldots, E_\ell$, and each corresponds to a generator of $\pi_1(M/T)$. The space $M/T - \left(\bigcup_{i=1}^{\ell} E_i\right)$ is simply connected. For each leg of each spider in our generating set, the paths γ in M/T that start at the body fiber and end at the respective 1-skeleton component will live entirely in $M/T - \left(\bigcup_{i=1}^{\ell} E_i\right)$. This makes the γ unique up to homotopy and fixes our choice of spiders $[s_j]$. The classes $[s_1], \ldots, [s_{m-2}]$ make up approximately half of the classes which will form our generating set for $H_2(M)$.

4.4.3 Origami Tori

Recall that $b_2(M) = 2\ell + \#M^T - 2$ where ℓ is the rank of $\pi_1(M/T)$ and $\#M^T$ is the number of fixed points of the torus action. Let m be the number of components of the 1-skeleton of M. Let $t \leq m$ be the number of components of the 1-skeleton of M which are tori rather than spheres. Then $\#M^T = m - t$ because the number of fixed points of the torus action is equal to the number of spheres in the 1-skeleton. Thus $b_2(M) = 2\ell + (m - t) - 2$. Since we have found $m - 2$ spider classes, we are looking for another $2\ell - t$ classes to round out our generating set for $H_2(M)$.

The other $2\ell - t$ classes will be embedded 2-tori in M, defined as follows. Let α_i for $1 \leq i \leq \ell$ be a loop in the interior of M/T that lifts to a representative of the generator in $\pi_1(M)$ corresponding to the folded facet E_i. That is, α_i passes through the the facet E_i exactly once, and does not pass through any facet E_j for $j \neq i$. Since α_i is a loop in the interior of M/T, the fiber above every point is a full 2-torus. Thus $q^{-1}(\alpha_i) \cong \alpha_i \times S_x^1 \times S_y^1 \cong T^3$, where S_x^1 is the $(1,0)$ loop in the torus fiber and S_y^1 is the $(0,1)$ loop in the torus fiber. Since the 3-torus T^3 has

$H_2(T^3) \cong \mathbb{Z}^3$, there are 3 possible classes that could include into $H_2(M)$. The full torus fiber $S^1_x \times S^1_y$ is trivial in $H_2(M)$ because it is the boundary of the 3-chain solid torus created by taking $q^{-1}(\gamma)$ for any path γ that starts at a generic fiber and ends in an unfolded facet.

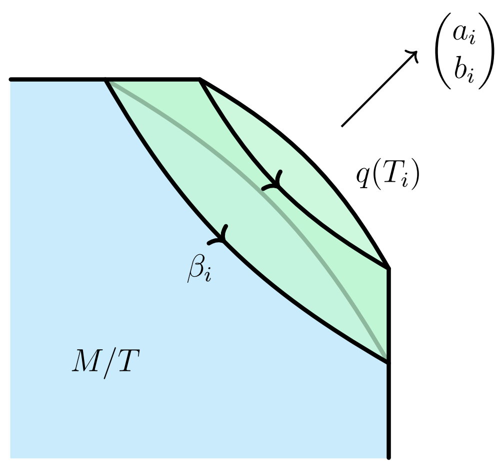


Figure 4.6: Showing that $[\tau_i] = a_i[\beta_i x] \times b_i[\beta_i y]$ is trivial in $H_2(M)$.

In general, the other two 2-tori, $\alpha_i x := \alpha_i \times S^1_x$ and $\alpha_i y := \alpha_i \times S^1_y$, are non-trivial in $H_2(M)$ and provide 2ℓ generators. However, each torus component of the 1-skeleton introduces a linear relation amongst the $\alpha_i x$ and $\alpha_i y$. In particular, let $T_1, \ldots T_t$ be the torus components of the 1-skeleton. Let $\beta_i = t^i_1\alpha_1 + \cdots + t^i_\ell\alpha_\ell$ for $t^i_j \in \{0, \pm 1\}$ be the loop in $H_1(M)$ that is homologous to the loop $q(T_i)$ which is the image of T_i in M/T. Let (a_i, b_i) be the normal vector to the 1-skeleton component T_i. Then for each $1 \leq i \leq t$,

$$[\tau_i] = a_i \left(t^i_1[\alpha_1 x] + \cdots + t^i_\ell[\alpha_\ell x]\right) + b_i \left(t^i_1[\alpha_1 y] + \cdots + t^i_\ell[\alpha_\ell y]\right) = 0$$

because $[\tau_i] \in H_2(M)$ is the boundary of the solid torus $S^1 \times D^2$ where the S^1 factor is the circle $t^i_1\alpha_1 + \cdots + t^i_\ell\alpha_\ell$ and the D^2 factor is the annular homotopy between β_i and $q(T_i)$ in M/T crossed with the circle (a, b) in the torus fibers above M/T. Since (a_i, b_i) is collapsed in T_i, the circle cross annulus is collapsed at one end forming a solid torus $S^1 \times D^2$. Since $[\tau_i]$ is exactly $a_i[\beta_i x] \times b_i[\beta_i y]$, it is the boundary of this $S^1 \times D^2$ and therefore trivial in $H_2(M)$. See Figure 4.6.

Therefore each loop α_i for $1 \leq i \leq \ell$ adds two elements to our generating set for $H_2(M)$, and each torus T_i in the 1-skeleton adds a single relation. Taken together, the tori add $2\ell - t$ generators to our generating set for $H_2(M)$, as desired.

4.4.4 Spider Moves

The set of $m-2$ spiders $[s_j]$ chosen in 4.4.2 along with the set of 2ℓ tori chosen in 4.4.3 will form our generating set $X \subseteq H_2(M)$. In order to show that $\text{span}(X) = H_2(M)$, we first need a better understanding of $\text{span}(X)$. In particular, this section discusses some "moves" that can be applied to spider and tori classes that change their projections to M/T but maintain their homology classes.

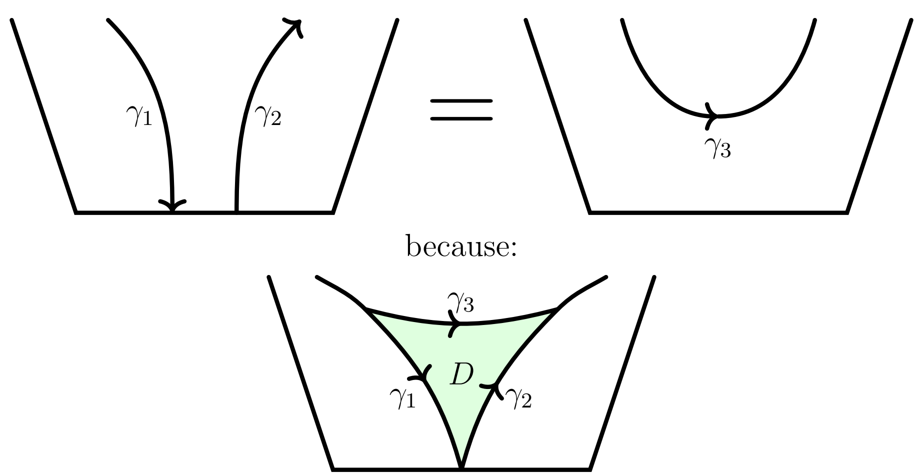


Figure 4.7: The 3-chain $D \times [\nu]$ shows that $(\gamma_1 \times [\nu]) + (\gamma_2 \times [\nu])$ is homologous to $\gamma_3 \times [\nu]$.

First, if one spider leg $\gamma_1 \times \nu$ is entering the facet F corresponding to ν and another spider leg $\gamma_2 \times \nu$ is leaving F, then these two legs are homologous to a third leg $\gamma_3 \times \nu$ that does not enter F. This can be shown by letting D be the

2-disk in M/T with $\partial D = \gamma_1 + \gamma_2 - \gamma_3$. Then the 3-chain $D \times [\nu]$ has boundary $\partial(D \times [\nu]) = (\gamma_1 \times [\nu]) + (\gamma_2 \times [\nu]) - (\gamma_3 \times [\nu])$, as desired. See Figure 4.7.

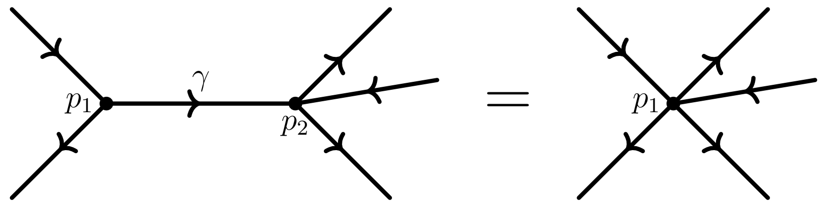


Figure 4.8: Consolidating body fibers joined by a single leg cylinder.

Second, suppose two spider body fibers above p_1 and p_2 are connected by a "leg cylinder" $\gamma \times [\nu]$ where γ is a path from p_1 to p_2 and $[\nu]$ is any loop in $H_1(T^2)$. Then contracting γ in a single point allows us to move the two body fibers into the fiber above a single point, say p_1. See Figure 4.8.

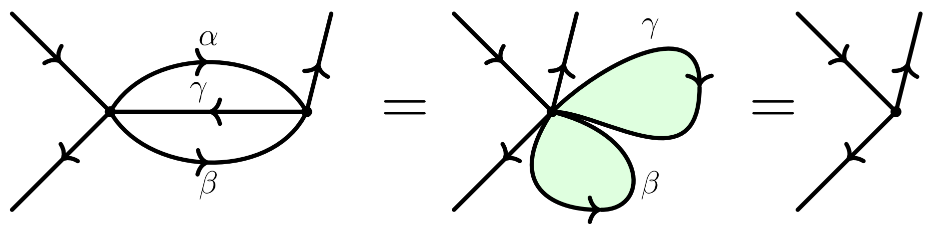


Figure 4.9: Consolidating body fibers joined by multiple leg cylinders.

In the situation that there are n leg cylinders connecting the same two body fibers, the body fibers can still be consolidated. First contract along a single leg cylinder. This will leave a single body fiber, by with $n-1$ leg loops connected to it. The disks bounded by each loop form null-homotopies of each leg loop, and thus they can be removed. Note that the orientations of the paths between the two body fibers are irrelevant once they become loops based at a single fiber. See Figure 4.9.

In general, given any linear relations amongst the normal vectors to the 1-skeleton components of M, there is a corresponding spider $[s]$ living in $H_2(M)$ with legs above the spanning tree of M/T. Since there can only $m-2$ linearly independent relations amongst $\nu_0, \ldots, \nu_{m-1}$, it follows that some combination of the relations defining the spiders $[s_1], \ldots, [s_{m-2}]$ must create the spider $[s]$. The moves outlined above show how a linear combination of the $[s_1], \ldots, [s_{m-2}]$ spiders combine to form the single spider $[s]$.

To create a spider whose legs stray outside the spanning tree, we simply use the tori $\alpha_i x$ and $\alpha_i y$. Suppose γ_1 is a leg path in the spanning tree starting at the body and ending in some facet F with normal vector (a, b). Let γ_2 be any other leg path starting at the body and ending at F. To replace the leg $\gamma_1 \times (a, b)$ with $\gamma_2 \times (a, b)$, we note that $\gamma_2\gamma_1^{-1}$ is a loop in $H_1(M)$ and is therefore homologous to some combination of the loops α_i. Let $\gamma_2\gamma_1^{-1} = \sum_{i=1}^{\ell} c_i\alpha_i$. Then the torus $\sum_{i=1}^{\ell} c_i(a[\alpha_i x] + b[\alpha_i y])$ added to the spider with leg $\gamma_1 \times (a, b)$ will be homologous to the same spider with leg $\gamma_2 \times (a, b)$.

The uptake from this is that if X is our set of $m-2$ basis spiders together with our $2\ell - t$ tori, then any spider representing any relations amongst the normal vectors corresponding to the 1-skeleton of M is in the span(X), regardless of the paths the legs take when projected to M/T.

4.4.5 Proof that X Generates $H_2(M)$

We will first prove a general lemma which gives two conditions under which our set X will generate $H_2(M)$. We will then show that X satisfies these two conditions, and thus generates $H_2(M)$.

Lemma 4.4.1. *Let A, B, C be $\mathbb{Z}$-modules with module homomorphisms $f : A \to B$ and $g : B \to C$ such that* $\operatorname{im}(f) = \ker(g)$. *Suppose that $X = \{x_1, \ldots, x_n\}$ is a set of elements in B. Then* $\operatorname{span}(X) = B$ *if:*

- $\operatorname{im}(f) \subseteq \operatorname{span}(X)$ *and*
- $\operatorname{im}(g) = g(\operatorname{span}(X))$.

Proof of Lemma 4.4.1. Suppose $\operatorname{im}(f) \subseteq \operatorname{span}(X)$ and $\operatorname{im}(g) = g(\operatorname{span}(X))$. Since $\operatorname{span}(X) \subseteq B$ by definition, to show that $\operatorname{span}(X) = B$ we will show that $B \subseteq \operatorname{span}(X)$.

Let $b \in B$. Then $g(b) \in \operatorname{im}(g) = g(\operatorname{span}(X))$, so $g(b) = g(x)$ for some $x \in \operatorname{span}(X)$. Then $g(b - x) = 0$, so $b - x \in \ker(g) = \operatorname{im}(f) \subseteq \operatorname{span}(X)$. Thus there exists $y \in \operatorname{span}(X)$ such that $b - x = y$. Then $b = x + y$, and both x and y are in $\operatorname{span}(X)$. Thus $b \in \operatorname{span}(X)$ and so $\operatorname{span}(X) = B$, as desired. $\square$

From the Mayer-Vietoris long exact sequence we have the maps

$$\bigoplus_i H_2(A_i) \xrightarrow{f} H_2(M) \xrightarrow{g} \bigoplus_{i \neq j} H_1(A_i \cap A_j)$$

where $\operatorname{im}(f) = \ker(g)$. Let X be our chosen set of tori and spider classes in $H_2(M)$. Then to show that $\operatorname{span}(X) = H_2(M)$ it suffices to show that $\operatorname{im}(f) \subseteq \operatorname{span}(X)$ and $\operatorname{im}(g) = g(\operatorname{span}(X))$.

Proof that $\operatorname{im}(f) \subseteq \operatorname{span}(X)$

The map $f : \bigoplus_{i=1}^n H_2(A_i) \to H_2(M)$ is induced by the inclusion maps $A_i \hookrightarrow M$. Each A_i is an open neighborhood of the toric symplectic submanifold with boundary M_i in M. Each connected component of $A_i \cap A_j$ is an open neighborhood of $q^{-1}(F)$ for some folded facet F gluing the polytope P_i to the polytope for P_j in M/T. We will show that the basis spiders in an arbitrary $H_2(M_i)$ map into the span of X. Fix $1 \leq i \leq n$. Let $C_0, \ldots, C_{m-1}$ be the 1-skeleton components of M, with normal vectors $\nu_0, \ldots, \nu_{m-1}$. Re-index the C_j so that the unfolded facets $F_0, \ldots, F_{k_i-1}$ in M_i include into the 1-skeleton components $C_0, \ldots, C_{k_i-1}$ respectively.

Recall from Section 4.4.2 that $H_2(A_i) \cong H_2(M_i)$ can be generated by a set of $k_i - 2$ spider classes where k_i is the number of unfolded facets of P_i. Let $[s] \in H_2(M_i)$ be one of these spider generators, represented by the relation

$$\sum_{j=1}^{k_i} c_j[\nu_j] = 0.$$

Thus $[s]$ is a spider with c_j legs to the facet F_j in M_i. Then $f([s])$ is the identical spider in M, with c_j legs to the 1-skeleton component C_j. By the arguments in Section 4.4.4, any such spider is contained in the span of X, as desired.

Proof that $\mathrm{im}(g) = g(\mathrm{span}(X))$

We now focus on the portion of the long exact sequence given by

$$H_2(M) \xrightarrow{g} \bigoplus_{i \neq j} H_1(M_i \cap M_j) \xrightarrow{h} \bigoplus_i H_1(M_i)$$

where we have replaced each A_i in the original sequence by the M_i manifold with boundary it deformation retracts onto. Since $\mathrm{im}(g) = \ker(h)$, we really need to show that $g(\mathrm{span}(X)) = \ker(h)$. Each connected component of $M_i \cap M_j$ is $q^{-1}(F)$ for some folded facet F with normal vector ν. Suppose the normal vectors to the facets adjacent to F are ν_0 and ν_1. Then $q^{-1}(F) = L$ is a lens space with $H_1(L) \cong \mathbb{Z}^2/\langle \nu_0, \nu_1 \rangle$. Note that if $\mathrm{span}(\nu_0, \nu_1) = \mathbb{Z}^2$, then $L \cong S^3$ and $H_1(L) \cong 0$. If $\mathrm{span}(\nu_0, \nu_1) = \mathbb{Z}$, then $L \cong S^1 \times S^2$ and $H_1(L) \cong \mathbb{Z}$. Further, if $\phi : L \to M_i$ is the inclusion map and β is a generator of $H_1(L)$, then $\phi_*(\beta) \in H_1(M_i)$ is exactly the loop $[\nu] \in H_1(M_i)$. This is because the loop $[\nu]$ spans $H_1(T^2)$ when paired with either $[\nu_0]$ or $[\nu_1]$ by the smoothness condition on the Delzant polytope for M_i. Therefore we can choose $[\nu]$ to be the generator of $H_1(L) \cong \mathbb{Z}^2/\langle \nu_0, \nu_1 \rangle$.

We enumerate the connected components of $\bigsqcup_{i \neq j} M_i \cap M_j$ as the lens spaces $L_1, \ldots, L_r$ corresponding to the folded facets of M (i.e. let r be the number of edges in the template graph G). Let $\phi_1, \phi_2 : \bigsqcup_{k=1}^r L_k \to \bigsqcup_{i=1}^n M_i$ be the maps that include the lens spaces into their corresponding two toric symplectic manifolds with boundary. For example, if L_3 is a lens space corresponding to a connected component of $M_2 \cap M_5$, then $\phi_1(L_3)$ is the image of the inclusion of L_3 into M_2 and $\phi_2(L_3)$ is the image of the inclusion of L_3 into M_5. In general, if $L_k \in M_i \cap M_j$ with $i < j$, then $\phi_1(L_k) \in M_i$ and $\phi_2(L_k) \in M_j$.

We need notation that lets us keep track of which L_k are included in which M_i. We will do this by creating an $n \times r$ matrix B. We define

$$B_{ik} = \begin{cases} 1, & \phi_1(L_k) \in M_i \\ -1, & \phi_2(L_k) \in M_i \\ 0, & \text{else.} \end{cases}$$

Thus each of the r columns of B will correspond to one of $L_1, \ldots, L_r$, and will have exactly one 1, exactly one -1, and the rest of its entries 0. Each of the n rows of B will correspond to one of the open toric manifolds $M_1, \ldots, M_n$ and will have non-zero entries corresponding to that manifold's boundary components.

Since $L_1, \ldots, L_r$ are the connected components of $\bigsqcup_{i \neq j} M_i \cap M_j$, it follows that $\bigoplus_{i \neq j} H_1(M_i \cap M_j) \cong \bigoplus_{k=1}^r H_1(L_k)$. Thus we can write h as $h : \bigoplus_{k=1}^r H_1(L_k) \to \bigoplus_{i=1}^n H_1(M_i)$.

To understand h, let $[\nu_k]$ be the generator of the cyclic group $H_1(L_k)$. Then

$h([\nu_k]) = (h_1([\nu_k]), \ldots, h_n([\nu_k])$ with $h_i([\nu_k]) \in H_1(M_i)$ such that

$$h_i([\nu_k]) = \begin{cases} (\phi_1)_*[\nu_k], & B_{ik} = 1 \\ -(\phi_2)_*[\nu_k], & B_{ik} = -1 \\ 0, & B_{ik} = 0. \end{cases}$$

Since $g(\mathrm{span}(X)) \subseteq \mathrm{im}(g) = \ker(h)$, it remains to show that $\ker(h) \subseteq g(\mathrm{span}(X))$. Let $\beta \in \ker(h)$. That is, let $\beta \in \bigoplus_{i \neq j} H_1(M_i \cap M_j) \cong \bigoplus_{k=1}^{n} H_1(L_k)$ with $h(\beta) = 0$.

Our goal is to construct a class $x \in \mathrm{span}(X)$ such that $g(x) = \beta$. We will do this by constructing classes $x_i \in H_2(M_i, \partial M_i)$ in relative homology such that the boundary components of the x_i and x_j match up in each connected component $\partial M_i \cap \partial M_j$ and they can therefore be glued together into a class $x \in H_2(M)$ with $g(x) = \beta$. We will then show that this class x is in $\mathrm{span}(X)$.

Write $\beta = (\beta[\nu_1], \ldots, \beta[\nu_r])$ with $\beta_k \in \mathbb{Z}$ and $[\nu_k] \in H_1(L_k)$. Then the i-th coordinate of $h(\beta)$ (i.e. the image of $h(\beta)$ in $H_1(M_i)$) is given by

$$h_i(\beta) = \sum_{k=1}^{r} \beta_k B_{ik}[\nu_k].$$

Since $h_i(\beta) = 0$ for all i and $H_1(M_i) \cong \mathbb{Z}^2/\langle\text{normals to unfolded facets in } M_i\rangle$, it follows that $h_i(\beta)$ is contained in the span of the normal vectors $\nu_1^i, \ldots, \nu_{k_i}^i$ to the unfolded facets $F_1^i, \ldots, F_{k_i}^i$ of M_i. Thus there is a linear relation

$$\sum_{j=1}^{k_i} c_j^i[\nu_j^i] + \sum_{k=1}^{r} \beta_k B_{ik}[\nu_k] = 0.$$

Create the spider x_i which has c_j^i legs to the facet F_j^i for each $1 \leq j \leq k_i$, and $\beta_k B_{ik}$ legs to the boundary component L_k of M_i for each non-zero B_{ik}. The legs to L_k will not be true spider legs, because they will be cylinders rather than disks as the circle $[\nu_k]$ is not collapsed above L_k. Thus the spider x_i is a relative spider, with boundary components $\beta_k B_{ik}[\nu_k]$ above L_k.

We create such a spider x_i for each $1 \leq i \leq n$. Therefore, in the intersection L_k of M_i and M_j for $i < j$, the spider x_i has boundary component $\beta_k[\nu_k]$ and the spider x_j has boundary component $-\beta_k[\nu_k]$. Thus the two leg cylinders can be glued together, creating a cylinder $\gamma \times [\nu_k]$ for γ a path starting and the body fiber of x_i and ending at the body fiber of x_j. After gluing the cylinder legs above each lens space intersection L_k for $1 \leq k \leq r$, the new 2-chain will be called x and will have no boundary components. Thus $x \in H_2(M)$.

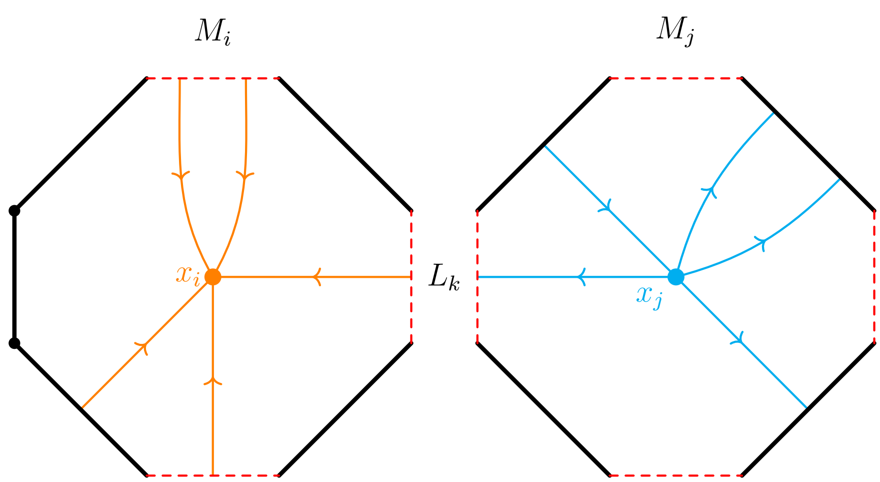


Figure 4.10: Combining the spiders x_i and x_j.

Further, by construction we have that $g(x) = \beta$ because the intersection of x with each L_k will be exactly $\beta_k[\nu_k]$. Finally, using the spider moves from Section 4.4.4, we can combine the n body fibers of x into a single body fiber, and move all the legs that connect body fibers. What remains will be a normal spider class connecting 1-skeleton components of M. Therefore $x \in \text{span}(X)$, and in fact $\text{span}(X) = H_2(M)$, as desired.

4.4.6 Non-Trivial Torsion

Let M be a toric origami 4-manifold, and let N_X be the span of the normal vectors to the 1-skeleton components of M. The case where N/N_X is finite cyclic deserves a short section of explanation. In general, if M has any fixed points under the T action, then $N/N_X \cong 0$. This is because a fixed point of the action implies two adjacent unfolded facets in some polytope P_i whose smoothness condition will immediately force $N_X \cong \mathbb{Z}^2$. In addition, fixed points of the action are paired with sphere components of the 1-skeleton. Therefore the $N/N_X \cong \mathbb{Z}/k\mathbb{Z}$ case is also the case where every component of the 1-skeleton is a torus.

We know $H_2(M) = \mathbb{Z}^{2\ell+(m-t)-2} \times N/N_X$ where m is the number of 1-skeleton components and t is the number of torus 1-skeleton components. In this case, there are still $m-2$ spider classes, 2ℓ torus classes, and t relations amongst the torus classes. However, the t relations don't cleanly remove t torus generators; they leave behind a $\mathbb{Z}/k\mathbb{Z}$. In this section we will describe exactly the linear combination of torus classes that becomes a generator for the $\mathbb{Z}/k\mathbb{Z}$ factor.

To do so, we will do a slight notation reset since we need to discuss the normal vectors at the component level. Let $x_1, \ldots, x_{\ell+1}, y_1, \ldots, y_{\ell+1}$ be the x and y torus generators corresponding to each of the $t = \ell + 1$ torus components of the 1-skeleton. Since $\pi_1(M/T)$ is free on ℓ generators and M/T is topologically an $\ell+1$ times punctured sphere, we can assume that $x_{\ell+1} = \sum_{i=1}^{\ell} x_i$ and $y_{\ell+1} = \sum_{i=1}^{\ell} y_i$. The $\ell+1$ torus relations give that $a_i x_i + b_i y_i = 0$ for each $1 \leq i \leq \ell+1$. By putting M/T into a standard position we can assume that $a_{\ell+1} = -1$ and $b_{\ell+1} = 0$. Combined, this gives us the following relation for the torus part of $H_2(M)$:

$$\left\langle \begin{array}{c|c} x_1, \ldots, x_\ell & a_i x_i + b_i y_i = 0 \text{ for all } 1 \leq i \leq \ell, \\ y_1, \ldots, y_\ell & x_1 + \cdots + x_\ell = 0 \end{array} \right\rangle$$

Since $(-1, 0)$ is unfolded, it follows that $\gcd(b_1, \ldots, b_\ell) = k$. We define $A = \operatorname{lcm}(a_1, \ldots, a_\ell)$ with $\hat{a}_i = A/a_i$ so that $a_i \hat{a}_i = A$. Finally, let $m = \min(a_1, \ldots, a_\ell)$. To find our torsion class we will work backwards, starting by adding up all the relations from the group presentation, and then working until we get k times a class equal to 0.

$$\begin{aligned} \sum_{i=1}^{\ell} a_i x_i + b_i y_i &= 0 \\ A\left[\sum_{i=1}^{\ell} (a_i - m) x_i + b_i y_i\right] &= 0 \\ \sum_{i=1}^{\ell} a_i (A - \hat{a}_i m) x_i + A b_i y &= 0 \\ \sum_{i=1}^{\ell} (A - \hat{a}_i m)(-b_i y_i) + A b_i y &= 0 \\ \sum_{i=1}^{\ell} \hat{a}_i b_i m y_i &= 0 \\ k\left[\sum_{i=1}^{\ell} \frac{b_i}{k} \hat{a}_i m y_i\right] &= 0. \end{aligned}$$

Therefore the class $\beta = \sum_{i=1}^{\ell} \frac{b_i}{k} \hat{a}_i m y_i$ is of order k or some divisor of k. However, no divisor of k can divide $\hat{a}_i$ or $m = a_j$ for some j without one of the (a_i, b_i) failing to be a primitive vector. Therefore β is truly of order k.

In addition, the presentation of the group ensures that β is non-trivial. The only way to destroy a generator y_i is to replace it with x_i and then use the $\sum x_i = 0$ relation. However, translating the y_i generators to x_i's creates the class $\frac{Am}{k} \sum_{i=1}^{\ell} x_i$. But divisor of k can divide A or m, so $\frac{Am}{k}$ is not an integer and no such translation into x_i terms is possible. Thus β is non-trivial of order k and must generate the $\mathbb{Z}/k\mathbb{Z}$ factor of $H_2(M)$.

4.5 Intersections and the Cohomology Ring

We now have embedded submanifold representatives of the generating classes for each homology group. By applying Poincaré duality, the dual classes will generate the cohomology groups. The cup product structure on the cohomology ring is dual to the oriented intersection of embedded submanifolds in the homology groups. Therefore by carefully describing the intersections of all the embedded submanifolds generating the homology of M we will be describing the ring structure on $H^*(M;\mathbb{Z})$. Recall that

$$\begin{array}{ll}
H_0(M) \cong \mathbb{Z} & H^0(M;\mathbb{Z}) \cong \mathbb{Z} \\
H_1(M) \cong \mathbb{Z}^\ell \times \mathbb{Z}/k\mathbb{Z} & H^1(M;\mathbb{Z}) \cong \mathbb{Z}^\ell \\
H_2(M) \cong \mathbb{Z}^{2\ell+\#M^T-2} \times \mathbb{Z}/k\mathbb{Z} & H^2(M;\mathbb{Z}) \cong \mathbb{Z}^{2\ell+\#M^T-2} \times \mathbb{Z}/k\mathbb{Z} \\
H_3(M) \cong \mathbb{Z}^\ell & H^3(M;\mathbb{Z}) \cong \mathbb{Z}^\ell \times \mathbb{Z}/k\mathbb{Z} \\
H_4(M) \cong \mathbb{Z} & H^4(M;\mathbb{Z}) \cong \mathbb{Z}.
\end{array}$$

The generators for $H_1(M) \cong H^3(M;\mathbb{Z})$ are the loops $\alpha_1, \dots, \alpha_\ell$ lifted from $\pi_1(M/T)$ along with the loop β in the torus fiber in the case that N/N_X is non-trivial. The generators of $H_2(M) \cong H^2(M;\mathbb{Z})$ are the $m-2$ spiders and the $2\ell - t$ tori. The generators of $H_3(M) \cong H^1(M;\mathbb{Z})$ are the ℓ lens spaces $L_i = q^{-1}(F_i)$ corresponding to the ℓ folded facets $F_1, \dots, F_\ell$ which are not part of the spanning tree for M/T.

Some of the intersections are easy. $L_1, \dots, L_\ell$ are pairwise disjoint and thus have no intersections. Taking $q^{-1}(F_i)$ of two parallel non-intersecting copies of F_i

for any single facet that shows that $[L_i] \smile [L_i]$ is also zero. Thus $[L_i] \smile [L_j] = 0$ for all i and j. Thus the cup product between any two classes in $H^1(M;\mathbb{Z})$ is 0.

The cup product on $H^3(M;\mathbb{Z}) \times H^1(M;\mathbb{Z})$ is given by

$$[\alpha_i] \smile [L_j] = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

This is because by definition the loop α_i passes through the lens space L_i exactly once in the positive direction, and passes through none of the other L_j. The intersection $[\beta] \cdot [L_j] = 0$ because a representative of $[\beta]$ can be chosen in a generic fiber away from L_j.

The intersection on $H^2(M;\mathbb{Z}) \times H^1(M;\mathbb{Z})$ is non-trivial but straightforward. Let $[s] \in H^2(M;\mathbb{Z})$ be dual to one of the spider classes $s \in X$ which generate $H_2(M)$. Let $[L_j] \in H^1(M;\mathbb{Z})$ be dual to one of the ℓ lens spaces generating $H_3(M)$. The spider classes in X are constructed to only intersect the lens spaces corresponding to edges in the spanning tree of the template graph. The generators $L_1, \ldots, L_\ell$ of $H_3(M)$ are the lens spaces corresponding to edges *not* in the spanning tree. Therefore $s \cap L_j = \emptyset$ for all j and so $[s] \smile [L_j] = 0$.

On the other hand, let $[\alpha_i x], [\alpha_i y] \in H^2(M)$ be dual to the tori $\alpha_i x$ and $\alpha_i y$, respectively. Then since α_i intersects L_j exactly once when $i = j$, and since the fibers above points in the facet corresponding to L_j are full torus orbits, we get that $\alpha_i x \cap L_i = x$ and $\alpha_i y \cap L_i = y$, where $x \in H_1(M)$ is the $(1,0)$ loop in the torus fiber and $y \in H_1(M)$ is the $(0,1)$ loop. Therefore

$$[\alpha_i x] \smile [L_j] = \begin{cases} [x], & i = j, \\ 0, & i \neq j, \end{cases} \qquad [\alpha_i y] \smile [L_j] = \begin{cases} [y], & i = j, \\ 0, & i \neq j, \end{cases}$$

where $[x] \in H^3(M;\mathbb{Z})$ is dual to x and $[y] \in H^3(M;\mathbb{Z})$ is dual to y. Notice that x and y live in the N/N_X portion of $H_1(M)$. In the case where $N = N_X$ (for example if there are any fixed points of the torus action), we have $x = y = 0$ and therefore $[x] = [y] = 0$. In this case it follows that the cup product on $H^2(M;\mathbb{Z}) \times H^1(M;\mathbb{Z})$ is trivial.

This leaves the cup product on $H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z})$, i.e. the intersection form of M. We need to figure out how to calculate intersections between spiders, between tori and between a spider and a torus. This is tricky because it requires careful tracking of orientations.

We start with understanding how to intersect two spiders. By moving the spiders bodies away from each other, it is sufficient to understand how to intersect the spiders legs. By ensuring only two legs intersect above any point in M/T, the leg intersections can be calculated 2 legs at a time and then summed.

To calculate the intersection of a leg $\gamma_0 \times \nu_0$ with a leg $\gamma_1 \times \nu_1$, first homotope γ_0 and γ_1 so that γ_0 is in the x_0 direction and γ_1 is in the x_1 direction, switching the labels 0 and 1 if that makes it easier. Thus we assume γ_0 has tangent direction $(-1)^{\epsilon_0} dx_0$ and γ_1 has tangent direction $(-1)^{\epsilon_1} dx_1$ where $\epsilon_0, \epsilon_1 \in \{0, 1\}$.

Let ν_0 be the curve (a, b) and let ν_1 be the curve (c, d) in the torus fiber. Then there are $|\det(\nu_0, \nu_1)| = |ad - bc|$ points of intersection between the two ν_0 and ν_1 in the torus fiber. The sign on each of those points of intersection is given by the sign for the oriented basis for $\gamma_0 \times \nu_0$ wedged with the oriented basis for $\gamma_1 \times \nu_1$ which is

$$[(-1)^{\epsilon_0}\, dx_0 \wedge (a\, dt_0 + b\, dt_1)] \wedge [(-1)^{\epsilon_1}\, dx_1 \wedge (c\, dt_0 + d\, dt_1)]$$
$$= (-1)^{\epsilon_0}(-1)^{\epsilon_1}(ad - bd)\, dx_0 dt_0 dx_1 dt_1.$$

Let $\epsilon_2 = 0$ if the intersection is occurring in M_+ and $\epsilon_2 = 1$ if the intersection is occurring in M_-. Then the intersection number of $\gamma_0 \times \nu_0$ with $\gamma_1 \times \nu_1$ where γ_0 is in the x_0 direction and γ_1 is in the s_1 direction is $(-1)^{\epsilon_0+\epsilon_1+\epsilon_2} \det(\nu_0, \nu_1)$.

Unfortunately there is not yet a formula that computes the intersection number of two spiders based only on the linear relations defining them. At this time it is still required to actually draw the spiders in M/T, figure out where all the leg intersections are, and then carefully determine orientations by hand. We will do this in an example in Section 4.6.

Since the tori $\alpha_i x$ and $\alpha_i y$ are topologically just like legs, their intersections are computed similarly. The pairwise intersection and self-intersection of any two tori generators will be zero by taking disjoint paths in M/T representing them. However, the intersection of a torus with a spider will be computed as though the torus were a leg of the spider.

When M is toric symplectic the intersection form computed using the spiders will be dual to the usual intersection form computed using the facet spheres.

To recap the results:

Theorem 4.5.1. *For a compact, orientable, 4-dimensional, toric origami manifold M, the loops $\alpha_1, \ldots, \alpha_\ell, \beta$ generate $H_1(M)$, the spiders $s_1, \ldots, s_{m-2}$ and tori $\alpha_1 x, \alpha_1 y, \ldots, \alpha_\ell x, \alpha_\ell y$ generate $H_2(M)$, and the lens spaces $L_1, \ldots, L_\ell$ generate $H_3(M)$. The dual classes in $H^*(M;\mathbb{Z})$ generate $H^*(M;\mathbb{Z})$ and allow for the computation of cup product. In particular, the cup product is dual to the oriented intersections of the embedded homology representatives. In general:*

- *The cup product $H^1(M;\mathbb{Z}) \times H^1(M;\mathbb{Z}) \xrightarrow{\smile} H^2(M;\mathbb{Z})$ is trivial.*

- *The cup product* $H^2(M;\mathbb{Z}) \times H^1(M;\mathbb{Z}) \xrightarrow{\smile} H^3(M;\mathbb{Z})$ *is trivial except for* $[\alpha_i x] \smile [L_i] = [x]$ *and* $[\alpha_i y] \smile [L_i] = [y]$ *for* $1 \leq i \leq \ell$.

- *The cup product* $H^3(M;\mathbb{Z}) \times H^1(M;\mathbb{Z}) \xrightarrow{\smile} H^4(M;\mathbb{Z})$ *is trivial except* $[\alpha_i] \smile [L_i] = 1$ *for* $1 \leq i \leq \ell$.

- *The cup product* $H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \xrightarrow{\smile} H^4(M;\mathbb{Z})$ *is trivial when both classes are tori. The cup product of a spider with a torus or a spider with a spider requires carefully drawing out the images of the classes in* M/T*, and calculating the oriented intersections of each leg with each leg or torus.*

4.6 A Worked Example

Let M be a toric origami manifold with template graph G two vertices connected by three edges. Let P_1 and P_2 be the Delzant polytopes corresponding to vertex v_1 and v_2 respectively. Suppose that $P_1 = P_2$ is the same Delzant polytope, with normal vectors as given in Figure 4.11, and dashed facets marked for folding.

Folding together P_1 and P_2 gives the orbit space M/T which is drawn topologically in Figure 4.12. The normal vectors to the 1-skeleton components are labeled.

We can flatten the view of M/T topologically, and simply remember the normal vectors associated to each 1-skeleton component. This will make it easier to draw spiders and tori in M/T. See Figure 4.13 for the flat view of M/T.

There are four 1-skeleton component, so there should be $4 - 2 = 2$ linear relations amongst the normal vectors. In particular, two linearly independent

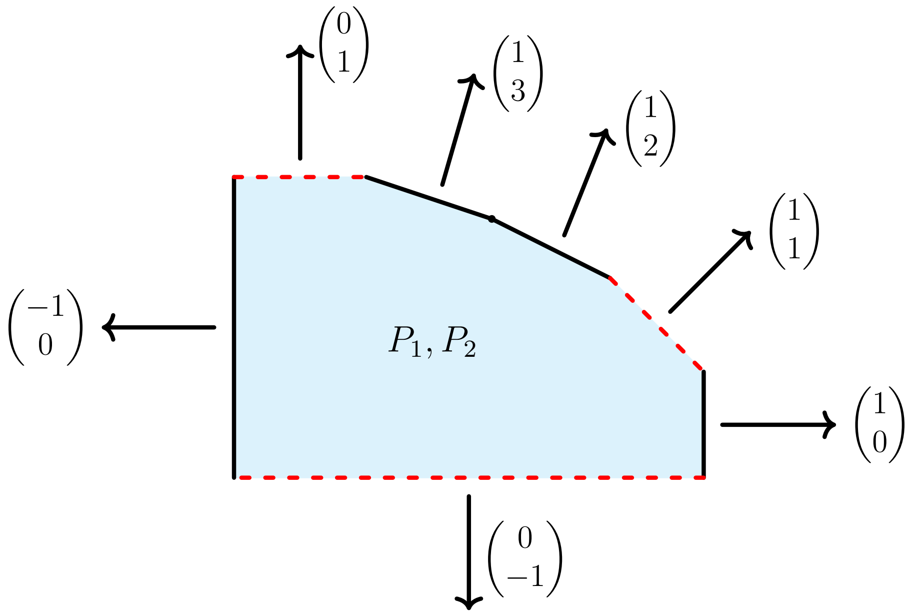


Figure 4.11: The Delzant polytope $P_1 = P_2$. Solid facets will be unfolded, dashed facets will be folded.

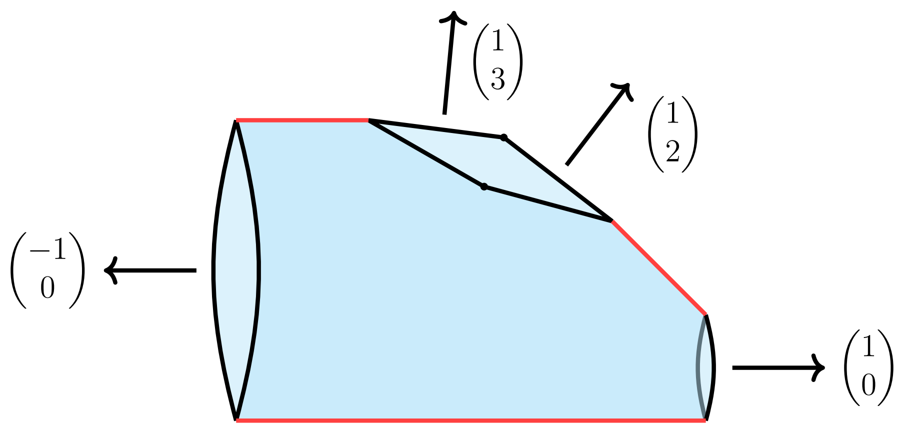


Figure 4.12: The orbit space M/T.

linear relations are given by

$$2 \cdot (1,3) - 3 \cdot (1,2) + (1,0) = 0,$$

$$(1,0) + (-1,0) = 0.$$

Let $[s_1]$ be the spider representing the first relation and let $[s_2]$ be the spider representing the second relation. The images of the spiders in M/T are drawn in Figure 4.14.

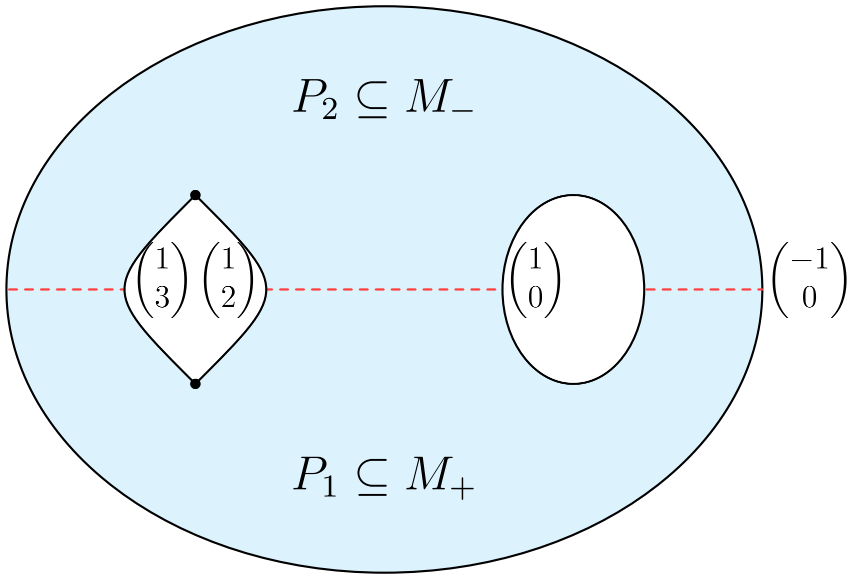


Figure 4.13: A flat view of M/T.

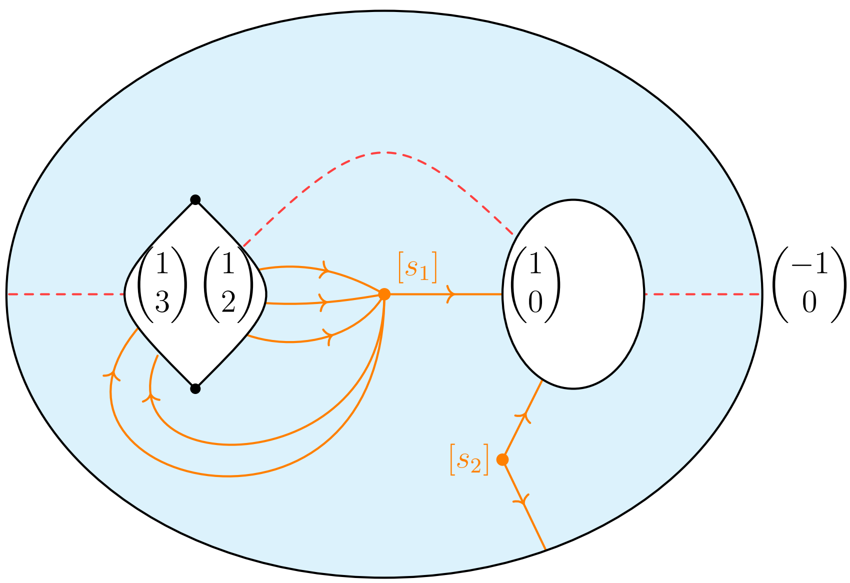


Figure 4.14: The spiders $[s_1]$ and $[s_2]$.

The fundamental group of M/T is free on two generators. Let α_1 and α_2 be those generators, oriented as shown in Figure 4.15. Then α_1 is the loop corresponding to the torus component with normal vector $(1, 0)$ and the loop $\alpha_1 - \alpha_2$ corresponds to the torus component with normal vector $(-1, 0)$. Therefore the two relations induced on our four generators are $\alpha_1 x = 0$ and $-(\alpha_1 - \alpha_2)x = 0$. The second relation reduces to $\alpha_2 x = \alpha_1 x$, so both are in fact trivial. Thus $\alpha_1 y$ and $\alpha_2 y$ are the only tori generators needed to span $H_2(M)$.

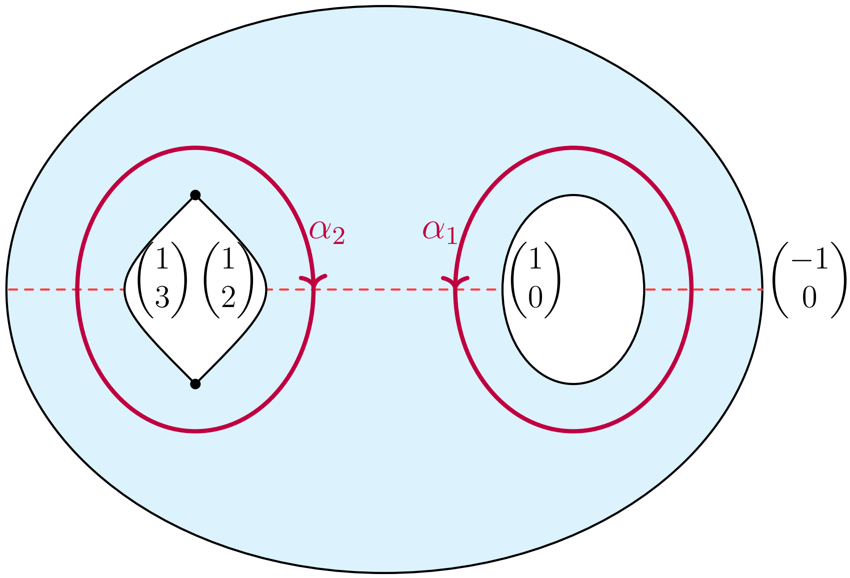


Figure 4.15: The images of the tori $\alpha_1 y$ and $\alpha_2 y$ in M/T.

Then to understand the intersection form on $H_2(M)$ we need to do some calculations of intersections. The spiders $[s_1]$ and $[s_2]$ do not intersect so $[s_1] \cdot [s_2] = 0$. The tori have $[\alpha_1 y] \cdot [\alpha_2 y] = 0$, $[\alpha_1 y] \cdot [\alpha_1 y] = 0$, and $[\alpha_2 y] \cdot [\alpha_2 y] = 0$ because the curves α_1 and α_2 don't intersect and have disjoint parallel copies of themselves. The interesting intersections are between the spiders and tori, and spider self-intersections. In Figure 4.16, the images of $[s_1]$, $[s_2]$, $[\alpha_1]$, and $[\alpha_2]$ are drawn in $P_1 \subseteq M_+$. We can therefore do some calculations.

First, $[s_1]$ and $[\alpha_1 y]$ have a single leg intersection along the $(0,1)$ leg of $[s_1]$. In the drawing we have $[s_1]$ in the horizontal coordinate with positive direction ($\epsilon_0 = 0$), and curve $(1,0)$ above it. We have $[\alpha_1 y]$ in the vertical coordinate with negative direction ($\epsilon_1 = 1$), and curve $(1,0)$ above it. Therefore we get that

$$[s_1] \cdot [\alpha_1 y] = (-1)^{\epsilon_0+\epsilon_1+\epsilon_2} \det((1,0),(0,1)) = (-1)^{0+1+0}(1) = -1.$$

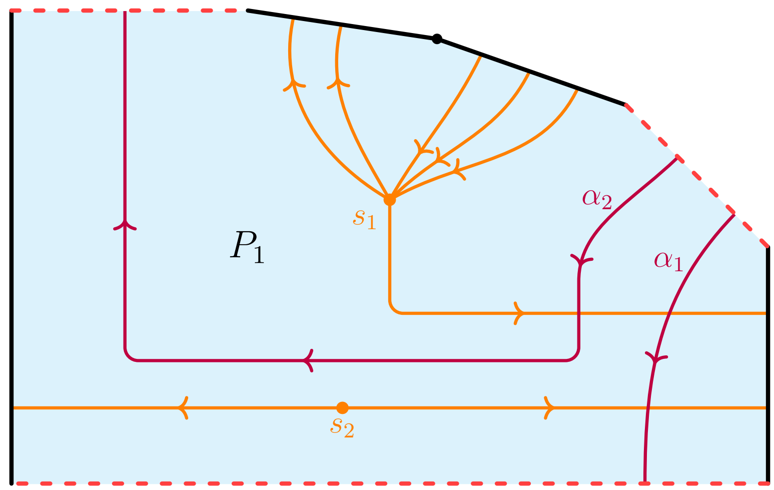


Figure 4.16: The intersections of the spiders and tori in P_1.

The intersection of $[s_1]$ with $[\alpha_2 y]$ is the identical calculation. Therefore $[s_1] \cdot [\alpha_2 y] = -1$.

The leg of $[s_2]$ which intersects $[\alpha_1 y]$ is identical to the leg of $[s_1]$ which was used in the previous calculations. Thus $[s_2] \cdot [\alpha_1 y] = -1$. Since $[s_2]$ does not intersect $[\alpha_2 y]$ we have that $[s_2] \cdot [\alpha_2 y] = 0$.

Finally, we need the spider self-intersections. Two parallel copies of the spider $[s_2]$ do not intersect, so $[s_2] \cdot [s_2] = 0$. However, the self-intersection of $[s_1]$ is more interesting. See Figure 4.17.

There are three points of intersection, with the horizontal paths in the negative direction ($\epsilon_0 = 1$) having the loop $(1, 2)$ above them. The vertical paths are also in the negative direction ($\epsilon_1 = 1$) and have the loop $(1, 0)$ above them. The intersections take place in M_+ so $\epsilon_2 = 0$. Thus the contribution of each point of intersection in M/T is

$$(-1)^{\epsilon_0+\epsilon_1+\epsilon_2} \det((1, 2), (1, 0)) = (-1)^{1+1+0}(-2) = -2.$$

There are three such points of intersection, so the total self-intersection number is

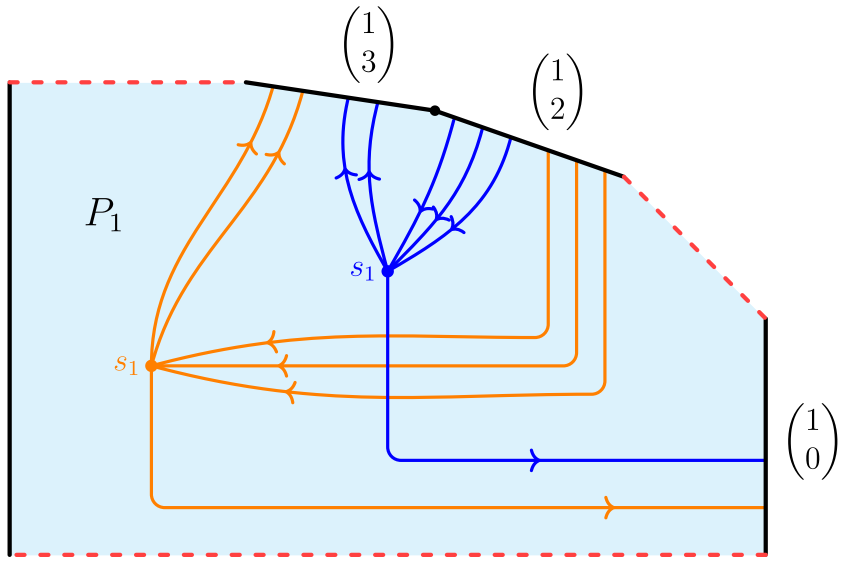


Figure 4.17: The self-intersection $[s_1] \cdot [s_1]$.

$[s_1] \cdot [s_1] = -6$.

Finally, we can write down the intersection form. Let the ordered basis for $H_2(M)$ be $\{[s_1], [s_2], [\alpha_1 y], [\alpha_2 y]\}$. Then the corresponding matrix Q for the intersection form is

$$Q = \begin{pmatrix} -6 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

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