# ETERNAL SOLUTIONS AND HETEROCLINIC ORBITS OF A SEMILINEAR PARABOLIC EQUATION

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ETERNAL SOLUTIONS AND HETEROCLINIC ORBITS OF A SEMILINEAR PARABOLIC EQUATION

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This dissertation describes the space of heteroclinic orbits for a class of semilinear parabolic equations, focusing primarily on the case where the nonlinearity is a second degree polynomial with variable coefficients. Along the way, a new and elementary proof of existence and uniqueness of solutions is given. Heteroclinic orbits are shown to be characterized by a particular functional being finite. A novel asymptotic-numeric matching scheme is used to uncover delicate bifurcation behavior in the equilibria. The exact nature of this bifurcation behavior leads to a demonstration that the equilibria are degenerate critical points in the sense of Morse. Finally, the space of heteroclinic orbits is shown to have a cell complex structure, which is finite dimensional when the number of equilibria is finite.

## BIOGRAPHICAL SKETCH

Michael Robinson entered the field of mathematics by a rather circuitous route. In high school, he learned computer programming as a hobby. It was there that his first exposure to partial differential equations occurred, when he wrote a solver for the invicid Navier-Stokes equations in the plane under the direction of Robert Ryder (who was then with Pratt & Whitney). Feeling that he ought to understand computer hardware at a deeper level, he enrolled at Rensselaer Polytechnic Institute and completed a Bachelor of Science in Electrical Engineering. His Master of Science degree in Mathematics at Rensselaer Polytechnic Institute was completed under the direction of Dr. Ashwani Kapila. His Master's thesis combined the Navier-Stokes equations and Maxwell's equations to examine the scattering of plasma waves. After completing his Master's degree, Robinson went to work for Syracuse Research Corporation and enrolled at Cornell University one year later.

This project is dedicated to my wife, Donna, who urged me to do the obvious thing and continue my graduate studies.

## ACKNOWLEDGEMENTS

I would like to thank all of the members of my thesis committee for their helpful and insightful discussions concerning this project. I would especially like to thank my advisor, Dr. John Hubbard for suggesting that I study $\frac{\partial u}{\partial t} = \Delta u - u^2 + \phi$ as a "summer project." This problem has led into a surprising variety of interesting mathematics.

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# INTRODUCTION

## 1.1 The Grand Plan

This dissertation presents some recent progress towards a rather lofty (and very difficult) goal. Specifically, there is great interest in understanding the topology of solution spaces of systems of semilinear parabolic equations,

$$\begin{cases} \frac{\partial u_1}{\partial t} = D_1(u_1, u_2, ...u_p) + f_1(t, x, u_1, u_2, ...u_p) \\ \frac{\partial u_2}{\partial t} = D_2(u_1, u_2, ...u_p) + f_2(t, x, u_1, u_2, ...u_p) \\ ...\frac{\partial u_p}{\partial t} = D_p(u_1, u_2, ...u_p) + f_p(t, x, u_1, u_2, ...u_p) \end{cases} \tag{1.1}$$

where $u_i \in C^1(\mathbb{R}, C^{0,\alpha}(\mathbb{R}^n))$, $D_i$ are densely-defined linear elliptic (diffusion) operators, and $f_i$ satisfy reasonable smoothness conditions. Of course, a major problem for anyone interested in (1.1) is existence and uniqueness of short-time solutions! Although the existence and uniqueness in general is daunting, many interesting and important problems have the form (1.1). (Fortunately, for some special cases, existence and uniqueness can be proven, as is done in Chapter 2.) The applications of (1.1) are numerous; for instance:

- The Navier-Stokes equations, which describe fluid flow, can be put in the form of (1.1) [20]. Understanding the topology of the solution space of the Navier-Stokes equations gives insight into the onset of turbulence. This has applications to fluid amplifiers (viscous fluid logic gates, with no moving parts) [21], in which a particular geometry and set of boundary conditions allows several semistable equilibria. The orbits which connect these equilibria (the *heteroclinic orbits* or *heteroclines*) can involve turbulent flows. As a result, understanding the topology of the space of heteroclines for such a situation might provide insight into turbulence phenomena.
- Many chemical reactions are of this form, in particular those describing com-

bustion. The precise nature of the ignition of a flame is encoded in the topology of the solution space.

- The combination of the Navier-Stokes equations with combustion equations can model turbulent combustion phenomena. Such turbulent, reacting flows are important in modeling the inside of internal combustion engines. The ignition of a turbulent combustion depends very delicately upon the exact nature of the flow, and a topological description for such events is lacking.

- Related to the Navier-Stokes and chemical reaction equations are nonlinear wave equations. Many nonlinear wave equations are of the form (1.1), and traveling waves appear as heteroclines connecting two equilibrium states. Traveling waves are often stable, in that perturbations of them tend to "wash out" over time. However, there are interesting situations where traveling waves are suddenly suppressed as a parameter is changed slightly. This is a consequence of an abrupt change in the topology of the solution space.

- In population biology, (1.1) describes a number of competing or cooperating species. One can ask about the kinds of bifurcations in stable populations when a new species is introduced, or when harvesting patterns are changed.

- One of the most pressing issues in population biology is that of non-native invasive species, which disrupt the ecology of many parts of our planet. In agriculture, they cause significant crop losses and threaten our protected areas. One of the important problems concerning invasive species is how to displace them with minimal ecological impact. Understanding the topology of the space of solutions for (1.1) would help find optimal control algorithms for eliminating (or limiting) the spread of invasive species. There is a vast literature on this subject, going back to Fisher [13], Kolmogorov, Petrovski, and Piskunov [27].

## 1.2 Specialization to a scalar gradient equation

Since the general setting of (1.1) is much too difficult to allow any kind of progress at this time, we must instead consider more specialized situations. To this end, we restrict attention to the case of

- a scalar equation (p=1 in (1.1)),
- in one spatial dimension, where
- the diffusion operator is the Laplacian, $D_1 = \frac{\partial^2}{\partial x^2}$, and
- the reaction term is a polynomial.

In other words, consider

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + \sum_{i=0}^{N} a_i(x) u^i(t,x) = \Delta u + P(u), \tag{1.2}$$

where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, and the $a_i$ are bounded and smooth. In this dissertation, we consider *eternal* solutions, those $u$ satisfying (1.2) that lie in $C^1(\mathbb{R}, C^{0,\alpha}(\mathbb{R}))$ for $0 < \alpha < 1$. In particular, observe that $\Delta : C^{0,\alpha}(\mathbb{R}) \to C^{0,\alpha}(\mathbb{R})$ is densely defined when $\alpha$ is not an integer.

This kind of equation provides a simple model for a number of physical phenomena. First, choosing the right side to be $\Delta u - u^2 + a_1 u$ results in an equation which can represent a model of the population of a single species with diffusion and a spatially-varying carrying capacity, $a_1(x)$. As a second application, this equation is a very simple model of combustion. If $a_1$ is a positive constant, then the equation supports traveling waves. Such traveling waves can model the propagation of a flame through a fuel source.

## 1.3 Discussion of the literature

Equations of the form (1.2) have been of interest to researchers for quite some time. Existence and uniqueness of solutions on short time intervals (on strips $(-t_0, t_0) \times \mathbb{R}^n$) can been shown using semigroup methods and are entirely standard [44]. However, there are obstructions to the existence of eternal solutions. Aside from the typical loss of regularity due to solving the backwards heat equation, there is also a blow-up phenomenon which can spoil existence in the forward-time solution to (1.2). Blow-up phenonmena in the forward time Cauchy problem (where one does not consider $t < 0$) have been studied by a number of authors [18] [10] [42] [26] [6] [46] [47]. More recently, Zhang *et al.* ([45] [38] [43]) studied global existence for the forward Cauchy problem for

$$\frac{\partial u}{\partial t} = \Delta u + u^p - V(x)u$$

for positive $u, V$. Du and Ma studied a related problem in [9] under more restricted conditions on the coefficients but they obtained stronger existence results. In fact, they found that all of the solutions which were defined for all $t > 0$ tended to equilibrium solutions.

Eternal solutions to (1.2) are rather rare. Most works which describe blow-up make the assumption that the solution is positive. Unfortunately, blow-up is much more difficult to characterize in the general situation, and understanding exactly what kind of initial conditions are responsible for blow-up in the Cauchy problem for (1.2) is an important part of the question.

The boundary value problem that results from taking $x \in \Omega \subset \mathbb{R}^n$ for some bounded $\Omega$ (instead of $x \in \mathbb{R}^n$) has also been discussed extensively in the literature [20] [24] [7]. For the boundary value problem, all bounded forward Cauchy problem solutions tend to limits as $|t| \to \infty$, and these limits are equilibrium solutions.

Almost all of the literature (including this dissertation) describing eternal solutions to (1.2) is restricted to discussing heteroclines. For unbounded domains and certain symmetries among the coefficients $a_i$, one can find traveling waves. Since the propagation of waves in nonlinear models is of great interest in applications, there is much written on the subject. The general idea is that one makes a change of variables $(t, x) \mapsto \xi = x - ct$ which reduces (1.2) to an ordinary differential equation. This ordinary differential equation describes the profile of a traveling wave. Powerful topologically-motivated techniques, such as the Leray-Schauder degree, can be used to prove existence of wave solutions to (1.2). Asymptotic methods can be used to determine the wave speed $c$, which is often of interest in applications. See [40] for a very thorough introduction to the subject of traveling waves in (1.2).

## 1.4 A Morse-theoretic approach

A somewhat less traditional approach to studying (1.2) exists. This method attempts to directly compute topological invariants for the space of heteroclinic orbits $\mathcal{H}$ of (1.2). It makes use of the fact that equation (1.2) defines the flow of the $L^2$ gradient of a certain *action* functional,

$$A(f) = \int_{\mathbb{R}^n} \frac{1}{2}\|\nabla f\|^2 + \sum_{i=0}^{N} \frac{a_i(x)}{i+1} f^{i+1}(x)dx. \tag{1.3}$$

It is then evident that along a solution $u(t)$ to (1.2), $A(u(t))$ is a monotonic function in $t$. As an immediate consequence, nonconstant $t$-periodic solutions to (1.2) do not exist. This kind of behavior suggests that a Morse-theoretic framework might be helpful.

Morse theory is concerned with the computation of homotopy or homology groups of a Riemannian or Hilbert manifold $M$ by "exploring" it with a suitable scalar function $f : M \to \mathbb{R}$. The function $f$ is selected to satisfy the Morse-Smale (-Floer) conditions, namely

- a nondegeneracy condition: if $x \in M$ is a critical point ($df(x) = 0$), then the Hessian at $x$ is nonsingular,
- the *Morse index*, which is the number of negative eigenvalues of the Hessian is finite for each critical point,
- stable and unstable manifolds for the gradient flow of $f$ are transverse (the Smale condition), and
- if $M$ is noncompact, there is a compact isolating neighborhood for each pair of critical points under the gradient flow [17].

The function $f$ can be thought of as a special kind of "height function" on $M$. One then examines the topology of sets $M^t = \{x \in M | f(x) \leq t\}$, which form a cover for $M$. It is straightforward to show that the homotopy class of $M^t$ remains constant on $t \in (t_0, t_1)$ when there are no critical points in $f^{-1}((t_0, t_1))$. The homotopy class of $M^t$ changes abruptly, however, when $f^{-1}(t)$ contains a critical point. Morse theory describes how this homotopy class changes by the attachment of handles to $M^t$. A very readable introduction to Morse theory is [30].

There is a dual formulation of Morse theory, which uses Witten's complex to compute homology instead of homotopy. This approach is better suited to understanding differential equations, as it focuses not on level sets, but rather on the flow of

$$\frac{dx}{dt} = \nabla f(x(t)). \tag{1.4}$$

Using this flow, one constructs a chain complex $(C_*, \partial_*)$ in which the $C_k$ are free modules generated by the critical points of Morse index $k$. The boundary maps $\partial_k$ are then constructed by the formula

$$\partial_k(q) = \sum_{p \in C_{k-1}} n(q,p)p,$$

where $n(q,p)$ is the number of heteroclinic orbits of (1.4) which connect $q$ to $p$, counted with sign. The surprising thing is that this chain complex computes the homology of $M$! A thorough, modern treatment of Morse theory can be found in [3].

## 1.5 Floer homology

A similar theory can be made to work even if the Morse index of all critical points is infinite. Instead of relying on the critical points to supply an index directly, one constructs a "relative" index based on the structure of connecting manifolds. This theory was first assembled by Floer for the purpose of understanding the homology of the space of orbits for an exact symplectomorphism [15]. More recently, Ghrist *et al.* [19] have done work on a similar theory for a certain evolution of braids. What is crucial to Floer theories is that the manifold of heteroclinic orbits which connect a given pair of equilibria (a connecting manifold) is finite dimensional, and compact modulo time translation. Suppose that the connecting manifold for a given pair of equilibria $x, y$ has dimension $\mu(x,y)$. One shows that the following two relations hold for this dimension:

$$\mu(x,x) = 0, \ \mu(x,z) = \mu(x,y) + \mu(y,z), \tag{1.5}$$

where $x, y, z$ are distinct critical points. This relation allows one to assign indices $I$ to the critical points such that $\mu(x,y) = I(x) - I(y)$. Evidently, $I$ is only defined uniquely up to an additive constant. However, one can use the index $I$ in place of the Morse index to construct a Witten complex in the usual way. However, the ambiguity in the definition of $I$ means that the degrees of the resulting complex are only defined up to this additive constant. In this way, one obtains a kind of ambiguous-degree homology theory, which is called "Floer homology." Alternatively, one may take the dual approach, and use the finite dimensional connecting manifolds to assemble a cell complex structure for the space of heteroclines. The attaching maps of this cell complex are evidently related to the boundary maps of the Witten complex.

## 1.6 How to construct a Floer theory for parabolic equations

For the case of semilinear parabolic equations, the following must be established in order to construct a Floer theory:

1. One must compute the Morse index of each critical point, or more properly, show that the Morse index is not well-defined due to degeneracy.
2. One must show that connecting manifolds are finite dimensional, and that they form a cell-complex structure for the space of heteroclinic orbits $\mathcal{H}$.
3. The connecting manifolds must obey the additivity relation (1.5).
4. The space of heteroclinic orbits $\mathcal{H}$ must be compact moduluo time translation.
5. One must construct the boundary maps in the Witten complex, and verify that they actually form a chain complex that computes the homology of $\mathcal{H}$.

This dissertation contains proofs of items 1, 2, and most of 3 for a special case of (1.2). (In the case of a bounded spatial domain, all but item 5 are standard [32], [24].) Rather than working with equation (1.2) in full generality, the later chapters use the following special case:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - u^2(t,x) + \phi(x), \tag{1.6}$$

where $\phi$ tends to zero as $|x| \to \infty$. The resulting questions and techniques have obvious generalizations to (1.2), though there are many technical obstacles. Higher degree polynomial nonlinearities may of course have more than two roots, which creates the possibility for more complicated equilibrium structure than we analyze here.

This simpler model still provides insight into applications, as it is still a model of the population of a single species, with a spatially-varying carrying capacity, $\phi$. Indeed, one easily finds that under certain conditions the behavior of solutions to (1.6) is reminiscent of the growth and (admittedly tenuous) control of invasive species [4]. It is the control of invasive species that is of most interest, and it is also what the structure of the boundary maps reveals. In the example given in Chapter 8, there is one more stable equilibrium, and several other less stable ones. The more stable equilibrium can be thought of as the situation where an invasive species dominates. The task, then, is to try to perturb the system so that it no longer is attracted to that equilibrium. An optimal control approach is to perturb the system so that it barely crosses the boundary of the stable manifold of the the undesired equilibrium, and thereby the invasive species is eventually brought under control with minimal disturbance to the rest of the environment.

## 1.7 Outline and prerequisite results

While Chapters 6 and 7 contain the results of most interest to constructing a Floer theory for (1.6), the other chapters provide a number of results that are prerequisites.

### 1.7.1 Prerequisites that concern the higher degree case

Chapters 2 and 3 are devoted to the general equation (1.2). The first of these provides a new proof of short-time existence and uniqueness for (1.2), and as a side-effect provides a numerical method for approximating its solutions. While existence and uniqueness for (1.2) is standard [44], the usual proofs are not suited to computation.

The first novel result for (1.2) is obtained in Chapter 3, where a decay condition on the $a_i$ allows us to classify heteroclinic orbits of (1.2). In particular, if an eternal solution $u$ exists and converges uniformly to equilibria as $t \to \pm\infty$ if and only if $u$ has finite energy (supremum of the difference in the action functional (1.3) over all time). Without the decay condition on the $a_i$, finite energy classifies those eternal solutions that connect finite action equilibria.

The result in Chapter 3 is actually quite important for connecting the Morse-theoretic results with the analysis. Morse theory, and in particular Witten's complex, requires the flow to have a gradient structure. As a result, the space on which the flow acts must have an inner product structure, so a natural solution space would be $C^1(\mathbb{R}, L^2(\mathbb{R}))$. However, the proofs of the cell-complex structure (Chapter 7) do not work with timeslices in $L^2(\mathbb{R})$. In particular, one needs this space to have a Banach algebra structure (in which the Laplacian is densely defined), so the Hölder space $C^{0,\alpha}(\mathbb{R})$ for $0 < \alpha < 1$ is more natural. What follows from the results of Chapter 3 is that the space of heteroclines $\mathcal{H}$ lies in the intersection $C^1(\mathbb{R}, C^{0,\alpha}(\mathbb{R})) \cap C^1(\mathbb{R}, L^2(\mathbb{R}))$, so in fact there is no difficulty (Corollary 30).

### 1.7.2 Prerequisites that concern the quadratic case

In the remaining chapters (Chapters 4 through 8), only the special case (1.6) is considered. This is quite sufficient to obtain interesting results about the structure of $\mathcal{H}$.

It is important to understand the collection of equilibrium solutions for (1.6), which are global solutions to

$$0 = \Delta u - u^2 + \phi \tag{1.7}$$

on all of $\mathbb{R}$. Like (1.2), there are obstructions to global existence in (1.7) [39]. Indeed, there are fairly few global solutions to (1.7). We examine solutions to this problem under asymptotic decay conditions for $\phi$ in Chapter 4. The solution reveals delicate bifurcation behavior in the number of equilbria as $\phi$ is varied. Further, the asymptotic behavior is such that all global solutions to (1.7) have finite action (see (1.3)).

Since they are rare, it is reassuring to construct an example of heteroclinic orbits, which is done in Chapter 5. This example makes specific use of the structure of the equilibrium solutions, in particular, their asymptotic decay is crucial.

In order to construct a Morse theory for (1.2), understanding the dimension of the stable, center, and unstable manifolds of equilibria is important. In Chapter

6 it is shown that the the center/stable manifold's dimension is typically infinite, and later in Chapter 7 it is shown that the unstable manifold has finite dimension. Each equilibrium solution is in fact unstable, even if its linearization is stable. This implies that each equilibrium is a degenerate critical point. This neatly derails any hope of using a standard Morse theory, or even using any of its extensions to infinite dimensional dynamical systems [31]. (In Chapter 9, Conjecture 104 suggests that restriction of the flow to $\mathcal{H}$ may correct the degeneracy.)

The most important result of this work is obtained in Chapter 7, where the space of heteroclinic orbits is shown to have a cell-complex structure (with finite dimensional cells). The dimension of each cell is determined, under a standing assumption of transversality (Conjecture 95). From the formula for the dimension of the cells, it is clear that an additivity rule like (1.5) will hold. This result is further explained by an example in Chapter 8. Finally, in Chapter 9, several important future directions are outlined.

CHAPTER 2

# SHORT-TIME EXISTENCE AND UNIQUENESS

## 2.1 Introduction

(This chapter has already been published as [35].)

Existence and uniqueness of solutions for (2.1) under reasonable initial conditions have been known for some time. For instance, [20] and [44] contain straightforward proofs using semigroup methods. The purpose of this chapter is to show how a *more elementary* proof can be obtained from a sequence of explicitly computed discrete-time approximations.

Due to their theoretical and computational stability, implicit iteration schemes are often prefered over their easier-to-implement explicit analogues. However, in the case of semilinear equations, one can form a hybrid implicit-explicit (IMEX) method which offers computational and theoretical benefits. The use of IMEX methods for approximating semilinear parabolic equations is well-established [2]. Many of the recent works on these methods employ discretizations in both space and time. These fully discrete approximations can be computed directly by a computer. However, one can obtain a stronger condition for convergence of the approximation if only the time dimension is discretized [8]. We show how an even stronger condition for convergence is met by the Cauchy problem for

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + \sum_{i=0}^{\infty} a_i(x) u^i(x,t), \tag{2.1}$$

where $a_i \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and how convergence of this method provides an elementary proof of existence and uniqueness of solutions.

The Cauchy problem for (2.1) arises in a variety of settings. Notably, some reaction-diffusion equations are of this form [12]. Another application is the special case

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) - u^2(x,t) + a_0(x),$$

where $a_0$ is a nonzero function of $x$. This situation corresponds to a spatially-dependent logistic equation with a diffusion term, which can be thought of as a toy model of population growth with migration.

Following [8], the approximation to be used is

$$u_{n+1} = (I - h\Delta)^{-1}(u_n + h\sum_{i=0}^{\infty} a_i u_n^i), \tag{2.2}$$

which is obtained by inverting the linear portion of a discrete version of (2.1). For brevity, we shall call (2.2) *the* implicit-explicit method. (In the summary paper [2], this is called an SBDF method, to distinguish it from other implicit-explicit methods.) One can compute the operator $(I - h\Delta)^{-1}$ explicitly using Fourier transform methods, and obtain a proof of the numerical stability of the iteration as a whole.

## 2.2 A version of the fundamental inequality

In order to simplify the algebraic expressions, we make the following definitions.

**Definition 1.** Let

$$F(u(x,t)) = \Delta u(x,t) + \sum_{i=0}^{\infty} a_i(x) u^i(x,t), \tag{2.3}$$

and

$$G(u(x,t)) = \sum_{i=0}^{\infty} a_i(x) u^i(x,t). \tag{2.4}$$

**Definition 2.** Define the analytic functions

$$g_1(z) = \sum_{i=0}^{\infty} \|a_i\|_1 z^i, \tag{2.5}$$

and

$$g_\infty(z) = \sum_{i=0}^{\infty} \|a_i\|_\infty z^i. \tag{2.6}$$

Since we do not discretize the spatial dimension, we can employ some of the theory of ordinary differential equations. We therefore first prove a variant of the fundamental (Gronwall) inequality for (2.1) as is done in [23]. The fundamental inequality gives a sufficient condition for approximate solutions to converge. A slightly weaker version of Lemma 3 was obtained in Theorem 3.1 of [8], where the existence of solutions was required.

**Lemma 3.** *Suppose $\{u_i\}_{i=1}^{\infty}$ is a sequence of piecewise $C^1$ functions $u_i : [0,T] \to C^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, such that*

1. *there exist $A, B > 0$ so that for each $i$ and $t \in [0,T]$, $\|u_i(t)\|_1 \leq A$ and $\|u_i(t)\|_\infty \leq B$,*
2. *for each $i$ and $t \in [0,T]$, the series $g_1(\|u_i(t)\|_1)$ and $g_\infty(\|u_i(t)\|_\infty)$ converge,*
3. *for each $t \in [0,T]$, $\|\frac{d}{dt}u_i(t) - F(u_i(t))\|_\infty < \epsilon_i$ and $\lim_{i\to\infty} \epsilon_i = 0$, and*
4. *$u_1(0) = u_i(0)$ for all $i \geq 0$*

*Then for each $t \in [0,T]$, $\{u_i(t)\}_{i=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$.*

*Proof.* Let $i, j > 0$ be given. Let $\eta(t) = \|u_i(t) - u_j(t)\|_2^2 = \int (u_i(t) - u_j(t))^2 dx$. Notice that the fourth condition in the hypothesis gives $\eta(0) = 0$.

$$\eta'(t) = 2 \int \left(u_i'(t) - u_j'(t)\right) \left(u_i(t) - u_j(t)\right) dx.$$

But, $\|\frac{d}{dt}u_i(t) - F(u_i(t))\|_\infty < \epsilon_i$ is equivalent to the statement that for each $t \in [0,T]$ and $x \in \mathbb{R}^n$,

$$F(u_i(x,t)) - \epsilon_i < u_i'(x,t) < F(u_i(x,t)) + \epsilon_i,$$

giving

$$\begin{aligned}
\eta'(t) &\le 2\int (F(u_i(t)) - F(u_j(t)))\,(u_i(t) - u_j(t))dx \\
&\quad +2(\epsilon_i + \epsilon_j)\int |u_i(t) - u_j(t)|dx \\
&\le 2\int (\Delta u_i(t) + G(u_i(t)) - \Delta u_j(t) - G(u_j(t)))\,(u_i(t) - u_j(t))dx \\
&\quad +2(\epsilon_i + \epsilon_j)\|u_i(t) - u_j(t)\|_1 \\
&\le 2\int (\Delta(u_i(t) - u_j(t)))\,(u_i(t) - u_j(t))dx \\
&\quad +2\int (G(u_i(t)) - G(u_j(t)))\,(u_i(t) - u_j(t))dx + 2(\epsilon_i + \epsilon_j)\|u_i(t) - u_j(t)\|_1 \\
&\le -2\int \|\nabla(u_i(t) - u_j(t))\|^2 dx + 2\|G(u_i(t)) - G(u_j(t))\|_2\|u_i(t) - u_j(t)\|_2 \\
&\quad +2(\epsilon_i + \epsilon_j)\|u_i(t) - u_j(t)\|_1 \\
&\le 2\|G(u_i(t)) - G(u_j(t))\|_2\|u_i(t) - u_j(t)\|_2 + 2(\epsilon_i + \epsilon_j)\|u_i(t) - u_j(t)\|_1.
\end{aligned}$$

Now also

$$\begin{aligned}
\|G(u_i(t)) &- G(u_j(t))\|_2 = \left\|\sum_{k=0}^{\infty} a_k(u_i^k(t) - u_j^k(t))\right\|_2 \\
&\le \sum_{k=0}^{\infty} \|a_k\|_\infty \left\|u_i^k(t) - u_j^k(t)\right\|_2 \\
&\le \sum_{k=0}^{\infty} \|a_k\|_\infty \sqrt{\int \left(u_i^k(x,t) - u_j^k(x,t)\right)^2 dx} \\
&\le \sum_{k=0}^{\infty} \|a_k\|_\infty \sqrt{\int (u_i(x,t) - u_j(x,t))^2 \left(\sum_{m=0}^{k-1} u_i^m(x,t) u_j^{k-m-1}(x,t)\right)^2 dx} \\
&\le \sum_{k=0}^{\infty} \|a_k\|_\infty \left\|\sum_{m=0}^{k-1} u_i^m(t) u_j^{k-m-1}(t)\right\|_\infty \|u_i(t) - u_j(t)\|_2 \\
&\le \left(\sum_{k=0}^{\infty} \|a_k\|_\infty k B^{k-1}\right) \|u_i(t) - u_j(t)\|_2 \\
&\le g'_\infty(B)\|u_i(t) - u_j(t)\|_2,
\end{aligned}$$

which allows

$$\begin{aligned}\eta'(t) &\leq 2g'_\infty(B)\|u_i(t)-u_j(t)\|_2^2 + 2(\epsilon_i+\epsilon_j)\|u_i(t)-u_j(t)\|_1. \\ &\leq 2g'_\infty(B)\eta(t) + 2(\epsilon_i+\epsilon_j)\|u_i(t)-u_j(t)\|_1.\end{aligned}$$

$$\begin{aligned}\eta'(t) - 2g'_\infty(B)\eta(t) &\leq 2(\epsilon_i+\epsilon_j)\|u_i(t)-u_j(t)\|_1 \\ \frac{d}{dt}\left(\eta(t)e^{-2g'_\infty(B)t}\right) &\leq 2(\epsilon_i+\epsilon_j)e^{-2g'_\infty(B)t}\|u_i(t)-u_j(t)\|_1,\end{aligned}$$

so (recall $\eta(0)=0$)

$$\begin{aligned}\eta(t) &\leq \left[2(\epsilon_i+\epsilon_j)\int_0^t e^{-2g'_\infty(B)s}\|u_i(s)-u_j(s)\|_1 ds\right] e^{2g'_\infty(B)t} \\ &\leq \left[2(\epsilon_i+\epsilon_j)\int_0^t \|u_i(s)-u_j(s)\|_1 ds\right] e^{2g'_\infty(B)t}. \\ &\leq 4(\epsilon_i+\epsilon_j)Ate^{2g'_\infty(B)t}.\end{aligned}$$

Hence as $i,j\to\infty$, $\eta(t)\to 0$ for each $t$. Thus for each $t$, $\{u_i(t)\}_{i=1}^\infty$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. □

**Remark 4.** Since $C^2(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\subseteq L^2(\mathbb{R}^n)$ and $L^2$ is complete, Lemma 3 gives conditions for existence and uniqueness of a short-time solution to (2.1).

**Lemma 5.** *Suppose $\{u_i(t)\}_{i=1}^\infty$ is the sequence of functions defined in Lemma 3, and that $u(t)=\lim_{i\to\infty}u_i(t)$ in $L^2(\mathbb{R}^n)$. Then*

$$u'(t,x) = \lim_{i\to\infty} u_i'(t,x) \text{ for almost every } x, \tag{2.7}$$

*wherever the limit exists.*

*Proof.* Notice that since each $u_i(t) \in L^\infty(\mathbb{R}^n)$ and $\|u_i(t)\|_\infty \leq B$, the dominated convergence theorem allows for each $x \in \mathbb{R}^n$

$$\begin{aligned}\int_0^t \lim_{i\to\infty} u_i'(\tau,x)d\tau &= \lim_{i\to\infty}\int_0^t u_i'(\tau,x)d\tau \\ &= \lim_{i\to\infty}(u_i(t,x)-u_i(0,x)) \\ &= u(t,x)-u(0,x) \text{ for almost every } x.\end{aligned}$$

Hence, by differentiating in $t$,

$$u'(t,x) = \lim_{i\to\infty} u_i'(t,x) \text{ for almost every } x.$$

□

## 2.3 The implicit-explicit approximation

In this section, we consider the case of a 1-dimensional spatial domain, that is, $x \in \mathbb{R}$. There is no obstruction to extending any of these results to higher dimensions, though it complicates the exposition unnecessarily.

As is usual, the first task is to define the function spaces to be used. Initial conditions will be drawn from a subspace of $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, as suggested by Lemma 3, and the first four spatial derivatives will be prescribed, for use in Lemma 10.

**Definition 6.** Let

$$W = L^1 \cap C^4(\mathbb{R}),$$

where we interpret $C^4(\mathbb{R})$ as being the space of bounded functions with four continuous bounded derivatives. For the remainder of this chapter, we consider the case where each of the coefficients $a_i \in W$. Then let $X = \{f \in W | g_1(\|f\|_1) < \infty \text{ and } g_\infty(\|f\|_\infty) < \infty\}$. We consider the case where the initial condition is drawn from $X$.

An approximate solution given by the implicit-explicit iteration will be the piecewise linear interpolation through the iterates computed by (2.2). A smoother approximation will prove to be unnecessary, as will be shown in Lemma 11.

**Definition 7.** Suppose $f_0$ and $h > 0$ are given. Put

$$f_{n+1} = (I - h\Delta)^{-1}(f_n + hG(f_n)). \tag{2.8}$$

The function

$$u(t) = \left(1 - \left(\frac{t}{h} - n(t)\right)\right) f_{n(t)} + \left(\frac{t}{h} - n(t)\right) f_{n(t)+1}, \tag{2.9}$$

where $n(t) = \lfloor \frac{t}{h} \rfloor$, is called the **implicit-explicit iteration of size $h$ beginning at $f_0$.**

**Calculation 8.** We explicitly compute the operator $(I - h\Delta)^{-1}$ using Fourier transforms. Suppose

$$(I - h\Delta)u(x) = u(x) - h\Delta u(x) = f(x).$$

Taking the Fourier transform (with transformed variable $\omega$) gives

$$\hat{u}(\omega) + h\omega^2 \hat{u}(\omega) = \hat{f}(\omega),$$
$$\hat{u}(\omega) = \frac{\hat{f}(\omega)}{1 + h\omega^2}.$$

The Fourier inversion theorem yields

$$\begin{aligned} u(x) &= \frac{1}{2\pi} \int \frac{e^{i\omega x}}{1 + h\omega^2} \int f(y) e^{-i\omega y} dy d\omega \\ &= \int f(y) \left( \frac{1}{2\pi} \int \frac{e^{i\omega(x-y)}}{1 + h\omega^2} d\omega \right) dy. \end{aligned}$$

Using the method of residues, this can be simplified to give

$$u(x) = \left((I - h\Delta)^{-1} f\right)(x) = \frac{1}{2\sqrt{h}} \int f(y) e^{-|y-x|/\sqrt{h}} dy. \tag{2.10}$$

**Calculation 9.** Bounds on the $L^1$ and $L^\infty$ operator norms of $(I-h\Delta)^{-1}$ are now computed. First, let $f \in L^\infty(\mathbb{R})$. Then

$$\begin{aligned}
|\left((I-h\Delta)^{-1}f\right)(x)| &= \left|\frac{1}{2\sqrt{h}}\int f(y)e^{-|y-x|/\sqrt{h}}dy\right| \\
&\leq \|f\|_\infty \frac{1}{2\sqrt{h}}\int e^{-|y-x|/\sqrt{h}}dy \\
&\leq \|f\|_\infty \frac{1}{\sqrt{h}}\int_0^\infty e^{-s/\sqrt{h}}ds \\
&\leq \|f\|_\infty,
\end{aligned}$$

so $\|(I-h\Delta)^{-1}\|_\infty \leq 1$.

Now, let $f \in L^1(\mathbb{R})$. So then

$$\begin{aligned}
\|(I-h\Delta)^{-1}f\|_1 &= \int_{-\infty}^{\infty}\left|\frac{1}{2\sqrt{h}}\int_{-\infty}^{\infty} f(y)e^{-|y-x|/\sqrt{h}}dy\right|dx \\
&\leq \frac{1}{2\sqrt{h}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(y)|e^{-|y-x|/\sqrt{h}}dydx \\
&\leq \frac{1}{\sqrt{h}}\int_{-\infty}^{\infty}|f(y)|\int_0^\infty e^{-|y-x|/\sqrt{h}}dxdy \\
&\leq \int_{-\infty}^{\infty}|f(y)|dy = \|f\|_1,
\end{aligned}$$

which means $\|(I-h\Delta)^{-1}\|_1 \leq 1$.

The third condition of Lemma 3 is a control on the slope error of the approximation. A bound on this error may be established for the implicit-explicit iteration as follows.

**Lemma 10.** *Suppose $f_0 \in X$, $h > 0$. Put $f(x,t) = f_0(x) + tD(x)$, where*

$$D = \frac{(I-h\Delta)^{-1}(f_0 + hG(f_0)) - f_0}{h}$$

*Then for every $0 < t < h$,*

$$\|f'(t) - F(f(t))\|_\infty = O(h). \tag{2.11}$$

*Proof.* Recall every function in $X$ will have bounded partial derivatives up to fourth order from Definition 6.

$$\begin{aligned}
\|f'(t) - F(f(t))\|_\infty &= \|D - (\Delta(f_0 + tD) + G(f_0 + tD))\|_\infty \\
&= \left\| D - \left( \Delta(f_0 + tD) + \sum_{i=0}^{\infty} a_i (f_0 + tD)^i \right) \right\|_\infty \\
&\leq \left\| D - \Delta f_0 - t\Delta D - \sum_{i=0}^{\infty} a_i \left( \sum_{j=0}^{i} \binom{i}{j} f_0^j (tD)^{i-j} \right) \right\|_\infty \\
&\leq \left\| D - \Delta f_0 - t\Delta D - \sum_{i=0}^{\infty} a_i f_0^i \right\|_\infty + O(h) \\
&\leq \left\| \frac{(I - h\Delta)^{-1} - I}{h} f_0 - \Delta f_0 \right. \\
&\quad \left. + ((I - h\Delta)^{-1} - I) G(f_0) \right\|_\infty + O(h)
\end{aligned}$$

Now, using the fact that $(I - h\Delta)^{-1} - I = (I - h\Delta)^{-1}(h\Delta)$,

$$\begin{aligned}
\|f'(t) - F(f(t))\|_\infty &\leq \left\| (I - h\Delta)^{-1} \Delta f_0 - \Delta f_0 \right. \\
&\quad \left. + (I - h\Delta)^{-1} (h\Delta) G(f_0) \right\|_\infty + O(h) \\
&\leq \|(I - h\Delta)^{-1}(h\Delta)(\Delta f_0 + G(f_0))\|_\infty + O(h) \\
&\leq h\|(I - h\Delta)^{-1}(\Delta F(f_0))\|_\infty + O(h) \\
&\leq h\|(I - h\Delta)^{-1}\|_\infty \|(\Delta F(f_0))\|_\infty + O(h) = O(h)
\end{aligned}$$

□

**Lemma 11.** *Suppose* $0 < h_i \to 0$. *Let* $u_i$ *be the implicit-explicit iteration of size* $h_i$ *beginning at* $f_0 \in X$ *on* $t \in [0, T]$. *Then provided there exist* $A, B > 0$ *such that for each* $i$ *and* $t \in [0, T]$, $\|u_i(t)\|_1 \leq A$ *and* $\|u_i(t)\|_\infty \leq B$, *then the sequence* $\{u_i(t)\}_{i=1}^{\infty}$ *converges pointwise to a function in* $t$. *The limit function is piecewise differentiable in* $t$.

*Proof.* Let $u_i$ be the implicit-explicit iteration of size $h_i$. By Lemma 10, the slope error is bounded:

$$\|u_i'(t) - F(u_i(t))\|_\infty = O(h_i) = \epsilon_i.$$

Notice that $\epsilon_i \to 0$. Then, since $X \subset C^2(\mathbb{R}^n)$, Lemma 3 applies, giving a pointwise limit function $u(t)$. Finally, since the slope error uniformly vanishes, Lemma 5 implies that the solution is piecewise differentiable. $\square$

## 2.4 "*A priori* estimates" for the approximate solutions

Now we demonstrate that the implicit-explicit method converges for all initial conditions in $X$. Specifically, for each $f_0 \in X$, there exist $A, B > 0$ such that for each $i$ and $t \in [0, T]$, $\|u_i(t)\|_1 \leq A$ and $\|u_i(t)\|_\infty \leq B$, given sufficiently small $T$. We begin by recalling that from Calculation 9, the $L^\infty$-norm of $(I - h\Delta)^{-1}$ is less than one. This means that for the implicit-explicit iteration,

$$\begin{aligned}
\|f_{n+1}\|_\infty &\leq \|f_n + hG(f_n)\|_\infty \\
&\leq \|f_n\|_\infty + h\left\|\sum_{i=0}^{\infty} a_i f_n^i\right\|_\infty \\
&\leq \|f_n\|_\infty + h\sum_{i=0}^{\infty} \|a_i\|_\infty \|f_n^i\|_\infty \\
&\leq \|f_n\|_\infty + h\sum_{i=0}^{\infty} \|a_i\|_\infty \|f_n\|_\infty^i \\
&\leq \|f_n\|_\infty + hg_\infty(\|f_n\|_\infty)
\end{aligned}$$

Hence the norm of each step of the implicit-explicit iteration will be controlled by the behavior of the recursion

$$f_{n+1} = f_n + hg_\infty(f_n), \tag{2.12}$$

for $f_n, h, a > 0$. Since we are only concerned with short-time existence and uniqueness, we look specifically at $h = T/N$ and $0 \leq n \leq N$, for fixed $T > 0$ and $N \in \mathbb{N}$.

**Remark 12.** The recursion defined by (2.12) is an Euler solver for

$$\frac{dy}{dt} = g_\infty(y), \text{ with } y(0) = f_0. \tag{2.13}$$

This equation is separable, and $g_\infty$ is analytic near $f_0$, so there exists a unique solution for the initial value problem (2.13) for sufficiently short time. Also, whenever $y(t) > 0$

$$\frac{d^2y}{dt^2} = g'_\infty(y(t)) > 0,$$

the function $y(t)$ is concave up. As a result, the exact solution to (2.13) provides an upper bound for the recursion (2.12). More precisely, we have the following result.

**Lemma 13.** *Suppose $y(0) = f_0 > 0$ in (2.13). Let $T > 0$ be given so that $y$ is continuous on $[0, T]$, and let $N \in \mathbb{N}$. Then for each $0 \leq n \leq N$, $f_n \leq y(T)$, where $f_n$ satisfies (2.12) with $h = T/N$.*

*Proof.* Since the right side of (2.13) is strictly positive, the maximum of $y$ is attained at $T$ on any interval $[0, T]$ where $y$ is continuous. Furthermore, since $y(0) > 0$, it follows from Remark 12 that $y$ is concave up on all of $[0, T]$. Therefore, $y$ is a convex function on $[0, T]$. Hence Euler's method, (2.12), will always underestimate the true value of $y$. Another way of stating this is that

$$f_n \leq y(nh) \leq y(T).$$

□

Using Lemma 13, the growth of iterates to (2.12) may be controlled independently of the step size. This provides a uniform bound on the sequence of implicit-explicit approximations.

**Lemma 14.** *Suppose* $0 < h_i = T/i$ *for* $i \in \mathbb{N}$. *Let* $u_i$ *be the implicit-explicit iteration of size* $h_i$ *beginning at* $f_0 \in X$ *on* $t \in [0,T]$. *Then there exists a* $B > 0$ *such that for each* $i$ *and* $t \in [0,T]$, *we have* $\|u_i(t)\|_\infty \leq B$ *for sufficiently small* $T > 0$.

*Proof.* Suppose $f_{in}$ is the $n$-th step of the implicit-explicit iteration of size $h_i$. If we let $y(0) = \|f_0\|_\infty$, Lemma 13 implies that for any $i$ and any $0 \leq n \leq i$

$$\|f_{in}\|_\infty \leq y(T)$$

for sufficiently small T. Hence by (2.9) and the triangle inequality, $\|u_i(t)\|_\infty \leq B$ for all $i$ and $t \in [0,T]$. □

With the bound on the suprema of the approximations, we can obtain a bound on the 1-norms.

**Lemma 15.** *Suppose* $0 < h_i = T/i$ *for* $i \in \mathbb{N}$. *Let* $u_i$ *be the implicit-explicit iteration of size* $h_i$ *beginning at* $f_0 \in X$ *on* $t \in [0,T]$. *Then there exists an* $A > 0$ *such that for each* $i$ *and* $t \in [0,T]$, *we have* $\|u_i(t)\|_1 \leq A$ *for sufficiently small* $T > 0$.

*Proof.* First, notice that Lemma 14 implies that there is a $B > 0$ such that for each $i$ and $t \in [0,T]$, we have $\|u_i(t)\|_\infty \leq A$ for sufficiently small $T > 0$. Again suppose $f_{in}$ is the $n$-th step of the implicit-explicit iteration of size $h_i$. Then we compute

$$
\begin{aligned}
\|f_{i,n+1}\|_1 &\leq \|f_{in}\|_1 + h_i\|G(f_{in})\|_1 \\
&\leq \|f_{in}\|_1 + h_i \sum_{k=0}^{\infty} \|a_k f_{in}^k\|_1 \\
&\leq \|f_{in}\|_1 + h_i \sum_{k=0}^{\infty} \int |a_k f_{in}^k| dx \\
&\leq \|f_{in}\|_1 + h_i \sum_{k=1}^{\infty} \|f_{in}\|_\infty^{k-1} \|a_k\|_\infty \|f_{in}\|_1 + h_i\|a_0\|_1 \\
&\leq \|f_{in}\|_1 \left(1 + h_i \sum_{k=1}^{\infty} \|a_k\|_\infty B^{k-1}\right) + h_i\|a_0\|_1 \\
&\leq \|f_{in}\|_1 \left(1 + \frac{h_i}{B} g_\infty(B) - \frac{h_i}{B}\|a_0\|_\infty\right) + h_i\|a_0\|_1 \\
&\leq \|f_{in}\|_1 \left(1 + h_i C\right) + h_i\|a_0\|_1
\end{aligned}
$$

This recurence leads to

$$
\begin{aligned}
\|f_{in}\|_1 &\leq \|f_0\|_1(1+h_iC)^n + h_i\|a_0\|_1 \sum_{m=0}^{n-1} (1+h_iC)^m \\
&\leq \|f_0\|_1(1+h_iC)^n + h_i\|a_0\|_1 \frac{(1+h_iC)^n - 1}{h_iC} \\
&\leq \left(\|f_0\|_1 + \frac{1}{C}\|a_0\|_1\right)(1+h_iC)^n - \frac{1}{C}\|a_0\|_1 \\
&\leq \left(\|f_0\|_1 + \frac{1}{C}\|a_0\|_1\right)\left(1+\frac{CT}{i}\right)^n - \frac{1}{C}\|a_0\|_1 \\
&\leq \left(\|f_0\|_1 + \frac{1}{C}\|a_0\|_1\right)\left(1+\frac{CT}{i}\right)^i - \frac{1}{C}\|a_0\|_1 \\
&\leq \left(\|f_0\|_1 + \frac{1}{C}\|a_0\|_1\right)e^{CT} - \frac{1}{C}\|a_0\|_1 = A.
\end{aligned}
$$

Once again, by referring to (2.9) and using the triangle inequality, it follows that $\|u_i(t)\|_1 \leq B$ for all $i$ and $t \in [0,T]$. □

**Theorem 16.** *Suppose $0 < h_i = T/i$ for $i \in \mathbb{N}$. Let $u_i$ be the implicit-explicit iteration of size $h_i$ beginning at $f_0 \in X$ on $t \in [0,T]$. Then, for sufficiently small*

*$T > 0$, the sequence $\{u_i(t)\}_{i=1}^{\infty}$ converges pointwise to a function in $t$. The limit function is piecewise differentiable in $t$.*

*Proof.* This compiles the results of Lemma 11, Lemma 14, and Lemma 15. $\square$

**Remark 17.** These proofs can be generalized further to handle all equations of the form

$$\frac{\partial u(t)}{\partial t} = L(u(t)) + G(u),$$

where $G$ is as in (2.4). If the operator $L$ satisfies

- $L : L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R})$ is a sectorial linear operator [20],
- $\|(I - hL)^{-1}\|_1 \leq 1$ and $\|(I - hL)^{-1}\|_\infty \leq 1$,

then the implicit-explicit iteration

$$f_{n+1} = (I - hL)^{-1}(f_n + hG(f_n))$$

converges whenever $f \in X$.

**Remark 18.** Additionally, the techniques can be easily extended to handle the initial boundary value problem

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + \sum_{i=0}^{\infty} a_i(x) u^i(x,t), \text{ for } x \in K \subset \mathbb{R}^n, t > 0$$

with $u(x,t) = v(x,t)$ a given Lipschitz function along $\partial K \times [0,\infty)$, for $K$ compact with smooth boundary. In this case, a boundary term appears in the estimate for $\eta'(t)$ in Lemma 3, which depends on the Lipschitz constant of $v$. Additionally, in Definition 7, one defines $f_{n+1}$ to be the unique solution to the linear elliptic boundary value problem

$$(I - h\Delta)f_{n+1} = f_n + hG(f_n)$$

with $f_{n+1}(x) = v(x, nh)$ for $x \in \partial K$.

## 2.5 Conclusions

The convergence proof for the implicit-explicit method presented here has a number of advantages. First of all, like all IMEX methods, each approximation to the solution is computed explicitly. As a result, a fully discretized version (as is standard in the literature) is easy to program on a computer. Theorem 16 therefore assures the convergence of these fully discrete methods.

However, since the implicit-explicit method presented here is discretized only in time, the convergence proof actually shows the existence of a semigroup of solutions. As a result, the convergence proof forms a bridge between the functional-analytic viewpoint of differential equations, namely that of semigroups, and the numerical methods used to approximate solutions. While the existence and uniqueness of solutions for (2.1) has been known via semigroup methods, the proof provided here gives a more elementary explanation of how this occurs. In particular, it approximates the semigroup action directly.

CHAPTER 3

# CLASSIFICATION OF HETEROCLINES

## 3.1 Introduction

(This chapter is available on the arXiv as [34].)

In this chapter, the global behavior of smooth solutions to the semilinear parabolic equation (1.2)

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) - u^N + \sum_{i=0}^{N-1} a_i(x) u^i(t,x) = \Delta u + P(u), \tag{3.1}$$

for $(t, x) \in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$ is considered, where $N \geq 2$ and $a_i \in L^\infty(\mathbb{R}^n)$ are smooth with all derivatives of all orders bounded.

The main result is that solutions to (3.1) which are heteroclinic orbits connecting two sufficiently regular equilibrium solutions of (3.1) are characterized by finite energy (Definition 20). That this characterization is necessary at all comes from the fact that the spatial domain of (3.1) is unbounded. For bounded spatial domains, all bounded global solutions converge to equilibria [24]. The strength of our result comes from the fact that the finite energy constraint makes solutions behave rather well. Therefore, this result is much sharper than what has typically been obtained in the past, and it applies to more complicated nonlinear terms.

The disadvantage is that in doing so, we cannot treat some of the more complicated aspects of the dynamics. In particular, traveling wave solutions do not have finite energy. Even though a traveling wave will often converge locally to equilibria, at least one of those equilibria will not be admissible in our analysis. On the other hand, we can exclude traveling waves if we require that all the coefficients $a_i$ decay fast enough and only consider one spatial dimension. Then our result establishes an equivalence between the heteroclinic orbits and the finite energy solutions.

For somewhat more restricted nonlinearities, Du and Ma were able to use squeezing methods to obtain similar results to what we obtain here. In particular, they also show that certain kinds of solutions approach equilibria [9]. In a somewhat different setting, Floer used a finite energy constraint for solutions and a regularity constraint on equilibria to characterize heteroclinic orbits of an elliptic problem [15]. The techniques of Floer were subsequently used by Salamon to provide a new characterization of solutions to gradient flows on finite-dimensional manifolds [37]. In this chapter, we recast some of Salamon's work into a parabolic setting, and of course work within an infinite-dimensional space.

## 3.2 Finite energy constraints

From Chapter 2, we have that solutions to (3.1) exist along strips of the form $(t, x) \in I \times \mathbb{R}^n$ for sufficiently small $t$-intervals $I$. One might hope to extend such solutions to all of $\mathbb{R}^{n+1}$, but for certain choices of initial conditions such eternal solutions may fail to exist. Fujita's classic paper [18] gives examples of this "blow-up" pathology. We will specifically avoid it by considering only eternal solutions to (3.1). By eternal solutions, we mean those which are defined for all $\mathbb{R}^{n+1}$, have one continous partial derivative in time, and two continous partial derivatives in space. It should be noted that eternal solutions to (3.1) are quite rare: the backwards-time Cauchy problem contains a heat operator, and so most solutions will not extend to all of $\mathbb{R}^{n+1}$.

**Definition 19.** Our analysis of (3.1) will make considerable use of the fact that it is a gradient differential equation. Recall that the right side of (3.1) is the $L^2(\mathbb{R}^n)$ gradient of the *action functional* (1.3), defined for all $f \in C^1(\mathbb{R}^n)$ by

$$A(f) = \int_{\mathbb{R}^n} \frac{1}{2}\|\nabla f(x)\|^2 - \frac{f^{N+1}(x)}{N+1} + \sum_{i=0}^{N-1} \frac{a_i(x)}{i+1} f^{i+1}(x) dx.$$

It is then evident that along a solution $u(t)$ to (3.1),

$$\begin{aligned} \frac{dA(u(t))}{dt} &= dA|_{u(t)}\left(\frac{\partial u}{\partial t}\right) \\ &= \left\langle \nabla A(u(t)), \frac{\partial u}{\partial t}\right\rangle \\ &= \left\langle \Delta u + P(u), \frac{\partial u}{\partial t}\right\rangle \\ &= \left\| \frac{\partial u}{\partial t}\right\|_2^2 \geq 0, \end{aligned}$$

so $A(u(t))$ is a monotone function. As an immediate consequence, nonconstant $t$-periodic solutions to (3.1) do not exist.

**Definition 20.** The *energy functional* is the following quantity defined on the space of functions $\mathbb{R}^{n+1} \to \mathbb{R}$ with one continuous partial derivative in the first variable ($t$), and two continuous partial derviatives in the rest ($x$):

$$E(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int \left|\frac{\partial u}{\partial t}\right|^2 + |\Delta u + P(u)|^2 \, dx\, dt. \tag{3.2}$$

**Calculation 21.** Suppose $u$ is in the domain of definition for the energy functional, then

$$\begin{aligned} E(u) &= \frac{1}{2}\int_{-\infty}^{\infty}\int \left|\frac{\partial u}{\partial t}\right|^2 + |\Delta u + P(u)|^2 \, dx\, dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\int \left(\frac{\partial u}{\partial t} - \Delta u - P(u)\right)^2 + 2\frac{\partial u}{\partial t}\left(\Delta u + P(u)\right) dx\, dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\int \left(\frac{\partial u}{\partial t} - \Delta u - P(u)\right)^2 dx\, dt + \int_{-\infty}^{\infty}\left\langle \frac{\partial u}{\partial t}, \Delta u + P(u)\right\rangle dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\int \left(\frac{\partial u}{\partial t} - \Delta u - P(u)\right)^2 dx\, dt + \int_{-\infty}^{\infty}\left\langle \frac{\partial u}{\partial t}, \nabla A(u(t))\right\rangle dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\int \left(\frac{\partial u}{\partial t} - \Delta u - P(u)\right)^2 dx\, dt + \int_{-\infty}^{\infty}\frac{d}{dt}A(u(t))dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\int \left(\frac{\partial u}{\partial t} - \Delta u - P(u)\right)^2 dx\, dt + A(u(T))\Big|_{T=-\infty}^{\infty}. \end{aligned}$$

This calculation shows that finite energy solutions to (3.1) minimize the energy functional. If a solution to (3.1) is a heteroclinic connection between two equilibria, then the energy functional measures the difference between the values of the action functional evaluated at the two equilibria. The main result of this chapter is the converse, so that finite energy characterizes the solutions which connect equilibria.

**Remark 22.** Finite energy solutions to (3.1) are even more rare than eternal solutions. However, the set of finite energy solutions is not entirely vacuous, as will be shown in Chapter 5.

It is well-known that when equations like (3.1) exhibit the correct symmetry, they can support traveling wave solutions [12]. A typical traveling wave solution $u$ has a symmetry like $u(t,x) = U(x-ct)$ for some $c \in \mathbb{R}$. As a result, it is immediate that traveling waves will have infinite energy. On the other hand, they also evidently connect equilibria. As a result, Calculation 21 shows that a necessary condition for traveling waves is that there exists at least one equilibrium whose action is infinite. In this chapter, we will consider only equilibria with finite action, and solutions with finite energy. As a result, we will not be working with traveling waves.

## 3.3 Convergence to equilibria

In this section, we show that finite energy solutions tend to equilibria as $|t| \to \infty$. In doing this, we follow Floer in [15] which leads us through an essentially standard parabolic bootstrapping argument.

**Lemma 23.** *Let $U \subseteq \mathbb{R}^n$ and $u \in W^{k,p}(U)$ satisfy $\|D^j u\|_\infty \le C < \infty$ for $0 \le j \le k$*

*(in particular, $u$ is bounded). If $P(u) = \sum_{i=1}^{N} a_i u^i$ with $a_i \in L^\infty(U)$ then there exists a $C'$ such that $\|P(u)\|_{k,p} \le C'\|u\|_{k,p}$.*

*Proof.* First, using the definition of the Sobolev norm,

$$\|P(u)\|_{k,p} = \sum_{j=0}^{k} \|D^j P(u)\|_p \le \sum_{j=0}^{k}\sum_{i=1}^{N} \|D^j a_i u^i\|_p.$$

Now $|D^j a_i u^i| \le P_{i,j}(u, Du, ..., D^j u)$ is a polynomial in $j$ variables with constant coefficients, which has no constant term. (It has constant coefficients because the derivatives of the $a_i$ are bounded.) Additionally,

$$\begin{aligned}\|(D^m u)^q D^j u\|_p &= \left(\int \left|(D^m u)^q D^j u\right|^p\right)^{1/p} \\ &\le \|D^m u\|_\infty^q \left(\int \left|D^j u\right|^p\right)^{1/p} \le C^q \|D^j u\|_p,\end{aligned}$$

so by collecting terms,

$$\|P(u)\|_{k,p} \le \sum_{j=0}^{k}\sum_{i=1}^{N} \|D^j a_i u^i\|_p \le \sum_{j=0}^{k} A_j \|D^j u\|_p \le C'\|u\|_{k,p}.$$

□

The following result is a parabolic bootstrapping argument that does most of the work. In it, we follow Floer in [15], replacing "elliptic" with "parabolic" as necessary.

**Lemma 24.** *If $u$ is a finite energy solution to (3.1) with $\|D^j u\|_{L^\infty((-\infty,\infty)\times V)} \le C < \infty$ for $0 \le j \le k$ with $k \ge 1$ on each compact $V \subset \mathbb{R}^n$, then each of $\lim_{t\to\pm\infty} u(t,x)$ exists, and converges with $k$ of its first derivatives uniformly on compact subsets of $\mathbb{R}^n$. Further, the limits are equilibrium solutions to (3.1).*

*Proof.* Define $u_m(t,x) = u(t+m,x)$ for $m = 0,1,2....$ Suppose $U \subset \mathbb{R}^{n+1}$ is a bounded open set and $K \subset U$ is compact. Let $\beta$ be a bump function whose support is in $U$ and takes the value 1 on $K$. We take $p > 1$ such that $kp > n+1$. Then we can consider $u_m \in W^{k,p}(U)$ (recall that $u$ and its first $k$ derivatives of $u$ are bounded on the closure of $U$), and we have

$$\|u_m\|_{W^{k+1,p}(K)} \leq \|\beta u_m\|_{W^{k+1,p}(U)}.$$

Then using the standard parabolic regularity for the heat operator,

$$\|\beta u_m\|_{W^{k+1,p}(U)} \leq C_1 \left\| \left( \frac{\partial}{\partial t} - \Delta \right) (\beta u_m) \right\|_{W^{k,p}(U)}.$$

Let $P'(u) = -u^N + \sum_{i=1}^{N-1} a_i u^i$, noting carefully that we have left out the $a_0$ term. The usual product rule, and a little work, as suggested in [37] yields the following sequence of inequalities

$$\begin{aligned}
\|u_m\|_{W^{k+1,p}(K)} &\leq C_1 \left\| \beta \left( \frac{\partial}{\partial t} - \Delta \right) u_m \right\|_{W^{k,p}(U)} + C_2 \|u_m\|_{W^{k,p}(U)} \\
&\leq C_1 \left\| \beta \left( \frac{\partial}{\partial t} - \Delta \right) u_m + \beta P'(u_m) - \beta P'(u_m) \right\|_{W^{k,p}(U)} \\
&\quad + C_2 \|u_m\|_{W^{k,p}(U)} \\
&\leq C_1 \|\beta a_0\|_{W^{k,p}(U)} + C_1 \|\beta P'(u_m)\|_{W^{k,p}(U)} + C_2 \|u_m\|_{W^{k,p}(U)} \\
&\leq C_1 \|\beta a_0\|_{W^{k,p}(U)} + C_3 \|u_m\|_{W^{k,p}(U)},
\end{aligned}$$

where the last inequality is a consequence of Lemma 23. By the hypotheses on $u$ and $a_0$, this implies that there is a finite bound on $\|u_m\|_{W^{k+1,p}(K)}$, which is independent of $m$. Now by our choice of $p$, the general Sobolev inequalities imply that $\|u_m\|_{C^{k+1-(n+1)/p}(K)}$ is uniformly bounded. By choosing $p$ large enough, there is a subsequence $\{v_{m'}\} \subset \{u_m\}$ such that $v_{m'}$ and its first $k$ derivatives converge uniformly on $K$, say to $v$. For any $T > 0$, we observe

$$\begin{aligned}\int_{-T}^{T} \int \left| \frac{\partial v}{\partial t} \right|^2 dx\, dt &= \lim_{m' \to \infty} \int_{-T}^{T} \int \left| \frac{\partial v_{m'}}{\partial t} \right|^2 dx\, dt \\ &= \lim_{m' \to \infty} \int_{m'-T}^{m'+T} \int \left| \frac{\partial u}{\partial t} \right|^2 dx\, dt = 0,\end{aligned}$$

where the last equality is by the finite energy condition. Hence $\left|\frac{\partial v}{\partial t}\right| = 0$ almost everywhere, which implies that $v$ is an equilibrium and that $\lim_{t \to \infty} u(t, x) = v(x)$. Similar reasoning works for $t \to -\infty$. □

Now we would like to relax the bounds on $u$ and its derivatives, by showing that they are in fact consequences of the finite energy condition.

**Lemma 25.** *Suppose that either $n = 1$ (one spatial dimension) or $N$ is odd, then we have the following. If $u$ is a finite energy solution to* (3.1), *then the the limits* $\lim_{t \to \pm\infty} u(t, x)$ *exist uniformly on compact subsets, and additionally,*

- *$u$ is bounded,*
- *the derivatives $Du$ are bounded,*
- *and therefore the limits are continuous equilibrium solutions.*

*Proof.* Note that since

$$E(u) = \frac{1}{2} \int_{-\infty}^{\infty} \int \left| \frac{\partial u}{\partial t} \right|^2 + |\Delta u + P(u)|^2 \, dx\, dt < \infty,$$

we have that for any $\epsilon > 0$,

$$\lim_{T \to \infty} \frac{1}{2} \int_{T-\epsilon}^{T+\epsilon} \int \left| \frac{\partial u}{\partial t} \right|^2 + |\Delta u + P(u)|^2 \, dx\, dt = 0,$$

whence $\lim_{t \to \infty} \left|\frac{\partial u}{\partial t}\right| = 0$ for almost all $x$. So this gives that the limit is an equilibrium almost everywhere. Of course, this argument works for $t \to -\infty$.

Now in the case of $N$ being odd, a comparison principle shows that solutions to (3.1) are always bounded. So we need to consider the case with $N$ even. In that case, a comparison principle on (3.1) shows that $u$ is bounded from *above*. On the other hand, if $N$ is even we have assumed that $n = 1$ in this case, and it follows from an easy ODE phase-plane argument that unbounded equilibria are bounded from *below*. (Here we have used that the coefficients $a_i$ are bounded.) As a result, we must conclude that if a solution to (3.1) tends to any equilibrium, that equilibrium (and hence $u$ also) must be bounded.

Now observe that $\left|\frac{\partial u}{\partial t}\right| \to 0$ as $t \to \infty$ on almost all of any compact $K \subset \mathbb{R}^n$, and that $\left|\frac{\partial u}{\partial t}\right| \le a < \infty$ for some finite $a$ on $\{(t,x)|t = 0, x \in K\}$ by the smoothness of $u$. By the compactness of $K$, this means that if $\left\|\frac{\partial u}{\partial t}\right\|_{L^\infty((-\infty,\infty)\times K)} = \infty$, there must be a $(t^*, x^*)$ such that $\lim_{(t,x)\to(t^*,x^*)} \left|\frac{\partial u}{\partial t}\right| = \infty$. This contradicts smoothness of $u$, so we conclude $\left|\frac{\partial u}{\partial t}\right|$ is bounded on the strip $(-\infty, \infty) \times K$. On the other hand, the finite energy condition also implies that for each $v \in \mathbb{R}^n$,

$$\lim_{s\to\infty} \int_{-\infty}^{\infty} \int_{K+sv} \left|\frac{\partial u}{\partial t}\right|^2 dx\, dt = 0,$$

whence we must conclude that $\lim_{s\to\infty} \left|\frac{\partial u(t,x+sv)}{\partial t}\right| = 0$ for almost every $t \in \mathbb{R}$ and $x \in K$. Thus the smoothness of $u$ implies that $\left|\frac{\partial u}{\partial t}\right|$ is bounded on all of $\mathbb{R}^{n+1}$.

Next, note that since $\left|\frac{\partial u}{\partial t}\right|$ and $u$ are both bounded, then so is $\Delta u$. (Use the boundedness of the coefficients of $P$.) Taken together, this implies that all the spatial first derivatives of $u$ are also bounded.

As a result, we have on $K$ a bounded equicontinuous family of functions, so Ascoli's theorem implies that they (after extracting a suitable subsequence) converge uniformly on compact subsets of $K$ to a continuous limit. $\square$

**Corollary 26.** *Suppose that either $n = 1$ or $N$ is odd. An eternal solution $u$ to*

(3.1) *has finite energy if and only if each of the following hold:*

- *each of* $U_\pm(x) = \lim_{t\to\pm\infty} u(t,x)$ *exists and converges with its first derivatives uniformly on compact subsets of* $\mathbb{R}^n$*,*
- $U_\pm$ *are bounded, continuous equilibrium solutions to* (3.1)*,*
- *and either* $|A(U_+) - A(U_-)| < \infty$ *or* $U_+ = U_-$*.*

**Remark 27.** If we consider the more limited case of (1.6), then the asymptotic decay rate for equilibria indicates that all equilibria have finite action (Corollary 48). In this case, Corollary 26 characterizes *all* heteroclinic orbits, not just those whose action difference is finite.

**Theorem 28.** *Suppose that* $n = 1$ *and that all equilibria have finite action. If* $u$ *is a finite energy solution then it converges uniformly to equilibria as* $|t| \to \infty$*.*

*Proof.* Suppose that $u$ tends to equilibrium solutions $f_\pm$ as $t \to \pm\infty$. Suppose that this convergence is not uniform, so that there exists an $\epsilon > 0$ such for each $T$, there is a $|t| > |T|$ with either $\|u(t) - f_-\|_\infty > \epsilon$ or $\|u(t) - f_+\|_\infty > \epsilon$. We therefore postulate the existence of a pair of sequences $\{t_j\}$, $\{x_j\}$ such that $|t_j| \to \infty$ and $\min\{|u(t_j,x_j) - f_-(x_j)|, |u(t_j,x_j) - f_+(x_j)|\} > \epsilon$ for all $j$. We assume that for each $j$, $x_j$ is chosen so that $\min\{|u(t_j,x_j) - f_-(x_j)|, |u(t_j,x_j) - f_+(x_j)|\}$ is maximized. Notice that since $u \to f_\pm$ uniformly on compact subsets, we must have $|x_j| \to \infty$.

To simplify the discussion, we find an $R > 0$ such that for all $|y| > \epsilon$ and $|x| > R$,

$$\left| \sum_{i=0}^{N-1} a_i(x) y^i \right| < \frac{1}{2}|y|^N. \tag{3.3}$$

We assume that $|x_j| > R$ for all $j$. This condition ensures that the leading nonlinear coefficient of (3.1) dominates.

We discern three cases, which we can consider without loss of generality after extracting a suitable subsequence of $\{(t_j, x_j)\}$. In each of the cases, we shall perform a coordinate transformation so that the equilibrium to which $u$ converges is the zero function. In particular, we start the sums at 1 rather than 0.

1. Suppose $t_j \to +\infty$ and $u(t_j, x_j) > \epsilon > 0$. Since $x_j$ is chosen at a maximum of $u(t_j)$ for each $j$, we have that $\frac{\partial^2 u(t_j,x_j)}{x^2} < 0$ by the maximum principle. As a result,

$$\begin{aligned}
\frac{\partial}{\partial t} u(t_j, x_j) &= \frac{\partial^2}{\partial x^2} u(t_j, x_j) - u^N(t_j, x_j) + \sum_{i=1}^{N-1} a_i(x_j) u^i(t_j, x_j) \\
&\leq -u^N(t_j, x_j) + \sum_{i=1}^{N-1} a_i(x_j) u^i(t_j, x_j) \\
&\leq -\frac{\epsilon^N}{2}, \text{ by (3.3).}
\end{aligned}$$

   Therefore we conclude that $\|u(t)\|_\infty \to 0$.

2. Suppose $t_j \to -\infty$ and $u(t_j, x_j) > \epsilon > 0$. Consider the time-reversed version of (3.1), namely

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + u^N - \sum_{i=1}^{N-1} a_i u^i. \tag{3.4}$$

   The comparison principle works in reverse for this equation! Suppose that $v(t, x) = U(t)$ is a spatially constant solution to (3.4) with $U(t_j) = v(t_j, x_j) = u(t_j, x_j) > 0$ for some $j$. Then, shortly thereafter, $\|u(t)\|_\infty > U(t)$, since

$$\begin{aligned}
\frac{\partial u(t_j, x_j)}{\partial t} &\geq U^N(t_j) - \sum_{i=1}^{N-1} a_i(x_j) U^i(t_j) \\
&\geq \frac{1}{2} U^N(t_j) > 0.
\end{aligned}$$

   On the other hand, this rate of growth indicates that $u$ blows up in finite time. This contradicts the fact that $u$ is an eternal solution.

3. Suppose $u(t_j, x_j) < -\epsilon < 0$ and that $t_j \to -\infty$ or $t_j \to +\infty$. If $N$ is odd, then this case can be covered by the previous ones, *mutatis mutandis*. Therefore, we assume $N$ is even. We assume that the limit as $t \to +\infty$ of $u$ is the zero function. From Lemma 25, we have a constant $A = \left\| \frac{\partial^2 u}{\partial x^2} \right\|_\infty < \infty$ which is independent of $t$. Thus for each $t_j$, we have an upper bound for $u(t_j)$ which looks like

$$U_j(x) = \min \left\{ \mu_j, u(t_j, x_j) + \frac{A}{2}(x - x_j)^2 \right\}, \tag{3.5}$$

for some $\mu_j > 0$. In particular, note that $\mu_j \to 0$ by the previous cases.

We show that the forward Cauchy problem for (3.1) started with $U_j$ as an initial condition blows up for sufficiently large $j$. By the comparison principle, this implies that $u$ cannot be an eternal solution, which is a contradiction. This can be shown using the method of Fujita, which we briefly sketch here.

Apply the coordinate transformation $w = u - \mu$ for some $\mu > \mu_j > 0$. Therefore, the initial condition can be made entirely negative, and by the previous cases, the solution *stays* negative for arbitrarily long future time (by taking $j$ large). (Notice that it may not remain negative for *all* future time in the case where $t_j \to -\infty$.) This transformation changes (3.1) into

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} - w^N + \sum_{i=1}^{N-1} a_i w^i + \sum_{k=0}^{N-1} w^k \left( - \begin{pmatrix} N \\ k \end{pmatrix} \mu^{N-k} + \sum_{i=k+1}^{N-1} \begin{pmatrix} i \\ k \end{pmatrix} a_i \mu^{i-k} \right).$$

As is usual for the Fujita method, we choose a solution $v$ to $\frac{\partial v}{\partial t} = -\frac{\partial^2 v}{\partial x^2}$. In particular, fix $T > 0$ and choose $\epsilon > 0$ to define

$$v_\epsilon(s, x) = \frac{1}{\sqrt{4\pi(T - s + \epsilon)}} e^{-\frac{1}{4(T-s+\epsilon)}(x - x_j)^2}.$$

Then we define $J(s) = \int v_\epsilon(s,x)w(s-t_j,x)dx$ and compute

$$\begin{aligned}
\frac{dJ}{ds} &= \int \frac{\partial v_\epsilon}{\partial s} w + v_\epsilon \frac{\partial w}{\partial s} dx \\
&= -\int v_\epsilon w^N dx + \sum_{i=1}^{N-1} \int a_i v_\epsilon w^i dx \\
&\quad + \sum_{k=0}^{N-1} \int v_\epsilon w^k \left( - \begin{pmatrix} N \\ k \end{pmatrix} \mu^{N-k} + \sum_{i=k+1}^{N-1} \begin{pmatrix} i \\ k \end{pmatrix} a_i \mu^{i-k} \right) dx \\
&\leq -J^N + \sum_{i=1}^{N-1} \|w\|_\infty^i \int a_i v_\epsilon dx \\
&\quad + \sum_{k=0}^{N-1} \|w\|_\infty^k \left( \begin{pmatrix} N \\ k \end{pmatrix} \mu^{N-k} + \sum_{i=k+1}^{N-1} \begin{pmatrix} i \\ k \end{pmatrix} \|a_i\|_\infty \mu^{i-k} \right)
\end{aligned}$$

where we have used Lemma 25 to bound $w$, and we have used the assumption that $N$ is even in the last step. Since $a_i$ decays to zero, the second term can be made arbitrarily small for an arbitrarily large $s$ by taking $j$ large as well (for fixed $T$ and $\epsilon$). (The second term may eventually grow larger.) The last term is a constant, independent of $T, \epsilon$ and can be made arbitrarily small by taking $j$ large. By (3.5), for sufficiently large $j$, $J(0) < 0$, and for larger $j$, $J(0)$ becomes more negative. Therefore, for a certain $T$ and sufficiently large $j$, $J(s)$ tends to $-\infty$ for some $0 < s < T$. However, this contradicts the fact that $w$ is bounded.

□

**Corollary 29.** *Suppose that $n = 1$ and that all equilibria have finite action. If $u$ is a finite energy solution then it converges uniformly to equilibria as $|x| \to \infty$.*

*Proof.* Really, the only thing that must be noticed is that Theorem 28 shows that there is uniform convergence in the time direction. For a given $\epsilon$, there is a $T > 0$

such that $|u(t,x) - f_{\pm}(x)| < \epsilon$ for all $|t| > T$. However, this means that for $t \in [-T, T]$, this does not hold. However, $[-T, T]$ is compact, and the proof of Lemma 24 indicates that there is uniform convergence to equilibria as $\|x\| \to \infty$ on compact subsets. $\square$

**Corollary 30.** *The above Corollary implies that the asymptotic spatial behavior of heteroclinic orbits is determined entirely by the asymptotic spatial behavior of equilibria. In particular, in Chapter 4, it is shown that the equilibria for the case of* (1.6),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u^2 + \phi$$

*with $\phi$ decaying to zero, all lie in $L^1(\mathbb{R})$. As a result, each timeslice of a heterocline lies in $L^1(\mathbb{R})$ as well.*

## 3.4 Discussion

The point of employing the bootstrapping argument of Lemma 24 is only to extract uniform convergence of the derivatives of the solution. As can be seen from the proof of Lemma 25, such regularity arguments are unneeded to obtain good convergence of the solution only.

While Corollary 26 is probably true for all spatial dimensions, the proof given here cannot be generalized to higher dimensions. In particular, Véron in [39] shows that in the case of $P(u) = -u^N$, there are solutions to the equilibrium equation $\Delta u - u^N = 0$ which are *unbounded below* and *bounded above* when the spatial dimension is greater than one. This breaks the proof of Lemma 25, that the limiting equilibria of finite energy solutions are bounded for $N$ even, since the proof requires exactly the opposite.

On the other hand, the case of $P(u) = -u|u|^{N-1} + \sum_{i=0}^{N-1} a_i u^i$ is considerably easier than what we have considered here. In particular, all solutions to (3.1) are then bounded. In that case, the proof of Lemma 25 works for all spatial dimensions.

# CHAPTER 4

# EQUILIBRIUM ANALYSIS

## 4.1 Introduction

(This chapter is available on the arXiv as [33], and has been accepted for publication in *Ergodic Theory and Dynamical Systems.*)

Since the dynamics of solutions to the semilinear parabolic equation (1.2) depend strongly on the equilibrium solutions, it is important to understand the number and structure of equilibrium solutions. As will be shown, this is a somewhat ill-defined and rather delicate goal. Therefore, to fix ideas and techniques, we shall focus on the specific case of the equilibria of (1.6)

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - u^2(t,x) + \phi(x), \tag{4.1}$$

where $\phi$ tends to zero as $|x| \to \infty$. The resulting questions and techniques we encounter have obvious generalizations to the more general equation. Therefore, we are faced with the task of analyzing a nonlinear *ordinary* differential equation, and finding its *global* solutions. Additionally, the asymptotic properties of such solutions will be crucial in Chapters 3, 5, and 7.

Finding global solutions to nonlinear ordinary differential equations on an infinite interval can be rather difficult. Numerical approximations can be particularly misleading, because they examine only a finite-dimensional portion of the infinite-dimensional space in which solutions lie. Additionally, the conditions for global existence can be rather delicate, which a numerical solver may have difficulty rigorously checking. In situations where there is well-defined asymptotic behavior for global solutions, it is possible to exploit the asymptotic information to answer questions about global existence and uniqueness of solutions directly. Additionally, more detailed information may be provided by using the asymptotic behavior to install artificial boundary conditions for use in a numerical solver. The numerical solver can then be used on the remaining (bounded) interval with boundary conditions that match the numerical approximation to an asymptotic expansion valid on the rest of the solution interval.

In this chapter, we consider the behavior of global solutions satisfying the equilibrium equation for (4.1), namely

$$0 = f''(x) - f^2(x) + \phi(x), \text{ for all } x \in \mathbb{R}. \tag{4.2}$$

In particular, we wish to know how many solutions there are for a given $\phi$. (There may be uncountably many solutions, as in the case where $\phi \equiv \text{const} > 0$.) This problem depends rather strongly on the asymptotic behavior of solutions to (4.2) as $|x| \to \infty$, so it is useful to study instead the pair of initial value problems

$$\begin{cases} 0 = f''(x) - f^2(x) + \phi(x) \text{ for } x > 0 \\ (f(0), f'(0)) \in Z, \end{cases} \tag{4.3}$$

and

$$\begin{cases} 0 = f''(x) - f^2(x) + \phi(x) \text{ for } x < 0 \\ (f(0), f'(0)) \in Z', \end{cases} \tag{4.4}$$

where $\phi \in C^\infty(\mathbb{R})$. The sets $Z$,$Z'$ supply the initial conditions for which solutions exist to (4.3) for all $x > 0$ and to (4.4) for all $x < 0$, respectively. Solutions to (4.2) will occur exactly when $Z \cap Z'$ is nonempty. Indeed, the theorem on existence and uniqueness for ODE gives a bijection between points in $Z \cap Z'$ and solutions to (4.2) [29]. Since (4.3) and (4.4) are related by reflection across $x = 0$, it is sufficient to study (4.3) only.

Due to the asympotic behavior of solutions to (4.3), the methods we employ here will be most effective in the specific cases where $\phi$ is nonnegative and monotonically decreasing to zero. (We denote the space of smooth functions that decay to zero as $C_0^\infty(\mathbb{R})$.) The decay condition on $\phi$ allows the differential operators in (4.2) through (4.4) to be examined with a perturbative approach as $x$ becomes large, and makes sense if one is looking for smooth solutions in $L^p(\mathbb{R})$ with bounded derivatives.

When $\phi$ is strictly negative, it happens that no solutions exist to (4.3) for all $x > 0$. The monotonicity restriction on $\phi$ provides some technical simplifications and sharpens the results that we obtain. This leads us to restrict $\phi$ to a class of functions that captures this monotonicity restriction but allows some flexibility, which we shall call the *M-shaped functions*.

It is unlikely that we will be able to solve (4.3) explicitly for arbitrary $\phi$, so one might think that numerical approximations might be helpful. However, most numerical approximations will not be able to count the number of global solutions accurately. For instance, finite-difference methods are typically only useful for finding solutions valid on finite intervals of $\mathbb{R}$. However, one cannot easily infer a solution's behavior for large values of $|x|$ when it is only known on a finite interval. In particular, global solutions to (4.3) must tend to zero (Theorem 38). All other solutions fail to exist for all of $\mathbb{R}$. Worse, the space of initial conditions which give rise to global solutions is at best a 1-dimensional submanifold of the 2-dimensional space of initial conditions (Theorem 55). Therefore, a typical finite-difference solution that *appears* to tend to zero may in fact not, and as a result fails to be a solution over all $x > 0$.

Because of this failure, we need to understand the asymptotic behavior of solutions to (4.3) as we take $x \to \infty$. Equivalently, since $\phi \in C_0^\infty(\mathbb{R})$, this means that we should examine solutions with $\phi$ small. The driving motivation for this discussion is that solutions to $0 = f''(x) - f^2(x) + \phi(x)$ for $\phi$ small behave much like solutions to $0 = f''(x) - f^2(x)$. In the latter case, we can completely characterize the solutions which exist on intervals like $[x_0, \infty)$.

In Section 4.2 we review what is known about the much simpler case where $\phi$ is a constant. Of course, then (4.3) is autonomous, and the results are standard. In Section 4.3, we establish the existence of solutions which are asymptotic to zero. Some of these solutions are computed explicitly using perturbation methods in Section 4.4, where low order approximations are used to gather qualitative information about the initial condition sets $Z$ and $Z'$. In Sections 4.5 and 4.6, these qualitative observations are made precise. Section 4.7 applies these observations about $Z$ and $Z'$ to give existence and uniqueness results for (4.2). Finally, in Section 4.8, we use the information gathered about $Z$ and $Z'$ to provide artificial boundary conditions to a numerical solver on a bounded interval, which sharpens the results from Section 4.7. We exhibit the numerical results for a typical family of $\phi$, showing bifurcations in the global solutions to (4.2).

## 4.2 Review of behavior of solutions to $0 = f''(x) - f^2(x) + P$

It will be helpful to review the behavior of

$$\begin{cases} 0 = f''(x) - f^2(x) + P \\ f(0), f'(0) \text{ given,} \end{cases} \tag{4.5}$$

where $P$ is a constant, since varying $\phi$ can be viewed as a perturbation on the case $\phi(x) = P$. In particular, we need to compute some estimates for later use. We shall typically take $P > 0$, as there do not exist solutions for all $x$ if $P < 0$.

**Lemma 31.** *Suppose $f$ is a solution to the initial value problem* (4.5) *with* $f(0) >$

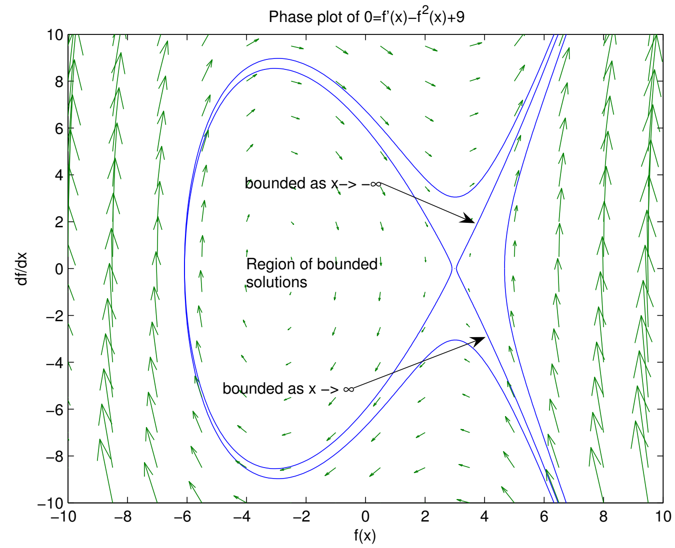


Figure 4.1: The phase plot of $f'' - f^2 + 9 = 0$. Bounded solutions live in a small region, the rest are unbounded.

*$\sqrt{P}$ and $f'(0) > 0$. Then there does not exist an upper bound on $f(x)$, when $x > 0$. Additionally, if $P < 0$, there does not exist an upper bound on $f(x)$.*

*Proof.* Observe that for $f > \sqrt{P}$ or if $P < 0$

$$f'' = f^2 - P > 0.$$

Hence, since $f'(0) > 0$, and $f'$ is monotonic increasing, $f(x)$ is monotonic increasing at an increasing rate. Thus it must be unbounded from above. $\square$

**Definition 32.** The differential equation (4.5) comes from a Hamiltonian, namely

$$H(f, f') = \frac{1}{3}f^3 - \frac{1}{2}f'^2 - fP + \frac{2}{3}P^{3/2}.$$

**Definition 33.** A useful tool in the study of smooth dynamical systems is the *funnel*. Suppose $\Phi$ is a local flow on a manifold $M$. A funnel $F$ is a set such that if $x \in F$, then $\Phi_x(t) \in F$ for all $t > 0$. A funnel $F$ with an oriented, piecewise $C^1$

boundary is characterized by having the vector field $\frac{d}{dt}\Phi_x$ being inward-pointing for all $x \in \partial F$.

**Lemma 34.** *Suppose $f$ is a solution to the equation* (4.5) *on* $\mathbb{R}$. *All bounded solutions lie in the funnel*

$$M = \{(f, f')|H(f, f') \geq 0 \text{ and } f \leq \sqrt{P}\}. \tag{4.6}$$

*Any solution which includes a point outside the closure of $M$ is unbounded, either for $x > 0$ or $x < 0$. (Note that $M$ is the teardrop-shaped region in Figure 4.1.)*

*Proof.*

- $M$ is a bounded set. Notice that $H(f, 0) \geq H(f, f')$, or in other words within $M$,

  $$0 < \frac{1}{3}f^3 - \frac{1}{2}f'^2 - fP + \frac{2}{3}P^{3/2} \leq \frac{1}{3}f^3 - fP + \frac{2}{3}P^{3/2}.$$

  Elementary calculus reveals that this inequality establishes a lower bound on $f$, namely that

  $$-\sqrt{3P} \leq f \leq \sqrt{P} \tag{4.7}$$

  On the other hand,

  $$|f'| < \sqrt{\frac{4}{3}P^{3/2} + \frac{2}{3}f^3 - 2fP} \leq \sqrt{\frac{8}{3}}P^{3/4} \tag{4.8}$$

  immediately establishes a bound on $f'$.

- $M$ is a funnel, from which solutions neither enter nor leave. This is immediate from the fact that $H$ is the Hamiltonian, and the definition of $M$ simply says that $H(f, f') \geq 0$. This suffices since solutions to (4.5) are tangent to level curves of $H$.

- If $(f(0), f'(0)) \notin M$ then $f$ is unbounded. Evidently if $f(0) > \sqrt{P}$ and $f'(0) > 0$, then Lemma 31 applies to give that $f$ is unbounded. For the remainder, discern two cases. First, suppose $f(0) > \sqrt{P}$ and $f'(0) < 0$. Evidently, $H(f(0), f'(0)) = H(f(0), -f'(0))$, so it's just a matter of verifying that a solution curve transports our solution to the first quadrant. But this is immediately clear from the formula for

$$f' = \pm\sqrt{\frac{2}{3}f^3 - 2fP - 2H(f(0), f'(0))},$$

which gives $f' = \pm f'(0)$ when $f = f(0)$. The other case is when $H(f(0), f'(0)) < 0$. Then we show that there is a point $(\sqrt{P}, g)$ on the same solution curve, and then Lemma 31 applies. So we try to satisfy

$$\begin{aligned} \frac{1}{3}P^{3/2} - \frac{1}{2}g^2 - P^{3/2} + \frac{2}{3}P^{3/2} &= H(f(0), f'(0)) < 0 \\ g^2 &= -2H(f(0), f'(0)) > 0, \end{aligned}$$

which clearly has a solution in $g$.

□

**Lemma 35.** *If $f$ is a solution to* (4.5) *with $f(0) > \sqrt{P}$, and $f'(0) > 0$ then there exists a $C$ such that* $\lim_{x\to C} f(x) = \infty$.

*Proof.* From Lemma 34, we have that $f$ is unbounded, and goes to $+\infty$. Using the Hamiltonian, we can solve for

$$\frac{df}{dx} = \pm\sqrt{\frac{2}{3}f^3 - 2fP - 2H(f(0), f'(0))},$$

or viewing $f$ as the independent variable,

$$\begin{aligned} \frac{dx}{df} &= \frac{1}{\pm\sqrt{\frac{2}{3}f^3 - 2fP - 2H(f(0), f'(0))}} \\ &\sim \sqrt{\frac{3}{2}}f^{-3/2}, \end{aligned}$$

as $f$ becomes large. Solving this asymptotic differential equation is easy, and leads to

$$\begin{aligned} x &\sim -\frac{1}{2}\sqrt{\frac{3}{2f}} + C, \\ f &\sim \frac{3}{8(x-C)^2}, \text{ (for } |x-C| \text{ small)} \end{aligned}$$

which has an asymptote at $x = C$. □

## 4.3 Existence of asymptotic solutions for $\phi \in C_0^\infty(\mathbb{R})$

The first collection of results we obtain will make the assumption that $\phi$ tends to zero. From this, a number of useful asymptotic results follow. Working in the phase plane will be useful for understanding (4.3). Of course (4.3) is not autonomous, but by adding an additional variable, it becomes so.

**Definition 36.** We think of (4.3) as a vector field $V$ on $\mathbb{R}^3$, defined by the formula

$$V(f, f', x) = \begin{pmatrix} f' \\ f^2 - \phi(x) \\ 1 \end{pmatrix}. \tag{4.9}$$

Notice that the first coordinate of an integral curve for this vector field solves (4.3).

**Definition 37.** Define $H(f, f', x) = \frac{1}{3}f^3 - \frac{1}{2}f'^2 - f\phi(x) + \frac{2}{3}\phi^{3/2}(x)$. Notice that for constant $\phi = P$, this reduces to a Hamiltonian for (4.5).

**Theorem 38.** *Suppose $f$ is a solution to the problem* (4.3) *where $\phi \in C_0^\infty(\mathbb{R})$. If $f$ does not tend to zero as $x \to \infty$, then there exists a $z$ such that* $\lim_{x\to z} f(x) = \infty$. *Stated another way, if $f$ solves* (4.3) *for all $x > 0$, then* $\lim_{x\to\infty} f(x) = 0$.

*Proof.* If $f$ does not tend to zero, this means that there is an $R > 0$ such that for each $x_0 > 0$, there is an $x > x_0$ so that $|f(x)| > R$. But since $\phi$ tends to zero as $x \to \infty$, for any $P > 0$ we can find an $x_1 > 0$ such that for all $x > x_1$, $|\phi(x)| < P$. Choose such a $P$ so that the set $M$ in Lemma 34 associated to (4.5) is contained entirely within the strip $-R < f < R$. We can do this since the set $M$ is bounded, and its radius decreases with decreasing $P$, as shown in (4.7) and (4.8). But this means that there is an $x_2 > x_1$ such that $|f(x_2)| > R$.

Construct the following regions (See Figure 4.2):

$$I = \{(f, f', x)|f \geq R \text{ and } f' \leq 0\},$$

$$II = \{(f, f', x)|f \geq R \text{ and } f' \geq 0\},$$

$$III = \{(f, f', x)|f \leq -R\},$$

and

$$IV = \left\{(f, f', x)|f' \geq 0 \text{ and } f \geq -R \text{ and } \left(\frac{1}{3}f^3 - \frac{1}{2}f'^2 - fP + \frac{2}{3}P^{3/2} \leq 0 \right.\right.$$
$$\left.\left. \text{if } f \leq \sqrt{P}\right)\right\}.$$

The following statements hold:

- Region $I$ is an antifunnel. Along $f = R$ and $f' = 0$, solutions must exit. Once a solution exits Region $I$, it cannot reenter. Also, because $f > \sqrt{P}$, $f'' = f^2 - \phi > f^2 - P > 0$, solutions must exit Region $I$ in finite $x$.
- Region $II$ is a funnel. Along $f = R$ and $f' = 0$, solutions enter. Now $f'' = f^2 - \phi > f^2 - P \geq 0$ and $f' \geq 0$, so solutions will increase at an increasing rate and so, they are unbounded.
- Solutions remain in Region $III$ for only finite $x$, after which they must enter Region $IV$. This occurs since $f \leq -\sqrt{P} < 0$, and so $f'$ always increases.

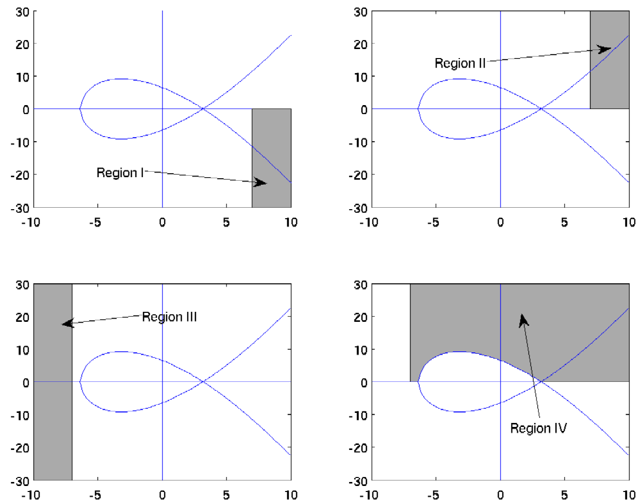


Figure 4.2: The Regions $I$, $II$, $III$, and $IV$ of Theorem 38

Note that for $f' < 0$, solutions will enter Region $III$ along $f = -R$, and for $f' > 0$, solutions exit along $f = -R$.

- Region $IV$ is a funnel. Solutions enter along $f = -R$ and along $f' = 0$ (note that $|f| \geq \sqrt{P}$ in both cases). Along the curve boundary of Region $IV$, we have that

$$\nabla\left(\frac{1}{3}f^3 - \frac{1}{2}f'^2 - fP + \frac{2}{3}P^{3/2}\right) \cdot V(f, f', x) = \begin{pmatrix} f^2 - P \\ -f' \\ 0 \end{pmatrix}^T \begin{pmatrix} f' \\ f^2 - \phi \\ 1 \end{pmatrix} = f'(P - \phi) < 0,$$

so that solutions enter.

Now suppose $(f(x_2), f'(x_2), x_2) \in I$. After finite $x$, say at $x = x_2'$, the solution through that point must exit Region $I$, never to return. Then, there is an $x_3 > x_2'$ such that $|f(x_3)| > R$. So this solution has either $(f(x_3), f'(x_3), x_3) \in II$ or $\in III$. The former gives the conclusion we want, so consider the latter case. The solution will only remain in Region $III$ for finite $x$, after which it enters Region $IV$, say at $x = x_3'$. Then there is an $x_4 > x_3'$ such that $|f(x_4)| > R$. Now the only possible location for $(f(x_4), f'(x_4), x_4)$ to be is within Region $II$, since it must also remain in Region $IV$. As a result, the solution is unbounded by an easy extension of Lemma 31. As $x$ becomes large, $\phi$ tends to zero, so the solution will be asymptotic to an unbounded solution of $0 = f'' - f^2$. But Lemma 31 above assures us that such a solution is unbounded from above, and Lemma 35 gives that it has an asymptote. Hence, our solution must blow up at a finite $x$. $\square$

This result indicates that solutions to (4.3) which exist for all $x > 0$ are rather rare. Those which exist for all $x > 0$ must tend to zero, and it seems difficult to "pin them down." We now apply topological methods, similar to those employed in [23], to "capture" the solutions we seek. The methods we use are due to Ważewski [41].

We begin by extending the usual definition of a flow slightly to the case of a manifold with boundary.

**Definition 39.** Suppose $M$ is a manifold with boundary. A *flow domain* $J$ is a subset of $\mathbb{R} \times M$ such that if $x \in M$ then $J_x = \mathrm{pr}_1(J \cap \mathbb{R} \times \{x\})$ is an interval containing 0, and if $x$ is in the interior of $M$ then 0 is in the interior of $J_x$. ($\mathrm{pr}_1 : \mathbb{R} \times M \to \mathbb{R}$ is projection onto the first factor)

**Definition 40.** A *(smooth) flow* is a smooth map $\Phi$ from a flow domain $J$ to a manifold with boundary $M$, satisfying

- $\Phi(0, x) = x$ for all $x \in M$ and
- $\Phi(t_1 + t_2, x) = \Phi(t_1, \Phi(t_2, x))$ whenever both sides are well-defined.

Additionally, we assume that flows are *maximal* in the sense that they cannot be written as a restriction of a map from a larger flow domain which satisfies the above axioms. We call the curve $\Phi_x : J_x \to M$ defined by $\Phi_x(t) = \Phi(t, x)$ the *integral curve through* $x$ for $\Phi$.

**Definition 41.** Suppose $\Phi : J \to M$ is a flow on $M$ and $x \in \partial M$. Then the flow at $x$ is said to be *inward-going* (or simply *inward*) if $J_x$ is an interval of the form $[0, a)$ or $[0, a]$ for some $0 < a \leq \infty$. Likewise, the flow at $x$ is *outward-going* if $J_x$ is of the form $(a, 0]$ or $[a, 0]$ for $-\infty \leq a < 0$.

**Theorem 42.** *(Ważewski's antifunnel theorem) Suppose $\Phi : J \to M$ is a flow on $M$ and that $\{A, B\}$ forms a partition of the boundary of $M$ such that the flow of $\Phi$ is inward along $A$ and outward along $B$. If every integral curve of $\Phi$ intersects $B$ in finite time (ie. $J_x$ is bounded for each $x$), then $A$ is diffeomorphic to $B$.*

*Proof.* For each $x \in A$, $J_x = [0, t_x]$, where $t_x$ is the time which the integral curve through $x$ intersects $B$. (We have that $\Phi(t_x, x)$ is outward-going, since $J_x$ is closed, so it is in $B$.)

Using this, we can define a map $F : A \to B$ by $F(x) = \Phi(t_x, x)$. $F$ takes $A$ smoothly and injectively into $B$. The smoothness follows from the smoothness of $\Phi$ and that $\partial M$ is a smooth submanifold. To see the injectivity, suppose $F(x) = F(y)$ for some $x, y \in A$, so $\Phi(t_x, x) = \Phi(t_y, y)$. Without loss of generality, suppose

$0 < t_x \leq t_y$. Then we have that

$$\begin{aligned} F(x) &= F(y) \\ \Phi(-t_x, F(x)) &= \Phi(-t_x, F(y)) \\ \Phi(-t_x, \Phi(t_x, x)) &= \Phi(-t_x, \Phi(t_y, y)) \\ \Phi(t_x - t_x, x) &= \Phi(t_y - t_x, y) \\ x &= \Phi(t_y - t_x, y). \end{aligned}$$

But the flow is inward at $x$, so it is also inward at $\Phi(t_y - t_x, y)$. This means that $(t_y - t_x - \epsilon, y) \notin J$ for every $\epsilon > 0$. But this contradicts the fact that $(t_y, y) \in J$ unless we have $t_y \leq t_x$. As a result, $t_y = t_x$, so $x = y$.

In just the same way as for $F$, we construct a map $G : B \to A$ so that $G$ takes $B$ smoothly and injectively into $A$. Namely, we suppose $J_y = [s_y, 0]$ for some $s_y$, and put $G(y) = \Phi(s_y, y)$. Notice that by maximality, if there were to be an $x \in A$ such that $F(x) = y$, $s_y = -t_x$.

Now we claim that $G$ is the inverse of $F$. We have that

$$\begin{aligned} (G \circ F)(x) &= \Phi(s_{F(x)}, F(x)) \\ &= \Phi(s_{F(x)}, \Phi(t_x, x)) \\ &= \Phi(s_{F(x)} + t_x, x) \\ &= \Phi(-t_x + t_x, x) = x, \end{aligned}$$

where we employ the remark about $s_y$ above. □

**Remark 43.** We can extend the Antifunnel theorem to a topological space $X$ on which a flow $\Phi : J \to X$ acts in the obvious way. In that case, there is no reasonable definition of the boundary of $X$. However, the notion of inward- and

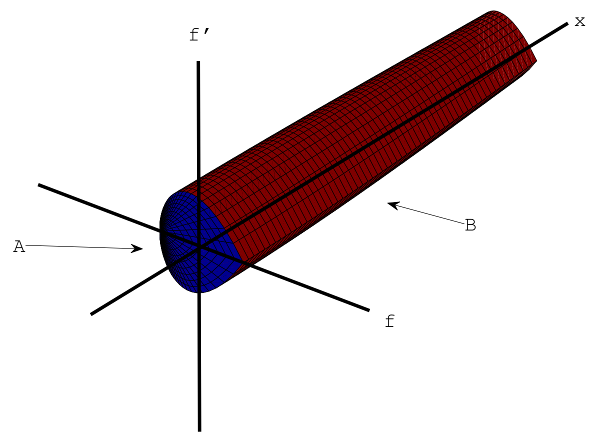


Figure 4.3: Schematic of the region $R_1$, showing the boundary partition $A$ and $B$.

outward-going points still makes sense. If we let $A$ be the set of inward-going points and $B$ be the set of outward-going points in $X$, then the conclusion is that $A$ is homeomorphic to $B$.

Now we employ the Antifunnel theorem to deduce the existence of a bounded solution to $0 = f'' - f^2 + \phi$ for $x > x_0$ for some $x_0 \geq 0$.

**Theorem 44.** *Suppose $0 \leq \phi(x) \leq K$ for all $x \geq x_0$ for some $x_0 > 0$ and $0 < K < \infty$, and that there exists an $x_1 \geq x_0$ such that for all $x > x_1$, $\phi(x) > 0$. Then the region $R_1$ given by $R_1 = \{(f, f', x) | H(f, f', x) \geq 0, x \geq 0, f \leq \sqrt{\phi(x)}\}$ contains a bounded solution to $0 = f''(x) - f^2(x) + \phi(x)$, which exists for all $x$ greater than some nonnegative $x_2$.*

*Proof.* Without loss of generality, we may take $x_0 = 0$, because otherwise solutions must exit the portions of $R_1$ in $\{(f, f', x) | x < x_0\}$ since the $x$-component of

$V(f, f', x)$ is equal to 1.

If $\phi(0) > 0$, partition the boundary of $R_1$ into two pieces: $A = \{(f, f', x)|x = 0\}$ and $B = \{(f, f', x)|H(f, f', x) = 0\}$ (See Figure 4.3). The flow of $V$ is evidently inward along $A$. As for $B$, notice that $\nabla H$ is an inward-pointing vector field normal to $B$. We compute

$$\begin{aligned} \nabla H \cdot V &= \begin{pmatrix} f^2 - \phi(x) \\ -f' \\ (-f + \sqrt{\phi(x)})\phi'(x) \end{pmatrix}^T \begin{pmatrix} f' \\ f^2 - \phi(x) \\ 1 \end{pmatrix} \\ &= (-f + \sqrt{\phi(x)})\phi'(x), \end{aligned}$$

which has the same sign as $\phi'(x)$ when $f < \sqrt{\phi(x)}$ in $R_1$. Finally, we must deal with the case where $f = \sqrt{\phi(x)} \in B$. But in this case, $f' = 0$ from the equation for $H$, so we see that $V(\sqrt{\phi(x)}, 0, x) = (0, 0, 1)^T$, so the flow is inward when $\phi'(x) < 0$ and outward when $\phi'(x) > 0$. This means that the portion of the boundary of $R_1$ on which the flow is outward is a disjoint union of annuli. On the other hand, the portion of the boundary of $R_1$ on which the flow is inward is the disjoint union of a disk (namely $R_1 \cap \{x = 0\}$) and some annuli.

We now consider the case of $\phi(0) = 0$, in which case the set $A$ above is just a point. Assume without loss of generality that $\phi(x)$ is strictly positive for all $x > 0$, so we let $x_1 = 0$. Let

$$x' = \begin{cases} \inf \{x \in (0, \infty)|\phi'(x) = 0\} \text{ or} \\ \infty \text{ if } \phi'(x) > 0 \text{ for all } x > 0. \end{cases}$$

In this case, the set $\{(f, f', x)|0 \leq x \leq x'\} \cap \partial R_1$, is a contractible (it may be a point if $\phi$ oscillates rapidly as $x \to 0$), connected component of the inflow portion of the boundary of $R_1$. It is obvious that the remainder of the inflow portion of the boundary is homeomorphic to a disjoint union of annuli, since $\phi$ is smooth and strictly positive.

We can apply the Antifunnel theorem to conclude that there is a solution which does not intersect either the inflow or outflow portions of the boundary. There is a lower bound on the $x$-coordinate of such a solution, since the $x$-component of $V(f, f', x)$ is equal to 1, and the Region $R_1$ lies within the half-space $x > 0$. Therefore, there must exist a solution which enters $R_1$, and remains inside the interior of $R_1$ for all larger $x$. That such a solution is bounded follows from the fact that each constant $x$ cross section of $R_1$ has a radius bounded by the inequalities (4.7) and (4.8), and the fact that $\phi(x) \leq K < \infty$. □

## 4.4 Asymptotic series solution

Theorem 44 ensures the existence of solutions to $0 = f'' - f^2 + \phi$ for $x$ sufficiently large. However, it does not give any description of the initial condition set $Z$ which leads to such solutions, nor does it give a description of the maximal intervals of existence. Fortunately, it is relatively easy to construct an asymptotic series for solutions to (4.3), which will provide a partial answer to this concern. In doing so, we essentially follow standard procedure, as outlined in [22], for example. However, our case is better than the standard situation, because under relatively mild restrictions this series *converges* to a true solution.

We begin by supposing that our solution has the form

$$f = \sum_{k=0}^{\infty} f_k, \tag{4.10}$$

where we temporarily assume $f_{k+1} \ll f_k$ and $f_0 \gg \phi$, as $x \to +\infty$. (This assumption will be verified in Lemma 45.) Substituting (4.10) into (4.3), we get

$$\begin{aligned}
0 &= \sum_{k=0}^{\infty}\left[f_k'' - \sum_{m=0}^{k} f_m f_{k-m}\right] + \phi \\
0 &= f_0'' - f_0^2 + (f_1'' - 2f_0 f_1 + \phi) + \sum_{k=2}^{\infty}\left[f_k'' - 2f_0 f_k - \sum_{m=1}^{k-1} f_m f_{k-m}\right].
\end{aligned}$$

We solve this equation by setting different orders to zero. Namely,

$$\begin{aligned}
0 &= f_0'' - f_0^2 \\
0 &= f_1'' - 2f_0 f_1 + \phi \\
0 &= f_k'' - 2f_0 f_k - \sum_{m=1}^{k-1} f_m f_{k-m}.
\end{aligned}$$

The equation for $f_0$ is integrable, and therefore easy to solve. (There are two families of solutions for $f_0$. We select the nontrivial one, because the other one simply results in $f(x) \sim -\int_x^\infty \int_t^\infty \phi(s) ds\, dt$.) The equations for $f_k$ are linear and can be solved by a reduction of order. Thus formally, the solutions are

$$\begin{cases}
f_0 = \frac{6}{(x-d)^2} \\
f_1 = \frac{1}{(x-d)^3}\left[K + \int^x (t-d)^6 \int_t^\infty \frac{\phi(s)}{(s-d)^3} ds\, dt\right] \\
f_k = -\frac{1}{(x-d)^3}\int^x (t-d)^6 \int_t^\infty \frac{\sum_{m=1}^{k-1} f_m(s) f_{k-m}(s)}{(s-d)^3} ds\, dt,
\end{cases} \tag{4.11}$$

for $d, K$ constants. Notice that these constants parametrize the set of initial conditions $Z$.

**Lemma 45.** *Suppose $f(x) = \sum_{k=0}^\infty f_k(x)$ where the $f_k$ are given by (4.11). If there exists an $M > 0$, an $R > 0$, and an $\alpha > 5$ such that*

$$|\phi(x)| < \frac{M}{(x-d)^\alpha} \text{ for all } |x-d| > R > 0, \tag{4.12}$$

*then $f(x)$ is bounded above by the power series*

$$|f(x)| \leq \frac{1}{(x-d)^2}\sum_{k=0}^{\infty}\left|\frac{A_k}{x-d}\right|^k. \tag{4.13}$$

*Proof.* We proceed by induction, and begin by showing that the $f_1$ term is appropriately bounded:

$$\begin{aligned}
|f_1(x)| &= \left| \frac{1}{(x-d)^3} \left[ K + \int^x (t-d)^6 \int_t^\infty \frac{\phi(s)}{(s-d)^3} ds\, dt \right] \right| \\
&\leq \left| \frac{1}{(x-d)^3} \left[ K + \int^x (t-d)^6 \int_t^\infty \frac{M}{(s-d)^{3+\alpha}} ds\, dt \right] \right| \\
&\leq \left| \frac{1}{(x-d)^3} \left[ K + \frac{M}{(2+\alpha)(5-\alpha)(x-d)^{\alpha-5}} \right] \right|
\end{aligned}$$

Now since $|x-d| > R$ and $\alpha > 5$, we have that

$$\begin{aligned}
|f_1(x)| &\leq \frac{1}{|x-d|^3} \left[ |K| + \frac{M}{(2+\alpha)|5-\alpha| R^{\alpha-5}} \right]. \\
&\leq \frac{A_1}{|x-d|^3}.
\end{aligned}$$

with

$$A_1 = |K| + \frac{M}{(2+\alpha)(\alpha-5)R^{\alpha-5}}. \tag{4.14}$$

For the induction hypothesis, we assume that $|f_i| \leq \frac{A_i}{|x-d|^{2+i}}$ with $A_i \geq 0$ and for all $i \leq k-1$. We have that

$$\begin{aligned}
\sum_{m=1}^{k-1} f_m f_{k-m} &\leq \sum_{m=1}^{k-1} \frac{A_m}{|x-d|^{2+m}} \frac{A_{k-m}}{|x-d|^{2+k-m}} \\
&\leq \frac{1}{|x-d|^{k+4}} \sum_{m=1}^{k-1} A_m A_{k-m},
\end{aligned}$$

so by the same calculation as for $f_1$, we obtain

$$f_k \leq \frac{\sum_{m=1}^{k-1} A_m A_{k-m}}{(k+6)(k-1)} \frac{1}{|x-d|^{k+2}}.$$

Hence we should take

$$A_k = \frac{\sum_{m=1}^{k-1} A_m A_{k-m}}{(k+6)(k-1)}. \tag{4.15}$$

Hence we have that

$$|f(x)| \leq \sum_{k=0}^{\infty} |f_k(x)| \leq \frac{1}{|x-d|^2} \sum_{k=0}^{\infty} A_k \left| \frac{1}{x-d} \right|^k .$$

□

**Lemma 46.** *The power series given by*

$$\sum_{k=0}^{\infty} \frac{A_k}{|x-d|^k},$$

*with* $A_0, A_1 \geq 0$ *given, and*

$$A_k = \frac{\sum_{m=1}^{k-1} A_m A_{k-m}}{(k+6)(k-1)} = \frac{\sum_{m=1}^{k-1} A_m A_{k-m}}{k^2+5k-6}$$

*converges for* $|x-d| > R$ *if* $A_1 \leq 8R$.

*Proof.* We show that under the conditions given, the series passes the usual ratio test. That is, we wish to show

$$\lim_{k\to\infty} \left| \frac{A_{k+1}}{A_k} \right| \leq R.$$

Proceed by induction. Take as the base case, $k = 1$: by the formula for $A_k$,

$$A_2 = \frac{A_1^2}{8}, \text{ so } \frac{A_2}{A_1} = \frac{A_1^2}{8A_1} = \frac{A_1}{8} \leq R.$$

Then for the induction step,

$$\begin{aligned}
\frac{A_{k+1}}{A_k} &= \frac{\sum_{m=1}^{k} A_m A_{k-m+1}}{A_k(k^2+7k)} \\
&= \frac{\sum_{m=2}^{k} A_m A_{k-m+1} + A_1 A_k}{A_k(k^2+7k)} \\
&= \frac{\sum_{m=1}^{k-1} A_{m+1} A_{k-m} + A_1 A_k}{A_k(k^2+7k)} \\
&\leq \frac{R\sum_{m=1}^{k-1} A_m A_{k-m} + A_1 A_k}{A_k(k^2+7k)} \\
&\leq \frac{RA_k(k^2+5k-6) + A_1 A_k}{A_k(k^2+7k)} \\
&\leq \frac{R(k^2+5k-6) + A_1}{(k^2+7k)} \\
&\leq \frac{R(k^2+5k-2)}{(k^2+7k)} \\
&\leq R,
\end{aligned}$$

since $A_1 \leq 8R$. Thus $\left|\frac{A_{k+1}}{A_k}\right| \leq R$ for all $k$, so the power series converges. □

Lemma 46 provides conditions for the convergence of the bounding series found in Lemma 45. Hence we have actually proven the following:

**Theorem 47.** *Suppose $f(x) = \sum_{k=0}^{\infty} f_k(x)$ where the $f_k$ are given by (4.11). If there exists an $M > 0$, an $R > 0$, an $\alpha > 5$ such that (4.12) holds, and furthermore*

$$M < 8(\alpha+2)(\alpha-5)R^{\alpha-4}, \tag{4.16}$$

*then the series for $f(x)$ converges for all $x$ such that $|x - d| > R$.*

*Proof.* Combining Lemmas 45 and 46, we find that the key condition is that $A_1 \leq 8R$, which by substitution into (4.14) yields

$$0 < |K| + \frac{M}{(\alpha+2)(\alpha-5)R^{\alpha-5}} \leq 8R.$$

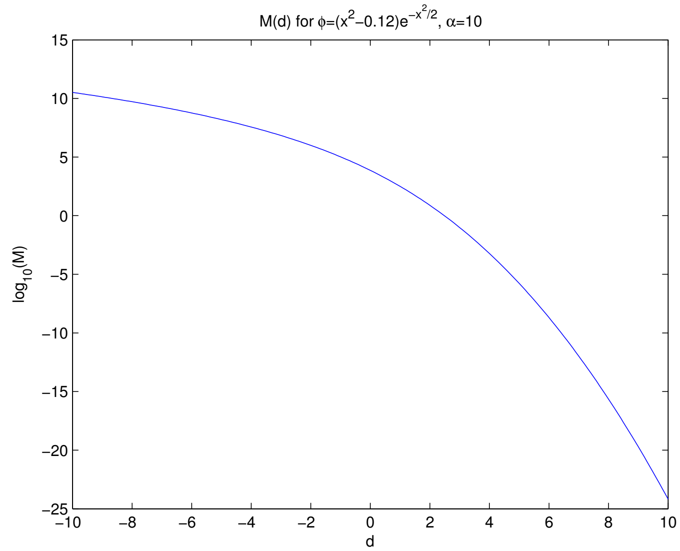


Figure 4.4: A typical $M(d)$ function

But in order to have $|K| \geq 0$, this gives

$$0 < 8R - \frac{M}{(\alpha+2)(\alpha-5)R^{\alpha-5}},$$

which leads immediately to the condition stated. $\square$

**Corollary 48.** *As an immediate consequence of Theorem 47, the action functional $A$ given by* (1.3) *evaluated at each equilibrium is finite provided the conditions* (4.12) *and* (4.16) *on $\phi$ hold.*

**Remark 49.** It is worth noting that if the spatial dimension $n > 1$ ((4.2) is now an elliptic *partial* differential equation), then the asymptotic decay rate will typically be slower than that of the series solution given here. As a result, Corollary 48 will not hold for higher spatial dimensions. Indeed, whether anything like Corollary 48 holds in higher spatial dimensions is an open question.

**Example 50.** It is important to notice that the $M$ defined above in Lemma 45 can depend crucially upon the value of $d$ and the shape of the curve $\phi(x)$. For the

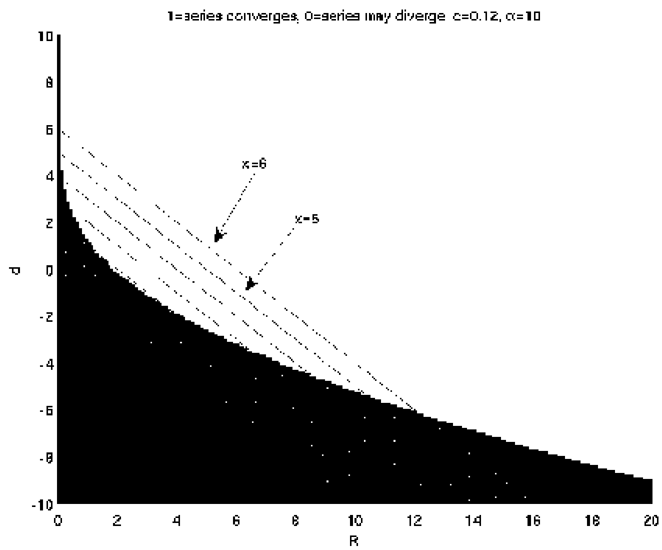


Figure 4.5: Series convergence test, for $\phi(x) = (x^2 - 0.12)e^{-x^2/2}$: white = series converges, black = series may diverge

case of $\phi(x) = (x^2 - c)e^{-x^2/2}$, a typical plot of $M(d)$ is shown in Figure 4.4. It should be noted that for various values of $c$, the $M(d)$ function is numerically very similar.

This also means that the condition (4.16) defines a somewhat complicated region over which parameters $d, K$ and $R$ yield convergent series solutions. An example with our given $\phi(x)$ function is shown in Figure 4.5. Thus it appears that our series solution converges if one goes out far enough, and specifies small enough initial conditions.

**Remark 51.** The convergence of the series solution is controlled by the convergence of a well-behaved power series. It follows that as the $\phi$ function becomes smaller, fewer terms in the series are needed to accurately approximate the solution. Indeed, each term in the series solution is asymptotically smaller than the previous one. Thus, we can gain some qualitative information from the leading two terms of the series, which are

$$f(x) \sim \frac{6}{(x-d)^2} + \frac{1}{(x-d)^3}\left[K + \int^x (t-d)^6 \int_t^\infty \frac{\phi(s)}{(s-d)^3} ds\, dt\right].$$

Taking a derivative by $x$ gives

$$f'(x) \sim \frac{-12}{(x-d)^3} + \frac{-3}{(x-d)^4}\left[K + \int^x (t-d)^6 \int_t^\infty \frac{\phi(s)}{(s-d)^3} ds\, dt\right] + \\ (x-d)^3 \int_x^\infty \frac{\phi(s)}{(s-d)^3} ds.$$

On the other hand, using the standard expansion for $(a+b)^{3/2}$, one obtains

$$\begin{aligned} f^{3/2}(x) &\sim \left(\frac{6^3}{(x-d)^6} + \frac{3\cdot 6^2}{(x-d)^7}\left[K + \int^x (t-d)^6 \int_t^\infty \frac{\phi(s)}{(s-d)^3} ds\, dt\right]\right)^{1/2} \\ &\sim \frac{6^{3/2}}{(x-d)^3} + \frac{(x-d)^3}{2\cdot 6^{3/2}} \frac{3\cdot 6^2}{(x-d)^7}\left[K + \int^x (t-d)^6 \int_t^\infty \frac{\phi(s)}{(s-d)^3} ds\, dt\right] \\ &\sim \frac{6^{3/2}}{(x-d)^3} + \frac{3\cdot 18}{6^{3/2}(x-d)^4}\left[K + \int^x (t-d)^6 \int_t^\infty \frac{\phi(s)}{(s-d)^3} ds\, dt\right] \end{aligned}$$

which leads to

$$f'(x) \sim -\sqrt{\frac{2}{3}} f^{3/2} + (x-d)^3 \int_x^\infty \frac{\phi(s)}{(s-d)^3} ds. \tag{4.17}$$

Notice that this equation depends only on $d$, not $K$. So from this we should expect that the initial data for solutions to be confined to a thin region in the plane $x = 0$. This will be confirmed in Theorem 55

Additionally, the relation $f' = -\sqrt{2/3} f^{3/2}$ holds exactly for the bounded solutions of $0 = f'' - f^2$. Indeed, in that case, the set $Z$ is $\{(f, f') | 3f'^2 = 2f^3, f' < 0\}$. So (4.17) indicates that the presence of $\phi \neq 0$ will deflect the set $Z$ largely in the $f'$ direction. This is exactly what we show in Section 4.6.

## 4.5 Restriction to $\phi$ nonnegative and monotonically decreasing

We now examine what stronger results can be obtained by requiring $\phi(x) \geq 0$ and $\phi'(x) < 0$ for all $x > 0$. This can be expected to provide stronger results, in particular because the region $R_1$ employed in Theorem 44 acquires a simpler inflow and outflow structure on the boundary, and in particular, solutions will exist for all $x > 0$. A collection of four results indicate that all bounded solutions to (4.3) lie within a narrow region.

**Lemma 52.** *Suppose $\phi(x) \geq 0$ and $\phi'(x) < 0$ for all $x \geq 0$. Then the region given by $R_1 = \{(f, f', x) | H(f, f', x) \geq 0, x \geq 0, f \leq \sqrt{\phi(x)}\}$ contains a bounded solution to* (4.3).

*Proof.* Following the proof of Theorem 44, we partition the boundary of $R_1$ into two pieces: $A = \{(f, f', x) | x = 0\}$ and $B = \{(f, f', x) | H(f, f', x) = 0\}$, noting that the flow of $V$ is inward along $A$. Reviewing the computation in Theorem 44, the flow is outward along all of $B$.

Now we employ the Antifunnel theorem, noting that while $A$ is simply-connected, $B$ is not. Hence they cannot be homeomorphic, and so there must be a solution that remains inside $R_1$ (which evidently starts on $A$). But the first coordinate of such an integral curve must obviously be bounded, since the $x$ cross-sections of $R_1$ form a decreasing sequence of sets, ordered by inclusion, and the cross-section for $x = 0$ is a bounded set. □

**Lemma 53.** *Suppose $\phi(x) \geq 0$ and $\phi'(x) < 0$ for all $x \geq 0$. Then the region given by $R_2 = \{(f, f', x) | H(f, f', x) \leq 0, \frac{1}{3}f^3 - \frac{1}{2}f'^2 \geq 0, x \geq 0, f' \leq 0\}$ contains a* *bounded solution to* (4.3)*.*

*Proof.* Partition the boundary of $R_2$ into two pieces:

$$A = \{(f, f', 0) | f' \leq 0\} \cup \{(f, f', x) | H(f, f', x) = 0, f \leq \sqrt{\phi(x)}, f' \leq 0\},$$

and

$$\begin{aligned} B = \{(f, f', x) | H(f, f', x) = 0, f \geq \sqrt{\phi(x)}, f' \leq 0\} \cup \\ \{(f, f', x) | \frac{1}{3} f^3 - \frac{1}{2} f'^2 = 0, f' \leq 0\}. \end{aligned}$$

By the calculation in Theorem 44, the flow along $A$ is inward-going. Additionally, the flow along the first connected component of $B$ is outward-going. Finally, we put $S(f, f', x) = \frac{1}{3} f^3 - \frac{1}{2} f'^2$ and observe that $\nabla S$ is an inward pointing normal vector field to $B$. We compute

$$\begin{aligned} \nabla S \cdot V &= \begin{pmatrix} f^2 \\ -f' \\ 0 \end{pmatrix}^T \begin{pmatrix} f' \\ f^2 - \phi(x) \\ 1 \end{pmatrix} \\ &= f' \phi(x) \leq 0, \end{aligned}$$

so the flow along this component of $B$ is outward-going. As a result, we can apply the Antifunnel theorem, noting that $A$ is connected, while $B$ is not. Therefore, there exists a solution to (4.3) that remains in $R_2$. Note that there is a lower bound on the $x$-coordinate of this solution, since the $x$-component of $V(f, f', x)$ is equal to 1, and the Region $R_2$ lies within the half-space $x > 0$. So this solution must enter $R_2$ through $A$, and then never intersect $B$. Additionally, notice that such a solution will have $f' \leq 0$ and $f \geq 0$, so it must be bounded. $\square$

**Lemma 54.** *Suppose $\phi(x) \geq 0$ and $\phi'(x) < 0$ for all $x \geq 0$. The complement of the set $A = R_1 \cup R_2$ consists of solutions which are unbounded, and blow up in finite $x$.*

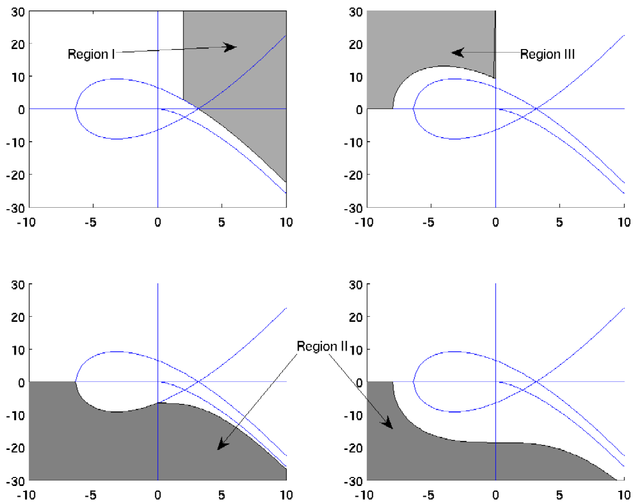


Figure 4.6: The Regions $I$, $II$, and $III$ of Lemma 54

*Proof.* Let the complement of the set $A$ be called $C$, namely $C = \{(f, f', x)|x > 0\} - A$. Now the calculations in Lemmas 52 and 53 show that $C$ is a funnel, in that the flow through the entire boundary of $C$ is inward. If $\phi$ does not tend to zero, then the argument in the proof of Theorem 38 completes the proof, as there is a tubular neighborhood about $\{f = f' = 0\}$ with strictly positive radius in which solutions in $C$ cannot remain. So without loss of generality, we assume $\phi \to 0$.

Define the Region $I$ by

$$I = \left\{(f, f', x)|f > \sqrt{\phi(x)} \text{ and } (f' > 0 \text{ or } H(f, f', x) > 0)\right\}.$$

There are two bounding faces of Region $I$, along which the flow is inward. The first is $S_1 = f - \sqrt{\phi(x)} = 0$, along which

$$\begin{aligned}
\nabla S_1 \cdot V(f, f', x) &= \begin{pmatrix} 1 \\ 0 \\ -\frac{\phi'(x)}{2\sqrt{\phi(x)}} \end{pmatrix}^T \begin{pmatrix} f' \\ f^2 - \phi(x) \\ 1 \end{pmatrix} \\
&= f' - \frac{\phi'(x)}{2\sqrt{\phi(x)}} > 0.
\end{aligned}$$

The second was computed already in the proof of Theorem 44. Notice that $f'' = f^2 - \phi(x) > 0$ in Region $I$, so $f(x)$ is concave-up, so solutions which enter Region $I$ are unbounded. Using similar reasoning to that of Theorem 38, such solutions blow up in finite $x$.

Now suppose we have a point $(a, a', x_0) \in C$ with $a' < 0$. We claim that for some $x_1 > x_0$, the integral curve through this point will cross the $f' = 0$ plane. To see this, construct Region $II$ by

$$II = \left\{ (f, f', x) | \frac{1}{3} f^3 - \frac{1}{2} f'^2 - \frac{1}{3} a^3 + \frac{1}{2} a'^2 \leq 0 \text{ and } f' \leq 0 \right\} \cap C.$$

Note that

$$\nabla \left( \frac{1}{3} f^3 - \frac{1}{2} f'^2 \right) \cdot V(f, f', x) = f' \phi(x) \leq 0,$$

so the flow is inward along Region $II$ except along $f' = 0$ (along which it is outward). Also note that Region $II$ excludes a tubular neighborhood of the line $f = f' = 0$ with strictly positive radius. As a result of this, the integral curve through $(a, a', x_0)$ proceeds at least as far as to allow $f < -\sqrt{\phi(x)}$, at which point, a finite amount of distance in $x$ takes it to $f' = 0$.

So at that point, the integral curve has entered Region $III$, say at $x = x_1$, where

$$III = \{(f, f', x) | H(f, f', x_1) \leq 0 \text{ and } f \leq 0 \text{ and } f' \geq 0\}.$$

The flow is evidently inward along $f' = 0$ and the curved portion by previous calculations, and outward along $f = 0$. Again, note that the line $f = f' = 0$ is excluded from Region $III$ by a tubular neighborhood of strictly positive radius, so there is an $x_2 > x_1$ where the integral curve exits Region $III$ through $f = 0$.

Now, consider a point $(0, c', x_2)$ along this integral curve with $c' > 0$. In this case, the flow moves such a point rightward. On the other hand, the left boundary of Region $I$ moves leftward, approaching $f = 0$. So there must be an $x_3 > x_2$ such that the integral curve through $(0, c', x_2)$ enters the Region $I$. Collecting our findings, we see that every point in $C$ has an integral curve which passes to Region $I$, and therefore corresponds to a solution which is unbounded, and blows up for some finite $x$. □

**Theorem 55.** *Suppose $\phi(x) \geq 0$ and $\phi'(x) < 0$ for all $x \geq 0$. The set $Z$ of initial conditions to* (4.3) *that lead to bounded solutions*

1. *lies within $A = R_1 \cup R_2$ and is*
2. *nonempty,*
3. *closed,*
4. *unbounded,*
5. *connected, and*
6. *simply connected.*
7. *Additionally, the portion of $Z$ corresponding to solutions that enter the interior of $R_2$ is a 1-dimensional submanifold of $\{(f, f', x) | x = 0\}$.*

*Proof.* 1. From Lemma 54, all bounded solutions must lie in $A$.

2. That there exist bounded solutions in $A$ is the content of Lemmas 52 and 53.

3. Now, put $A_0 = A \cap \{(f, f', 0)\}$ and $B_0 = \partial A - A_0$. Observe that from the proofs of the previous theorems, the flow of $V$ along $A_0$ is inward, and the flow along $B_0$ is outward. Since the last component of $V$ does not vanish, the flow of $V$ causes each point of $B_0$ to lie on an integral curve starting on $A_0$. This establishes a homeomorphism $\Omega$ from $B_0$ into a subset of $A_0$. In particular, $\Omega$ is an open map. Now every solution passing through $B_0$ is of course unbounded, so $Z = A_0 - \Omega(B_0)$ is evidently closed (it is the complement of an open set).

4. $B_0$ clearly has the topology of $\mathbb{R} \times [0, \infty)$, so $\pi_1(B_0) = 0$. Hence, $\pi_1(\Omega(B_0)) = 0$ also, but notice that $\Omega(B_0)$ contains $\partial A_0$. Suppose $Z$ were a bounded set. Then it is contained in some disk $D$. But $\partial D$ is homotopic to a loop in $A_0 - Z$, which either lies in $\text{int}(A_0 - Z)$ (in which case the homotopy need not move it) or in $\partial A_0$. But this means that the loop encloses all of $Z$, and so cannot be contractible in $\Omega(B_0)$, which contradicts the triviality of $\pi_1(\Omega(B_0))$. Hence $Z$ is unbounded.

5. We first show that the portion of $Z$ lying in the region $R_2$ satisfies the horizontal line test. First, note that a solution starting in $Z \cap R_2$ cannot exit $R_2$. For one, it cannot enter $R_1$, since $R_1$ is an antifunnel. Secondly, it cannot exit into $\mathbb{R}^3 - (R_1 \cup R_2)$ since solutions there are all nonglobal. Suppose that $f_1(0) \geq f_2(0) \geq 0$ and $f_1'(0) = f_2'(0)$ with $(f_1(0), f_1'(0))$ and $(f_2(0), f_2'(0))$ both in $Z \cap R_2$. But then

$$\begin{aligned} \frac{d}{dx}(f_1'(x) - f_2'(x)) &= f_1''(x) - f_2''(x) \\ &= f_1^2(x) - f_2^2(x) \geq 0, \end{aligned}$$

with equality only if $f_1(0) = f_2(0)$. Hence, $\frac{d}{dx}(f_1(x) - f_2(x)) \geq 0$ for $x > 0$, again with equality only if $f_1(0) = f_2(0)$. Now all solutions which remain in

$R_2$ are monotonic decreasing and bounded from below, so they must have limits. On the other hand, the only possible limit is $(0, \lim_{x\to\infty} \sqrt{\phi(x)})$, so therefore all bounded solutions in $R_2$ must have a common limit. Therefore, we must have that $f_1(0) = f_2(0)$. Now this means that the portion of $Z$ in the region $R_2$ can be realized as the graph of a function from the $f'$ coordinate to the $f$ coordinate. Therefore, if $Z$ were not connected, at least one component of $Z$ would be a bounded subset, which is a contradiction.

6. Finally, if $Z$ were not simply connected, the Jordan curve theorem gives that there are two (or more) path components to $\Omega(B_0) = A_0 - Z$, which contradicts the continuity of $\Omega$.

7. By the connectedness of $Z$ and the horizontal line test in $R_2$, the function from the $f'$ coordinate to the $f$ coordinate whose graph is $Z \cap \text{int } R_2$ must be continuous. Additionally, by the connectedness of $Z$ and the uniquenss of solutions to ODE, this implies that the rest of $Z$ whose solutions enter the interior of $R_2$ is also a 1-manifold.

$\square$

**Definition 56.** It is convenient to define, in addition to the initial condition set $Z$, other sets $Z_{x_0} \subset \{(f, f', x) | x = x_0\}$ such that any integral curve passing through a point in $Z_{x_0}$ exists for all $x > 0$. Similarly, one can define $Z'_{x_0}$.

**Remark 57.** If $\phi \to 0$ as $x \to \infty$, we conjecture that $Z$ acquires the structure of a 1-manifold with boundary. The series solution (4.11) is not valid at such a boundary of $Z$, since such a solution must remain in $R_1$ and therefore decays quicker than the leading coefficient of (4.11). Indeed, by analogy with the case where $\phi \equiv 0$, the leading term $f_0$ of the series solution would vanish, and the solution is then asymptotic to $-\int_x^\infty \int_t^\infty \phi(s) ds\, dt$.

All solutions in the form of the series solution (4.11) enter $R_2$, so a result of this theorem is that one of the two parameters $d$ or $K$ in the series solution is superfluous. Since $d$ parametrizes solutions when $\phi \equiv 0$, we conventionally take $K = 0$. Using this, (4.17) indicates that a good approximation (as $x_0 \to \infty$, locally near $f = f' = 0$) to the set $Z_{x_0}$ is the set

$$\{H(f, f') = 0\} = \{(f, f')|\frac{1}{3}f^3 = \frac{1}{2}f'^2\}.$$

**Remark 58.** If $\phi \to P > 0$ as $x \to \infty$, then it is not true that $Z$ is a 1-manifold (with boundary). Indeed, $Z$ has the structure of a 1-manifold attached to the teardrop-shaped set $M$ from Lemma 34.

## 4.6 Geometric properties of the initial condition set $Z$

**Lemma 59.** *Suppose $\phi(x) > 0$, $\phi'(x) < 0$ for all $x > 0$ and $\phi \to 0$ as $x \to \infty$. Then the set $Z$ intersects $\{(f, f', x)|f' = 0\}$.*

*Proof.* First, observe that $Z$ intersects the boundary of $R_1$ in $x = 0$, since we have by Lemmas 52 and 53 solutions entirely within $R_1$ and its complement. Using the fact that $Z$ is connected and the Jordan curve theorem, $Z$ must intersect the boundary of $R_1$ in the plane $x = 0$. This reasoning also applies for each $Z_{x_0}$ with $x_0 \geq 0$, so that we can find points in the intersections $Z_{x_0} \cap \partial R_1$ for each $x_0 \geq 0$. Also note that for the backwards flow associated to our equation (ie. the flow of $-V$), solutions which enter $R_1$ must exit through the plane $x = 0$. Hence there exists a sequence of points $\{F_n\} \subset Z$ with $F_n = (f_n, f_n', 0)$ such that the integral curve through $F_n$ passes through $G_n = (g_n, g_n', n) \in Z_n \cap \partial R_1$ for each integer $n \geq 0$.

Discern three cases:

1. If any $F_n$ are in Quadrants I or II, then since $Z$ is connected, it must intersect $\{f' = 0\}$.
2. If any $F_n$ are in Quadrant III, observe that the flow across the surface $S = \left\{(f, f', x) | \frac{1}{3} f^3 = \frac{1}{2} f'^2, f' \leq 0\right\}$ is right-to-left. Thus the integral curve must cross into Quadrant II on its way to $G_n$. Therefore, the set $Z$ cannot intersect the surface $S$, and so it must intersect $\{f' = 0\}$.
3. Assume all the $F_n$ lie in Quadrant IV. Observe that $\{F_n\}$ is a closed subset of $R_1 \cap \{x = 0\}$, which is compact. Hence some subsequence of $\{F_n\}$ must have a limit, say $F$. Since $Z$ is closed, $F \in Z$. But in the portion of $R_1$ lying in the $x = 0$ plane and in Quadrant IV, we have that

$$\frac{d}{dx} f' = f^2 - \phi < 0$$

and

$$\frac{d}{dx} f = f' < 0.$$

Hence $f'_n \geq g'_n$. But since $\phi \to 0$, $g'_n \to 0$, so $F$ lies on $\{f' = 0\}$.

□

**Lemma 60.** *Under the same hypotheses as Lemma 59, $Z$ also intersects the half plane $\{f = 0, f' > 0\}$.*

*Proof.* Using Lemma 59, we form a sequence $\{F_n\} \subset Z$ such that the integral curve through $F_n$ passes through $\{f' = 0, f \geq 0, x = n\}$ for each integer $n$. (This can be done without loss of generality, because if any integral curves pass through $\{f' = 0, f < 0\}$, then the proof is complete by connectedness of $Z$.) Note that this sequence is entirely contained within $R_1$ by Lemma 54.

Discern three cases:

1. There exists an $F_n$ in either of Quadrants II or III. The result follows by the connectedness of $Z$.
2. There exists $F_n$ in Quadrant IV. This cannot occur unless the integral curve through $F_n$ passes through Quadrant III since the flow along $\{f' = 0\}$ points inward into the portion of Quadrant IV inside $R_1$.
3. Otherwise, we assume $\{F_n\}$ is entirely contained within Quadrant I. In this case, note that
$$\frac{d}{dx} f' = f^2 - \phi < 0.$$
Hence the $f'$-coordinate of the integral curve through each $F_n$ is positive on the interior of Quadrant I. Hence
$$\frac{d}{dx} f = f' > 0,$$
so $f_n \leq g_n$. But $g_n \to 0$ since $\phi \to 0$, so any limit point of $\{F_n\}$ will have $f$-coordinate equal to zero. By the compactness of $R_1 \cap \{x = 0\}$ and the closedness of $Z$, this implies that $Z$ intersects $\{f = 0, f' > 0\}$.

□

**Lemma 61.** *Suppose $\phi(x) > 0$ for all $x > 0$, $\phi \to 0$ as $x \to \infty$, and that there exists an $x_0 \geq 0$ such that for all $x > x_0$, $\phi'(x) < 0$. Then the set $Z$ intersects $\{f = 0, f' > 0\}$.*

*Proof.* We follow the pattern of proving the existence of an intersection for an open interval in $x$ containing $x_0$, and then constructing an *a priori* estimate for the $f'$-coordinate of this intersection.

Apply Lemma 60 to $x_0$, we have that $Z_{x_0}$ intersects $\{f = 0, f' > 0\}$. Let $(0, f_0', x_0)$ lie in this intersection. Note that

$$\frac{d}{dx} f = f' > 0$$

and

$$\frac{d}{dx} f' = f^2 - \phi = -\phi < 0$$

when evaluated there. As a result, the integral curve passing through $(0, f_0', x_0)$ must pass through Quadrant II first, say for $x \in (x_1, x_0)$. Then evidently, $Z_{x_1}$ must intersect $\{f = 0, f' > 0\}$.

Now since $\phi(x) > 0$ between $x_1$ and $x_0$, and $[x_1, x_0]$ is compact, there is an open set in $\mathbb{R}^3$ containing the intersection of each $Z_x$ with $\{f = 0, f' > 0\}$ for each $x \in [x_1, x_0]$, such that in this open set $\frac{d}{dx} f' \leq K < 0$. As a result, $f_1' \geq f_0'$. Hence the $f'$-coordinate of the intersection point of $Z_x$ with $\{f = 0, f' > 0\}$ is decreasing with increasing $x$. (Since we have $f^2 - \phi > -\phi$, it is decreasing at a rate no faster than $\phi$. This implies that this intersection point has $f'$-coordinate no larger than $\int_0^{x_0} \phi(x) dx + f_0'$ at $x = 0$.) Now since solutions through $Z_{x_1}$ exist for all $x > 0$ by definition, this suffices to show that $Z$ intersects $\{f = 0, f' > 0\}$. □

**Remark 62.** The line of reasoning used in the third case of each of Lemmas 59 and 60 (and also in 61) fails if we try to continue $Z$ much farther. This is due to the nonmonotonicity of $df'/dx$ in Quadrants II and III. More delicate control of $\phi$ must be exercised to say more.

**Calculation 63.** Towards the end of the more delicate results mentioned in Remark 62, it is useful to know the maximum speed along integral curves on points in the region $R_1$ in the $f$- and $f'$-directions. By this we mean to compute for fixed

$x$ the maximum values of

$$\begin{cases} |f'| \text{ for the } f\text{-direction} \\ |f^2 - \phi(x)| \text{ for the } f'\text{-direction} \end{cases} \tag{4.18}$$

in $R_1$. The first is easy to maximize: we simply look for the maximum value of $f'$ in $R_1$, which is a maximum of

$$f' = \sqrt{\frac{2}{3}f^3 - 2f\phi(x) + \frac{4}{3}\phi^{3/2}(x)},$$

for $-2\sqrt{\phi(x)} \leq f \leq \sqrt{\phi(x)}$. This occurs at $f = -\sqrt{\phi(x)}$, and has the value of $\sqrt{8/3}\phi^{3/4}$. For the second part of (4.18), it is easy to see that the maximum is $3\phi(x)$. In summary,

$$\begin{cases} |f'| \leq \sqrt{\frac{8}{3}}\phi^{3/4}(x) \text{ for the } f\text{-direction} \\ |f^2 - \phi(x)| \leq 3\phi(x) \text{ for the } f'\text{-direction} \end{cases} \tag{4.19}$$

on $R_1$.

Using this calculation, we can impose a stronger bound on the decay of $\phi(x)$, and constrain the set $Z$ further.

**Lemma 64.** *Suppose $\phi(x) > 0$, $\phi'(x) < -D\frac{4\sqrt{2}}{k\sqrt{3}}\phi^{5/4}(x)$ for all $x > 0$ for some $0 < k < 1$ and $D > 1$. Then the set $Z$ is contained within $\{f \geq -k\sqrt{\phi(0)}\}$ and intersects each vertical and horizontal line in $\{f \geq 0\}$ exactly once, and intersects $\{f' = 0\}$ only once.*

*Proof.* That $Z$ intersects $\{f = 0, f' \geq 0\}$ and $\{f' = 0, f \geq 0\}$ at all follows from Lemmas 59 and 61. Now consider the region $A \subset R_1$ shown in Figure 4.7 and defined by

$$A = R_1 \cap \left(\{f' \geq 0, f \leq k\sqrt{\phi(x)}\} \cup \{f' \leq 0, 2f^3 \leq 3f'^2\}\right).$$

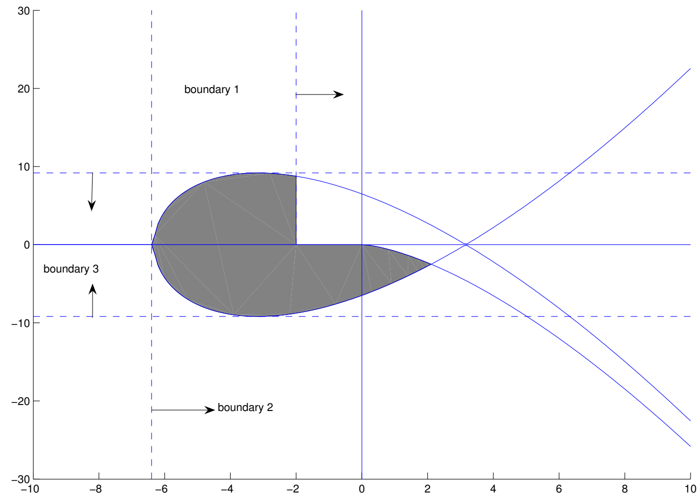


Figure 4.7: The region $A$ of Lemma 64

The boundary segments strictly to the right of the boundary labelled 1 in Figure 4.7 are evidently inflow, so long as $\phi > 0$. The boundary labelled as 1 in the figure moves with speed

$$\frac{d}{dx}(-k\sqrt{\phi(x)}) = \frac{-k}{2\sqrt{\phi(x)}}\phi'(x) > D\frac{2\sqrt{2}}{\sqrt{3}}\phi^{3/4}(x)$$

which is greater than maximum speed in the $f$-direction given in (4.19). This implies that the boundary moves faster than any solution inside $R_1$. Hence it is an inflow portion of the boundary. On the other hand, the curved segment of the boundary to the left has been shown to be outflow, in Lemma 52.

We observe that the boundary marked 2 in Figure 4.7 moves with speed

$$\frac{d}{dx}(-2k\sqrt{\phi(x)}) = \frac{-k}{\sqrt{\phi(x)}}\phi'(x),$$

which is strictly faster than the boundary marked 1 in Figure 4.7, and the boundary marked 3 in Figure 4.7 moves with speed

$$\frac{d}{dx}\left(\pm\sqrt{\frac{8}{3}}\phi^{3/4}(x)\right) = \pm\frac{\sqrt{3}}{\sqrt{2}\phi^{1/4}(x)}\phi'(x),$$

noting that $f'(-\sqrt{\phi(x)}) = \pm\sqrt{8/3}\phi^{3/4}(x)$ is the value of the maximum $f'$-coordinate of $R_1$ at a given $x$ value. This last speed is greater than the maximum speed in the $f'$-direction given by (4.19) since $\phi'(x) < -D\sqrt{6}\phi^{5/4}(x)$. (Notice that $\sqrt{6} < \frac{4\sqrt{2}}{k\sqrt{3}}$, since $0 < k < 1$.)

Since $D > 1$, this means that both the boundaries marked 2 and 3 in Figure 4.7 overtake any solution constrained to be within $R_1$. As a result, every solution within the region $A$ must leave it within finite $x$. But the only way to leave $A$ causes a solution to enter $\mathbb{R}^3 - (R_1 \cup R_2)$, so every solution which contains a point in $A$ cannot exist for all $x > 0$ by Lemma 54. Therefore, $Z$ is contained within $(R_1 \cup R_2) - A$.

Now consider the region $B$ which is defined by

$$B = R_1 \cap \left(\{f' \geq 0, f \leq 0\} \cup \{f' \leq 0, 2f^3 \leq 3f'^2\}\right),$$

which is simply the region $A$, with $k$ taken to be zero. The portion of the boundary of $B$ lying in the $\{f = 0\}$ plane is inflow. We can therefore apply the reasoning of the vertical line test: Suppose $(f_1, f_1', 0), (f_2, f_2', 0) \in Z$ with $f_1 = f_2 > 0$ and $f_1' \geq f_2'$. Then we have both (at $x = 0$)

$$\frac{d}{dx}(f_1' - f_2') = f_1^2 - f_2^2 = 0$$

and

$$\frac{d}{dx}(f_1 - f_2) = f_1' - f_2' \geq 0,$$

which gives that $f_1^2 - f_2^2 \geq 0$ for some open interval about $x = 0$. Then, $\frac{d}{dx}(f_1' - f_2') \geq 0$, which implies that in fact $\frac{d}{dx}(f_1 - f_2) \geq 0$. However, since all bounded solutions tend to the common limit of zero, we have that this implies $f_1' = f_2'$ at $x = 0$. (Note that since each solution starts in $Z \cap (R_1 - B)$, we have that neither solution can become negative, since that would involve entering $B \subset A$ or leaving $R_1 \cup R_2$.) This implies that there is a unique intersection of $Z$ with each vertical line. The same reasoning applies in the case of the horizontal line test, as in Theorem 55. $\square$

**Lemma 65.** *Suppose $\phi(x) > 0$ for all $x > 0$, and that $\phi'(x) < -D\frac{4\sqrt{2}}{k\sqrt{3}}\phi^{5/4}(x)$ for all $x > x_0 \geq 0$ for some $0 < k < 1$ and $D > 1$. Additionally, suppose that for all $x \in [0, x_0]$,*

$$x_0 - x < \frac{\sqrt{\phi(x)} - k\sqrt{\phi(x_0)}}{\sqrt{\frac{8}{3}}P^{3/4}}, \tag{4.20}$$

*where $P = \max_{x\in[0,x_0]} \phi(x)$. Then the set $Z$ is contained within $\{f \geq -\sqrt{\phi(0)}\}$ and intersects each vertical and horizontal line in $\{f \geq 0\}$ exactly once, and intersects $\{f' = 0\}$ only once.*

*Proof.* The set $Z_{x_0}$ is constrained to lie within the set $\{f' \geq -k\sqrt{\phi(x_0)}\}$, by Lemma 64 (replacing $x_0$ by zero). Now using the $f$-direction part of (4.19), the smallest $f$-value attained in $Z_x$ is

$$\int_{x_0}^{x} \sqrt{\frac{8}{3}}\phi^{3/4}(x)dx - k\sqrt{\phi(x_0)}.$$

If $x < x_0$, we have

$$\begin{aligned}\int_{x_0}^{x} \sqrt{\frac{8}{3}}\phi^{3/4}(x)dx - k\sqrt{\phi(x_0)} &\geq \sqrt{\frac{8}{3}}P^{3/4}(x - x_0) - k\sqrt{\phi(x_0)} \\ &> -\sqrt{\phi(x)},\end{aligned}$$

by (4.20). As a result, $Z_x \subset \{f \geq -\sqrt{\phi(x)}\}$ for each $x < x_0$. This additionally means that in the backwards flow, the entire portion of $Z_x$ contained in $\{f \leq 0\}$ is moving away from the plane $\{f' = 0\}$, which completes the proof. $\square$

**Remark 66.** The condition that $\phi'(x) < C\phi^{5/4}(x)$ implies

$$\begin{aligned} \phi^{-5/4}\phi'(x) &< C \\ -\frac{1}{4}\phi^{-1/4}(x) &< Cx + C' \\ \phi(x) &< \frac{C'''}{(C''-x)^4} \end{aligned}$$

for some $C''$ and $C'''$. Notice that this condition is satisfied when the series solution converges by Theorem 47.

## 4.7 Solutions on the entire real line

We now combine the results for (4.3) and (4.4) to discuss properties of the solutions to (4.2). When $\phi(x)$ is monotonically decreasing for $x \geq 0$, we have by Lemma 54 that the initial condition set stays within $R_1 \cup R_2$. In particular, $Z \subset \{f' \leq \sqrt{\frac{8}{3}}\phi^{3/4}(0)\}$. If we relax the restriction of monotonicity, we obtain a similar result.

**Lemma 67.** *If $f = f(x)$ is a bounded solution to the initial value problem* (4.3) *with $\phi \in C^\infty \cap L^\infty(\mathbb{R})$ then $f'(0) < \sqrt{8/3}\|\phi\|_\infty^{3/4}$.*

*Proof.* Since $f$ is a solution to (4.3), then it must satisfy

$$f'' = f^2 - \phi(x) \geq f^2 - \|\phi\|_\infty.$$

Now Lemma 34 shows that all bounded solutions to $g'' = g^2 - \|\phi\|_\infty$ lie in the closure of the set $M$ given by

$$M = \left\{(g, g') | \frac{1}{3}g^3 - \frac{1}{2}g'^2 - g\|\phi\|_\infty + \frac{2}{3}\|\phi\|_\infty^{3/2} > 0, g < \sqrt{\|\phi\|_\infty}\right\}.$$

Since this set $M$ is bounded, we can find the maximum value of $f'$, which is $f'_{max} = \sqrt{8/3}\|\phi\|_\infty^{3/4}$. ∎

**Lemma 68.** *Consider solutions to* (4.2) *on the real line, with* $\phi \in C_0^\infty \cap L^\infty(\mathbb{R})$. *If for some* $-\infty < A < B < \infty$,

$$-\int_A^B \phi(x)dx > \sqrt{\frac{8}{3}}\left( (\sup_{x\in(-\infty,A]} |\phi(x)|)^{3/4} + (\sup_{x\in[B,\infty)} |\phi(x)|)^{3/4} \right)$$

*then no bounded solutions exist.*

*Proof.* For a solution $f$, we have that $f'' = f^2 - \phi(x) \geq -\phi(x)$. Integrating both sides we have

$$f'(B) - f'(A) \geq -\int_A^B \phi(x)dx.$$

By Lemma 67, bounded solutions on

- $x > B$ have $f'(B) < \sqrt{\frac{8}{3}}(\sup_{x\in(-\infty,A]} |\phi(x)|)^{3/4}$, and
- on $x > A$, they have $f'(A) < (\sup_{x\in[B,\infty)} |\phi(x)|)^{3/4}$,

so a necessary condition for there to be a bounded solution is that

$$-\int_A^B \phi(x)dx \leq \sqrt{\frac{8}{3}}\left( (\sup_{x\in(-\infty,A]} |\phi(x)|)^{3/4} + (\sup_{x\in[B,\infty)} |\phi(x)|)^{3/4} \right).$$

□

**Corollary 69.** *A necessary condition for bounded solutions to* (4.2) *to exist if* $\phi \in C_0^\infty \cap L^\infty(\mathbb{R})$ *is* $\int_{-\infty}^\infty \phi(x)dx > 0$.

*Proof.* Suppose bounded solutions exist. By the proof of Lemma 68, if we let

$$g_n = -\int_{-n}^n \phi(x)dx,$$

and

$$h_n = \sqrt{\frac{8}{3}}\left( (\sup_{x\in(-\infty,-n]} |\phi(x)|)^{3/4} + (\sup_{x\in[n,\infty)} |\phi(x)|)^{3/4} \right),$$

then $g_n < h_n$ for each positive integer $n$. But the continuity of limits gives

$$-\int_{-\infty}^{\infty} \phi(x)dx = \lim_{n\to\infty} g_n < \lim_{n\to\infty} h_n = 0.$$

□

**Definition 70.** A function $\phi \in C_0^\infty \cap L^\infty(\mathbb{R})$ will be called *M-shaped* if there exists an $x_0 > 0$ such that for all $|x| > x_0$, $\phi(x) > 0$ and

- $\phi$ is monotonic increasing for $x < -x_0$ and
- $\phi$ is monotonic decreasing for $x > x_0$.

**Theorem 71.** *Suppose $\phi$ is a positive M-shaped function, then solutions exist to* (4.2).

*Proof.* Observe that by Lemma 61, we have that the set $Z$ intersects $\{f = 0, f' > 0\}$. Additionally, by Theorem 55, we have that $Z$ also lies in $R_2$, which is unbounded in Quadrant IV. Likewise, the set $Z'$ (for (4.4)) intersects $\{f = 0, f' < 0\}$, and becomes unbounded in Quadrant I, so $Z \cap Z'$ must be nonempty, and at least one point in this intersection is in the half-plane $\{x = 0, f > 0\}$. □

**Theorem 72.** *Suppose $\phi$ is a positive M-shaped function which additionally satisfies the decay constraints of Lemma 65 for $x > 0$ and $x < 0$ separately, then a unique positive solution exists to* (4.2). *(Note that for $x < 0$, the inequalities and signs in Lemma 65 must be reversed, mutatis mutandis.)*

*Proof.* By the Theorem 71, there exist solutions to (4.2), one of which comes from the intersection of $Z \cap Z'$ in the half-plane $\{x = 0, f > 0\}$. The vertical-line test in Lemma 65 allows one to conclude that the solution which passes through that

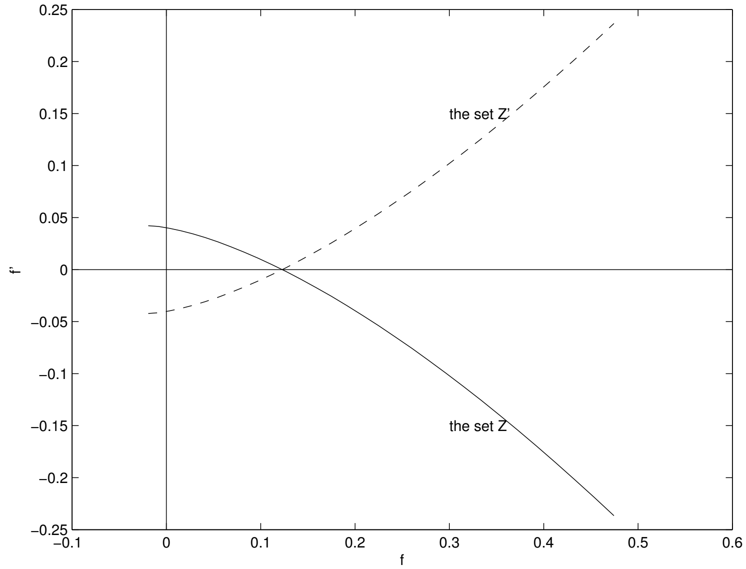


Figure 4.8: The sets $Z$ and $Z'$ in Example 73

half-plane must continue directly to the region $R_2$ of Lemma 53, without crossing the plane $\{f = 0\}$. Thus this solution is strictly positive.

On the other hand, Lemma 65 indicates that $Z$ may lie only in Quadrants I, II, and IV, while the set $Z'$ must lie in Quadrants I, III, and IV. On the other hand, the vertical- and horizontal-line tests ensure a unique intersection of $Z$ and $Z'$ in Quadrants I and IV, so the solution is unique. $\square$

**Example 73.** We examine the family $\phi_c(x) = ce^{-x^2/2}$, which is M-shaped when $c \geq 0$. Notice that when $c < 0$, then the necessary condition of Corollary 69 is not met, so solutions do not exist for all $x \in \mathbb{R}$. When $c = 0$, then the trivial solution $f = 0$ is the only solution. For $c > 0$, we examine $\phi_c'(x) = -xce^{-x^2/2}$. Figure 4.8 shows the sets $Z$ and $Z'$ for the case when $c = 0.05$. In particular, one notes that there appears to be a unique point of intersection.

We find the $x_0$ for which larger $x$ satisfy $\phi'(x) < -4\sqrt{2}\phi^{5/4}(x)/(k\sqrt{3})$:

$$\begin{aligned} -xce^{-x^2/2} &< -\frac{4\sqrt{2}}{k\sqrt{3}}c^{5/4}e^{-5x^2/8} \\ xe^{x^2/8} &> \frac{4\sqrt{2}}{k\sqrt{3}}c^{1/4}, \end{aligned}$$

which occurs if $x > \frac{4\sqrt{2}}{k\sqrt{3}}c^{1/4}$, so we may take $x_0 = \frac{4\sqrt{2}}{k\sqrt{3}}c^{1/4}$.

By way of example, if we fix $x_0 = 4/3$, then $k = \sqrt{6}c^{1/4}$. (We enforce $0 < k < 1$ by taking $c$ small.) Now we must check to see if (4.20) holds. In this case, we need to see if $c$ can be chosen so that $x_0 - x = 4/3 - x$ is bounded above by

$$\begin{aligned} \frac{\sqrt{\phi(x)} - k\sqrt{\phi(x_0)}}{\sqrt{\frac{8}{3}}P^{3/4}} &= \frac{\sqrt{c}e^{-x^2/4} - \sqrt{6}c^{3/4}e^{-16/36}}{\sqrt{8/3}c^{3/4}} \\ &= \frac{e^{-x^2/4} - \sqrt{6}c^{1/4}e^{-16/36}}{\sqrt{8/3}c^{1/4}} \\ &\geq \frac{e^{-16/36} - \sqrt{6}c^{1/4}e^{-16/36}}{\sqrt{8/3}c^{1/4}}, \end{aligned}$$

which can be made as large as one likes by taking $c$ sufficiently small. Noting that this last line is a constant in $x$ completes the bound. Therefore, there is a unique positive solution for $0 = f'' - f^2 + ce^{-x^2/2}$ with $c \in [0, \epsilon)$ for some $\epsilon > 0$.

**Remark 74.** Taken together, the results of Corollary 69 and Theorems 71 and 72 for M-shaped $\phi$ provide the following story about solutions to the equation $0 = f'' - f^2 + \phi$ on the real line:

- If the portion of $\phi$ where it is allowed to be negative is sufficiently negative, then no solutions exist,
- If $\phi$ is positive, then a solution will exist. There is no particular reason to believe that this solution will be strictly positive or unique.
- If the decay in the monotonic portions of $\phi$ is fast enough, there is exactly one solution, which is strictly positive.

## 4.8 Numerical examination

### 4.8.1 Computational framework

Notice that the results of Remark 74 are not sharp: nothing is said if $\phi$ has a portion which is negative, but still satisfies the necessary condition of Corollary 69. Further, if $\phi$ is positive, but does not satisfy the decay rate conditions, nothing is said about the number of global solutions that exist. Answers to these questions can be obtained by combining the asymptotic information we have collected about the sets $Z$ and $Z'$ with a numerical solver. In particular, we can obtain information about the number of global solutions to (4.2) for any M-shaped $\phi$.

Implicit in the use of a numerical solver is the following Conjecture:

**Conjecture 75.** There are only finitely many smooth global solutions to (4.2).

Suppose that $\phi$ is an M-shaped function, and that $x_0$ is such that $\phi(x)$ is monotonic decreasing for all $x > x_0$ and is monotonic increasing for all $x < -x_0$. (If $\phi$ decreases fast enough, we can choose $x_0$ so that the series solution converges on the complement of $(-x_0, x_0)$ for sufficiently small initial conditions.) Then we have the sets $Z'_{-x_0}$ and $Z_{x_0}$ of initial conditions to ensure existence of solutions on $(-\infty, -x_0]$ and $[x_0, \infty)$ respectively. Then any solution to the boundary value problem

$$\begin{cases} 0 = f''(x) - f^2(x) + \phi(x) \text{ for } -x_0 < x < x_0 \\ (f(-x_0), f'(-x_0)) \in Z'_{-x_0} \\ (f(x_0), f'(x_0)) \in Z_{x_0} \end{cases} \tag{4.21}$$

extends to a global solution of (4.2). So all one must do is solve (4.21) numerically.

An easy way to do this is to numerically extend the sets $Z'_{-x_0}$ and $Z_{x_0}$ to $Z'$ and $Z$ respectively (ie. extend them to the plane $x = 0$) and compute $Z' \cap Z$.

In order to analyze (4.2) numerically, it is necessary to make a choice of $\phi$. Evidently, the numerical results for that particular choice of $\phi$ cannot be expected to apply in general. However, a good choice of $\phi$ will suggest features in the solutions that are common to a larger class of $\phi$. We shall use

$$\phi(x; c) = (x^2 - c)e^{-x^2/2}, \tag{4.22}$$

where $c$ is taken to be a fixed parameter. (See Figure 4.9) This choice of $\phi$ has the following features which make for interesting behavior in solutions to (4.2):

- $\phi(x; c) > 0$ for $c < 0$. In this case, there are solutions to (4.2), by Theorem 71. On the other hand, the decay rate conditions are not met over all of $\mathbb{R}$ so the uniqueness result of Theorem 72 does not apply. Inded, the decay rate conditions are met only for sufficiently large $|x|$, but not for $|x|$ small.
- If $c > 0$ is large enough, it should happen that no solutions to (4.2) exist, since the necessary condition of Corollary 69 is not met. Indeed, the integral of $\phi$ vanishes when $c = 1$.

### 4.8.2 Bifurcations in the global solutions

Once computed, the numerical solutions can then be tabulated conveniently in a bifurcation diagram. That is, consider the set in $\mathbb{R}^3$ given by $(c, f(0), f'(0))$ for each solution $f$. Evidently, by existence and uniqueness for ordinary differential equations, each solution can be uniquely represented by such a point. The results

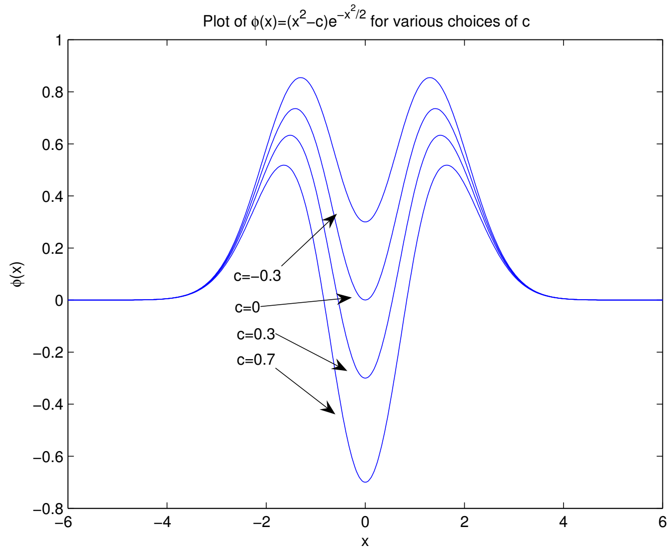


Figure 4.9: The function $\phi(x;c)$ for various $c$ values

of such a computation are shown in Figure 4.10. In this diagram, the solutions are color-coded by the number of positive eigenvalues of $\frac{d^2}{dx^2} - 2f$ as an operator $C^2(\mathbb{R}) \to C^0(\mathbb{R})$, which will be shown in Chapter 7 to be the dimension of the unstable manifold of an equilibrium solution $f$. (It should be noted that the green curve continues for $c < -1.2$, but was stopped for display reasons.)

Considering the bifurcation diagram, it appears to indicate that (4.3) undergoes a saddle node bifurcation at approximately $c = 0.7706$, and a subcritical pitchfork bifurcation at $c = 0.0501$. The results agree with Theorem 71, in that solutions do exist when $c < 0$. The saddle node bifurcation was anticipated by the general shape of $\phi$. For $c > 0.7706$, global solutions do not exist, which was qualitatively predicted by Corollary 69.

However, there are some stranger features of the bifurcation diagram. Most prominently, the bifurcation diagram appears simply to *end* near $c = -0.4652$, and at each branch of the pitchfork at $c = 0.0740$. It is important to verify that

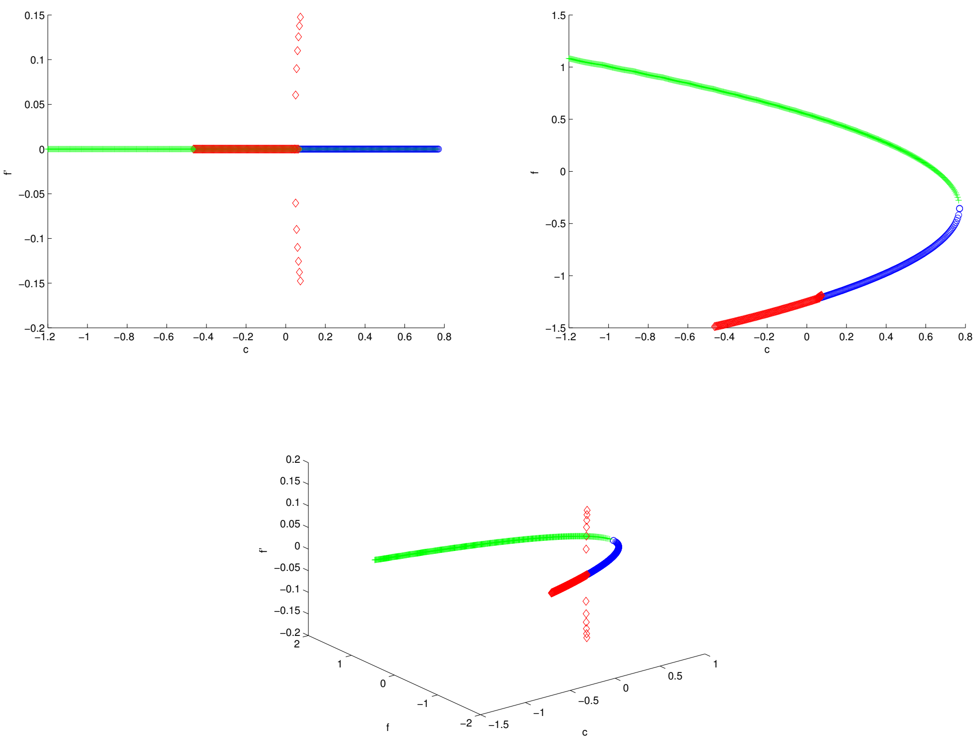


Figure 4.10: Bifurcation diagram, coded by spectrum of $\frac{d^2}{dx^2} - 2f$: green = nonpositive spectrum, blue = one positive eigenvalue, red = two positive eigenvalues

these are not numerical or discretization errors. If these *ends* are to be thought of as valid bifurcations, very likely, $\frac{d^2}{dx^2} - 2f$ acquires a zero eigenvalue there. Plotting the smallest magnitude eigenvalue gives some credence to this possibility. (See Figure 4.12)

As another check, one can measure the size of the existence interval for solutions to (4.2), centered at $x = 0$. Looking in the $(c, f(0))$-plane (taking $f'(0) = 0$), one can find the first $x$ such that the solution exceeds a particular value. This is shown in Figure 4.13, in which one sees the same general shape as in the bifurcation diagram. (The jagged nature of the graph along the actual bifurcation diagram is due to aliasing.) However, for $c < -0.4652$, the lower branch clearly continues

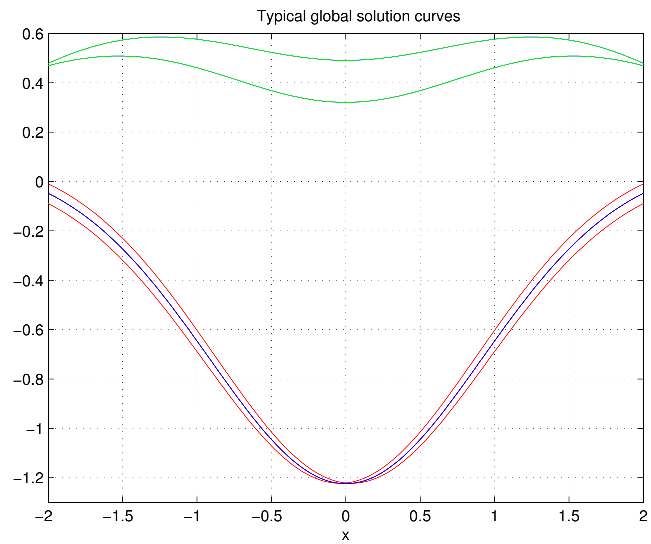


Figure 4.11: Typical global solutions: green are from the positive branch, the blue one is taken from the lower branch with $f'(0) = 0$, and the red ones are from the fork arms

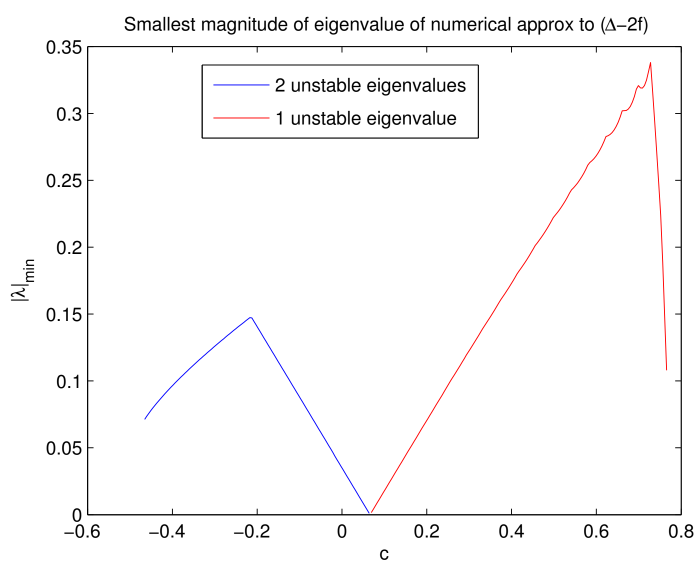


Figure 4.12: Smallest-magnitude eigenvalue measured along the lower branch with $f'(0) = 0$

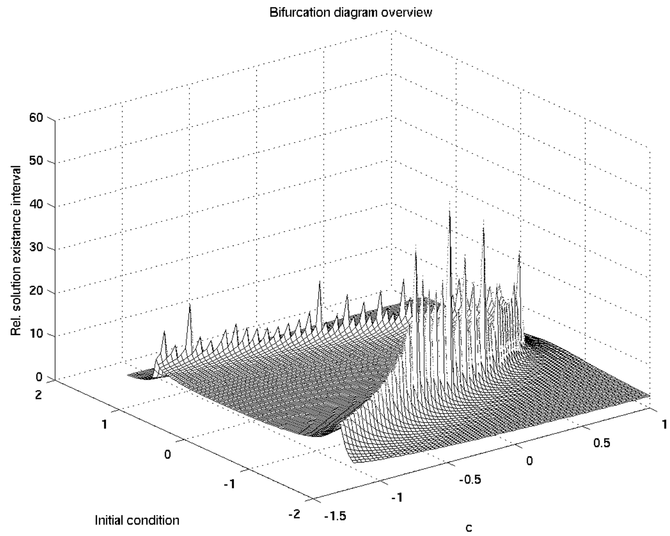


Figure 4.13: Estimate of existence interval length

into solutions that exist for only finite $x$. So the end bifurcation indicates a failure of the solutions to (4.2) to exist for all $x$.

From the point of view of (4.1) (the parabolic problem), the end bifurcations indicate that the equilibria are degenerate in the sense of Morse. It is easy to construct a 1-parameter family of 2-dimensional flows for which end bifurcations occur. The resulting equilibrium solutions in that case always acquire a center manifold – essentially a zero eigenvalue as noted above. In an infinite-dimensional flow, however, equilibria can be degenerate without having a center manifold. This is a manifestation of the fact that infinite-dimensional spaces are not locally compact. In Chapter 6, we show that in fact all equilibrium solutions are asymptotically unstable. However, in Chapter 7, we find that all equilibria have finite-dimensional unstable manifolds whose dimension is determined by the dimension of the positive eigenspace of $\frac{d^2}{dx^2} - 2f$. In particular, there are equilibria whose unstable manifolds are empty, yet they are unstable.

## 4.9 Conclusions

In this chapter, an approach for counting and approximating global solutions to a nonlinear, nonautonomous differential equation was described that combines asymptotic and numerical information. The asymptotic information alone is enough to give necessary and sufficient (but not sharp) conditions for solutions to exist, and provides a fairly weak uniqueness condition. More importantly, the asymptotic approximation can be used to supply enough information to pose a boundary value problem on a bounded interval containing a smaller interval where asymptotic approximation is not valid. This boundary value problem is well-suited for numerical examination, and the combined approach yields much more detailed results than either method alone.

The techniques and results of this chapter should apply to more general kinds of differential equations. Indeed, there should be no particular obstruction to extending any of the analysis to equations of the form

$$0 = f''(x) - f^N(x) + \phi(x),$$

where $N > 2$. Regions $R_1$ and $R_2$ are then relatively easy to construct, and similar results hold for them as are shown here. Additionally, the asymptotic series for large $x$ can be obtained since $0 = f'' - f^N$ has easily-found explicit solutions.

There is considerably more difficulty in trying to understand solutions to equations like

$$0 = f''(x) + G(f) + \phi(x), \tag{4.23}$$

where $G$ is some polynomial with $G(0) = 0$. In this more general setting, explicit solutions to $0 = f'' + G(f)$ are significantly harder to find and work with. There is also no reason to expect that Theorem 38 will hold, since $G$ may have several zeros.

As a result, the asymptotic analysis becomes essentially unavailable. Therefore, the only remaining tool is Ważewski's antifunnel theorem, for which one still needs a good description of the asymptotic behavior of solutions.

CHAPTER 5

# EXISTENCE OF NONTRIVIAL ETERNAL SOLUTIONS

## 5.1 Introduction

The existence of eternal solutions poses a potentially difficult problem, because the backward-time Cauchy problem is well known to be ill-posed. Obviously, equilibrium solutions are trivial examples of such eternal solutions, and in Chapter 4 we showed that they exist. It is not at all clear that there are other eternal solutions, and indeed there may not be. In this chapter we use a pair of nonintersecting equilibrium solutions to construct a heteroclinic orbit which connects them. Therefore, the set of heteroclinic orbits is generally nonempty.

As has been done in previous chapters, we will work with the more limited equation (1.6)

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - u^2(t,x) + \phi(x), \tag{5.1}$$

where $\phi$ is a certain smooth function which decays to zero. In particular, when there are at least two equilibria whose difference is never zero, there exist nonequilibrium eternal solutions to (5.1). By an eternal solution, we mean a classical solution that is defined for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$. We follow the general technique for constructing "ancient solutions," which was used in a different context by Perelman.

## 5.2 Equilibrium solutions

We choose $\phi(x) = (x^2 - 0.4)e^{-x^2/2}$. It has been shown in Chapter 4 (see Figure 4.10), that in this situation, there exists a pair of equilibrium solutions $f_+, f_-$ with the following properties:

1. $f_+$ and $f_-$ are smooth and bounded,
2. $f_+$ and $f_-$ have bounded first and second derivatives,
3. $f_+$ and $f_-$ are asymptotic to $6/x^2$ for large $x$, and so both belong to $L^1(\mathbb{R})$,
4. $f_+(x) > f_-(x)$ for all $x$,
5. there is no equilibrium solution $f_2$ with $f_+(x) > f_2(x) > f_-(x)$ for all $x$,

and additionally, there exists a one-parameter family $g_c$ of solutions to

$$0 = g_c''(x) - g_c^2(x) + \phi_c(x) \tag{5.2}$$

with

1. $c \in [0, 1)$,
2. $g_0 = f_-$ and $\phi_0 = \phi$,
3. $\phi_a(x) < \phi_b(x)$ and $g_a(x) > g_b(x)$ for all $x$ if $a > b$.

The latter set of properties can occur as a consequence of the specific structure of $f_-$. For instance, consider the following result.

**Proposition 76.** *Suppose $f_- \in C^{2,\alpha}(\mathbb{R})$ satisfies the above conditions and additionally, there is a compact $K \subset \mathbb{R}$ with nonempty interior such that $f_-$ is negative on the interior of $K$ and is nonnegative on the complement of $K$. Then such a family $g_c$ above exists.*

*Proof.* (Sketch) Work in $T_{f_-}C^{2,\alpha}(\mathbb{R})$, the tangent space at $f_-$. Then (5.2) becomes its linearization (for $h_c$, say), namely

$$0 = h_c''(x) - 2f_-(x)h_c(x) + (\phi_c - \phi). \tag{5.3}$$

Consider the slighly different problem,

$$0 = y''(x) - 2f_-(x)y(x) + v(x)y(x), \tag{5.4}$$

where $v$ is a smooth function to be determined. If we can find a $v \leq 0$ such that $y > 0$ and $y \to 0$ as $|x| \to \infty$, then we are done, because we simply let $vy = \phi_c - \phi$ in (5.3). In that case, $h_c = y$ has the required properties. We sketch why such a $v$ exists:

- If $v \equiv 0$, then $y \equiv 0$ is a solution, giving $g_c = f_-$ as a base case.
- If $v(x) = -2\|u\|_\infty \beta(x)$ for $\beta$ is a smooth bump function with compact support and $\beta|K = 1$, then the Sturm-Liouville comparison theorem implies that $y$ has no sign changes. We can take $y$ strictly positive. However, in this case, the Sturm-Liouville theorem imples that ther are no critical points of $y$ either, so $y$ may not tend to zero as $|x| \to \infty$.
- Hence there should exist an $s$ with $0 < s < 2\|u\|_\infty$ such that if $v(x) = -s\beta(x)$, then $y$ has no sign changes, one critical point, and tends to zero as $|x| \to \infty$. This choice of $v$ is what is required. (The precise details of this argument fall under standard Sturm-Liouville theory, which are omitted here.)

□

In what follows, we shall not be concerned with the exact form of $\phi$, but rather we shall assume that the above properties of the equilibria hold. Many other choices of $\phi$ will allow a similar construction.

**Lemma 77.** *The set*

$$W = \{v \in C^2(\mathbb{R}) | f_-(x) < v(x) < f_+(x) \textit{ for all } x\} \tag{5.5}$$

*is a forward invariant set for* (5.1)*. That is, if $u$ is a solution to* (5.1) *and $u(t_0) \in W$, then $u(t) \in W$ for all $t > t_0$.*

*Proof.* We show that the flow of (5.1) is inward whenever a timeslice is tangent to either $f_-$ or $f_+$. To this end, define the set $B$

$$\begin{aligned} B &= \{v \in C^2(\mathbb{R}) | f_-(x) \leq v(x) \leq f_+(x) \text{ for all } x, \text{ and there exists an } x_0 \\ &\quad \text{such that } v(x_0) = f_+(x) \text{ or } v(x_0) = f_-(x)\}. \end{aligned}$$

Without loss of generality, consider a $v \in B$ with a single point of tangency, $v(x_0) = f_-(x_0)$. At such a point $x_0$, the smoothness of $v$ and $f_-$ implies that $\Delta v(x_0) \geq \Delta f_-(x_0)$ using the maximum principle. Then, if $u$ is a solution to (5.1) with $u(0, x) = v(x)$, we have that

$$\begin{aligned} \frac{\partial u(0, x_0)}{\partial t} &= \Delta v(x_0) - v^2(x_0) + \phi(x_0) \\ &\geq \Delta f_-(x_0) - f_-^2(x_0) + \phi(x_0) = 0, \end{aligned}$$

hence the flow is inward. One can repeat the above argument for each point of tangency, and for tangency with $f_+$ as well. $\square$

**Lemma 78.** *Solutions to the Cauchy problem*

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - u^2(t, x) + \phi(x), \\ u(0, x) = U(x) \in W_c \end{cases} \tag{5.6}$$

*where*

$$W_c = \{v \in C^2(\mathbb{R}) | g_c(x) < v(x) < f_+(x) \textit{ for all } x\}$$

*for $c \in [0, 1)$ have the property that they lie in $L^1 \cap L^\infty(\mathbb{R})$ for all $t > 0$. We shall assume that $U$ has bounded first and second derivatives.*

*Additionally, when $c \in (0,1)$, solutions to (5.6) cannot have $f_-$ as a limit as $t \to \infty$.*

*Proof.* The fact that solutions lie in $L^1 \cap L^\infty(\mathbb{R})$ is immediate from Lemma 77 and the asymptotic behavior of $f_+, f_-$ (Section 4.4). Observe that for each $c \in [0,1)$, $W_c$ is forward invariant, and that $W_a \subset W_b$ if $a > b$. Since $f_-$ is not in $W_c$ for $c$ strictly larger than 0, the proof is completed. □

The following is an outline for the rest of the chapter. All solutions to (5.6) have bounded first and second spatial derivatives. This implies that all of their first partial derivatives are bounded (the time derivative is controlled by (5.1)). Using the fact that (5.1) is autonomous in time, time translations of solutions are also solutions. We therefore construct a sequence of solutions $\{u_k\}$ to Cauchy problems started at $t = 0, T_1, T_2, ...$ which tend to $f_+$ as $t \to +\infty$, but their initial conditions tend to $f_-$ as $k \to \infty$. By Ascoli's theorem, this sequence converges uniformly on compact subsets to a continuous eternal solution.

## 5.3 Integral equation formulation

In order to estimate the derivatives of a solution to (5.6), it is more convenient to work with an integral equation formulation of (5.6). This is obtained in the usual way.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u - u^2 + \phi \\
\left(\frac{\partial}{\partial t} - \Delta\right) u &= -u^2 + \phi \\
u &= \left(\frac{\partial}{\partial t} - \Delta\right)^{-1} (\phi - u^2)
\end{aligned}$$

$$u(t,x) = \int_{-\infty}^{\infty} H(t, x-y) U(y) dy + \int_0^t \int_{-\infty}^{\infty} H(t-s, x-y) \left(\phi(y) - u^2(s,y)\right) dy\, ds, \tag{5.7}$$

where $H(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ is the usual heat kernel.

**Calculation 79.** We begin by estimating the first derivative of $u$ for a short time. Let $T > 0$ be given, and consider $0 \le t \le T$. The key fact is that $\int H(t,x)dx = 1$ for all $t$. Using (5.7)

$$\begin{aligned}
\left\|\frac{\partial u}{\partial x}\right\|_\infty &\le \left\|\frac{\partial U}{\partial x}\right\|_\infty + \left|\int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial x} H(t-s, x-y) \left(\phi(y) - u^2(s,y)\right) dy\, ds\right| \\
&\le \left\|\frac{\partial U}{\partial x}\right\|_\infty + \int_0^t \int_{-\infty}^{\infty} \left|\frac{\partial}{\partial y}(H(t-s, x-y)) \left(\phi(y) - u^2(s,y)\right)\right| dy\, ds \\
&\le \left\|\frac{\partial U}{\partial x}\right\|_\infty + \int_0^t \int_{-\infty}^{\infty} \left|H(t-s, x-y) \left(\frac{\partial \phi}{\partial y} - 2u\frac{\partial u}{\partial y}\right)\right| dy\, ds \\
&\le \left\|\frac{\partial U}{\partial x}\right\|_\infty + T\left\|\frac{\partial \phi}{\partial x}\right\|_\infty + 2\|u\|_\infty \int_0^t \left\|\frac{\partial u}{\partial x}\right\|_\infty ds.
\end{aligned}$$

This integral equation fence is easily solved to give

$$\begin{aligned}
\left\|\frac{\partial u}{\partial x}\right\|_\infty &\le \left(\left\|\frac{\partial U}{\partial x}\right\|_\infty + T\left\|\frac{\phi}{\partial x}\right\|_\infty\right) e^{2t \max\{\|f_+\|_\infty, \|f_-\|_\infty\}} \\
&\le K_1 e^{K_2 T}.
\end{aligned}$$

**Calculation 80.** With the same choice of $T$ as above, we find a bound for the second derivative in the same way:

$$
\begin{aligned}
\left\|\frac{\partial^2 u}{\partial x^2}\right\|_\infty &\le \left\|\frac{\partial^2 U}{\partial x^2}\right\|_\infty + T\left\|\frac{\partial^2 \phi}{\partial x^2}\right\|_\infty + \int_0^t \left\|\frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y}\right)\right\|_\infty ds \\
&\le \left\|\frac{\partial^2 U}{\partial x^2}\right\|_\infty + T\left\|\frac{\partial^2 \phi}{\partial x^2}\right\|_\infty + \int_0^t 2\left\|\frac{\partial u}{\partial x}\right\|_\infty^2 + 2\|u\|_\infty \left\|\frac{\partial^2 u}{\partial x^2}\right\|_\infty ds \\
&\le K_3 e^{K_2 T}
\end{aligned}
$$

for some $K_3$ which depends on $U$, $\phi$, and $T$.

**Calculation 81.** Now, we extend Calculation 80 to handle $t > T$,

$$
\begin{aligned}
\left\|\frac{\partial^2 u}{\partial x^2}\right\|_\infty &\le \left\|\frac{\partial^2 U}{\partial x^2}\right\|_\infty + \left|\frac{\partial^2}{\partial x^2}\int_0^T\int_{-\infty}^{\infty} H(t-s,x-y)(\phi(y)-u^2(s,y))dy\,ds\right| + \\
&\quad \left|\frac{\partial^2}{\partial x^2}\int_T^t\int_{-\infty}^{\infty} H(t-s,x-y)(\phi(y)-u^2(s,y))dy\,ds\right| \\
&\le K_3 e^{K_2 T} + \int_T^t \left\|\frac{\partial^2}{\partial x^2}H(t-s,x)\right\|_\infty (\|\phi\|_1 + \|u^2\|_1)ds \\
&\le K_3 e^{K_2 T} + K_4\int_T^t \frac{1}{s\sqrt{s}}ds + K_4'\int_T^t \frac{1}{s^2\sqrt{s}}ds \\
&\le K_3 e^{K_2 T} + K_5\left(\frac{1}{\sqrt{T}} - \frac{1}{\sqrt{t}}\right) + K_5'\left(\frac{1}{T\sqrt{T}} - \frac{1}{t\sqrt{t}}\right) \\
&\le K_3 e^{K_2 T} + K_6,
\end{aligned}
$$

hence there is a uniform upper bound on $\left\|\frac{\partial^2 u}{\partial x^2}\right\|_\infty$ which depends only on the initial conditions, $\phi$, and $T$.

**Lemma 82.** *Let $f \in C^2(\mathbb{R})$ be a bounded function with a bounded second derivative. Then the first derivative of $f$ is also bounded, and the bound depends only on $\|f\|_\infty$ and $\|f''\|_\infty$.*

*Proof.* The proof is elementary. The key fact is that at its maxima and minima, $f$ has a horizontal tangent. From a horizontal tangent, the quickest $f'$ can grow is at a rate of $\|f''\|_\infty$. However, since $f$ is bounded, there is a maximum amount that this growth of $f'$ can accrue. Indeed, a sharp estimate is

$$\|f'\|_\infty \leq \sqrt{2\|f\|_\infty \|f''\|_\infty}.$$

□

Using the fact that $u$ is bounded, Lemma 82 implies that the first spatial derivative of $u$ is bounded. By (5.1), it is clear that the first time derivative of $u$ is also bounded.

**Lemma 83.** *As an immediate consequence of Lemmas 78 and 82, the action integral*

$$A(u(t)) = \int_{-\infty}^{\infty} \frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 + \frac{1}{3}u^3(t,x) - u(t,x)\phi(x)dx$$

*is bounded. Therefore, the solutions to the Cauchy problem* (5.6) *all tend to limits as $t \to \infty$ (Corollary 26). By Lemma 78, we conclude that they all tend to the common limit of $f_+$ when $c > 0$.*

*Proof.* The latter two terms are bounded due to the fact that $u$ lies in $L_1 \cap L^\infty(\mathbb{R})$ for all $t$. The bound on the first term comes from combining the fact that $u$ and its first two spatial derivatives are bounded with the asymptotic decay of $f_\pm$, and is otherwise straightforward (use L'Hôpital's rule). □

## 5.4 Construction of an eternal solution

Let

$$U_k(x) = (1 - 2^{-k-1})g_{1/k}(x) + 2^{-k-1}f_+(x), \text{ for } k \geq 0$$

noting that $U_k \to f_-$ as $k \to \infty$. Since $U_k$ is a convex combination of $f_+$ and $g_{1/k}$, it follows that $U_k \in W_{1/k}$ for all $k$. Also, since $f_+$ and $f_-$ have bounded first and second derivatives, the $\{U_k\}$ have a common bound for their first and second derivatives.

Now consider solutions to the following set of Cauchy problems

$$\begin{cases} \frac{\partial u_k(t,x)}{\partial t} = \frac{\partial^2 u_k(t,x)}{\partial x^2} - u_k^2(t,x) + \phi(x), \\ u_k(T_k, x) = U_k(x). \end{cases} \tag{5.8}$$

We choose $T_k$ so that for all $k > 0$, $u_k(0,0) = u_0(0,0)$. We can do this using the continuity of the solution and Lemma 83. As $k \to \infty$, solutions are started nearer and nearer to the equilibrium $f_-$, so we are forced to choose $T_k \to -\infty$ as $k \to \infty$.

It's clear that each solution $u_k$ is defined for only $t > T_k$. However, for each compact set $S \subset \mathbb{R}^2$, there are infinitely many elements of $\{u_k\}$ which are defined on it. The results of the previous section imply that $\{u_k\}$ is a bounded, equicontinous family. As a result, Ascoli's theorem implies that $\{u_k\}$ converges uniformly on compact subsets to a continous $u$, which is an eternal solution to (5.1).

Our constructed eternal solution will have the value $u(0,0) = u_0(0,0)$, which is strictly between $f_+$ and $f_-$. As a result, the eternal solution we have constructed is not an equilibrium solution. By Lemma 83, it is a finite energy solution, so it must be a heteroclinic orbit connecting $f_-$ to $f_+$.

CHAPTER 6

# INSTABILITY OF EQUILIBRIA

## 6.1 Introduction

(This chapter is available on the arXiv as [36].)

If we try to apply standard Morse theory to our semilinear parabolic equation, we encounter a serious difficulty. In particular, the stability of the linearization about an equilibrium of our system is not sufficient to ensure that the equilibrium is stable, even though there may not be a zero eigenvalue. The instability of equilibria for (1.2) is not a new fact, having been studied carefully in the 1980s. This chapter is included for completeness, providing an explicit construction of a sequence of solutions starting near the equilibrium which all blow up. Indeed, there is a complementary result of *stability* in certain weighted norms, described in [45] and [38].

Note that the right side of (1.2) is an operator which has a spectrum which includes zero, so stability is possible (as in the unforced heat equation), though not guaranteed. This is in stark contrast to the situation in finite-dimensional settings, where asymptotic stability of the linearized system implies stability of the equilibrium (in particular, zero is not an eigenvalue of the linearized operator). (See [5], for instance.) Essentially, this difficulty suggests that each critical point is degenerate. (This line of reasoning is completed in Chapter 7.) This implies that Morse theory (even when strengthened to its natural infinite-dimensional form [31]) cannot be used to study the dynamics of our system.

Again, we study the general situation by working with the simpler Cauchy problem (1.6). (The computations we exhibit in this chapter will carry over *mutatis mutandis* to (1.2).) However, it is useful to center on an equilibrium solution. That is, we apply the change of variables $u(t,x) \mapsto u(t,x) - f(x)$ to obtain

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) - 2f(x)u(t,x) - u^2(t,x) \\ u(0,x) = h(x) \in C^\infty(\mathbb{R}) \\ t > 0, x \in \mathbb{R}, \end{cases} \tag{6.1}$$

where $f \in C_0^\infty(\mathbb{R})$ is a positive function with two bounded derivatives. (By $C_0^\infty$, we mean the space of smooth functions which decay to zero.) We interpret $f$ as being an equilibrium of the original problem (1.6).

The assumption that $f$ be *positive* requires some motivation. With this assumption, the zero function is an asymptotically stable equilibrium for the linearized problem,

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) - 2f(x)u(t,x). \tag{6.2}$$

by a standard comparison principle argument. This corresponds neatly to the case of zero positive eigenvalues of $\frac{\partial^2}{\partial x^2} - 2f$ which was found numerically in Figure 4.10. Intuition would suggest that this implies $f$ is a stable equilibrium of (1.6). However, using a technique pioneered by Fujita in [18], we will show that this equilibrium is not stable in the nonlinear problem, even if the initial condition has small $p$-norm for every $1 \leq p \leq \infty$.

Fujita showed that if $f \equiv 0$, then the zero function is an unstable equilibrium of (6.1). The cause of the instability in (6.1) is the decay of $f$, for if $f = \text{const} > 0$, then the comparison principle shows that the zero function is stable. We extend Fujita's result, so that roughly speaking, since $f \to 0$ away from the origin, the system is less stable to perturbations away from the origin. Another indication that there may be instability lurking (though not conclusive proof) is that the decay of $f$ means that the spectrum of the linearized operator on the right side of (6.2) includes zero.

## 6.2 Motivation

The problem (6.1) describes a reaction-diffusion equation [12], or a diffusive logistic population model with a spatially-varying carrying capacity. The choice of $f$ positive means that the equilibrium $u \equiv 0$ describes a population saturated at its carrying capacity. Without the diffusion term, this situation is well known to be stable. The decay condition on $f$ means that the carrying capacity diminishes away from the origin.

The spatial inhomogeneity of $f$ makes the analysis of (6.1) much more complicated than that of typical reaction-diffusion equations. The existence of additional equilibria for (6.1) is a fairly difficult problem, which depends delicately on $f$. (See [6] for a proof of existence of equilibria in a related setting.)

## 6.3 Instability of the equilibrium

Given an $\epsilon > 0$, we will construct an initial condition $h \in C^\infty(\mathbb{R})$ for the problem (6.1), with $\|h\|_p < \epsilon$ for each $1 \le p \le \infty$, such that $\|u(t)\|_\infty \to \infty$ as $t \to T < \infty$. In particular, this implies that $u \equiv 0$ is not a stable equilibrium of (6.1), at least insofar as classical solutions are concerned. We employ a technique of Fujita, which provides sufficient conditions for equations like (6.1) to blow up [18]. (Additionally, [11] contains a more elementary discussion of the technique with a similar construction.) Our choice for $h$ can be thought of as a sequence of progressively shifted gaussians, and we will demonstrate that though each has smaller $p$-norm than the previous, the solution started at $h$ still blows up.

### 6.3.1 The technique of Fujita

The technique of Fujita examines the blow-up behavior of nonlinear parabolic equations by treating them as ordinary differential equations on a Hilbert space. Suppose $u(t)$ solves

$$\frac{\partial u(t)}{\partial t} = Lu(t) + N(u(t), t), \tag{6.3}$$

where $L$ is a linear operator not involving $t$, and $N$ may be nonlinear and may depend on $t$. Suppose that $v(t)$ solves

$$\frac{\partial v(t)}{\partial t} = -L^* v(t), \tag{6.4}$$

where $L^*$ is the adjoint of $L$. Let $J(t) = \langle v(t), u(t)\rangle$. We observe that if $|J(t)| \to \infty$ then either $\|v(t)\|$ or $\|u(t)\|$ also does. So if $v(t)$ does not blow up, then we can show that $\|u(t)\|$ blows up, and perhaps more is true. If we differentiate $J(t)$, we obtain the identity

$$\begin{aligned}
\frac{d}{dt}J(t) &= \frac{d}{dt}\langle v(t), u(t)\rangle \\
&= \left\langle \frac{dv}{dt}, u(t)\right\rangle + \left\langle v(t), \frac{du}{dt}\right\rangle \\
&= \langle -L^* v(t), u(t)\rangle + \langle v(t), Lu(t) + N(u(t), t)\rangle \\
&= \langle v(t), N(u(t), t)\rangle,
\end{aligned}$$

where there is typically a technical justification required for the second equality. It is often possible to find a bound for $\langle v(t), N(u(t), t)\rangle$ in terms of $J(t)$. So then the method provides a fence (in the sense of [23]) for $J(t)$, which we can solve to give a bound on $|J(t)|$. As a result, the blow-up behavior of $u(t)$ is controlled by the solution of an *ordinary* differential equation (for $J(t)$) and a *linear* parabolic equation (for $v(t)$), both of which are much easier to examine than the original nonlinear parabolic equation.

### 6.3.2 Instability in $L^p$ for $1 \leq p \leq \infty$

We begin our application of the method of Fujita by working with $L = \frac{\partial^2}{\partial x^2} - 2f$ and $N(u) = -u^2$ in (6.3). Since (6.4) is then not well-posed for all $t$, we must be a little more careful than the method initially suggests. For this reason, we consider a family of solutions $v_\epsilon$ to (6.4) that have slightly extended domains of definition. It will also be important, for technical reasons, to enforce the assumption that the first and second derivatives of $f$ are bounded.

**Definition 84.** Suppose $w = w(t, x)$ solves

$$\begin{cases} \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} - 2f(x)w(t, x) \\ w(0, x) = w_0(x) \geq 0. \end{cases} \tag{6.5}$$

Define $v_\epsilon(s, x) = w(t - s + \epsilon, x)$ for fixed $t > 0$ and $s < t + \epsilon$. Notice that by the comparison principle, $v_\epsilon(s, x) \geq 0$.

**Lemma 85.** *Suppose that $w$ solves* (6.5)*. Then $w, \frac{\partial w}{\partial x} \in C_0(\mathbb{R})$.*

*Proof.* The standard existence and regularity theorems for linear parabolic equations (see [44], for example) give that $w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2} \in L^2(\mathbb{R})$ and that $w \in C^2(\mathbb{R})$. The comparison principle, applied to $\frac{\partial}{\partial t}\frac{\partial w}{\partial x}$ and $\frac{\partial}{\partial t}\frac{\partial^2 w}{\partial x^2}$ gives that the first and second derivatives of $w$ are bounded for each fixed $t$. (This uses our assumption that $f$ has two bounded derivatives.)

The lemma follows from a more general result: if $g \in C^1 \cap L^p(\mathbb{R})$ for $1 \leq p < \infty$ and $g' \in L^\infty(\mathbb{R})$, then $g \in C_0(\mathbb{R})$. To show this, we suppose the contrary, that $\lim_{x\to\infty} g(x) \neq 0$ (and possibly doesn't exist). By definition, this implies that there is an $\epsilon > 0$ such that for all $x > 0$, there is a $y$ satisfying $y > x$ and $|g(y)| > \epsilon$. Let $S = \{y | \, |g(y)| > \epsilon\}$, which is a union of open intervals, is of finite measure, and has $\sup S = \infty$. Let $T = \{y | \, |g(y)| > \epsilon/2\}$. Note that $T$ contains $S$, but since $g'$ is bounded, for each $x \in S$, there is a neighborhood of $x$ contained in $T$ of measure at least $\epsilon/\|g'\|_\infty$. Hence, since $\sup T = \sup S = \infty$, $T$ cannot be of finite measure, which contradicts the fact that $g \in L^p(\mathbb{R})$ with $1 \leq p < \infty$. $\square$

**Lemma 86.** *Suppose $u : [0,T) \times \mathbb{R} \to \mathbb{R}$ is a classical solution to (6.1) with $u \leq 0$ and $u(t) \in L^\infty(\mathbb{R})$ for each $t \in [0,T)$. Then*

$$-\int w(t,x)h(x)dx \leq \left( \int_0^t \frac{1}{\|w(s)\|_1} ds \right)^{-1}, \tag{6.6}$$

*where $w$ is defined as in Definition 84.*

*Proof.* Define

$$J_\epsilon(s) = \int v_\epsilon(s,x)u(s,x)dx. \tag{6.7}$$

First of all, we observe that since $u \in L^\infty(\mathbb{R})$, $v_\epsilon(s,\cdot)u(s,\cdot)$ is in $L^1(\mathbb{R})$ for each $s < t$.

Now suppose we have a sequence $\{m_n\}$ of compactly supported smooth functions with the following properties: [29]

- $m_n \in C^\infty(\mathbb{R})$,
- $m_n(x) \geq 0$ for all $x$,
- $\mathrm{supp}(m_n)$ is contained in the interval $(-n-1, n+1)$, and
- $m_n(x) = 1$ for $|x| \leq n$.

Then it follows that

$$J_\epsilon(s) = \lim_{n\to\infty} \int v_\epsilon(s,x)u(s,x)m_n(x)dx.$$

Now

$$\begin{aligned}
\frac{d}{ds}J_\epsilon(s) &= \frac{d}{ds}\lim_{n\to\infty}\int v_\epsilon(s,x)u(s,x)m_n(x)dx \\
&= \lim_{h\to 0}\lim_{n\to\infty}\frac{1}{h}\int \left(v_\epsilon(s+h,x)u(s+h,x) - v_\epsilon(s,x)u(s,x)\right)m_n(x)dx.
\end{aligned}$$

We'd like to exchange limits using uniform convergence. To do this we show that

$$\lim_{n\to\infty}\lim_{h\to 0}\frac{1}{h}\int \left(v_\epsilon(s+h,x)u(s+h,x) - v_\epsilon(s,x)u(s,x)\right)m_n(x)dx \tag{6.8}$$

exists and the inner limit is uniform. We show both together by a little computation, using uniform convergence and LDCT:

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{h\to 0}\frac{1}{h}\int \left(v_\epsilon(s+h,x)u(s+h,x) - v_\epsilon(s,x)u(s,x)\right)m_n(x)dx \\
= &\lim_{n\to\infty}\int \left(\frac{\partial}{\partial s}v_\epsilon(s,x)u(s,x) + v_\epsilon(s,x)\frac{\partial}{\partial s}u(s,x)\right)m_n(x)dx \\
= &\lim_{n\to\infty}\int \left(-\frac{\partial^2}{\partial x^2}v_\epsilon(s,x) + 2f(x)v_\epsilon(s,x)\right)u(s,x)m_n(x) + \\
&v_\epsilon(s,x)\left(\frac{\partial^2}{\partial x^2}u(s,x) - u^2(s,x) - 2f(x)u(x)\right)m_n(x)dx \\
= &\lim_{n\to\infty}\int -v_\epsilon(s,x)u^2(s,x)m_n(x)dx.
\end{aligned}$$

Minkowski's inequality has that

$$\left|\int v_\epsilon u m_n dx\right| \leq \int v_\epsilon |u| m_n dx \leq \left(\int v_\epsilon m_n dx\right)^{1/2}\left(\int v_\epsilon u^2 m_n dx\right)^{1/2},$$

since $v_\epsilon, m_n \geq 0$. This gives that

$$
\begin{aligned}
&\int -v_\epsilon(s,x)u^2(s,x)m_n(x)dx \\
\leq\ & -\frac{(\int v_\epsilon u m_n dx)^2}{\int v_\epsilon m_n dx} \\
\leq\ & -\frac{\left(\int v_\epsilon u dx\right)^2}{\int v_\epsilon m_1 dx},
\end{aligned}
$$

hence the inner limit of (6.8) is uniform. On the other hand,

$$
|v_\epsilon(s,x)u^2(s,x)m_n(x)| \leq v_\epsilon(s,x)\|u(s)\|_\infty^2 \in L^1(\mathbb{R})
$$

so the double limit of (6.8) exists by dominated convergence. Thus we have the fence

$$
\frac{dJ_\epsilon(s)}{ds} \leq -\frac{(J_\epsilon(s))^2}{\|v_\epsilon(s)\|_1}. \tag{6.9}
$$

We solve the fence (6.9) to obtain (note $J_\epsilon \leq 0$)

$$
\begin{aligned}
\frac{1}{\|v_\epsilon(s)\|_1} &\leq -\frac{dJ_\epsilon(s)}{ds}\frac{1}{(J_\epsilon(s))^2} \\
\int_0^t \frac{1}{\|v_\epsilon(s)\|_1}ds &\leq \frac{1}{J_\epsilon(t)} - \frac{1}{J_\epsilon(0)} \\
\int_0^t \frac{1}{\|v_\epsilon(s)\|_1}ds &\leq -\frac{1}{J_\epsilon(0)}.
\end{aligned}
$$

Taking the limit as $\epsilon \to 0$ of both sides of the inequality yields

$$
-\int w(t,x)h(x)dx \leq \left(\int_0^t \frac{1}{\|w(t-s)\|_1}ds\right)^{-1} = \left(\int_0^t \frac{1}{\|w(s)\|_1}ds\right)^{-1},
$$

as desired. $\square$

**Remark 87.** Since we are interested in proving the instability of the zero function in (6.1), consider $u(0,x) = h(x) = -\epsilon$ for $\epsilon > 0$. Then (6.6) takes on the simple form

$$
\epsilon \int_0^t \frac{\|w(t)\|_1}{\|w(s)\|_1}ds \leq 1. \tag{6.10}
$$

So in particular, $\|u(t)\|_\infty$ blows up if there exists a $T > 0$ such that $\epsilon \int_0^T \frac{\|w(T)\|_1}{\|w(s)\|_1} ds > 1$.

The stability of the zero function in (6.1) depends on the stability of the zero function in (6.5) – the linearized problem. If the zero function in the linearized problem is very strongly attractive, say $\|w(t)\|_1 \sim e^{-t}$, then

$$\int_0^t \frac{e^{-t}}{e^{-s}} ds = (1 - e^{-t}) < 1,$$

and so a small choice of $\epsilon < 1$ does not cause blow-up via a violation of (6.10). On the other hand, blow-up occurs if it is less attractive, say $\|w(t)\|_1 \sim t^{-\alpha}$ for $\alpha \geq 0$. Because then

$$\int_0^t \frac{s^\alpha}{t^\alpha} ds = \frac{t}{\alpha + 1},$$

whence blow-up occurs before $t = \frac{\alpha+1}{\epsilon}$.

In the particular case of $f(x) = 0$ for all $x$, we note that $w$ is simply a solution to the heat equation, which has $\|w(t)\|_1 = \|w_0\|_1$ for all $t$ (by direct computation using the fundamental solution, say), so blow up occurs. Thus we can recover a special case of the original blow-up result of Fujita in [18].

**Theorem 88.** *Suppose a sufficiently small $\epsilon > 0$ is given. Then for a certain choice of initial condition $h(x)$ with $\|h\|_p < \epsilon$ for all $1 \leq p \leq \infty$, there exists a $T > 0$ for which $\lim_{t \to T^-} \|u(t)\|_\infty = \infty$.*

*Proof.* First, it suffices to choose $\|u(0)\|_1 < \epsilon$ and $\|u(0)\|_\infty < \epsilon$, since

$$\|u\|_p = \left( \int |u|^p dx \right)^{1/p} \leq \|u\|_\infty^{(p-1)/p} \|u\|_1^{1/p} < \epsilon.$$

We assume, contrary to what is to be proven, that $\|u(t)\|_\infty$ does not blow up for any finite $t$. In other words, assume that $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a classical solution to (6.1), with $\|u(t)\|_\infty < \infty$ for all $t$. We make several definitions:

- Choose $0 < \beta < \min\left\{\epsilon, \frac{\epsilon^4}{16\pi^2}\right\}$.
- Choose $\gamma > 0$ small enough so that

$$\frac{\beta}{27\gamma^2} = K, \tag{6.11}$$

  for some some arbitrary $K > 1$.
- Since $0 \leq f \in C_0^\infty(\mathbb{R})$, we can choose an $x_1$ such that

$$f(x) \leq \gamma \text{ when } x < x_1. \tag{6.12}$$

- Next, we choose $x_0 < x_1$ so that

$$\sqrt{t}\|f\|_\infty \left(1 - \operatorname{erf}\left(\frac{x_1 - x_0}{2\sqrt{t}}\right)\right) < \gamma \tag{6.13}$$

  for all $0 < t < \frac{1}{4\gamma^2}$. Notice that any choice less than $x_0$ will also work.
- Choose the initial condition for (6.1) to be

$$u(0,x) = h(x) = -\beta e^{\beta^{3/2}(x-x_0)^2}. \tag{6.14}$$

  This choice of initial condition has $\|u(0)\|_\infty = \beta < \epsilon$, $\|u(0)\|_1 = 2\pi^{1/2}\beta^{1/4} < \epsilon$, and $\left\|\frac{\partial^2 u(0)}{\partial x^2}\right\|_\infty = \mu = 2\beta^{5/2}$. (The value of $\mu$ will be important shortly.)
- Finally, let $w_0(y) = \delta(y - x_0)$ (the Dirac $\delta$-distribution), and suppose that $w$ solves (6.5). In other words, choose $w$ to be the fundamental solution to (6.5) concentrated at $x_0$. Note that the maximum principle ensures both that $w(t,x) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}$ and that $\|w(t)\|_1 \leq \|w(0)\|_1 = 1$ for all $t > 0$. This allows us to rewrite (6.6) as

$$-t \int w(t,x)h(x)dx \leq 1. \tag{6.15}$$

Now we estimate the integral in (6.15). Notice that

$$\begin{aligned}\frac{d}{dt}\int w(t,x)\,(-h(x))\,dx &= \int\left(\frac{\partial^2 w}{\partial x^2} - 2f(x)w(t,x)\right)(-h(x))\,dx\\ &= \int\left(-\frac{\partial^2 u}{\partial x^2} + 2f(x)h(x)\right)w(t,x)dx,\end{aligned}$$

where Lemma 85 eliminates the boundary terms. Now suppose $z$ solves the heat equation with the same initial condition as $w$, namely

$$\begin{cases}\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}\\ z(0,x) = w_0(x) = \delta(x-x_0).\end{cases} \tag{6.16}$$

The comparison principle estabilishes that $z(t,x) \geq w(t,x)$ for all $t > 0$ and $x \in \mathbb{R}$, since $f, w \geq 0$. As a result, we have that

$$\begin{aligned}\frac{d}{dt}\int w(t,x)\,(-h(x))\,dx &\geq \int\left(-\left|\frac{\partial^2 u}{\partial x^2}\right| + 2f(x)h(x)\right)z(t,x)dx\\ &\geq -\mu - 2\beta\int f(x)z(t,x)dx,\end{aligned}$$

where $\mu = \left\|\frac{\partial^2 u}{\partial x^2}(0)\right\|_\infty$ and $\beta = \|u(0)\|_\infty$, which is an integrable equation. As a result,

$$\int w(t,x)\,(-h(x))\,dx \geq \beta - \mu t - 2\beta\int_0^t\int\int f(x)\frac{1}{\sqrt{4\pi s}}e^{-\frac{(x-y)^2}{4s}}w_0(y)dy\,dx\,ds. \tag{6.17}$$

On the other hand using our choice for $w_0$,

$$\begin{aligned}
\int_0^t \int\int \quad f(x) \quad & \frac{1}{\sqrt{4\pi s}} e^{-\frac{(x-y)^2}{4s}} w_0(y) dy\, dx\, ds = \int_0^t \int f(x) \frac{1}{\sqrt{4\pi s}} e^{-\frac{(x-x_0)^2}{4s}} dx\, ds \\
\leq \quad & \int_0^t \frac{1}{\sqrt{4\pi s}} \left( \gamma \int_{-\infty}^{x_1} e^{-\frac{(x-x_0)^2}{4s}} dx + \|f\|_\infty \int_{x_1}^{\infty} e^{-\frac{(x-x_0)^2}{4s}} dx \right) ds \\
\leq \quad & \frac{\gamma\sqrt{t}}{4} + \frac{1}{2}\|f\|_\infty \int_0^t 1 - \operatorname{erf}\left( \frac{x_1 - x_0}{2\sqrt{s}} \right) ds \\
\leq \quad & \frac{\gamma\sqrt{t}}{4} + \frac{1}{2}\|f\|_\infty \int_0^t 1 - \operatorname{erf}\left( \frac{x_1 - x_0}{2\sqrt{t}} \right) ds \\
\leq \quad & \frac{\gamma\sqrt{t}}{4} + \frac{1}{2} t \|f\|_\infty \left( 1 - \operatorname{erf}\left( \frac{x_1 - x_0}{2\sqrt{t}} \right) \right) \\
\leq \quad & \frac{3\gamma\sqrt{t}}{4} \leq \gamma\sqrt{t},
\end{aligned}$$

we have used (6.12), (6.13), and assumed that $0 < t < \frac{1}{4\gamma^2}$. Then (6.15) becomes

$$1 \geq t \int w(t,x)\,(-h(x))\,dx \geq \beta t - \mu t^2 - 2\beta\gamma t\sqrt{t} = -2\beta^{5/2}t^2 - \frac{2\beta^{3/2}t^{3/2}}{\sqrt{27K}} + \beta t,$$

using our choices of $\mu$, $\gamma$, and initial condition. Maple reports that the maximum of $A(t) = -2\beta^{5/2}t^2 - \frac{2\beta^{3/2}t^{3/2}}{\sqrt{27K}} + \beta t$ is unique, occurs at $0 < t_0 < \frac{1}{4\gamma^2}$, and has the asymptotic expansion

$$A(t_0) \sim K - 18K\sqrt{\beta} + 432K^3\beta + O(\beta^{3/2}).$$

Thus for all small enough $\epsilon > \beta$, we obtain a contradiction to (6.15) since $K > 1$. Thus, for some $T < t_0 < \infty$, $\lim_{t\to T^-} \|u(t)\|_\infty = \infty$. □

## 6.4 Discussion

Theorem 88 gives a fairly strong instability result. No matter how small an initial condition to (6.1) is chosen, even with all $p$-norms chosen small, solutions can blow up so quickly that they fail to exist for all $t$. This precludes any kind of stability for classical solutions. Like the analogous result in Fujita's paper, the kind of initial conditions which can be responsible for blow up are of the nicest kind imaginable – gaussians in either case!

It must be understood that the argument in Theorem 88 depends crucially on the decay of $f$. Without it, the lower bound on $\int w(t,x)(-h(x))dx$ decreases too quickly. Indeed, if $f = \text{const} > 0$ and $h(x) > -f$, then the comparison principle demonstrates that the zero function is asymptotically stable. On the other hand, any rate of decay for $f$ satisfies the hypotheses of Theorem 88, and so will cause (6.1) to exhibit instability.

Finally, although we have examined the case where the nonlinearity in (6.1) is due to $u^2$, there is no obstruction to extending the analysis to any nonlinearity like $|u|^k$, with degree $k$ greater than 2. A higher-degree nonlinearity would result in a somewhat different form for (6.6), but this presents no further difficulties to the argument. Indeed, by analogy with Fujita's work, higher-degree nonlinearities would result in significantly faster blow-up.

CHAPTER 7

# CELL COMPLEX STRUCTURE FOR THE SPACE OF HETEROCLINES

## 7.1 Introduction

In this chapter, we determine that all unstable manifolds of (1.6)

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - u^2(t,x) + \phi(x) \tag{7.1}$$

are finite dimensional. This is not a particularly new result, indeed Theorem 5.2.1 in [20] can easily be made to apply with the Banach spaces we shall choose. Theorem 5.2.1 in [20] shows the existence of a smooth finite dimensional unstable manifold locally at an equilibrium. One can then use the iterated time-1 map of the flow for (7.1) to extend this local manifold to a maximal unstable manifold. There are also finite Hausdorff dimensional attractors for the forward Cauchy problem on bounded domains [32]. However, we shall exhibit a more global approach to the finite dimensionality of the unstable manifolds. This approach allows us to examine the finite dimensionality of the space of heteroclinic orbits connecting a pair of equilibria, which is a new result in the spirit of [15]. The techniques used here depend rather delicately on both the degree of the nonlinearity (quadratic) and the spatial dimension (1). Both of these are important in the standard methodology as well, as the portion of the spectrum of the linearization in the right half-plane needs to be bounded away from zero. In the case of (7.1), the spectrum in the right-half plane is discrete and consists of a finite number of points.

## 7.2 The linearization and its kernel

We begin by considering an equilibrium solution $f$ to (7.1). As discussed in Chapter 4, this solution has asymptotic behavior which places it in $C^2 \cap L^1 \cap L^\infty(\mathbb{R})$. We are particularly interested in solutions which lie in the $\alpha$-limit set of $f$, those solutions which are defined for all $t < 0$ and tend to $f$. As in previous chapters, center on this equilibrium by applying the change of variables $u(t,x) \mapsto u(t,x) - f(x)$ to obtain

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) - 2f(x)u(t,x) - u^2(t,x) \\ u(0,x) = h(x) \in C^2(\mathbb{R}) \\ \lim_{t\to-\infty} u(t,x) = f(x) \\ t < 0, x \in \mathbb{R}. \end{cases} \tag{7.2}$$

Thus we have a final value problem for our nonlinear equation. All solutions to (7.2) will tend to zero as $t \to -\infty$ uniformly by Theorem 28. Of course, (7.2) is ill-posed. We show that there is only a finite dimensional manifold of choices of $h$ for which a solution exists.

### 7.2.1 Backward time decay

The decay of solutions to zero is a crucial part of the analysis, as it provides the ability to perform Laplace transforms. In the forward time direction, one obtains upper bounds for solutions by way of maximum principles, and lower bounds for the upper bounds by way of Harnack estimates. In the backward time direction, these tools reverse roles. Harnack estimates provide upper bounds, while the maximum principle provides lower bounds for the upper bound. In the proof of Theorem 28, the latter was used to some advantage. In this section, we briefly apply a standard Harnack estimate to obtain an exponentially decaying upper bound.

Harnack estimates for a very general class of parabolic equations are discussed in [28] and [1]. In those articles, the authors examine positive solutions to

$$\text{div } \mathbf{A}(x,t,u,\nabla u) - \frac{\partial u}{\partial t} = B(x,t,u,\nabla u),$$

where $x \in \mathbb{R}^n$, and $\mathbf{A} : \mathbb{R}^{2n+2} \to \mathbb{R}^n$ and $B : \mathbb{R}^{2n+2} \to \mathbb{R}$ satisfy

$$\begin{aligned} |\mathbf{A}(x,t,u,p)| &\leq a|p| + c|u| + e \\ |B(x,t,u,p)| &\leq b|p| + d|u| + f \\ p \cdot \mathbf{A}(x,t,u,p) &\geq \frac{1}{a}|p|^2 - d|u|^2 - g, \end{aligned}$$

for some $a > 0$ and $b, ...g$ are measurable functions. For a solution $u$ defined on a rectangle $R$, the authors define a pair of congruent, disjoint closed rectangles $R^+, R^- \subset R$ with $R^-$ being a backward time translation of $R^+$. The main result is the Harnack inequality

$$\max_{R^-} u \leq \gamma \left( \min_{R^+} u + L \right), \tag{7.3}$$

where $\gamma > 0$ depends only on geometry and $a$ (but not $b, ...g$) and $L$ is a linear combination of $e, f, g$ whose coefficients depend on geometry.

In the case of (7.2), or indeed of the analogous equation with higher degree terms, we have that (7.3) will apply with $L = 0$. Notice that the conditions on $A, B$ are satisfied because any solution to (7.2) is automatically a finite energy solution, and therefore is bounded and has bounded first derivatives. The only difficulty is that (7.3) applies for *positive* solutions, while (7.2) may have solutions with negative portions. However, one can pose the problem for the (weak) solution of

$$\begin{aligned} \frac{\partial |u|}{\partial t} &= \text{sgn } (u) \left( \Delta u - u^2 - 2fu \right) \\ &= \Delta |u| - u|u| - 2f|u| \\ &\geq \Delta |u| - |u|^2 - 2|f||u| \end{aligned}$$

for which we only get positive solutions. By iterating (7.3) we have that solutions to (7.2) decay exponentially as $t \to -\infty$.

### 7.2.2 Topological considerations

**Definition 89.** Let $Y_a(X)$ be the subspace of $C^1(X, C^{0,\alpha}(\mathbb{R}))$ which consists of functions which decay exponentially to zero like $e^{at}$, where $0 < \alpha \leq 1$. We define the weighted norm

$$\|u\|_{Y_a} = \left\|e^{-at}\|u(t)\|_{C^{0,\alpha}(\mathbb{R})}\right\|_{C^1}$$

and the space

$$Y_a(X) = \left\{u = u(t,x) \in C^1(X, C^{0,\alpha}(\mathbb{R})) | \|u\|_{Y_a} < \infty\right\}.$$

In a similar way, we can define the weighted Banach space $Z_a(X)$ as a subspace of $C^0(X, C^{0,\alpha}(\mathbb{R}))$. It is quite important that $Y_a$ and $Z_a$ are Banach algebras under pointwise multiplication.

In light of the previous section, solutions to (7.2) are zeros of the densely defined nonlinear operator $N : Y_a((-\infty, 0]) \to Z_a((-\infty, 0])$ given by

$$N(u) = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u^2 + 2fu. \tag{7.4}$$

About the zero function, the linearization of $N$ is the densely defined linear map $L : Y_a((-\infty, 0]) \to Z_a((-\infty, 0])$ given by

$$L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + 2f = \frac{\partial}{\partial t} - H, \tag{7.5}$$

where we define $H = \frac{\partial^2}{\partial x^2} - 2f$. Also note that $L$ is the Frechét derivative of $N$, which follows from the fact that $Y_a$ and $Z_a$ are Banach algebras.

**Remark 90.** We are using $C^{0,\alpha}(\mathbb{R})$ instead of $C^0(\mathbb{R})$ to ensure that $N$ and $L$ be densely defined. We could use space of continous functions which decay to zero, or the space of uniformly continous functions equally well.

**Convention 91.** *We shall conventionally take $a > 0$ to be smaller than the smallest eigenvalue of $H$.*

We show two things: that the kernel of $L$ is finite dimensional, and that $L$ is surjective. These two facts enable us to use the implicit function theorem to conclude that the space of solutions comprising the $\alpha$-limit set of an equilibrium is a finite dimensional submanifold of $Y_a((-\infty, 0])$.

### 7.2.3 Dimension of the kernel

**Lemma 92.** *If $f$ is an equilibrium solution, then the operator $L : Y_a((-\infty, 0]) \to Z_a((-\infty, 0])$ in (7.5) has a finite dimensional kernel.*

*Proof.* Notice that the operator $L$ is separable, so we try the usual separation $h(t, x) = T(t)X(x)$. Substituting into (7.5) gives

$$\begin{aligned}
0 &= Lh = \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + 2f\right) h \\
&= T'X + T\left(-\frac{\partial^2}{\partial x^2} + 2f\right) X \\
\frac{T'}{T} &= \frac{\left(\frac{\partial^2}{\partial x^2} - 2f\right) X}{X} = \lambda
\end{aligned}$$

for some $\lambda \in \mathbb{C}$. The separated equation for $T$ yields $T = C_x e^{\lambda t}$. Since we are looking for the kernel of $L$ in $Y_a \subset L^\infty(\mathbb{R}^2)$, we must conclude that $\lambda$ must have nonnegative real part. On the other hand, the spectrum of $H = \left(\frac{\partial^2}{\partial x^2} - 2f\right)$ is strictly real, so $\lambda \geq 0$. Indeed, there are finitely many positive possibilities for $\lambda$ each with finite-dimensional eigenspace. This is a standard fact about the Schrödinger operator $H$ since $f$ is an equilibrium (Proposition 111). Thus $L$ has a finite dimensional kernel. $\square$

### 7.2.4 Surjectivity of the linearization

In order to show the surjectivity of $L$, we will construct a map $\Gamma : Z_a((-\infty, 0]) \to Y_a((-\infty, 0])$ for which $L \circ \Gamma = \mathrm{id}_{Z_a}$. That is, we construct a right-inverse to $L$, noting of course that $L$ is typically not injective. We shall derive a formula for $\Gamma$ using the Laplace transform $v \mapsto \overline{v}$

$$\overline{v}(s, x) = \int_{-\infty}^{0} e^{st} v(t, x) dt,$$

where $\Re(s) > -a$ and $v \in Z_a((-\infty, 0])$.

Since Lemma 92 essentially solves (7.2), we will be solving the inhomogeneous problem with zero final condition

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} - \frac{\partial^2 v(t,x)}{\partial x^2} + 2f(x)v(t,x) = -w(t,x) \in Z_a((-\infty, 0]) \\ v(0, x) = 0 \end{cases} \tag{7.6}$$

for $t < 0$. The Laplace transform of this problem is

$$\begin{aligned} s\overline{v}(s, x) + \frac{\partial^2 \overline{v}(s, x)}{\partial x^2} - 2f(x)\overline{v}(s, x) &= \overline{w}(s, x) \\ (H + s)\overline{v}(s, x) &= \overline{w}(s, x). \end{aligned}$$

Choose a vertical contour $C$ with $0 > \Re(s) > -a$, so that the Laplace transforms are well-defined, and that the contour remains entirely in the resolvent set of $-H$.

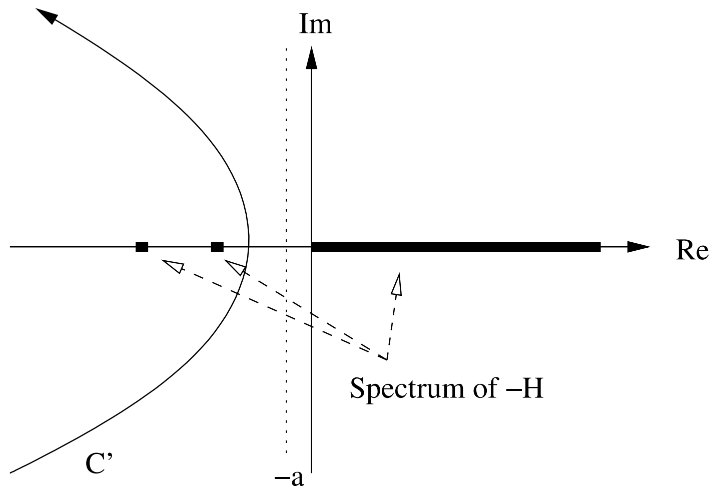


Figure 7.1: Definition of the contour $C'$

Then we can invert to obtain

$$\overline{v}(s,x) = (H+s)^{-1}\overline{w}(s,x).$$

Using the inversion formula for the Laplace transform yields

$$\begin{aligned} v(t,x) &= \frac{1}{2\pi i}\int_C e^{-st}(H+s)^{-1}\overline{w}(s,x)ds \\ &= \frac{1}{2\pi i}\int_C e^{-st}(H+s)^{-1}\int_t^0 e^{s\tau}w(\tau,x)d\tau\, ds \\ &= \int_t^0 \left(\frac{1}{2\pi i}\int_C e^{s(\tau-t)}(H+s)^{-1}ds\right) w(\tau,x)d\tau. \end{aligned}$$

We can obtain operator convergence of the operator-valued integral in parentheses if we deflect the contour $C$. Choose instead the portion $C'$ of the hyperbola (See Figure 7.1)

$$(\Re(s))^2 - (\Im(s))^2 = \frac{1}{4}(\lambda - a)^2 \tag{7.7}$$

(where $\lambda$ is the smallest magnitude eigenvalue of $-H$) which lies in the left half-plane as our new contour. Then, since $-H : C^{0,\alpha} \to C^{0,\alpha}$ is sectorial about

$(\lambda - a)/2$ (Proposition 112), Theorem 1.3.4 in [20] implies that the integral

$$\left( \frac{1}{2\pi i} \int_{C'} e^{s(\tau - t)} (H + s)^{-1} ds \right)$$

defines an operator-valued semigroup $e^{-H(\tau - t)}$, so the formula for $\Gamma$ is given by

$$\Gamma(w)(t, x) = \int_t^0 e^{-H(\tau - t)} w(\tau, x) d\tau. \tag{7.8}$$

It remains to show that the image of $\Gamma$ is in fact $Y_a$, as it is easy to see that its image is in $L^\infty$. That the image is as advertised is not immediately obvious because the contour deflection $C \to C'$ changes the domain of the Laplace transform. In particular, the derivation given above is no longer valid with the new contour.

Therefore, we must estimate $\|v\|_{Z_a}$ (recall that $\lambda$ is the smallest magnitude eigenvalue of $-H$)

$$\begin{aligned}
\|e^{-at} v(t, x)\|_{C^0} &= \left\| \frac{1}{2\pi i} \int_{C'} (s + H)^{-1} \int_t^0 e^{-(s+a)(t-\tau)} e^{a\tau} w(\tau, x) d\tau \, ds \right\|_{C^0} \\
&\leq \frac{1}{2\pi} \int_{C'} \frac{K_1}{|s - \lambda|} e^{-\Re(s+a)t} \int_t^0 e^{\Re(s+a)\tau} \|w\|_{Z_a} d\tau \, ds \\
&\leq \frac{K_1 \|w\|_{Z_a}}{2\pi} \int_{C'} \frac{1}{|s - \lambda|} e^{-\Re(s+a)t} \frac{1}{\Re(s+a)} \left(1 - e^{\Re(s+a)t}\right) ds \\
&\leq \frac{K_1 \|w\|_{Z_a}}{\pi} \int_{C'} \frac{ds}{|s - \lambda| |\Re(s+a)|} \\
&\leq K_2 \|w\|_{Z_a},
\end{aligned}$$

where $0 < K_1, K_2 < \infty$ are independent of $t$ and $w$. We have made use of the estimate in Proposition 112 of the norm of $(H + s)^{-1} : C^{0,\alpha} \to C^{0,\alpha}$ when $s$ is in the resolvent set of $-H$. In particular, note that the choice of $C'$ being to the left of $-a$ is crucial to the convergence of the integrals. Thus the image of $\Gamma$ lies in $Z_a$. The backward-time decay of $\frac{\partial v}{\partial t}$ is immediate from the Harnack inequality, so in fact the image of $\Gamma$ lies in $Y_a$.

**Theorem 93.** *The linear map* $L : Y_a((-\infty, 0]) \to Z_a((-\infty, 0])$ *is surjective and has a finite dimensional kernel. Therefore the set* $N^{-1}(0)$ *is a finite dimensional manifold, which is the unstable manifold of the equilibrium* $f$. *The dimension of* $N^{-1}(0)$ *is precisely the dimension of the positive eigenspace of* $H$.

*Proof.* The only thing which remains to be shown is that the domain $Y_a$ splits into a pair of closed complementary subspaces: the kernel of $L$ and its complement. That its complement is closed follows immediately from a standard application of the Hahn-Banach theorem. (Extend $\mathrm{id}_{\ker L}$ to all of $Y_a$.) □

Combining the fact that an equilibrium solution can have an empty unstable manifold (we numerically computed the dimension of the eigenspaces of $L$ in Chapter 4) and is yet unstable, we have proven the following result.

**Theorem 94.** *All equilbrium solutions to* (7.1) *are degenerate critical points in the sense of Morse.*

## 7.3 Linearization about heteroclinic orbits

We can extend the technique of the previous section to the linearization about a heteroclinic orbit. The resulting generalization of Theorem 93 is that the connecting manifolds of (7.1) are all finite dimensional.

Suppose that $u$ is a heteroclinic orbit of (7.1). Let $f_-, f_+$ be the equilibrium solutions of (1.2) to which $u$ converges as $t \to -\infty$ and $t \to +\infty$ respectively.

Suppose that $\lambda_0 : \mathbb{R} \to (0, \infty)$ is the smallest positive eigenvalue of $H(t)$. It is easy to see that $\lambda_0$ is piecewise $C^1$, for instance, see Proposition I.7.2 in [25].

Propostion 110 ensures that $\lambda_0$ is a bounded function. We will define a pair of bounded, piecewise $C^1$ functions $\lambda_1$ and $\lambda_2$ which will aid us in defining a two more pairs of function spaces. Let $\lambda_1 : \mathbb{R} \to (0, \infty)$ be a bounded, piecewise $C^1$ function with bounded derivative which has the following properties:

- $\lambda_1(t)$ is never an eigenvalue of $H(t)$,
- $\lim_{t\to\infty} \frac{\lambda_1(t)}{\lambda_0(t)} < 1$,
- $\lim_{t\to-\infty} \frac{\lambda_1(t)}{\lambda_0(t)} < 1$, and
- since $u \to f_\pm$ uniformly, for a sufficiently large $R > 0$, $\lambda_1$ can be chosen so that there are no jumps on its restriction to $\mathbb{R} - [-R, R]$.

Defining $\lambda_2$ is a somewhat more delicate problem. We would like to exclude the solutions which lie in the unstable manifold of $f_+$, since they cannot lie in the space of heteroclines from $f_- \to f_+$. We do this by separating the eigenvalues corresponding to the intersection of the unstable manifolds of $f_-$ and $f_+$ from those which lie in the stable manifold of $f_+$. However, there is an obstruction to this technique. In particular, the eigenvalues of $H(t) = \frac{\partial^2}{\partial x^2} - 2u(t)$ vary with time, and can bifurcate. To avoid this issue, we need some kind of regularity for the eigenvalues to prevent them from bifurcating. We follow Floer [14] in the following way:

**Conjecture 95.** There is a generic subset (a Baire subset) of choices for the coefficients $a_i$ in (1.2) so that if $u$ is a heteroclinic orbit, all of the eigenvalues of $H(t)$ are simple.

Numerical evidence, as exhibited in Chapters 4 and 8 suggests that the above Conjecture is true. When we assume that all of the eigenvalues of $H(t)$ are simple,

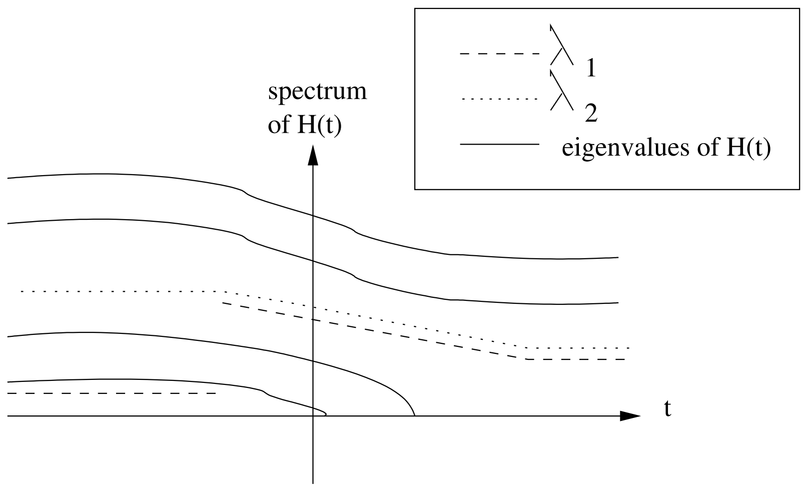


Figure 7.2: Definition of $\lambda_1$ and $\lambda_2$

and therefore do not undergo any bifurcations other than passing through zero, we shall say $u$ is a heterocline contained in $U_{reg}$.

Let $\lambda_2$ be in $C^1(\mathbb{R})$ such that

- $\lambda_2 = \lambda_1$ on $[R, \infty)$, and
- $\lambda_2(t)$ is not an eigenvalue of $H(t)$ for any $t$.

We can do this when $u \in U_{reg}$. See Figure 7.2.

**Definition 96.** Define the Banach algebra $Y_{\lambda_i}(X)$ (for $i = 1, 2$) to be the set of $u$ in $C^1(X, C^{0,\alpha}(\mathbb{R}))$ such that the norm

$$\left\| e^{-\int_0^t \lambda_i(\tau)d\tau} \|u(t)\|_{C^{0,\alpha}} \right\|_{C^1} < \infty,$$

where $X$ is an interval containing zero. Likewise, we can define the spaces $Z_{\lambda_i}(X) \subset C^0(X, C^{0,\alpha}(\mathbb{R}))$ in a similar way. That these are Banach spaces follows from the boundedness of the $\lambda_i$. It is also elementary to see that these are Banach algebras.

We then consider $N_i, L_i$ as $Y_{\lambda_i}(\mathbb{R}) \to Z_{\lambda_i}(\mathbb{R})$, where $L_i$ is the linearization of $N_i$ about $u$ for $i = 1, 2$. (Again, since $Y_{\lambda_i}$ and $Z_{\lambda_i}$ are Banach algebras, $L_i$ is the Frechét derivative of $N_i$.) For a $i \in \{1, 2\}$, consider the restriction $L_i^-$ of $L_i$ to a map $Y_{\lambda_i}((-\infty, 0]) \to Z_{\lambda_i}((-\infty, 0])$. We rewrite

$$L_i^- = \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + 2f_-\right) + (2f_- - 2u). \tag{7.9}$$

Likewise, we can define $L_i^+ : Y_{\lambda_i}([0, \infty)) \to Z_{\lambda_i}([0, \infty))$.

We define the positive eigenspaces $V^+$ for the equilibria as well

$$V^+(f_\pm) = \text{span}\left\{v \in C^{0,\alpha}(\mathbb{R}) | \text{ there is a } \lambda > 0 \text{ with } \left(\frac{\partial^2}{\partial x^2} - 2f_\pm\right) v = \lambda v\right\}. \tag{7.10}$$

Note in particular that dim $V^+(f_\pm) < \infty$.

**Lemma 97.** *If $u \in U_{reg}$ is a heterocline that converges to $f_\pm$ as $t \to \pm\infty$, then the operator $L_i$ has a finite dimensional kernel for $i \in \{1, 2\}$, and in particular*

$$\lim_{t \to -\infty} \textit{dim } V^+(u(t)) - \lim_{t \to +\infty} \textit{dim } V^+(u(t)) \leq \textit{dim ker } L_i \leq \textit{dim ker } L_i^- < \infty.$$

*(The condition $u \in U_{reg}$ is only necessary for the $i = 2$ case.)*

*Proof.* Notice that the first term of (7.9) has finite dimensional kernel by Lemma 92 and closed image by Theorem 93. The second term of (7.9) is a compact operator since $u \to f_-$ uniformly. Thus $L_i^-$ has a finite dimensional kernel. Let span$\{v_m\}_{m=1}^M = \ker L_i^-$ and consider the set of Cauchy problems

$$\begin{cases} \frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} - 2uh \text{ for } t > 0 \\ h(0, x) = v_m(0, x). \end{cases} \tag{7.11}$$

Standard parabolic theory gives uniqueness of solutions to (7.11), and that a solution $h$ lies in the kernel of $L_i^+$, the restriction of $L_i$ to $[0, \infty) \times \mathbb{R}$. Therefore dim ker $L_i \leq$ dim ker $L_i^- < \infty$.

For the other inequality, modify $u$ outside of $[-R,R]\times\mathbb{R}$ to get a $\bar{u}$ so that the linearization $\overline{L_i}$ of $N$ about $\bar{u}$ satisfies

- ker $\overline{L_i}$ is isomorphic to ker $L_i$ as vector spaces,
- $\bar{u}|_{(-\infty,-R)\times\mathbb{R}} = f_-$, and
- $\bar{u}|_{(R,\infty)\times\mathbb{R}} = f_+$.

We can do this for a sufficiently large $R$, since $u$ tends uniformly to equilibria. Then the flow of

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + 2\bar{u}h$$

defines an injective linear map from the timeslice at $-R$ to the timeslice at $R$. (That is, it gives an injective map from $C^{0,\alpha}(\mathbb{R})$ to itself – injectivity being an expression of the uniqueness of solutions.) Each element $v$ of the kernel of $\overline{L_i}$ evidently must have $v(-R)\in V^+(f_-)$ and $v(R)\notin V^+(f_+)$. Therefore, the injectivity ensures that the intersection of the image under the flow of $V^+(f_-)$ with the complement of $V^-(f_+)$ has at least dimension dim $V^+(f_-)$ − dim $V^+(f_+)$. $\square$

**Remark 98.** Multiplication by $u$, $C^1(\mathbb{R}^2, C^{0,\alpha}(\mathbb{R})) \to C^0(\mathbb{R}^2)$ is not a compact operator, in particular note that dim ker $L_i^+ = \infty$.

**Theorem 99.** *Let $u$ be a heterocline of* (7.1) *which connects equilibria $f_\pm$. There exists a union $\bigcup M_u$ of finite dimensional submanifolds $M_u$ of $C^1(\mathbb{R}, C^{0,\alpha}(\mathbb{R}))$ which*

- *contains $u$ and*
- *consists of heteroclines connecting $f_-$ to $f_+$.*

*If $u\in U_{reg}$, then $M_u$ has dimension $\lim_{t\to-\infty}$ dim $V^+(u(t)) - \lim_{t\to\infty}$ dim $V^+(u(t))$, and this is maximal among such submanifolds $M_u$.*

*Proof.* Observe that $L_1$ is surjective, since it is easy to show that the formula

$$\Gamma_1(w)(t) = \int_t^0 e^{-\int_0^{T-t} H(\tau)d\tau} w(T,x)dT$$

is a well defined right inverse of $L_1$. This involves showing that

$$e^{-\int_0^t H(\tau)d\tau} = \frac{1}{2\pi i}\int_{C(t)} e^{st}(H(t)+s)^{-1}ds$$

converges, where we note that the contour changes with time. As it happens, the computation in [20] goes through with the only change that at $t = 0$, we deflect the contour to the right, rather than the left (as in Figure 7.1). Since Lemma 97 shows that $L_1$ has finite dimensional kernel, then it follows that $M_u = N_1^{-1}(0)$ is a union of finite dimensional manifolds, with a finite maximal dimension. It is obvious that $M_u$ consists entirely of heteroclinic orbits and contains $u$.

It remains to show that the dimension of $M_u$ is as advertised and maximal. Observe that $L_2$ is a compact perturbation of an operator $L_2' : Y_{\lambda_2}(\mathbb{R}) \to Z_{\lambda_2}(\mathbb{R})$ which is time-translation invariant. This follows from the precise choice of $\lambda_2$ being continous and not intersecting the eigenvalues of $H$. $L_2$ and $L_2'$ are both surjective by exactly the same reasoning as for $L_1$. $L_2'$ is injective by using separation of variables as in Lemma 92 (noting that all nontrivial solutions blow up in the $Y_{\lambda_2}$ norm). Therefore the Fredholm index of $L_2'$, hence $L_2$ is zero. However, this implies that $L_2$ is injective.

Since $L_2$ is bijective, any solution to $L_2 u = 0$ which decays faster than $e^{\int \lambda_2(t)dt}$ as $t \to -\infty$ ends up growing faster than $e^{\int \lambda_2(t)dt}$ as $t \to +\infty$, and in particular does not tend to zero. As a result, such a solution cannot be in ker $L_1$. This implies that dim ker $L_1 \leq \lim_{t\to-\infty} \dim V^+(u(t)) - \lim_{t\to\infty} \dim V^+(u(t))$, which with the estimate in Lemma 97 completes the proof. □

**Remark 100.** Even if $u \notin U_{reg}$ (when there exist nonsimple eigenvalues of $H(t)$), the function $\lambda_1$ can still be constructed. As a result, we *always* get that the connecting manifold $M_u$ is finite-dimensional.

**Corollary 101.** *The space of heteroclinic orbits has the structure of a cell complex with finite dimensional cells. This cell complex structure is evidently finite dimensional if there exist only finitely many equilibria for* (7.1)*.*

## 7.4 Conclusions

We have shown that the tangent space at an equilibrium splits into a finite dimensional unstable subspace, and infinite dimensional center and stable subspaces. However, it is quite clear by Chapter 6 that the center subspace is nonempty and large. Indeed, considering the work of [38], the center and stable subspaces are not closed complements of each other. Additionally, we have given conditions for the space of heteroclinic orbits to have a finite dimensional cell complex structure.

CHAPTER 8

# AN EXTENDED EXAMPLE

## 8.1 Introduction

Consider the following equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u^2 + (x^2 - c)e^{-x^2/2}, \tag{8.1}$$

where the choice of $\phi$ in (1.6) has been fixed. The bifurcation diagram for the equilibria of (8.1) can be found in Figure 4.10. Based on the Theorem 93, the number of positive eigenvalues shown in Figure 4.10 corresponds exactly to the dimension of the unstable manifold of each equilibrium.

## 8.2 Frontier of the stable manifold

According to Figure 4.10, when $c = -1.2$, there is only one equilibrium, $f_0$. It has empty unstable manifold, though of course it is asymptotically unstable (as is shown in Chapter 6). On the other hand, $f_0$ has an infinite dimensional stable manifold, which is not all of $C^{0,\alpha}(\mathbb{R})$, as a consequence of the asymptotic instability. As a result, its stable manifold has a frontier in $C^{0,\alpha}(\mathbb{R})$ (which may not be a boundary in the sense of a manifold with boundary). We are interested in the qualitative behavior of solutions near and along this frontier. We know by Theorem 28 that if they tend to $f_0$ uniformly on compact subsets, then they do so uniformly. It is enlightening to use a numerical procedure to this end. We start solutions at the following family of initial conditions

$$u_A(x) = f_0(x) + Ae^{-x^2/10}. \tag{8.2}$$

Using the Fujita technique (exactly as shown in Chapter 6), we can show that for sufficiently negative $A$, the solution started at $u_A$ will not be eternal. As a result,

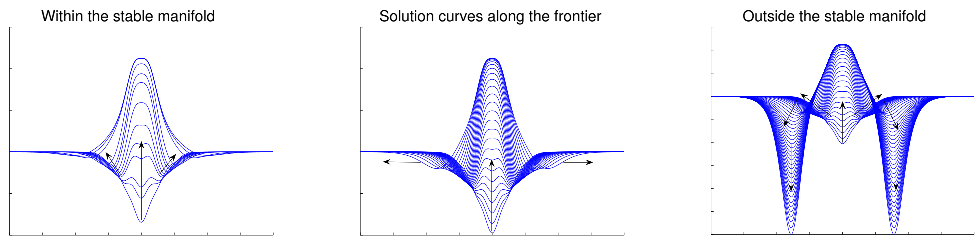


Figure 8.1: Behavior of solutions near the frontier of the stable manifold of $f_0$ (horizontal axis is $x$)

the family of initial conditions $u_A$ intersects the frontier of the stable manifold of $f_0$. An approximation to the value of $A$ which corresponds to the frontier can be easily found using a binary search. Some typical such solutions are shown in Figure 8.1, and the approximate value of $A$ corresponding to the frontier is $A \approx -2.15$

The qualitative behavior shown in Figure 8.1 indicates that there is some kind of traveling disturbance in the frontier solutions, which seems like a traveling wave. However, such a solution also appears to tend uniformly on compact subsets to $f_0$, so in fact it converges uniformly. (The uniform convergence is not obvious from the figure, due to the numerical solution being truncated at a finite time.) The leading edge of this disturbance collapses to $-\infty$ in finite time for solutions just outside the stable manifold of $f_0$.

## 8.3 Flow near equilibria with two-dimensional unstable manifolds

Also of interest is the structure of the flow in the unstable manifold of the "fork arms" which occur at $c = 0.0740$, as they approach the pitchfork bifurcation at

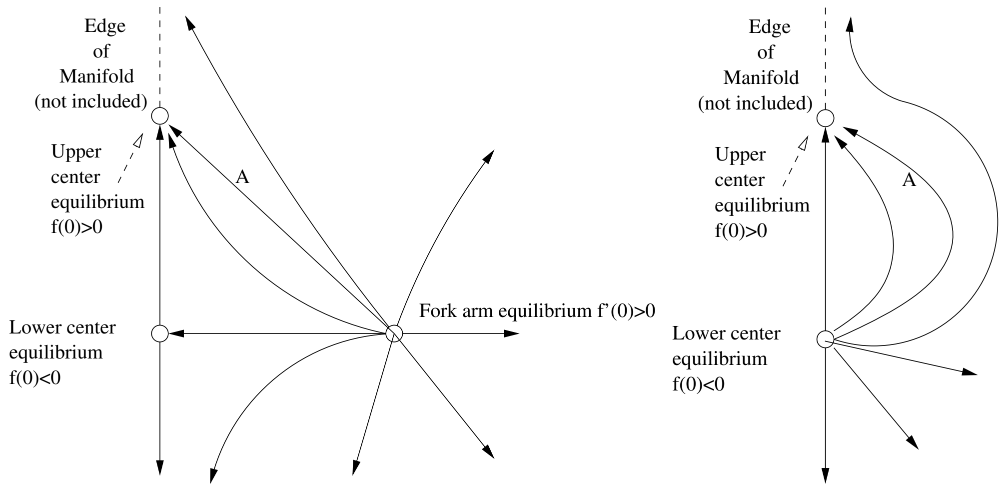


Figure 8.2: Flow in the unstable manifold of a "fork arm." $c = 0.0600$ (left); $c = 0.0501$ (right)

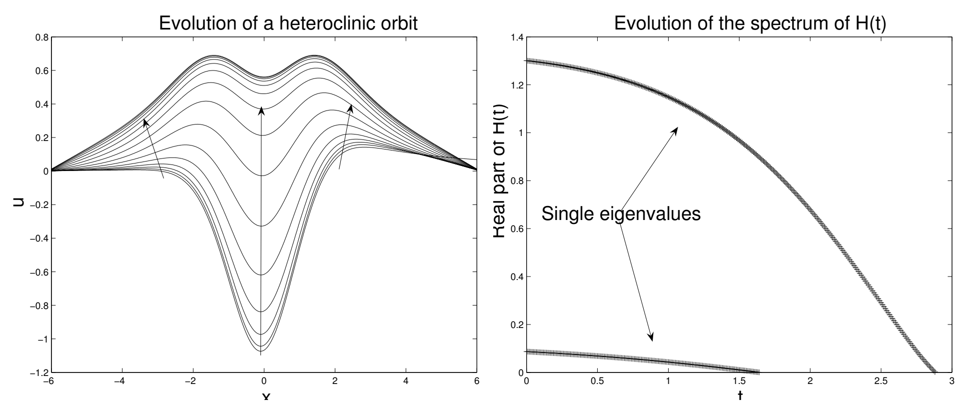


Figure 8.3: A typical heteroclinic orbit to the left of boundary A, with the spectrum of $H(t)$ as a function of $t$.

$c = 0.0501$. Figure 8.2 shows a schematic of the flow based on numerical evidence. Of particular interest is the behavior near the boundary marked A. Solutions to the right of the boundary are not eternal solutions – they fail to exist for all $t$. Solutions to the left of A are heteroclinic orbits connecting the equilibrium with an unstable manifold of dimension 2 to the equilibrium with an unstable manifold of dimension zero. A typical such solution is shown in Figure 8.3.

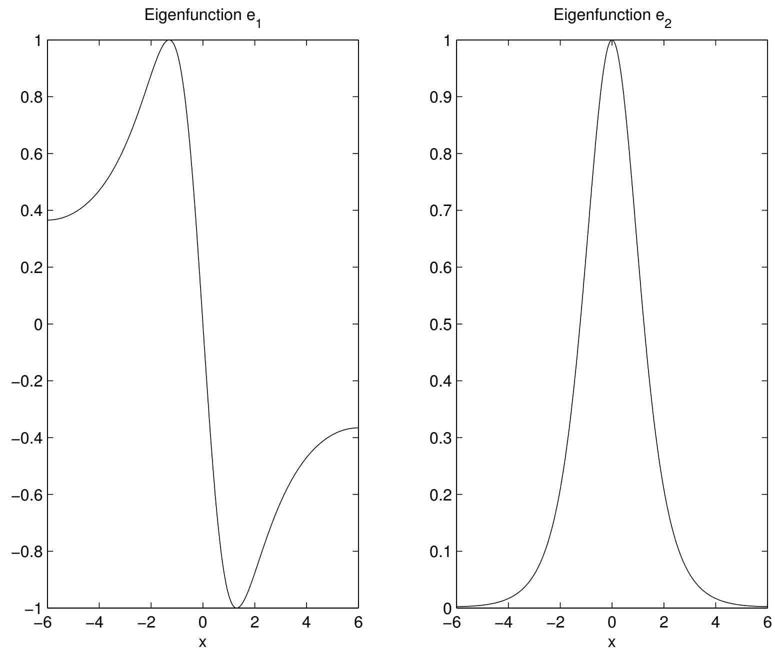


Figure 8.4: Eigenfunctions describing unstable directions at $f_1$

To examine solutions near the boundary A, we center our attention on the case $c = 0$, which has two equilibria, one of which (call it $f_1$) has a 2-dimensional unstable manifold. (This corresponds to the right pane of Figure 8.2.) If we linearize about $f_1$, the operator $H = \frac{\partial^2}{\partial x^2} - 2f_1 : C^{0,\alpha}(\mathbb{R}) \to C^{0,\alpha}(\mathbb{R})$ has a pair of simple eigenvalues, as is easily seen in the right pane of Figure 8.3 at $t = 0$. One of these eigenvalues is smaller, to which is associated the eigenfunction $e_1$ in Figure 8.4. The eigenfunction $e_2$ is associated to the larger eigenvalue. In Figure 8.2, $e_1$ corresponds to the horizontal direction, and $e_2$ corresponds to the vertical direction. From the proof of Lemma 92, it is clear that $\{e_1, e_2\}$ spans the tangent space of the unstable manifold at $f_1$. Therefore, we specify initial conditions $u_{A,\theta}(x)$ for a numerical solver using

$$u_{A,\theta}(x) = f_1(x) + A\left(e_1(x)\cos\theta + e_2(x)\sin\theta\right). \tag{8.3}$$

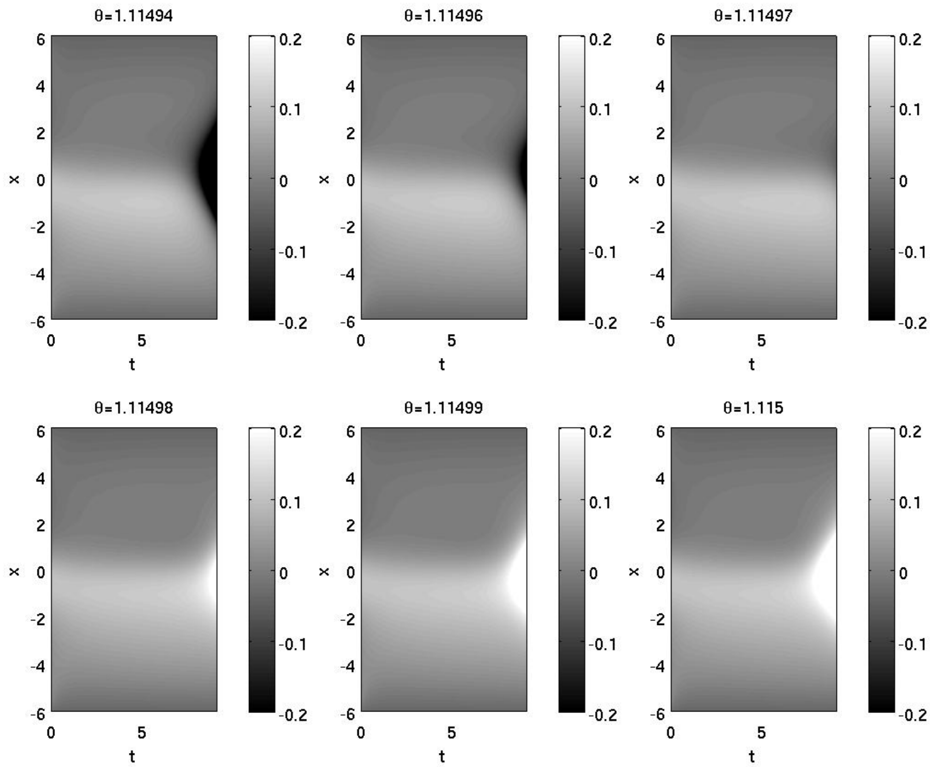


Figure 8.5: Difference between equilibrium $f_1$ and the numerical solution started at $u_{A,\theta}$, where black indicates a value of -0.2, and white indicates 0.2. The horizontal axis represents $t$, and the vertical axis represents $x$. $A = 0.1$ in all figures. Starting from the upper left, $\theta = 1.11494, 1.11496, 1.11497, 1.11498, 1.11499, 1.115$.

(Taking $A$ small allows us to approximate solutions which tend to $f_1$ in backwards time.) Since the perturbations along $e_1, e_2$ are quite small, and indeed the eigenvalue associated to $e_1$ is much smaller than that associated to $e_2$, examining the numerical results of evolving $u_{A,\theta}$ is quite difficult. The behavior along the boundary occurs at a much smaller scale than $f_1$, yet is crucial in determining the long-time behavior of the solution. To remedy this, the boundary behavior is better emphasized by plotting $u_{A,\theta}(t, x) - f_1(x)$ instead. Figure 8.5 shows the results of evolving initial conditions (8.3) for $A = 0.1$ and various values of $\theta$.

Solutions in Figure 8.5 show a similar kind of behavior as in the case of the frontier of $f_0$. There is a traveling front, which moves very slowly in the negative $x$-direction. However, the behavior is quite a bit more delicate. The determining factor in locating the frontier of $f_0$ is the perturbation in a direction roughly like $e_2$, which has a large eigenvalue. On the other hand, for $f_1$, Figure 8.2 indicates that such a direction is not parallel to the boundary of the connecting manifold. (The boundary direction is some linear combination of $e_1$ and $e_2$, with a numerical value for the angle $\theta$ being roughly 1.114975 radians.) The eigenvalue associated to $e_1$ is roughly ten times smaller, and therefore perturbations in that direction are much more sensitive. Additionally, the action of the flow is therefore primarily in the direction of $e_1$, which tends to mask effects in other directions. For this reason, it was visually necessary to postprocess the numerical solutions by subtracting $f_1$ from them. Otherwise the presence of the traveling front was unclear.

CHAPTER 9

# CONJECTURES AND FUTURE WORK

## 9.1 Conjectures about the present problem

### 9.1.1 Analytical conjectures

It seems that under reasonable conditions on the coefficients of (1.2), all eternal solutions ought to be heteroclinic orbits. An easy calculation with the formula (1.3) for the action $A(u(t))$ shows that if $|A(u(t))|$ blows up, then one of the following is true:

1. $\|u(t)\|_1 \to \infty$,
2. $\|u(t)\|_\infty \to \infty$, or
3. $\|Du(u)\|_2 \to \infty$.

Essentially, eternal solutions which are not heteroclinic orbits are big in some sense. Of course, traveling fronts satisfy the first condition. On the other hand, the Harnack inequality seems to imply that eternal solutions do not blow up in the $\infty$-norm as $t \to -\infty$. More intriguingly, [45] and [38] show that under certain conditions on the coefficients of (1.2), global solutions to the forward Cauchy problem have a universal bound on their $\infty$-norm. However, these results are obtained under the hypothesis that the solution $u$ is strictly negative, a condition that is essential to their analysis. Relaxing this condition leads to currently open problems.

**Conjecture 102.** Suppose all of the coefficients $a_i$ in (1.2) decay sufficiently fast as $|x| \to \infty$. Then all eternal solutions are bounded in the $\infty$-norm by a universal bound, which depends only on the $a_i$.

More ambitious is the following (which involves proving a universal bound for the 1-norm as well):

**Conjecture 103.** Suppose all of the coefficients $a_i$ in (1.2) decay sufficiently fast as $|x| \to \infty$. Then all eternal solutions are heteroclinic orbits.

Related to both of these conjectures is the conjecture that under suitable decay conditions on the $a_i$, there exist only finitely many equilibria (Conjecture 75).

### 9.1.2 Conjectures related to the topology of the space of heteroclinic orbits

Much of what remains to be understood about the space of heteroclinic orbits of (1.2) and (1.6) involves a more precise understanding of the gluing maps between the cells in its cell complex structure. The eventual goal is to construct a homology theory, called a Floer homology, for the space of heteroclinic orbits. This would allow the space of heteroclinic orbits to be decomposed as a complex of connecting manifolds (without boundary) and boundary maps which associate higher dimensional manifolds to lower dimensional ones. From the outset, degeneracy in the sense of Morse provides the biggest obstacle to this kind of theory. In particular, *nondegeneracy* allows one to show that generically, connecting manifolds can only have boundaries of one dimension lower. However, in the example of the previous chapter, namely that of (1.6) with $\phi = (x^2 - c)e^{-x^2/2}$, such a statement is still true. Perhaps it is possible that one can find conditions for connecting manifolds to have codimension-1 boundaries, even in the face of degeneracy in the equilibria. Or put another way,

**Conjecture 104.** When the flow of (1.2) is restricted to the space of heteroclinic orbits $\mathcal{H}$, all of the equilibria become nondegenerate critical points in the sense of Morse.

Another obstacle is that there needs to be some kind of compactness result for the space of heteroclinic orbits modulo time translation (or perhaps modulo action of the flow). In Floer's case, he was able to employ Gromov's compactness results for pseudoholomorphic curves. This leads to his "no bubbling theorem". However, no such result is known in our case. The closest available results are those of [45] and [38], which only hold for *positive* solutions to (1.2) that have been centered on an equilibrium. It is easy to show that if there exists a positive equilibrium for (1.6), then their results suffice to show compactness, but the situation of general sign is currently an open problem.

To summarize, we have the following conjectures:

**Conjecture 105.** The space of heteroclinic orbits of (1.2) modulo time translation is compact in $Y_\lambda$.

**Conjecture 106.** There is a generic subset (a Baire subset) of choices for the coefficients $a_i$ in (1.2) so that if $u$ is a heteroclinic orbit, all of the eigenvalues of $H(t)$ are simple.

We then define $C_k(R)$ to be the free $R$-module generated by the $k$-dimensional connecting manifolds of (1.2). Probably it is best to think of $R = \mathbb{Z}/2$, at least to fix ideas.

**Conjecture 107.** For a generic subset of coefficients $a_i$ in (1.2), there is a collection of maps $\partial_k : C_k \to C_{k-1}$ such that

- $\partial_k$ is an $R$-module homomorphism, for each $k \geq 0$,
- $\partial_0 = 0$,
- $\partial_{k-1} \circ \partial_k = 0$ for each $k \geq 0$, and
- $v = \partial_k(u)$ if and only if roughly speaking $v$ is a sum of boundary elements of $u$, obtained by a deformation retraction of a neighborhood of $v$ in $u$ onto $v$. In Floer's case, as is likely in ours, this is a rather involved construction called the "gluing theorem" described in [16].

This would turn $(C_*(R), \partial_*)$ into what is most reasonably called a "Floer complex." One can then define the Floer homology modules, $F_k = \ker \partial_k / \mathrm{im}\ \partial_{k+1}$ and formulate the following (reasonable) conjecture:

**Conjecture 108.** Let $E$ be the space of heteroclinic orbits of (1.2). For a generic subset of $a_i$ in (1.2), $F_k \cong H_k(E; R)$. That is, the Floer complex computes the homology of the space of heteroclinic orbits.

## 9.2 Future work on related problems

### 9.2.1 Higher spatial dimensions, with decay conditions enforced

Of course, the most obvious dependence on 1-dimensional space is the equilibrium analysis of Chapter 4. The analogous nonlinear elliptic problem (1.7) is not well understood. Indeed, very little is known about (1.7) at all, especially if the solutions are allowed to be of general sign.

One thing is likely: the spatial decay of heteroclinic orbits is much slower – not in $L^1(\mathbb{R}^n)$. Worse, Sturm-Liouville theory is no longer available to control the eigenvalues of $H = \Delta - 2u$. Therefore, there might be infinitely many positive eigenvalues of the operator $H$, which accumulate at zero. As a result, equilibria might have infinite dimensional unstable manifolds. The analysis of the structure of the connecting manifolds will therefore not work, though there should be a filtration structure based on Lyapunov exponent for the unstable manifolds, which should allow for an infinite dimensional cell complex with finite dimensional cells. Additionally, the slower spatial decay will disrupt the finite energy classification scheme in Chapter 3.

### 9.2.2 Relaxation of decay conditions on the coefficients

If we no longer require that the coefficients $a_i$ decay to zero as $|x| \to \infty$, then (1.2) can support traveling wave solutions. Indeed, there can be extremely complicated and delicate traveling wave structures if the spatial dimension is also greater than 1. This will remove the uniform convergence to equilibria, of course. Also, likely is that what will be found is that the space of heteroclinic orbits is an infinite-dimensional cell complex, perhaps were the "cells" are Banach manifolds. The resulting dynamics can therefore be expected to become extremely complicated.

APPENDIX A

# SPECTRUM OF SCHRÖDINGER OPERATORS

## A.1 Introduction

This appendix recounts a few standard facts about the structure of the spectrum of the Laplacian and Schrödinger operators $\Delta, H : C^2(\mathbb{R}) \to C^0(\mathbb{R})$ respectively. Nothing in this appendix is original, but it is useful to have the facts and the requisite calculations available for reference.

### A.1.1 Spectrum of the Laplacian operator

**Proposition 109.** *The spectrum of* $\Delta : C^2(\mathbb{R}) \to C^0(\mathbb{R})$ *is the closure of the negative real axis.*

*Proof.* The spectrum of $\Delta$ contains all $\lambda \in \mathbb{C}$ for which $(\Delta - \lambda)$ is not injective. In other words, it contains the solutions to the equation

$$u'' - \lambda u = 0.$$

An elementary calculation yields that $u = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$. If $\lambda$ is real and non-positive, there are nontrivial bounded solutions (which are oscillatory or constant). Otherwise the nontrivial $C^2$ solutions are unbounded. Hence the spectrum must contain the closed negative real axis.

Next, we show $(\Delta - \lambda)^{-1}$ exists and is bounded away from the closed negative real axis. To show that $(\Delta - \lambda)^{-}$ exists, we find an inversion formula, which is valid when $\lambda$ does not lie on the closed negative real axis. To this end one can solve

$$u'' - \lambda u = f$$

using a slightly modified version of Calculation 8,

$$u(x) = \frac{1}{2\sqrt{\lambda}} \int f(y) \begin{Bmatrix} (-1)e^{-\sqrt{\lambda}(y-x)} & \text{if } \Re(\sqrt{\lambda})(y-x) > 0 \\ e^{-\sqrt{\lambda}(y-x)} & \text{otherwise} \end{Bmatrix} dy.$$

This inversion formula defines a bounded inverse for $(\Delta - \lambda)$ when $\lambda$ is not on the closed negative real axis. To see this, simply observe that

$$\frac{1}{2|\sqrt{\lambda}|} \int e^{-|\Re(\sqrt{\lambda})||y-x|} dy$$

is independent of $x$. This fact implies that $\|u\|_{C^2} \le K_\lambda \|f\|_{C^0}$ for some finite $K$ by differentiation under the integral. Therefore the complement of the closed negative real axis is in the complement of the spectrum of $\Delta$. □

## A.2 Spectrum of Schrödinger operators

The previous section can be generalized to the case of Schrödinger operators $H = (\Delta - V) : C^2(\mathbb{R}) \to C^0(\mathbb{R})$ to obtain a few results of interest. Assume that $V$ is a smooth function which satisfies $\lim_{|x|\to\infty} V(x) = A$.

**Proposition 110.** *The spectrum of $H$ contains the portion of the real axis less than or equal to $-A$. All of the eigenvalues of $H$ are contained in the portion of the real axis less than or equal to the supremum of $V$.*

*Proof.* Of course, the eigenvalues $\lambda$ are those where there are nontrivial solutions to the equation

$$u''(x) - (V(x) + \lambda)u(x) = 0. \tag{A.1}$$

Recast (A.1) as a first-order system, namely

$$\frac{d}{dx}\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ V(x)+\lambda & 0 \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix} = T(x)\begin{pmatrix} u \\ u' \end{pmatrix}. \tag{A.2}$$

Observe that the eigenvalues of $T(x)$ are purely imaginary if $\lambda$ is real and less than $-V(x)$. Thus, the flow restricted to a plane of constant $x$ consists of periodic orbits if and only if $\lambda \leq -V(x)$ (ignoring the origin, of course).

Now suppose that $\lambda$ is real and $\lambda < -A$. Then there exists an $R > 0$ such that for all $|x| > R$, $|f(x) - A| < \frac{1}{2}|\lambda + A|$. So on $\mathbb{R} - [-R, R]$, solutions to (A.1) will all tend to limiting cycles. On $[-R, R]$, solutions grow exponentially fast, at a rate of no more than $\sqrt{\|v\|_\infty + |\lambda|}$. Thus there exist nontrivial bounded solutions to (A.1), which are obviously in $C^2$. Hence the spectrum of $H$ contains $(-\infty, -A]$.

On the other hand, if $\lambda$ is real and $A \leq \sup V < \lambda$, the origin is always a saddle point for the first order system (A.2). Thus all solutions to (A.1) are unbounded. Likewise, if $\lambda \in \mathbb{C} - \mathbb{R}$, then the system (A.2) is a spiral, so $(H - \lambda)$ is injective for $\lambda \notin (-\infty, \sup V]$. □

It is best to treat the portion of the spectrum lying between $-A$ and $\sup V$ using Sturm-Liouville theory. Indeed, (A.1) on $[0, \infty)$ with boundary conditions

$$u(0) = a, \ \lim_{x\to\infty} u(x) = 0$$

is a classic Sturm-Liouville problem. It is known that the eigenvalues of this problem are discrete and accumulate only at zero.

**Proposition 111.** *Suppose $V \in C^0(\mathbb{R})$ is positive outside a compact interval. Then the operator $H$ has finitely many eigenvalues greater than $-A$.*

*Proof.* It suffices to show that any solution $u$ to

$$u'' - (V + A)u = 0$$

has finitely many zeros. Notice that (assuming by hypothesis that $\lambda > -A$)

$$-V - \lambda < -V - A$$

so the Sturm-Liouville comparision theorem states that any eigenfunction $v$ of $H$ with eigenvalue $\lambda$ has strictly fewer zeros than solutions to the $\lambda = -A$ case. Also, the zeros of solutions which are bounded for half intervals $(-\infty, x_0)$ are monotonic in $\lambda$.

By hypothesis, $V$ is positive on $\mathbb{R} - [-R, R]$ for some $R > 0$. By comparision with the case of $V \equiv 0$ on $\mathbb{R} - [-R, R]$, zeros of $u$ only occur on $[-R, R]$. By comparison with $\|V\|_\infty$, the number of zeros is proportional to $R\sqrt{\|V\|_\infty}$, and therefore finite. $\square$

It is Proposition 111 that ensures that the finite dimensionality results of Chapter 7 hold. In the case of higher spatial dimensions, Sturm-Liouville theory does not apply (at least if there is no assumed symmetries in the equilibria). It is therefore possible that there is no finite dimensionality for the space of heteroclines for higher spatial dimensions.

It will be technically important in Chapter 7 that $(H + s) = (\Delta - (V + s))$ is sectorial about any real $s$ not in the spectrum of $H$. It is then useful to consider the densely defined operator $(H + s) : C^{0,\alpha}(\mathbb{R}) \to C^{0,\alpha}(\mathbb{R})$ intead of $C^2 \to C^0$, where $0 < \alpha \leq 1$.

**Proposition 112.** *If $s \in \mathbb{R}$ is not in the spectrum of $H = (\Delta - V)$, then $(H + s) : C^{0,\alpha}(\mathbb{R}) \to C^{0,\alpha}(\mathbb{R})$ is sectorial for $0 < \alpha \leq 1$.*

*Proof.* We have already shown that the structure of the spectrum is favorable for the sectoriality of $(H+s) = (\Delta - (V+s))$, and the operator is densely defined if $\alpha > 0$. (It is *not* densely defined if $\alpha = 0$.) What remains is that the operator norm of the inverse must decay like $K/|s|$, and that its image must consist of continous functions. For the former:

$$\begin{aligned}
u'' - (V+s)u &= f \\
s\left(\frac{1}{s}\Delta - I\right)u &= Vu + f \\
u &= \frac{1}{s}\left(\frac{1}{s}\Delta - I\right)^{-1}(Vu+f) \\
u(x) &= -\frac{1}{2\sqrt{s}}\int e^{-|x-y|\sqrt{s}}(V(y)u(y)+f(y))dy,
\end{aligned}$$

where $s$ is not in the spectrum. Using Calculation 9,

$$\begin{aligned}
\|u\|_\infty &\leq \frac{1}{|s|}\left\|\left(\frac{1}{s}\Delta - I\right)^{-1}(Vu+f)\right\|_\infty \\
&\leq \frac{1}{|s|}\|Vu\|_\infty + \frac{1}{|s|}\|f\|_\infty \\
&\leq \frac{\frac{1}{|s|}}{1-\frac{1}{|s|}\|V\|_\infty}\|f\|_\infty.
\end{aligned}$$

Since $V$ and $f$ are assumed to decay to zero and the kernel $e^{-|x-y|\sqrt{s}}$ is in $L^1$, it is immediate that $u$ must also decay to zero. The following calculation shows that image of $(\Delta - (V+s))^{-1}$ consists of Lipschitz functions. Assume $x_1 < x_0$, so that

$$
\begin{aligned}
|u(x_0)-u(x_1)| &= \frac{1}{2\sqrt{s}}\left|\int_{-\infty}^{x_0} e^{(y-x_0)\sqrt{s}}(Vu+f)dy - \int_{x_0}^{\infty} e^{(x_0-y)\sqrt{s}}(Vu+f)dy\right. \\
&\quad \left. -\int_{-\infty}^{x_1} e^{(y-x_1)\sqrt{s}}(Vu+f)dy + \int_{x_1}^{\infty} e^{(x_1-y)\sqrt{s}}(Vu+f)dy\right| \\
&= \frac{1}{2\sqrt{s}}\left|\int_{-\infty}^{x_1}\left(e^{(y-x_0)\sqrt{s}} - e^{(y-x_1)\sqrt{s}}\right)(Vu+f)dy\right. \\
&\quad +\int_{x_1}^{x_0} e^{(y-x_0)\sqrt{s}}(Vu+f)dy \\
&\quad +\int_{x_0}^{\infty}\left(e^{(x_1-y)\sqrt{s}} - e^{(x_0-y)\sqrt{s}}\right)(Vu+f)dy \\
&\quad \left. +\int_{x_1}^{x_0} e^{(x_1-y)\sqrt{s}}(Vu+f)dy\right| \\
&\le \frac{1}{2\sqrt{s}}\|Vu+f\|_\infty\left(\int_{-\infty}^{x_1}\left|e^{(y-x_0)\sqrt{s}} - e^{(y-x_1)\sqrt{s}}\right|dy\right. \\
&\quad \left. +\int_{x_0}^{\infty}\left|e^{(x_1-y)\sqrt{s}} - e^{(x_0-y)\sqrt{s}}\right|dy + \frac{2}{\sqrt{s}}(1-e^{(x_1-x_0)\sqrt{s}})\right) \\
&\le \frac{1}{2\sqrt{s}}\|Vu+f\|_\infty\left(\frac{1}{\sqrt{s}}e^{x_1\sqrt{s}}\left|e^{-x_0\sqrt{s}} - e^{-x_1\sqrt{s}}\right|\right. \\
&\quad \left. +\frac{1}{\sqrt{s}}{}^{-x_0\sqrt{s}}\left|e^{x_1\sqrt{s}} - e^{x_0\sqrt{s}}\right| + \frac{2}{\sqrt{s}}(1-e^{(x_1-x_0)\sqrt{s}})\right) \\
&\le \frac{2}{|s|}\|Vu+f\|_\infty\left|e^{(x_1-x_0)\sqrt{s}}-1\right| \\
&\quad \to 0 \text{ as } |x_0-x_1|\to 0.
\end{aligned}
$$

But since we've chosen $x_1 < x_0$, the above calculation proves that $u$ is Lipschitz. Looking at the Lipschitz constant and the bound on $u$, it is immediate that the $C^{0,1}$-operator norm of $(H+s)^{-1}$ decays like $1/|s|$. Thus $(H+s)$ is sectorial. $\square$

It is important to remark that the above proof shows that in fact $(\Delta-(V+s))^{-1}$ is a bounded operator $C^0(\mathbb{R}) \to C^2(\mathbb{R})$. However, the norm of the operator does not decay as $\Im(s)\to\infty$ as required for a sectorial operator. In particular,

$$\left|\frac{du}{dx}\right| \le \frac{K_1\|Vu+f\|_\infty}{\sqrt{s}},$$

and

$$\left|\frac{d^2u}{dx^2}\right| \leq K_2\|Vu + f\|_\infty,$$

for $K_1, K_2$ independent of $s$ and $f$. Examples can be constructed to show that these bounds are tight.

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