RATE DISTORTION APPROACH TO BROADCASTING WITH SIDE INFORMATION

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We consider data dissemination from a single transmitter to multiple receivers with side information, which is possibly due to prior transmissions. Side information at receivers can be utilized to reduce the broadcasting rate at the transmitter. How to accomplish this is the main focus of this dissertation. We address this problem in three parts from an information theoretic point of view.

First we model the source as uniform vector of bits and each side information is an arbitrary subset of the source. Known as index coding problem [1], we approach this problem as a special case of rate-distortion with multiple receivers, each with side information. Specifically, using techniques developed for the rate-distortion problem, we provide two upper bounds and one lower bound on the optimal index coding rate. The upper bounds are based on specific choices of the auxiliary random variables in the best existing scheme for the rate-distortion problem [2], which is shown invalid for the general rate-distortion problem and improved in our work [3] later. The lower bound is based on a new lower bound for the general rate-distortion problem. The bounds are shown to coincide for a number of (groupcast) index coding instances, including all instances for which the number of decoders does not exceed three.

Then we consider rate-distortion with two decoders, each with distinct side information. This problem is well understood when the side information at the various decoders satisfies a certain degradedness condition. We consider cases

in which this degradedness condition is violated but the source and the side information consist of jointly Gaussian vectors. We provide a hierarchy of four lower bounds on the optimal rate. These bounds are then used to determine the optimal rate for several classes of instances.

Lastly, we consider a rate distortion problem with side information at multiple decoders. We provide an upper bound for general instances of this problem by utilizing random binning and simultaneous decoding techniques [4] and compare it with the existing bounds. Also, we provide a lower bound for the general problem, which was inspired by a linear-programming lower bound for index coding, and show that it subsumes most of the lower bounds in literature including the ones we used for the index coding and rate distortion with two decoders problems. Using these upper and lower bounds, we explicitly characterize the rate distortion function of a problem which can be seen as a Gaussian analogue of the "odd-cycle" index coding problem.

BIOGRAPHICAL SKETCH

Sinem Unal was born and raised in Eskisehir, where she attended Eskisehir Science High School. She received her BSc. degree in Electrical and Electronics Engineering with minor in Mechatronics Engineering from Middle East Technical University (METU), Ankara, Turkey in 2011 and received her MSc. degree in Electrical and Computer Engineering from Cornell University, Ithaca, NY in 2015. She was a summer intern at Bell Labs, Holmdel, NJ in 2014. Her research interests include coding for wireless networks, information theory and rate distortion theory.

She was a finalist for the Qualcomm Innovation Fellowship in 2014 and received Irwin D. Jacobs fellowship in 2011. Also, she ranked *second* among $\sim$ 2400 students in METU graduating class of 2011, got the best project award from METU Senior Design Projects Competition, and she was ranked 54^{th} among $\sim$ 1.5 million entrants in University Entrance Exam Turkey, 2006.

Dedicated to my family and Caner.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

With the increase on the number of smart mobile devices, mobile data traffic has been increasing and this trend is expected to continue in near future [7]. How to satisfy this increasing demand on data using the available resources effectively is a very broad question. One aspect of this question we consider is how to disseminate the data from a single transmitter to multiple receivers with side information. This side information can occur if users cached prior transmissions from the transmitter when media is distributed over a rate-constrained downlink.

One can utilize the side information at receivers to reduce the broadcasting rate at the transmitter. We can illustrate this on the following toy example. There are two receivers demanding the same file and suppose that first receiver is already downloaded the first half of the file, denoted as A, whereas second receiver has the second half of the file, denoted as B, on its cache. Instead of sending A and B separately, transmitter can send $A \oplus B$ to receivers by reducing the required rate to satisfy their demands. Motivated by this, the goal of this thesis is to investigate the fundamental limits of lossy data compression with side information at receivers in the framework of information theory. Our contributions are on the following three problems.

1.1 Prior Work and Overview of the Contributions

Index Coding

First we consider a general version of the *index coding* problem, in which a single encoder observes a vector-valued source, the components of which are i.i.d., uniformly-distributed, binary random variables. There are several decoders, each of which has a subset of the source components as side information, and seeks to losslessly reproduce a disjoint subset of source components. The encoder must broadcast a single message to all of the decoders, which allows all of them to reproduce their desired source components. We seek to understand what rate is required of the encoder's message when the encoder may code over many i.i.d. realizations of the source vector. Specifically, we seek estimates of the minimal rate that are both efficiently computable and provably close to the true minimal rate, at least under some conditions.

The index coding problem has attracted considerable attention since it was introduced (e.g., [1, 8, 9, 10, 11, 12, 13, 14]). The formulations studied for index coding vary along at least three different axis. First, one can impose structure on the demands and the side information. Birk and Kol's paper [1] introducing the problem focused on the case in which each source bit is demanded by exactly one decoder, and if Decoder i has Decoder j's demand as side information then Decoder j must have Decoder i's demand as side information. The problem instance can then be represented as an undirected graph, with the nodes representing the source bits (or equivalently, the decoders) and the edges representing the side information pattern. Slightly more generally, one can relax the symmetry assumption to obtain a directed, instead of undirected, graph.

Shanmugam *et al.* [10] call this *unicast* index coding. We shall consider here the more general version in which each decoder can demand any number of source components and the demands may be overlapping among the decoders. Shanmugam *et al.* call this *groupcast* index coding; we shall simply call it *index coding*.

A second independent axis along which index coding formulations vary is whether one allows for coding over time (vector codes) or whether the code must operate on each time instant separately (scalar codes). We shall focus on the former here, due to its intrinsic importance and its connection with rate-distortion theory. Furthermore, block coding can yield improved rates when compared to scalar codes [15].

Third, and finally, some works require the decoders to reproduce their demands with zero error [8, 10, 11, 12, 16] while others require that the block-error probability vanish [13, 14]. Yet another possibility is to require that the bit-error probability vanish. This work shall focus on the latter two.

Irrespective of the formulation, most work on index coding views the problem graph-theoretically [9, 10, 16], as in Birk and Kol's original paper. One can then lower and upper bound the optimal rate using graph-theoretic quantities such as the independence number, the clique-cover number, fractional clique-cover number (e.g. [11]), the min-rank [8] and others [9, 10, 16]. This approach has proven to be successful for showing the utility of coding over blocks for this problem [15], and for showing the utility of nonlinear codes [17]. Many of these graph-theoretic quantities are known to be NP-hard to compute, however, and for the others there is no apparent polynomial-time algorithm. Thus these bounds are only useful theoretically or when numerically solving small examples. A noteworthy exception is Theorem 2 of Blasiak *et al.*, which provides a polynomial-time-computable bound that is within a nontrivial factor of the optimal rate for arbitrary instances. The factor in question is quite large, however.

We approach index coding as a special case of the problem of lossy compression with a single encoder and multiple decoders, each with side information. This more general problem was introduced by Heegard-Berger [18] (but see Kaspi [19]) and is sometimes referred to as the *Heegard-Berger problem.* Index coding can be viewed as the special case in which the source, at each time, is a vector of i.i.d. uniform bits, the side information of each decoder consists of a subset of the source bits, and the distortion measure for Decoder i is the Hamming distortion between the subset of the source bits that Decoder i seeks to reproduce and Decoder i's reproduction of that subset. We then consider the minimum rate possible so that all of the decoders can achieve zero distortion.

Viewing the problem in this way allows us to apply tools from network information theory, such as random coding techniques, binning, the use of auxiliary random variables, etc. Using this approach, we prove two achievable bounds and an impossibility (or "converse") bound [20]. Furthermore, we characterize the optimal rate explicitly for the special case in which each source bit is present at either all of the decoders, none of the decoders, all but one of the decoders, or all but two of the decoders and other special cases [20, 21, 22], which we shall present in Chapter 2.

Vector Gaussian Rate Distortion with Side Information

In index coding problem, we model the source as an i.i.d. vector of uniform bits. Treating the source like this is appropriate if the source is first compressed by an optimal rate-distortion encoder. Thus index coding implicitly assumes a separation-based architecture in which lossy compression is performed first and then the broadcasting with side information is performed at the bit level. Ideally, one would perform both types of coding jointly. Indeed when we consider the index coding problem we do not fully exploit benefits network information theoretic tools. One of the advantages of these tools is that it allows one to consider the problems of lossy compression and coding for side information together, by allowing for a richer class of source models and distortion constraints.

Hence as a second problem, we consider a special case of the Heegard-Berger problem. We shall focus on the case in which the source and the side information at the decoders are all jointly Gaussian vectors. This class of instances is important in applications, since vector Gaussian sources are natural stepping stones on the path from discrete memoryless sources to more sophisticated models of multimedia. The vector Gaussian setup can also be motivated theoretically since, like index coding, it is one of the simplest classes of instances that does not have degraded side information structure in general. [1] We shall focus on the case of two decoders; unlike index coding, for vector Gaussian problems even the two-decoder case is nontrivial.

In Chapter 3, we provide a hierarchy of four lower bounds on the optimal rate for this problem. For three separate special cases, we show that at least one of the lower bounds matches the achievable rate [2, 18], thereby determining the optimal rate [23, 24, 25].

[1] The problem is well understood when it is *degraded*, i.e., the side information at one of the decoders is stochastically degraded with respect to the other's [18].

General Rate Distortion with Side Information

Now we consider the Heegard-Berger [18] problem, which is essentially the multiple-decoder extension of the Wyner-Ziv [26] problem. As we briefly mention in the index coding problem, in Heegard-Berger problem, an encoder with access to a source of interest broadcasts a single message to multiple decoders, each endowed with side information about the source. Each decoder then seeks to reproduce the source subject to a distortion constraint.

Even for two decoders, characterization of the rate-distortion function is a long-standing open problem. The rate-distortion function has been determined in several special cases, however, including when the side information at the various decoders can be ordered according to stochastic degradedness [18], when there are two decoders whose side information is "mismatch degraded" [5], when there are two decoders and the side information at decoder 2 is "conditionally less noisy" than the side information at decoder 1 and decoder 1 seeks to losslessly reproduce a deterministic function of the source [6]. Also, instead of imposing some degraded structure on side information, Benammar *et al.* considers degraded reconstruction sets at two decoders and characterize the rate distortion function when one component of the source is reconstructed at both decoders with vanishing block error probability and the other component of the source is only reconstructed at a single decoder [27]. Furthermore, various vector Gaussian instances of the problem [23, 24], shown in Chapter 3, are solved. Several important instances of the index coding problem have also been solved (e.g., [8, 11, 20]).

As a main contribution, we present new linear programming (LP) type upper bound and lower bound to the general instances of rate distortion problem with side information at multiple decoders by utilizing various tools such as random binning and simultaneous decoding. We also compare these bounds with the existing bounds and provide optimality results for several special cases of the problem [28, 29, 3] in Chapter 4.

1.2 Outline of the Dissertation

The dissertation is outlined as follows. In Chapter 2 we consider general index coding problem from a rate distortion point of view. Section 2.2 and 2.4 present problem formulations while in the remaining sections in Chapter 2 we present our main results. Chapter 3 considers vector Gaussian rate-distortion problem with side information at decoders. In section 3.3, we introduce the four lower bounds and section 3.4 presents main optimality results. Lastly, in Chapter 4 we focus on Heegard-Berger problem with multiple decoders. Section 4.3 and 4.3.1 contain the achievability (upper bound) results. In Section 4.4 we introduce the LP type lower bound. The remaining sections include the comparison of new bounds with the existing bounds and optimality results obtained by using these new bounds.

CHAPTER 2

A RATE-DISTORTION APPROACH TO INDEX CODING

2.1 Introduction

Here, we view the index coding problem as a special case of the rate distortion function with side information at decoders. First we provide two upper bounds to the optimal rate for the index coding problem. Both of the achievable bounds are built upon the achievable bound for the Heegard-Berger problem, which is due to Timo *et al.* [2]. The Timo *et al.* scheme involves an optimization over the joint distribution of a large number of auxiliary random variables; we provide two methods for selecting this distribution, the first of which is polynomial-time computable but only yields integer rate bounds, while the second is more complex but can yield fractional rates. The achievability results in this work are thus unusual in that the emphasis is on algorithms for selecting the joint distribution of auxiliary random variables rather than proving new coding theorems *per se*. It is worth noting that the Timo *et al.* result is representative of many achievability results in network information theory that take the form of optimization problems over the joint distribution of auxiliary random variables (e.g., [4]). The task of solving these optimization problems has received little attention in the literature.[1]

Our impossibility result is related to the "degraded-same-marginals" (DSM) impossibility result for broadcast channels [31, 32]. The idea is that the optimal rate can be computed exactly when the source S and the side information

[1]Indeed, the prospect that some of these "single-letter" optimization problems might be intrinsically hard to compute is intriguing and seemingly unexplored (though see Arikan [30]).

variables $Y_1, \ldots, Y_m$ can be coupled in such a way that

$$S \leftrightarrow Y_{\sigma(1)} \leftrightarrow Y_{\sigma(2)} \leftrightarrow \cdots \leftrightarrow Y_{\sigma(m)},$$

meaning that the random variables form a Markov chain in this order, where $\sigma(\cdot)$ is an arbitrary permutation [2]. We call such an instance one with *degraded side information*. One may then lower bound a given problem by providing (for example), Y_1 to Decoder 2, Y_1 and Y_2 to Decoder 3, etc., to form a degraded instance whose optimal rate is only lower than that of the original problem. We provide a lower bound in this spirit for the general Heegard-Berger problem that improves somewhat on that obtained via a direct application of the above technique. We shall call it the *Maximin Lower Bound*, abbreviated as *MLB*. When applied to the index coding problem, the *MLB* provides the same conclusion as a lower bound due to Blasiak *et al.*, although under slightly weaker hypotheses.

We use the *MLB* to show that our low-complexity achievable bound equals the optimal rate for any number of source components, so long as the number of decoders does not exceed three.[2] In fact, we show the more general result that the achievable bound equals the optimal rate for any number of source components and any number of decoders so long as each source component is present as side information at all of the decoders, none of the decoders, all but one of the decoders, or all but two of the decoders. It is apparent that every problem with three or fewer decoders must be of this form. We also show that the achievable bound coincides with the optimal rate when none of the source components are "excess," a concept that plays an important role in our achievable scheme and that shall be defined later.

The virtue of our low-complexity scheme is that its performance is

[2]Recall that we allow each decoder to demand more than one source component and each source component to be demanded by more than one decoder.

polynomial-time computable, whereas the relevant graph-theoretic performance characterizations are not apparently so, even when they are finite-blocklength. Our low-complexity achievable scheme bears some resemblance to the *partition multicast* scheme of Tehrani, Dimakis, and Neely [12]. For a comparison of partition multicast and other graph theoretic quantities such as hyperclique cover, local hyperclique cover etc., one can see [10]. Although our scheme does not subsume *partition multicast* (which is NP-hard to compute [12]), we do show that it is optimal in all explicit instances of the problem for which Tehrani, Dimakis, and Neely show that partition multicast is optimal.[3]

Although we are focused mainly on index coding, the results herein also have some significance for the Heegard-Berger problem. The *MLB*, mentioned earlier, is the best general lower bound for this problem; however, we will introduce lower bounds subsuming *MLB* in Chapter 3 and 4. Our conclusive results for the index problem represent some of the few nondegraded instances of the Heegard-Berger problem for which the optimal rate is known (see [5, 33, 23, 34, 35, 36] others). This work is also the first work that considers algorithms for selecting the distribution of the auxiliary random variables in the Timo *et al.* scheme.

As noted earlier, this work differs from much of the literature on index coding by approaching the problem as one of rate-distortion, or source coding. Some recent works have also approached the problem as one of channel coding [13, 14], and in particular, interference alignment. One of the advantages of the source coding approach espoused here is that it can readily accommodate richer source models and distortion constraints, including sources with mem-

[3]Tehrani *et al.* also show that partition multicast is optimal for the implicitly-defined class of instances for which clique cover is optimal.

ory, lossy reconstruction of analog sources, etc. The very formulation of index coding presumes that the sources have already been compressed down to i.i.d. uniform bits. Thus the index coding is "separated" from the underlying compression, when in fact there might be some advantage to combining the two, a topic that we shall consider in subsequent work. See the discussion after Corollary 1 for additional differentiation between this work and above approaches. Also, we will investigate rate distortion function with side information more in Chapter 3 and 4.

This chapter is outlined as follows. Section 2.2 formulates the Heegard-Berger problem and Section 2.3 provides the *MLB* for it. Section 2.4 formulates the index coding problem. Section 2.5 and 2.6 provide a lower bound and an upper bound for the problem respectively. Section 2.7 describes our first scheme of index coding, and Section 2.8 provides several optimality results for this scheme, including our results for three decoders. Section 2.9 describes our second scheme.

2.2 Problem Definition

We begin by considering the general form of the Heegard-Berger problem, as opposed to the index-coding problem in particular. There is a single encoder with source S and there are m decoders. Decoder i has a side information Y_i that in general depends on S. The encoder sends a message at rate R to the decoders, and Decoder i wishes to reconstruct the source with a given distortion constraint D_i. The objective is to find the rate distortion tradeoff for this problem setup. This is made precise via the following definitions.

Definition 1. *An* $(n, M, D_1, \ldots, D_m)$ code *consists of mappings*

$$
\begin{aligned}
&f : \mathcal{S}^n \to \{1, \ldots, M\} \\
&g_1 : \{1, \ldots, M\} \times \mathcal{Y}_1^n \to \hat{\mathcal{S}}_1^n \\
&g_2 : \{1, \ldots, M\} \times \mathcal{Y}_2^n \to \hat{\mathcal{S}}_2^n \\
&\vdots \\
&g_m : \{1, \ldots, M\} \times \mathcal{Y}_m^n \to \hat{\mathcal{S}}_m^n,
\end{aligned}
$$

$$
\mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n} d(S_k, \hat{S}_{(i)k})\right] \leq D_i, \ \forall i \in [m]
$$

where $\mathcal{S}$ *denotes the source alphabet,* $\mathcal{Y}_1, \ldots, \mathcal{Y}_m$ *denote the side information alphabets at Decoder* 1 *through* m *and* $\hat{\mathcal{S}}_1^n, \ldots, \hat{\mathcal{S}}_m^n$ *denote the reconstruction alphabets at Decoder* 1 *through* m *and* $d(.,.) \in [0\ \infty)$ *denotes a distortion measure and* $[m] = \{1, \ldots, m\}$*. Lastly, we call* f *the* encoding function at the encoder *and* g_i *the* decoding function at Decoder i *where* $i \in [m]$.

Definition 2. *A* rate distortion pair (R, D)*, where* $D = (D_1, \ldots, D_m)$ *is* achievable *if for every* $\epsilon > 0$*, there exists an* $(n, M, D_1+\epsilon, \ldots, D_m+\epsilon)$ *code such that* $n^{-1} \log M \leq R+\epsilon$.

Definition 3. *The* rate distortion function $R(D)$ *is defined as*

$$
R(D) = \inf\{R | (R, D) \textit{ is achievable}\}.
$$

Finding a computable characterization $R(D)$ is a long-standing open problem in network information theory. Currently, such a characterization is only available for a few special cases. Heegard and Berger themselves [18] provided one when the side information at the decoders is degraded. Watanabe [5] provided one for the case that the source consists of two independent components, the distortion constraints for both decoders are decoupled across the two components, and the side information at the two decoders is degraded "in mismatched order" (see [5] for the precise setup). Sgarro's result [34] implies a characterization for the problem in which two decoders both wish to reproduce the source losslessly, without any assumption on their side information (see also [35]). Timo *et al.* [36] provide a characterization for the two-decoder case when one decoder's side information is "conditionally less noisy" than the other's and the weaker decoder seeks to losslessly reproduce a deterministic function of the source. Timo *et al.* [33] solve various two-decoder cases in which the source consists of two components, say (X, Y), and one decoder has X as side information and wants to reconstruct Y while the other has Y as side information and wants to reconstruct X. The present authors determined the rate distortion region for the two-decoder problem with vector Gaussian sources and side information, subject to a constraint on the error covariance matrices at the two decoders [23]. Several (nondegraded) special cases in which both decoders wish to losslessly reproduce a function of the source have been solved by Laich and Wigger [37]. Of course, several instances of index coding that are not degraded have also been solved.

A general achievable result, i.e., an upper bound on $R(D)$, was provided by Heegard and Berger [18], which was corrected and extended by Timo *et al.* [2]. Later in Chapter 4, we propose a new general achievable scheme and show that scheme by Timo *et al.* is not correct for the general case either. Additional conditions on the messages are required to make the scheme valid. However, we would like to point out that versions of the scheme by Timo *et al.* that we utilize or base on in this thesis are all valid achievable schemes. We provide a computable lower bound on $R(D)$ for general instances of the problem in this section. This lower bound will be used later in Chapter 2 to solve several index coding instances.

2.3 Lower Bound for a Rate Distortion Function

We start our analysis by providing a lower bound to the general rate distortion problem.

Theorem 1 (Maximin Lower Bound, MLB). *Let the pmf's $P(S, Y_i)$ for all $i \in [m]$ be given. $R(D)$ is lower bounded by*

$$R_{MLB}(D) = \sup_{\bar{P}} \max_{\sigma} \bar{R}_{\sigma}(D) \tag{2.1}$$

where

$$\begin{aligned}\bar{R}_{\sigma}(D) = \min_{U_1,\ldots,U_m} [&I(S\,; U_{\sigma(1)}|Y_{\sigma(1)}) + I(S\,; U_{\sigma(2)}|U_{\sigma(1)}, Y_{\sigma(1)}, Y_{\sigma(2)}) + \cdots \\ &+ I(S\,; U_{\sigma(m)}|U_{\sigma(1)}, \ldots, U_{\sigma(m-1)}, Y_{\sigma(1)}, \ldots, Y_{\sigma(m)})]\end{aligned} \tag{2.2}$$

and

1) $\sigma(.)$ denotes a permutation on integers $[m]$

2) $\bar{P} = \{P(S, Y_1, \ldots, Y_m)| \sum_{Y_j: j\neq i} P(S, Y_1, \ldots, Y_m) = P(S, Y_i), \forall i \in [m]\}$

3) $(U_1, \ldots, U_m)$ is jointly distributed with $S, Y_1, \ldots, Y_m$ such that

$$(Y_1, \ldots, Y_m) \leftrightarrow S \leftrightarrow (U_1, \ldots, U_m)$$

and

4) there exist functions $g_1, \ldots, g_m$ such that

$$\mathbb{E}[d(S, g_{\sigma(i)}(U_{\sigma(i)}, Y_{\sigma(i)}))] \leq D_{\sigma(i)} \forall i \in [m], \tag{2.3}$$

5) $|\mathcal{U}_{\sigma(i)}| \leq |\mathcal{S}| \prod_{j=1}^{i-1} |\mathcal{U}_{\sigma(j)}| + (m + 2 - i)$ for all $i \in [m]$.

Proof of Theorem 1. The proof is given in Appendix 5.1 $\square$

The idea behind the proof was described in the introduction. Note that since the optimal rate only depends on the source and side information through the "marginals"

$$(S, Y_i) \quad i \in [m],$$

we may couple the Y_i variables to form a joint distribution $\bar{P}(S, Y_1, \ldots, Y_m)$ as we please, leading to the outer optimization in (2.1). Also note that a direct application of the DSM idea would yield the weaker bound in which $\cup_{j \leq i} Y_{\sigma(j)}$ appears as an argument to $g_{\sigma(i)}$ in (2.3).

Remark 1. *Since the proof constructs the U_i variables in a way that the joint distribution of $(U_1, \ldots, U_m)$ does not depend on the permutation $\sigma(\cdot)$, one could state the bound with the minimum over $U_1, \ldots, U_m$ outside the maximum over $\sigma(\cdot)$. This complicates the proof of the cardinality bounds in 5), however, and the maximin form of the bound is sufficient for the purposes of this chapter, so we shall defer consideration of this potential strengthening to later chapters.*

Next, we turn our focus to the *index coding* problem which can be viewed as a special case of the Heegard-Berger problem.

2.4 Index coding : Problem Formulation

For the m user index coding problem, each decoder α wants to reconstruct $\mathbf{f}_\alpha$, which is an arbitrary subset of the source $\mathbf{S}$, that is, a collection of i.i.d. Bernoulli($\frac{1}{2}$) bits at the encoder. There may be *overlapping demands*, i.e., more than one decoder may demand the same bit. Also, each Decoder α has side information $\mathbf{Y}_\alpha$ consisting of an arbitrary subset of the source. We assume that decoders do not demand a component of their own side information since they already have it, and we assume that $\mathbf{Y}_\alpha \neq \mathbf{Y}_\beta$, for all $\alpha \neq \beta$ since we can combine two decoders if they have the same side information. We may also assume that every source bit is demanded by at least one decoder, for otherwise that bit may be completely purged from the system.

Let $\mathbf{S_J}$ denote the part of the source which each decoder in a subset J of $[m]$ does not have and all decoders in $[m]\backslash J$ have as side information. If $J = \{\alpha\}$, i.e., a singleton, then for ease of notation we use $\mathbf{S}_\alpha$ instead of $\mathbf{S}_{\{\alpha\}}$. Since there are m decoders, we group the elements of $\mathbf{S}$ into 2^m disjoint sets such that $\mathbf{S} = \cup_{J\subseteq[m]}\mathbf{S_J}$. Note that each $\mathbf{S_J}$ may be empty, may consist of a single bit, or may consist of multiple bits.

Let $G_0 = \mathbf{S_{[m]}}$ denote the elements of the source that none of the decoders have, $G_m = \mathbf{S}_\emptyset$ denote the elements all decoders have, $G_{m-1} = \cup_{\alpha\in[m]}\mathbf{S}_\alpha$ denote elements that $m-1$ of the decoders have, $G_{m-2} = \cup_{\substack{\{\alpha,\beta\}\subseteq[m] \\ \alpha\neq\beta}}\mathbf{S}_{\{\alpha,\beta\}}$ denote elements that $m-2$ of the decoders have and so on. To ease the notation for the rest of the chapter, whenever we write a set $\{\alpha,\beta\}$, we assume $\alpha \neq \beta$ unless otherwise stated. Then $\mathbf{S}$ can be represented as $\mathbf{S} = \{G_0, G_m, G_{m-1}, \ldots, G_1\}$, as shown in Fig. 2.1.

The demand $\mathbf{f}_\alpha$ at Decoder α can be written in terms of components $\mathbf{S_J}$ of $\mathbf{S}$. For this, we introduce the following notation.

Let f_{IJ} denote the demand that is a subset of source $\mathbf{S_J}$ and is required by each decoder in a subset I of $[m]$ and by no decoders in $[m]\backslash I$. If $I = \{\alpha\}$, then for ease of notation we use $f_{\alpha J}$ instead of $f_{\{\alpha\}J}$. We will generally assume that $I \subseteq J$ since only decoders in J may have a demand about $\mathbf{S_J}$ and decoders in $[m]\backslash J$

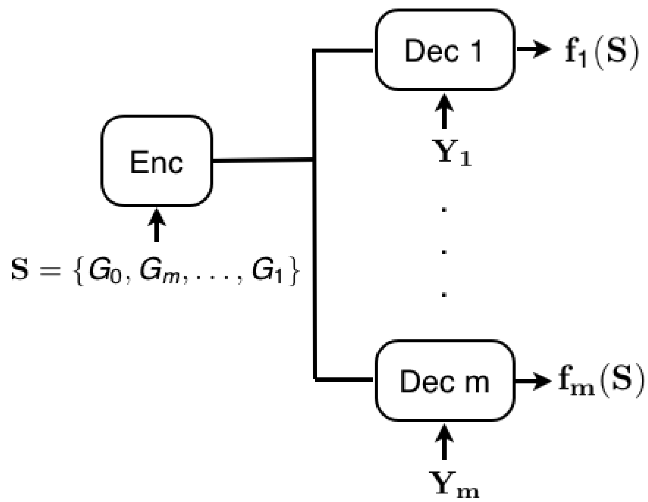

Figure 2.1: Index coding with m users.

already have $\mathbf{S_J}$ as side information. If $I \nsubseteq J$, f_{IJ} is empty. Also, f_{IJ} and f_{KJ} are independent (i.e., $f_{IJ} \perp f_{KJ}$) for all possible choices of I, K and J with $I \neq K$ since $f_{IJ} \cap f_{KJ} = \emptyset$ unless $I = K$. Lastly, each f_{IJ} may be empty, a single bit or may consist of multiple bits.

We have written the source as $\mathbf{S} = \{G_0, G_m, G_{m-1}, \ldots, G_1\}$ and the demands in terms of $\mathbf{S_J}$'s. From now on, we consider an ordered set structure on $\mathbf{S}$ which naturally induces orders on $\mathbf{S_J}$'s. Then each demand f_{IJ} is also an ordered set that can also be viewed as a vector. In fact, we shall find it convenient to view f_{IJ}, $\mathbf{S}$, and other similar quantities at times as sets and at times as vectors.

Since this problem can be considered as a special case of the Heegard-Berger problem, we use a similar definition for the code except for the distortion. Specifically, we consider block error probabilities instead of the distortion constraints stated in Definition 1. Hence, we use the following definitions for the code, error and optimal rate.

Definition 4. *Let $\mathcal{S}$ denote the alphabet of* $\mathbf{S}$*. An* (n, M) code *consists of mappings*

$$
\begin{aligned}
&f : \mathcal{S}^n \to \{1, \ldots, M\} \\
&g_1 : \{1, \ldots, M\} \times \mathcal{Y}_1^n \to \mathcal{F}_1^n \\
&g_2 : \{1, \ldots, M\} \times \mathcal{Y}_2^n \to \mathcal{F}_2^n \\
&\vdots \\
&g_m : \{1, \ldots, M\} \times \mathcal{Y}_m^n \to \mathcal{F}_m^n,
\end{aligned}
$$

where f denotes the encoding function at the encoder, g_α denotes the decoding function at Decoder α where $\alpha \in [m]$, and $\mathcal{F}_\alpha$ denotes the reconstruction alphabet at Decoder α.

Definition 5. *The* probability of error *for a given code is defined as*

$$P_e = Pr\{g_1(f(\mathbf{S^n}), \mathbf{Y_1^n}) \neq \mathbf{f_1^n}(\mathbf{S^n}) \cup g_2(f(\mathbf{S^n}), \mathbf{Y_2^n}) \neq \mathbf{f_2^n}(\mathbf{S^n}), \ldots, \cup g_m(f(\mathbf{S^n}), \mathbf{Y_m^n}) \neq \mathbf{f_m^n}(\mathbf{S^n})\},$$

where $\mathbf{f_i^n}(\mathbf{S^n}) \subseteq \mathbf{S^n}$ is the demand of decoder i that decoder i wants to reconstruct, $i \in [m]$.

Then achievability and optimal rate can be defined as follows.

Definition 6. *The rate R is* achievable *if there exists a sequence of (n, M) codes with rate $n^{-1} \log M \leq R$ such that the probability of error, P_e, tends to zero as n tends to infinity.*

Definition 7. *The* optimal rate R_{opt} *is defined as*

$$R_{opt} = \inf\{R | R \text{ is achievable}\}.$$

We shall call the problem defined in this section *index coding*, although most existing work on index coding requires the code to achieve zero, as opposed to vanishing, block error [10, 11, 12]. For index coding problem β is also used to denote the optimal rate [11],[15]. In support of the definitions adopted here, see [13, 14] for works that use vanishing block error probability and [38] for results connecting the two formulations.

2.5 Lower Bound for Index Coding

The next theorem gives a lower bound to the index coding problem using the *MLB* from Section 2.3.

Theorem 2. *The optimal rate of the index coding problem is lower bounded by*

$$R_{MLB} = \max_{\sigma} [H(\mathbf{f}_{\sigma(1)}|\mathbf{Y}_{\sigma(1)}) + H(\mathbf{f}_{\sigma(2)}|\mathbf{f}_{\sigma(1)}, \mathbf{Y}_{\sigma(1)}, \mathbf{Y}_{\sigma(2)}) + \cdots + H(\mathbf{f}_{\sigma(m)}|\mathbf{f}_{\sigma(1)}, \ldots, \mathbf{f}_{\sigma(m-1)}, \mathbf{Y}_{\sigma(1)}, \ldots, \mathbf{Y}_{\sigma(m)})] \tag{2.4}$$

where $\sigma(.)$ denotes a permutation on integers $[m]$.

Proof of Theorem 2. We will use the lower bound in Theorem 1 to prove the theorem. Note that this lower bound is for per-letter distortion constraints but it can be adapted to handle block error probabilities in the following way. Vanishing error probability, P_e, for index coding problem implies vanishing block error probability for each Decoder i, i.e., $Pr(g_i(f(\mathbf{S^n}), \mathbf{Y_i^n}) \neq \mathbf{f_i^n}(\mathbf{S^n}))$, which implies vanishing distortion with respect to Hamming distortion measure for Decoder i. Also, note that lower bound in Theorem 1 is continuous from right by Lemma 12. Hence, the optimal rate for the index coding problem, R_{opt}, is lower bounded by

$$R_{opt} \geq \max_{\sigma} \min_{U_1,\ldots,U_m} [I(\mathbf{S}; U_{\sigma(1)}|\mathbf{Y}_{\sigma(1)}) + I(\mathbf{S}; U_{\sigma(2)}|U_{\sigma(1)}, \mathbf{Y}_{\sigma(1)}, \mathbf{Y}_{\sigma(2)}) + \cdots + I(\mathbf{S}; U_{\sigma(m)}|U_{\sigma(1)}, \ldots, U_{\sigma(m-1)}, \mathbf{Y}_{\sigma(1)}, \ldots, \mathbf{Y}_{\sigma(m)})]$$

such that

1) $\sigma(.)$ denotes a permutation on integers $[m]$

2) $(U_1, \ldots, U_m)$ jointly distributed with $\mathbf{S}, \mathbf{Y_1}, \ldots, \mathbf{Y_m}$ such that

$$(U_1, \ldots, U_m) \leftrightarrow \mathbf{S} \leftrightarrow (\mathbf{Y_1}, \ldots, \mathbf{Y_m})$$

and

$$E\left[d(\mathbf{f}_{\sigma(\mathbf{i})}, g_{\sigma(i)}(U_{\sigma(i)}, \mathbf{Y}_{\sigma(\mathbf{i})}))\right] = 0, \forall i \in [m],$$

where $d(\cdot,\cdot)$ is Hamming distortion measure, giving

$$H(\mathbf{f}_{\sigma(\mathbf{i})}|U_{\sigma(i)}, \mathbf{Y}_{\sigma(\mathbf{i})}) = 0, \forall i \in [m]. \tag{2.5}$$

Note that since $\mathbf{Y_i} \subseteq \mathbf{S}$ for all $i \in [m]$, there is only one possible joint distribution of $(\mathbf{S}, \mathbf{Y_1}, \dots, \mathbf{Y_m})$. Hence, the maximum over $\bar{P}$ in (2.1) is degenerate for index coding. First we consider the permutation $\sigma(i) = i$ for all $i \in [m]$. Then we have,

$$\begin{aligned} R_{opt} \geq \min_{U_1,\dots,U_m} & [I(\mathbf{S}; U_1|\mathbf{Y_1}) + I(\mathbf{S}; U_2|U_1, \mathbf{Y_1}, \mathbf{Y_2}) + \cdots + \\ & I(\mathbf{S}; U_m|U_1, \dots, U_{m-1}, \mathbf{Y_1}, \dots, \mathbf{Y_m})] \end{aligned} \tag{2.6}$$

To find an explicit expression for (2.6), we use the following lemma.

Lemma 1. *For $j \in [m]$ we define*

$$K_j = \sum_{i=1}^{j-1} I(\mathbf{f_i}; U_1^i|\mathbf{Y_1^i}, \mathbf{f_1^{i-1}}) + I(\mathbf{S}; U_1^j|\mathbf{Y_1^j}, \mathbf{f_1^{j-1}}) + \sum_{i=j+1}^{m} I(\mathbf{S}; U_i|\mathbf{Y_1^i}, U_1^{i-1}),$$

where $\mathbf{f_1^i} = (\mathbf{f_1}, \dots, \mathbf{f_i})$ and likewise for U_1^i etc. Then $K_1 \geq K_2 \geq \ldots \geq K_m$.

Proof of Lemma 1. We fix any $j \in [m-1]$ and write,

$$\begin{aligned}
&K_j - K_{j+1} \\
&= -I(\mathbf{f_j}; U_1^j|\mathbf{Y_1^j}, \mathbf{f_1^{j-1}}) + I(\mathbf{S}; U_1^j|\mathbf{Y_1^j}, \mathbf{f_1^{j-1}}) - I(\mathbf{S}; U_1^{j+1}|\mathbf{Y_1^{j+1}}, \mathbf{f_1^j}) + I(\mathbf{S}; U_{j+1}|\mathbf{Y_1^{j+1}}, U_1^j) \\
&\overset{a}{=} I(\mathbf{S}; U_1^j|\mathbf{Y_1^j}, \mathbf{f_1^j}) - I(\mathbf{S}; U_1^{j+1}|\mathbf{Y_1^{j+1}}, \mathbf{f_1^j}) + I(\mathbf{S}; U_{j+1}|\mathbf{Y_1^{j+1}}, U_1^j) \\
&= I(\mathbf{S}; U_1^j|\mathbf{Y_1^j}, \mathbf{f_1^j}) - I(\mathbf{S}; U_1^j|\mathbf{Y_1^{j+1}}, \mathbf{f_1^j}) - I(\mathbf{S}; U_{j+1}|\mathbf{Y_1^{j+1}}, \mathbf{f_1^j}, U_1^j) + I(\mathbf{S}; U_{j+1}|\mathbf{Y_1^{j+1}}, U_1^j) \\
&\overset{b}{\geq} 0,
\end{aligned}$$

where

a: is due to the chain rule and reconstructions being subsets of the source, $\mathbf{S}$.

b: is due to the side information and reconstructions being subsets of the source, $\mathbf{S}$, and $(U_1, \ldots, U_m) \leftrightarrow \mathbf{S} \leftrightarrow (\mathbf{Y_1}, \ldots, \mathbf{Y_m})$. $\square$

Then (2.6) becomes,

$$\begin{aligned}
&R_{opt} \\
&\geq K_1 \\
&\geq K_m, \text{ by Lemma 1} \\
&= \sum_{i=1}^{m-1} I(\mathbf{f_i}; U_1^i | \mathbf{Y_1^i}, \mathbf{f_1^{i-1}}) + I(\mathbf{S}; U_1^m | \mathbf{Y_1^m}, \mathbf{f_1^{m-1}}) \\
&\geq \sum_{i=1}^{m} I(\mathbf{f_i}; U_1^i | \mathbf{Y_1^i}, \mathbf{f_1^{i-1}}) \\
&= \sum_{i=1}^{m} H(\mathbf{f_i} | \mathbf{Y_1^i}, \mathbf{f_1^{i-1}}) - H(\mathbf{f_i} | U_1^i, \mathbf{Y_1^i}, \mathbf{f_1^{i-1}}) \\
&= \sum_{i=1}^{m} H(\mathbf{f_i} | \mathbf{Y_1^i}, \mathbf{f_1^{i-1}}), \text{ from (2.5).}
\end{aligned} \tag{2.7}$$

We can apply the same procedure to all $m!$ permutations which gives the result.

$\square$

Remark 2. *Evidently the proof shows that the conclusion holds even if one only requires that the bit-error probability, as opposed to the block-error probability, vanish.*

Remark 3. *Let us consider one of the $m!$ expressions of the lower bound in Theorem 2, say the one in (2.7). We can rewrite it as*[4]

$$\begin{aligned}
&H(\mathbf{f_1} \setminus \mathbf{Y_1}) + H(\mathbf{f_2} \setminus \{\mathbf{f_1} \cup \mathbf{Y_1} \cup \mathbf{Y_2}\}) + \cdots + H(\mathbf{f_m} \setminus \{\mathbf{Y_m}, \cup_{i=1}^{m-1} \{\mathbf{Y_i}, \mathbf{f_i}\}\}) \qquad (2.8) \\
&= |\mathbf{f_1} \setminus \mathbf{Y_1}| + |\mathbf{f_2} \setminus \{\mathbf{f_1} \cup \mathbf{Y_1} \cup \mathbf{Y_2}\}| + \cdots + |\mathbf{f_m} \setminus \{\mathbf{Y_m}, \cup_{i=1}^{m-1} \{\mathbf{Y_i}, \mathbf{f_i}\}\}|.
\end{aligned}$$

[4]Recall that we assume an ordered set structure on $\mathbf{S}$ which naturally induces orders on $\mathbf{S_J}$'s and demands $\mathbf{f_i}$'s.

Blasiak et al. *[11] define an* expanding sequence *of decoders as one for which each decoder in the sequence demands a bit that is not contained in the union of the demands and the side information of the decoders that appear earlier in the sequence. Blasiak* et al. *prove that the size of a largest expanding sequence is a lower bound on the optimal rate. Writing the above bound as in (2.8) shows that it coincides with the Blasiak* et al. *bound when each decoder demands a single bit. Of course, the more general case in which a decoder may demand multiple bits can be obtained from the Blasiak* et al. *result by replacing each such decoder with multiple decoders that each demand a single bit. The Blasiak* et al. *result does not quite imply Theorem 2, however, since the former assumes a zero-error formulation (though one could appeal to a result of Langberg and Effros [38] to relate the two formulations).*

2.6 Achievable Scheme for Index Coding

For our achievable scheme for index coding, we rely on an achievability result of Timo *et al.* [2], mentioned earlier, for the general Heegard-Berger problem (see also Heegard and Berger [18]). Since the Timo *et al.* scheme is rather complicated, we shall state it in a substantially weakened form that will be sufficient for our purposes.

Proposition 1. *(cf. [2, Theorem 2]) The optimal rate R_{opt} of an index coding problem is upper bounded by*

$$\min \sum_{I \subseteq [m]} \left[\max_{i \in I} H(U_I | \mathbf{Y_i}) \right] \tag{2.9}$$

where the minimization is over the set of all random variables U_I jointly distributed with $\mathbf{S}$ such that

1) There exist functions

$g_1(\cup_{I:1\in I}U_I, \mathbf{Y_1}), \ldots, g_m(\cup_{I:m\in I}U_I, \mathbf{Y_m})$ *such that*

$$g_i(\cup_{I:i\in I}U_I, \mathbf{Y_i}) = \mathbf{f_i}(\mathbf{S}), \textit{ for all } i \in [m].$$

2) Each U_I is a (possibly empty) vector of bits, each of which is the mod-2 sum of a set (possibly singleton) of source components.

The full-strength version of Timo *et al.*'s result omits the condition 2) but replaces the rate expression in (2.9) with one that is more complex. Under the condition 2), however, their expression is upper bounded by (2.9). Also Timo *et al.* state their result as an upper bound on $R(D)$ defined in Section 2.2, as opposed to R_{opt} as defined in Section 2.4. That is, they provide a guarantee on the expected time-average distortion, instead of on the block error probability that we use to define index coding. Their proof technique can be used to bound the block error probability with minimal modification, however.

One way of interpreting U_I is that it is a "message" that is "sent" to all Decoders i such that $i \in I$. That is, U_I includes some information about the source that is decoded by all of the decoders in I but is not available to any of the decoders in I^c. The contribution of U_I to the overall rate in (2.9) is simply the rate needed to send U_I to all of the decoders in I using standard binning arguments (and relying on the fact that U_I is a deterministic function of the source $\mathbf{S}$). Specifically, consider encoding and decoding a message U_I for a given $I \subseteq [m]$. We randomly bin the different realizations of U_I. The encoder then broadcasts the index of the bin containing the observed realization to all of the decoders. The decoders in I then identify the correct realization within the bin using typicality considerations [39, Section 15]. The required rate for sending U_I to decoder i, $i \in I$ is $H(U_I|\mathbf{Y_i})$. Since U_I needs to be obtained by all decoder i's where $i \in I$, the resulting rate for U_I to be successfully transmitted to all decoder i's,

$i \in I$, is $\max_{i \in I} H(U_I|\mathbf{Y_i})$. The other messages are handled similarly and decoder i then reconstructs its demand $\mathbf{f_i}(\mathbf{S})$ using messages $\cup_{I:i\in I} U_I$ and its side information $\mathbf{Y_i}$.

Evaluating this upper bound requires finding the optimal joint distribution of the U_I auxiliary random variables. Since each U_I is a deterministic function of $\mathbf{S}$, this is equivalent to finding the optimal such functions. Such an optimization problem is evidently quite complicated. We shall provide a polynomial-time heuristic for finding a feasible choice of the U_Is. Of the many different index coding schemes that have been proposed (e.g., [10, 8, 12, 13, 14]), ours most closely resembles the *partition multicast* of Tehrani, Dimakis, and Neely [12]. In the language of our setup, their scheme amounts to finding the optimal choice of the U_I subject to the constraint that each U_I must be a vector consisting of a (possibly empty) subset of the source components. Tehrani *et al.* show that finding this optimal choice is NP-hard [12]. Our scheme, in contrast, consists of three steps, the first two of which amount to a polynomial-time heuristic for finding a reasonable and feasible (but not necessarily optimal) choice of auxiliary random variables subject to the constraint that each U_I must be a vector consisting of a subset of the source components. Thus the output of the second step of our heuristic is a feasible solution to the optimization problem for which partition multicast is optimal. Our third step, however, replaces some of the U_I variables with ones that are more general functions of source, i.e., not just subsets of the source variables. Due to the similarity between our heuristic and partition multicast, we call our heuristic *coded approximate partition multicast (CAPM)*. Although CAPM is not guaranteed to be never worse than partition multicast, we shall show that it is optimal for all of the explicit scenarios for which Tehrani *et al.* show that partition multicast is optimal as well as some other, more general scenarios.

2.7 CAPM: Selection of U_I's in the Achievable Scheme for Index Coding

CAPM is a method for choosing a feasible choice of the auxiliary random variables U_I for $I \subseteq [m]$. Note that the number of auxiliary random variables is exponential in the number of decoders, although in typical instances most of these random variables will be null. To minimize the worst-case complexity of CAPM, therefore, we shall work with a linked list of the auxiliary random variables that are not null, which shall begin empty. We shall call all U_I auxiliary random variables for which $|I| = i$ "*level i messages*."

Step 1 : Beginning with an empty linked list of auxiliary random variables, we sequence through the vector of source bits. Any given bit must be in f_{KJ} for some $K \subseteq J \subseteq [m]$. So long as $J \neq [m]$, we seek to include this bit in $U_{K \cup J^c}$: if $U_{K \cup J^c}$ does not exist in our linked list of auxiliary random variables, then we add it to the list and set it equal to the source bit in question. If it already exists in the list, then we locate it, and we set $U_{K \cup J^c}$ to be a vector of bits consisting of all source bits that were included previously along with this newly included source bit. For a source bit in f_{KJ} where $J = [m]$, we include the bit in the auxiliary random variable $U_{[m]}$, i.e., the auxiliary random variable that is decoded by all of the decoders. This process is repeated until all of the source bits have been included in an auxiliary random variable. Note that each nonvoid auxiliary random variable is then simply a vector of source bits. Also note that each source bit will be included in exactly one auxiliary random variable.

We now sort the linked list so that all level-2 messages appear first, followed by all level-3 messages, etc. Note that all level-1 messages are necessarily empty (assuming there is more than one decoder), by virtue of the fact that every source bit is assumed to be demanded by at least one decoder, and source bits that no decoder has as side information are placed in $U_{[m]}$. The complexity of Step 1 is at most $O(s^2 \cdot m)$, where $s = |\mathbf{S}|$.

Remark 4. *See Proposition 3 to follow for a justification of this particular approach to allocating the source components among the different auxiliary random variables.*

Step 2 : Let U_I denote the first auxiliary random variable in the linked list. If $I = [m]$, i.e., this first auxiliary random variable is decoded by all of the decoders, then this U_I must be the only non-null auxiliary random variable (since they are sorted by level), in which case we skip Step 2 and proceed to Step 3. Suppose instead that $|I| < m$. Note that U_I's contribution to the overall rate is

$$\max_{i \in I} H(U_I|\mathbf{Y_i}).$$

In many cases $H(U_I|\mathbf{Y_i})$ will not be constant over $i \in I$. That is, some decoders in I will require a higher rate to decode U_I than others. When this happens we move some of the source bits in U_I to a higher-level message. Define the two decoder indices

$$i^* = \min\{i : H(U_I|\mathbf{Y_i}) = \min_{l \in I} H(U_I|\mathbf{Y_l})\} \tag{2.10}$$

and

$$j^* = \min\{j : H(U_I|\mathbf{Y_j}) = \max_{l \in I} H(U_I|\mathbf{Y_l})\}. \tag{2.11}$$

If $H(U_I|\mathbf{Y_{i^*}}) < H(U_I|\mathbf{Y_{j^*}})$, then there must exist a source bit in U_I that is contained in $\mathbf{Y_{i^*}}$ but not in $\mathbf{Y_{j^*}}$. We select the lowest-index source bit with this property and move it from U_I to some U_J such that $I \subset J$ and $|J| = |I| + 1$. If $|I| < m - 1$, then there are many such choices of J; J can be chosen arbitrarily, but for concreteness we shall assume the following. First we look for nonempty U_J's such that $I \subset J$ and $|J| = |I| + 1$. If we can find such a message or messages, we select the J with the lowest index that is not already in I. If that is not the case, J is obtained by adding to I the lowest index that is not already in I. We call the bit that is moved *leftover* or *excess.* We then recompute i^* and j^* according to (2.10) and (2.11), respectively, and move an additional bit to a higher-level message if necessary, repeating this process until U_I is such that $H(U_I|\mathbf{Y_{i^*}}) = H(U_I|\mathbf{Y_{j^*}})$. Note that this condition must eventually be satisfied, since after sufficiently many iterations, U_I will become null. Once this condition is satisfied for U_I, we apply the same procedure to the next auxiliary random variable in the linked list, and so on until this procedure has been applied to every variable in the linked list. It is possible that some auxiliary random variables in the linked list are made null through this procedure, in which case they are removed from the linked list. The complexity of Step 2 is $O(m^2 \cdot s^3)$.

Remark 5. *The rationale for moving source bits up to higher-level messages is as follows. A bit that is excess contributes to the maximum*

$$\max_{l \in I} H(U_I|\mathbf{Y_l}), \tag{2.12}$$

which is U_I's contribution to the overall rate. Thus removing this bit from U_I has the potential to reduce U_I's contribution to the rate (although it will not necessarily do so, if there are multiple l that achieve the maximum in (2.12); see the next remark). Of course, including this bit in a higher-level message, U_J, will tend to increase U_J's contribution to the rate. But it will only do so the source bit in question is not in the side information of one of the decoders l that achieve the maximum in

$$\max_{l \in J} H(U_I|\mathbf{Y_J}).$$

Thus moving the bit up one level often yields a rate reduction, and even if it does not, it may yield a rate reduction upon being elevated again during a later iteration.

Remark 6. *If there exists a unique $j \in I$ such that $H(U_I|\mathbf{Y_j}) = \max_l H(U_I|\mathbf{Y_l})$, then moving an excess bit to a higher-level message cannot increase the overall rate, and in some cases it may strictly decrease the rate. If the decoder with maximum rate is not unique, then moving an excess bit to a higher-level message can increase the rate, as in Example 1 to follow, although this increase is sometimes offset during later movements of excess bits, or during Step 3 (again as in Example 1). For this reason we move excess bits according to the procedure outlined in Step 2 even when such movements have the immediate effect of increasing the overall rate.*

Remark 7. *Finding the feasible allocation of source components among the various U_I variables that minimizes the rate in (2.9) is NP-hard, as shown by Tehrani* et al. *[12].*

Step 3 : In the final step, we exclusive-OR (XOR) some of the bits included in the auxiliary random variables. Let U_I denote the first auxiliary random variable in the linked list, and suppose that $V_1, \ldots, V_l$ denote the excess source bits that are included in U_I. Recall that bits placed in $U_{[m]}$ during Step 1 are not considered excess. For each i, let N_i denote the set of decoders that need (i.e., demand) V_i and let H_i denote the set of decoders that have V_i has side information. We search for a pair of components V_i and V_j such that $N_i \subset H_j$, $N_j \subset H_i$, and V_i and V_j were included in the same auxiliary random variable in Step 1 (that is, $N_i \cup H_i = N_j \cup H_j$). If there are no such V_i and V_j then we proceed to the next U variable in the linked list. Otherwise, we delete V_j from U_I, we replace V_i in U_I with $V_i \oplus V_j$, we replace N_i with $N_i \cup N_j$ and H_i with $H_i \cap H_j$. Since both V_i and V_j were placed in the same auxiliary random variable in Step 1, we view the new V_i as also being placed in that variable in Step 1, although of course the auxiliary random variables constructed in Step 1 did not involve taking the XOR of any of the source components. We then repeat this process, again looking for V_i and V_j such that $N_i \subset H_j$, $N_j \subset H_i$, and V_i and V_j were included in the same auxiliary random variable in Step 1. If we find such a pair, we replace them with their exclusive-OR. We repeat this process until there are no such pairs remaining. We then apply this procedure to all of the other auxiliary random variables in the linked list. The complexity of Step 3 is $O(m \cdot s^3)$.

Remark 8. *Evidently Step 3 will never increase the rate. Moreover, one could certainly exclusive-OR bits V_i and V_j satisfying $N_i \subset H_j$ and $N_j \subset H_i$ but for which V_i and V_j are not included in the same auxiliary random variable in Step 1 or for which either V_i or V_j are not excess bits. Choosing to exclusive-OR certain pairs of bits can foreclose other such choices, however, and the latter choices may ultimately lead to lower rates. The restriction that we only exclusive-OR bits that are excess and that originated in the same auxiliary random variable in Step 1 is intended to guide the process toward the most productive exclusive-OR choices. Of course, once the above process exhausts all of its exclusive-OR possibilities, one could look for exclusive-OR opportunities among bits that are not excess or that did not originate in the same auxiliary random variable. We shall not include this step in the heuristic, however, since it is not necessary in any of our optimality results or any of our examples.*

One can verify that this selection procedure provides a feasible choice of the U_I variables as follows. First note that the choice will be feasible after each step 1. This is because each source component is included in a U_I variable that is decoded by all of the decoders that demand it. Thus condition 1) in Proposition 1 is satisfied. Condition 2) is satisfied because each U_I consists of a subset of the source components. Step 2 only moves source components from a U_I to a U_J for which $I \subseteq J$, so it is evident that conditions 1) and 2) continue to hold. Finally, the exclusive-OR operation applied in Step 3 evidently never violates condition 2), and the specific conditions under which the exclusive-OR operation is applied ensures that condition 1) continues to hold.

Notation 1. *The achievable rate provided by CAPM is denoted by R_{CAPM}.*

Remark 9. *Note that CAPM specifies that each message U_I consists of a subset of the source components and possibly bits obtained by applying exclusive-OR to the source components, all of which are i.i.d Bernoulli random variables. Since each side information $\mathbf{Y_i}$ is a subset of the source $\mathbf{S}$, after applying CAPM, the resulting $H(U_I|\mathbf{Y_i})$ end up being the entropy of set of independent Bernoulli random variables. Then, CAPM can only give integer rates since the entropy of set of i.i.d Bernoulli random variables must be an integer.*

To illustrate CAPM, we provide three examples.

Example 1. *Consider the 4-decoder index coding problem instance with demands $f_{12^c}, f_{32^c}, f_{3\{1,2\}^c}, f_{1\{2,3\}^c}, f_{4[4]}, f_{2[4]}$ where each demand is one bit and $a^c = [m] \setminus \{a\}$. Now we show each step of CAPM.*

Step 1: At the end of this step we have

$U_{12} = f_{12^c}, U_{23} = f_{32^c},$

$U_{123} = f_{3\{1,2\}^c}, f_{1\{2,3\}^c},$

$U_{1234} = f_{4[4]}, f_{2[4]}.$

Step 2: We start with level-2 messages. The first level-2 message is U_{12}. f_{12^c} in U_{12} is an excess bit and since we already have level-3 message U_{123} which f_{12^c} can be placed we move f_{12^c} to U_{123}. The next message is U_{23}. f_{32^c} in U_{23} is an excess bit and it is also placed to U_{123}. The messages at this point are

$U_{123} = f_{3\{1,2\}^c}, f_{1\{2,3\}^c}, f_{12^c}, f_{32^c},$

$U_{1234} = f_{4[4]}, f_{2[4]}$

and we move on to level-3 messages. Note that there is only one level-3 message, U_{123}*. All demands in it are excess bits since* $H(U_{123}|\mathbf{Y_2}) = 0$*. We move all excess bits to* U_{1234}*, which is the only one level-4 message. This completes the Step 2 and we have*

$U_{1234} = f_{4[4]}, f_{2[4]}, f_{3\{1,2\}^c}, f_{1\{2,3\}^c}, f_{12^c}, f_{32^c}$

at the end of this step.

Step 3: Note that $(f_{3\{1,2\}^c}, f_{1\{2,3\}^c})$ *are the only excess bits that were in the same message at Step 1 and* $f_{3\{1,2\}^c} \oplus f_{1\{2,3\}^c}$ *is decodable at the respective decoders. Hence at the end of Step 3, selection of the messages is the following:*

$U_{1234} = f_{4[4]}, f_{2[4]}, f_{3\{1,2\}^c} \oplus f_{1\{2,3\}^c}, f_{12^c}, f_{32^c}$

and all others are empty.

Note that after Step 2 the total rate is 6 bits, whereas after Step 3 the rate is 5 bits. The lower bound in Theorem 2 also gives 5 bits, showing that CAPM achieves the optimal rate for this example.

Remark 10. *Let* R_a *be a rate obtained by placing* f_{IJ}*'s in messages (*U_K*'s) by applying Step 1. Let* R_a^* *be a rate obtained such that* f_{IJ}*'s are placed in messages (*U_K^**'s) by following the Step 1, and applying the Step 2 only for level- 2 messages.*

Note that $U_K = U_K^*$*, for all level-i messages where* $i > 3$*. Also, each possible excess bit (or bits) which we will denote by* $f_{IJ}^* \subseteq f_{IJ}$*, coming from a level-2 message* U_K^* *is such that either* $I = \{\alpha\}$ *or* $I = \{\beta\}$ *where* $K = \{\alpha, \beta\}$*. Then, when level-2 messages* U_K *and*

U_K^*, where $K = \{\alpha, \beta\}$, are not the same, we can write $U_K^* \cup f_{IJ}^* = U_K$ and

$$\begin{aligned}
\max_{i \in K}\{H(U_K|\mathbf{Y_i})\} &= \max_{i \in K}\{H(U_K^* \cup f_{IJ}^*|\mathbf{Y_i})\} \\
&= \max_{i \in K}\{H(U_K^*|\mathbf{Y_i}) + H(f_{IJ}^*|\mathbf{Y_i})\} \\
&\overset{a}{=} H(U_K^*|\mathbf{Y_i}) + \max_{i \in K}\{H(f_{IJ}^*|\mathbf{Y_i})\} \\
&\overset{b}{=} H(U_K^*|\mathbf{Y_i}) + H(f_{IJ}^*) \qquad (2.13)
\end{aligned}$$

a: Since all $H(U_K^*|\mathbf{Y_i})$ for $i \in K$ are the same.

b: Since f_{IJ}^* is such that either $I = \{\alpha\}$ or $I = \{\beta\}$.

Hence, we can write $R_a - R_a^*$ as

$$\left(\sum_{|I|=3} \max_{i \in I}\{H(U_I|\mathbf{Y_i})\} - \sum_{|I|=3} \max_{i \in I}\{H(U_I^*|\mathbf{Y_i})\}\right) + \left(\sum_{|K|=2} \max_{i \in K}\{H(U_K|\mathbf{Y_i})\} - \sum_{|K|=2} \max_{i \in K}\{H(U_K^*|\mathbf{Y_i})\}\right),$$

which is equal to

$$\left(\sum_{|I|=3} \max_{i \in I}\{H(U_I|\mathbf{Y_i})\} - \sum_{|I|=3} \max_{i \in I}\{H(U_I^*|\mathbf{Y_i})\}\right) + \sum_{f_{IJ}^*} H(f_{IJ}^*), \quad \text{from (2.13).}$$

Since $U_I^* = U_I \cup f_I^*$ where f_I^* denotes all of the excess bits in U_I^* and

$$\max_{i \in I}\{H(U_I \cup f_I^*|\mathbf{Y_i})\} \leq \max_{i \in I}\{H(U_I|\mathbf{Y_i})\} + H(f_I^*),$$

we can write

$$R_a - R_a^* \geq -\sum_{|I|=3} H(f_I^*) + \sum_{f_{IJ}^*} H(f_{IJ}^*) = 0.$$

Note that since we apply Step 2 once to the level-2 messages, the leftover bits that we get are unique, i.e., independent of the sorted demand sequence given at the beginning of Step 1 and different leftover bits coming from previous levels (since there is no leftover bit coming to level-2 messages). However, if we apply CAPM for an arbitrary instance of an index coding problem, this may not be the case. In other words, at Step 2 of the CAPM, we may get different excess bits due to differently sorted demand sequence given at the beginning of Step 1 or different leftover bits coming from previous levels and this may affect the resulting rate. Also, when there are multiple options for leftover bits to be moved, one may get different rates due to the selection of different next level messages to move the leftover bits. Lastly, there may be instances of index coding problem where not moving the bits to the next level gives a lower rate. We would like to point out that for our optimality results given in the next section, these issues either do not occur at all or do not affect the rate obtained by applying CAPM. To illustrate some of these issues, however, we provide the following examples. For the examples, $f_{IJ}^{a} \backslash U_K$ denotes the leftover bits (a bits) of f_{IJ} from U_K. If all f_{IJ} are leftover bits then we remove the superscript a.

Example 2. *Consider the* 4*-decoder index coding problem instance with demands* $f_{12^c}, f_{23^c}, f_{31^c}, f_{3\{1,2\}^c}$, $f_{4[4]}$ *where* f_{12^c}, f_{23^c} *are two bits and the rest are one bit. Now, we explain each step of the CAPM for this example.*

Step 1: At the end of this step we have

$U_{12} = f_{12^c}$, $U_{23} = f_{23^c}$, $U_{13} = f_{31^c}$,

$U_{123} = f_{3\{1,2\}^c}$, $U_{1234} = f_{4[4]}$.

*Step 2: We begin with level-*2 *messages. Note that all demands at level-*2 *messages are excess bits and can be moved to level-*3 *message,* U_{123}*. Then we have,*

$U_{123} = f_{3\{1,2\}^c}, f_{12^c}, f_{23^c}, f_{31^c}$,

$U_{1234} = f_{4[4]}$.

*We move on to level-*3 *messages. There is only one level-*3 *message,* U_{123}*, and one bit of* f_{12^c}*, denoted as* $f^1_{12^c}$*, is an excess bit. Then we move it to* U_{1234}*, concluding Step* 2*.*

Hence, messages at the end of this step are

$U_{123} = f_{3\{1,2\}^c}, f^1_{12^c}, f_{23^c}, f_{31^c},$

$U_{1234} = f_{4[4]}, f^1_{12^c}.$

Step 3: Since there is no $\oplus$ opportunity as described in Step 3, the messages at the end of Step 2 remains the same, giving a total rate of 5 bits.

Note that without loss of generality we can label Decoder 3 as 1 and Decoder 1 as 3. Then, if we apply CAPM with this relabeling we get the following messages at each step.

Step 1: At the end of this step we have

$U_{32} = f_{32^c}, U_{12} = f_{21^c}, U_{13} = f_{13^c},$

$U_{123} = f_{1\{2,3\}^c}, U_{1234} = f_{4[4]}.$

Step 2: We begin with level-2 messages. Similar to previous case all demands at level-2 messages are excess bits and moved to U_{123}. Then we have,

$U_{123} = f_{1\{2,3\}^c}, f_{32^c}, f_{21^c}, f_{13^c},$

$U_{1234} = f_{4[4]}.$

We move on to level-3 messages. As in the previous case, there is only one level-3 message, U_{123}. However, now the excess bits of U_{123} are $f_{1\{2,3\}^c}, f^1_{21^c}, f^1_{32^c}$. Then we move these to U_{1234}, concluding Step 2. Hence, messages at the end of this step are

$U_{123} = f^1_{32^c}, f^1_{21^c}, f_{13^c},$

$U_{1234} = f_{4[4]}, f_{1\{2,3\}^c}, f^1_{21^c}, f^1_{32^c},$

Step 3: Since there is no $\oplus$ opportunity as described in Step 3, the messages at the end of Step 2 remains the same, giving a total rate of 6 bits. Thus rate achieved by the heuristic depends on the indexing of the decoders.

Example 3. *Consider the* 5*-decoder index coding problem instance with demands* $f_{1[5]}, f_{5[5]}, f_{2\{1,4\}^c}, f_{3\{1,2\}^c}, f_{4\{1,3\}^c}$ *where each demand is one bit. Now we show each step of the CAPM.*

Step 1: At the end of this step we have,
$U_{124} = f_{2\{1,4\}^c}$, $U_{123} = f_{3\{1,2\}^c}$, $U_{134} = f_{4\{1,3\}^c}$,
$U_{12345} = f_{1[5]}, f_{5[5]}$.

*Step 2: We begin with the lowest level, i.e., level-*3 *for this example. Note that all of the demands in level-*3 *messages are excess bits and they are moved to* U_{1234}*. Then we have*
$U_{1234} = f_{2\{1,4\}^c}, f_{3\{1,2\}^c}, f_{4\{1,3\}^c}$,
$U_{12345} = f_{1[5]}, f_{5[5]}$.
Note that total rate is 4 *bits at this state. We move on to level-*4 *messages. There is only one level-*4 *message,* U_{1234}*. Since* $H(U_{1234}|\mathbf{Y_1}) = 0$*, all demands in* U_{1234} *are excess bits and they are moved to* U_{12345}*. Then we have*
$U_{12345} = f_{1[5]}, f_{5[5]}, f_{2\{1,4\}^c}, f_{3\{1,2\}^c}, f_{4\{1,3\}^c}$*, where the total rate is* 5 *bits.*

Step 3: Since there is no $\oplus$ *opportunity as described in Step* 3*, the messages at the end of Step* 2 *remains the same and total rate is* 5 *bits. Note that if we did not move the excess bits at level-*4 *message, the total rate would be* 4 *bits.*

In the next section, we show that applying CAPM gives us the optimal rate for several specific cases of the index coding problem.

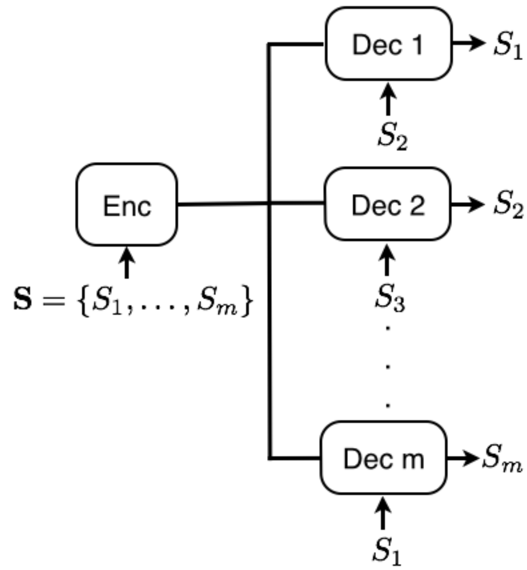

Figure 2.2: Index coding instance which is also called "directed cycle"

2.8 Optimality Results for Index Coding

We shall show that CAPM yields the optimal rate for several scenarios. Since the partition multicast scheme of Tehrani *et al.* [12] is the most direct antecedent of CAPM, we begin by showing that CAPM coincides with the *MLB*, and is thus optimal, for all of the explicit scenarios for which Tehrani *et al.* show that partition multicast is optimal.

First, consider the case depicted in Fig. 2.2, in which there are m decoders, m source bits, and Decoder k demands source bit k and has source bit $k+1$ as side information, for $k \in \{1, \dots, m-1\}$. Decoder m demands source bit m and has the first source bit as side information. Such an instance is typically called a "directed cycle" after its graph-theoretic description.

Proposition 2. *For the instance depicted in Fig. 2.2, the achievable rate provided by*

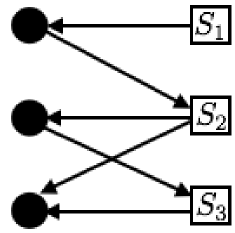

Figure 2.3: Bipartite graph representation of index coding example with $m = 3$ and $\mathbf{S} = \{S_1, S_2, S_3\}$. Circle nodes represent users while square nodes denote source bits.

CAPM and the lower bound provided by the MLB coincide. In fact

$$R_{\text{CAPM}} = R_{MLB} = m - 1.$$

Proof of Proposition 2. First we show that $R_{\text{CAPM}} = m - 1$. After Step 1 of CAPM, the messages are $U_{\{i,i+1\}} = S_{i+1}$ for all $i \in [m-1]$ and $U_{\{1,m\}} = S_1$. Observe that at any point of the algorithm, for any non-empty message U_I where $|U_I| = k$, there exists $\mathbf{Y_j} \in I$ such that $H(U_I|\mathbf{Y_j}) = k - 1$ and $k \geq H(U_I|\mathbf{Y_i}) \geq k - 1$ for all $i \in I$. Hence after Step 2 of CAPM, $H(U_I|\mathbf{Y_j})$ will be equal to $k - 1$ for all $j \in I$ for any nonempty message U_I. Now we show that for any non-empty message U_I with $|U_I| = k$, $H(U_I|\mathbf{Y_j}) = k - 1$ for all $j \in I$ if and only if $|U_I| = m$. This will imply that $R_{\text{CAPM}} \leq m - 1$. Consider a U_I such that $|U_I| = k$ and $H(U_I|\mathbf{Y_j}) = k - 1$ for all $j \in I$. Let $i = \min\{j : S_j \in U_I\}$. Then by virtue of Steps 1) and 2) of CAPM, we must have $i \in I$. Since $H(U_I|\mathbf{Y_i}) = k - 1$, we must have $S_{(i+1)} \in U_I$ as well, where $(j) = ((j-1) \mod m) + 1$. Likewise, $(i+1) \in I$, which implies that $(i+2) \in I$, etc. It follows, then, that $|I| = |U_I| = m$.

Conversely, selecting the permutation $\sigma(i) = m - i + 1$ in Theorem 2 shows that $R_{MLB} \geq m - 1$. $\square$

We can represent any (groupcast) index coding problem as a bipartite graph $G = (\mathbf{M}, \mathbf{S}, E)$ where $\mathbf{M}$, $\mathbf{S}$, and E denote the set of user nodes, the set of source bit nodes and the set of edges respectively [12]. There is a directed edge (m_i, S_i), $m_i \in \mathbf{M}$, $S_i \in \mathbf{S}$ if and only if m_i has S_i as side information (i.e., $S_i \in \mathbf{Y_{m_i}}$) and there is a directed edge (S_i, m_i), $S_i \in \mathbf{S}$, $m_i \in \mathbf{M}$ if and only if m_i demands S_i (i.e., $S_i \in \mathbf{f_{m_i}}$ and see Figure 2.3 for an example).

Second, we consider an index coding instance represented by a directed acyclic graph (DAG). Tehrani *et al.* [12] show that partition multicast achieves the optimal rate for DAGs and the optimal rate equals to total number of demanded bits, $s = |\mathbf{S}|$. Note that for any given instance of an index coding problem, the rate achieved by CAPM cannot be more than the number of demanded bits. Thus it suffices to show that the rate s is optimal. Tehrani *et al.* show this under the zero-error formulation. Using the *MLB*, one can show that the optimal rate is also s under the vanishing block error probability assumption.

Lemma 2. *For an instance of the index coding problem represented by a DAG, there exists a permutation, $\sigma(\cdot)$, on $[m]$ such that*

$$\mathbf{Y}_{\sigma(\mathbf{i})} \subseteq \cup_{j=1}^{i-1} \mathbf{f}_{\sigma(\mathbf{j})}, \textit{ for all } i \in [m]. \tag{2.14}$$

Proof of Lemma 2. We follow Neely *et al.* [40]. Observe that every index coding problem represented by a DAG must have a node in the graph with no outgoing edges. This node must represent some decoder ℓ since every source bit is assumed to be demanded by at least one decoder (see Section 2.4). Decoder ℓ must then have no side information. Let $\sigma(1) = \ell$.

We then proceed by induction. Suppose the containment in (2.14) holds for all i in $[k]$ with $k < m$. Consider the modified index coding instance in which we delete Decoders $\sigma(1), \ldots, \sigma(k)$ and all of their incoming and outgoing edges in the graph. We also delete any source components that are left with no edges. This instance must again be a DAG, and since $k < m$ it must have at least one decoder node, so there must be a decoder v that has no side information. It follows that in the original instance, $\mathbf{Y}_v \subset \cup_{j=1}^{k} \mathbf{f}_{\sigma(j)}$. We then set $\sigma(k+1) = v$. □

Proposition 3. *For DAGs, $R_{CAPM} = R_{MLB} = s$.*

Proof of Proposition 3. By Lemma 2, we have that the Maximin lower bound R_{MLB} is greater than or equal to

$$
\begin{aligned}
&H(\mathbf{f}_{\sigma(1)}|\mathbf{Y}_{\sigma(1)}) + H(\mathbf{f}_{\sigma(2)}|\mathbf{f}_{\sigma(1)}, \mathbf{Y}_{\sigma(1)}, \mathbf{Y}_{\sigma(2)}) + \cdots \\
&\quad + H(\mathbf{f}_{\sigma(m)}|\mathbf{f}_{\sigma(1)}, \ldots, \mathbf{f}_{\sigma(m-1)}, \mathbf{Y}_{\sigma(1)}, \ldots, \mathbf{Y}_{\sigma(m)}) \\
&= H(\mathbf{f}_{\sigma(1)}) + H(\mathbf{f}_{\sigma(2)}|\mathbf{f}_{\sigma(1)}) + \cdots + H(\mathbf{f}_{\sigma(m)}|\mathbf{f}_{\sigma(1)}, \ldots, \mathbf{f}_{\sigma(m-1)}),
\end{aligned}
$$

giving $R_{MLB} \geq s$. Since the rate achieved by CAPM cannot be more than s, it gives the optimal rate for DAGs. □

Finally, Tehrani *et al.* [12] show that partition multicast achieves the optimal rate when each decoder demands a single bit and has as side information all of the other source bits. Note that under these assumptions one may, without loss of generality, assume that each source bit is demanded by at most one decoder; two decoders that demand the same source bit must have the same side information and therefore one of the two can be deleted without affecting the rate. Then each source component must be present as side information at either all or all but one of the decoders.

We shall prove that CAPM is optimal for the more general scenario in which each source bit is present at none of the decoders, all of the decoders, all but one, or all but two. That is, $\mathbf{S}$ consists of $\{G_0, G_m, G_{m-1}, G_{m-2}\}$. We do not assume that each decoder demands a single bit or that each bit is demanded by at most one decoder.

Theorem 3. *The optimal rate, R_{opt}, for the index coding problem where $\mathbf{S} = \{G_0, G_m, G_{m-1}, G_{m-2}\}$ is*

$$R_{opt} = \max\{R_1, \ldots, R_m\} \tag{2.15}$$

where

$$\begin{aligned} R_i =& H(\mathbf{f_1} \backslash \{\cup_{\{1,\beta\}\subseteq[m]} f_{1\{1,\beta\}}\}, \ldots, \mathbf{f_{i-1}} \backslash \{\cup_{\{i-1,\beta\}\subseteq[m]} f_{i-1\{i-1,\beta\}}\}, \\ & \mathbf{f_i}, \mathbf{f_{i+1}} \backslash \{\cup_{\{i+1,\beta\}\subseteq[m]} f_{i+1\{i+1,\beta\}}\}, \ldots, \mathbf{f_m} \backslash \{\cup_{\{m,\beta\}\subseteq[m]} f_{m\{m,\beta\}}\} | \mathbf{Y_i}) + \max_{j, j\in[m]\backslash i} f_{j\{i,j\}}, \end{aligned} \tag{2.16}$$

and is achieved by CAPM.

Proof of Theorem 3. The proof is given in Appendix. □

Corollary 1. *For any index coding problem with three or fewer decoders, the optimal rate is given by (2.15), and is achieved by CAPM.*

Proof of Corollary 1. Any index coding problem with three or fewer decoders must have the property that each source component is present as side information at either all of the decoders, none of the decoders, all but one, or all but two. □

Note that since CAPM and the *MLB* only give integer-valued bounds, it follows that the optimal rate is integer-valued for the scenario described in Theorem 3, and in particular, in Corollary 1. Moreover, Corollary 1 solves the index coding problem with three decoders and any number of source components. In contrast, Arbabjolfaei *et al.* [14] solve the index coding problem with up to five source components and any number of decoders. Evidently neither of these results implies the other, even if one ignores slight differences in the problem formulation between the two works. It is also worth noting that the Arbabjolfaei *et al.* result is numerical while Corollary 1 is analytical. Now we move on to the proof of Theorem 3.

The following result illustrates the importance of excess bits.

Proposition 4. *If the demands of the m-user index coding problem are such that there are no excess bits after Step* 1 *of CAPM then the rate obtained by following only Step* 1 *is optimal. The optimal rate* R_* *can be written as*

$$\begin{aligned} R_* = \max_{\pi}\{&H(\mathbf{f}_{\pi(\mathbf{1})}|\mathbf{Y}_{\pi(\mathbf{1})}) + H(\mathbf{f}_{\pi(\mathbf{2})}|\mathbf{f}_{\pi(\mathbf{1})}, \mathbf{Y}_{\pi(\mathbf{1})}, \mathbf{Y}_{\pi(\mathbf{2})}) \\ &+ \cdots + \\ &H(\mathbf{f}_{\pi(\mathbf{m})}|\mathbf{f}_{\pi(\mathbf{1})}, \mathbf{Y}_{\pi(\mathbf{1})}, \ldots, \mathbf{f}_{\pi(\mathbf{m-1})}, \mathbf{Y}_{\pi(\mathbf{m-1})}, \mathbf{Y}_{\pi(\mathbf{m})})\} \end{aligned} \tag{2.17}$$

where $\pi(.)$ *denotes the following m permutations on* $[m]$*:*

$$(1, 2, \ldots, m), (2, 1, 3, \ldots, m), \ldots, (m, 1, \ldots, m-1).$$

Proof of Proposition 4. First we show that the achievable rate we get by applying Step 1 of CAPM gives the expression in (2.17). We begin with the following three observations. Firstly, all demands of each Decoder i, $\mathbf{f_i}$, are in $\cup_{i\in I} U_I$. Secondly, since the demands are such that there are no excess bits after Step 1, $H(U_I|\mathbf{Y_i}) = H(U_I|\mathbf{Y_j})$, for all $i, j \in I \subset [m]$. Lastly, demands placed in $U_{[m]}$ at Step 1 cannot be excess bits since $U_{[m]}$ is the highest level message. Hence $H(U_{[m]}|\mathbf{Y_i})$ does not have to be equal for all $i \in [m]$.

We can write the achievable rate R_{CAPM} as

$$R_{\text{CAPM}} = \max\{R_1, \dots, R_m\}, \text{where}$$

$$\begin{aligned} R_i &= H(U_{[m]}|\mathbf{Y_i}) + \sum_{I\subset[m]} \max_{j\in I}\{H(U_I|\mathbf{Y_j})\} \\ &\overset{a}{=} H(U_{[m]}|\mathbf{Y_i}) + \sum_{I\subset[m]} H(U_I|\mathbf{Y_{i_I}}), \end{aligned} \tag{2.18}$$

where i_I is an arbitrary element of I and a is due to the assumption that $H(U_I|\mathbf{Y_i}) = H(U_I|\mathbf{Y_j})$, for all $i, j \in I \subset [m]$.

Let us focus on R_1. From (2.18), we can write R_1 as

$$R_1 = \sum_{C_1} H(U_I|\mathbf{Y_1}) + \sum_{C_2} H(U_I|\mathbf{Y_2}) + \cdots + \sum_{C_m} H(U_I|\mathbf{Y_m})$$

where $C_1 = \{I \subseteq [m]|1 \in I\}$, $C_2 = \{I \subseteq [m]|2 \in I, 1 \notin I\}$, ..., $C_m = \{I \subseteq [m]|m \in I, 1 \notin I, \dots, m-1 \notin I\}$.

Since all U_I's are independent, and for all collections of subsets $J_1, \dots J_j$, $K_1, \dots K_k$, $L_1, \dots, L_l$, and all subsets $\{i_1, \dots, i_p\} \subseteq [m]$, we have that $(U_{J_1}, \dots, U_{J_j})$ and $(U_{K_1}, \dots, U_{K_k})$ are conditionally independent given $(U_{L_1}, \dots, U_{L_l})$, and $(\mathbf{Y_{i_1}}, \dots, \mathbf{Y_{i_p}})$, provided that the collections $J_1, \dots J_j$ and $K_1, \dots K_k$ are disjoint, R_1 equals

$$\begin{aligned} &H(\cup_{C_1} U_I|\mathbf{Y_1}) + H(\cup_{C_2} U_I|\mathbf{Y_2}) + \cdots + H(\cup_{C_m} U_I|\mathbf{Y_m}) \\ =&H(\mathbf{U_1}|\mathbf{Y_1}) + H(\mathbf{U_2} \setminus \mathbf{U_1}|\mathbf{Y_2}) + \cdots + H(\mathbf{U_m} \setminus \{\mathbf{U_1}, \dots, \mathbf{U_{m-1}}\}|\mathbf{Y_m}), \end{aligned}$$

where $\mathbf{U_i}$ is defined as $\cup_{I\subseteq[m]:i\in I} U_I$. By Step 1, no decoder in I^c can demand any source bit in U_I or have it as side information. Then we can write R_1 as

$$R_1 = H(\mathbf{U_1}|\mathbf{Y_1}) + H(\mathbf{U_2} \setminus \mathbf{U_1}|\mathbf{Y_1}, \mathbf{Y_2}) + \cdots + H(\mathbf{U_m} \setminus \{\mathbf{U_1}, \dots, \mathbf{U_{m-1}}\}|\mathbf{Y_1}, \dots, \mathbf{Y_m}). \tag{2.19}$$

Also, by Step 1, each U_I consists of those source bits such that, for each decoder i in I, Decoder i either demands the bit or has it as side information. Then (2.19) becomes

$$\begin{aligned} &H(\mathbf{f_1}|\mathbf{Y_1}) + H(\mathbf{f_2} \setminus \mathbf{f_1}|\mathbf{Y_1}, \mathbf{Y_2}) + \cdots + H(\mathbf{f_m} \setminus \{\mathbf{f_1}, \ldots, \mathbf{f_{m-1}}\}|\mathbf{Y_1}, \ldots, \mathbf{Y_m}) \\ &= H(\mathbf{f_1}|\mathbf{Y_1}) + H(\mathbf{f_2}|\mathbf{f_1}, \mathbf{Y_1}, \mathbf{Y_2}) + \cdots + H(\mathbf{f_m}|\mathbf{f_1}, \ldots, \mathbf{f_{m-1}}, \mathbf{Y_1}, \ldots, \mathbf{Y_m}). \end{aligned} \tag{2.20}$$

Note that the expression for R_1 in (2.20) is equivalent to first expression of the R_*. Applying the procedure above to the other R_i's similarly, we see that R_{CAPM} gives the expression in (2.17). Evidently this expression cannot exceed the lower bound in Theorem 2, so the proof is complete. □

The *coded caching* problem, which was introduced by Maddah-Ali and Niesen [41], is closely related to the index coding problem. The coded caching problem consists of two phases, called the *cache allocation* phase and the *delivery* phase. During the cache allocation phase, the server can decide how to populate the caches of the various users. Each user then selects some content to demand, and during the delivery phase the server must broadcast a common message to all of the clients that allows each one to meet its demand, given its cache contents. Thus the delivery phase of the coded caching problem can be viewed as an index coding problem.

If we perform the cache allocation as in [42] and each user demands a different file at the delivery phase, then the instance of the index coding problem that results during the delivery phase satisfies the conditions in Proposition 4 in a certain asymptotic sense. Therefore, CAPM gives the optimal rate for the delivery phase in this case.

2.9 S-CAPM: A Heuristic Achieving Fractional Rates

Recall from Remark 9 that CAPM gives only integer rates. However, some instances of the index coding problem are known to have non-integer optimal rates. We next show how CAPM can be modified to give non-integer rate bounds, and this modification performs strictly better than CAPM in some examples. The extension is not polynomial-time computable, however. The following multi-letter extension of Proposition 1 is necessary.

Proposition 5. *Let t be a positive integer. The optimal rate R_{opt} of an index coding problem is upper bounded by*

$$\min \frac{1}{t} \sum_{I \subseteq [m]} \left[\max_{i \in I} H(U_I^t | \mathbf{Y_i^t}) \right]$$

where the minimization is over the set of all random variables U_I^t jointly distributed with $\mathbf{S^t}$ such that

1) There exist functions

$g_1(\cup_{I:1 \in I} U_I^t, \mathbf{Y_1^t}), \ldots, g_m(\cup_{I:m \in I} U_I^t, \mathbf{Y_m^t})$ *such that*

$$g_i(\cup_{I:i \in I} U_I^t, \mathbf{Y_i^t}) = \mathbf{f_i^t}(\mathbf{S}), \text{ for all } i \in [m].$$

2) Each U_I^t is a (possibly empty) vector of bits, each of which is the mod-2 sum of a set (possibly singleton) of the bits in $\mathbf{S^t}$.

Proposition 1 can evidently be recovered from Proposition 5 by taking $t = 1$. Similar to Proposition 1, Proposition 5 can also be obtained from an achievability result of Timo *et al.* [2]. In this case, we consider a revised setup where each t consecutive symbols is taken as a single symbol. Then we apply Proposition 1 to this revised setup in order to obtain Proposition 5.

Now we provide a heuristic, which we call *Split Coded Approximate Partition Multicast (S-CAPM)*, for selecting the auxiliary random variables in Proposition 5. The steps for S-CAPM are very similar to the ones for CAPM in Section 2.7 except for the placement of leftover bits.

Step 1 (Initialization) : This step is exactly the same as in CAPM, except that we shall parametrize the solution differently. For each $k \in \{1, \ldots, |\mathbf{S}|\}$ and each subset $I \subseteq [m]$, let $\theta(I, k)$ denote a variable in the interval $[0, 1]$. We shall interpret $\theta(I, k)$ as the "fraction" of source bit S_k that is allocated to the auxiliary random variable U_I. All such variables are initially zero.

For each source component k we set $\theta(K \cup J^c, k) = 1$, where K and J are chosen so that S_k is in f_{KJ}. This is assuming that $J \neq [m]$. As in CAPM, if $J = [m]$ then we set $\theta([m], k) = 1$. Note that after this has been done for each k, we have

$$\sum_{I \subset [m]} \theta(I, k) = 1$$

for each k. This equality will remain true after Step 2.

Step 2 : As with CAPM, the goal of Step 2 is to promote "excess bits" to a higher-level message. Since each auxiliary random variable now stores fractional bits, however, both the notion of "excess" and the promotion process are more involved.

Given the variables $\{\theta(I, k)\}$, let us define the "conditional entropy" of U_I given $\mathbf{Y_j}$ as

$$H(U_I|\mathbf{Y_j}) = \sum_{k : S_k \notin \mathbf{Y_j}} \theta(I, k). \tag{2.21}$$

Note that if $\theta(I, k) \in \{0, 1\}$ for all I and k, then this reduces to the conditional entropy examined in Step 2 of CAPM. We shall be most interested in $H(U_I|\mathbf{Y_j})$

when $j \in I$, although the definition in (2.21) does not require this.

We then perform the following procedure for each subset I. The order in which we process the different subsets I is not specified by the heuristic, except that if $|I_1| < |I_2|$ then I_1 must be processed prior to I_2. For a given subset I, we define

$$i^* = \min\{i : H(U_I|\mathbf{Y_i}) = \min_{l \in I} H(U_I|\mathbf{Y_l})\} \tag{2.22}$$

and

$$j^* = \min\{j : H(U_I|\mathbf{Y_j}) = \max_{l \in I} H(U_I|\mathbf{Y_l})\}. \tag{2.23}$$

If $H(U_I|\mathbf{Y_{i^*}}) = H(U_I|\mathbf{Y_{j^*}})$ the we are done with this subset and may move to the next one. If $H(U_I|\mathbf{Y_{i^*}}) < H(U_I|\mathbf{Y_{j^*}})$, then let E denote the set of source bits that are "excess"

$$E = \{k : \theta(I,k) > 0 \text{ and } S_k \in \mathbf{Y_{i^*}} \text{ but } S_k \notin \mathbf{Y_{j^*}}\}.$$

We then select a source bit in E to promote to higher-level messages. Consider the set

$$\{k \in E : \theta(I,k) \leq H(U_I|\mathbf{Y_{j^*}}) - H(U_I|\mathbf{Y_{i^*}})\}. \tag{2.24}$$

If this set is nonempty, then there is at least one source bit that is "entirely excess." We shall select one such bit to promote. Choose an arbitrary

$$k^* \in \arg\max\{\theta(I,k) : k \in E \text{ and } \theta(I,k) \leq H(U_I|\mathbf{Y}^*_\mathbf{j}) - H(U_I|\mathbf{Y}^*_\mathbf{i})\}.$$

We then set $\theta(I,k^*) = 0$ and we increment $\theta(I',k^*)$ for all I' such that $I \subseteq I'$ and $|I'| = |I| + 1$ by the amount

$$\frac{\theta(I,k^*)}{|I^c|}.$$

In words, we view $\theta(I, k^*)$ as an amount of fluid that is removed from U_I and divided equally among the I^c sets I'.

If there are no bits that are entirely excess, i.e., the set in (2.24) is empty, then choose an arbitrary

$$k^* \in \arg\min\{\theta(I, k) : k \in E\}.$$

We then promote only the portion of $\theta(I, k)$ that is excess. That is, we replace $\theta(I, k)$ with

$$H(U_I|\mathbf{Y}^*_{\mathbf{j}}) - H(U_I|\mathbf{Y}^*_{\mathbf{i}})$$

and divide the remaining part,

$$\theta(I, k) - (H(U_I|\mathbf{Y}^*_{\mathbf{j}}) - H(U_I|\mathbf{Y}^*_{\mathbf{i}}))$$

equally among all of the sets I' such that $I \subset I'$ and $|I'| = |I| + 1$. Observe that since $\theta(I, k)$ must be rational for all I and k, the process will eventually terminate.

Step 3 : As in CAPM, we now look for opportunities to exclusive-OR source bits included in the same auxiliary random variable. First we convert the fractional bits described by the $\theta(\cdot, \cdot)$ variables to an integral number by increasing the parameter t. Observe that $\theta(I, k)$ must be rational for each I and k; let t denote the smallest positive integer so that $\theta(I, k) \cdot t$ is an integer for all I and k. Next recall that for each k

$$\sum_{I \subseteq [m]} \theta(I, k) \cdot t = t.$$

We then divide the block of t bits corresponding to source component k among the U_I variables so that the number of bits that U_I receives is $\theta(I, k) \cdot t$. One can verify that the resulting U_I variables satisfy conditions 1) and 2) in Proposition 5. For each U_I variable, we then look for exclusive-OR opportunities as in Step 3 of CAPM, resulting in revised U_I variables that remain feasible.

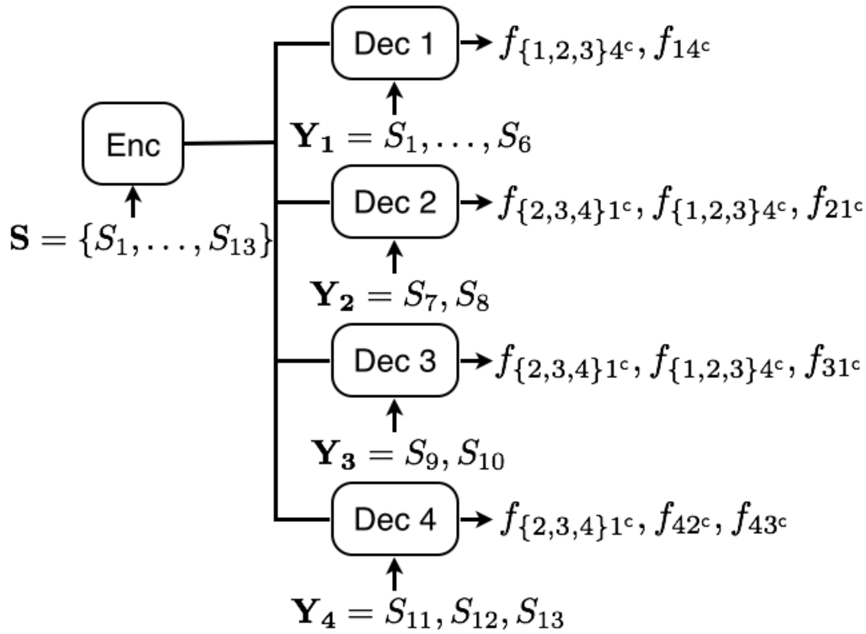

Figure 2.4: Index coding example with 4 users

We next illustrate S-CAPM with two examples.

Example 4. *In this case, there are* 4 *decoders with side information and demands as shown in Fig. 2.4, where* $f_{\{2,3,4\}1^c} = (S_1, S_2)$*,* $f_{21^c} = (S_3, S_4)$*,* $f_{31^c} = (S_5, S_6)$*,* $f_{42^c} = (S_7, S_8)$*,* $f_{43^c} = (S_9, S_{10})$*,* $f_{\{1,2,3\}4^c} = (S_{11}, S_{12})$ *and* $f_{14^c} = S_{13}$*. By using S-CAPM, we determine the messages and* t *of the achievable scheme.*

Step 1 : At the end of this step, all of the following $\theta(I, k)$*'s are unity:*

$\theta(\{1,4\}, 13)$, $\theta(\{1,2\}, 3)$, $\theta(\{1,2\}, 4)$, $\theta(\{1,3\}, 5)$, $\theta(\{1,3\}, 6)$,

$\theta(\{2,4\}, 7)$, $\theta(\{2,4\}, 8)$, $\theta(\{3,4\}, 9)$, $\theta(\{3,4\}, 10)$,

$\theta([4], 1)$, $\theta([4], 2)$, $\theta([4], 11)$, $\theta([4], 12)$.

*Step 2: We start with level-*2 *messages. Note that all demands in level-*2 *messages are excess bits. Since there are two possible level-*3 *messages that each demand can move,* *we set all the corresponding $\theta(I,k)$s to 0.5. At this point the nonzero $\theta(I,k)$'s are*

$$\theta(\{1,2,4\},13), \theta(\{1,2,4\},7), \theta(\{1,2,4\},8), \theta(\{1,2,4\},3),$$
$$\theta(\{1,2,4\},4),$$
$$\theta(\{1,3,4\},13), \theta(\{1,3,4\},9), \theta(\{1,3,4\},10), \theta(\{1,3,4\},5),$$
$$\theta(\{1,3,4\},6),$$
$$\theta(\{1,2,3\},5), \theta(\{1,2,3\},6), \theta(\{1,2,3\},3), \theta(\{1,2,3\},4),$$
$$\theta(\{2,3,4\},7), \theta(\{2,3,4\},8), \theta(\{2,3,4\},9), \theta(\{2,3,4\},10),$$
$$\theta([4],1), \theta([4],2), \theta([4],11), \theta([4],12),$$

where $\theta(I,k) = 0.5$ for all $|I| = 3$ and $\theta(I,k) = 1$ for all $|I| = 4$. Now we move on to level-3 messages. Since there is only one level-4 message, U_{1234}, all possible excess bits at this stage will be moved to U_{1234}. We start with U_{124}. Since $H(U_{124}|\mathbf{Y_1}) = 1.5$, $H(U_{124}|\mathbf{Y_2}) = 1.5$, $H(U_{124}|\mathbf{Y_4}) = 2$, we have $i^* = 1$, $j^* = 4$. We declare, say, S_3 to be excess and we move all of $\theta(\{1,2,4\},3)$ to U_{1234}. Then we recalculate $H(U_{124}|\mathbf{Y_i})$, for $i \in \{1,2,4\}$. Now $i^* = 2$, $j^* = 1$ and the fraction of S_7, i.e., $\theta(\{1,2,4\},7)$, becomes an excess bit. We recalculate $H(U_{124}|\mathbf{Y_i})$, for $i \in \{1,2,4\}$ and all are equal. Hence we move on to another level-3 message, say U_{134}. For this message fraction of S_5 and S_9 become excess bits and are moved to U_{1234}. Lastly, all demands in U_{123}, U_{234} are excess bits and moved to U_{1234}. This concludes Step 2.

Step 3: Since there is no XOR opportunities as described in this step, we only require t to be 2. Then the nonzero $\theta(I,k)$'s are

$$\theta(\{1,2,4\},13), \theta(\{1,2,4\},8), \theta(\{1,2,4\},4),$$
$$\theta(\{1,3,4\},13), \theta(\{1,3,4\},10), \theta(\{1,3,4\},6),$$
$$\theta([4],1), \theta([4],2), \theta([4],11), \theta([4],12), \theta([4],3), \theta([4],7),$$
$$\theta([4],5), \theta([4],9), \theta([4],6), \theta([4],4), \theta([4],8), \theta([4],10),$$

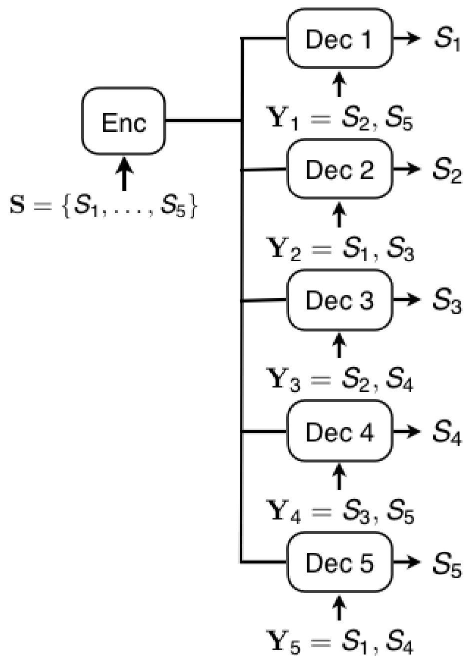

Figure 2.5: Index coding example with 5 users

where $\theta(I, k) = 0.5$ *for* $k \in \{4, 6, 8, 10, 13\}$ *and the rest are* 1*. As a result, the rate coming from level-*3 *and level-*4 *messages are* 2 *and* 8.5 *bits respectively and the total rate for this problem is* 10.5 *bits.*

From the linear programming lower bound[5] *stated in [11], we get* 10.5 *bits showing that S-CAPM is optimal.*

Example 5. *We consider the "5-cycle" index coding problem shown in Fig. 2.5. Its optimal rate is found in [11]. When we apply S-CAPM, we determine the messages and t as follows:*

Step 1 : After this step we have the following nonzero $\theta(I, k)$*'s.*

$\theta(\{1, 2, 5\}, 1)$, $\theta(\{1, 2, 3\}, 2)$, $\theta(\{2, 3, 4\}, 3)$, $\theta(\{3, 4, 5\}, 4)$, $\theta(\{1, 4, 5\}, 5)$, *where all* $\theta(I, k) =$

[5]This lower bound is for the zero-error setting. However, it can be modified to handle vanishing block error probabilities.

1 *for all* $k \in [5]$.

Step 2 *: We start with level-*3 *messages. Note that all demands in level-*3 *messages are excess bits to be moved to level-*4 *messages. Since each leftover bit has two possible level-*4 *messages to go,* $\theta(I,k) = 0.5$ *for all* $k \in [5]$*. Then the nonzero* $\theta(I,k)$*'s are*

$\theta(\{1,2,3,5\},1), \theta(\{1,2,3,5\},2)$,

$\theta(\{1,2,4,5\},1), \theta(\{1,2,4,5\},5)$,

$\theta(\{1,2,3,4\},2), \theta(\{1,2,3,4\},3)$,

$\theta(\{1,3,4,5\},4), \theta(\{1,3,4,5\},5)$,

$\theta(\{2,3,4,5\},3), \theta(\{2,3,4,5\},4)$.

*Now, we move on to level-*4 *messages. Since there are no leftover bits at level-*4 *messages and all nonzero* $\theta(I,k) = 0.5$*, we set* $t = 2$ *concluding S-CAPM. Hence, the total rate becomes* 2.5 *bits which is the optimal rate.*

CHAPTER 3

VECTOR GAUSSIAN RATE-DISTORTION WITH VARIABLE SIDE INFORMATION

3.1 Introduction

We investigate the special case of Heegard-Berger problem by considering two decoders. We also take source and side information as jointly Gaussian random vectors. We obtain four lower bounds using variations on the following argument. Since the rate-distortion function is known when the side information is degraded [18], a natural approach to proving lower bounds is to *enhance* the side information of one encoder or the other in order to make the problem degraded. The optimal rate for the newly-obtained instance is thus known and provides a lower bound on the optimal rate for the original instance. This idea can be applied several ways, leading to lower bounds of varying strength and usability. The weakest of these bounds is quite weak but also quite simple. The strongest, on the other hand, is quite strong but also difficult to apply. The intermediate bounds attempt to provide the best attributes of both.

We consider three different distortion constraints, all phrased as constraints on the error covariance matrices, averaged over the block, at the two decoders. The first stipulates an upper bound on the mean square error of the reproduction of each component of the source; this can be viewed as constraints on the diagonal elements of the time-average error covariance matrix. The second requires that the average error covariance matrix itself must be dominated, in a positive definite sense, by a given scaled identity matrix. In the final case, we require the trace of the average error covariance matrix to be upper bounded by a constant. For each of the three distortion measures, we solve a class of instances using the lower bounds developed in this chapter. The necessary achievability arguments are standard, although our analysis does provide insight into how the auxiliary random variables therein should be chosen. Specifically, we show how to divide the signal space into "regions," in which the side information at one decoder is "stronger" than that of the other. We then show that it is optimal for certain auxiliary random variables to live in certain of these regions.

The balance of the chapter is organized as follows. The next two sections provide the problem formulation and the four lower bounds on the optimal rate and a discussion of how they interrelate, respectively. Section 3.4 contains statement of the main results. All of the achievability analysis is presented in Section 3.5. Section 3.6 contains the proofs of our main results. Section 3.7 contains a brief epilogue describing a conjectured difference among the lower bounds.

3.2 Problem Definition

Let $\mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2}$ [1] be correlated vector Gaussian sources [2] of size $k \times 1$, $k_1 \times 1$ and $k_2 \times 1$ respectively where $\mathbf{X}$ is the source to be compressed at the encoder and $\mathbf{Y_1}$ and $\mathbf{Y_2}$ comprise the side information at Decoder 1 and Decoder 2, respectively. We assume that conditional covariance matrix of $\mathbf{X}$ given $\mathbf{Y_i}$, $K_{\mathbf{X}|\mathbf{Y_i}}$, $i \in \{1, 2\}$ are invertible matrices. Both Decoder 1 and 2 want to reconstruct $\mathbf{X}$ subject to given distortion constraints. The objective is to characterize the rate distortion function for this setting. The following definitions are used to formulate the problem precisely.

[1] We use bold letters to denote vectors.
[2] Unless otherwise is stated, we assume that all Gaussian random variables are zero mean.

Definition 8. *Γ_i, $i \in \{1,2\}$ is defined as a mapping from the set of all $k \times k$ positive semi-definite (PSD) matrices to the set of $k_0 \times k_0$ PSD matrices such that*
1) $\Gamma_i(\cdot)$ is linear,
2) $A \preceq B$[3] implies that $\Gamma_i(A) \preceq \Gamma_i(B)$.

Definition 9. *An (n, M, D_1, D_2)* code *where D_1 and D_2 are positive definite matrices, is composed of*

- *an* encoding function

$$f : \mathbb{R}^{kn} \to \{1, ..., M\}$$

- *and* decoding functions

$$g_1 : \{1, ..., M\} \times \mathbb{R}^{k_1 n} \to \mathbb{R}^{kn}$$
$$g_2 : \{1, ..., M\} \times \mathbb{R}^{k_2 n} \to \mathbb{R}^{kn}$$

satisfying the distortion constraints

$$E\left[\frac{1}{n}\sum_{k=1}^{n}\Gamma_i\left((\mathbf{X}_k - \widehat{\mathbf{X}}_{\mathbf{i}k})(\mathbf{X}_k - \widehat{\mathbf{X}}_{\mathbf{i}k})^T\right)\right] \preceq D_i, \quad i \in \{1,2\}$$

where $\widehat{\mathbf{X}}_{\mathbf{1}}^{\mathbf{n}} = g_1(f(\mathbf{X}^{\mathbf{n}}), \mathbf{Y}_{\mathbf{1}}^{\mathbf{n}})$, and $\widehat{\mathbf{X}}_{\mathbf{2}}^{\mathbf{n}} = g_2(f(\mathbf{X}^{\mathbf{n}}), \mathbf{Y}_{\mathbf{2}}^{\mathbf{n}})$. We call n the block length *and M the* message size *of the code.*

Definition 10. *A rate R is (D_1, D_2)*-achievable *if for every $\epsilon > 0$, there exists an $(n, M, D_1 + \epsilon I, D_2 + \epsilon I)$* code *such that $n^{-1}\log M \le R + \epsilon$.*

Definition 11. *The* rate-distortion function *is defined as*

$$R(\mathbf{D}) = \inf\{R : R \textit{ is } \mathbf{D}\textit{-achievable}\},$$

where $\mathbf{D} = (D_1, D_2)$.

[3] $A \preceq B$ means that $B - A$ is a positive semidefinite matrix.

We shall prove our lower bounds for arbitrary distortion measures Γ satisfying the requirements of Definition 8. We conclude this section by introducing the following notations used in rest of this chapter.

Notation 2. *Let $\mathbf{X}$ be a $k \times 1$ vector where $k = l_1 + l_2$. Then $(\mathbf{X})_{l_1}$ denotes the $l_1 \times 1$ vector consisting of the first $l_1 \times 1$ components of $\mathbf{X}$ and $[\mathbf{X}]_{l_2}$ denotes the remaining part of $\mathbf{X}$.*

Notation 3. *Let E be a $p \times p$ matrix. Then $(E)_{ij}$ denotes the element of E which is in i^{th} row and j^{th} column of E.*

Notation 4. *Let E and F be $p \times p$ and $r \times r$ matrices where $p \geq l_1$ and $r \geq l_2$. Then $(E)_{l_1}$ denotes upper-left $l_1 \times l_1$ submatrix of E and $[F]_{l_2}$ denotes lower-right $l_2 \times l_2$ submatrix of F.*

Notation 5. *Let E and F be $p \times p$ and $r \times r$ matrices where $p \geq l_1$ and $r \geq l_2$. Then $(E)_{diag}$ denotes $p \times p$ diagonal matrix whose diagonal elements are the same as that of E. Also, $(E)_{l_1 diag}$ denotes $l_1 \times l_1$ diagonal matrix whose diagonal elements are the same as that of upper-left $l_1 \times l_1$ submatrix of E and $[F]_{l_2 diag}$ denotes $l_2 \times l_2$ diagonal matrix whose diagonal elements are the same as that of lower-right $l_2 \times l_2$ submatrix of F.*

Notation 6. *Let E and F be $p \times p$ diagonal matrices. Then $\min\{E, F\}$ denotes the $p \times p$ diagonal matrix whose each diagonal entry is the minimum of corresponding diagonal entries of E and F.*

Notation 7. *Let $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ be a random vector. Then $\mathbf{X} \perp \mathbf{Y} | \mathbf{Z}$ denotes that $\mathbf{X}$ and $\mathbf{Y}$ are independent given $\mathbf{Z}$, $\mathbf{X} \leftrightarrow \mathbf{Y} \leftrightarrow \mathbf{Z}$ denotes that $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ forms a Markov chain, and $K_{\mathbf{X}}$ denotes the covariance matrix of $\mathbf{X}$.*

3.3 Lower Bounds

We turn to lower bounds on the optimal rate. We shall provide four such bounds. In order of strongest (largest) to weakest (smallest), these are

1. The *Minimax bound* (*mLB*);
2. The *Maximin bound* (*MLB*);
3. The *Enhanced Enhancement bound* (Enhanced *ELB*);
4. The *Enhancement bound* (*ELB*).

Although the Maximin bound, the Enhanced Enhancement bound, and the Enhancement bound are never larger than the Minimax bound, they are useful in that they are simpler to work with in some respects. We begin with the simplest, and weakest, of the bounds. This bound is folklore, and it turns out to be quite weak indeed.

3.3.1 Enhancement Lower Bound

If the side information at the decoders is *degraded*, meaning that we can find a joint distribution of $(\mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2})$ such that

$$\mathbf{X} \leftrightarrow \mathbf{Y}_{\sigma(\mathbf{1})} \leftrightarrow \mathbf{Y}_{\sigma(\mathbf{2})} \tag{3.1}$$

for some permutation $\sigma(.)$, then the rate distortion function is known [18, 2]. Hence a natural way to obtain a lower bound to $R(\mathbf{D})$ is to create degraded problems by providing extra side information to one decoder so that the problem becomes degraded. We call this lower bound *enhancement lower bound*, abbreviated as *ELB*, due to its similarity to the converse results for broadcast channels [31]. Proposition 6 states this lower bound.

Proposition 6. *The rate distortion function $R(\mathbf{D})$ is lower bounded by*

$$R_{ELB}(\mathbf{D}) = \max\{\sup_{S_G} \inf_{\widetilde{C}_{l1}(\mathbf{D})} R_{lo1}, \sup_{S_G} \inf_{\widetilde{C}_{l2}(\mathbf{D})} R_{lo2}\}, \tag{3.2}$$

where

$$R_{lo1} = I(\mathbf{X}; \mathbf{W}, \mathbf{U}|\mathbf{Y_1}) + I(\mathbf{X}; \mathbf{V}|\mathbf{W}, \mathbf{U}, \mathbf{Y}), \tag{3.3}$$

$$R_{lo2} = I(\mathbf{X}; \mathbf{W}, \mathbf{V}|\mathbf{Y_2}) + I(\mathbf{X}; \mathbf{U}|\mathbf{W}, \mathbf{V}, \mathbf{Y}), \tag{3.4}$$

$S_G = \{\mathbf{Y}$ *jointly Gaussian with* $(\mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2})|\mathbf{X} \leftrightarrow \mathbf{Y} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2})\}$, *and*

$\widetilde{C}_{l1}(\mathbf{D})$ *is the set of* $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ *such that*	$\widetilde{C}_{l2}(\mathbf{D})$ *is the set of* $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ *such that*
$(\mathbf{W}, \mathbf{U}, \mathbf{V}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y}, \mathbf{Y_1}, \mathbf{Y_2})$	$(\mathbf{W}, \mathbf{U}, \mathbf{V}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y}, \mathbf{Y_1}, \mathbf{Y_2})$
$\Gamma_1\left(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}\right) \leq D_1,\ \Gamma_2\left(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}\right) \leq D_2$	$\Gamma_1\left(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}\right) \leq D_1,\ \Gamma_2\left(K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}\right) \leq D_2.$

The *ELB* is quite weak. Consider, for example, what is arguably the simplest nontrivial instance of the problem: the source $\mathbf{X}$ is bivariate, $K_{\mathbf{X}}$, $K_{\mathbf{X}|\mathbf{Y_1}}$, and $K_{\mathbf{X}|\mathbf{Y}_2}$ are all diagonal, and reconstructions at decoders are subject to component-wise MSE distortion constraint. This is essentially the parallel scalar Gaussian version of the problem. If the overall problem is degraded then the *ELB* is of course tight. But if one of the two components is degraded in one direction and the other component is degraded in the other, then Watanabe [5] has shown that the *ELB* is not tight, at least for the natural choice of $\mathbf{Y}$ that has

$$K_{\mathbf{X}|\mathbf{Y}} = \min\left(K_{\mathbf{X}|\mathbf{Y_1}}, K_{\mathbf{X}|\mathbf{Y}_2}\right).$$

Comparing the *ELB* against the achievable bound in Theorem 8, one sees several potential sources of looseness. We shall see that the culprit is that the distortion constraints

$$\Gamma_1\left(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}\right) \preceq D_1$$
$$\Gamma_2\left(K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}\right) \preceq D_2$$

in the achievable bound in Theorem 8 have been weakened to

$$\Gamma_1\left(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}\right) \preceq D_1$$
$$\Gamma_2\left(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}\right) \preceq D_2$$

here. Weakening the constraints in this way allows less informative $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ to be feasible, because one can make use of the enhanced side information $\mathbf{Y}$ for estimation purposes. We shall make this intuition precise by showing that the Maximin and Enhanced Enhancement lower bound, which differ from the *ELB* only in the distortion constraints, are tight for this problem. For reasons of expeditiousness, we shall state and prove the Minimax lower bound first, and then weaken it to obtain the Maximin and Enhanced Enhancement lower bound.

3.3.2 Minimax Lower Bound

Theorem 4 states the *Minimax lower bound*, abbreviated as *mLB*, to the rate distortion problem.

Theorem 4. *The rate distortion function, $R(\mathbf{D})$, is lower bounded by*

$$R_{mLB}(\mathbf{D}) = \sup_{S} \inf_{C_l(\mathbf{D})} \max\{R_{lo1}, R_{lo2}\} \tag{3.5}$$

where R_{lo1} and R_{lo2} are as in (3.3) and (3.4), and

$$
\begin{aligned}
& S = \{\mathbf{Y} | \mathbf{X} \leftrightarrow \mathbf{Y} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2})\} \\
& C_l(\mathbf{D})\text{: \textit{the set of} } (\mathbf{W}, \mathbf{U}, \mathbf{V}) \text{ \textit{such that}} \\
& (\mathbf{W}, \mathbf{U}, \mathbf{V}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y}) \\
& \Gamma_1 \left(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}\right) \le D_1, \Gamma_2 \left(K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}\right) \le D_2.
\end{aligned}
$$

Proof of Theorem 4 . By definition, for any $\mathbf{D}-$achievable rate, R, and for all $\epsilon > 0$, we can find a $(n, 2^{n(R+\epsilon)}, \mathbf{D} + \epsilon(I, I))$ code. Let $\epsilon > 0$ be given and J denote the output of the encoder. Also let $\mathbf{Y}$ be an auxiliary source in S. Then, we can write

$$
\begin{aligned}
n(R+\epsilon) &\ge H(J) \\
&\ge I(\mathbf{X}^n, \mathbf{Y}_\mathbf{1}^n, \mathbf{Y}^n; J) \\
&\overset{a}{=} I(\mathbf{Y}_\mathbf{1}^n; J) + I(\mathbf{Y}^n; J | \mathbf{Y}_\mathbf{1}^n) + I(\mathbf{X}^n; J | \mathbf{Y}_1^n, \mathbf{Y}^n) \\
&\ge I(\mathbf{Y}^n; J | \mathbf{Y}_\mathbf{1}^n) + I(\mathbf{X}^n; J | \mathbf{Y}_\mathbf{1}^n, \mathbf{Y}^n) \\
&\overset{b}{\ge} \sum_{i=1}^{n} \left[I(\mathbf{Y}_i; J, \mathbf{Y}_{\mathbf{1}\not{i}} | \mathbf{Y}_{\mathbf{1}i}) + I(\mathbf{X}_i; J, \mathbf{Y}_{\mathbf{1}\not{i}}, \mathbf{Y}_{\not{i}} | \mathbf{Y}_{\mathbf{1}i}, \mathbf{Y}_i) \right] \qquad (3.6)
\end{aligned}
$$

where $\mathbf{Y}_{\mathbf{1}\not{i}}$ denotes all $\mathbf{Y}_\mathbf{1}^n$ except $\mathbf{Y}_{1i}$ and a is due to the chain rule, and b is due to the chain rule and that conditioning reduces entropy. Then if we apply chain rule to the last term above, the right hand side of (3.6) equals

$$
\begin{aligned}
& \sum_{i=1}^{n} \left[I(\mathbf{Y}_i; J, \mathbf{Y}_{\mathbf{1}\not{i}} | \mathbf{Y}_{\mathbf{1}i}) + I(\mathbf{X}_i; J, \mathbf{Y}_{\mathbf{1}\not{i}} | \mathbf{Y}_{\mathbf{1}i}, \mathbf{Y}_i) + I(\mathbf{X}_i; \mathbf{Y}_{\not{i}} | J, \mathbf{Y}_{\mathbf{1}\not{i}}, \mathbf{Y}_{\mathbf{1}i}, \mathbf{Y}_i) \right] \qquad (3.7) \\
&= \sum_{i=1}^{n} \left[I(\mathbf{X}_i, \mathbf{Y}_i; J, \mathbf{Y}_{\mathbf{1}\not{i}} | \mathbf{Y}_{\mathbf{1}i}) + I(\mathbf{X}_i; \mathbf{Y}_{\not{i}} | J, \mathbf{Y}_{\mathbf{1}\not{i}}, \mathbf{Y}_{\mathbf{1}i}, \mathbf{Y}_i) \right] \\
&\ge \sum_{i=1}^{n} \left[I(\mathbf{X}_i; J, \mathbf{Y}_{\mathbf{1}\not{i}} | \mathbf{Y}_{1i}) + I(\mathbf{X}_i; \mathbf{Y}_{\not{i}} | J, \mathbf{Y}_{\mathbf{1}\not{i}}, \mathbf{Y}_{\mathbf{1}i}, \mathbf{Y}_i) \right]. \qquad (3.8)
\end{aligned}
$$

Also, since $\mathbf{X} \leftrightarrow \mathbf{Y} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2})$ the right hand side of (3.8) is equal to

$$\begin{aligned}
&\sum_{i=1}^{n} [I(\mathbf{X}_i; J, \mathbf{Y}_{\mathbf{1}\check{i}}|\mathbf{Y}_{\mathbf{1}i}) + I(\mathbf{X}_i; \mathbf{Y}_{\check{i}}|J, \mathbf{Y}_{\mathbf{1}\check{i}}, \mathbf{Y}_i)] \\
&\geq \sum_{i=1}^{n} [I(\mathbf{X}_i; J, \mathbf{Y}_{\mathbf{1}\check{i}}|\mathbf{Y}_{\mathbf{1}i}) + I(\mathbf{X}_i; \mathbf{Y}_{\mathbf{2}\check{i}}|J, \mathbf{Y}_{\mathbf{1}\check{i}}, \mathbf{Y}_i)] \\
&= \sum_{i=1}^{n} [I(\mathbf{X}_i; \mathbf{W'}_i, \mathbf{U'}_i|\mathbf{Y}_{\mathbf{1}i}) + I(\mathbf{X}_i; \mathbf{V'}_i|\mathbf{W'}_i, \mathbf{U'}_i, \mathbf{Y}_i)] \qquad (3.9)
\end{aligned}$$

where $\mathbf{W'}_i = J$, $\mathbf{U'}_i = \mathbf{Y}_{\mathbf{1}\check{i}}$ and $\mathbf{V'}_i = \mathbf{Y}_{\mathbf{2}\check{i}}$. Note that $(\mathbf{W'}_i, \mathbf{U'}_i, \mathbf{V'}_i) \leftrightarrow \mathbf{X}_i \leftrightarrow (\mathbf{Y}_{\mathbf{1}i}, \mathbf{Y}_{\mathbf{2}i}, \mathbf{Y}_i)$ for all $i \in [n]$. Let T be a random variable uniformly distributed on $[n]$ and independentthe source, side information and all $(\mathbf{W}'_i, \mathbf{U}'_i, \mathbf{V}'_i)$, $i \in [n]$. Then we can write the right hand side of (3.9) as

$$\begin{aligned}
&\sum_{i=1}^{n} [I(\mathbf{X}_i; \mathbf{W'}_i, \mathbf{U'}_i|\mathbf{Y}_{\mathbf{1}i}, T = i) + I(\mathbf{X}_i; \mathbf{V'}_i|\mathbf{W'}_i, \mathbf{U'}_i, \mathbf{Y}_i, T = i)] \\
&= n[I(\mathbf{X}; \mathbf{W}', \mathbf{U}', T|\mathbf{Y_1}) + I(\mathbf{X}; \mathbf{V}', T|\mathbf{W}', \mathbf{U}', T, \mathbf{Y})] \\
&= nR_{lo1}, \text{ by denoting } (\mathbf{W}', T), (\mathbf{U}', T), (\mathbf{V}', T) \text{ as } \mathbf{W}, \mathbf{U}, \mathbf{V} \text{ respectively.} \qquad (3.10)
\end{aligned}$$

If we swap the role of $\mathbf{Y_1}$ and $\mathbf{Y_2}$ and apply the same procedure above, we can get

$$\begin{aligned}
R + \epsilon &\geq I(\mathbf{X}; \mathbf{W}, \mathbf{V}|\mathbf{Y}_2) + I(\mathbf{X}; \mathbf{U}|\mathbf{W}, \mathbf{V}, \mathbf{Y}) \\
&= R_{lo2}. \qquad (3.11)
\end{aligned}$$

Note that since $(\mathbf{W'}_i, \mathbf{U'}_i, \mathbf{V'}_i) \leftrightarrow \mathbf{X}_i \leftrightarrow (\mathbf{Y}_{\mathbf{1}i}, \mathbf{Y}_{\mathbf{2}i}, \mathbf{Y}_i)$ for all $i \in [n]$, we have $(\mathbf{W}, \mathbf{U}, \mathbf{V}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$. Moreover since $(\mathbf{W'}_i, \mathbf{U'}_i, \mathbf{Y}_{\mathbf{1}i}) = (J, \mathbf{Y}_\mathbf{1}^n)$ and $(\mathbf{W'}_i, \mathbf{V'}_i, \mathbf{Y}_{\mathbf{2}i}) = (J, \mathbf{Y}_\mathbf{2}^n)$, given $(\mathbf{W'}_i, \mathbf{U'}_i, \mathbf{Y}_{\mathbf{1}i})$ Decoder 1 can reconstruct the source, $\mathbf{X}_i$, subject to its distortion constraint. Similarly, Decoder 2 can reconstruct the source, $\mathbf{X}_i$ given $(\mathbf{W'}_i, \mathbf{V'}_i, \mathbf{Y}_{\mathbf{2}i})$. Hence, $(\mathbf{W}, \mathbf{U}, \mathbf{V}) \in C_l(\mathbf{D} + \epsilon(I, I))$ and we have

$$R(\mathbf{D}) \geq \inf_{C_l(\mathbf{D}+\epsilon(I,I))} \max\{R_{lo1}, R_{lo2}\} - \epsilon. \qquad (3.12)$$

Let $R'_{lo}(\mathbf{D} + \epsilon(I, I), \mathbf{Y})$ denote the right hand side of (3.12). Note that (3.12) holds for any $\mathbf{Y} \in S$, where S as in Theorem 4. Hence we can write

$$R(\mathbf{D}) \geq \sup_{S} R'_{lo}(\mathbf{D} + \epsilon(I, I), \mathbf{Y}) - \epsilon. \tag{3.13}$$

Note that from Lemma 16 in Appendix 6.2, $R'_{lo}(\mathbf{D}, \mathbf{Y})$ is convex in $\mathbf{D}$. Since $0 \prec D_i$, $i \in \{1, 2\}$ we can find $\delta(D_1, D_2) > 0$ such that $0 \prec D_i - \delta(D_1, D_2)I$ for $i \in \{1, 2\}$. Hence $R'_{lo}(\mathbf{D} + \gamma(I, I), \mathbf{Y})$ is also convex in γ, where $\gamma \geq -\delta(D_1, D_2)$. Note that $\sup_S R'_{lo}(\mathbf{D} + \epsilon(I, I), \mathbf{Y})$ is also convex since supremum of convex functions is convex. Then, we can conclude that $\sup_S R'_{lo}(\mathbf{D} + \epsilon(I, I), \mathbf{Y})$ is continuous at $\epsilon = 0$ since a convex function on an open set is continuous. Lastly, since ϵ was arbitrary, letting $\epsilon \to 0$ gives the result. □

It is worth noting that one can prove a bound similar to *mLB* for non-Gaussian sources and general additive distortion constraints. Although the *mLB* is quite powerful, it can be difficult to apply. In particular, it is not clear that it is sufficient to consider $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ that are jointly Gaussian with $(\mathbf{X}, \mathbf{Y}_1, \mathbf{Y}_2)$. Similarly, when considering the analogous form of this bound for discrete memoryless sources, it it not clear how to obtain cardinality bounds on the auxiliary random variables $(\mathbf{W}, \mathbf{U}, \mathbf{V})$. As such, it is not clear how to compute this bound in general. We shall therefore consider a slightly weakened form of the bound that is easier to apply. It turns out that simply swapping the min and the max in the objective and adding that $\mathbf{Y}$ is jointly Gaussian with $(\mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2})$ to S yields a bound that is significantly more tractable.

3.3.3 Maximin Lower Bound

Next proposition gives us *Maximin lower bound*, abbreviated as *MLB*.

Proposition 7. *The rate distortion function, $R(\mathbf{D})$, is lower bounded by*

$$R_{MLB}(\mathbf{D}) = \max\{\sup_{S_G} \inf_{C_{l1}(\mathbf{D})} R_{lo1}, \sup_{S_G} \inf_{C_{l2}(\mathbf{D})} R_{lo2}\}, \tag{3.14}$$

where R_{lo1} and R_{lo2} are as in (3.3) and (3.4) respectively, S_G as in Proposition 6, and

$C_{l1}(\mathbf{D})$: *the set of* $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ *such that*	$C_{l2}(\mathbf{D})$: *the set of* $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ *such that*
$(\mathbf{W}, \mathbf{U}, \mathbf{V}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$	$(\mathbf{W}, \mathbf{U}, \mathbf{V}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$
$\Gamma_1\left(K_{\mathbf{X}\mid\mathbf{W},\mathbf{U},\mathbf{Y_1}}\right) \leq D_1,\ \Gamma_2\left(K_{\mathbf{X}\mid\mathbf{W},\mathbf{V},\mathbf{Y_2}}\right) \leq D_2$	$\Gamma_1\left(K_{\mathbf{X}\mid\mathbf{W},\mathbf{U},\mathbf{Y_1}}\right) \leq D_1,\ \Gamma_2\left(K_{\mathbf{X}\mid\mathbf{W},\mathbf{V},\mathbf{Y_2}}\right) \leq D_2.$

Proof. This follows directly from the *mLB*, Theorem 4, by moving the inf in the objective inside the maximization over the bounds in (3.3) and (3.4) and replacing the set S with S_G. □

Although numerical evidence suggests that the *MLB* can be strictly weaker than the *mLB* (see the discussion in Section 3.7 to follow), the *MLB* does have certain advantages. For the analogous bound for discrete memoryless sources with additive distortion measures, one can obtain cardinality bounds on the alphabets of **W**, **U**, and **V** using straightforward techniques [43]. And we shall show that, for the Gaussian form examined here, one may restrict attention to **W**, **U**, and **V** that are jointly Gaussian with $(\mathbf{X}, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y})$.

Evidently the *MLB* differs from the *ELB* in Proposition 6 only in that the distortion constraints are replaced with those that appear in the achievable upper bound presented in Theorem 8, Section 3.5. In Section 3.6, we shall see that this improvement suffices to make the bound tight for the rate distortion problem with MSE distortion constraints stated in Section 3.4. Now, we consider the fourth and final lower bound.

3.3.4 Enhanced Enhancement Lower Bound

Proposition 8. *The rate distortion function, $R(\mathbf{D})$, is lower bounded by*

$$R_{E^2LB}(\mathbf{D}) = \max\{\sup_{S_G} \inf_{\bar{C}_{l1}(\mathbf{D})} R_{lo1}, \sup_{S_G} \inf_{\bar{C}_{l2}(\mathbf{D})} R_{lo2}\}, \tag{3.15}$$

where R_{lo1} and R_{lo2} are as in (3.3) and (3.4) respectively, S_G as in Proposition 6, and

$\bar{C}_{l1}(\mathbf{D})$: *the set of* $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ *such that*	$\bar{C}_{l2}(\mathbf{D})$: *the set of* $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ *such that*
$(\mathbf{W}, \mathbf{U}, \mathbf{V}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$	$(\mathbf{W}, \mathbf{U}, \mathbf{V}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$
$\Gamma_1 (K_{\mathbf{X}\mid\mathbf{W},\mathbf{U},\mathbf{Y_1}}) \leq D_1,$	$\Gamma_1 \left((K^{-1}_{\mathbf{X}\mid\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}} - \widehat{K})^{-1}\right) \leq D_1,$
$\Gamma_2 \left((K^{-1}_{\mathbf{X}\mid\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}} - \widetilde{K})^{-1}\right) \leq D_2$	$\Gamma_2 (K_{\mathbf{X}\mid\mathbf{W},\mathbf{V},\mathbf{Y_2}}) \leq D_2$

and $\widetilde{K} = K^{-1}_{\mathbf{X}\mid\mathbf{Y}} - K^{-1}_{\mathbf{X}\mid\mathbf{Y_2}}$, $\widehat{K} = K^{-1}_{\mathbf{X}\mid\mathbf{Y}} - K^{-1}_{\mathbf{X}\mid\mathbf{Y_1}}$.

Proof. Note that only difference between *MLB* and Enhanced *ELB* is the optimization sets where the infima are taken. Hence it is enough to show that $C_{li}(\mathbf{D}) \subseteq \bar{C}_{li}(\mathbf{D})$ for $i \in \{1,2\}$. Let $(\mathbf{W}, \mathbf{U}, \mathbf{V}) \in C_{l1}(\mathbf{D})$. Then $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ satisfy the Markov chain condition $(\mathbf{W}, \mathbf{U}, \mathbf{V}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$ and we have $\Gamma_1 (K_{\mathbf{X}\mid\mathbf{W},\mathbf{U},\mathbf{Y_1}}) \leq D_1$. Also, the inequalities $K_{\mathbf{X}\mid\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y_2}} \leq K_{\mathbf{X}\mid\mathbf{W},\mathbf{V},\mathbf{Y_2}}$ and $K^{-1}_{\mathbf{X}\mid\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y_2}} \leq K^{-1}_{\mathbf{X}\mid\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}} - \widetilde{K}$ imply, by the Gaussian variance-drop lemma (Lemma 14 in Appendix 6.1), that $\Gamma_2 \left((K^{-1}_{\mathbf{X}\mid\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}} - \widetilde{K})^{-1}\right) \leq D_2$. Hence $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ is also in $\bar{C}_{l1}(\mathbf{D})$, giving $C_{l1}(\mathbf{D}) \subseteq \bar{C}_{l1}(\mathbf{D})$. We can apply similar procedure to get $C_{l2}(\mathbf{D}) \subseteq \bar{C}_{l2}(\mathbf{D})$, which concludes the proof. □

Comparing the Enhanced *ELB* against the *ELB* in (3.2) shows that the differences lie entirely in the distortion constraints. The *ELB* effectively allows the decoders to use their "enhanced" side information for the purposes of estimating the source. The achievable bound, by contrast, does not. The Enhanced *ELB* allows the decoders to use their enhanced side information, but it also tightens the constraint to account for this extra information, as justified by the Gaussian variance-drop lemma. We shall see in the next subsection that the Enhanced *ELB* actually coincides with the *MLB* for all of the problems considered in this chapter. We mention the Enhanced *ELB* only because the idea of using the Gaussian variance-drop lemma to tighten the distortion constraints at decoders that are provided with improved side information may prove useful in other contexts.

3.3.5 Properties of the Lower Bounds

It is evident from the proofs in this section that the four lower bounds can be ordered as follows

$$R_{ELB}(\mathbf{D}) \leq R_{E^2LB}(\mathbf{D}) \leq R_{MLB}(\mathbf{D}) \leq R_{mLB}(\mathbf{D}).$$

We shall show that Gaussian auxiliary random variables are optimal for *MLB*, Enhanced *ELB*, and *ELB*, and that the *MLB* and Enhanced *ELB* are in fact equal. We begin by showing that Gaussian auxiliary random variables are optimal for the *ELB* and Enhanced *ELB*.

Lemma 3. *One may add the constraint that* $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ *is jointly Gaussian with* $(\mathbf{X}, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y})$ *to the optimization problem in the ELB in (3.2) and the Enhanced ELB in (3.15) without affecting the optimal value.*

Proof. See Appendix 6.1. □

Proposition 9. *The Maximin bound and Enhanced ELB in Proposition 7 and 8, respectively, coincide:*

$$R_{MLB}(\mathbf{D}) = R_{E^2LB}(\mathbf{D}). \tag{3.16}$$

Proof. It suffices to show that

$$R_{MLB}(\mathbf{D}) \leq R_{E^2LB}(\mathbf{D})$$

By Lemma 3, $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ in $\bar{C}_{l1}(\mathbf{D})$ or $\bar{C}_{l2}(\mathbf{D})$ can be restricted to vector Gaussian random variables without loss of optimality. Furthermore, any $\mathbf{U} \in \bar{C}_{l1}$ can be lumped into $\mathbf{W} \in \bar{C}_{l1}(\mathbf{D})$, i.e. $\mathbf{U}$ is deterministic, without loss of optimality since $\mathbf{W}$ and $\mathbf{U}$ always appear together both in the objective and the conditions. The same argument holds when we swap the roles of $\mathbf{U}$ and $\mathbf{V}$ in $\bar{C}_{l2}(\mathbf{D})$. Hence, with those additional conditions we can write the optimizing sets, $\bar{C}_{l1}(\mathbf{D})$ and $\bar{C}_{l2}(\mathbf{D})$, as

$\bar{C}_{l1}(\mathbf{D})$:	$\bar{C}_{l1}(\mathbf{D})$:
$(\mathbf{W}, \mathbf{U}, \mathbf{V}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$	$(\mathbf{W}, \mathbf{U}, \mathbf{V}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$
$(\mathbf{W}, \mathbf{U}, \mathbf{V}, \mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$ jointly Gaussian	$(\mathbf{W}, \mathbf{U}, \mathbf{V}, \mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$ jointly Gaussian
$\mathbf{U} = \emptyset$	$\mathbf{V} = \emptyset$
$K_{\mathbf{X}\|\mathbf{W},\mathbf{Y_1}} \preceq D_1, \; K_{\mathbf{X}\|\mathbf{W},\mathbf{V},\mathbf{Y_2}} \preceq D_2$	$K_{\mathbf{X}\|\mathbf{W},\mathbf{U},\mathbf{Y_1}} \preceq D_1, \; K_{\mathbf{X}\|\mathbf{W},\mathbf{Y_2}} \preceq D_2$

Then any such $(\mathbf{W}, \mathbf{U}, \mathbf{V}) \in \bar{C}_{l1}(\mathbf{D})$ (or $(\mathbf{W}, \mathbf{U}, \mathbf{V}) \in \bar{C}_{l2}(\mathbf{D})$) is also in $C_{l1}(\mathbf{D})$ (or $C_{l2}(\mathbf{D})$). Hence, $R_{MLB}(\mathbf{D}) \leq R_{E^2LB}(\mathbf{D})$. □

It follows from the two previous results that Gaussian auxiliary random variables are optimal for the *MLB*. To see this, let $R^{G}_{E^2LB}(\mathbf{D})$ denote the Enhanced

ELB with the auxiliary random variables constrained to be jointly Gaussian with $(\mathbf{X}, \mathbf{Y}_1, \mathbf{Y}_2)$. Define $R^G_{MLB}(\mathbf{D})$ likewise. Then we have

$$\begin{aligned} R^G_{MLB}(\mathbf{D}) &\geq R_{MLB}(\mathbf{D}) \\ &\stackrel{a}{=} R_{E^2LB}(\mathbf{D}) \\ &\stackrel{b}{=} R^G_{E^2LB}(\mathbf{D}) \\ &\stackrel{c}{=} R^G_{MLB}(\mathbf{D}), \end{aligned}$$

where a follows from Proposition 9, b follows from Lemma 3, and c is straightforward to verify.

We now proceed to state our main results, characterization of rate distortion functions subject to three different distortion constraints.

3.4 Main Results

We shall determine the optimal rate only for the following choices of Γ_1, Γ_2, D_1, and D_2:

1. *Mean square error (MSE):* Γ_1 and Γ_2 are chosen as

$$\Gamma_i(K) = (K)_{diag} \quad i \in \{1, 2\}, \tag{3.17}$$

   and D_1 and D_2 are diagonal matrices satisfying

$$D_1 \preceq K_{\mathbf{X}|\mathbf{Y_1}} \text{ and } D_2 \preceq K_{\mathbf{X}|\mathbf{Y_2}}. \tag{3.18}$$

2. *Error covariance matrix:* Γ_1 and Γ_2 are chosen as

$$\Gamma_i(K) = K \quad i \in \{1, 2\} \tag{3.19}$$

and D_1 and D_2 are scaled identity matrices satisfying

$$D_1 \preceq K_{\mathbf{X}|\mathbf{Y}_1} \text{ and } D_2 \preceq K_{\mathbf{X}|\mathbf{Y}_2}. \tag{3.20}$$

Note that scaled identity matrix constraints on the error covariance matrix enable us to bound the MSE of the reconstruction vector uniformly from all directions.

3. *Trace of the error covariance matrix:* Γ_1 and Γ_2 are chosen as

$$\Gamma_i(K) = Tr(K) \quad i \in \{1, 2\}, \tag{3.21}$$

and D_1 and D_2 are scalars satisfying

$$D_1 I \preceq K_{\mathbf{X}|\mathbf{Y}_1} \text{ and } D_2 I \preceq K_{\mathbf{X}|\mathbf{Y}_2}. \tag{3.22}$$

Most of the prior work on Heegard-Berger problem assumes some sort of degradedness structure between the source and the side information at the two decoders (e.g. [18, 5, 6]). Watanabe [5], in particular, assumes that the source and the side information all consist of two components, and the first components of all three variables are independent of the second components of all three variables. The two components are "mismatched degraded," i.e., each component is individually degraded, but the two components are degraded in opposite order. Although we do not assume any degradedness structure, we shall reduce our problem to one that resembles Watanabe's. Specifically, we shall decompose the signal space into "*regions*," one of which is such that the side information at Decoder 1 is "*stronger*" than that of Decoder 2 and one such that the reverse is true. Many such candidate decompositions are possible; we shall use the following approach.

Recall that we assume that $K_{\mathbf{X}|\mathbf{Y_i}}$, $i \in \{1,2\}$ are invertible matrices.[4] Now consider the matrix $K^{-1}_{\mathbf{X}|\mathbf{Y_2}} - K^{-1}_{\mathbf{X}|\mathbf{Y_1}}$. Since it is symmetric we can find an orthogonal matrix Q_1 such that $Q_1(K^{-1}_{\mathbf{X}|\mathbf{Y_2}} - K^{-1}_{\mathbf{X}|\mathbf{Y_1}})Q_1^T$ is diagonal. Furthermore, we can find another orthogonal matrix Q_2 such that $Q_2Q_1(K^{-1}_{\mathbf{X}|\mathbf{Y_2}} - K^{-1}_{\mathbf{X}|\mathbf{Y_1}})Q_1^TQ_2^T$ is of the form

$$K = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{3.23}$$

where $A \geq 0$ is an $l_1 \times l_1$ diagonal matrix, $B \prec 0$ is an $l_2 \times l_2$ diagonal matrix and $l_1 + l_2 = k$.

Let $Q = Q_2Q_1$. Note that $QDQ^T = D$ when D is a scaled identity matrix and distortion measure in (3.21) is invariant under $(\mathbf{X}, \widehat{\mathbf{X}}_\mathbf{i}) \to (Q\mathbf{X}, Q\widehat{\mathbf{X}}_\mathbf{i})$.

Note that MSE distortion measure is not invariant under $(\mathbf{X}, \widehat{\mathbf{X}}_\mathbf{i}) \to (Q\mathbf{X}, Q\widehat{\mathbf{X}}_\mathbf{i})$. Then for MSE and any Γ_i such that it is not invariant under $(\mathbf{X}, \widehat{\mathbf{X}}_\mathbf{i}) \to (Q\mathbf{X}, Q\widehat{\mathbf{X}}_\mathbf{i})$, we restrict our attention to the source $\mathbf{X}$ and side information $\mathbf{Y_i}$ such that

$$K^{-1}_{\mathbf{X}|\mathbf{Y_2}} - K^{-1}_{\mathbf{X}|\mathbf{Y_1}} = K.$$

Therefore, the rate-distortion problems where $Q\mathbf{X}$ is the source, $\mathbf{Y_i}$ is side information at Decoder i subject to the distortion constraints D_i, $i \in \{1,2\}$ are equivalent to the problems that we defined at the beginning. For the rest of the chapter, we assume that $Q\mathbf{X}$ is the source and we relabel $Q\mathbf{X}$ as $\mathbf{X}$ for the ease of notation, $\mathbf{Y_1}$ and $\mathbf{Y_2}$ are side information and D_1 and D_2 distortion constraints for Decoder 1 and 2 respectively as shown in Figure 3.1. Note that we have not entirely reduced the problem to that of Watanabe because the components of $\mathbf{X}$ may be dependent.

[4] Distortion constraints in (3.20), (3.22), and (3.18) also imply that $K_{\mathbf{X}|\mathbf{Y_1}}$ and $K_{\mathbf{X}|\mathbf{Y_2}}$ are positive definite matrices.

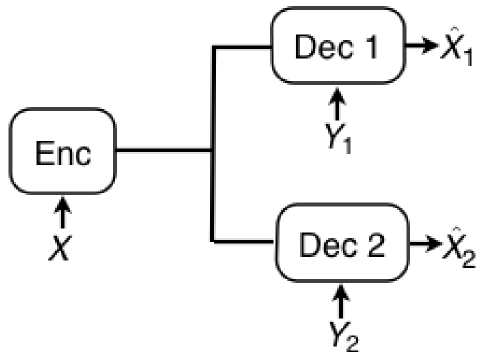

Figure 3.1: Problem Setup

From now on we use the abbreviation $RDSI$ for the problem of finding the rate distortion function where reconstructions at decoders are subjected to error covariance distortion constraints that are scaled identity matrices as in (3.20) and denote the corresponding rate distortion function as $R^{Sc}(\mathbf{D})$, where $\mathbf{D} = (D_1, D_2)$. Also $RDTr$ and $RDmse$ denote the rate distortion problems where decoders have distortion constraints as in (3.22) on the trace of error covariance matrices and (3.18) componentwise MSE constraints respectively. Corresponding rate distortion functions of $RDTr$ and $RDmse$ are denoted by $R^{Tr}(\mathbf{D})$ and $R^{MSE}(\mathbf{D})$ respectively.

Remark 11. *Notice that*

$$K^{-1}_{\mathbf{X}|\mathbf{Y_2}} - K^{-1}_{\mathbf{X}|\mathbf{Y_1}} = K. \tag{3.24}$$

Since $(K^{-1}_{\mathbf{X}|\mathbf{Y_2}})_{l_1} \succeq (K^{-1}_{\mathbf{X}|\mathbf{Y_1}})_{l_1}$, we say that $\mathbf{Y_2}$ is "stronger" than $\mathbf{Y_1}$ in the "region" involving upper left part of the inverse covariance matrices. Similarly $\mathbf{Y_1}$ is "stronger" than $\mathbf{Y_2}$ in lower right part of the inverse covariance matrices since $[K^{-1}_{\mathbf{X}|\mathbf{Y_2}}]_{l_2} \preceq [K^{-1}_{\mathbf{X}|\mathbf{Y_1}}]_{l_2}$.

Now we are ready to state our main results.

Theorem 5. *Let $K_{\mathbf{X}|\mathbf{Y_i}}$, $i \in \{1,2\}$ be diagonal matrices. Then the rate distortion function of RDmse, $R^{MSE}(\mathbf{D})$, can be written as*

$$R^{MSE}(\mathbf{D}) = \max\{R_1^{MSE}(\mathbf{D}), R_2^{MSE}(\mathbf{D})\},$$

where

$$R_1^{MSE}(\mathbf{D}) = \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|(D_1)_{l_1}||\min\{[D_1]_{l_2}, [\widetilde{D}_2]_{l_2}\}|} + \frac{1}{2}\log\frac{|(\widehat{D}_1)_{l_1}|}{|\min\{(\widehat{D}_1)_{l_1}, (D_2)_{l_1}\}|} \tag{3.25}$$

$$R_2^{MSE}(\mathbf{D}) = \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_2}}|}{|[D_2]_{l_2}||\min\{(\widehat{D}_1)_{l_1}, (D_2)_{l_1}\}|} + \frac{1}{2}\log\frac{|[\widetilde{D}_2]_{l_2}|}{|\min\{[D_1]_{l_2}, [\widetilde{D}_2]_{l_2}\}|}, \tag{3.26}$$

and[5] $\widehat{D}_1 = (D_1^{-1} + K)^{-1}$, $\widetilde{D}_2 = (D_2^{-1} - K)^{-1}$.

To prove Theorem 5, first we find an upper bound based on the achievable scheme in [2] in Section 3.5 and then we utilize Enhanced *ELB* in the previous section, matching the upper bound.

Remark 12. *Theorem 5 subsumes the Gaussian version of Watanabe's result [5] by allowing for* $\mathbf{X}$ *to have dimension exceeding two. Watanabe points out that the rate-distortion for his problem, and thus for ours, does not in general equal the sum of the individual rate-distortion functions across the components of* $\mathbf{X}$*, even though they are independent, independent given either side information vector, and subject to separate distortion constraints. Thus, even in this case, it is necessary to code across the different components of* $\mathbf{X}$*.*

Theorem 6. *The rate-distortion function for RDSI, $R^{Sc}(\mathbf{D})$, can be expressed as*

$$R^{Sc}(\mathbf{D}) = \max\{R_1^{Sc}(\mathbf{D}), R_2^{Sc}(\mathbf{D})\},$$

[5]Note that $\widehat{D}_1$ and $\widetilde{D}_2$ are positive definite since $D_1^{-1} \succeq K_{\mathbf{X}|\mathbf{Y_1}}^{-1} \succ 0$, $D_2^{-1} \succeq K_{\mathbf{X}|\mathbf{Y_2}}^{-1} \succ 0$, and $K_{\mathbf{X}|\mathbf{Y_2}}^{-1} = K_{\mathbf{X}|\mathbf{Y_1}}^{-1} + K$.

where

$$R_1^{Sc}(\mathbf{D}) = \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|(D_1)_{l_1}||\min\{[D_1]_{l_2},[\widetilde{D}_2]_{l_2}\}|} + \frac{1}{2}\log\frac{|(\widehat{D}_1)_{l_1}|}{|\min\{(\widehat{D}_1)_{l_1},(D_2)_{l_1}\}|} \tag{3.27}$$

$$R_2^{Sc}(\mathbf{D}) = \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_2}}|}{|[D_2]_{l_2}||\min\{(\widehat{D}_1)_{l_1},(D_2)_{l_1}\}|} + \frac{1}{2}\log\frac{|[\widetilde{D}_2]_{l_2}|}{|\min\{[D_1]_{l_2},[\widetilde{D}_2]_{l_2}\}|}, \tag{3.28}$$

and[6] $\widehat{D}_1 = (D_1^{-1}+K)^{-1}$, $\widetilde{D}_2 = (D_2^{-1}-K)^{-1}$.

For the direct part of the proof of Theorem 6, we utilize the achievable scheme in Section 3.5. For the converse result presented in Section 3.6, we use Enhanced *ELB*.

Theorem 7. *The rate distortion function for RDTr, $R^{Tr}(\mathbf{D})$, can be characterized as*

$$R^{Tr}(\mathbf{D}) = \min_{C^{Tr}(\mathbf{D})} \max\{R_1^{Tr}(\mathbf{D}), R_2^{Tr}(\mathbf{D})\}$$

where

$$R_1^{Tr}(\mathbf{D}) = \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|I+A(K_{\mathbf{X}|\mathbf{W},\mathbf{Y_1}})_{l_1}|} + \frac{1}{2}\log\frac{1}{|(K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}})_{l_1}||[K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}]_{l_2}|}, \tag{3.29}$$

$$R_2^{Tr}(\mathbf{D}) = \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_2}}|}{|I-B[K_{\mathbf{X}|\mathbf{W},\mathbf{Y_2}}]_{l_2}|} + \frac{1}{2}\log\frac{1}{|(K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}})_{l_1}||[K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}]_{l_2}|} \tag{3.30}$$

and $C^{Tr}(\mathbf{D})$ denotes

$$(\mathbf{W},\mathbf{U},\mathbf{V}) \text{ jointly Gaussian with } (\mathbf{X},\mathbf{Y_1},\mathbf{Y_2}) \tag{3.31}$$

$$(\mathbf{W},\mathbf{U},\mathbf{V}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1},\mathbf{Y_2}) \tag{3.32}$$

$$\mathbf{U} \perp (\mathbf{X})_{l_1}|(\mathbf{W},\mathbf{Y_1}),\ \mathbf{V} \perp [\mathbf{X}]_{l_2}|(\mathbf{W},\mathbf{Y_2}) \tag{3.33}$$

$$K_{\mathbf{X}|\mathbf{W},\mathbf{Y_1}}, K_{\mathbf{X}|\mathbf{W},\mathbf{Y_2}}, K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}, K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}, \text{ diagonal} \tag{3.34}$$

$$Tr((K_{\mathbf{X}|\mathbf{W},\mathbf{Y_1}})_{l_1}) + Tr([K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}]_{l_2}) \le D_1 \tag{3.35}$$

$$Tr((K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}})_{l_1}) + Tr([K_{\mathbf{X}|\mathbf{W},\mathbf{Y_2}}]_{l_2}) \le D_2 \tag{3.36}$$

[6]Note that $\widehat{D}_1$ and $\widetilde{D}_2$ are positive definite due to similar reasoning in Theorem 5.

Remark 13. *Let* $\mathbf{W}$ *be jointly Gaussian with* $(\mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2})$ *such that* $\mathbf{W} \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2})$. *Due to (3.24),* $K_{\mathbf{X}|\mathbf{W},\mathbf{Y_1}}$ *is a diagonal matrix if and only if* $K_{\mathbf{X}|\mathbf{W},\mathbf{Y_2}}$ *is diagonal.*

Similar to the proof of Theorem 6 and 5, we begin with proving the direct part using the same achievable scheme for *RDSI* by changing the distortion measure. For the converse part; however, we utilize *mLB* that is better than the Enhanced *ELB* in general.

3.5 Achievable Scheme

Here, we utilize the random coding arguments similar to [2, 18] to obtain the following achievable rate.

Theorem 8. *Rate distortion function,* $R(\mathbf{D})$*, is upper bounded by*

$$R_{ach}(\mathbf{D}) = \inf_{C_u(\mathbf{D})} \max\{I(\mathbf{X}; \mathbf{W}, \mathbf{U}|\mathbf{Y_1}) + I(\mathbf{X}; \mathbf{V}|\mathbf{W}, \mathbf{Y_2}), I(\mathbf{X}; \mathbf{W}, \mathbf{V}|\mathbf{Y_2}) + I(\mathbf{X}; \mathbf{U}|\mathbf{W}, \mathbf{Y_1})\} \tag{3.37}$$

where

$$\begin{aligned}
&C_u(\mathbf{D}) : \text{ set of } (\mathbf{W}, \mathbf{U}, \mathbf{V}) \text{ such that}\\
&(\mathbf{W}, \mathbf{U}, \mathbf{V}) \text{ jointly Gaussian with } (\mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2})\\
&(\mathbf{W}, \mathbf{U}, \mathbf{V}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2})\\
&\Gamma_1\left(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}\right) \leq D_1, \Gamma_2\left(K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}\right) \leq D_2,
\end{aligned}$$

and Γ_i *can be equal to one of the mappings in (3.17), (3.19), and (3.21) and the corresponding distortion constraints are as in (3.18), (3.20), and (3.22) respectively.*

Note that since feasible $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ should be jointly Gaussian with $(\mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2})$ in (3.37), we can write $R_{ach}(\mathbf{D})$ as

$$R_{ach}(\mathbf{D}) = \inf_{C_u(\mathbf{D})} \max\{R_1, R_2\}$$

where

$$R_1 = \frac{1}{2} \log \frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}|} \frac{|K_{\mathbf{X}|\mathbf{W},\mathbf{Y_2}}|}{|K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}|}, \tag{3.38}$$

$$R_2 = \frac{1}{2} \log \frac{|K_{\mathbf{X}|\mathbf{Y_2}}|}{|K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}|} \frac{|K_{\mathbf{X}|\mathbf{W},\mathbf{Y_1}}|}{|K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}|}. \tag{3.39}$$

Here $\mathbf{W}$ can be viewed as a common message to both decoders, and $\mathbf{U}$ and $\mathbf{V}$ are private messages for Decoder 1 and 2 respectively. The encoder first creates $\mathbf{W}$ via vector quantization with a given Gaussian test channel and then generates $\mathbf{U}$ and $\mathbf{V}$ with respect to the source and $\mathbf{W}$. Then $\mathbf{W}$ is sent to both decoders and $\mathbf{U}$ and $\mathbf{V}$ are sent to Decoder 1 and Decoder 2, respectively. At the Decoder side, Decoder 1 decodes $\mathbf{W}$ and $\mathbf{U}$ by using its side information $\mathbf{Y_1}$. Similarly, Decoder 2 decodes $\mathbf{W}$ and $\mathbf{V}$ using $\mathbf{Y_2}$.

To get an explicit expression for the upper bounds we need to specify the properties of the auxiliary random variables. Next three propositions give an explicit upper bound on the $R^{MSE}(\mathbf{D})$, $R^{Sc}(\mathbf{D})$, and properties of $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ in the optimizing set $C_u(\mathbf{D})$ for trace distortion constraints.

Proposition 10. *$R^{MSE}(\mathbf{D})$ is upper bounded by*

$$R_u^{MSE}(\mathbf{D}) = \max\{R_1^{MSE}(\mathbf{D}), R_2^{MSE}(\mathbf{D})\}$$

where $R_1^{MSE}(\mathbf{D})$ and $R_2^{MSE}(\mathbf{D})$ are as in (3.25) and (3.26) respectively.

Proof. We start the proof by showing that

$$G = \begin{pmatrix} (\widehat{D}_1)_{l_1} & 0 \\ 0 & [D_2]_{l_2} \end{pmatrix},$$

where $\widehat{D}_1$ as in Theorem 5, is dominated by $K_{\mathbf{X}|\mathbf{Y_2}}$. Since $K_{\mathbf{X}|\mathbf{Y_2}}$ and G are diagonal matrices and $D_2 \preceq K_{\mathbf{X}|\mathbf{Y_2}}$, it is enough to show that $(\widehat{D}_1)_{l_1} \preceq (K_{\mathbf{X}|\mathbf{Y_2}})_{l_1}$. Note that $\widehat{D}_1 = (D_1^{-1} + K)^{-1} \preceq K_{\mathbf{X}|\mathbf{Y_2}}$ since $D_1 \preceq K_{\mathbf{X}|\mathbf{Y_1}}$. Thus, $(\widehat{D}_1)_{l_1} \preceq (K_{\mathbf{X}|\mathbf{Y_2}})_{l_1}$ and $G \preceq K_{\mathbf{X}|\mathbf{Y_2}}$. Then we can select $\mathbf{W}$ such that it is jointly Gaussian with $\mathbf{X}$ and $K_{\mathbf{X}|\mathbf{W},\mathbf{Y_2}} = G$. This implies

$$\begin{aligned}
K_{\mathbf{X}|\mathbf{W},\mathbf{Y_1}} &= (K_{\mathbf{X}|\mathbf{W},\mathbf{Y_2}}^{-1} - K)^{-1} \\
&= (G^{-1} - K)^{-1} \\
&= \begin{pmatrix} (D_1)_{l_1} & 0 \\ 0 & [\widetilde{D}_2]_{l_2} \end{pmatrix},
\end{aligned}$$

where $\widetilde{D}_2$ is as in Theorem 5.

Lastly, we select $\mathbf{U}$ and $\mathbf{V}$ jointly Gaussian with $\mathbf{X}$ and $\mathbf{W}$ such that

$$\begin{aligned}
K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}} &= \begin{pmatrix} \min\{(\widehat{D}_1)_{l_1}, (D_2)_{l_1}\} & 0 \\ 0 & [D_2]_{l_2} \end{pmatrix}, \\
K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}} &= \begin{pmatrix} (D_1)_{l_1} & 0 \\ 0 & \min\{[D_1]_{l_2}, [\widetilde{D}_2]_{l_2}\} \end{pmatrix},
\end{aligned}$$

satisfying the distortion constraints. Evaluating R_1 and R_2 for this choice of $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ gives us $R_1^{MSE}(\mathbf{D})$ and $R_2^{MSE}(\mathbf{D})$. □

If we take a closer look on the selection of "common" and "private" messages, we can make the following observation. "common" message is used to hit the distortion constraint of each decoder with equality on the region where it is "weaker" in the scheme for *RDmse*. On the other hand, using both the "common" and "private" messages, each decoder may undershoot its distortion constraint where it is "stronger" depending on D_1, D_2 and K. Now, we provide the following proposition which gives an explicit upper bound on $R^{Sc}(\mathbf{D})$.

Proposition 11. *$R^{Sc}(\mathbf{D})$ is upper bounded by*

$$R_u^{Sc}(\mathbf{D}) = \max\{R_1^{Sc}(\mathbf{D}), R_2^{Sc}(\mathbf{D})\}$$

where $R_1^{Sc}(\mathbf{D})$ and $R_2^{Sc}(\mathbf{D})$ are as in (3.27) and (3.28) respectively.

Proof. We follow similar approach in the proof of Proposition 10. We take a particular feasible choice of $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ in $R_{ach}(\mathbf{D})$ to get an explicit upper bound on the rate-distortion function, $R^{Sc}(\mathbf{D})$. We would like to choose $\mathbf{W}$ jointly Gaussian with $\mathbf{X}$ so that $K_{\mathbf{X}|\mathbf{W},\mathbf{Y_2}}$ is equal to

$$G = \begin{pmatrix} (\widehat{D}_1)_{l_1} & 0 \\ 0 & [D_2]_{l_2} \end{pmatrix}.$$

This is possible if and only if G is dominated by $K_{\mathbf{X}|\mathbf{Y_2}}$. To see that this is the case, note that $K^{-1}_{\mathbf{X}|\mathbf{Y_1}} \preceq D_1^{-1}$ so we have $K^{-1}_{\mathbf{X}|\mathbf{Y_1}} + K \preceq D_1^{-1} + K$, where K is in (3.24). This implies that $K^{-1}_{\mathbf{X}|\mathbf{Y_2}} \preceq \widehat{D}_1^{-1}$ since $D_1^{-1} + K = \widehat{D}_1^{-1}$.

Now since D_1 and D_2 are scaled identity matrices, we must have either $D_1 \preceq D_2$ or $D_1 \succeq D_2$. We shall show that we have $K^{-1}_{\mathbf{X}|\mathbf{Y_2}} \preceq G^{-1}$ in both cases.

Case 1: $D_1 \preceq D_2$.

Note that $(\widehat{D}_1^{-1})_{l_1} \succeq (D_1^{-1})_{l_1} \succeq (D_2^{-1})_{l_1}$. Then

$$\begin{aligned} G^{-1} - D_2^{-1} &= \begin{pmatrix} (\widehat{D}_1^{-1})_{l_1} - (D_2^{-1})_{l_1} & 0 \\ 0 & 0 \end{pmatrix} \\ &\succeq 0. \end{aligned}$$

So $G^{-1} \succeq D_2^{-1} \succeq K^{-1}_{\mathbf{X}|\mathbf{Y_2}}$.

Case 2: $D_1 \succeq D_2$.

Note that $[\widehat{D}_1^{-1}]_{l_2} \preceq [D_1^{-1}]_{l_2} \preceq [D_2^{-1}]_{l_2}$. Then

$$\begin{aligned} G^{-1} - \widehat{D}_1^{-1} &= \begin{pmatrix} 0 & 0 \\ 0 & [D_2^{-1}]_{l_2} - [\widehat{D}_1^{-1}]_{l_2} \end{pmatrix} \\ &\succeq 0. \end{aligned}$$

So $G^{-1} \succeq \widehat{D}_1^{-1} \succeq K_{\mathbf{X}|\mathbf{Y_2}}^{-1}$. This shows that $K_{\mathbf{X}|\mathbf{Y_2}} \succeq G$ as desired. Hence we can select $K_{\mathbf{X}|\mathbf{W},\mathbf{Y_2}} = G$.

Now for any $\mathbf{W}$ that is jointly Gaussian with $\mathbf{X}$ and has the specified $K_{\mathbf{X}|\mathbf{W},\mathbf{Y_2}}$, we will have

$$\begin{aligned} K_{\mathbf{X}|\mathbf{W},\mathbf{Y_1}} &= (K_{\mathbf{X}|\mathbf{W},\mathbf{Y_2}}^{-1} - K)^{-1} \\ &= \left(\begin{pmatrix} (\widehat{D}_1^{-1})_{l_1} & 0 \\ 0 & [D_2^{-1}]_{l_2} \end{pmatrix} - \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right)^{-1} \\ &= \begin{pmatrix} (D_1)_{l_1} & 0 \\ 0 & [\widetilde{D}_2]_{l_2} \end{pmatrix}. \end{aligned}$$

Then select $\mathbf{U}$ and $\mathbf{V}$ jointly Gaussian with $\mathbf{X}$ and $\mathbf{W}$ so that

$$\begin{aligned} K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}} &= \begin{pmatrix} \min\{(\widehat{D}_1)_{l_1}, (D_2)_{l_1}\} & 0 \\ 0 & [D_2]_{l_2} \end{pmatrix}, \\ K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}} &= \begin{pmatrix} (D_1)_{l_1} & 0 \\ 0 & \min\{[D_1]_{l_2}, [\widetilde{D}_2]_{l_2}\} \end{pmatrix}. \end{aligned}$$

Note that $K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}} \preceq D_1$ and $K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}} \preceq D_2$ as required. Evaluating R_1 and R_2 for this choice of $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ gives us $R_1^{Sc}(\mathbf{D})$ and $R_2^{Sc}(\mathbf{D})$. □

As in achievable scheme for *RDmse* in Proposition 10, each decoder hits its own distortion constraint with equality on the region where it is "weaker" while each may undershoot its distortion constraint where it is "stronger" depending on D_1, D_2 and K. Now, we provide the following proposition giving additional constraints on the optimizers in the optimization set $C_u(\mathbf{D})$ when we have trace distortion constraints.

Proposition 12. *$R^{Tr}(\mathbf{D})$ is upper bounded by*

$$R_u^{Tr}(\mathbf{D}) = \min_{C^{Tr}(\mathbf{D})} \max\{R_1^{Tr}(\mathbf{D}), R_2^{Tr}(\mathbf{D})\} \tag{3.40}$$

where $R_1^{Tr}(\mathbf{D})$, $R_2^{Tr}(\mathbf{D})$ and $C^{Tr}(\mathbf{D})$ as in Theorem 7.

Proof. Notice that we can include the conditions

$$\mathbf{U} \perp (\mathbf{X})_{l_1} | (\mathbf{W}, \mathbf{Y_1}), \ \mathbf{V} \perp [\mathbf{X}]_{l_2} | (\mathbf{W}, \mathbf{Y_2}) \tag{3.41}$$

$$K_{\mathbf{X}|\mathbf{W},\mathbf{Y_1}}, K_{\mathbf{X}|\mathbf{W},\mathbf{Y_2}}, K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}, K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}} \text{ diagonal} \tag{3.42}$$

to $C_u(\mathbf{D})$ of $R_{ach}(\mathbf{D})$, which gives the result. □

3.6 Converse Results

3.6.1 Converse for *RDmse* and *RDSI*

It turns out that the Enhancement *ELB* is sufficient for the *RDmse* and *RDSI*, so we will use that bound. We start our analysis by selecting $\mathbf{Y}$ in Enhancement *ELB* with properties stated in Lemma 4 below.

Lemma 4. *Let joint distribution of source and side information pairs $(\mathbf{X}, \mathbf{Y_i})$, $i \in \{1, 2\}$ be given. We can find a random vector, $\mathbf{Y}$, jointly Gaussian with $(\mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2})$ such that*

$$\mathbf{X} \leftrightarrow \mathbf{Y} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2}) \tag{3.43}$$

and

$$K^{-1}_{\mathbf{X}|\mathbf{Y}} = K^{-1}_{\mathbf{X}|\mathbf{Y_1}} + \widehat{K} \tag{3.44}$$

$$= K^{-1}_{\mathbf{X}|\mathbf{Y_2}} + \widetilde{K} \tag{3.45}$$

where $\widehat{K} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\widetilde{K} = \begin{pmatrix} 0 & 0 \\ 0 & -B \end{pmatrix}$.

Proof. Observe that if $(\mathbf{X}, \mathbf{Y}, \mathbf{Y_i})$ can be coupled so that $\mathbf{X} \leftrightarrow \mathbf{Y} \leftrightarrow \mathbf{Y_i}$ holds for $i \in \{1, 2\}$ and $(\mathbf{X}, \mathbf{Y})$ has the same distribution under both couplings then it is possible to couple all four variables such that $\mathbf{X} \leftrightarrow \mathbf{Y} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2})$ holds.

Next note that the matrix $K^{-1}_{\mathbf{X}|\mathbf{Y_1}} - K^{-1}_{\mathbf{X}} + \widehat{K} = K^{-1}_{\mathbf{X}|\mathbf{Y_2}} - K^{-1}_{\mathbf{X}} + \widetilde{K}$ is positive semidefinite. Thus, we can find a matrix M such that $M^T M = K^{-1}_{\mathbf{X}|\mathbf{Y_1}} - K^{-1}_{\mathbf{X}} + \widehat{K} = K^{-1}_{\mathbf{X}|\mathbf{Y_2}} - K^{-1}_{\mathbf{X}} + \widetilde{K}$. Then, let $\mathbf{N}$ be a Gaussian random vector, independent of $\mathbf{X}$, with covariance matrix $K_{\mathbf{N}} = I$ and let $\mathbf{Y} = M\mathbf{X} + \mathbf{N}$. Then, $K^{-1}_{\mathbf{X}|\mathbf{Y}} = K^{-1}_{\mathbf{X}} + M^T M = K^{-1}_{\mathbf{X}|\mathbf{Y_1}} + \widehat{K} = K^{-1}_{\mathbf{X}|\mathbf{Y_2}} + \widetilde{K}$. Since we have $K_{\mathbf{X}|\mathbf{Y}} \preceq K_{\mathbf{X}|\mathbf{Y_i}}$, $i \in \{1, 2\}$, we can couple $(\mathbf{X}, \mathbf{Y}, \mathbf{Y_i})$ so that $\mathbf{X} \leftrightarrow \mathbf{Y} \leftrightarrow \mathbf{Y_i}$. $\square$

Let $\mathbf{Y}$ be selected as in Lemma 4. By Lemma 3 we can add the condition that $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ is jointly Gaussian with the source and side information at decoders to optimization sets $\bar{C}_{l1}$ and $\bar{C}_{l2}$ in Enhanced *ELB*. Then we can write R_{lo1} in (3.15) as

$$R_{lo1} = \frac{1}{2} \log \frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}|} \frac{|K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}}|}{|K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}|}.$$

Likewise, R_{lo2} in (3.15) can be written as

$$R_{lo2} = \frac{1}{2} \log \frac{|K_{\mathbf{X}|\mathbf{Y_2}}|}{|K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}|} \frac{|K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y}}|}{|K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}|}.$$

We can further write,

$$
\begin{aligned}
R_{lo1} &= \frac{1}{2}\log \frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}} + \widehat{K}|} \frac{|K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}|}{|K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}|} \\
&= \frac{1}{2}\log \frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|I + \widehat{K}K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}|} \frac{1}{|K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}|} \\
&\geq \frac{1}{2}\log \frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{\prod_{i=1}^{l_1+l_2}(1 + (\widehat{K})_{ii}(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{ii})} \frac{1}{\prod_{i=1}^{l_1+l_2}(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}})_{ii}},
\end{aligned} \tag{3.46}
$$

Now we focus on *RDmse* where $K_{\mathbf{X}|\mathbf{Y_i}}$, $i \in \{1, 2\}$ are diagonal matrices and D_i, $i \in \{1, 2\}$ are as in (3.18). Since $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ is jointly Gaussian with $(\mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$, we can write $K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y_2}} = K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}} - \widehat{K}$, where $\widehat{K}$ as in Lemma 4. Then we can write $(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{diag} \preceq D_1$ and $((K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}} - \widehat{K})^{-1})_{diag} \preceq D_2$, the constraints at $\bar{C}_{l1}$, as $(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{diag} \preceq D_1$ and $(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y_2}})_{diag} \preceq D_2$.

The following lemma will be useful for matching the distortion constraints in the achievable scheme and the Enhanced *ELB*.

Lemma 5. *Let $A \succeq 0$ be a $m \times m$ diagonal matrix, $M \succ 0$ be a $m \times m$ matrix and M_{diag} denote $(M)_{diag}$. Then $[(M_{diag})^{-1} + A]^{-1} \succeq ([M^{-1} + A]^{-1})_{diag}$.*

Proof. See Appendix 6.3. □

From $(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{diag} \preceq D_1$ and $(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y_2}})_{diag} \preceq D_2$, the constraints at $\bar{C}_{l1}$, and by Lemma 5 we can get

$$
\begin{aligned}
(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}})_{diag} &\preceq (D_1^{-1} + \widehat{K})^{-1} \\
(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}})_{diag} &\preceq (D_2^{-1} + \widetilde{K})^{-1}
\end{aligned}
$$

which implies

$$
(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}})_{diag} \preceq \min((D_1^{-1} + \widehat{K})^{-1}, (D_2^{-1} + \widetilde{K})^{-1}).
$$

Let $\widehat{D}_1$ and $\widetilde{D}_2$ be as in Theorem 5. Note that $((D_1^{-1} + \widehat{K})^{-1})_{l_1} = (\widehat{D}_1)_{l_1}$ and $[(D_1^{-1} + \widehat{K})^{-1}]_{l_2} = [D_1]_{l_2}$ Also, $((D_2^{-1} + \widetilde{K})^{-1})_{l_1} = (D_2)_{l_1}$ and $[(D_2^{-1} + \widetilde{K})^{-1}]_{l_2} = [\widetilde{D}_2]_{l_2}$. Then the right hand side of (3.46) is lower bounded by

$$\frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|I + A(D_1)_{l_1}|}\frac{1}{|\min((\widehat{D}_1)_{l_1},(D_2)_{l_1})|}\frac{1}{|\min([D_1]_{l_2},[\widetilde{D}_2]_{l_2})|}.$$

Since

$$\frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|I + A(D_1)_{l_1}|} = \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_1}}|\cdot|(\widehat{D}_1)_{l_1}|}{|(D_1)_{l_1}|},$$

we have $R_{lo1} \geq R_1^{MSE}(\mathbf{D})$. If we follow a similar procedure for R_{lo2}, we obtain

$$\begin{aligned}
R_{lo2} &= \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_2}}|}{|I + \widetilde{K}K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}|}\frac{1}{|K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}|}\\
&\geq \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_2}}|}{\prod_{i=1}^{l_1+l_2}(1+(\widetilde{K})_{ii}(K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}})_{ii})}\frac{1}{\prod_{i=1}^{l_1+l_2}(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}})_{ii}}\\
&\geq \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_2}}|}{|I - B[D_2]_{l_2}|}\frac{1}{|\min((\widehat{D}_1)_{l_1},(D_2)_{l_1})|}\frac{1}{|\min([D_1]_{l_2},[\widetilde{D}_2]_{l_2})|}.
\end{aligned}$$

Since $\frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_2}}|}{|I-B[D_2]_{l_2}|} = \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_2}}|\cdot|[\widetilde{D}_2]_{l_2}|}{|[D_2]_{l_2}|}$, we have $R_{lo2} \geq R_2^{MSE}(\mathbf{D})$. Hence together with Proposition 10, this proves Theorem 5.

Note that for $RDSI$ we can lower bound the right hand side of (3.46) by

$$\frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|I + A(D_1)_{l_1}|}\frac{1}{|\min((\widehat{D}_1)_{l_1},(D_2)_{l_1})|}\frac{1}{|\min([D_1]_{l_2},[\widetilde{D}_2]_{l_2})|},$$

where D_i, $i \in \{1,2\}$, $\widehat{D}_1$ and $\widetilde{D}_2$ are as in Theorem 6. Since

$$\frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|I + A(D_1)_{l_1}|} = \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_1}}|\cdot|(\widehat{D}_1)_{l_1}|}{|(D_1)_{l_1}|},$$

we have $R_{lo1} \geq R_1^{Sc}(\mathbf{D})$. If we follow a similar procedure for R_{lo2}, we obtain

$$\begin{aligned}
R_{lo2} &= \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_2}}|}{|I + \widetilde{K}K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}|}\frac{1}{|K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}|}\\
&\geq \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_2}}|}{\prod_{i=1}^{l_1+l_2}(1+(\widetilde{K})_{ii}(K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}})_{ii})}\frac{1}{\prod_{i=1}^{l_1+l_2}(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}})_{ii}}\\
&\geq \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_2}}|}{|I - B[D_2]_{l_2}|}\frac{1}{|\min((\widehat{D}_1)_{l_1},(D_2)_{l_1})|}\frac{1}{|\min([D_1]_{l_2},[\widetilde{D}_2]_{l_2})|}.
\end{aligned}$$

Since $\frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_2}}|}{|I-B[D_2]_{l_2}|} = \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_2}}|\cdot|[\widetilde{D}_2]_{l_2}|}{|[D_2]_{l_2}|}$, we have $\bar{R}_{lo2} \geq R_2^{Sc}(\mathbf{D})$. Hence together with Proposition 11, this proves Theorem 6.

3.6.2 Converse for *RDTr*

For *RDTr*, we utilize the *mLB*. Similar to the converse of *RDmse* and *RDSI*, Let $\mathbf{Y}$ in *mLB* be selected as in Lemma 4. Then, by Lemma 15 in Appendix 6.1 we can create a $\widehat{\mathbf{Y}}_\mathbf{i}$, $i \in \{1,2\}$ so that $(\mathbf{X}, \mathbf{Y}, \mathbf{Y_1}, \widehat{\mathbf{Y}}_\mathbf{i})$ jointly Gaussian, $\widehat{\mathbf{Y}}_\mathbf{i} \leftrightarrow \mathbf{X} \leftrightarrow \mathbf{Y_i}$ and $E[\mathbf{X}|\mathbf{Y_i}, \widehat{\mathbf{Y}}_\mathbf{i}] = E[\mathbf{X}|\mathbf{Y_i}, \mathbf{Y}]$ almost surely. Since $\widehat{\mathbf{Y}}_\mathbf{i} \leftrightarrow \mathbf{X} \leftrightarrow \mathbf{Y_i}$, we can write

$$\widehat{\mathbf{Y}}_\mathbf{i} = A_{\widehat{\mathbf{Y}}_\mathbf{i}}\mathbf{X} + \mathbf{N}_{\widehat{\mathbf{Y}}_\mathbf{i}}, i \in \{1,2\},$$

where $\mathbf{N}_{\widehat{\mathbf{Y}}_\mathbf{i}}$ is independent of $\mathbf{X}$ and $\mathbf{Y_i}$.

Then,

$$K^{-1}_{\mathbf{X}|\widehat{\mathbf{Y}}_\mathbf{i},\mathbf{Y_i}} = K^{-1}_{\mathbf{X}|\mathbf{Y_i}} + A^T_{\widehat{\mathbf{Y}}_\mathbf{i}} K^{-1}_{\mathbf{N}_{\widehat{\mathbf{Y}}_\mathbf{i}}} A_{\widehat{\mathbf{Y}}_\mathbf{i}}. \tag{3.47}$$

Also, since $E[\mathbf{X}|\mathbf{Y_i}, \widehat{\mathbf{Y}}_\mathbf{i}] = E[\mathbf{X}|\mathbf{Y_i}, \mathbf{Y}]$ almost surely $K^{-1}_{\mathbf{X}|\mathbf{Y}} = K^{-1}_{\mathbf{X}|\mathbf{Y},\mathbf{Y_i}} = K^{-1}_{\mathbf{X}|\widehat{\mathbf{Y}}_\mathbf{i},\mathbf{Y_i}}$. Then, from (3.47), $K^{-1}_{\mathbf{X}|\mathbf{Y}} - K^{-1}_{\mathbf{X}|\mathbf{Y_1}} = \widehat{K}$ and $K^{-1}_{\mathbf{X}|\mathbf{Y}} - K^{-1}_{\mathbf{X}|\mathbf{Y_2}} = \widetilde{K}$,

$$A^T_{\widehat{\mathbf{Y}}_\mathbf{1}} K^{-1}_{\mathbf{N}_{\widehat{\mathbf{Y}}_\mathbf{1}}} A_{\widehat{\mathbf{Y}}_\mathbf{1}} = \widehat{K}. \tag{3.48}$$

$$A^T_{\widehat{\mathbf{Y}}_\mathbf{2}} K^{-1}_{\mathbf{N}_{\widehat{\mathbf{Y}}_\mathbf{2}}} A_{\widehat{\mathbf{Y}}_\mathbf{2}} = \widetilde{K}. \tag{3.49}$$

Now, we consider any feasible variable satisfying the constraints in the optimization of $R_{lo}(\mathbf{D})$ in Theorem 4. We can rewrite R_{lo1} in (3.3) as

$$\begin{aligned}
R_{lo1} &= I(\mathbf{X};\mathbf{W},\mathbf{U}|\mathbf{Y_1}) + I(\mathbf{X};\mathbf{V}|\mathbf{W},\mathbf{U},\mathbf{Y}) \\
&= h(\mathbf{X}|\mathbf{Y_1}) - h(\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}) + h(\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}) - h(\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}) \\
&= h(\mathbf{X}|\mathbf{Y_1}) - h(\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}) + h(\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1},\mathbf{Y}) - h(\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}),
\end{aligned}$$

since $\mathbf{X} \leftrightarrow \mathbf{Y} \leftrightarrow \mathbf{Y_1}$. Furthermore, since $\mathbf{X} \leftrightarrow E[\mathbf{X}|\mathbf{Y_1}, \mathbf{Y}] \leftrightarrow (\mathbf{Y_1}, \mathbf{Y})$ and $\mathbf{X} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y}) \leftrightarrow E[\mathbf{X}|\mathbf{Y_1}, \mathbf{Y}]$, $h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{Y_1}, \mathbf{Y}) = h(\mathbf{X}|\mathbf{W}, \mathbf{U}, E[\mathbf{X}|\mathbf{Y}, \mathbf{Y_1}])$. Furthermore, we can write $h(\mathbf{X}|\mathbf{W}, \mathbf{U}, E[\mathbf{X}|\mathbf{Y}, \mathbf{Y_1}]) = h(\mathbf{X}|\mathbf{W}, \mathbf{U}, E[\mathbf{X}|\mathbf{Y_1}, \widehat{\mathbf{Y}}_\mathbf{1}])$, since $E[\mathbf{X}|\mathbf{Y_1}, \widehat{\mathbf{Y}}_\mathbf{1}] = E[\mathbf{X}|\mathbf{Y_1}, \mathbf{Y}]$ almost surely. Then we can write

$$\begin{aligned}
R_{lo1} &= h(\mathbf{X}|\mathbf{Y_1}) - h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{Y_1}) + h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{Y_1}, \widehat{\mathbf{Y}}_\mathbf{1}) - h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{V}, \mathbf{Y}) \\
&= h(\mathbf{X}|\mathbf{Y_1}) - I(\mathbf{X}; \widehat{\mathbf{Y}}_\mathbf{1}|\mathbf{W}, \mathbf{U}, \mathbf{Y_1}) - h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{V}, \mathbf{Y}) \\
&= h(\mathbf{X}|\mathbf{Y_1}) + h(\widehat{\mathbf{Y}}_\mathbf{1}|\mathbf{X}, \mathbf{Y_1}) - h(\widehat{\mathbf{Y}}_\mathbf{1}|\mathbf{W}, \mathbf{U}, \mathbf{Y_1}) - h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{V}, \mathbf{Y}) \\
&\geq \frac{1}{2} \log \frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|K_{\widehat{\mathbf{Y}}_\mathbf{1}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}|} \frac{|K_{\widehat{\mathbf{Y}}_\mathbf{1}|\mathbf{X},\mathbf{Y_1}}|}{|K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}|} \qquad (3.50)
\end{aligned}$$

with equality if $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ is Gaussian achieving the given covariance matrices. Now, let us focus on the ratio $\frac{|K_{\widehat{\mathbf{Y}}_\mathbf{1}|\mathbf{X},\mathbf{Y_1}}|}{|K_{\widehat{\mathbf{Y}}_\mathbf{1}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}|}$. Since $\widehat{\mathbf{Y}}_\mathbf{1} \leftrightarrow \mathbf{X} \leftrightarrow \mathbf{Y_1}$ we can write

$$\frac{|K_{\widehat{\mathbf{Y}}_\mathbf{1}|\mathbf{X},\mathbf{Y_1}}|}{|K_{\widehat{\mathbf{Y}}_\mathbf{1}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}|} = \frac{|K_{\mathbf{N}_{\widehat{\mathbf{Y}}_\mathbf{1}}}|}{|K_{\mathbf{N}_{\widehat{\mathbf{Y}}_\mathbf{1}}} + A_{\widehat{\mathbf{Y}}_\mathbf{1}} K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}} A^T_{\widehat{\mathbf{Y}}_\mathbf{1}}|}.$$

Since $K_{N_{\widehat{\mathbf{Y}}_\mathbf{1}}}$ is positive definite we can write it as $S_{\widehat{\mathbf{Y}}_\mathbf{1}} S_{\widehat{\mathbf{Y}}_\mathbf{1}}$ where $S_{\widehat{\mathbf{Y}}_\mathbf{1}}$ is an invertible matrix. Then we can write,

$$\begin{aligned}
\frac{|K_{\widehat{\mathbf{Y}}_\mathbf{1}|\mathbf{X},\mathbf{Y_1}}|}{|K_{\widehat{\mathbf{Y}}_\mathbf{1}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}|} &= \frac{1}{|I + S^{-1}_{\widehat{\mathbf{Y}}_\mathbf{1}} A_{\widehat{\mathbf{Y}}_\mathbf{1}} K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}} A^T_{\widehat{\mathbf{Y}}_\mathbf{1}} S^{-1}_{\widehat{\mathbf{Y}}_\mathbf{1}}|} \\
&= \frac{1}{|I + K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}} A^T_{\widehat{\mathbf{Y}}_\mathbf{1}} S^{-1}_{\widehat{\mathbf{Y}}_\mathbf{1}} S^{-1}_{\widehat{\mathbf{Y}}_\mathbf{1}} A_{\widehat{\mathbf{Y}}_\mathbf{1}}|}, \text{ by Sylvester's determinant identity} \\
&= \frac{1}{|I + K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}} A^T_{\widehat{\mathbf{Y}}_\mathbf{1}} K^{-1}_{\mathbf{N}_{\widehat{\mathbf{Y}}_\mathbf{1}}} A_{\widehat{\mathbf{Y}}_\mathbf{1}}|} \\
&= \frac{1}{|I + K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}|}
\end{aligned}$$

where the last equality is due to (3.48). Then we can write (3.50) as

$$\begin{aligned}
R_{lo1} &\geq \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|I+K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}\begin{pmatrix} A & 0 \\ 0 & 0\end{pmatrix}|}\frac{1}{|K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}|} \\
&= \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|\begin{pmatrix} I+(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{l_1}A & 0 \\ (K_{[\mathbf{X}]_{l_2}(\mathbf{X})_{l_1}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{l_1}A & I\end{pmatrix}|}\frac{1}{|K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}|} \\
&= \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{|I+(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{l_1}A|}\frac{1}{|K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}|} \\
&\geq \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{\prod_{i=1}^{l_1}(1+(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{ii}(A)_{ii})}\frac{1}{\prod_{i=1}^{k}(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}})_{ii}}, \text{ by Hadamard inequality,}
\end{aligned}$$

with equality if $(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{l_1}$ and $K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}$ are diagonal matrices. Since $K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}} \preceq K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}}$ and $K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}} \preceq K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y}}$ imply $(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}})_{ii} \leq \min\{(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}})_{ii}, (K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y}})_{ii}\}$ for all $i \in [k]$, we can further write

$$R_{lo1} \geq \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_1}}|}{\prod_{i=1}^{l_1}(1+(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{ii}(A)_{ii})} + \frac{1}{2}\log\frac{1}{\prod_{i=1}^{k}\min\{(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}})_{ii}, (K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y}})_{ii}\}}. \tag{3.51}$$

By applying the same procedure as above for the R_{lo2} we can get

$$R_{lo2} \geq \frac{1}{2}\log\frac{|K_{\mathbf{X}|\mathbf{Y_2}}|}{\prod_{i=1}^{l_2}(1-([K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}]_{l_2})_{ii}(B)_{ii})} + \frac{1}{2}\log\frac{1}{\prod_{i=1}^{k}\min\{(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}})_{ii}, (K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y}})_{ii}\}}. \tag{3.52}$$

We denote right hand sides of (3.51) and (3.52) as $\widehat{R}_{lo1}$ and $\widehat{R}_{lo2}$ respectively. Next proposition gives a tight lower bound to $R^{Tr}(\mathbf{d})$ by specifying the properties of the optimizers in *mLB*.

Proposition 13. *Rate distortion function of RDTr, $R^{Tr}(\mathbf{D})$, is lower bounded by*

$$\min_{C^{Tr}(\mathbf{D})} \max\{R_1^{Tr}(\mathbf{D}), R_2^{Tr}(\mathbf{D})\} \tag{3.53}$$

where $C^{Tr}(\mathbf{D})$, $R_1^{Tr}(\mathbf{D})$ and $R_2^{Tr}(\mathbf{D})$ are as in Theorem 7.

The proof follows from the next 4 lemmas. At each lemma, we show that without loss of optimality we can add a constrain to the optimization set, $C_l(\mathbf{D})$ of Theorem 4 for the trace constraints. With those additional constraints $C_l(\mathbf{D})$ becomes $C^{Tr}(\mathbf{D})$ and $\widehat{R}_{loi} = R_i^{Tr}(\mathbf{D})$ for $i \in \{1, 2\}$.

Lemma 6. *There exist a feasible $(\mathbf{W_G}, \mathbf{U_G}, \mathbf{V_G})$ for $R_{lo}(\mathbf{D})$ such that $(\mathbf{W_G}, \mathbf{U_G}, \mathbf{V_G})$ are jointly Gaussian with $(\mathbf{X}, \mathbf{Y}, \mathbf{Y_1}, \mathbf{Y_2})$. Furthermore, such $(\mathbf{W_G}, \mathbf{U_G}, \mathbf{V_G})$ do not increase $\widehat{R}_{lo1}$ and $\widehat{R}_{lo2}$.*

Proof. Let $(\mathbf{W_G}, \mathbf{U_G}, \mathbf{V_G})$ be jointly Gaussian with $(\mathbf{X}, \mathbf{Y}, \mathbf{Y_1}, \mathbf{Y_2})$ such that

$$\begin{aligned}
&(\mathbf{W_G}, \mathbf{U_G}, \mathbf{V_G}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y}, \mathbf{Y_1}, \mathbf{Y_2}),\\
&K_{\mathbf{X}|\mathbf{W_G},\mathbf{U_G},\mathbf{Y_1}} = K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}},\\
&K_{\mathbf{X}|\mathbf{W_G},\mathbf{V_G},\mathbf{Y_2}} = K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}.
\end{aligned}$$

By Lemma 14, we have $K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}} \preceq K_{\mathbf{X}|\mathbf{W_G},\mathbf{U_G},\mathbf{Y}}$ and $K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y}} \preceq K_{\mathbf{X}|\mathbf{W_G},\mathbf{V_G},\mathbf{Y}}$. This implies

$$\min\{(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}})_{ii}, (K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y}})_{ii}\} \le \min\{(K_{\mathbf{X}|\mathbf{W_G},\mathbf{U_G},\mathbf{Y}})_{ii}, (K_{\mathbf{X}|\mathbf{W_G},\mathbf{V_G},\mathbf{Y}})_{ii}\} \text{ for all } i \in [k].$$

Hence, we can conclude that $(\mathbf{W_G}, \mathbf{U_G}, \mathbf{V_G})$ is feasible for $R_{lo}(\mathbf{D})$ and replacing the $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ with $(\mathbf{W_G}, \mathbf{U_G}, \mathbf{V_G})$ on (3.51) and (3.52) does not increase $\widehat{R}_{lo1}$ and $\widehat{R}_{lo2}$. □

Then by Lemma 6 we can write the lower bound

$$R_{lo}(\mathbf{D}) \ge \widehat{R}_{lo}(\mathbf{D}) \tag{3.54}$$

where

$$\widehat{R}_{lo}(\mathbf{D}) = \inf_{\widehat{C}_l(\mathbf{D})} \max\{\widehat{R}_{lo1}, \widehat{R}_{lo2}\}$$

and $\widehat{C}_l(\mathbf{D}) = \{(\mathbf{W}, \mathbf{U}, \mathbf{V}) \in C_l(\mathbf{D}) | (\mathbf{W}, \mathbf{U}, \mathbf{V})$ jointly Gaussian with $(\mathbf{X}, \mathbf{Y}, \mathbf{Y_1}, \mathbf{Y_2})\}$.

The following lemmas show that without loss of optimality we can add the conditions $\mathbf{U} \perp (\mathbf{X})_{l_1} | (\mathbf{W}, \mathbf{Y_1})$, $\mathbf{V} \perp [\mathbf{X}]_{l_2} | (\mathbf{W}, \mathbf{Y_2})$, and $K_{\mathbf{X}|\mathbf{W},\mathbf{Y_1}}, K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}, K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}$ diagonal matrices to $\widehat{C}_l(\mathbf{D})$.

Lemma 7. *One can add the constraint that $K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}, K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}$ are diagonal matrices to $\widehat{C}_l(\mathbf{D})$ without increasing the optimal value, $\widehat{R}_{lo}(\mathbf{D})$.*

Proof. Note that for each feasible $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ in $\widehat{C}_l(\mathbf{D})$, we can find a $(\mathbf{W}', \mathbf{U}', \mathbf{V}')$ jointly Gaussian with $(\mathbf{X}, \mathbf{Y}, \mathbf{Y_1}, \mathbf{Y_2})$ and $(\mathbf{W}', \mathbf{U}', \mathbf{V}') \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$ such that

$$K_{\mathbf{X}|\mathbf{W}',\mathbf{U}',\mathbf{Y_1}} = (K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{diag}$$
$$K_{\mathbf{X}|\mathbf{W}',\mathbf{V}',\mathbf{Y_2}} = (K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}})_{diag}$$

since $K_{\mathbf{X}|\mathbf{W}',\mathbf{U}',\mathbf{Y_i}} \preceq D_i I \preceq K_{\mathbf{X}|\mathbf{Y_i}}$ for $i \in \{1, 2\}$. Also notice that $(\mathbf{W}', \mathbf{U}', \mathbf{V}')$ satisfies the corresponding distortion constraints. Lastly we need to check that $(K_{\mathbf{X}|\mathbf{W}',\mathbf{U}',\mathbf{Y}})_{diag} \succeq (K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}})_{diag}$ and $(K_{\mathbf{X}|\mathbf{W}',\mathbf{V}',\mathbf{Y}})_{diag} \succeq (K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y}})_{diag}$. Since

$$\begin{aligned} K_{\mathbf{X}|\mathbf{W}',\mathbf{U}',\mathbf{Y}} &= [K^{-1}_{\mathbf{X}|\mathbf{W}',\mathbf{U}',\mathbf{Y_1}} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}]^{-1} \\ &= [((K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{diag})^{-1} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},]^{-1}. \end{aligned}$$

from Lemma 5 we have $K_{\mathbf{X}|\mathbf{W}',\mathbf{U}',\mathbf{Y}} \succeq (K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}})_{diag}$ and similarly $K_{\mathbf{X}|\mathbf{W}',\mathbf{V}',\mathbf{Y}} \succeq (K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y}})_{diag}$. Hence, without loss of optimality we can add the condition that $K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}, K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}$ are diagonal matrices to $\widehat{C}_l(\mathbf{D})$. □

By Lemma 7, we can write

$$\widehat{R}_{lo}(\mathbf{D}) = \inf_{\widehat{\widehat{C}}_l(\mathbf{D})} \max\{\widehat{R}_{lo1}, \widehat{R}_{lo2}\} \tag{3.55}$$

where $\widehat{\widehat{C}}_l(\mathbf{D}) = \{(\mathbf{W}, \mathbf{U}, \mathbf{V}) \in \widehat{C}_l(\mathbf{D}) | K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}, K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}$ are diagonal $\}$.

Lemma 8. *One may add the constraints*

$$\mathbf{U} \perp (\mathbf{X})_{l_1} | \mathbf{W}, \mathbf{Y_1}, \tag{3.56}$$

$$\mathbf{V} \perp [\mathbf{X}]_{l_2} | \mathbf{W}, \mathbf{Y_2}, \tag{3.57}$$

$$K_{\mathbf{X}|\mathbf{W_G},\mathbf{Y_1}}, K_{\mathbf{X}|\mathbf{W_G},\mathbf{Y_2}} \textit{ are diagonal matrices.} \tag{3.58}$$

to the optimization set $\widehat{\widehat{C}}_l(\mathbf{D})$ *without increasing the optimal value,* $\widehat{R}_{lo}(\mathbf{D})$*.*

Proof. Let $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ be feasible for $\widehat{R}_{lo}(\mathbf{D})$, i.e $(\mathbf{W}, \mathbf{U}, \mathbf{V}) \in \widehat{\widehat{C}}_l(\mathbf{D})$. From these, we shall construct $(\widetilde{\mathbf{W}}, \widetilde{\mathbf{U}}, \widetilde{\mathbf{V}})$ that are feasible for $\widehat{R}_{lo}(\mathbf{D})$ and also satisfy the conditions in (3.56), (3.57), (3.58) and for which the objective is only lower.

First suppose that $d_2 \leq d_1$. Then note that

$$\begin{aligned}
\begin{pmatrix} (K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{l_1}^{-1} & 0 \\ 0 & [K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}]_{l_2}^{-1} - B \end{pmatrix} &\geq \begin{pmatrix} d_1^{-1} I & 0 \\ 0 & d_2^{-1} I - B \end{pmatrix} \\
&\geq d_1^{-1} I \\
&\geq K_{\mathbf{X}|\mathbf{Y_1}}^{-1},
\end{aligned}$$

Then we may choose $\widetilde{\mathbf{W}}$ such that

$$K_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_1}}^{-1} = \begin{pmatrix} (K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{l_1}^{-1} & 0 \\ 0 & [K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}]_{l_2}^{-1} - B \end{pmatrix} \tag{3.59}$$

in which case we have

$$K_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_2}}^{-1} = K_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_1}}^{-1} + K = \begin{pmatrix} (K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{l_1}^{-1} + A & 0 \\ 0 & [K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}]_{l_2}^{-1} \end{pmatrix}. \tag{3.60}$$

Likewise, if $d_1 < d_2$, we have

$$\begin{aligned}
\begin{pmatrix} (K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{l_1}^{-1} + A & 0 \\ 0 & [K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}]_{l_2}^{-1} \end{pmatrix} &\geq \begin{pmatrix} d_1^{-1}I + A & 0 \\ 0 & d_2^{-1}I \end{pmatrix} \\
&\geq d_2^{-1}I \\
&\geq K_{\mathbf{X}|\mathbf{Y_2}}^{-1}.
\end{aligned}$$

Hence we may choose $\widetilde{\mathbf{W}}$ such that

$$K_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_2}}^{-1} = \begin{pmatrix} (K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{l_1}^{-1} + A & 0 \\ 0 & [K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}]_{l_2}^{-1} \end{pmatrix}$$

in which case

$$K_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_1}}^{-1} = K_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_2}}^{-1} - K = \begin{pmatrix} (K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{l_1}^{-1} & 0 \\ 0 & [K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}]_{l_2}^{-1} - B \end{pmatrix}.$$

Thus either way, we may choose $\widetilde{\mathbf{W}}$ such that (3.59) and (3.60) hold, and so $K_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_1}}$ and $K_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_2}}$ are both diagonal.

Next we choose $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{V}}$ such that $(\widetilde{\mathbf{W}}, \widetilde{\mathbf{U}}, \widetilde{\mathbf{V}}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y}, \mathbf{Y_1}, \mathbf{Y_2})$ and

$$\begin{aligned}
K_{\mathbf{X}|\widetilde{\mathbf{W}},\widetilde{\mathbf{U}},\mathbf{Y_1}} &= \begin{pmatrix} (K_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_1}})_{l_1} & 0 \\ 0 & \min\{[K_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_1}}]_{l_2}, [K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}]_{l_2}\} \end{pmatrix} \\
&= \begin{pmatrix} (K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{l_1} & 0 \\ 0 & \min\{[K_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_1}}]_{l_2}, [K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}]_{l_2}\} \end{pmatrix} \qquad (3.61)
\end{aligned}$$

and

$$\begin{aligned}
K_{\mathbf{X}|\widetilde{\mathbf{W}},\widetilde{\mathbf{V}},\mathbf{Y_2}} &= \begin{pmatrix} \min\{(K_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_2}})_{l_1}, (K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}})_{l_1}\} & 0 \\ 0 & [K_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_2}}]_{l_2} \end{pmatrix} \\
&= \begin{pmatrix} \min\{(K_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_2}})_{l_1}, (K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}})_{l_1}\} & 0 \\ 0 & [K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}]_{l_2} \end{pmatrix}. \qquad (3.62)
\end{aligned}$$

Evidently we have $\widetilde{\mathbf{U}} \perp (\mathbf{X})_{l_1}|\widetilde{\mathbf{W}}, \mathbf{Y_1}$ and $\widetilde{\mathbf{V}} \perp [\mathbf{X}]_{l_2}|\widetilde{\mathbf{W}}, \mathbf{Y_2}$, and $(\widetilde{\mathbf{W}}, \widetilde{\mathbf{U}}, \widetilde{\mathbf{V}})$ satisfy the distortion constraints.

Finally, from (3.59) we have

$$\begin{aligned} K^{-1}_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y}} &= K^{-1}_{\mathbf{X}|\widetilde{\mathbf{W}},\mathbf{Y_1}} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}})_{l_1} + A & 0 \\ 0 & [K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y_2}}]_{l_2} - B \end{pmatrix} \\ &= \begin{pmatrix} (K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}})_{l_1} & 0 \\ 0 & [K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y}}]_{l_2} \end{pmatrix}. \end{aligned} \tag{3.63}$$

Similarly,

$$K^{-1}_{\mathbf{X}|\widetilde{\mathbf{W}},\widetilde{\mathbf{U}},\mathbf{Y}} = K^{-1}_{\mathbf{X}|\widetilde{\mathbf{W}},\widetilde{\mathbf{U}},\mathbf{Y_1}} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

Substituting (3.61) into this equation gives,

$$K_{\mathbf{X}|\widetilde{\mathbf{W}},\widetilde{\mathbf{U}},\mathbf{Y}} = \begin{pmatrix} (K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}})_{l_1} & 0 \\ 0 & \min\{[K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}}]_{l_2}, [K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y}}]_{l_2}\} \end{pmatrix}. \tag{3.64}$$

Likewise,

$$K_{\mathbf{X}|\widetilde{\mathbf{W}},\widetilde{\mathbf{V}},\mathbf{Y}} = \begin{pmatrix} \min\{(K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}})_{l_1}, (K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y}})_{l_1}\} & 0 \\ 0 & [K_{\mathbf{X}|\mathbf{W},\mathbf{V},\mathbf{Y}}]_{l_2} \end{pmatrix}. \tag{3.65}$$

From (3.61), (3.62), (3.64) and (3.65), we see that the objective for $(\widetilde{\mathbf{W}}, \widetilde{\mathbf{U}}, \widetilde{\mathbf{V}})$ is equal to the objective for $(\mathbf{W}, \mathbf{U}, \mathbf{V})$. $\square$

By Lemma 8, we can conclude that $\widehat{R}_{lo}(\mathbf{D})$ is equal to $R_u^{Tr}(\mathbf{D})$.

3.7 Concluding Remarks

Recall that we used the Enhanced ELB to prove the converse for the *RDmse* the *RDSI* problems, while for the *RDTr* problem we used the *mLB*. It appears that the other lower bounds are in fact insufficient for the *RDTr* problem.

Conjecture 1. *There exists an instance of the RDTr such that Minimax lower bound is strictly greater than the Maximin lower bound (and hence the Enhanced ELB and ELB).*

To support this conjecture, one can apply the same arguments in the proof of Proposition 13 to the each minimization in the *MLB* separately. This way we obtain a lower bound, which is the same as in (3.53) except that the minimization and maximization are swapped. Consider the case where the vectors $\mathbf{X}$, $\mathbf{Y_1}$, $\mathbf{Y_2}$ are bivariate Gaussian random vectors such that

$$K_{\mathbf{X}|\mathbf{Y_1}} = \begin{pmatrix} \frac{4}{9} & 0 \\ 0 & \frac{4}{9} \end{pmatrix}, \quad K_{\mathbf{X}|\mathbf{Y_2}} = \begin{pmatrix} \frac{4}{17} & 0 \\ 0 & \frac{4}{5} \end{pmatrix}$$

and the distortion constraints are $d_1 = d_2 = 0.15$. When we use *CVX*, a package for solving convex programs [44, 45], and the *sqp* function of Octave [46] to solve for the minimum rate using Theorem 7 we get a solution of 1.7808784 while we get 1.7802127 when we swap the min and max in (3.53) from both solvers. Thus it appears that there are instances for which the added strength provided by the *mLB* is necessary.

CHAPTER 4

LP LOWER AND UPPER BOUNDS FOR RATE-DISTORTION WITH VARIABLE SIDE INFORMATION

4.1 Introduction

Characterizing the rate distortion function can be divided into two main parts, namely finding an upper bound and a lower bound. Random binning idea is commonly used as the main ingredient of finding an upper bound (achievable scheme) and in most of the existing achievable schemes, the encoder encodes the messages in a certain order and each decoder decodes its received messages with the help of side information in the same order that they are encoded [18],[5],[2]. Simultaneous decoding used in some of the channel coding problems [4] and it decodes the received messages altogether, without imposing any order on decoding. We adopt this simultaneous decoding idea to rate distortion problem. Our first main contribution is to provide such an achievable scheme.

An upper bound to the general rate distortion problem was provided by Heegard *et al.* in [18] which is proven wrong by Timo *et al.* by providing a counter example for 3 decoder case [2]. In the same paper, Timo *et al.* introduced another achievable scheme for the general case and it was believed to be the state of the art scheme for the general rate distortion problem. However, as we discuss in Section 4.5, this upper bound is not correct for the general case either. We also compare our scheme with the one introduced by Timo *et al.* [2] (for the cases that it gives a valid upper bound) and discuss the advantage of our scheme over this scheme.

One natural way of obtaining a lower bound; on the other hand, is to consider a relaxed instance of the problem in which the side information at some of the decoders is improved in such a way that the problem becomes stochastically degraded. Indeed, most existing lower bounds adopt this approach in some form [20, 24]. For the special case of index coding, Blasiak *et al.* [11] provide a lower bound that takes the form of a linear program (LP), the constraints for which are derived from properties of the entropy functional, such as submodularity. This raises the question of whether a similar-style bound can be obtained for more general instances of the problem. The second main contribution of this work is such a bound. It is obtained by introducing a notion of "*generalized side information*" and capturing the properties of mutual information in the form of a linear program. We show that this lower bound subsumes several existing lower bounds.

To demonstrate the efficacy of the upper and lower bounds that we introduced, we consider a rate distortion problem obtained by extending the odd-cycle index coding problem to Gaussian sources with mean squared error (MSE) distortion constraints. We find an explicit expression for its rate distortion function.

The outline of this chapter is as follows. Section 4.2 formulates the general rate distortion problem. Section 4.3 presents the LP-type upper bound based on simultaneous decoding while Section 4.3.1 provides the extension of this upper bound to Gaussian sources. In Section 4.4, we provide the LP type lower bound and in Section 4.5 we show that the LP-type upper and lower bounds subsume several existing lower bounds. We conclude the chapter by finding the explicit characterization of odd-cycle Gaussian index coding problem utilizing our upper and lower bounds in Section 4.6.

4.2 Problem Description

Let X denote the source at the encoder and $\mathcal{X}$ denote the source alphabet. Also, $Y_l \in \mathcal{Y}_l$, $l \in [m]$ denotes the side information at decoder l and Y_l is jointly distributed with the source, X. Lastly, $\widehat{X}_l \in \widehat{\mathcal{X}}_l$ denotes the reconstruction of the X at decoder l and D_l denotes the corresponding distortion constraint. Each decoder wants to reconstruct the source, X, subject to its distortion constraint and we assume initially that the source alphabet, $\mathcal{X}$, the side information' alphabets, $\mathcal{Y}_l$, $l \in [m]$, and the reconstruction alphabets $\widehat{\mathcal{X}}_l$, $l \in [m]$, are finite. For the sake of completeness, we state the following definitions to formulate the problem as we did in Chapter 2.

Definition 12. *An $(n, M, \mathbf{D})$ code where n denotes the block length and M denotes the message size and $\mathbf{D} = (D_1, \ldots, D_m)$ is composed of*

- *an encoding function*

$$f : \mathcal{X}^n \to \{1, ..., M\}$$

- *and decoding functions*

$$\begin{aligned} g_1 &: \{1, ..., M\} \times \mathcal{Y}_1^n \to \widehat{\mathcal{X}}_1^n \\ &\vdots \\ g_m &: \{1, ..., M\} \times \mathcal{Y}_m^n \to \widehat{\mathcal{X}}_m^n \end{aligned}$$

satisfying the distortion constraints

$$E\left[\frac{1}{n}\sum_{k=1}^{n} d_l(X_k, \widehat{X}_{lk})\right] \le D_l, \textit{ for } l \in [m]$$

where

$$\widehat{X}_l^n = g_l(f(X^n), Y_l^n), \textit{ for } l \in [m]$$

and $d_l(\cdot,\cdot) \in [0,\infty)$ *is the distortion measure for decoder* l*.*

Definition 13. *A rate* R *is* **D**-achievable*, if for every* $\epsilon > 0$ *there exists an* $(n, M, \mathbf{D} + \epsilon\mathbf{1})$ *(where* $\mathbf{1}$ *is the all-ones vector)* code *such that for sufficiently large* n*, we have* $n^{-1}\log M \le R + \epsilon$*.*

We define the *rate-distortion function* as

$$R(\mathbf{D}) = \inf\{R : R \text{ is } \mathbf{D}\text{-achievable}\}.$$

4.3 Simultaneous Decoding Based Upper Bound to $R(\mathbf{D})$

Here, we present our first main result of this chapter, an upper bound to rate distortion function $R(\mathbf{D})$. The following notation, which is similar to that in [2], will be useful to represent the results.

Notation 8. *Let* (X, Y, Z) *be a random vector. Then* $X \perp Y$ *denotes that* X *and* Y *are independent,* $X \perp Y|Z$ *denotes that* X *and* Y *are independent given* Z*, and* $X \leftrightarrow Y \leftrightarrow Z$ *denotes that* X*,* Y *and* Z *forms a Markov chain.*

Notation 9. $v = \mathscr{S}_1, \ldots, \mathscr{S}_{2^m-1}$ *denotes a list of all possible nonempty subsets of* $[m]$*, where each* $\mathscr{S}_i$ *denotes a different subset.* $\mathscr{V}$ *denotes the set of all possible such* v*.*

Notation 10. *Let $v \in \mathscr{V}$ be fixed. $\mathcal{U}_{\mathscr{S}_1}, \ldots, \mathcal{U}_{\mathscr{S}_{2^m-1}}$ denote finite-alphabet random variables $U_{\mathscr{S}_1}, \ldots, U_{\mathscr{S}_{2^m-1}}$ respectively. $\mathscr{P}_v$ denotes the set of all distributions on $\mathcal{U}_v^* \times \mathcal{X} \times \mathcal{Y}^*$ where $\mathcal{U}_v^* = \mathcal{U}_{\mathscr{S}_1} \times \cdots \times \mathcal{U}_{\mathscr{S}_{2^m-1}}$ and $\mathcal{Y}^* = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m$.*

Notation 11. *Let $\mathscr{U} = \{U_{\mathscr{S}_1}, U_{\mathscr{S}_2}, \ldots, U_{\mathscr{S}_{2^m-1}}\}$, $\mathscr{D}_l = \{\mathscr{S}_i | \, l \in \mathscr{S}_i\}$, and $\mathscr{D}'_l$ be a nonempty subset of $\mathscr{D}_l$. Then we define*

$$
\begin{aligned}
U_{\mathscr{D}'_l} &= \{U_{\mathscr{S}_i} \in \mathscr{U} \,|\, \mathscr{S}_i \in \mathscr{D}'_l\}, \\
U^-_{\mathscr{S}_j} &= \left\{U_{\mathscr{S}_i} \in \mathscr{U} \mid i < j\right\} \\
U^-_{\mathscr{S}_j, \mathscr{D}'_l} &= \left\{U_{\mathscr{S}_i} \in U^-_{\mathscr{S}_j} | \, \mathscr{S}_i \in \mathscr{D}'_l\right\}.
\end{aligned}
$$

Theorem 9. *The rate distortion function, $R(\mathbf{D})$, is upper bounded by*

$$
R_{ach}(\mathbf{D}) = \min_{v \in \mathscr{V}} \inf_{C_{ach,v}(\mathbf{D})} \inf_{C^{LP}_{ach}} \sum_{j=1}^{2^m-1} R_{\mathscr{S}_j} \tag{4.1}
$$

where

$C_{ach,v}(\mathbf{D}) : p \in \mathscr{P}_v,$ *such that*

1)$p(x, y_1, \ldots, y_m)$ *equals to joint distribution of* $(X, Y_1, \ldots, Y_m)$

2)$\mathscr{U} \leftrightarrow X \leftrightarrow (Y_1, \ldots, Y_m)$

3) *There exist functions* $g_l(U_{\mathscr{D}_l}, Y_l)$ *such that*

$E\left[d_l(X, g_l(U_{\mathscr{D}_l}, Y_l))\right] \leq D_l$ *for all* $l \in [m]$,

and

$C^{LP}_{ach} : R_{\mathscr{S}_j}, R'_{\mathscr{S}_j},$ *where* $\mathscr{S}_j \in v,$ *such that*

1)$R_{\mathscr{S}_j} \geq 0, R'_{\mathscr{S}_j} \geq 0$ *for all* $j \in [2^m - 1]$

2)$R_{\mathscr{S}_j} \geq I(X, U^-_{\mathscr{S}_j}; U_{\mathscr{S}_j}) - R'_{\mathscr{S}_j}$ *for all* $j \in [2^m - 1]$

3) *For each decoder* $l, l \in [m]$,

$$
\sum_{\mathscr{S}_j \in \mathscr{D}'_l} R'_{\mathscr{S}_j} \leq \Big(\sum_{\mathscr{S}_j \in \mathscr{D}'_l} H(U_{\mathscr{S}_j}) \Big) - H(U_{\mathscr{D}'_l} | U_{\mathscr{D}_l \setminus \mathscr{D}'_l}, Y_l), \text{ for all } \mathscr{D}'_l \subseteq \mathscr{D}_l.
$$

Proof of Theorem 9. See Appendix 7.1. □

Remark 14. *Using the chain rule, we can rewrite the condition 3) of C_{ach}^{LP} in Theorem 9 as*

$$\text{for each decoder } l, l \in [m],$$
$$\sum_{\mathscr{S}_j \in \mathscr{D}'_l} R'_{\mathscr{S}_j} \le \sum_{\mathscr{S}_j \in \mathscr{D}'_l} I\left(U_{\mathscr{S}_j}; U^{-}_{\mathscr{S}_j, \mathscr{D}'_l}, U_{\mathscr{D}_l \setminus \mathscr{D}'_l}, Y_l\right), \text{ for all } \mathscr{D}'_l \subseteq \mathscr{D}_l. \tag{4.2}$$

This representation will become useful when we consider rate distortion problems with continuous sources. Hence, from now on we consider the condition 3) of C_{ach}^{LP} in the form of (4.2).

Remark 15. *Result of Theorem 9 can be generalized to the case where the source and side information are random vectors and distortion constraints are component-wise distortion constraints.*

Remark 16. *Since $R_{ach}(\mathbf{D})$ is an upper bound to the rate distortion function, $R(\mathbf{D})$, we can obtain a computable upper bound to $R(\mathbf{D})$ by imposing cardinality bounds on the alphabets of auxiliary random variables $U_{\mathscr{S}_j}$ in Theorem 9.*

Overall scheme can be described as follows. Each $U_{\mathscr{S}_j}$ in Theorem 9 can be considered as a message for decoder i, $i \in \mathscr{S}_j$. The encoder encodes each message $U_{\mathscr{S}_j}$ with respect to the order $v \in \mathscr{V}$, using random binning argument. Here, $R_{\mathscr{S}_j}$ and $R'_{\mathscr{S}_j}$ can be interpreted as the number of bins in the codebook of message $U_{\mathscr{S}_j}$ and the number of codewords per bin respectively. Then each decoder i decodes its messages using simultaneous decoding and reconstructs the source using these messages and side information Y_i subject to its own distortion constraint.

4.3.1 Rate-Distortion Function with Gaussian Source and Side Information

Here, we extend the achievable scheme in Theorem 9 to the rate distortion problem with vector Gaussian sources. More specifically, we are interested in the following rate distortion problem. Source and side information at decoders, $(\mathbf{X}, \mathbf{Y_1}, \ldots, \mathbf{Y_m})$, are zero mean jointly Gaussian vectors. Source $\mathbf{X} = (X_1, \ldots, X_k)$ has length k and length of $\mathbf{Y_i}$ is k_i, $i \in [m]$.

Notation 12. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be* $k \times 1$ *vectors.* $\mathbf{v}$ *is lower than or equal to* $\mathbf{w}$, $\mathbf{v} \leq \mathbf{w}$, *denotes that* i^{th} *component of* $\mathbf{v}$, *denoted by* v_i, *is lower than or equal to* w_i *for all* $i \in [k]$.

Notation 13. *Let* M *be* $m \times m$ *matrix.* $(M)_d$ *denotes the vector where* i^{th} *component of* $(M)_d$ *is equal to* i^{th} *diagonal element of* M, $i \in [m]$.

Notation 14. $K_{\mathbf{X}}$ *denotes the covariance matrix of* $\mathbf{X}$. $K_{\mathbf{X}|\mathbf{Y}}$ *is conditional covariance matrix of* $\mathbf{X}$ *conditioned on* $\mathbf{Y}$.

Let $\mathbf{D_i} > 0$ for all $i \in [m]$. Distortion constraints are

$$\left(\frac{1}{n} \sum_{k=1}^{n} E\left[(\mathbf{X}_k - \widehat{\mathbf{X}}_{\mathbf{i}k})(\mathbf{X}_k - \widehat{\mathbf{X}}_{\mathbf{i}k})^T \right] \right)_d \leq \mathbf{D_i}, \text{ for all } i \in [m], \tag{4.3}$$

i.e., component-wise mean square error (MSE) distortion constraints. Since we have MSE distortion constraints, without loss of generality we can take the reconstruction at each decoder as conditional expectation of source given the output of the encoder and the corresponding side information. From now on, we denote the rate distortion function of this problem as $R^{MSE}(\mathbf{D})$. Now that we have the necessary definitions and notations, we are ready to state our achievability result.

Theorem 10. *Let joint distribution of* $(\mathbf{X}, \mathbf{Y_i})$*,* $i \in [m]$ *be given. Then rate distortion function,* $R^{MSE}(\mathbf{D})$ *is upper bounded by*

$$R^G_{ach}(\mathbf{D}) = \min_{v \in \mathscr{V}} R^G_{ach,v}(\mathbf{D}) \tag{4.4}$$

where

$$R^G_{ach,v}(\mathbf{D}) = \inf_{C^G_{ach,v}(\mathbf{D})} \inf_{C^{LP}_{ach}} \sum_{j=1}^{2^m-1} R_{\mathscr{S}_j}$$

$C^G_{ach,v}(\mathbf{D}) : p \in \mathscr{P}_v$*, such that*

1)$p(\mathbf{x}, \mathbf{y_1}, \ldots, \mathbf{y_m})$ *equals to joint distribution of* $(\mathbf{X}, \mathbf{Y_1}, \ldots, \mathbf{Y_m})$

2)T *is a discrete random variable over* $[\tau]$ *for some positive integer* τ

such that $T \perp (\mathbf{X}, \mathbf{Y_1}, \ldots, \mathbf{Y_m})$*.*

3)$\mathscr{U} \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1}, \ldots, \mathbf{Y_m})$

4)$K_{\mathbf{X}|\mathbf{U}_{\mathscr{D}_l}, \mathbf{Y_l}} \leq \mathbf{D_l}$ *for all* $l \in [m]$

5)$\mathbf{U}_{\mathscr{S}_j} = (\mathbf{U}_{\mathscr{S}_j,t}, T)$ *such that* $\mathbf{U}_{\mathscr{S}_j} = \mathbf{U}_{\mathscr{S}_j,t}$ *if* $T = t$ *and all* $\mathbf{U}_{\mathscr{S}_j,t}$

jointly Gaussian with $(\mathbf{X}, \mathbf{Y_1}, \ldots, \mathbf{Y_m})$*,*

and $I(\mathbf{U}_{\mathscr{S}_j}; \mathbf{U}_{\mathscr{S}_i}, \mathbf{X}) < \infty$ *for all* $\mathscr{S}_j \in [2^m - 1], \mathscr{S}_i \in [2^m - 1]$ *and* $i \neq j$*,*

and C^{LP}_{ach} *is the set of conditions obtained by replacing each* X*,* Y_i*, and* $U_{\mathscr{S}_j}$ *in the conditions of* C^{LP}_{ach} *in Theorem 9 by* $\mathbf{X}$*,* $\mathbf{Y_i}$*, and* $\mathbf{U}_{\mathscr{S}_j}$ *respectively.*

Remark 17. *Since all feasible messages* $\mathbf{U}_{\mathscr{S}_j}$ *in (4.4) are Gaussian mixtures and source and side information are jointly Gaussian, minimum mean square error (MMSE) estimator becomes linear MMSE estimator. In other words, we can write* $\hat{\mathbf{X}}_\mathbf{l} = A_{l,t}\mathbf{U}_{\mathscr{D}_l} + B_{l,t}\mathbf{Y_l}$ *if* $T = t$*, where value of* $A_{l,t}$ *and* $B_{l,t}$ *are determined by the joint distribution* $p \in C^G_{ach,v}(\mathbf{D})$*.*

Proof of Theorem 10. The argument is based on a quantization of source and messages similar to the procedure in [4, Section 3]. First we quantize the source, all messages and side information. Then we apply the achievable scheme in the proof of Theorem 9 to these quantized variables and show that the rate in (4.4) is **D**-achievable for our problem.

Let $v \in \mathscr{V}$ be fixed and $\epsilon > 0$ be given. Also let $(\mathbf{X}, \mathscr{U}, \mathbf{Y_1}, \ldots, \mathbf{Y_m})$ be such that joint distribution of it, denoted by p, is in $C^G_{ach,v}(\mathbf{D})$. Note that we can represent each message $\mathbf{U}_{\mathscr{S}_j} = A_{\mathscr{S}_j,t}\mathbf{X} + \mathbf{N}_{\mathscr{S}_j,t}$, $\mathscr{S}_j \in v$, if $T = t$ where $\mathbf{N}_{\mathscr{S}_j,t} \perp (\mathbf{Y_1}, \ldots, \mathbf{Y_m})$ and represent side information $\mathbf{Y_l} = B_l\mathbf{X} + \mathbf{N}_l$, $l \in [m]$ where $\mathbf{N}_l \perp (\mathscr{U}, \mathbf{X})$. Now we quantize $\mathbf{X}$ and all $\mathbf{Y_l}$, $l \in [m]$, and we use $\overline{\mathbf{X}}$ notation to denote the quantized version of $\mathbf{X}$. We do the quantization such that

$$E\left[(X_i - \overline{X}_i)^2\right] \le \delta(\epsilon) \min_{l\in[m]} D_{li} \text{ for all } i \in [k] \tag{4.5}$$

$$E\left[(\overline{X}_i - \overline{\overline{X}}_{li})^2\right] \le D_{li} + \delta(\epsilon)D_{li} \text{ for all } l \in [m] \text{ and } i \in [k], \tag{4.6}$$

$$|I(\mathbf{X}, \mathbf{U}^-_{\mathscr{S}_j}; \mathbf{U}_{\mathscr{S}_j}) - I(\overline{\mathbf{X}}, \overline{\mathbf{U}^-_{\mathscr{S}_j}}; \overline{\mathbf{U}}_{\mathscr{S}_j})| \le \delta(\epsilon), \text{ for all } \mathscr{S}_j \in [2^m - 1] \tag{4.7}$$

$$|\sum_{\mathscr{S}_j\in\mathscr{D}'_l} I\left(\mathbf{U}_{\mathscr{S}_j}; \mathbf{U}^-_{\mathscr{S}_j,\mathscr{D}'_l}, \mathbf{U}_{\mathscr{D}_l\setminus\mathscr{D}'_l}, \mathbf{Y_l}\right) - \sum_{\mathscr{S}_j\in\mathscr{D}'_l} I\left(\overline{\mathbf{U}}_{\mathscr{S}_j}; \overline{\mathbf{U}^-_{\mathscr{S}_j,\mathscr{D}'_l}}, \overline{\mathbf{U}}_{\mathscr{D}_l\setminus\mathscr{D}'_l}, \overline{\mathbf{Y}}_\mathbf{l}\right)| \le \delta(\epsilon), \tag{4.8}$$

for all $l \in [m]$ and $\mathscr{D}'_l \subseteq \mathscr{D}_l$, where $\delta(\epsilon) > 0$ be specified later, and

$$\overline{\mathscr{U}} \leftrightarrow \overline{\mathbf{X}} \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{Y_1}, \ldots, \mathbf{Y_m}) \leftrightarrow (\overline{\mathbf{Y}}_\mathbf{1}, \ldots, \overline{\mathbf{Y}}_\mathbf{m}).$$

Let $\overline{p}$ denote the joint distribution of $(\overline{\mathscr{U}}, \overline{\mathbf{X}}, \overline{\mathbf{Y}}_\mathbf{1}, \ldots, \overline{\mathbf{Y}}_\mathbf{m})$. Now, we form a new problem where the source is $\overline{\mathbf{X}}$, side information at decoder i is $\overline{\mathbf{Y}}_\mathbf{i}$, $i \in [m]$, and distortion constraints are as in (4.6). Note that for this problem, $\overline{p}$ is in $C_{ach,v}((1 + \delta(\epsilon))\mathbf{D})$ in (4.1). Then we can apply the achievable scheme in the proof Theorem 9 to the new problem.

Let $R_{LP}((1 + \delta(\epsilon))\mathbf{D}, \overline{p})$ denote the result of the linear program $\inf_{C^{LP}_{ach}} \sum_{j=1}^{2^m-1} R_{\mathscr{S}_j}$ in Theorem 9 when joint distribution is $\overline{p}$. Then from Theorem 1, rate $R_{LP}((1 +$

$\delta(\epsilon))\mathbf{D}, \overline{p})$ is $(1+\delta(\epsilon))\mathbf{D}$-achievable for the new problem. In other words, we can find a $(n, M, (1+\delta(\epsilon))\mathbf{D} + \epsilon'\mathbf{1})$, $\epsilon'(\epsilon) > 0$ (specified later), code with rate

$$R_{LP}\left((1+\delta(\epsilon))\mathbf{D}, \overline{p}\right) + \epsilon'(\epsilon) \tag{4.9}$$

and

$$\left(\frac{1}{n}\sum_{j=1}^{n} E\left[(\overline{\mathbf{X}}_j - \overline{\widehat{\mathbf{X}}}_{\mathbf{i}j})(\overline{\mathbf{X}}_j - \overline{\widehat{\mathbf{X}}}_{\mathbf{i}j})^T\right]\right)_d \leq (1+\delta(\epsilon))\mathbf{D}_{\mathbf{i}} + \epsilon'\mathbf{1} \text{ for all } i \in [m], \tag{4.10}$$

when blocklength, n is sufficiently large.

For our original problem, first we quantize the source, side information and all the messages distributed by p as described above and then we apply the $(n, M, (1+\delta(\epsilon))\mathbf{D} + \epsilon'\mathbf{1})$ code with rate (4.9) to these quantized variables, joint distribution of which is $\bar{p}$. Let $R^G_{LP}(\mathbf{D}, p)$ denote the result of the linear program $\inf_{C^{LP}_{ach}} \sum_{j=1}^{2^m-1} R_{\mathscr{S}_j}$ in Theorem 10 when joint distribution is p. Note that the linear programs defining both $R_{LP}\left((1+\delta(\epsilon))\mathbf{D}, \overline{p}\right)$ and $R^G_{LP}(\mathbf{D}, p)$ are finite. Thus by (4.7), (4.8) and standard results on the continuity of linear programs [47], we have that

$$|R^G_{LP}(\mathbf{D}, p) - R_{LP}\left((1+\delta(\epsilon))\mathbf{D}, \overline{p}\right)| \leq \gamma(\epsilon),$$

where $\gamma(\epsilon) \to 0$ as $\delta(\epsilon) \to 0$. Lastly utilizing the Cauchy and Jensen inequalities and using (4.5) and (4.10) as in [4, Section 3], we can obtain[1]

$$\begin{aligned}
&\left(\frac{1}{n}\sum_{j=1}^{n} E\left[(\mathbf{X}_j - \widehat{\mathbf{X}}_{\mathbf{i}j})(\mathbf{X}_j - \widehat{\mathbf{X}}_{\mathbf{i}j})^T\right]\right)_d \\
&\leq \delta(\epsilon)\mathbf{D}_{\mathbf{i}} + (1+\delta(\epsilon))\mathbf{D}_{\mathbf{i}} + \epsilon'\mathbf{1} + 2\sqrt{(\delta(\epsilon))\mathbf{D}_{\mathbf{i}}((1+\delta(\epsilon))\mathbf{D}_{\mathbf{i}} + \epsilon'\mathbf{1})} \\
&= \mathbf{D}_{\mathbf{i}} + 2\delta(\epsilon)\mathbf{D}_{\mathbf{i}} + \epsilon'\mathbf{1} + 2\sqrt{(\delta(\epsilon))\mathbf{D}_{\mathbf{i}}((1+\delta(\epsilon))\mathbf{D}_{\mathbf{i}} + \epsilon'\mathbf{1})} \text{ for all } i \in [m],
\end{aligned} \tag{4.11}$$

for sufficiently large n.

[1]When $\mathbf{v}$, $\mathbf{w}$ are $k \times 1$ vectors, $\mathbf{u} = \mathbf{v}\mathbf{w}$ is also $k \times 1$ vector such that $u_i = v_i w_i$, $i \in [k]$.

Thus for all sufficiently large n, there exists a code whose rate does not exceed

$$R_{LP}^{G}(\mathbf{D}, p) + \epsilon'(\epsilon) + \gamma(\epsilon)$$

and whose distortion at the decoder i is dominated by the expression in (4.11). It follows that $R_{LP}^{G}(\mathbf{D}, p)$ is $\mathbf{D}$-achievable. □

4.4 An LP Lower Bound to $R(\mathbf{D})$

We present our second main result, namely a lower bound with a LP structure to rate distortion function $R(\mathbf{D})$ of the problem where source $\mathbf{X}$ and side information $\mathbf{Y_i}$ are random vectors and the distortion constraint for each decoder i is $\mathbf{d_i}(\mathbf{X}, \widehat{\mathbf{X}}_\mathbf{i}) \leq \mathbf{D_i}$. Same definitions for the scalar case are used to formulate this problem by replacing the scalar source, side information and distortion constraints by the vector ones given above. We utilize the following definitions to represent our results.

Definition 14. *[6]* $\mathbf{B}$ *is* conditionally less noisy *than* $\mathbf{A}$ *given* $\mathbf{C}$*, denoted as* $(\mathbf{B} \geq \mathbf{A}|\mathbf{C})$*, if* $I(\mathbf{W};\mathbf{B}|\mathbf{C}) \geq I(\mathbf{W};\mathbf{A}|\mathbf{C})$ *for all* $\mathbf{W}$ *such that* $\mathbf{W} \leftrightarrow (\mathbf{X}, \mathbf{C}) \leftrightarrow (\mathbf{A}, \mathbf{B})$.

Definition 15. *Given a random vector* $\mathbf{W}$*,* $C(\mathbf{W})$ *denotes the set of joint distributions over two vectors where the first vector has the same marginal distribution as* $\mathbf{W}$*.*

We informally refer to $C(\mathbf{W})$ as the "set of random vectors coupled to $\mathbf{W}$" and we sometimes write $\mathbf{V} \in C(\mathbf{W})$ to denote such a random vector.

Definition 16. *Given* $\mathbf{V} \in C(\mathbf{X})$ *and a mapping* $\mathbf{U}_\cdot : C(\mathbf{X}) \to C(\mathbf{X}, \mathbf{V})$*, let* $R_{lb}^{LP}(\epsilon)$ *denote the infinite dimensional LP in Table 4.1, where* $K(\cdot)$ *varies over all maps from*

Table 4.1: LP for Rate Distortion Problem

$\inf K(\emptyset) - \epsilon$ *subject to*

$K(\mathbf{X}) = 0$ *(initialize)*

$K(\mathbf{A}) \geq 0,$ for all $\mathbf{A}$ *(non-negativity)*

$K(\mathbf{B}) + I(\mathbf{B}; \mathbf{V}, \mathbf{U_B}|\mathbf{A}) \geq K(\mathbf{A}),$ for all $(\mathbf{A}, \mathbf{B})$ couplings $: \mathbf{A} \leftrightarrow \mathbf{B} \leftrightarrow \mathbf{X}$ *(slope)*

$K((\mathbf{A}, \mathbf{C})) \geq K((\mathbf{B}, \mathbf{C})),$ for all $(\mathbf{A}, \mathbf{B})$ couplings $: (\mathbf{B} \geq \mathbf{A}|\mathbf{C})$ *(monotonicity)*

$K(\mathbf{A}) \geq K(\mathbf{B}) + I(\mathbf{B}; \mathbf{V}, \mathbf{U_A}|\mathbf{A}),$ for all $(\mathbf{A}, \mathbf{B})$ couplings $: \mathbf{A} \leftrightarrow \mathbf{B} \leftrightarrow \mathbf{X}$ *(monotonicity+)*

$K(\mathbf{A}) + K(\mathbf{B}) \geq K(\mathbf{C}) + K((\mathbf{A}, \mathbf{B})),$ for all $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ couplings $: \mathbf{B} \leftrightarrow \mathbf{C} \leftrightarrow \mathbf{A}$ and $\mathbf{C} = f_1(\mathbf{A})$ or $\mathbf{C} = f_2(\mathbf{B})$ *(submodularity)*

$C(\mathbf{X})$ to $[0, \infty)$, $f_1(\cdot)$ and $f_2(\cdot)$ are deterministic functions. $K(\cdot)$ assign the same number to all deterministic random variables and $K(\emptyset)$ denotes this common number. Also whenever $(\mathbf{U_A}, \mathbf{V}, \mathbf{X}, \mathbf{A}, \mathbf{B})$ appear together, their joint distribution satisfies $(\mathbf{U_A}, \mathbf{V}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{A}, \mathbf{B})$.

Theorem 11. *For any $\epsilon > 0$, $R(\mathbf{D})$ is lower bounded by*

$$R_{lb}(\mathbf{D} + \epsilon\mathbf{1}) = \inf_{\mathbf{V} \in C(\mathbf{X})} \inf_{\mathbf{U}_{\cdot}: C(\mathbf{X}) \to C(\mathbf{X}, \mathbf{V})} R_{lb}^{LP}(\epsilon) \tag{4.12}$$

where $\mathbf{V}$ and $\mathbf{U}_{\cdot}$ in the infima must satisfy

1) For all $\mathbf{B} \in C(\mathbf{X})$, $\mathbf{U_B}$ is independent of $\mathbf{X}$;

2) If $\{\mathbf{B}, A_1, \ldots, A_s\}$ are all elements of $C(\mathbf{X})$ and can be coupled so that $\mathbf{X} \leftrightarrow \mathbf{B} \leftrightarrow (A_1, \ldots, A_s)$ then it must be possible to couple $\mathbf{U_B}$ and $(U_{A_1}, \ldots, U_{A_s})$ to $(\mathbf{X}, \mathbf{V})$ such that $\mathbf{X} \leftrightarrow (\mathbf{V}, \mathbf{U_B}) \leftrightarrow (U_{A_1}, \ldots, U_{A_s})$.

3) There exists functions $g_1(\mathbf{V}, \mathbf{U_{Y_1}}, \mathbf{Y_1})$,…,$g_m(\mathbf{V}, \mathbf{U_{Y_m}}, \mathbf{Y_m})$ such that $E[d_i(\mathbf{X}, g_i(\mathbf{V}, \mathbf{U_{Y_i}}, \mathbf{Y_i}))] \leq \mathbf{D_i} + \epsilon\mathbf{1}$, for all $i \in [m]$.

Proof. Let R be a $\mathbf{D}$-achievable rate, $\epsilon > 0$, $p(\mathbf{x})$ be given and $p(\mathbf{y_i}|\mathbf{x})$, $i \in [m]$ be fixed. Then there exists a $(n, M, \mathbf{D} + \epsilon\mathbf{1})$ code for some n such that $H(I_0) \le n(R + \epsilon)$, where I_0 is the output of the encoder. Also, let $K(\mathbf{A}) = \frac{I(\mathbf{X}^n; I_0|\mathbf{A}^n)}{n}$, where $\mathbf{A}$ is a random vector with pmf $\sum_{\mathbf{x}\in\mathcal{X}} p(\mathbf{a}|\mathbf{x})p(\mathbf{x})$, i.e., $\mathbf{A} \in \mathcal{C}(\mathbf{X})$. We call such $\mathbf{A}$ *generalized side information*. Lastly, let $\mathbf{V'}_i = I_0$, $\mathbf{U'}_{\mathbf{A}i} = (\mathbf{A}^-{}_i, \mathbf{A}^+{}_i)$, where $\mathbf{A}^-{}_i = (\mathbf{A}_1, \ldots, \mathbf{A}_{i-1})$ and $\mathbf{A}^+{}_i = (\mathbf{A}_{i+1}, \ldots, \mathbf{A}_n)$ for $i \in [n]$, and let T denote a random variable that is uniformly distributed on $[n]$ such that it is independent of the source $\mathbf{X}$, all generalized side information $\mathbf{A}$, $\mathbf{U'}_{\mathbf{A}i}$, and $\mathbf{V'}_i$. Define $\mathbf{U}_\mathbf{A} = (\mathbf{U}'_\mathbf{A}, T)$, $\mathbf{V} = (\mathbf{V}', T)$. Note that we have

$$R + \epsilon \ge K(\emptyset).$$

Also, we can write $I(\mathbf{X}^n; I_0|\mathbf{X}^n) = 0$ and $I(\mathbf{X}^n; I_0|\mathbf{A}^n) \ge 0$, for all $\mathbf{A}$, giving the *(initialize)* and *(non-negativity)* conditions in the LP.

Let $\mathbf{A} \leftrightarrow \mathbf{B} \leftrightarrow \mathbf{X}$. For any such $\mathbf{A}$ and $\mathbf{B}$ we can write $n(K(\mathbf{A}) - K(\mathbf{B}))$ as

$$\begin{aligned} I(\mathbf{B}^n; I_0|\mathbf{A}^n) &= \sum_{i=1}^n I(\mathbf{B}_i; I_0, \mathbf{B}^-{}_i, \mathbf{A}^-{}_i, \mathbf{A}^+{}_i|\mathbf{A}_i) \\ &\le \sum_{i=1}^n I(\mathbf{B}_i; I_0, \mathbf{B}^-{}_i, \mathbf{B}^+{}_i|\mathbf{A}_i) \\ &= \sum_{i=1}^n I(\mathbf{B}_i; \mathbf{V'}_i, \mathbf{U'}_{\mathbf{B}i}|\mathbf{A}_i). \end{aligned}$$

Since T is independent of $\mathbf{X}, \mathbf{V}'$, all generalized side information $\mathbf{A}$ and all $\mathbf{U}'_\mathbf{B}$, we can write

$$\begin{aligned} n(K(\mathbf{A}) - K(\mathbf{B})) &\le \sum_{i=1}^n I(\mathbf{B}_i; \mathbf{V'}_i, \mathbf{U'}_{\mathbf{B}i}|\mathbf{A}_i, T = i) \\ &= nI(\mathbf{B}; \mathbf{V}', \mathbf{U'}_\mathbf{B}|\mathbf{A}, T) \\ &= nI(\mathbf{B}; \mathbf{V}, \mathbf{U}_\mathbf{B}|\mathbf{A}), \end{aligned}$$

which gives the (*slope*) constraints in the LP.

Let $(\mathbf{B} \geq \mathbf{A}|\mathbf{C})$. Then for each such coupling of $(\mathbf{B},\mathbf{A},\mathbf{C})$, $n(K((\mathbf{A},\mathbf{C})) - K((\mathbf{B},\mathbf{C})))$ is equal to

$$H(I_0|\mathbf{A}^n,\mathbf{C}^n) - H(I_0|\mathbf{B}^n,\mathbf{C}^n) \geq 0, \text{ by [6, Lemma 1]},$$

giving the (*monotonicity*) constraints in the LP.

Now we obtain the *monotonicity*+ conditions in the LP. Let $\mathbf{A} \leftrightarrow \mathbf{B} \leftrightarrow \mathbf{X}$. By utilizing the chain rule again, we can write $n(K(\mathbf{A}) - K(\mathbf{B}))$ as

$$\begin{aligned} I(\mathbf{B}^n; I_0|\mathbf{A}^n) &\geq \sum_{i=1}^{n} I(\mathbf{B}_i; I_0, \mathbf{A}^-{}_i, \mathbf{A}^+{}_i|\mathbf{A}_i) \\ &= \sum_{i=1}^{n} I(\mathbf{B}_i; \mathbf{V}'_i, \mathbf{U}'_{\mathbf{A}i}|\mathbf{A}_i) \\ &= nI(\mathbf{B}; \mathbf{V}, \mathbf{U}_\mathbf{A}|\mathbf{A}), \end{aligned} \tag{4.13}$$

giving (*monotonicity*+) conditions.

Let $\mathbf{A},\mathbf{B},\mathbf{C}$ be such that $\mathbf{A} \leftrightarrow \mathbf{C} \leftrightarrow \mathbf{B}$ and $\mathbf{C} = f_1(\mathbf{A})$ for some deterministic mapping $f_1(\cdot)$. By the chain rule, $n(K(\mathbf{A}) + K(\mathbf{B}))$ is equal to

$$\begin{aligned} & I(\mathbf{B}^n; I_0|\mathbf{A}^n) + I(\mathbf{X}^n; I_0|\mathbf{B}^n,\mathbf{A}^n) + I(\mathbf{X}^n; I_0|\mathbf{B}^n) \\ & \geq I(\mathbf{B}^n; I_0|\mathbf{C}^n) + I(\mathbf{X}^n; I_0|\mathbf{B}^n,\mathbf{A}^n) + I(\mathbf{X}^n; I_0|\mathbf{B}^n) \\ & \geq I(\mathbf{B}^n; I_0|\mathbf{C}^n) + I(\mathbf{X}^n; I_0|\mathbf{B}^n,\mathbf{A}^n) + I(\mathbf{X}^n; I_0|\mathbf{B}^n,\mathbf{C}^n) \\ & = I(\mathbf{X}^n,\mathbf{B}^n; I_0|\mathbf{C}^n) + I(\mathbf{X}^n; I_0|\mathbf{B}^n,\mathbf{A}^n) \\ & = I(\mathbf{X}^n; I_0|\mathbf{C}^n) + I(\mathbf{X}^n; I_0|\mathbf{B}^n,\mathbf{A}^n), \end{aligned}$$

By setting $\mathbf{C} = f_2(\mathbf{B})$ and swapping the role of $\mathbf{A}$ and $\mathbf{B}$ in the procedure above, we get the (*submodularity*) conditions.

Now we find the properties of $\mathbf{V}$ and $\mathbf{U}_\mathbf{A}$ that give us the conditions 1)–3) in Theorem 11 and the Markov chain property in Definition 5. Let $\mathbf{A}$,

$\bar{\mathbf{A}} = (A_1, \ldots, A_s)$ for some s be such that $\mathbf{X} \leftrightarrow \mathbf{A} \leftrightarrow (A_1, \ldots, A_s)$. Firstly, since any set of $\mathbf{U'}_{\mathbf{A}i}$ is independent of $\mathbf{X}_i$ and of any set of generalized side information A_i's, all $\mathbf{U_A}$'s are independent of $\mathbf{X}$ and all $\mathbf{A}$'s. Secondly, note that $\mathbf{X}_i \leftrightarrow (\mathbf{V'}_i, \mathbf{U'}_{\mathbf{A}i}) \leftrightarrow (\mathbf{V'}_i, U'_{A_1 i}, \ldots U'_{A_s i})$ since

$$
\begin{aligned}
H(\mathbf{V'}_i, \mathbf{U'}_{\bar{\mathbf{A}}i} | \mathbf{V'}_i, \mathbf{U'}_{\mathbf{A}i}, \mathbf{X}_i) &= H(\bar{\mathbf{A}}_i^-, \bar{\mathbf{A}}_i^+ | I_0, \mathbf{A^-}_i, \mathbf{A^+}_i, \mathbf{X}_i) \\
&= H(\bar{\mathbf{A}}_i^-, \bar{\mathbf{A}}_i^+ | I_0, \mathbf{A^-}_i, \mathbf{A^+}_i).
\end{aligned}
$$

Then $\mathbf{X} \leftrightarrow (\mathbf{V'}, \mathbf{U'_A}) \leftrightarrow (\mathbf{V'}, U'_{A_1}, \ldots U'_{A_s})$ implies $\mathbf{X} \leftrightarrow (\mathbf{V}, \mathbf{U_A}) \leftrightarrow (\mathbf{V}, U_{A_1}, \ldots U_{A_s})$. Furthermore, given $(\mathbf{V}, \mathbf{U}_{\mathbf{Y_i}})$ and $\mathbf{Y_i}$, $i \in [m]$, decoder i can reconstruct the source subject to its own distortion constraint. Lastly, $(\mathbf{V}, \mathbf{U_A}) \leftrightarrow \mathbf{X} \leftrightarrow (\mathbf{A}, \mathbf{B})$ since $(\mathbf{V'}_i, \mathbf{U'}_{\mathbf{A}i}) \leftrightarrow \mathbf{X}_i \leftrightarrow (\mathbf{A}_i, \mathbf{B}_i)$ for all $i \in [n]$. □

We can interpret $K(\mathbf{A})$ in the LP as the amount of information that a hypothetical decoder with side information $\mathbf{A}$ receives about $\mathbf{X}$ from the broadcasted message. We can also view $\mathbf{U_A}$ as a quantized representation of the source that the hypothetical decoder can extract from the message with the help of its side information $\mathbf{A}$ and $\mathbf{V}$ as a common message to all decoders.

The *(submodularity)* condition is so named for the following reason. Let $\mathbf{X} = (X_1, \ldots, X_k)$, where X_i's are all independent random variables and let $\mathbf{A} \subseteq \mathbf{X}$, $\mathbf{B} \subseteq \mathbf{X}$ [2]. Then we can write the *(submodularity)* condition for such $\mathbf{A}$ and $\mathbf{B}$ as $K(\mathbf{A}) + K(\mathbf{B}) \geq K(\mathbf{A} \cap \mathbf{B}) + K(\mathbf{A} \cup \mathbf{B})$.

Remark 18. *The lower bound in Theorem 11 can be generalized to continuous sources with well behaved distortion constraints such as Gaussian sources subject to component-wise mean square error (MSE) distortion constraints.*

[2] Although $\mathbf{X}$ is a vector, we can consider it as an ordered set which also induces an ordered set structure on the subsets. Hence, we can use the set notation whenever it is convenient.

The lower bound in Theorem 11 is not evidently computable, since the infimum over $K(\cdot)$ is subject to a continuum of constraints and there are no cardinality bounds on the V and $U_{\cdot}$ variables. We next provide a weakened lower bound that is computable. For this we need the following notation.

Notation 15. *Let* $\mathbf{A} \leftrightarrow \mathbf{B} \leftrightarrow \mathbf{X}$ *and* $\mathbf{D_A} = \{\mathbf{D_i} | \mathbf{Y_i} \leftrightarrow \mathbf{A} \leftrightarrow \mathbf{X}\}$. *Then* $R(\mathbf{D_A})$ *denotes the result of the following optimization problem:*

$$\min_{C_{\mathbf{A}}} I(\mathbf{B}; \mathbf{V}|\mathbf{A})$$

where

$C_{\mathbf{A}}$: $\mathbf{V} \in C(\mathbf{X})$ *such that*

there exists functions $g_i(\mathbf{V}, \mathbf{Y_i})$ *such that* $E[d_i(\mathbf{X}, g_i(\mathbf{V}, \mathbf{Y_i}))] \leq \mathbf{D_i}$ *for all* $\mathbf{D_i} \in \mathbf{D_A}$.

Theorem 12. *Let* $S_{\mathbf{A}}$ *be a finite set of generalized side information variables* $\mathbf{A} \in C(\mathbf{X})$ *and impose* $K(\mathbf{A})$ *on the elements of* $S_{\mathbf{A}}$. *For any* $\epsilon > 0$, $R_{lb}(\mathbf{D} + \epsilon\mathbf{1})$ *is lower bounded by* $R'_{lb}(\mathbf{D} + \epsilon\mathbf{1})$ *where* $R'_{lb}(\mathbf{D} + \epsilon\mathbf{1})$ *is equal to*

$$\inf K(\emptyset) - \epsilon, \tag{4.14}$$

where the infimum is over all $K(\cdot) : S_{\mathbf{A}} \to [0, \infty)$ *such that*

$K(\mathbf{X}) = 0$ *(initialize)*

$K(\mathbf{A}) \geq 0$, *for all* $\mathbf{A}$ *(non-negativity)*

$K((\mathbf{A}, \mathbf{C})) \geq K((\mathbf{B}, \mathbf{C}))$, *for all* $(\mathbf{A}, \mathbf{B}) : (\mathbf{B} \geq \mathbf{A}|\mathbf{C})$ *(monotonicity)*

$K(\mathbf{A}) \geq K(\mathbf{B}) + R(\mathbf{D_A} + \epsilon\mathbf{1})$, *for all* $(\mathbf{A}, \mathbf{B}) : \mathbf{A} \leftrightarrow \mathbf{B} \leftrightarrow \mathbf{X}$ *(monotonicity+)*

$K(\mathbf{A}) + K(\mathbf{B}) \geq K(\mathbf{C}) + K((\mathbf{A}, \mathbf{B}))$, *for all* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ *couplings* : $\mathbf{B} \leftrightarrow \mathbf{C} \leftrightarrow \mathbf{A}$ *and* $\mathbf{C} = f_1(\mathbf{A})$ *or* $\mathbf{C} = f_2(\mathbf{B})$ *(submodularity)*

Proof of Theorem 12. Let $\epsilon > 0$ and $\mathbf{V} \in C(\mathbf{X})$, and $\mathbf{U}_{\cdot} : C(\mathbf{X}) \to C(\mathbf{X}, \mathbf{V})$ satisfying the conditions 1)–3) in Theorem 11 be given. Also, let LP1 be the linear program in Table 4.1 when $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are in $S_{\mathbf{A}}$ and solution of LP1 is denoted by $\bar{R}^{LP}_{lb}(\epsilon)$. Then $R^{LP}_{lb}(\epsilon)$ in Theorem 11 is lower bounded by $\bar{R}^{LP}_{lb}(\epsilon)$. Therefore it is enough to show $\bar{R}^{LP}_{lb}(\epsilon) \geq R'_{lb}(\mathbf{D} + \epsilon\mathbf{1})$. Note that the conditions of LP1 and that of the LP in Theorem 12, denoted by LP2, are the same except the *monotonicity*+ condition and that there is no *slope* condition in LP2. Furthermore, for any $\mathbf{A} \leftrightarrow \mathbf{B} \leftrightarrow \mathbf{X}$ the *monotonicity*+ condition in LP1 implies the *monotonicity*+ condition in LP2 since $I(\mathbf{B}; \mathbf{V}, \mathbf{U_A}|\mathbf{A}) \geq R(\mathbf{D_A} + \epsilon\mathbf{1})$ by condition 2) and 3) in Theorem 11. Hence, $\bar{R}^{LP}_{lb}(\epsilon) \geq R'_{lb}(\mathbf{D} + \epsilon\mathbf{1})$. □

Note that $R'_{lb}(\mathbf{D}+\epsilon\mathbf{1})$ is computable since we have finite number of conditions in LP and each $R(\mathbf{D_A})$ can be computed by finding the cardinality constraints on auxiliary random variables $\mathbf{V}, \mathbf{U_A}$ using standard techniques [43].

4.5 Comparison with Other Bounds

4.5.1 Upper Bound

Although there are several achievable schemes for various forms of rate distortion with side information (e.g. [5],[20],[2],[18]), most are for special cases of the problem. The two exceptions, both of which purport to provide achievable schemes for the general problem considered here, are Heegard and Berger [18] and Timo *et al.* [2]. Heegard and Berger's achievable result was shown to be incorrect via a counterexample by Timo *et al.*, who also to provided a corrected scheme. In fact, the proof of Timo *et al.*'s achievable result contains an error that is similar to that of Heegard and Berger. To see this, let us state Timo *et al.*'s achievable result.[3]

Notation 16. *$\bar{v} = \mathscr{S}_1, \ldots, \mathscr{S}_{2^m-1}$ denotes a list of all possible nonempty subsets of $[m]$, where each $\mathscr{S}_i$ denotes a different subset such that $|\mathscr{S}_i| \geq |\mathscr{S}_j|$ for all $i < j$. $\bar{\mathscr{V}}$ denotes the set of all possible such $\bar{v}$.*

Notation 17.

$$\begin{aligned}
U^{-'}_{\mathscr{S}_j} &= \left\{ U_{\mathscr{S}_i} \in \mathscr{U} \mid i < j,\ \mathscr{S}_i \not\supseteq \mathscr{S}_j \right\}, \\
U^{\supset}_{\mathscr{S}_j} &= \left\{ U_{\mathscr{S}_i} \in \mathscr{U} \mid \mathscr{S}_i \supset \mathscr{S}_j \right\}, \\
U^{+}_{\mathscr{S}_j} &= \left\{ U_{\mathscr{S}_k} \in \mathscr{U} \mid k > j,\ \mathscr{S}_k \cap \mathscr{S}_j \neq \emptyset \right\}, \\
U^{\dagger}_{\mathscr{S}_j} &= \left\{ U_{\mathscr{S}_i} \in U^{-'}_{\mathscr{S}_j} \middle| \begin{array}{c} \exists U_{\mathscr{S}_k} \in U^{+}_{\mathscr{S}_j}, \\ \mathscr{S}_i \cap \mathscr{S}_k \neq \emptyset \end{array} \right\}, \text{ and} \\
U^{\ddagger}_{\mathscr{S}_j,l} &= \left\{ U_{\mathscr{S}_i} \in U^{\dagger}_{\mathscr{S}_j} \ :\ \mathscr{S}_i \ni l \right\} \text{ when } l \in \mathscr{S}_j.
\end{aligned}$$

Claim 1 (Theorem 2,[2]). *The rate distortion function $R(\mathbf{D})$ is upper bounded by*

$$R^{T}_{ach}(\mathbf{D}) = \min_{\bar{v} \in \bar{\mathscr{V}}} \inf_{C_{ach,v}(\mathbf{D})} \inf_{C^{LP}_{T}} \sum_{j=1}^{2^m-1} R_{\mathscr{S}_j}, \tag{4.15}$$

where $C_{ach,\bar{v}}(\mathbf{D})$ is as in Theorem 9 and

$$\begin{aligned}
C^{LP}_{T} :\ & 1) R_{\mathscr{S}_j} \geq 0, R'_{\mathscr{S}_j} \geq 0 \textit{ for all } j \in [2^m - 1] \\
& 2) R_{\mathscr{S}_j} \geq I(X, U^{\dagger}_{\mathscr{S}_j}, U^{\supset}_{\mathscr{S}_j}; U_{\mathscr{S}_j}) - R'_{\mathscr{S}_j} \textit{ for all } j \in [2^m - 1] \\
& 3) R'_{\mathscr{S}_j} \leq \min_{l \in \mathscr{S}_j} I(U_{\mathscr{S}_j}; U^{\ddagger}_{\mathscr{S}_j,l}, U^{\supset}_{\mathscr{S}_j}, Y_l) \textit{ for all } j \in [2^m - 1].
\end{aligned}$$

The proof given by Timo *et al.* begins as follows. Let $\bar{v} \in \bar{\mathscr{V}}$ be given. The codebook generation is the same as in the proof of Theorem 9. Encoding is almost the same except that at each stage j, we select the codeword that it is

[3] This problem also afflicts Theorem 1 in Timo *et al.*, although we shall focus our discussion on Theorem 2 of that paper, which is simpler and directly comparable to Theorem 9 in the present.

jointly typical with only those already-selected codewords that correspond to the messages $U^{\dagger}_{\mathscr{S}_j}, U^{\supset}_{\mathscr{S}_j}$ and the source, instead of messages $U^{-}_{\mathscr{S}_j}$ and the source as in Theorem 9. This creates an issue, however, because if the encoding proceeds in this fashion then there is no guarantee that the variables $U^{\dagger}_{\mathscr{S}_j}, U^{\supset}_{\mathscr{S}_j}$ are themselves jointly typical.

To illustrate this, consider the case in which there are six decoders and suppose that $\bar{\nu} = [6], \ldots, \{1,2\}, \{5,6\}, \{3,4\}, \{2,3\}, \{4,5\}, \{6\}, \{5\}, \{4\}, \{3\}, \{2\}, \{1\}$. Choose $\mathscr{U}$ such that all $U_{\mathscr{S}_j} = \emptyset$ except $U_{\{i,i+1\}}$, for $i \in [5]$. Then the encoding order of the nontrivial messages is $(U_{\{1,2\}}, U_{\{5,6\}}, U_{\{3,4\}}, U_{\{2,3\}}, U_{\{4,5\}})$. When the message $U_{\{3,4\}}$ is encoded, the encoder selects a codeword that is jointly typical with the codewords related to messages $U^{\dagger}_{\{3,4\}} = (U_{\{1,2\}}, U_{\{5,6\}})$ and the source (note that $U^{\supset}_{\{3,4\}} = \emptyset$). However, in previous stages $U_{\{1,2\}}$ and $U_{\{5,6\}}$ were not selected in a way that guarantees that they are jointly typical, since $U_{\{1,2\}} \notin \{U^{\dagger}_{\{5,6\}} \cup U^{\supset}_{\{5,6\}}\}$ and $U_{\{5,6\}} \notin \{U^{\dagger}_{\{1,2\}} \cup U^{\supset}_{\{1,2\}}\}$. The rate analysis in Timo *et al.*, specifically the use of Lemma 3 in that paper, presumes that the codewords corresponding to $U_{\{1,2\}}$ and $U_{\{5,6\}}$ are jointly typical when the codeword for $U_{\{3,4\}}$ is chosen. This error is similar to the one in Heegard and Berger [18].[4] For the two-decoder case, this issue does not arise, and the Timo *et al.* rate is indeed achievable, as is that of Heegard and Berger.

This error could be fixed in several ways. Our scheme in Theorem 9 avoids this issue by requiring that each codeword be jointly typical with all of the previously-selected codewords. If a certain pair of auxiliary random variables never appear together in any of the mutual information expressions, then one can impose a conditional independence condition between them without loss

[4]Unlike the Heegard-Berger result, however, the rate promised by Timo *et al.*'s achievable result is not known to be unachievable in general at this point.

of generality, which is tantamout, from a rate perspective, to not requiring that they be chosen in a way that ensures their joint typicality.

Our scheme in Theorem 9 differs from the achievable scheme in [2] in two other respects as well. We do not require that the sets in v be ordered so that their cardinalities are nonincreasing. Arguably the most notable difference is in the decoding. While in [2], each decoder decodes its messages sequentially in the same order that they are encoded, in our scheme we apply simultaneous decoding, i.e., we decode all messages for decoder i together. We shall see later, when discussing the odd-cycle index coding problem in Section 4.6, that for a given class of auxiliary random variables, simultaneous decoding can yield a strict rate improvement.

We conclude this subsection by showing that for the two-decoder case in which Claim 1 is valid, the upper bound in [2] is no worse than that of Theorem 9.

Lemma 9. *When there are two decoders, $R_{ach}(\mathbf{D})$ is upper bounded by*

$$R^T(\mathbf{D}) = \min_{C_{ach,v}(\mathbf{D})} \max_{i \in \{1,2\}} \{I(X; U_{\{1,2\}}|Y_i)\} + I(X; U_{\{1\}}|U_{\{1,2\}}, Y_1) + I(X; U_{\{2\}}|U_{\{1,2\}}, Y_2), \quad (4.16)$$

where $C_{ach,v}(\mathbf{D})$ is in Theorem 9.

Proof of Lemma 9. Firstly notice that $U_{\{1\}}$ and $U_{\{2\}}$ never appear together on the right-hand side of (4.16). Hence without loss of optimality we can add the condition $U_{\{1\}} \perp U_{\{2\}}|X, U_{\{1,2\}}$ to $C_{ach,v}(\mathbf{D})$ in the right-hand side of (4.16). Let $v = \{\{1, 2\}, \{1\}, \{2\}\}$ and $U_{\mathscr{S}_j} \in C_{ach,v}(\mathbf{D})$ with $U_{\{1\}} \perp U_{\{2\}}|X, U_{\{1,2\}}$. From the LP conditions in Theorem 9, we can write

$$R_{\{1,2\}} + R'_{\{1,2\}} \geq I(X; U_{\{1,2\}}) \tag{4.17}$$

$$R_{\{1\}} + R'_{\{1\}} \geq I(X, U_{\{1,2\}}; U_{\{1\}}) \tag{4.18}$$

$$R_{\{2\}} + R'_{\{2\}} \geq I(X, U_{\{1,2\}}, U_{\{1\}}; U_{\{2\}}) \tag{4.19}$$

$$R'_{\{i\}} \leq I(U_{\{i\}}; U_{\{1,2\}}, Y_i), \text{ for all } i \in \{1, 2\} \tag{4.20}$$

$$R'_{\{1,2\}} \leq \min_{i \in \{1,2\}} \{I(U_{\{1,2\}}; U_{\{i\}}, Y_i)\} \tag{4.21}$$

$$R'_{\{1,2\}} + R'_{\{i\}} \leq I(U_{\{1,2\}}; Y_i) + I(U_{\{i\}}; U_{\{1,2\}}, Y_i), \text{ for all } i \in \{1, 2\} \tag{4.22}$$

Then, $R'_{\{1,2\}} = \min_{i\in\{1,2\}}\{I(U_{\{1,2\}}; Y_i)\}$, $R'_{\{i\}} = I(U_{\{i\}}; U_{\{1,2\}}, Y_i)$, $R_{\{1,2\}} + R'_{\{1,2\}} = I(X; U_{\{1,2\}})$, $R_{\{1\}} + R'_{\{1\}} = I(X, U_{\{1,2\}}; U_{\{1\}})$, and $R_{\{2\}} + R'_{\{2\}} = I(X, U_{\{1,2\}}, U_{\{1\}}; U_{\{2\}})$ are feasible and we can upper bound $\inf_{C_{ach}^{LP}} \sum_{j=1}^{3} R_{\mathscr{S}_j}$ in (4.1) by

$$\max_{i \in \{1,2\}} \{I(X; U_{\{1,2\}}|Y_i)\} + I(X; U_{\{1\}}|U_{\{1,2\}}, Y_1) + I(X; U_{\{2\}}|U_{\{1,2\}}, Y_2) + I(U_{\{1\}}; U_{\{2\}}|X, U_{\{1,2\}}), \tag{4.23}$$

which is equal to the mutual information expression in Lemma 9 when $U_{\{1\}} \perp U_{\{2\}}|X, U_{\{1,2\}}$. Therefore, $R^T(\mathbf{D}) \geq R_{ach}(\mathbf{D})$. □

4.5.2 Lower Bounds

minimax-type Lower Bound

First we compare the general lower bound, $R_{lb}(\mathbf{D}+\epsilon\mathbf{1})$, with the minimax version of the lower bound in [20]. For completeness, we state the minimax version of the theorem below.

Theorem 13. *Let the pmf's $p(X, Y_i)$ for all $i \in [m]$ be given. Then $R(\mathbf{D})$ is lower bounded* *by*

$$R_{lb}^m(\mathbf{D} + \epsilon\mathbf{1}) = \sup_{\bar{P}} \inf_{\bar{C}} \bar{R}_{lb} - \epsilon, \tag{4.24}$$

$$\begin{aligned} \textit{where } \bar{R}_{lb} = \max_{\sigma} \, [&I(X; V, U_{Y_{\sigma(1)}}|Y_{\sigma(1)}) \\ &+ I(X; U_{Y_{\sigma(2)}}|V, U_{Y_{\sigma(1)}}, Y_{\sigma(1)}, Y_{\sigma(2)}) + \cdots \\ &+ I(X; U_{Y_{\sigma(m)}}|V, U_{Y_{\sigma(1)}}, \ldots, U_{Y_{\sigma(m-1)}}, Y)], \end{aligned} \tag{4.25}$$

and $Y = (Y_{\sigma(1)}, \ldots, Y_{\sigma(m)})$*, and*

1) $\bar{P} = \{p(X, Y_1, \ldots, Y_m)| \sum_{Y_j : j \neq i} p(X, Y_1, \ldots, Y_m) = p(X, Y_i), \forall i \in [m]\}$*.*

2) $\bar{C}$ *denotes the set of* $(V, U_{Y_1}, \ldots, U_{Y_m})$ *jointly distributed with* $X, Y_1, \ldots, Y_m$ *such that* $(Y_1, \ldots, Y_m) \leftrightarrow X \leftrightarrow (V, U_{Y_1}, \ldots, U_{Y_m})$ *and there exists functions* $g_1, \ldots, g_m$ *with the property that*

$\mathbb{E}[d_i(X, g_i(V, U_{Y_i}, Y_i))] \leq D_i + \epsilon, \forall i \in [m]$*.*

3) $\sigma(.)$ *denotes a permutation on integers* $[m]$*.*

The minimax lower bound in Theorem 13 is the state of the art for the general rate distortion problem with side information at multiple decoders. Note that in Theorem 13, one can absorb V into U_{Y_i}, $i \in [m]$ without loss of optimality. For the ease of comparison with $R_{lb}(\mathbf{D} + \epsilon\mathbf{1})$ we do not combine V with U_{Y_i}, $i \in [m]$, however.

Theorem 14. $R_{lb}(\mathbf{D} + \epsilon\mathbf{1}) \geq R_{lb}^m(\mathbf{D} + \epsilon\mathbf{1})$*, where* $\epsilon > 0$*.*

Proof. Consider $R_{lb}(\mathbf{D} + \epsilon\mathbf{1})$. Note that LP constraints of $R_{lb}(\mathbf{D} + \epsilon\mathbf{1})$ is for all couplings of the random variables in the $R_{lb}(\mathbf{D} + \epsilon\mathbf{1})$. Hence, we can write,

$$R_{lb}(\mathbf{D} + \epsilon\mathbf{1}) \geq \sup_{\bar{P}} \inf_{V \in C(X)} \inf_{U_{\cdot} : C(X) \to C(X,V)} R_{lb}^{LP}(\epsilon) \tag{4.26}$$

where $\bar{P}$ is as in Theorem 13, and V and U in the infima satisfy the conditions 1)–3) in Theorem 11 for a fixed coupling of the random variables. Now we find a lower bound to the $R_{lb}^{LP}(\epsilon)$ in (4.26) by utilizing the *monotonicity* and *monotonicity+* constraints of the LP in Table 4.1. We can write the following series of inequalities:

$$K(\emptyset) \geq K(Y_1) \text{ by } \textit{(monotonicity)} \tag{4.27}$$

$$K(Y_1) \geq K(Y_1, Y_2) + I(Y_2; V, U_{Y_1}|Y_1) \tag{4.28}$$

$$\vdots$$

$$K(Y_1, \ldots, Y_m) \geq K(Y_1, \ldots, Y_m, X) + I(X; V, U_{Y_1}, \ldots, U_{Y_m}|Y_1, \ldots, Y_m) \tag{4.29}$$

$$K(Y_1, \ldots, Y_m, X) = 0. \tag{4.30}$$

where (4.28) is from *monotonicity+* and (4.29) is from *monotonicity+* and $(Y_1, \ldots, Y_m) \leftrightarrow X \leftrightarrow (V, U_{Y_1 \ldots Y_m}) \leftrightarrow (V, U_{Y_1}, \ldots, U_{Y_m})$. If we add all these inequalities side by side we obtain

$$\begin{aligned} K(\emptyset) \geq\ & I(Y_2; V, U_{Y_1}|Y_1) + \cdots \\ & + I(Y_m; V, U_{Y_1}, \ldots, U_{Y_{m-1}}|Y_1, \ldots, Y_{m-1}) \\ & + I(X; V, U_{Y_1}, \ldots, U_{Y_m}|Y_1, \ldots, Y_m). \end{aligned} \tag{4.31}$$

By applying a series of chain rules and combining terms, we can write the right hand side of (4.31) as

$$\begin{aligned} I(X; V, U_{Y_1}|Y_1) + \cdots + I(X; U_{Y_2}|V, U_{Y_1}, Y_1, Y_2) \\ + I(X; U_{Y_m}|V, U_{Y_1}, \ldots, U_{Y_{m-1}}, Y_1, \ldots, Y_m) \end{aligned}$$

Let us define

$$\begin{aligned} \Gamma_k = & \sum_{i=2}^{k} I(Y_i; V, U_{Y_1}, \ldots, U_{Y_{i-1}}|Y_1, \ldots, Y_{i-1}) + \\ & + I(X; V, U_{Y_1}, \ldots, U_{Y_k}|Y_1, \ldots, Y_k) \\ & + \sum_{i=k+1}^{m} I(X; U_{Y_i}|V, U_{Y_1}, \ldots, U_{Y_{i-1}}, Y_1, \ldots, Y_i) \end{aligned}$$

for $k \in [m]$ where "empty" sums are zero. Note that Γ_m is equal to right hand side of (4.31). One can show that $\Gamma_1 = \Gamma_2 = \ldots = \Gamma_m$. Hence $K(\emptyset) \geq \Gamma_1$.

Also since there are m decoders, we can get $m!$ lower bounds on $K(\emptyset)$ by considering all possible permutations on integers $[m]$. Hence, we have $K(\emptyset) \geq \bar{R}_{lb}$, and from (4.26) we can write

$$R_{lb}(\mathbf{D} + \epsilon\mathbf{1}) \geq \sup_{\bar{P}} \inf_{V \in C(X)} \inf_{U.:C(X) \to C(X,V)} \bar{R}_{lb} - \epsilon \tag{4.32}$$

$$\geq \sup_{\bar{P}} \inf_{\bar{C}} \bar{R}_{lb} - \epsilon, \tag{4.33}$$

where $\bar{C}$ as in Theorem 13. Lastly, we have (4.33) since each feasible set of random variables in the infima (4.32) is also feasible for $\bar{C}$. Hence, $R_{lb}(\mathbf{D} + \epsilon\mathbf{1}) \geq R^m_{lb}(\mathbf{D} + \epsilon\mathbf{1})$. □

LP Lower Bound for the Index Coding Problem

We next compare the general lower bound, $R_{lb}(\mathbf{D} + \epsilon\mathbf{1})$ with the linear programming lower bound in [11] for the index coding problem [8]. In the index coding problem, the source $\mathbf{X} = (X_1, \ldots, X_k)$ is such that X_i, $i \in [k]$ are independent and identically distributed (i.i.d) Bernoulli $\left(\frac{1}{2}\right)$ random variables and each side information $\mathbf{Y_i}$ at decoder i is an arbitrary subset[5] of the source $\mathbf{X}$. Each decoder i wants to reconstruct an arbitrary subset of the source, $\widehat{\mathbf{X}}_\mathbf{i} \subseteq \mathbf{X} \setminus \mathbf{Y_i}$. The reconstructions can either be required to be zero error [11] or such that the block error probability vanishes [20]. Both formulations are more stringent than considering the problem with Hamming distortion in the limit in which the distortion goes to zero, so $R_{lb}(\epsilon\mathbf{1})$ is a valid lower bound to the index coding problem in all three cases. We first state the LP lower bound in [11], originally stated for

[5] Although $\mathbf{X}$ is a vector, we can consider it as an ordered set which also induces an ordered set structure on the subsets. Hence, we can use the set notation whenever it is convenient.

Table 4.2: LP Bound for Index Coding Problem

$\min \widehat{K}(\emptyset)$ *subject to*

$\widehat{K}(\mathbf{X}) \geq |\mathbf{X}|$ *(initialize)*

$\widehat{K}(\mathbf{A}) + |\mathbf{B} \setminus \mathbf{A}| \geq \widehat{K}(\mathbf{B}),$ for all $\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{X}$ *(slope)*

$\widehat{K}(\mathbf{B}) \geq \widehat{K}(\mathbf{A}),$ for all $\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{X}$ *(monotonicity)*

$\widehat{K}(\mathbf{A}) = \widehat{K}(\mathbf{B}),$ for all $\mathbf{A}, \mathbf{B} \subseteq \mathbf{X} : \mathbf{A} \rightsquigarrow \mathbf{B}$ *(decode)*

$\widehat{K}(\mathbf{A}) + \widehat{K}(\mathbf{B}) \geq \widehat{K}(\mathbf{A} \cap \mathbf{B}) + \widehat{K}(\mathbf{A} \cup \mathbf{B}),$

for all $\mathbf{A}, \mathbf{B} \subseteq \mathbf{X}$ *(submodularity)*.

the zero-error form of the problem. Then we show that $\lim_{\epsilon \to 0} R_{lb}(\epsilon \mathbf{1})$ is equal to this bound when we restrict the generalized side information, $\mathbf{A}$, in $R_{lb}(\epsilon \mathbf{1})$ to be subset of the source, $\mathbf{X}$. From now on we denote this weakened form of $R_{lb}(\epsilon \mathbf{1})$ obtained by restricting the generalized side information to be subset of the source by $R^I_{lb}(\epsilon \mathbf{1})$.

Notation 18. $\mathbf{A} \rightsquigarrow \mathbf{B}$ *denotes "$\mathbf{A}$ decodes $\mathbf{B}$," meaning that $\mathbf{A} \subseteq \mathbf{B}$ and for every source component $X_i \in \mathbf{B} \setminus \mathbf{A}$ there is a decoder j who reconstructs X_i and $\mathbf{Y_j} \subseteq \mathbf{A}$. Also $S(\mathbf{A}) = \{X_i |$ decoder j reconstructs $X_i \in \mathbf{X}$ and $\mathbf{Y_j} \subseteq \mathbf{A}\}$*

Theorem 15 (LP lower bound [11])**.** *The optimal value for the linear program in Table 4.2 [6], denoted by $\widehat{R}^{LP}_{lb}$, is a lower bound to the index coding problem.*

Following two lemmas will be useful to prove weakened lower bound $R^I_{lb}(\epsilon \mathbf{1})$ is equal to the LP lower bound in Theorem 15.

Lemma 10. *Without loss of optimality we can replace the (initialize) and (slope)*

[6]The statement of the result in [11] does not contain the (monotonicity) condition, although it is clear from the proof that it was intended to be included. The condition is present in the preprint version of the paper [48].

conditions to the LP in Table 4.2 with

$$\widehat{K}(\mathbf{X}) = |\mathbf{X}| \text{ (initialize*)}$$

$$\widehat{K}(\mathbf{A}) + |\mathbf{B} \setminus \{S(\mathbf{A}) \cup \mathbf{A}\}| \geq \widehat{K}(\mathbf{B}), \text{ for all } \mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{X} \text{ (slope*)}$$

respectively.

Proof. First we show without loss of optimality we can add the *initialize** and *slope** conditions to the LP in Table 4.2. Since they are more stringent than *initialize* and *slope* conditions, the result follows. We begin with *initialize** condition. Let $\widehat{K}(\mathbf{A})$, $\mathbf{A} \subseteq \mathbf{X}$ be feasible to the LP in Table 4.2 such that $\widehat{K}(\mathbf{X}) > |\mathbf{X}|$. Then there exists $\epsilon > 0$ such that $\widehat{K}(\mathbf{X}) = |\mathbf{X}| + \epsilon$. Note that $\widehat{K}(\mathbf{A}) - \epsilon$, $\mathbf{A} \subseteq \mathbf{X}$ is also feasible to the LP in Table 4.2 giving lower objective $\widehat{K}(\emptyset) - \epsilon$. Hence, without loss of optimality we can insert *initialize** condition to the LP in Table 4.2. Now we show that *slope* and *decode* conditions of the LP in Table 4.2 imply the *slope** condition. Let $\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{X}$. If $\mathbf{B} \cap S(\mathbf{A}) = \emptyset$ then *slope* and *slope** conditions are equivalent. Otherwise, i.e. if $\mathbf{B} \cap S(\mathbf{A}) = \mathbf{C} \neq \emptyset$, from *decode* and *slope* conditions we have

$$\widehat{K}(\mathbf{C} \cup \mathbf{A}) = \widehat{K}(\mathbf{A}),$$

$$\widehat{K}(\mathbf{C} \cup \mathbf{A}) + |\mathbf{B} \setminus \{\mathbf{C} \cup \mathbf{A}\}| \geq \widehat{K}(\mathbf{B})$$

respectively. Since $\mathbf{B} \setminus \{\mathbf{C} \cup \mathbf{A}\} = \mathbf{B} \setminus \{S(\mathbf{A}) \cup \mathbf{A}\}$, *decode* and *slope* conditions imply *slope** condition. □

Lemma 11. *Let* $\epsilon > 0$ *and* $\bar{R}_{lb}^{LP}(\epsilon\mathbf{1})$ *be optimal value of the LP in Table 4.3. Then* $R_{lb}^{I}(\epsilon\mathbf{1}) \geq \bar{R}_{lb}^{LP}(\epsilon\mathbf{1})$ *and* $\lim_{\epsilon\to 0} R_{lb}^{I}(\epsilon\mathbf{1}) = \bar{R}_{lb}^{LP}(\mathbf{0})$.

Proof. Since the random variables $\mathbf{A}, \mathbf{B}$ in $R_{lb}^{I}(\epsilon\mathbf{1})$ are such that $\mathbf{A}, \mathbf{B} \subseteq \mathbf{X}$, Markov chain $\mathbf{A} \leftrightarrow \mathbf{B} \leftrightarrow \mathbf{X}$ is equivalent to $\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{X}$. Then the *slope* constraints of the

Table 4.3: Relaxation of the LP in Table 4.1

$\min K(\emptyset) - \epsilon$ *subject to*

$K(\mathbf{X}) = 0$ *(initialize)*

$K(\mathbf{A}) \geq 0,$ for all $\mathbf{A} \subseteq \mathbf{X}$ *(non-negativity)*

$K(\mathbf{B}) + H(\mathbf{B}|\mathbf{A}) \geq K(\mathbf{A}),$ for all $\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{X}$ *(slope)*

$K(\mathbf{A}) \geq K(\mathbf{B}),$ for all $\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{X}$ *(monotonicity)*

$K(\mathbf{A}) \geq K(\mathbf{B}) + H(\mathbf{B}|\mathbf{A}) - H(\mathbf{B}|S(\mathbf{A}), \mathbf{A}) - \epsilon \log |S(\mathbf{A})|,$ for all $\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{X}$ *(monotonicity+)*

$K(\mathbf{A}) + K(\mathbf{B}) \geq K(\mathbf{A} \cap \mathbf{B}) + K(\mathbf{A} \cup \mathbf{B}),$

for all $\mathbf{A}, \mathbf{B} \subseteq \mathbf{X}$ *(submodularity)*.

LP in $R^I_{lb}(\epsilon\mathbf{1})$ imply the *slope* constraints of $\bar{R}^{LP}_{lb}(\epsilon\mathbf{1})$, since $H(\mathbf{B}|\mathbf{A}) \geq I(\mathbf{B}; \mathbf{V}, \mathbf{U_B}|\mathbf{A})$. Furthermore, using Fano's inequality, it can be seen that *monotonicity+* conditions of the LP in $R^I_{lb}(\epsilon\mathbf{1})$ are the same as *monotonicity+* conditions of $\bar{R}^{LP}_{lb}(\epsilon\mathbf{1})$ as well as the rest of the conditions. Hence, we have $R^I_{lb}(\epsilon\mathbf{1}) \geq \bar{R}^{LP}_{lb}(\epsilon\mathbf{1})$. Now we select $\mathbf{V} = \mathbf{Z}$ where $\mathbf{Z}$ is a vector of i.i.d Bernoulli($\frac{1}{2}$) bits of the same length as $\mathbf{X}$, $\mathbf{Z} \perp \mathbf{X}$, and we select $\mathbf{U_A} = (S(\mathbf{A}), \mathbf{A}) \oplus \mathbf{Z}$,[7] $\mathbf{A} \subseteq \mathbf{X}$. Note that this selection of $\mathbf{V}$ and $\mathbf{U_A}$ satisfy the conditions 1)–3) in Theorem 11. Then the solution of the resulting LP is equal to LP in Table 4.3 where $\epsilon \log |S(\mathbf{A})| = 0$, giving $\bar{R}^{LP}_{lb}(\mathbf{0}) - \epsilon \geq R^I_{lb}(\epsilon\mathbf{1})$. Since $\bar{R}^{LP}_{lb}(\epsilon\mathbf{1})$ is right continuous at $\epsilon = 0$ [47], letting $\epsilon \to 0$ gives the result. □

Theorem 16. $\lim_{\epsilon \to 0} R^I_{lb}(\epsilon\mathbf{1}) = \widehat{R}^{LP}_{lb}$.

Proof. Let LP_1 and LP_2 denote the LPs in Theorem 15 and Table 4.3 with $\epsilon = 0$, respectively. By Lemma 10, without loss of optimality we can add *initialize** and *slope** conditions in Lemma 10 to LP_1 and consider LP_1 of this form. Notice that

[7] $\mathbf{a} \oplus \mathbf{b}$ denotes componentwise exclusive-OR operation where the shorter vector is zero padded.

$\bar{R}^{LP}_{lb}(\mathbf{0})$ is the solution of LP_2 and from Lemma 11, $\lim_{\epsilon\to 0} R^{I}_{lb}(\epsilon\mathbf{1}) = \bar{R}^{LP}_{lb}(\mathbf{0})$. Hence, it is enough to show that $\widehat{R}^{LP}_{lb} = \bar{R}^{LP}_{lb}(\mathbf{0})$. We show it by reparametrizing the LP_2 in terms of $\widehat{K}(\mathbf{A})$ where $\widehat{K}(\mathbf{A}) = K(\mathbf{A}) + H(\mathbf{A})$. Note that $\widehat{K}(\emptyset) = K(\emptyset)$. Hence, the objective of LP_2 is $\widehat{K}(\emptyset)$ same as the objective of LP_1. Now we show that the conditions of LP_2 and the conditions of LP_1 are the same. We can rewrite the *initialize* and *non-negativity* conditions of LP_2 as
$\widehat{K}(\mathbf{X}) = H(\mathbf{X})$
$\widehat{K}(\mathbf{A}) \geq H(\mathbf{A})$ respectively. Together those two conditions are equivalent to *initialize** and *slope* conditions of LP_1.

When we rewrite *slope* condition of LP_2, we get
$\widehat{K}(\mathbf{B}) \geq \widehat{K}(\mathbf{A})$, *monotonicity* condition of LP_1.

When we rewrite *monotonicity* and *monotonicity+* conditions of LP_2, we get
$\widehat{K}(\mathbf{A}) + H(\mathbf{B}|\mathbf{A}) \geq \widehat{K}(\mathbf{B})$
$\widehat{K}(\mathbf{A}) + H(\mathbf{B}|S(\mathbf{A}),\mathbf{A}) \geq \widehat{K}(\mathbf{B})$ respectively and they are equivalent to *slope* and *slope** conditions of LP_1.

Also, combining *submodularity* condition of LP_2 and $H(\mathbf{A}) + H(\mathbf{B}) = H(\mathbf{B} \cap \mathbf{A}) + H(\mathbf{B} \cup \mathbf{A})$ we can get the same *submodularity* condition of LP_1.

Lastly, from *monotonicity+* and *slope* conditions of LP_2, we can obtain $K(\mathbf{A}) + H(\mathbf{A}) = K(\mathbf{B}) + H(\mathbf{B}|\mathbf{A}) + H(\mathbf{A})$ for all $A \rightsquigarrow B$, which is the *decode* condition of LP_1. Hence, each condition (or combinations) of LP_2 corresponds to a condition of LP_1 and vice versa. Since objective of LP_1 and LP_2 are the same, we can conclude that $\widehat{R}^{LP}_{lb} = \bar{R}^{LP}_{lb}(\mathbf{0})$. □

4.6 Optimality Results

The LP upper and lower bounds are tight in several instances[8]. We begin with two classes of instances for which the rate-distortion function is already known. Then we continue with odd-cycle index coding problem, which can be considered as a special case of Heegard-Berger problem. We conclude this section by finding the explicit characterization of the odd-cycle "Gaussian index coding" problem using the upper and lower bounds in Theorem 9 and 12 respectively.

4.6.1 Rate Distortion Function with Mismatched Side Information at Decoders [5]

In this problem, there is one encoder with source $\mathbf{X} = (X_1, X_2)$ and two decoders with side information $\mathbf{Y_1} = (Y_{11}, Y_{12})$ and $\mathbf{Y_2} = (Y_{21}, Y_{22})$, respectively. The source and side information satisfy the following relation

$$(X_1, Y_{11}, Y_{21}) \perp (X_2, Y_{12}, Y_{22}) \tag{4.34}$$

$$X_1 \leftrightarrow Y_{11} \leftrightarrow Y_{21} \text{ and } X_2 \leftrightarrow Y_{22} \leftrightarrow Y_{12} \tag{4.35}$$

[8]In a recent work [27], rate distortion problem with two decoders having a degraded reconstruction sets is considered and the corresponding rate distortion function is characterized. Slightly different from our problem setting, one component of the source is reconstructed at both decoders with vanishing block error probability and the other component of the source is only reconstructed at a single decoder. The construction of auxiliary random variables in the converse result of Benammar *et al.* [27] is crafted for this specific problem setting and not directly extendable to multiple decoders whereas our LP lower bound is for general rate distortion problem with side information at multiple decoders. An interesting future direction would be to investigate whether the LP lower bound subsumes this lower bound too.

and the reconstructions at decoders, $\widehat{\mathbf{X}}_{\mathbf{1}} = (\widehat{X}_{11}, \widehat{X}_{12})$ and $\widehat{\mathbf{X}}_{\mathbf{2}} = (\widehat{X}_{21}, \widehat{X}_{22})$, are such that

$$\mathbb{E}[d_{1i}(X_1, \widehat{X}_{1i})] \le D_{1i} \tag{4.36}$$

$$\mathbb{E}[d_{2i}(X_2, \widehat{X}_{2i})] \le D_{2i} \text{ for } i \in [2]. \tag{4.37}$$

We denote the rate distortion function of this problem as $R^M(\mathbf{D})$. Theorem 18 shows that the minimax lower bound in Theorem 13 is greater than or equal to $R^M(\mathbf{D})$, the rate distortion function characterized by Watanabe [5]. Hence, it implies that lower bounds in both Theorems 13 and 11 are tight for this problem.

Theorem 17 ([5])**.** *Rate distortion function, $R^M(\mathbf{D})$, is*

$$R^M(\mathbf{D}) = \min[\max\{R_1^M, R_2^M\}], \text{ where}$$
$$R_1^M = I(X_1; W_1|Y_{11}) + I(X_2; W_2|Y_{12}) + I(X_1; U_1|Y_{11}, W_1) + I(X_2; U_2|Y_{22}, W_2)$$
$$R_2^M = I(X_1; W_1|Y_{21}) + I(X_2; W_2|Y_{22}) + I(X_1; U_1|Y_{11}, W_1) + I(X_2; U_2|Y_{22}, W_2),$$

and the minimization is taken over all auxiliary random variables W_1, W_2, U_1, U_2 satisfying the following:
1) $(W_i, U_i) \leftrightarrow X_i \leftrightarrow (Y_{1i}, Y_{2i})$ for $i = 1, 2$.
2) $(W_1, U_1, X_1, Y_{11}, Y_{21})$ and $(W_2, U_2, X_2, Y_{12}, Y_{22})$ are independent of each other.
3) There exist functions $g_{11}(W_1, U_1, Y_{11}) = \widehat{X}_{11}$, $g_{12}(W_2, Y_{12}) = \widehat{X}_{12}$, $g_{21}(W_1, Y_{21}) = \widehat{X}_{21}$, and $g_{22}(W_2, U_2, Y_{22}) = \widehat{X}_{22}$ such that they satisfy (4.36) and (4.37).
4) $|\mathcal{W}_i| \le |\mathcal{X}_i| + 3$ and $|\mathcal{U}_i| \le |\mathcal{X}_i| \cdot (|\mathcal{X}_i| + 3) + 1$ for $i = 1, 2$, where $\mathcal{W}_i$ and $\mathcal{U}_i$ are alphabets of W_i and U_i respectively.

Theorem 18. $\liminf_{\epsilon \to 0} R_{lb}^m(\mathbf{D} + \epsilon \mathbf{1}) \ge R^M(\mathbf{D})$ *and* $R_{ach}(\mathbf{D}) \le R^M(\mathbf{D})$.

Proof. We select the joint distribution of $(\mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2})$ such that it satisfies (4.35). First we show $\liminf_{\epsilon \to 0} R_{lb}^m(\mathbf{D} + \epsilon \mathbf{1}) \ge R^M(\mathbf{D})$. Let $U_Y = (V, U_{Y_1})$ and $U_Z = (V, U_{Y_2})$.

Then $\bar{R}_{lb}$ in Theorem 13 can be written as[9] $\bar{R}_{lb} = \max\{\bar{R}_{lb1}, \bar{R}_{lb2}\}$,

$$\bar{R}_{lb1} = I(\mathbf{X}; U_Y|\mathbf{Y_1}) + I(\mathbf{X}; U_Z|U_Y, \mathbf{Y_1}, \mathbf{Y_2})$$

$$\bar{R}_{lb2} = I(\mathbf{X}; U_Z|\mathbf{Y_2}) + I(\mathbf{X}; U_Y|U_Z, \mathbf{Y_1}, \mathbf{Y_2}).$$

By the chain rule and using (4.35), R_{lb1} can be rewritten as

$$\begin{aligned}
& I(X_2; U_Y, Y_{11}|Y_{12}) + I(X_1; U_Y, Y_{12}, X_2|Y_{11}) \\
& \quad + I(X_2; U_Z|U_Y, Y_{11}, Y_{22}) + I(X_1; U_Z|U_Y, Y_{22}, X_2, Y_{11}) \\
& \overset{a}{=} I(X_2; U_Y, Y_{11}|Y_{12}) + I(X_1; U_Y, Y_{22}, X_2|Y_{11}) \\
& \quad + I(X_2; U_Z|U_Y, Y_{11}, Y_{22}) + I(X_1; U_Z|U_Y, Y_{22}, X_2, Y_{11}) \\
& \overset{b}{=} I(X_2; U_Y, Y_{11}|Y_{12}) + I(X_1; U_Y, Y_{22}, X_2, U_Z|Y_{11}) + I(X_2; U_Z|U_Y, Y_{11}, Y_{22}) \\
& \geq I(X_2; U_Y, Y_{11}|Y_{12}) + I(X_1; U_Y, Y_{22}, U_Z|Y_{11}) + I(X_2; U_Z|U_Y, Y_{11}, Y_{22}),
\end{aligned}$$

which equals to $I(X_2; W_2|Y_{12}) + I(X_1; W_1, U_1|Y_{11}) + I(X_2; U_2|W_2, Y_{22}) = R_1^M$, where $W_2 = (V, U_{Y_1}, Y_{11})$, $W_1 = (V, U_{Y_2}, Y_{22})$, $U_1 = U_{Y_1}$ and $U_2 = U_{Y_2}$.

a: since $I(X_1; Y_{12}, Y_{22}|X_2, U_Y, U_Z, Y_{11}) = 0$.

b: combining the second and last term

Similarly we have, $\bar{R}_{lb2} \geq R_2^M$.

Note that $(U_Y, U_Z) \leftrightarrow (X_1, X_2) \leftrightarrow (Y_{11}, Y_{12}, Y_{21}, Y_{22})$ implies the first condition of the minimization in Theorem 17. Also, distortion constraints of $R_{lb}^m(\mathbf{D} + \epsilon\mathbf{1})$ implies the third condition of the minimization with ϵ added to distortion constraints in Theorem 17. Hence, we can write

$$R_{lb}^m(\mathbf{D} + \epsilon\mathbf{1}) \geq \inf[\max\{R_1^M, R_2^M\}] - \epsilon, \tag{4.38}$$

where the minimization is over (W_1, U_1, W_2, U_2) such that they satisfy first and third conditions of the minimization in Theorem 17. Also, since (W_1, U_1) and

[9] Note that Theorem 13 can be applied to vector valued source and side information at decoders.

(W_2, U_2) do not appear together, we can add the condition 2) in Theorem 17 to the minimization in (4.38). Lastly cardinality bounds on (W_1, W_2, U_1, U_2) can be obtained as in $R^M(\mathbf{D})$ and right hand side of (4.38) can be shown to be continuous in ϵ using the same procedure as in [5].

Now we show that $R_{ach}(\mathbf{D}) \leq R^M(\mathbf{D})$. In [5], $R^T(\mathbf{D})$ in Lemma 9 is used to obtain $R^M(\mathbf{D})$. Hence, from Lemma 9, we have $R_{ach}(\mathbf{D}) \leq R^M(\mathbf{D})$. □

4.6.2 Rate Distortion Function with Conditionally Less Noisy Side Information [6]

There are two decoders, and the distortion measure at decoder 1, $d_1(.,.)$, is such that $d_1(X, \widehat{X}) = 0$ if $\widehat{X} = a(X)$ and $d_1(X, \widehat{X}) = 1$ otherwise, where $a(X)$ is a deterministic map. Also the allowable distortion at decoder 1, D_1, is taken as zero. Timo *et al.* [6] show that their lower bound for this problem is tight if Y_2 is conditionally less noisy than Y_1, i.e., $(Y_2 \succeq Y_1 | a(X))$, and $H(a(X)|Y_1) \geq H(a(X)|Y_2)$. Although whether the minimax lower bound in Theorem 13 is tight for this problem is not known, the next theorem shows that $R_{lb}(\mathbf{D} + \epsilon\mathbf{1})$ subsumes the lower bound in [6] when $(Y_2 \succeq Y_1 | a(X))$.

Theorem 19. $\liminf_{\epsilon \to 0} R_{lb}(\mathbf{D} + \epsilon\mathbf{1}) \geq R^{LN}(\mathbf{D})$ *and* $R_{ach}(\mathbf{D}) \leq R^{LN}(\mathbf{D})$ *where*

$$R^{LN}(\mathbf{D}) = H(a(X)|Y_1) + \min_{\substack{W \leftrightarrow X \leftrightarrow (a(X), Y_2) \\ \mathbb{E}[d_2(X, g_2(W, a(X), Y_2))] \leq D_2 \\ |\mathcal{W}| \leq |\mathcal{X}|+1,}} I(X; W | a(X), Y_2)$$

is the lower bound in [6, Lemma 5] when $(Y_2 \succeq Y_1 | a(X))$.

Proof. We begin with showing $\liminf_{\epsilon \to 0} R_{lb}(\mathbf{D} + \epsilon\mathbf{1})$. Similar to proof of Theorem 13, first we consider $R_{lb}(\mathbf{D} + \epsilon\mathbf{1})$. Note that LP constraints of $R_{lb}(\mathbf{D} + \epsilon\mathbf{1})$ is for all couplings of the random variables and we can include any valid generalized side information to the optimization. Hence for a given $\epsilon > 0$ we can write,

$$R_{lb}(\mathbf{D} + \epsilon\mathbf{1}) \geq \inf_{V \in C(X)} \inf_{U.:C(X) \to C(X,V)} R_{lb}^{LP}(\epsilon) \tag{4.39}$$

where the *LP* constraints for the random variables $(X, a(X), Y_1, Y_2)$, are as in the problem description. Now we find a lower bound to the $R_{lb}^{LP}(\epsilon)$ in (4.39) by utilizing some of the LP constraints. Note that we can write

$K(\emptyset) \geq K(Y_1)$ by *(monotonicity)*

$K(Y_1) \geq K(a(X), Y_1) + H(a(X)|Y_1) - \delta(\epsilon)$, by *(monotonicity+)*,

Fano's inequality, and $\delta(\epsilon) > 0$,

$K(a(X), Y_1) \geq K(a(X), Y_2)$ by *(monotonicity)*,

$K(a(X), Y_2) \geq I(X; V, U_{a(X)Y_2}|a(X), Y_2)$ by *(monotonicity+)* and $K(X, a(X), Y_2) = 0$.

Hence, $R_{lb}(\mathbf{D} + \epsilon\mathbf{1})$ is lower bounded by

$$\inf_{V \in C(X)} \inf_{U.:C(X) \to C(X,V)} H(a(X)|Y_1) + I(X; V, U_{a(X)Y_2}|a(X), Y_2) - \delta(\epsilon).$$

By finding the cardinality constraint on $(V, U_{a(X)Y_2})$ and letting $\epsilon \to 0$, we have the result.

Now we show that $R_{ach}(\mathbf{D}) \leq R^{LN}(\mathbf{D})$. By selecting the auxiliary random variables $U_{\{1,2\}} = a(X)$, $U_{\{1\}} = \emptyset$ and $U_{\{2\}} = W$ in Lemma 9 and imposing cardinality bound constraint $|\mathcal{W}| \leq |\mathcal{X}| + 1$, we have $R_{ach}(\mathbf{D}) \leq R^{LN}(\mathbf{D})$. □

4.6.3 Odd-cycle Index Coding Problem

The source $\mathbf{X} = (X_1, \ldots, X_m)$, where $m \geq 5$ is an odd number, is i.i.d Bernoulli $(\frac{1}{2})$ bits. The side information at decoder i, $i \in [m]$ is $\mathbf{Y_i} = (X_{i-1}, X_{i+1})$, where + and

– in subscripts are modulo-m operations, and decoder i wishes to reconstruct X_i with a vanishing block error probability.

Although the achievability result Theorem 9 is for per-letter distortion constraints, it can be modified to handle block error probabilities. Let $v \in \mathscr{V}$ be as follows. When we sort the elements $\mathscr{S}_i$ and $\mathscr{S}_j$ in increasing order, the resulting number obtained by concatenating the sorted elements of $\mathscr{S}_i$ is greater than that of $\mathscr{S}_j$ for all $i < j$. To illustrate, let us consider three decoder case. Then $v = \{\{1,2,3\},\{2,3\},\{1,3\},\{1,2\},\{3\},\{2\},\{1\}\}$. Then we select the messages $\mathbf{U}_{\mathscr{S}_j}$, $\mathscr{S}_j \in v$ such that

$$\mathbf{U}_{jk} = (X_j, X_k) \text{ for } j \in [m], k \equiv j+1 \mod m \tag{4.40}$$

and all the other messages $\mathbf{U}_{\mathscr{S}_\mathbf{j}} = \emptyset$. [10] Let $j \in [m], i \equiv j-1 \mod m, k \equiv j+1 \mod m, l \equiv k+1 \mod m$. Then from the conditions in C_{ach}^{LP}, we can write

$$\begin{aligned}
R_{jk} &\geq I(\mathbf{X};\mathbf{U}_{jk}) - R'_{jk}, \text{ from condition 2) of } C_{ach}^{LP} \\
&= 2 - R'_{jk}, \\
R'_{jk} &\leq \min\{I\left(\mathbf{U}_{jk};\mathbf{U}_{ij},\mathbf{Y_j}\right), I\left(\mathbf{U}_{jk};\mathbf{U}_{kl},\mathbf{Y_k}\right)\}, \text{ from condition 3) of } C_{ach}^{LP} \\
&= 2, \\
R'_{ij} + R'_{jk} &\leq I\left(\mathbf{U}_{ij};\mathbf{Y_j}\right) + I\left(\mathbf{U}_{jk};\mathbf{U}_{ij},\mathbf{Y_j}\right), \text{ from condition 3) of } C_{ach}^{LP} \\
&= 3.
\end{aligned} \tag{4.41}$$

Then selecting $R'_{jk} = \frac{3}{2}$ and $R_{jk} = \frac{1}{2}$ satisfies the conditions of C_{ach}^{LP}. Hence, rate $\frac{m}{2}$ is achievable. Also, in [11] it is shown that LP lower bound in Theorem 15 gives $\frac{m}{2}$ for the zero error case. From Theorem 16, we can conclude that R_{lb}^{I} lower bound also gives $\frac{m}{2}$ which is the optimal rate for this problem.

[10] We represent $\mathbf{U}_{\{j,k\}}$ as $\mathbf{U}_{jk}$ for ease of notation.

4.6.4 Odd-cycle Gaussian Rate Distortion Problem

We finish with an instance that seems not to be solvable with using existing lower bounds discussed in Section 4.5. The problem setting we consider is analogous to the odd-cycle index coding problem [11], by taking each source component as an independent Gaussian random variable instead of uniform binary bits and considering mean square error (MSE) distortion constraint on reconstructions. Hence, we call it the *odd-cycle Gaussian* problem from now on. Specifically, the source $\mathbf{X} = (X_1, \ldots, X_m)$, where $m \geq 5$ is an odd number, is a vector Gaussian such that each component is independent of others with unit variance. The side information at decoder i, $i \in [m]$ is $\mathbf{Y_i} = (X_{i-1}, X_{i+1})$, where $+$ and $-$ in subscripts are modulo-m operations, and decoder i wishes to reconstruct X_i subject to MSE distortion constraints, i.e. $E[(X_i - \widehat{X}_i)^2] \leq D$ for all $i \in [m]$.

Theorem 20. *The rate distortion function, $R^{IG}(\mathbf{D})$, is*

$$R^{IG}(\mathbf{D}) = \frac{m}{4} \log \frac{1}{D}. \tag{4.42}$$

Proof of Theorem 20. Achievability: The achievability argument is obtained by using Theorem 10. Let $v \in \mathscr{V}$ be as in odd-cycle index coding problem subsection of Section 4.3. We select the messages $U_{\mathscr{S}_j}$ such that

$$\mathbf{U}_{jk} = (X_j + N_j, X_k + \bar{N}_k) \text{ for } j \in [m], k \equiv j+1 \mod m \tag{4.43}$$

and all the other messages $U_{\mathscr{S}_j}$ are degenerate.[11] Here $(N_i, \bar{N}_i)$, $i \in [m]$ are Gaussian random variables with variance $K_{N_i} = K_{\bar{N}_i} = \frac{2D}{1-D}$ and all $N_i, \bar{N}_i$'s are independent of each other and the source $\mathbf{X}$. All $\mathbf{U}_{\mathscr{S}_j}$ satisfy conditions 1), 2) and 3) of $C^G_{ach,v}(\mathbf{D})$ as well as condition 4) of $C^G_{ach,v}(\mathbf{D})$ since $K_{X_j|\mathbf{U}_{jk},\mathbf{U}_{ij},\mathbf{Y_j}} = (K_{X_j}^{-1} + K_{N_j}^{-1} + K_{\bar{N}_j}^{-1})^{-1} =$

[11] We represent $U_{\{j,k\}}$ as U_{jk} for ease of notation.

D, where $i = j-1 \mod m$. Let $j \in [m], i \equiv j-1 \mod m, k \equiv j+1 \mod m, l \equiv k+1 \mod m$. Then from the conditions in C_{ach}^{LP}, we can write

$$R_{jk} \geq I(\mathbf{X};\mathbf{U}_{jk}) - R'_{jk}, \text{ from condition 2) of } C_{ach}^{LP} \tag{4.44}$$

and since any disjoint sets of $\mathbf{U}_{\mathscr{S}_j}$ are conditionaly independent of each other given $\mathbf{X}$.

$$R'_{jk} \leq \min\{I\left(\mathbf{U}_{jk};\mathbf{U}_{ij},\mathbf{Y_j}\right), I\left(\mathbf{U}_{jk};\mathbf{U}_{kl},\mathbf{Y_k}\right)\}, \tag{4.45}$$

from condition 3) of C_{ach}^{LP}.

$$R'_{ij} + R'_{jk} \leq I\left(\mathbf{U}_{ij};\mathbf{Y_j}\right) + I\left(\mathbf{U}_{jk};\mathbf{U}_{ij},\mathbf{Y_j}\right), \text{ by condition 3) of } C_{ach}^{LP}. \tag{4.46}$$

Note that the terms inside the minimum in (4.45) are equal to each other and also the encoding order of the messages does not affect the right hand side of (4.46). Then using the chain rule, mutual information terms in (4.44)–(4.46) can be written as

$$\begin{aligned}
I(\mathbf{X};\mathbf{U}_{jk}) &= I(X_j; X_j + N_j) + I(X_k; X_k + \bar{N}_k) \\
&= \log \frac{1+D}{2D}. \\
I\left(\mathbf{U}_{jk};\mathbf{U}_{ij},\mathbf{Y_j}\right) &= I\left(\mathbf{U}_{jk};\mathbf{U}_{ij},X_i,X_k\right) \\
&= I(X_k + \bar{N}_k; X_k) + I\left(X_j + N_j; X_j + \bar{N}_j\right) \\
&= \frac{1}{2}\log\frac{1+D}{2D} + \frac{1}{2}\log\frac{(1+D)^2}{4D}. \\
I\left(\mathbf{U}_{ij};\mathbf{Y_j}\right) + I\left(\mathbf{U}_{jk};\mathbf{U}_{ij},\mathbf{Y_j}\right) &= I\left(\mathbf{U}_{ij};X_i,X_k\right) + I\left(\mathbf{U}_{jk};\mathbf{U}_{ij},\mathbf{Y_j}\right) \\
&= \frac{1}{2}\log\frac{1+D}{2D} + \frac{1}{2}\log\frac{1+D}{2D} + \frac{1}{2}\log\frac{(1+D)^2}{4D}.
\end{aligned}$$

Then selecting $R'_{jk} = \frac{1}{2}\log\frac{1+D}{2D} + \frac{1}{4}\log\frac{(1+D)^2}{4D}$ and $R_{jk} = \log\frac{1+D}{2D} - R'_{jk}$, $j \in [m]$, $k = j+1 \mod m$ satisfies (4.44)–(4.46) and we take all other rates $R_{\mathscr{S}_j}, R'_{\mathscr{S}_j}$ as 0.

Hence, the achievable rate is

$$\begin{aligned}\sum_{i=1}^{m} R_{ij} &= m\left(\log\frac{1+D}{2D} - \frac{1}{2}\log\frac{1+D}{2D} - \frac{1}{4}\log\frac{(1+D)^2}{4D}\right) \\ &= \frac{m}{4}\left(2\log\frac{1+D}{2D} - \log\frac{(1+D)^2}{4D}\right) \\ &= \frac{m}{4}\log\frac{1}{D}.\end{aligned}$$

Converse: We utilize the computable relaxation of $R_{lb}(\mathbf{D}+\epsilon\mathbf{1})$ in Theorem 12. Similar to the proof of [11, Theorem 5.1] we define the ordered sets:

$\mathbf{O} = \{X_i : i \equiv 1 \mod 2, i \neq m\}$, $\mathbf{O}^+ = \{X_i : i \leq m-2\}$

$\mathbf{E} = \{X_i : i \equiv 0 \mod 2\}$, $\mathbf{E}^+ = \{X_i : 2 \leq i \leq m-1\}$

$\mathbf{M} = \{X_i : 2 \leq i \leq m-2\}$, and $\mathbf{S} = \mathbf{X} \setminus (\mathbf{M} \cup X_m)$. Note that $(\mathbf{O}^+ \setminus \mathbf{O}) \cap (\mathbf{E}^+ \setminus \mathbf{E}) = \emptyset$ and $\mathbf{M} = (\mathbf{O}^+ \setminus \mathbf{O}) \cup (\mathbf{E}^+ \setminus \mathbf{E})$. Also, define $R(D) = \frac{1}{2}\log\frac{1}{D}$. Then using the conditions of the LP in Theorem 12 we can get the following inequalities

$$K(\emptyset) \geq K(\mathbf{O}) \quad \text{by } (\textit{monotonicity}) \tag{4.47}$$

$$K(\emptyset) \geq K(\mathbf{E}) \quad \text{by } (\textit{monotonicity}) \tag{4.48}$$

$$K(\emptyset) \geq K(X_m) \quad \text{by } (\textit{monotonicity}) \tag{4.49}$$

$$K(\mathbf{O}) \geq K(\mathbf{O}^+) + \sum_{X_i \in \mathbf{O}^+ \setminus \mathbf{O}} R(D+\epsilon) \tag{4.50}$$

$$K(\mathbf{E}) \geq K(\mathbf{E}^+) + \sum_{X_i \in \mathbf{E}^+ \setminus \mathbf{E}} R(D+\epsilon) \tag{4.51}$$

$$K(\mathbf{O}^+) + K(\mathbf{E}^+) \geq K(\mathbf{M}) + K(\mathbf{X}) + R(D+\epsilon) \tag{4.52}$$

$$K(\mathbf{M}) + K(X_m) \geq K(\emptyset) + K(\mathbf{X}) + \sum_{X_i \in \mathbf{S}} R(D+\epsilon) \tag{4.53}$$

where (4.50) is due to *monotonicity+* (i.e, $K(\mathbf{O}) \geq K(\mathbf{O}^+) + R(D_{\mathbf{O}}+\epsilon)$) and $R(D_{\mathbf{O}}+\epsilon) = \sum_{X_i \in \mathbf{O}^+ \setminus \mathbf{O}} R(D+\epsilon)$. Also (4.51) is due to *monotonicity+* (4.52) and (4.53) are due to *submodularity* and *monotonicity+*. If we add inequalities (4.47) - (4.53) side by side, we obtain $2K(\emptyset) \geq mR(D+\epsilon)$. Taking $\epsilon \to 0$ gives the result. □

CHAPTER 5

CHAPTER 2 OF APPENDIX

5.1

Proof of Theorem 1. Let $P(S, Y_i)$ for all $i \in [m]$ be given and let permutation $\sigma(i) = i$ for all $i \in [m]$. Let (R, D) be an achievable rate distortion pair and $\epsilon > 0$. Then there exists a $(n, M, D_1 + \epsilon, \ldots, D_m + \epsilon)$ code for some n such that $\log M \leq n(R + \epsilon)$.

We can write,

$$\begin{aligned} n(R + \epsilon) &\geq H(J) \\ &\geq I(S^n, Y_1^n, \ldots, Y_m^n; J) \end{aligned} \tag{5.1}$$

where J is the output of the encoder, $Y_1^n = (Y_{11}, \ldots, Y_{1n})$ (for the ease of notation we drop the parentheses around the index of the random variable unless it causes ambiguity), Y_{1i}^- denotes $(Y_{11}, \ldots, Y_{1(i-1)})$, and $Y_{1\tilde{i}}$ denotes all Y_1^n but Y_{1i}. Then if we apply the chain rule to $I(S^n, Y_1^n, \ldots, Y_m^n; J)$, right hand side of (5.1) equals

$$I(Y_1^n; J) + I(Y_2^n; J|Y_1^n) + \cdots + I(Y_m^n; J|Y_1^n, \ldots, Y_{m-1}^n) + I(S^n; J|Y_1^n, \ldots, Y_m^n) \tag{5.2}$$

$$\geq I(Y_2^n; J|Y_1^n) + \cdots + I(Y_m^n; J|Y_1^n, \ldots, Y_{m-1}^n) + I(S^n; J|Y_1^n, \ldots, Y_m^n)$$

$$\stackrel{a}{=} \sum_{i=1}^{n} \big[I(Y_{2i}; J|Y_{1\tilde{i}}, Y_{2i}^-, Y_{1i}) + \cdots + I(Y_{mi}; J|Y_{1\tilde{i}}, \ldots, Y_{(m-1)\tilde{i}}, Y_{mi}^-, Y_{1i}, \ldots, Y_{(m-1)i})$$

$$+ I(S_i; J|Y_{1\tilde{i}}, \ldots, Y_{m\tilde{i}}, S_i^-, Y_{1i}, \ldots, Y_{mi}) \big] \tag{5.3}$$

$$\stackrel{b}{=} \sum_{i=1}^{n} \big[I(Y_{2i}; J, Y_{1\tilde{i}}, Y_{2i}^-|Y_{1i}) + \cdots + I(Y_{mi}; J, Y_{1\tilde{i}}, \ldots, Y_{(m-1)\tilde{i}}, Y_{mi}^-|Y_{1i}, \ldots, Y_{(m-1)i})$$

$$+ I(S_i; J, Y_{1\tilde{i}}, \ldots, Y_{m\tilde{i}}, S_i^-|Y_{1i}, \ldots, Y_{mi}) \big] \tag{5.4}$$

$$\geq \sum_{i=1}^{n} \big[I(Y_{2i}; J, Y_{1\tilde{i}}|Y_{1i}) + \cdots + I(Y_{mi}; J, Y_{1\tilde{i}}, \ldots, Y_{(m-1)\tilde{i}}|Y_{1i}, \ldots, Y_{(m-1)i})$$

$$+ I(S_i; J, Y_{1\tilde{i}}, \ldots, Y_{m\tilde{i}}|Y_{1i}, \ldots, Y_{mi}) \big] \tag{5.5}$$

$$\stackrel{c}{=} \sum_{i=1}^{n} \big[I(Y_{2i}; J, Y_{1\tilde{i}}|Y_{1i}) + \cdots + I(Y_{mi}; J, Y_{1\tilde{i}}, \ldots, Y_{(m-1)\tilde{i}}|Y_{1i}, \ldots, Y_{(m-1)i})$$

$$+ I(S_i; J, Y_{1\tilde{i}}, \ldots, Y_{(m-1)\tilde{i}}|Y_{1i}, \ldots, Y_{mi})$$

$$+ I(S_i; Y_{m\tilde{i}}|J, Y_{1\tilde{i}}, \ldots, Y_{(m-1)\tilde{i}}, Y_{1i}, \ldots, Y_{mi}) \big]$$

where a is obtained by the chain rule, b is due to the fact that $(Y_{1i}, \ldots, Y_{mi}, S_i) \perp (Y_{1\tilde{i}}, \ldots, Y_{m\tilde{i}}, S_{1\tilde{i}})$, and c is due to the chain rule applied to the last term. When we combine the second-to-last and third-to-last term above, we get

$$
\begin{aligned}
& n(R+\epsilon) \\
& \geq \sum_{i=1}^{n} \left[I(Y_{2i}; J, Y_{1\tilde{i}} | Y_{1i}) + \cdots + I(Y_{(m-1)i}; J, Y_{1\tilde{i}}, \ldots, Y_{(m-2)\tilde{i}} | Y_{1i}, \ldots, Y_{(m-2)i}) \right. \\
& \qquad + I(S_i, Y_{mi}; J, Y_{1\tilde{i}}, \ldots, Y_{(m-1)\tilde{i}} | Y_{1i}, \ldots, Y_{(m-1)i}) \\
& \qquad \left. + I(S_i; Y_{m\tilde{i}} | J, Y_{1\tilde{i}}, \ldots, Y_{(m-1)\tilde{i}}, Y_{1i}, \ldots, Y_{mi}) \right] \\
& \geq \sum_{i=1}^{n} \left[I(Y_{2i}; J, Y_{1\tilde{i}} | Y_{1i}) + \cdots + I(Y_{(m-1)i}; J, Y_{1\tilde{i}}, \ldots, Y_{(m-2)\tilde{i}} | Y_{1i}, \ldots, Y_{(m-2)i}) \right. \\
& \qquad + I(S_i; J, Y_{1\tilde{i}}, \ldots, Y_{(m-1)\tilde{i}} | Y_{1i}, \ldots, Y_{(m-1)i}) \\
& \qquad \left. + I(S_i; Y_{m\tilde{i}} | J, Y_{1\tilde{i}}, \ldots, Y_{(m-1)\tilde{i}}, Y_{1i}, \ldots, Y_{mi}) \right]. \qquad (5.6)
\end{aligned}
$$

Now we apply the chain rule on the second-to-last term, giving

$$
\begin{aligned}
& n(R+\epsilon) \\
& \geq \sum_{i=1}^{n} \left[I(Y_{2i}; J, Y_{1\tilde{i}} | Y_{1i}) + \cdots + I(Y_{(m-1)i}; J, Y_{1\tilde{i}}, \ldots, Y_{(m-2)\tilde{i}} | Y_{1i}, \ldots, Y_{(m-2)i}) \right. \\
& \qquad + I(S_i; J, Y_{1\tilde{i}}, \ldots, Y_{(m-2)\tilde{i}} | Y_{1i}, \ldots, Y_{(m-1)i}) \\
& \qquad + I(S_i; Y_{(m-1)\tilde{i}} | J, Y_{1\tilde{i}}, \ldots, Y_{(m-2)\tilde{i}}, Y_{1i}, \ldots, Y_{(m-1)i}) \\
& \qquad \left. + I(S_i; Y_{m\tilde{i}} | J, Y_{1\tilde{i}}, \ldots, Y_{(m-1)\tilde{i}}, Y_{1i}, \ldots, Y_{mi}) \right] \qquad (5.7)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{a}{=} \sum_{i=1}^{n} [I(Y_{2i}; J, Y_{1\tilde{i}}|Y_{1i}) + \cdots + I(Y_{(m-2)i}; J, Y_{1\tilde{i}}, \ldots, Y_{(m-3)\tilde{i}}|Y_{1i}, \ldots, Y_{(m-3)i}) \\
&\qquad + I(S_i, Y_{(m-1)i}; J, Y_{1\tilde{i}}, \ldots, Y_{(m-2)\tilde{i}}|Y_{1i}, \ldots, Y_{(m-2)i}) \\
&\qquad + I(S_i; Y_{(m-1)\tilde{i}}|J, Y_{1\tilde{i}}, \ldots, Y_{(m-2)\tilde{i}}, Y_{1i}, \ldots, Y_{(m-1)i}) \\
&\qquad + I(S_i; Y_{m\tilde{i}}|J, Y_{1\tilde{i}}, \ldots, Y_{(m-1)\tilde{i}}, Y_{1i}, \ldots, Y_{mi})] \\
&\geq \sum_{i=1}^{n} [I(Y_{2i}; J, Y_{1\tilde{i}}|Y_{1i}) + \cdots + I(Y_{(m-2)i}; J, Y_{1\tilde{i}}, \ldots, Y_{(m-3)\tilde{i}}|Y_{1i}, \ldots, Y_{(m-3)i}) \\
&\qquad + I(S_i; J, Y_{1\tilde{i}}, \ldots, Y_{(m-2)\tilde{i}}|Y_{1i}, \ldots, Y_{(m-2)i}) \\
&\qquad + I(S_i; Y_{(m-1)\tilde{i}}|J, Y_{1\tilde{i}}, \ldots, Y_{(m-2)\tilde{i}}, Y_{1i}, \ldots, Y_{(m-1)i}) \\
&\qquad + I(S_i; Y_{m\tilde{i}}|J, Y_{1\tilde{i}}, \ldots, Y_{(m-1)\tilde{i}}, Y_{1i}, \ldots, Y_{mi})] \qquad (5.8)
\end{aligned}
$$

where, a is obtained by combining third-to-last and fourth-to-last terms in (5.7). Note that (5.6) is obtained from (5.5) by applying a series of chain rules and term combinations. If we continue this procedure as we did while obtaining (5.8) from (5.6), we get

$$
\begin{aligned}
&R + \epsilon \\
&\geq \frac{1}{n} \sum_{i=1}^{n} [I(S_i; J, Y_{1\tilde{i}}|Y_{1i}) + I(S_i; Y_{2\tilde{i}}|J, Y_{1\tilde{i}}, Y_{1i}, Y_{2i}) + \cdots \\
&\qquad + I(S_i; Y_{m\tilde{i}}|J, Y_{1\tilde{i}}, \ldots, Y_{(m-1)\tilde{i}}, Y_{1i}, \ldots, Y_{mi})] \qquad (5.9) \\
&\overset{a}{=} \frac{1}{n} \sum_{i=1}^{n} [I(S_i; U'_{1i}|Y_{1i}) + I(S_i; U'_{2i}|U'_{1i}, Y_{1i}, Y_{2i}) + \cdots \\
&\qquad + I(S_i; U'_{mi}|U'_{1i}, \ldots, U'_{(m-1)i}, Y_{1i}, \ldots, Y_{mi})]
\end{aligned}
$$

$$
\begin{aligned}
&\overset{b}{=} \frac{1}{n}\sum_{i=1}^{n} [I(S_i; U'_{1i}|Y_{1i}, T = i) + I(S_i; U'_{2i}|U'_{1i}, Y_{1i}, Y_{2i}, T = i) + \cdots \\
&\qquad\qquad + I(S_i; U'_{mi}|U'_{1i}, \ldots, U'_{(m-1)i}, Y_{1i}, \ldots, Y_{mi}, T = i)] \\
&= I(S; U'_1|Y_1, T) + I(S; U'_2|U'_1, Y_1, Y_2, T) + \cdots + I(S; U'_m|U'_1, \ldots, U'_{(m-1)}, Y_1, \ldots, Y_m, T) \\
&\overset{c}{=} I(S; U'_1, T|Y_1) + I(S; U'_2, T|U'_1, Y_1, Y_2, T) + \cdots \\
&\qquad\qquad + I(S; U'_m, T|U'_1, \ldots, U'_{(m-1)}, Y_1, \ldots, Y_m, T) \\
&\overset{d}{=} I(S; U_1|Y_1) + I(S; U_2|U_1, Y_1, Y_2) + \cdots + I(S; U_m|U_1, \ldots, U_{(m-1)}, Y_1, \ldots, Y_m) \qquad (5.10)
\end{aligned}
$$

where,

a : $U'_{ji} = (J, Y_{ji})$, $\forall j \in [m]$.

b : T is a random variable uniformly distributed on $[n]$ and independent of all source and side information variables and the U'_{ji}s.

c : T is independent of all source and side information variables.

d: Denote (U'_i, T) as U_i for all $i \in [m]$.

Note that $(U'_{1i}, \ldots, U'_{mi}) \leftrightarrow S_i \leftrightarrow (Y_{1i}, \ldots, Y_{mi})$ for all $i \in [n]$, giving $(U'_1, \ldots, U'_m) \leftrightarrow S \leftrightarrow (Y_1, \ldots, Y_m)$. Hence, $(U_1, \ldots, U_m) \leftrightarrow S \leftrightarrow (Y_1, \ldots, Y_m)$, i.e., $(U_1, \ldots, U_m)$ satisfies the condition 3). Also, since $(U'_{ji}, Y_{ji}) = (J, Y_j^n)$, there exists functions $g'_j(U'_j, Y_j)$ such that $\mathbb{E}[d(S, g'_j(U'_j, Y_j))] \leq D_j + \epsilon$, for all $j \in [m]$. Then we can conclude that $(U_1, \ldots, U_m)$ satisfies the condition 4). By Lemma 13 in the Appendix 5.1, we can obtain the cardinality bounds on $(U_1, \ldots, U_m)$ as in condition 5). Then we minimize the right hand side of (5.10) over $(U_1, \ldots, U_m)$. Also, note that, we fixed the permutation as $\sigma(i) = i$ for all $i \in [m]$ and to get (5.2), we applied the chain rule to (5.1) in the following order. We started with Y_1^n then continued with $Y_2^n, \ldots, Y_m^n$ and lastly we had S^n. Since we have m decoders with side information, we can get $m!$ different permutations. Hence, applying a similar procedure to all permutations, we get $m!$ lower bounds. By taking their maximum, we obtain a lower bound to $R(D)$. Lastly, since the problem can be described by specifying only the marginal $P(S, Y_i)$'s, we optimize it over the set of joint distributions such that the marginal $P(S, Y_i)$'s are the same. This gives us $R_{MLB}(D + \epsilon\mathbf{1})$, where $\mathbf{1}$ denotes the $m \times 1$ vector with all components 1.

Hence we have,

$$R(D) \geq R_{MLB}(D + \epsilon\mathbf{1}) - \epsilon. \tag{5.11}$$

Lemma 12. *$R_{MLB}(D + \epsilon\mathbf{1})$ of Theorem 1 is continuous in ϵ from the right at $\epsilon = 0$.*

Proof of Lemma 12. First we show that for a given joint distribution of source and side information $P(S, Y_1, \ldots, Y_m)$, $\bar{R}_\sigma(D + \epsilon\mathbf{1})$ is continuous in ϵ from right for a given permutation $\sigma(.)$.

Let ϵ_k be a monotonically decreasing sequence converging to 0 and let $U_1(D + \epsilon_k\mathbf{1}), \ldots, U_m(D+\epsilon_k\mathbf{1})$ denote an optimal $(U_1, \ldots, U_m)$ which gives $\bar{R}_\sigma(D+\epsilon_k\mathbf{1})$. Since the cardinalities of $(U_1, \ldots, U_m)$ are finite, together with the conditions 3) and 4), we have an optimization over a compact set. Then, we can find a convergent subsequence ϵ_{s_k} such that $U_1(D+\epsilon_{s_k}\mathbf{1}), \ldots, U_m(D+\epsilon_{s_k}\mathbf{1})$ converges to a $(U_1, \ldots, U_m)$ which is feasible for the distortion D. Hence we have

$$\liminf_{\epsilon \to 0} \bar{R}_\sigma(D + \epsilon\mathbf{1}) \geq \bar{R}_\sigma(D).$$

Also, since $\bar{R}_\sigma(D)$ is non increasing function of D we can write

$$\limsup_{\epsilon \to 0} \bar{R}_\sigma(D + \epsilon\mathbf{1}) \leq \bar{R}_\sigma(D),$$

concluding that $\bar{R}_\sigma(D+\epsilon\mathbf{1})$ is continuous in ϵ from the right for a given permutation $\sigma(.)$. Then, since finite maximum of functions that are continuous from the right is also continuous from the right, $\max_{\sigma(.)} \bar{R}_\sigma(D+\epsilon\mathbf{1})$ is continuous in ϵ from right.

Hence, now we show that $\sup_{\bar{P}} \max_{\sigma(.)} \bar{R}_\sigma(D + \epsilon\mathbf{1})$ is continuous in ϵ from the right. Let us temporarily denote $\max_{\sigma(.)} \bar{R}_\sigma(D)$ as $\bar{R}(\bar{P}, D)$ to indicate the dependence on $\bar{P}$. Then for any $\epsilon > 0$ and any $\bar{P}$, we have $\bar{R}(\bar{P}, D+\epsilon\mathbf{1}) \leq \bar{R}(\bar{P}, D)$. Hence we have

$$\limsup_{\epsilon\to 0} \sup_{\bar{P}} \bar{R}(\bar{P}, D + \epsilon\mathbf{1}) \leq \sup_{\bar{P}} \bar{R}(\bar{P}, D). \tag{5.12}$$

Now, we fix any $\delta > 0$. Then there exists P' such that $\bar{R}(P', D) \geq \sup_{\bar{P}} \bar{R}(\bar{P}, D) - \frac{\delta}{2}$. Let $\epsilon > 0$ be such that $\bar{R}(P', D + \epsilon\mathbf{1}) \geq \bar{R}(P', D) - \frac{\delta}{2}$. Then for any $0 < \epsilon' < \epsilon$ we can write

$$\begin{aligned}
\sup_{\bar{P}} \bar{R}(\bar{P}, D) &\leq \bar{R}(P', D) + \frac{\delta}{2} \\
&\leq \bar{R}(P', D + \epsilon'\mathbf{1}) + \delta \\
&\leq \sup_{\bar{P}} \bar{R}(\bar{P}, D + \epsilon'\mathbf{1}) + \delta,
\end{aligned}$$

implying that

$$\sup_{\bar{P}} \bar{R}(\bar{P}, D) - \delta \leq \liminf_{\epsilon'\to 0} \sup_{\bar{P}} \bar{R}(\bar{P}, D + \epsilon'\mathbf{1}). \tag{5.13}$$

Since $\delta > 0$ was arbitrary, (5.12) and (5.13) give the result. □

By Lemma 12, when ϵ goes to 0, (5.11) becomes

$$R(D) \geq R_{MLB}(D). \tag{5.14}$$

□

Proof of Theorem 3.

1. Achievability:

We utilize the achievable scheme of Proposition 1 and apply the CAPM for selection of U_I to get an explicit expression.

Step 1: Note that there is no demand related to G_m since all decoders have it as side information. We place all demands of all decoders related to G_{m-1} and G_0 into $U_{[m]}$. The remaining demands are subsets of G_{m-2}, i.e., components that $m-2$ of decoders have. For any $\mathbf{S_J} \in G_{m-2}$, where $J = \{\alpha, \beta\}$, $\alpha \neq \beta$ Decoder α and Decoder β are the two decoders that do not have $\mathbf{S_J}$ as side information. Then we place $f_{\{\alpha,\beta\}J}$ to $U_{[m]}$. Also, we place $f_{\alpha J}$ and $f_{\beta J}$ to level $m-1$ messages. Since there is no demand related to $G_i, \forall i \in [m-3]$, all messages $U_I, |I| \leq m-2$ will be empty. This completes the Step 1.

Step 2 *and* 3: To determine the leftover bits in Step 2 and bits to be XORed in Step 3 we write the demands in the following way: Note that there are $m(m-1)$ different non overlapping pairs of demands related to G_{m-2} since $|G_{m-2}| = \frac{m(m-1)}{2}$ and there are two demands $f_{\alpha J}$ and $f_{\beta J}$ for each $\mathbf{S}_{\{\alpha,\beta\}} \in G_{m-2}$. Also, note that for each Decoder α there are $m-1$ non overlapping demands, $f_{\alpha J}$ related to G_{m-2}. Therefore, we can put all those non overlapping demands into a matrix A with m rows and $m-1$ columns in the following way. α^{th} row, denoted by A_α, consists of demands $f_{\beta\{\alpha,\beta\}}$ where β runs over the set $[m]\backslash\{\alpha\}$. Note that A_α does not contain any demand from the Decoder α. Also, for each $f_{\beta\{\alpha,\beta\}}$, all entries of A_α other than $f_{\beta\{\alpha,\beta\}}$ exist as side information at Decoder β. Hence, we observe that all non overlapping demands which are related to G_{m-2} and placed in U_{α^c} at Step 1 are in A_α. Also, there is no other type of demand in level $m-1$ messages.

Lastly, as the size of each demand in A_α can be different, we arrange the entries in A_α in an increasing order with respect to their sizes. If two demands are in equal size, which one is put first does not matter. This completes the construction of the matrix A.

For each A_α in A we apply the following $\oplus$ operation.

Definition 17. *Let $a_i, i \in \{1, \ldots, m-1\}$ be vectors. Assume without loss of generality that $l_1 \leq l_2 \leq, \ldots, \leq l_{m-1}$ where $l_i = |a_i|$ denotes the number of elements in a_i. Then,*

$$(a_1, a_2, \ldots, a_{m-1})^{\oplus} = (a_1^{\oplus}, \ldots, a_{m-1}^{\oplus})$$

where

$$\begin{aligned}(a_1^{\oplus}, \ldots, a_{m-1}^{\oplus}) = (&a_1 \oplus (a_2)_{l_1} \oplus \cdots \oplus (a_{m-1})_{l_1},\\ &(a_2)_{l_2-l_1} \oplus \cdots \oplus (a_{m-1})_{l_2-l_1},\\ &\cdots,\\ &(a_{m-1})_{l_{m-1}-l_{m-2}}),\end{aligned}$$

and where $(a_i)_{l_j-l_k}$ denotes the vector consisting of components of a_i from $(l_k+1)^{th}$ to l_j^{th} component.

Note that all the components in $A_{\alpha 2}^{\oplus}, \ldots, A_{\alpha(m-1)}^{\oplus}$ are leftover bits and moved to $U_{[m]}$. Then for each $A_{\alpha}^{\oplus} = (A_{\alpha 1}^{\oplus}, \ldots, A_{\alpha(m-1)}^{\oplus})$, all the components in $A_{\alpha 1}^{\oplus}$ (*i.e.*, $A_{\alpha 1}, (A_{\alpha 2})_{|A_{\alpha 1}|}, \ldots, (A_{\alpha(m-1)})_{|A_{\alpha 1}|}$) remain in U_{α^c}. This concludes Step 2.

Lastly by Step 3, we have $U_{\alpha^c} = A_{\alpha 1}^{\oplus}$, for all $\alpha \in [m]$, and $A_{\alpha 2}^{\oplus}, \ldots, A_{\alpha(m-1)}^{\oplus}$ are in $U_{[m]}$. This concludes CAPM. Also for the ease of notation, when we write $A^{\oplus} \backslash \{\cup_{\alpha \in [m]} A_{\alpha 1}^{\oplus}\}$, we will mean the vector $A^{\oplus}$ with the components $A_{\alpha 1}^{\oplus}$ for all $\alpha \in [m]$ removed.

Hence,

$$\begin{aligned}&U_{[m]} = \mathbf{f_1} \backslash \{\cup_{\{1,\beta\} \subseteq [m]} f_{1\{1,\beta\}}\}, \mathbf{f_2} \backslash \{\cup_{\{2,\beta\} \subseteq [m]} f_{2\{2,\beta\}}\}, \ldots, \mathbf{f_m} \backslash \{\cup_{\{m,\beta\} \subseteq [m]} f_{m\{m,\beta\}}\}, A^{\oplus} \backslash \{\cup_{\alpha \in [m]} A_{\alpha 1}^{\oplus}\}\\ &U_{\alpha^c} = A_{\alpha 1}^{\oplus} \quad \forall \alpha \in [m],\end{aligned}$$

and we can write the achievable rate R_{CAPM} as

$$\begin{aligned}
&\max_{k\in[m]}\{H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]}f_{1\{1,\beta\}}\}, \mathbf{f_2}\backslash\{\cup_{\{2,\beta\}\subseteq[m]}f_{2\{2,\beta\}}\}, \dots, \\
&\qquad \mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]}f_{m\{m,\beta\}}\}, A^{\oplus}\backslash\{\cup_{\alpha\in[m]}A^{\oplus}_{\alpha 1}\}|\mathbf{Y_k})\} \\
&+\max_{k\in\{1\}^{\mathsf{c}}}\{H(A^{\oplus}_{11}|\mathbf{Y_k})\}+\cdots+\max_{k\in\{m\}^{\mathsf{c}}}\{H(A^{\oplus}_{m1}|\mathbf{Y_k})\}.
\end{aligned} \tag{5.15}$$

Note that

$$H(A_{\alpha 1}, (A_{\alpha 2})_{|A_{\alpha 1}|}, \dots, (A_{\alpha(m-1)})_{|A_{\alpha 1}|}|\mathbf{Y_k}) \tag{5.16}$$

$$\overset{a}{=} H(A_{\alpha 1}\oplus(A_{\alpha 2})_{|A_{\alpha 1}|}\oplus, \cdots, \oplus(A_{\alpha(m-1)})_{|A_{\alpha 1}|}|\mathbf{Y_k}) \tag{5.17}$$

$$= H(A^{\oplus}_{\alpha 1}|\mathbf{Y_k}), \tag{5.18}$$

a: $A_{\alpha i}\subseteq\mathbf{Y_k}$, for all $i\in[m-1]$ and $i\neq j$, where $A_{\alpha j}$ is the demand at Decoder k related to G_{m-2}.

Before rearranging the terms in R_{CAPM} further, we would like to show the following relations.

From the Definition 17, we know that for all $k\in[m]\setminus\{\alpha\}$ the conditional entropy

$$\begin{aligned}
&H(A^{\oplus}_{\alpha 1}|\mathbf{Y_k}) \\
&= H((A_{\alpha j})_{|A_{\alpha 1}|}|\mathbf{Y_k}) \\
&= H((A_{\alpha j})_{|A_{\alpha 1}|}) = |A_{\alpha 1}| = \min_{l\in[m-1]}|A_{\alpha l}| \text{ bits},
\end{aligned} \tag{5.19}$$

where $A_{\alpha j}$ is the demand at Decoder k related to G_{m-2}. Hence we get,

$$\begin{aligned}
H(U_{\alpha^{\mathsf{c}}}|\mathbf{Y_k}) &= H(U_{\alpha^{\mathsf{c}}}|\mathbf{Y_j}) \\
&= \min_{l\in[m-1]}|A_{\alpha l}| \text{ bits}, \forall k, j\in\{\alpha\}^{\mathsf{c}}.
\end{aligned} \tag{5.20}$$

Moreover, by Definition 17, for all $\alpha \in [m]$, we can write

$$\begin{aligned}
H(A^{\oplus}\backslash A^{\oplus}_{\alpha 1}|\mathbf{Y}_{\alpha}) &= H(A^{\oplus}\backslash A^{\oplus}_{\alpha}, (A^{\oplus}_{\alpha 2}, \ldots, A^{\oplus}_{\alpha(m-1)})|\mathbf{Y}_{\alpha}) \\
&= H(A^{\oplus}\backslash A^{\oplus}_{\alpha}|\mathbf{Y}_{\alpha}) + H(A^{\oplus}_{\alpha 2}, \ldots, A^{\oplus}_{\alpha(m-1)}|\mathbf{Y}_{\alpha}) \\
&= H(\cup_{\{\alpha,\beta\}\subseteq[m]} f_{\alpha\{\alpha,\beta\}}|\mathbf{Y}_{\alpha}) + H(A^{\oplus}_{\alpha 2}, \ldots, A^{\oplus}_{\alpha(m-1)}) \\
&= H(\cup_{\{\alpha,\beta\}\subseteq[m]} f_{\alpha\{\alpha,\beta\}}|\mathbf{Y}_{\alpha}) + (\max_{j} |A_{\alpha j}| - \min_{j} |A_{\alpha j}|). \qquad (5.21)
\end{aligned}$$

From (5.20), $H(U_{\alpha^{\mathsf{c}}}|\mathbf{Y_k}) = H(U_{\alpha^{\mathsf{c}}}|\mathbf{Y_j})$ for all $k, j \in \{\alpha\}^{\mathsf{c}}$. When we expand the terms inside $\max_{k\in[m]}$, we can write

$$\begin{aligned}
R_{CAPM} = \max \Big\{ & H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]} f_{1\{1,\beta\}}\}, \ldots, \mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]} f_{m\{m,\beta\}}\}, A^{\oplus}\backslash\{\cup_{\alpha\in[m]} A^{\oplus}_{\alpha 1}\}|\mathbf{Y_1}) \\
& + \sum_{\alpha\in\{1\}^{\mathsf{c}}} H(U_{\alpha^{\mathsf{c}}}|\mathbf{Y_1}) + \max_{k\in\{1\}^{\mathsf{c}}}\{H(U_{1^{\mathsf{c}}}|\mathbf{Y_k})\}, \\
& H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]} f_{1\{1,\beta\}}\}, \ldots, \mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]} f_{m\{m,\beta\}}\}, A^{\oplus}\backslash\{\cup_{\alpha\in[m]} A^{\oplus}_{\alpha 1}\}|\mathbf{Y_2}) \\
& + \sum_{\alpha\in\{2\}^{\mathsf{c}}} H(U_{\alpha^{\mathsf{c}}}|\mathbf{Y_2}) + \max_{k\in\{2\}^{\mathsf{c}}}\{H(U_{2^{\mathsf{c}}}|\mathbf{Y_k})\} \\
& , \ldots, \\
& H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]} f_{1\{1,\beta\}}\}, \ldots, \mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]} f_{m\{m,\beta\}}\}, A^{\oplus}\backslash\{\cup_{\alpha\in[m]} A^{\oplus}_{\alpha 1}\}|\mathbf{Y_m}) \\
& + \sum_{\alpha\in\{m\}^{\mathsf{c}}} H(U_{\alpha^{\mathsf{c}}}|\mathbf{Y_m}) + \max_{k\in\{m\}^{\mathsf{c}}}\{H(U_{m^{\mathsf{c}}}|\mathbf{Y_k})\}\Big\}.
\end{aligned}$$

Since $H(U_{\alpha^{\mathsf{c}}}|\mathbf{Y_k}) = H(A^{\oplus}_{\alpha 1}|\mathbf{Y_k})$ and $H(U_{\alpha^{\mathsf{c}}}|\mathbf{Y_k}) = |A^{\oplus}_{\alpha 1}|$ for all $k \in \{\alpha\}^{\mathsf{c}}$ from (5.19) and (5.20), we can further write R_{CAPM} as

$$\max\Big\{H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]}f_{1\{1,\beta\}}\},\ldots,\mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]}f_{m\{m,\beta\}}\},A^{\oplus}\backslash\{\cup_{\alpha\in[m]}A^{\oplus}_{\alpha 1}\}|\mathbf{Y_1})$$
$$+\sum_{\alpha\in\{1\}^c}H(A^{\oplus}_{\alpha 1}|\mathbf{Y_1})+|A^{\oplus}_{11}|,$$
$$H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]}f_{1\{1,\beta\}}\},\ldots,\mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]}f_{m\{m,\beta\}}\},A^{\oplus}\backslash\{\cup_{\alpha\in[m]}A^{\oplus}_{\alpha 1}\}|\mathbf{Y_2})$$
$$+\sum_{\alpha\in\{2\}^c}H(A^{\oplus}_{\alpha 1}|\mathbf{Y_2})+|A^{\oplus}_{21}|$$
$$,\ldots,$$
$$H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]}f_{1\{1,\beta\}}\},\ldots,\mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]}f_{m\{m,\beta\}}\},A^{\oplus}\backslash\{\cup_{\alpha\in[m]}A^{\oplus}_{\alpha 1}\}|\mathbf{Y_m})$$
$$+\sum_{\alpha\in\{m\}^c}H(A^{\oplus}_{\alpha 1}|\mathbf{Y_m})+|A^{\oplus}_{m1}|\Big\}. \tag{5.22}$$

Note that all $U_{i^c} = A^{\oplus}_{i1}$ are independent and for all collections of subsets $J_1,\ldots J_j$, $K_1,\ldots K_k$, $L_1,\ldots,L_l$, and all subsets $\{i_1,\ldots,i_p\} \subseteq [m]$, we have that $(U_{J_1},\ldots,U_{J_j})$ and $(U_{K_1},\ldots,U_{K_k})$ are conditionally independent given $((U_{L_1},\ldots,U_{L_l}),(\mathbf{Y_{i_1}},\ldots,\mathbf{Y_{i_p}}))$, provided that the collections $J_1,\ldots J_j$ and $K_1,\ldots K_k$ are disjoint. Then by applying chain rule to the expression in (5.22), we have

$$R_{CAPM}=\max\Big\{H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]}f_{1\{1,\beta\}}\},\ldots,\mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]}f_{m\{m,\beta\}}\},A^{\oplus}\backslash\{\cup_{\alpha\in[m]}A^{\oplus}_{\alpha 1}\}|\mathbf{Y_1})$$
$$+H(\cup_{\alpha\in\{1\}^c}A^{\oplus}_{\alpha 1}|\mathbf{Y_1})+|A^{\oplus}_{11}|,$$
$$H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]}f_{1\{1,\beta\}}\},\ldots,\mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]}f_{m\{m,\beta\}}\},A^{\oplus}\backslash\{\cup_{\alpha\in[m]}A^{\oplus}_{\alpha 1}\}|\mathbf{Y_2})$$
$$+H(\cup_{\alpha\in\{2\}^c}A^{\oplus}_{\alpha 1}|\mathbf{Y_2})+|A^{\oplus}_{21}|$$
$$,\ldots,$$
$$H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]}f_{1\{1,\beta\}}\},\ldots,\mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]}f_{m\{m,\beta\}}\},A^{\oplus}\backslash\{\cup_{\alpha\in[m]}A^{\oplus}_{\alpha 1}\}|\mathbf{Y_m})$$
$$+H(\cup_{\alpha\in\{m\}^c}A^{\oplus}_{\alpha 1}|\mathbf{Y_m})+|A^{\oplus}_{m1}|\Big\}.$$

Since all $U_{i^c} = A^{\oplus}_{i1}$ are independent and have the conditional independence properties as stated in the previous paragraph, by applying chain rule once more, we get that R_{CAPM} is equal to

$$\max\Big\{H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]}f_{1\{1,\beta\}}\},\ldots,\mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]}f_{m\{m,\beta\}}\}|\mathbf{Y_1})+H(A^{\oplus}\backslash A^{\oplus}_{11}|\mathbf{Y_1})+|A^{\oplus}_{11}|,$$
$$H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]}f_{1\{1,\beta\}}\},\ldots,\mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]}f_{m\{m,\beta\}}\}|\mathbf{Y_2})+H(A^{\oplus}\backslash A^{\oplus}_{21}|\mathbf{Y_2})+|A^{\oplus}_{21}|$$
$$,\ldots,$$
$$H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]}f_{1\{1,\beta\}}\},\ldots,\mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]}f_{m\{m,\beta\}}\}|\mathbf{Y_m})+H(A^{\oplus}\backslash A^{\oplus}_{m1}|\mathbf{Y_m})+|A^{\oplus}_{m1}|\Big\}.$$

Finally, from (5.21), that $|A_{\alpha 1}| = \min_j |A_{\alpha j}|$, and chain rule, we have the following expression for R_{CAPM}:

$$\max\Big\{H(\mathbf{f_1},\mathbf{f_2}\backslash\{\cup_{\{2,\beta\}\subseteq[m]}f_{2\{2,\beta\}}\},\ldots,\mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]}f_{m\{m,\beta\}}\}|\mathbf{Y_1})+\max_j|A_{1j}|,$$
$$H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]}f_{1\{1,\beta\}}\},\mathbf{f_2},\mathbf{f_3}\backslash\{\cup_{\{3,\beta\}\subseteq[m]}f_{3\{3,\beta\}}\},\ldots,\mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\subseteq[m]}f_{m\{m,\beta\}}\}|\mathbf{Y_2})+\max_j|A_{2j}|$$
$$,\ldots,$$
$$H(\mathbf{f_1}\backslash\{\cup_{\{1,\beta\}\subseteq[m]}f_{1\{1,\beta\}}\},\ldots,\mathbf{f_{m-1}}\backslash\{\cup_{\{m-1,\beta\}\subseteq[m]}f_{m-1\{m-1,\beta\}}\},\mathbf{f_m}|\mathbf{Y_m})+\max_j|A_{mj}|\Big\}.$$

Then we can write this achievable rate for the problem, R_{CAPM} in (5.15) as

$$R_{CAPM} = \max\{R_1,\ldots,R_m\}. \tag{5.23}$$

2. Converse:

Now, we find a lower bound which matches R_{CAPM} above by utilizing the converse result in section 2.5. Let us focus on (2.7). Here, we can write

$$\begin{aligned}
&H(\mathbf{f_2}|\mathbf{Y_2}, \mathbf{f_1}, \mathbf{Y_1}) \\
&= H(\mathbf{f_2}|\mathbf{f_1}, \mathbf{Y_1}) \\
&= H(\mathbf{f_2}\backslash\{\cup_{\{2,\beta\}\in[m]} f_{2\{2,\beta\}}\}, \{\cup_{\{2,\beta\}\in[m]} f_{2\{2,\beta\}}\}|\mathbf{f_1}, \mathbf{Y_1}) \\
&= H(\mathbf{f_2}\backslash\{\cup_{\{2,\beta\}\in[m]} f_{2\{2,\beta\}}\}, f_{2\{2,1\}}\}|\mathbf{f_1}, \mathbf{Y_1}) \\
&= H(\mathbf{f_2}\backslash\{\cup_{\{2,\beta\}\in[m]} f_{2\{2,\beta\}}\}, A_{1j}|\mathbf{f_1}, \mathbf{Y_1}), \\
&\text{where } A_{1j} = f_{2\{2,1\}} \\
&= H(\mathbf{f_2}\backslash\{\cup_{\{2,\beta\}\in[m]} f_{2\{2,\beta\}}\}|\mathbf{f_1}, \mathbf{Y_1}) + H(A_{1j}) \\
&= H(\mathbf{f_2}\backslash\{\cup_{\{2,\beta\}\in[m]} f_{2\{2,\beta\}}\}|\mathbf{f_1}, \mathbf{Y_1}) + |A_{1j}| \qquad (5.24)
\end{aligned}$$

and

$$\begin{aligned}
&H(\mathbf{f_3}|\mathbf{Y_3}, \mathbf{f_2}, \mathbf{Y_2}, \mathbf{f_1}, \mathbf{Y_1}) \\
&= H(\mathbf{f_3}|\mathbf{f_2}, \mathbf{Y_2}, \mathbf{f_1}, \mathbf{Y_1}) \\
&= H(\mathbf{f_3}\backslash\{\cup_{\{3,\beta\}\in[m]} f_{3\{3,\beta\}}\}, \{\cup_{\{3,\beta\}\in[m]} f_{3\{3,\beta\}}\}|\mathbf{f_2}, \mathbf{Y_2}, \mathbf{f_1}, \mathbf{Y_1}) \qquad (5.25) \\
&= H(\mathbf{f_3}\backslash\{\cup_{\{3,\beta\}\in[m]} f_{3\{3,\beta\}}\}|\mathbf{f_2}, \mathbf{Y_2}, \mathbf{f_1}, \mathbf{Y_1}), \\
&\text{since } \{\cup_{\{3,\beta\}\in[m]} f_{3\{3,\beta\}}\} \subset \{\mathbf{Y_1}, \mathbf{Y_2}\} \\
&= H(\mathbf{f_3}\backslash\{\cup_{\{3,\beta\}\in[m]} f_{3\{3,\beta\}}\}|\mathbf{f_2}\backslash\{\cup_{\{2,\beta\}\in[m]} f_{2\{2,\beta\}}\}, \mathbf{Y_2}, \mathbf{f_1}, \mathbf{Y_1}), \qquad (5.26) \\
&\text{since } \{\cup_{\{2,\beta\}\in[m]} f_{2\{2,\beta\}}\} \perp \mathbf{f_3}\backslash\{\cup_{\{3,\beta\}\in[m]} f_{3\{3,\beta\}}\}|\mathbf{Y_i}, \mathbf{f_j} \text{ for all } i, j \in [m].
\end{aligned}$$

Note that $\mathbf{f_3}$ can be written as

$$\{f_{3[m]}, f_{33}, \cup_{\{3,\beta\}\in[m]} f_{3\{3,\beta\}}, \cup_{\{3,\beta\}\in[m]} f_{\{3,\beta\}\{3,\beta\}}\}.$$

Then we get the following equality:

$$\mathbf{f_3}\backslash\{\cup_{\{3,\beta\}\in[m]} f_{3\{3,\beta\}}\} = \{f_{3[m]}, f_{33}, \cup_{\{3,\beta\}\in[m]} f_{\{3,\beta\}\{3,\beta\}}\}.$$

Note that $f_{\{3,1\}\{3,1\}} \subseteq \mathbf{f_1}$ and $f_{\{3,\beta\}\{3,\beta\}} \subseteq \mathbf{Y_1}$, for all $\beta \in [m]\backslash\{3\}$ and $\beta \neq 1$. Also, $f_{3[m]} \notin \mathbf{Y_2}$, $f_{33} \subseteq \mathbf{Y_1}$, $f_{33} \subseteq \mathbf{Y_2}$. As a result, (5.26) can be written as

$$H(\mathbf{f_3}\backslash\{\cup_{\{3,\beta\}\in[m]} f_{3\{3,\beta\}}\}|\mathbf{f_2}\backslash\{\cup_{\{2,\beta\}\in[m]} f_{2\{2,\beta\}}\}, \mathbf{f_1}, \mathbf{Y_1}).$$

By similar arguments as above, for $p > 2$ we can write

$$\begin{aligned}
&H(\mathbf{f_p}|\mathbf{Y_p}, \mathbf{f_{p-1}}, \mathbf{Y_{p-1}}, \ldots, \mathbf{f_1}, \mathbf{Y_1}) \\
&= H(\mathbf{f_p}|\mathbf{f_{p-1}}, \mathbf{Y_{p-1}}, \ldots, \mathbf{f_1}, \mathbf{Y_1}) \\
&= H(\mathbf{f_p}\backslash\{\cup_{\{p,\beta\}\in[m]} f_{p\{p,\beta\}}\}, \cup_{\{p,\beta\}\in[m]} f_{p\{p,\beta\}}|\mathbf{f_{p-1}}, \mathbf{Y_{p-1}}, \ldots, \mathbf{f_1}, \mathbf{Y_1}) \\
&\overset{a}{=} H(\mathbf{f_p}\backslash\{\cup_{\{p,\beta\}\in[m]} f_{p\{p,\beta\}}\}|\mathbf{f_{p-1}}, \mathbf{Y_{p-1}}, \ldots, \mathbf{f_1}, \mathbf{Y_1}) \\
&\overset{b}{=} H(\mathbf{f_p}\backslash\{\cup_{\{p,\beta\}\in[m]} f_{p\{p,\beta\}}\}|\mathbf{f_{p-1}}\backslash\{\cup_{\{p-1,\beta\}\in[m]} f_{p-1\{p-1,\beta\}}\}, \mathbf{Y_{p-1}}, \ldots, \\
&\qquad\qquad\qquad \mathbf{f_2}\backslash\{\cup_{\{2,\beta\}\in[m]} f_{2\{2,\beta\}}\}, \mathbf{Y_2}, \mathbf{f_1}, \mathbf{Y_1}) \\
&\overset{c}{=} H(\mathbf{f_p}\backslash\{\cup_{\{p,\beta\}\in[m]} f_{p\{p,\beta\}}\}|\mathbf{f_{p-1}}\backslash\{\cup_{\{p-1,\beta\}\in[m]} f_{p-1\{p-1,\beta\}}\}, \ldots, \\
&\qquad\qquad\qquad \mathbf{f_2}\backslash\{\cup_{\{2,\beta\}\in[m]} f_{2\{2,\beta\}}\}, \mathbf{f_1}, \mathbf{Y_1})
\end{aligned} \tag{5.27}$$

a: Since $\{\cup_{\{p,\beta\}\in[m]} f_{p\{p,\beta\}}\} \subset \{\mathbf{Y_i}, \mathbf{Y_j}\}$, $\forall i, j \in [m]$, $p \neq i, j$ and $i \neq j$.

b: Since $\{\cup_{\{\alpha,\beta\}\in[m]} f_{\alpha\{\alpha,\beta\}}\} \perp \mathbf{f}_\gamma\backslash\{\cup_{\{\gamma,\beta\}\in[m]} f_{\gamma\{\gamma,\beta\}}\} \quad |\mathbf{Y_i}, \mathbf{f_j}$, $\forall i, j, \alpha, \gamma \in [m]$.

c: Since $\mathbf{f_p} = \{f_{p[m]}, f_{pp}, \{\cup_{\{p,\beta\}\in[m]} f_{p\{p,\beta\}}\}, \{\cup_{\{p,\beta\}\in[m]} f_{\{p,\beta\}\{p,\beta\}}\}, \}$ and $\mathbf{f_p}\backslash\{\cup_{\{p,\beta\}\in[m]} f_{p\{p,\beta\}}\}$ equals $\{f_{p[m]}, f_{pp}, \{\cup_{\{p,\beta\}\in[m]} f_{\{p,\beta\}\{p,\beta\}}\}\}$, where
$f_{p[m]} \notin \mathbf{Y_i}, \forall i \in [m]$ and $f_{pp} \subseteq \mathbf{Y_i}$ $\forall i \in [m], i \neq p$. Also, $f_{\{p,1\}\{p,1\}} \subseteq \mathbf{f_1}$ and $f_{\{p,\beta\}\{p,\beta\}} \subseteq \mathbf{Y_1}$, $\forall \beta \in [m]\backslash\{p\}, \beta \neq 1$.

Hence, from (5.24) and (5.27), (2.7) can be written as

$$R \geq H(\mathbf{f_1}, \mathbf{f_2}\backslash\{\cup_{\{2,\beta\}\in[m]} f_{2\{2,\beta\}}\}, \ldots, \mathbf{f_m}\backslash\{\cup_{\{m,\beta\}\in[m]} f_{m\{m,\beta\}}\}|\mathbf{Y_1}) + |A_{1j}|, \tag{5.28}$$

where $A_{1j} = f_{2\{2,1\}}$.

By similar arguments used to obtain (5.28), we can write

$$R \geq H(\mathbf{f_1}, \mathbf{f_2} \backslash \{\cup_{\{2,\beta\}\in[m]} f_{2\{2,\beta\}}\}, \ldots, \mathbf{f_m} \backslash \{\cup_{\{m,\beta\}\in[m]} f_{m\{m,\beta\}}\} | \mathbf{Y_1}) + \max_j |A_{1j}|. \tag{5.29}$$

Note that the right hand side of (5.29) is R_1. Since the problem is symmetric, similarly we can get all R_i's. Then,

$$R \geq \max_i R_i = R_{CAPM} \tag{5.30}$$

proving that R_{CAPM} is optimal. This concludes the proof of the Theorem 3. □

Lemma 13. *The minimum in (2.2) is unaffected by the presence of the cardinality bounds in condition 5).*

Proof of Lemma 13. Without loss of generality let $\sigma(i) = i$ for all $i \in [m]$. Let $P_{U_1,\ldots,U_m,S}(u_1, \ldots, u_m, s)$ denote the joint distribution of $(U_1, \ldots, U_m, S)$. We follow a procedure similar to that of [5]. First we find the bound on the cardinality of U_1, then U_2, etc.

To begin with, we consider the following $(|\mathcal{S}| - 1) + 1 + m$ functions of $P_{U_2,\ldots,U_m,S|U_1}(., \ldots, .|u_1)$, denoted as g_s^0, $s \in |\mathcal{S}| - 1$ and $g_{lo}^0, g_{d_1}^0, \ldots, g_{d_m}^0$.

$$g_s^0(P_{U_2,\ldots,U_m,S|U_1}(., \ldots, .|u_1)) = \sum_{u_2,\ldots,u_m} P_{U_2,\ldots,U_m,S|U_1}(u_2, \ldots, u_m, s|u_1), \tag{5.31}$$

for $s = 1, \ldots, |\mathcal{S}| - 1$ and

$$\begin{aligned} &g_{lo}^0(P_{U_2,\ldots,U_m,S|U_1}(., \ldots, .|u_1)) \\ &= H(S|Y_1) - H(S|U_1 = u_1, Y_1) + I(S; U_2|U_1 = u_1, Y_1, Y_2) + \cdots \\ &\quad + I(S; U_m|U_1 = u_1, \ldots, U_{(m-1)}, Y_1, \ldots, Y_m), \end{aligned} \tag{5.32}$$

and

$$
\begin{aligned}
&g^0_{d_1}(P_{U_2,\dots,U_m,S|U_1}(.,\dots,.|u_1)) = \mathbb{E}[d(S, g_1(u_1, Y_1))|U_1 = u_1]\\
&\vdots\\
&g^0_{d_m}(P_{U_2,\dots,U_m,S|U_1}(.,\dots,.|u_1)) = \mathbb{E}[d(S, g_m(U_m, Y_m))|U_1 = u_1]
\end{aligned}
$$

Then by Carathéodory's theorem [49, Theorem 17.1] we can find a random variable U_1^1 with $|\mathcal{U}_1^1| \leq |\mathcal{S}| + m + 1$ and random variables $U_2^1, \dots, U_m^1$ where $P_{U_1^1,\dots,U_m^1,S}(u_1,\dots,u_m,s) = P_{U_1^1}(u_1)P_{U_2,\dots,U_m,S|U_1}(u_2,\dots,u_m,s|u_1)$ such that from (5.31) P_S is preserved and from (5.32)

$$
\begin{aligned}
&I(S;U_1^1|Y_1) + I(S;U_2^1|U_1^1,Y_1,Y_2) + \cdots + I(S;U_m^1|U_1^1,\dots,U_{(m-1)}^1,Y_1,\dots,Y_m)\\
&= I(S;U_1|Y_1) + I(S;U_2|U_1,Y_1,Y_2) + I(S;U_m|U_1,\dots,U_{(m-1)},Y_1,\dots,Y_m),
\end{aligned}
$$

and we have

$$
\begin{aligned}
&\mathbb{E}[d(S, g_1(U_1^1, Y_1)] = \mathbb{E}[d(S, g_1(U_1, Y_1))]\\
&\vdots\\
&\mathbb{E}[d(S, g_m(U_m^1, Y_1)] = \mathbb{E}[d(S, g_m(U_m, Y_1))].
\end{aligned}
$$

Now take the following $|\mathcal{U}_1||\mathcal{S}| + (m-1)$ functions of $P_{U_1^1,U_3^1,\dots,U_m^1,S|U_2^1}(.,\dots,.|u_2)$.

$$
g^1_s(P_{U_1^1,U_3^1,\dots,U_m^1,S|U_1^1}(.,\dots,.|u_2)) = \sum_{u_3,\dots,u_m} P_{U_1^1,U_3^1,\dots,U_m^1,S|U_2^1}(.,\dots,.|u_2), \tag{5.33}
$$

for $(u_1, s) = 1, \dots, |\mathcal{U}_1||\mathcal{S}| - 1$ and

$$
\begin{aligned}
&g^1_{lo}(P_{U_1^1,U_3^1,\dots,U_m^1,S|U_2^1}(.,\dots,.|u_2))\\
&= -H(S|U_1^1, U_2^1 = u_2, Y_1, Y_2) + I(S;U_3^1|U_1^1, U_2^1 = u_2, Y_1, Y_2, Y_3)\cdots\\
&\quad + I(S;U_m^1|U_1^1, U_2^1 = u_2, U_3^1 \dots, U_{(m-1)}^1, Y_1, \dots, Y_m),
\end{aligned} \tag{5.34}
$$

and

$$g^1_{d_2}(P_{U^1_1,U^1_3,\ldots,U^1_m,S|U^1_2}(.,\ldots,.|u_2)) = \mathbb{E}[d(S,g_2(u_2,Y_2)|U^1_2 = u_2]$$
$$\vdots$$
$$g^1_{d_m}(P_{U^1_1,U^1_3,\ldots,U^1_m,S|U^1_2}(.,\ldots,.|u_2)) = \mathbb{E}[d(S,g_m(U^1_m,Y_m)|U^1_2 = u_2].$$

Again by Carathéodory's theorem, there is a random variable U^2_2 with $|\mathcal{U}^2_2| \leq |\mathcal{U}_1||\mathcal{S}| + m$ and random variables $U^2_3,\ldots,U^2_m$ where $P_{U^1_1,U^2_2,\ldots,U^2_m,S}(u_1,\ldots,u_m,s)$ is equal to $P_{U^2_2}(u_2)P_{U^1_1,U^1_3,\ldots,U^1_m,S|U^1_2}(u_1,u_3,\ldots,u_m,s|u_2)$ such that $P_{U^1_1S}$ is preserved (from 5.33).

Since $P_{U^1_1S}$ is preserved, $\mathbb{E}[d(S,g_2(U^1_1,Y_1)]$, $H(S|U^1_1,Y_1,Y_2)$, and $I(S;U^1_1|Y_1)$ are preserved. Also, from (5.34) we have

$$\begin{aligned}
&I(S;U^1_1|Y_1) + I(S;U^2_2|U^1_1,Y_1,Y_2) + \cdots + I(S;U^2_m|U^1_1,U^2_2,\ldots,U^2_{(m-1)},Y_1,\ldots,Y_m)\\
&= I(S;U^1_1|Y_1) + I(S;U^1_2|U^1_1,Y_1,Y_2) + I(S;U^1_m|U^1_1,\ldots,U^1_{(m-1)},Y_1,\ldots,Y_m).
\end{aligned}$$

Lastly, we have the following equalities.

$$\mathbb{E}[d(S,g_2(U^2_2,Y_2)] = \mathbb{E}[d(S,g_2(U^1_2,Y_2))]$$
$$\vdots$$
$$\mathbb{E}[d(S,g_m(U^1_m,Y_m)] = \mathbb{E}[d(S,g_m(U^1_mY_m))].$$

By applying the above procedure to $U^2_3,\ldots,U^2_m$ consecutively and relabeling $(U^1_1,U^2_2,\ldots)$ as $(U_1,\ldots,U_m)$ we obtain the cardinality bounds as stated in the condition 5) of Theorem 2.1. □

CHAPTER 6

CHAPTER 3 OF APPENDIX

6.1

The aim of this appendix is to prove the following lemma.

Lemma 14 (Gaussian Variance-Drop Lemma)**.** *Let* $(\mathbf{W}, \mathbf{W_G}, \mathbf{X}, \widetilde{\mathbf{Z}}, \mathbf{Z})$ *be random vectors such that* $(\mathbf{W_G}, \mathbf{X}, \widetilde{\mathbf{Z}}, \mathbf{Z})$ *is jointly Gaussian,* $(\mathbf{W}, \mathbf{W_G}) \leftrightarrow \mathbf{X} \leftrightarrow \widetilde{\mathbf{Z}} \leftrightarrow \mathbf{Z}$ *and* $K_{\mathbf{X}|\mathbf{W},\widetilde{\mathbf{Z}}} \succ 0$. *If* $K_{\mathbf{X}|\mathbf{W_G},\mathbf{Z}} = K_{\mathbf{X}|\mathbf{W},\mathbf{Z}}$ *then* $K_{\mathbf{X}|\mathbf{W},\widetilde{\mathbf{Z}}} \preceq K_{\mathbf{X}|\mathbf{W_G},\widetilde{\mathbf{Z}}}$. *Also, if* $K_{\mathbf{X}|\mathbf{W},\widetilde{\mathbf{Z}}} = K_{\mathbf{X}|\mathbf{W_G},\widetilde{\mathbf{Z}}}$ *then* $K_{\mathbf{X}|\mathbf{W_G},\mathbf{Z}} \preceq K_{\mathbf{X}|\mathbf{W},\mathbf{Z}}$.

This lemma can be interpreted as follows. We view $\mathbf{X}$ as an underlying source of interest and $\mathbf{W}$, $\mathbf{W}_G$, $\tilde{\mathbf{Z}}$, and $\mathbf{Z}$ as "noisy observations" of $\mathbf{X}$. All except possibly $\mathbf{W}$ are jointly Gaussian. If $(\mathbf{W}, \mathbf{Z})$ and $(\mathbf{W}_G, \mathbf{Z})$ are equally-good observations, in terms of their error covariance matrix, then $(\mathbf{W}, \tilde{\mathbf{Z}})$ can only be better than $(\mathbf{W}_G, \tilde{\mathbf{Z}})$. That is, replacing $\mathbf{Z}$ with $\tilde{\mathbf{Z}}$ results in a "variance drop," and this drop is smallest in the Gaussian case.

To prove this result we will make use of the following technical lemma.

Lemma 15. *Let* $(\mathbf{X}, \widetilde{\mathbf{Z}}, \mathbf{Z})$ *be jointly Gaussian random vectors such that* $\mathbf{X} \leftrightarrow \widetilde{\mathbf{Z}} \leftrightarrow \mathbf{Z}$ *and* $K_{\mathbf{X}|\widetilde{\mathbf{Z}}} \succ 0$. *We can form a* $\widehat{\mathbf{Z}}$ *such that* $(\mathbf{X}, \widetilde{\mathbf{Z}}, \mathbf{Z}, \widehat{\mathbf{Z}})$ *is jointly Gaussian,* $\widehat{\mathbf{Z}} \leftrightarrow \mathbf{X} \leftrightarrow \mathbf{Z}$, *and* $E[\mathbf{X}|\mathbf{Z}, \widehat{\mathbf{Z}}] = E[\mathbf{X}|\mathbf{Z}, \widetilde{\mathbf{Z}}]$ *almost surely.*

Proof. Given such $(\mathbf{X}, \widetilde{\mathbf{Z}}, \mathbf{Z})$, we can create a $\bar{\mathbf{Z}}$ such that $\bar{\mathbf{Z}} = A_{\bar{z}}\mathbf{X} + \mathbf{N}_{\bar{\mathbf{z}}}$ where $\mathbf{N}_{\bar{\mathbf{z}}}$ is Gaussian, independent of $(\mathbf{X}, \mathbf{Z})$ and $K_{\mathbf{X}|\mathbf{Z},\bar{\mathbf{Z}}} = K_{\mathbf{X}|\mathbf{Z},\widetilde{\mathbf{Z}}} = K_{\mathbf{X}|\widetilde{\mathbf{Z}}}$. Since

$(\mathbf{X}, \mathbf{Z}, \bar{\mathbf{Z}}, E[\mathbf{X}|\mathbf{Z}, \bar{\mathbf{Z}}])$ are jointly Gaussian, we can write

$$\bar{\mathbf{Z}} = B \begin{pmatrix} \mathbf{X} \\ \mathbf{Z} \\ E[\mathbf{X}|\mathbf{Z}, \bar{\mathbf{Z}}] \end{pmatrix} + \mathbf{N}_{\bar{\mathbf{Z}}}',$$

for some matrix B where $\mathbf{N}_{\bar{\mathbf{Z}}}'$ is independent of $(\mathbf{X}, \mathbf{Z}, E[\mathbf{X}|\mathbf{Z}, \bar{\mathbf{Z}}])$ and Gaussian with some covariance matrix $K_{\mathbf{N}_{\bar{\mathbf{Z}}}'}$.

Observe that the orthogonality principle and the equation $K_{\mathbf{X}|\mathbf{Z},\bar{\mathbf{Z}}} = K_{\mathbf{X}|\mathbf{Z},\widetilde{\mathbf{Z}}}$ together imply that

$$K_{E[\mathbf{X}|\mathbf{Z},\bar{\mathbf{Z}}]} = K_{E[\mathbf{X}|\mathbf{Z},\widetilde{\mathbf{Z}}]}. \tag{6.1}$$

Orthogonality also implies that $K_{\mathbf{X}E[\mathbf{X}|\mathbf{Z},\bar{\mathbf{Z}}]} = K_{E[\mathbf{X}|\mathbf{Z},\bar{\mathbf{Z}}]}$ and $K_{\mathbf{X}E[\mathbf{X}|\mathbf{Z},\widetilde{\mathbf{Z}}]} = K_{E[\mathbf{X}|\mathbf{Z},\widetilde{\mathbf{Z}}]}$. Hence,

$$K_{\mathbf{X}E[\mathbf{X}|\mathbf{Z},\bar{\mathbf{Z}}]} = K_{\mathbf{X}E[\mathbf{X}|\mathbf{Z},\widetilde{\mathbf{Z}}]}. \tag{6.2}$$

Likewise, orthogonality implies that $K_{E[\mathbf{X}|\mathbf{Z},\widetilde{\mathbf{Z}}]\mathbf{Z}} = K_{\mathbf{X}\mathbf{Z}}$ and $K_{E[\mathbf{X}|\mathbf{Z},\bar{\mathbf{Z}}]\mathbf{Z}} = K_{\mathbf{X}\mathbf{Z}}$. Thus,

$$K_{E[\mathbf{X}|\mathbf{Z},\widetilde{\mathbf{Z}}]\mathbf{Z}} = K_{E[\mathbf{X}|\mathbf{Z},\bar{\mathbf{Z}}]\mathbf{Z}}. \tag{6.3}$$

Then (6.1), (6.2), and (6.3) imply that $(\mathbf{X}, \mathbf{Z}, E[\mathbf{X}|\mathbf{Z}, \bar{\mathbf{Z}}])$ and $(\mathbf{X}, \mathbf{Z}, E[\mathbf{X}|\mathbf{Z}, \widetilde{\mathbf{Z}}])$ are equal in distribution. Now given $(\mathbf{X}, \widetilde{\mathbf{Z}}, \mathbf{Z})$, create $\widehat{\mathbf{Z}}$ via

$$\widehat{\mathbf{Z}} = B \begin{pmatrix} \mathbf{X} \\ \mathbf{Z} \\ E[\mathbf{X}|\mathbf{Z}, \widetilde{\mathbf{Z}}] \end{pmatrix} + \mathbf{N}_{\bar{\mathbf{Z}}}',$$

where $\mathbf{N}_{\widehat{\mathbf{Z}}}'$ is Gaussian with covariance matrix $K_{\mathbf{N}_{\widehat{\mathbf{Z}}}'}$ and is independent of $(\mathbf{X}, E[\mathbf{X}|\mathbf{Z}, \widetilde{\mathbf{Z}}], \mathbf{Z})$. Then,

$$(\mathbf{X}, \mathbf{Z}, \widehat{\mathbf{Z}}, E[\mathbf{X}|\mathbf{Z}, \widetilde{\mathbf{Z}}]) = (\mathbf{X}, \mathbf{Z}, \bar{\mathbf{Z}}, E[\mathbf{X}|\mathbf{Z}, \bar{\mathbf{Z}}]), \text{in distribution}$$

and so $\widehat{\mathbf{Z}} \leftrightarrow \mathbf{X} \leftrightarrow \mathbf{Z}$, and $E[\mathbf{X}|\mathbf{Z}, \widehat{\mathbf{Z}}] = E[\mathbf{X}|\mathbf{Z}, \widetilde{\mathbf{Z}}]$ almost surely. □

Proof of Lemma 14. Let $(\mathbf{W}, \mathbf{W_G}, \mathbf{X}, \widetilde{\mathbf{Z}}, \mathbf{Z})$ be as in the statement. Then by Lemma 15, we can form a random vector $\widehat{\mathbf{Z}} = A_{\widehat{z}}\mathbf{X} + \mathbf{N}_{\widehat{\mathbf{Z}}}$, where $\mathbf{N}_{\widehat{\mathbf{Z}}}$ is independent of $(\mathbf{X}, \mathbf{Z})$, such that $\widehat{\mathbf{Z}} \leftrightarrow \mathbf{X} \leftrightarrow \mathbf{Z}$, $K_{\mathbf{X}|\mathbf{Z},\widehat{\mathbf{Z}}} = K_{\mathbf{X}|\mathbf{Z},\widetilde{\mathbf{Z}}} = K_{\mathbf{X}|\widetilde{\mathbf{Z}}}$ and $E[\mathbf{X}|\mathbf{Z}, \widetilde{\mathbf{Z}}] = E[\mathbf{X}|\mathbf{Z}, \widehat{\mathbf{Z}}]$ almost surely. Since for any $\mathbf{W}$ such that $\mathbf{W} \leftrightarrow \mathbf{X} \leftrightarrow (\widetilde{\mathbf{Z}}, \widehat{\mathbf{Z}}, \mathbf{Z})$ we have $K_{\mathbf{X}|\mathbf{W},\widetilde{\mathbf{Z}}} = K_{\mathbf{X}|\mathbf{W},\mathbf{Z},\widetilde{\mathbf{Z}}} = K_{\mathbf{X}|\mathbf{W},E[\mathbf{X}|\mathbf{Z},\widetilde{\mathbf{Z}}]} = K_{\mathbf{X}|\mathbf{W},E[\mathbf{X}|\mathbf{Z},\widehat{\mathbf{Z}}]} = K_{\mathbf{X}|\mathbf{W},\mathbf{Z},\widehat{\mathbf{Z}}}$, it suffices to prove the result for the special case in which $\widetilde{\mathbf{Z}} = (\widehat{\mathbf{Z}}, \mathbf{Z})$ so we shall assume that $\widetilde{\mathbf{Z}}$ has this form. Also, let $\widehat{\mathbf{X}} = E[\mathbf{X}|\mathbf{W}, \mathbf{Z}]$. We will write the covariance matrix of the best linear estimate of $\mathbf{X}$ using $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{Z}}$ in terms of $K_{\mathbf{X}|\mathbf{W},\mathbf{Z}}$ and $K_{\mathbf{X}|\widehat{\mathbf{Z}}}$ by applying the procedure of [50]. Then we can write

$$K(\mathbf{X}, \widehat{\mathbf{X}}, \widehat{\mathbf{Z}}) = \begin{pmatrix} K_{\mathbf{X}} & K_{\widehat{\mathbf{X}}} & K_{\mathbf{X}}A_{\widehat{z}}^T \\ K_{\widehat{\mathbf{X}}} & K_{\widehat{\mathbf{X}}} & K_{\widehat{\mathbf{X}}}A_{\widehat{z}}^T \\ A_{\widehat{z}}K_{\mathbf{X}} & A_{\widehat{z}}K_{\widehat{\mathbf{X}}} & A_{\widehat{z}}K_{\mathbf{X}}A_{\widehat{z}}^T + K_{\mathbf{N}_{\widehat{z}}} \end{pmatrix}$$

where $K_{\widehat{\mathbf{X}}} = (K_{\mathbf{X}} - K_{\mathbf{X}|\mathbf{W},\mathbf{Z}})$. Note that $K_{\widehat{\mathbf{X}}}$ may not be invertible meaning that some of the elements of $\widehat{\mathbf{X}}$ can be determined as linear combinations of others. Thus it is enough to consider only the components of $\widehat{\mathbf{X}}$ or linear combinations of them, denoted by $\bar{\mathbf{X}} = Q\widehat{\mathbf{X}}$, such that the resulting covariance matrix, denoted as $K_{\bar{\mathbf{X}}} = QK_{\widehat{\mathbf{X}}}Q^T$, is invertible. Then we can write,

$$K(\mathbf{X}, \bar{\mathbf{X}}, \widehat{\mathbf{Z}}) = \begin{pmatrix} K_{\mathbf{X}} & K_{\widehat{\mathbf{X}}}Q^T & K_{\mathbf{X}}A_{\widehat{z}}^T \\ QK_{\widehat{\mathbf{X}}} & K_{\bar{\mathbf{X}}} & QK_{\widehat{\mathbf{X}}}A_{\widehat{z}}^T \\ A_{\widehat{z}}K_{\mathbf{X}} & A_{\widehat{z}}K_{\widehat{\mathbf{X}}}Q^T & A_{\widehat{z}}K_{\mathbf{X}}A_{\widehat{z}}^T + K_{\mathbf{N}_{\widehat{z}}} \end{pmatrix}.$$

The covariance matrix of a linear estimation of $\mathbf{X}$ using $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{Z}}$ is

$$K_{(\mathbf{X}|\widehat{\mathbf{X}},\widehat{\mathbf{Z}})_L} = K_{(\mathbf{X}|\bar{\mathbf{X}},\widehat{\mathbf{Z}})_L} = K_{\mathbf{X}} - \begin{pmatrix} K_{\widehat{\mathbf{X}}}Q^T & K_{\mathbf{X}}A_{\widehat{z}}^T \end{pmatrix} C^{-1} \begin{pmatrix} QK_{\widehat{\mathbf{X}}} \\ A_{\widehat{z}}K_{\mathbf{X}} \end{pmatrix}$$

where

$$C = \begin{pmatrix} K_{\bar{\mathbf{X}}} & QK_{\widehat{\mathbf{X}}}A_{\widehat{z}}^T \\ A_{\widehat{z}}K_{\widehat{\mathbf{X}}}Q^T & A_{\widehat{z}}K_{\mathbf{X}}A_{\widehat{z}}^T + K_{\mathbf{N}_{\widehat{\mathbf{z}}}} \end{pmatrix}.$$

By matrix inversion lemma we have

$$K^{-1}_{(\mathbf{X}|\widehat{\mathbf{X}},\widehat{\mathbf{Z}})_L} = K_{\mathbf{X}}^{-1} + K_{\mathbf{X}}^{-1} \begin{pmatrix} K_{\widehat{\mathbf{X}}}Q^T & K_{\mathbf{X}}A_{\widehat{z}}^T \end{pmatrix} E^{-1} \begin{pmatrix} QK_{\widehat{\mathbf{X}}} \\ A_{\widehat{z}}K_{\mathbf{X}} \end{pmatrix} K_{\mathbf{X}}^{-1}$$

where

$$\begin{aligned}
E &= C - \begin{pmatrix} QK_{\widehat{\mathbf{X}}} \\ A_{\widehat{z}}K_{\mathbf{X}} \end{pmatrix} K_{\mathbf{X}}^{-1} \begin{pmatrix} K_{\widehat{\mathbf{X}}}Q^T & K_{\mathbf{X}}A_{\widehat{z}}^T \end{pmatrix} \\
&= C - \begin{pmatrix} Q(I - K_{\mathbf{X}|\mathbf{W},\mathbf{Z}}K_{\mathbf{X}}^{-1}) \\ A_{\widehat{z}} \end{pmatrix} \begin{pmatrix} K_{\widehat{\mathbf{X}}}Q^T & K_{\mathbf{X}}A_{\widehat{z}}^T \end{pmatrix} \\
&= C - \begin{pmatrix} Q(K_{\widehat{\mathbf{X}}} - K_{\mathbf{X}|\mathbf{W},\mathbf{Z}}K_{\mathbf{X}}^{-1}K_{\widehat{\mathbf{X}}})Q^T & Q\widehat{K}_x A_{\widehat{z}}^T \\ A_{\widehat{z}}K_{\widehat{\mathbf{X}}}Q^T & A_{\widehat{z}}K_{\mathbf{X}}A_{\widehat{z}}^T \end{pmatrix} \\
&= \begin{pmatrix} K_{\bar{\mathbf{X}}} & QK_{\widehat{\mathbf{X}}}A_{\widehat{z}}^T \\ A_{\widehat{z}}K_{\widehat{\mathbf{X}}}Q^T & A_{\widehat{z}}K_{\mathbf{X}}A_{\widehat{z}}^T + K_{\mathbf{N}_{\widehat{\mathbf{z}}}} \end{pmatrix} - \begin{pmatrix} Q(K_{\widehat{\mathbf{X}}} - K_{\mathbf{X}|\mathbf{W},\mathbf{Z}}K_{\mathbf{X}}^{-1}K_{\widehat{\mathbf{X}}})Q^T & QK_{\widehat{\mathbf{X}}}A_{\widehat{z}}^T \\ A_{\widehat{z}}K_{\widehat{\mathbf{X}}}Q^T & A_{\widehat{z}}K_{\mathbf{X}}A_{\widehat{z}}^T \end{pmatrix} \\
&= \begin{pmatrix} Q(K_{\mathbf{X}|\mathbf{W},\mathbf{Z}} - K_{\mathbf{X}|\mathbf{W},\mathbf{Z}}K_{\mathbf{X}}^{-1}K_{\mathbf{X}|\mathbf{W},\mathbf{Z}})Q^T & 0 \\ 0 & K_{\mathbf{N}_{\widehat{\mathbf{z}}}} \end{pmatrix}.
\end{aligned}$$

Then

$$
\begin{aligned}
K^{-1}_{(\mathbf{X}|\widehat{\mathbf{X}},\widehat{\mathbf{Z}})_L} &= K^{-1}_{\mathbf{X}} + K^{-1}_{\mathbf{X}} \begin{pmatrix} K_{\widehat{\mathbf{X}}} Q^T & K_{\mathbf{X}} A^T_{\widehat{z}} \end{pmatrix} \begin{pmatrix} K^{-1}_{(\bar{\mathbf{X}}|\mathbf{X})_L} & 0 \\ 0 & K^{-1}_{\widehat{\mathbf{Z}}|\mathbf{X}} \end{pmatrix} \begin{pmatrix} QK_{\widehat{\mathbf{X}}} \\ A_{\widehat{z}} K_{\mathbf{X}} \end{pmatrix} K^{-1}_{\mathbf{X}} \\
&= K^{-1}_{\mathbf{X}} + K^{-1}_{\mathbf{X}} \begin{pmatrix} K_{\widehat{\mathbf{X}}} Q^T K^{-1}_{(\bar{\mathbf{X}}|\mathbf{X})_L} & K_{\mathbf{X}} A^T_{\widehat{z}} K^{-1}_{\widehat{\mathbf{Z}}|\mathbf{X}} \end{pmatrix} \begin{pmatrix} QK_{\widehat{\mathbf{X}}} \\ A_{\widehat{z}} K_{\mathbf{X}} \end{pmatrix} K^{-1}_{\mathbf{X}} \\
&= K^{-1}_{\mathbf{X}} + K^{-1}_{\mathbf{X}} \begin{pmatrix} K_{\widehat{\mathbf{X}}} Q^T K^{-1}_{(\bar{\mathbf{X}}|\mathbf{X})_L} QK_{\widehat{\mathbf{X}}} & K_{\mathbf{X}} A^T_{\widehat{z}} K^{-1}_{\widehat{\mathbf{Z}}|\mathbf{X}} A_{\widehat{z}} K_{\mathbf{X}} \end{pmatrix} K^{-1}_{\mathbf{X}} \\
&= K^{-1}_{(\mathbf{X}|\bar{\mathbf{X}})_L} + K_{\mathbf{X}|\widehat{\mathbf{Z}}} - K^{-1}_{\mathbf{X}}, \text{ by matrix inversion lemma} \\
&= K^{-1}_{\mathbf{X}|\bar{\mathbf{X}}} + K_{\mathbf{X}|\widehat{\mathbf{Z}}} - K^{-1}_{\mathbf{X}} \\
&= K^{-1}_{\mathbf{X}|\widehat{\mathbf{X}}} + K_{\mathbf{X}|\widehat{\mathbf{Z}}} - K^{-1}_{\mathbf{X}} \\
&= K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{Z}} + K_{\mathbf{X}|\widehat{\mathbf{Z}}} - K^{-1}_{\mathbf{X}}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
K^{-1}_{(\mathbf{X}|\widehat{\mathbf{X}},\widehat{\mathbf{Z}})_L} &= K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{Z}} + K^{-1}_{\mathbf{X}|\widehat{\mathbf{Z}}} - K^{-1}_{\mathbf{X}} + K^{-1}_{\mathbf{X}|\mathbf{W}_\mathbf{G},Z} - K^{-1}_{\mathbf{X}|\mathbf{W}_\mathbf{G},\mathbf{Z}} \\
&= K^{-1}_{\mathbf{X}|\mathbf{W}_\mathbf{G},\mathbf{Z},\widehat{\mathbf{Z}}} + K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{Z}} - K^{-1}_{\mathbf{X}|\mathbf{W}_\mathbf{G},\mathbf{Z}}. \qquad (6.4)
\end{aligned}
$$

Note that $K_{\mathbf{X}|\mathbf{W},\mathbf{Z},\widehat{\mathbf{Z}}} \preceq K_{(\mathbf{X}|\widehat{\mathbf{X}},\widehat{\mathbf{Z}})_L}$ so $K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{Z},\widehat{\mathbf{Z}}} \succeq K^{-1}_{(\mathbf{X}|\widehat{\mathbf{X}},\widehat{\mathbf{Z}})_L}$. Then, from (6.4) we have

$$
K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{Z},\widehat{\mathbf{Z}}} \succeq K^{-1}_{\mathbf{X}|\mathbf{W}_\mathbf{G},\mathbf{Z},\widehat{\mathbf{Z}}} + K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{Z}} - K^{-1}_{\mathbf{X}|\mathbf{W}_\mathbf{G},\mathbf{Z}}. \qquad (6.5)
$$

Thus, by (6.5) if $K_{\mathbf{X}|\mathbf{W}_\mathbf{G},\mathbf{Z}} = K_{\mathbf{X}|\mathbf{W},\mathbf{Z}}$ then $K_{\mathbf{X}|\mathbf{W},\widetilde{\mathbf{Z}}} \preceq K_{\mathbf{X}|\mathbf{W}_\mathbf{G},\widetilde{\mathbf{Z}}}$ and if $K_{\mathbf{X}|\mathbf{W},\widetilde{\mathbf{Z}}} = K_{\mathbf{X}|\mathbf{W}_\mathbf{G},\widetilde{\mathbf{Z}}}$ then $K_{\mathbf{X}|\mathbf{W}_\mathbf{G},\mathbf{Z}} \preceq K_{\mathbf{X}|\mathbf{W},\mathbf{Z}}$. □

Lemma 14 leads us to the following corollary.

Corollary 2. *Let* $(\mathbf{W}, \mathbf{X}, \mathbf{Z}, \widetilde{\mathbf{Z}})$ *be random vectors such that* $\mathbf{X}$, $\mathbf{Z}$, *and* $\widetilde{\mathbf{Z}}$ *are jointly Gaussian,* $\mathbf{W} \leftrightarrow \mathbf{X} \leftrightarrow \widetilde{\mathbf{Z}} \leftrightarrow \mathbf{Z}$ *and* $K_{\mathbf{X}|\mathbf{W},\widetilde{\mathbf{Z}}} \succ 0$. *Also, let* $\widetilde{D} = (D^{-1} + K^{-1}_{\mathbf{X}|\widetilde{\mathbf{Z}}} - K^{-1}_{\mathbf{X}|\mathbf{Z}})^{-1}$. *If* $K_{\mathbf{X}|\mathbf{W},\mathbf{Z}} = D$ *then* $K_{\mathbf{X}|\mathbf{W},\widetilde{\mathbf{Z}}} \preceq \widetilde{D}$.

Proof. We can find $\mathbf{W_G}$ jointly Gaussian with $(\mathbf{X}, \widetilde{\mathbf{Z}}, \mathbf{Z})$ such that $(\mathbf{W}, \mathbf{W_G}) \leftrightarrow \mathbf{W} \leftrightarrow \widetilde{\mathbf{Z}} \leftrightarrow \mathbf{Z}$, and $K_{\mathbf{X}|\mathbf{W_G},\mathbf{Z}} = K_{\mathbf{X}|\mathbf{W},\mathbf{Z}} = D$. Then $K^{-1}_{\mathbf{X}|\mathbf{W_G},\widetilde{\mathbf{Z}}} = K^{-1}_{\mathbf{X}|\mathbf{W_G},\mathbf{Z}} + K^{-1}_{\mathbf{X}|\widetilde{\mathbf{Z}}} - K^{-1}_{\mathbf{X}|\mathbf{Z}} = K^{-1}_{\mathbf{X}|\mathbf{W},\mathbf{Z}} + K^{-1}_{\mathbf{X}|\widetilde{\mathbf{Z}}} - K^{-1}_{\mathbf{X}|\mathbf{Z}}$. Lemma 14 then implies the result. □

Proof of Lemma 3. We show that without loss of optimality, the auxiliary random vectors can be chosen jointly Gaussian with $(\mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$ in *ELB* and Enhanced *ELB*. Let $\mathbf{Y} \in S_G$, $(\mathbf{W}, \mathbf{U}, \mathbf{V}) \in \widetilde{C}_{l1}$ ($(\mathbf{W}, \mathbf{U}, \mathbf{V}) \in \bar{C}_{l1}$ for Enhanced *ELB*) be given and $R_{lo1} = I(\mathbf{X}; \mathbf{W}, \mathbf{U}|\mathbf{Y_1}) + I(\mathbf{X}; \mathbf{V}|\mathbf{W}, \mathbf{U}, \mathbf{Y})$ as defined before. Note that without loss of generality we can write $\mathbf{Y} = A_{\mathbf{Y}}\mathbf{X} + B_{\mathbf{Y}}\mathbf{Y_1} + \mathbf{N_Y}$, where $\mathbf{N_Y}$ is a Gaussian vector that is independent of the pair $(\mathbf{X}, \mathbf{Y_1})$. Then we have

$$\begin{aligned} R_{lo1} &= h(\mathbf{X}|\mathbf{Y_1}) - h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{Y_1}) + h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{Y}) - h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{V}, \mathbf{Y}) \\ &= h(\mathbf{X}|\mathbf{Y_1}) - h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{Y_1}) + h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{Y_1}, \mathbf{Y}) - h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{V}, \mathbf{Y_1}, \mathbf{Y}), \end{aligned}$$

since $\mathbf{X} \leftrightarrow \mathbf{Y} \leftrightarrow \mathbf{Y_1}$. Then we can further write

$$\begin{aligned} R_{lo1} &= h(\mathbf{X}|\mathbf{Y_1}) - I(\mathbf{X}; \mathbf{Y}|\mathbf{W}, \mathbf{U}, \mathbf{Y_1}) - h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{V}, \mathbf{Y_1}, \mathbf{Y}) \\ &= h(\mathbf{X}|\mathbf{Y_1}) + h(\mathbf{Y}|\mathbf{X}, \mathbf{Y_1}) - h(\mathbf{Y}|\mathbf{W}, \mathbf{U}, \mathbf{Y_1}) - h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{V}, \mathbf{Y_1}, \mathbf{Y}) \\ &= h(\mathbf{X}|\mathbf{Y_1}) + h(\mathbf{Y}|\mathbf{X}, \mathbf{Y_1}) - h(A_{\mathbf{Y}}\mathbf{X} + \mathbf{N_Y}|\mathbf{W}, \mathbf{U}, \mathbf{Y_1}) \\ &\quad - h(\mathbf{X}|\mathbf{W}, \mathbf{U}, \mathbf{V}, \mathbf{Y_1}, \mathbf{Y}), \text{ since } \mathbf{Y} = A_{\mathbf{Y}}\mathbf{X} + B_{\mathbf{Y}}\mathbf{Y_1} + \mathbf{N_Y} \\ &\geq \frac{1}{2} \log \frac{|K_{\mathbf{X}|\mathbf{Y_1}}||K_{\mathbf{Y}|\mathbf{X},\mathbf{Y_1}}|}{|K_{A_{\mathbf{Y}}\mathbf{X}+\mathbf{N_Y}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}||K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y_1},\mathbf{Y}}|} \end{aligned}$$

where $K_{A_{\mathbf{Y}}\mathbf{X}+\mathbf{N_Y}|\mathbf{W},\mathbf{U},\mathbf{Y_1}} = A_{\mathbf{Y}} K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}} A_{\mathbf{Y}}^T + K_{\mathbf{N_Y}}$ and equality holds if $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ is Gaussian. We can find $(\mathbf{W_G}, \mathbf{U_G})$ that are jointly Gaussian with $(\mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$ such that $(\mathbf{W_G}, \mathbf{U_G}) \leftrightarrow \mathbf{X} \leftrightarrow \mathbf{Y} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2})$ and $K_{\mathbf{X}|\mathbf{W_G},\mathbf{U_G},\mathbf{Y_1}} = K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y_1}}$. Then by Lemma 14, $K_{\mathbf{X}|\mathbf{W_G},\mathbf{U_G},\mathbf{Y}} \geq K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{Y}} \geq K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}$. Thus we can find a $\mathbf{V_G}$ that is jointly Gaussian with $(\mathbf{W_G}, \mathbf{U_G}, \mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$ such that $(\mathbf{W_G}, \mathbf{U_G}, \mathbf{V_G}) \leftrightarrow \mathbf{X} \leftrightarrow \mathbf{Y} \leftrightarrow (\mathbf{Y_1}, \mathbf{Y_2})$ and $K_{\mathbf{X}|\mathbf{W_G},\mathbf{U_G},\mathbf{V_G},\mathbf{Y}} = K_{\mathbf{X}|\mathbf{W},\mathbf{U},\mathbf{V},\mathbf{Y}}$, giving $(\mathbf{W_G}, \mathbf{U_G}, \mathbf{V_G}) \in \widetilde{C}_{l1}$, ($(\mathbf{W_G}, \mathbf{U_G}, \mathbf{V_G}) \in \bar{C}_{l1}$ for Enhanced *ELB*). Therefore, one can choose the auxiliary random vectors to be jointly Gaussian with $(\mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$ without loss of optimality in R_{lo1}. The same argument applies to R_{lo2} as well. □

6.2

Lemma 16. *$R'_{lo}(\mathbf{D}, \mathbf{Y})$ is a convex function with respect to* $\mathbf{D}$.

Proof of Lemma 16. To prove the lemma, we use a similar argument to [51]. Let $\epsilon > 0$ be given. We can find $(\widetilde{\mathbf{W}}, \widetilde{\mathbf{U}}, \widetilde{\mathbf{V}})$ and $(\widehat{\mathbf{W}}, \widehat{\mathbf{U}}, \widehat{\mathbf{V}})$ in $C_l(\widetilde{\mathbf{D}})$ and $C_l(\widehat{\mathbf{D}})$ respectively such that

$$\begin{aligned}
&R'_{lo}(\widetilde{\mathbf{D}}, \mathbf{Y}) + \epsilon \\
&\geq \max\{I(\mathbf{X}; \widetilde{\mathbf{W}}, \widetilde{\mathbf{U}}|\mathbf{Y_1}) + I(\mathbf{X}; \widetilde{\mathbf{V}}|\widetilde{\mathbf{W}}, \widetilde{\mathbf{U}}, \mathbf{Y}), I(\mathbf{X}; \widetilde{\mathbf{W}}, \widetilde{\mathbf{V}}|\mathbf{Y_2}) + I(\mathbf{X}; \widetilde{\mathbf{U}}|\widetilde{\mathbf{W}}, \widetilde{\mathbf{V}}, \mathbf{Y})\} \text{ and} \\
&R'_{lo}(\widehat{\mathbf{D}}, \mathbf{Y}) + \epsilon \\
&\geq \max\{I(\mathbf{X}; \widehat{\mathbf{W}}, \widehat{\mathbf{U}}|\mathbf{Y_1}) + I(\mathbf{X}; \widehat{\mathbf{V}}|\widehat{\mathbf{W}}, \widehat{\mathbf{U}}, \mathbf{Y}), I(\mathbf{X}; \widehat{\mathbf{W}}, \widehat{\mathbf{V}}|\mathbf{Y_2}) + I(\mathbf{X}; \widehat{\mathbf{U}}|\widehat{\mathbf{W}}, \widehat{\mathbf{V}}, \mathbf{Y})\}.
\end{aligned}$$

Now we construct $(\mathbf{W}, \mathbf{U}, \mathbf{V})$ and show that it is in $C_l(\lambda\widetilde{\mathbf{D}} + (1-\lambda)\widehat{\mathbf{D}})$. Let T be a binary random variable with $P(T = 1) = \lambda$ and independent of $(\widetilde{\mathbf{W}}, \widetilde{\mathbf{U}}, \widetilde{\mathbf{V}}, \widehat{\mathbf{W}}, \widehat{\mathbf{U}}, \widehat{\mathbf{V}}, \mathbf{X}, \mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y})$. Then we define

$$\begin{aligned}
\mathbf{W} &= (\widetilde{\mathbf{W}}, T) \text{ if } T = 1, & \mathbf{W} &= (\widehat{\mathbf{W}}, T) \text{ if } T = 0, \\
\mathbf{U} &= (\widetilde{\mathbf{U}}, T) \text{ if } T = 1, & \mathbf{U} &= (\widehat{\mathbf{U}}, T) \text{ if } T = 0, \\
\mathbf{V} &= (\widetilde{\mathbf{V}}, T) \text{ if } T = 1, & \mathbf{V} &= (\widehat{\mathbf{V}}, T) \text{ if } T = 0
\end{aligned}$$

and

$$\begin{aligned}
g_1(\mathbf{W}, \mathbf{U}, \mathbf{Y_1}) &= E[\mathbf{X}|\widetilde{\mathbf{W}}, \widetilde{\mathbf{U}}, \mathbf{Y_1}] \text{ if } T = 1, & g_1(\mathbf{W}, \mathbf{U}, \mathbf{Y_1}) &= E[\mathbf{X}|\widehat{\mathbf{W}}, \widehat{\mathbf{U}}, \mathbf{Y_1}] \text{ if } T = 0, \\
g_2(\mathbf{W}, \mathbf{V}, \mathbf{Y_2}) &= E[\mathbf{X}|\widetilde{\mathbf{W}}, \widetilde{\mathbf{V}}, \mathbf{Y_2}] \text{ if } T = 1, & g_2(\mathbf{W}, \mathbf{V}, \mathbf{Y_2}) &= E[\mathbf{X}|\widehat{\mathbf{W}}, \widehat{\mathbf{V}}, \mathbf{Y_2}] \text{ if } T = 0.
\end{aligned}$$

Note that $K_{\mathbf{X}|g_1(\mathbf{W},\mathbf{U},\mathbf{Y_1})} = \lambda K_{\mathbf{X}|\widetilde{\mathbf{W}},\widetilde{\mathbf{U}},\mathbf{Y_1}} + (1-\lambda) K_{\mathbf{X}|\widehat{\mathbf{W}},\widehat{\mathbf{U}},\mathbf{Y_1}}$ and since Γ_1 is a linear operator, $\Gamma_1(K_{\mathbf{X}|g_1(\mathbf{W},\mathbf{U},\mathbf{Y_1})}) \preceq \lambda \widetilde{D}_1 + (1-\lambda)\widehat{D}_1$. Similarly, that $K_{\mathbf{X}|g_2(\mathbf{W},\mathbf{V},\mathbf{Y_2})} = \lambda K_{\mathbf{X}|\widetilde{\mathbf{W}},\widetilde{\mathbf{V}},\mathbf{Y_2}} + (1-\lambda) K_{\mathbf{X}|\widehat{\mathbf{W}},\widehat{\mathbf{V}},\mathbf{Y_2}}$ gives $\Gamma_2(K_{\mathbf{X}|g_2(\mathbf{W},\mathbf{V},\mathbf{Y_2})}) \preceq \lambda \widetilde{D}_2 + (1-\lambda)\widehat{D}_2$. Hence, $(\mathbf{W},\mathbf{U},\mathbf{V}) \in C_l(\lambda\widetilde{\mathbf{D}} + (1-\lambda)\widehat{\mathbf{D}})$. We can write

$$
\begin{aligned}
&R'_{lo}(\lambda\widetilde{\mathbf{D}} + (1-\lambda)\widehat{\mathbf{D}}, \mathbf{Y}) \\
&\leq \max\{I(\mathbf{X};\mathbf{W},\mathbf{U}|\mathbf{Y_1}) + I(\mathbf{X};\mathbf{V}|\mathbf{W},\mathbf{U},\mathbf{Y}), I(\mathbf{X};\mathbf{W},\mathbf{V}|\mathbf{Y_2}) + I(\mathbf{X};\mathbf{U}|\mathbf{W},\mathbf{V},\mathbf{Y})\} \\
&= \max\{I(\mathbf{X};\mathbf{W},\mathbf{U},T|\mathbf{Y_1}) + I(\mathbf{X};\mathbf{V}|\mathbf{W},\mathbf{U},T,\mathbf{Y}), I(\mathbf{X};\mathbf{W},\mathbf{V},T|\mathbf{Y_2}) + I(\mathbf{X};\mathbf{U}|\mathbf{W},\mathbf{V},T,\mathbf{Y})\} \\
&= \max\{I(\mathbf{X};\mathbf{W},\mathbf{U}|\mathbf{Y_1},T) + I(\mathbf{X};\mathbf{V}|\mathbf{W},\mathbf{U},T,\mathbf{Y}), I(\mathbf{X};\mathbf{W},\mathbf{V}|\mathbf{Y_2},T) + I(\mathbf{X};\mathbf{U}|\mathbf{W},\mathbf{V},T,\mathbf{Y})\} \\
&= \max\{\lambda I(\mathbf{X};\widetilde{\mathbf{W}},\widetilde{\mathbf{U}}|\mathbf{Y_1}) + (1-\lambda)I(\mathbf{X};\widehat{\mathbf{W}},\widehat{\mathbf{U}}|\mathbf{Y_1}) + \lambda I(\mathbf{X};\widetilde{\mathbf{V}}|\widetilde{\mathbf{W}},\widetilde{\mathbf{U}},\mathbf{Y}) \\
&\qquad\qquad + (1-\lambda)I(\mathbf{X};\widehat{\mathbf{V}}|\widehat{\mathbf{W}},\widehat{\mathbf{U}},\mathbf{Y}), \\
&\qquad \lambda I(\mathbf{X};\widetilde{\mathbf{W}},\widetilde{\mathbf{V}}|\mathbf{Y_2}) + (1-\lambda)I(\mathbf{X};\widehat{\mathbf{W}},\widehat{\mathbf{V}}|\mathbf{Y_2}) + \lambda I(\mathbf{X};\widetilde{\mathbf{U}}|\widetilde{\mathbf{W}},\widetilde{\mathbf{V}},\mathbf{Y}) \\
&\qquad\qquad + (1-\lambda)I(\mathbf{X};\widehat{\mathbf{U}}|\widehat{\mathbf{W}},\widehat{\mathbf{V}},\mathbf{Y})\} \\
&\leq \lambda \max\{I(\mathbf{X};\widetilde{\mathbf{W}},\widetilde{\mathbf{U}}|\mathbf{Y_1}) + I(\mathbf{X};\widetilde{\mathbf{V}}|\widetilde{\mathbf{W}},\widetilde{\mathbf{U}},\mathbf{Y}), I(\mathbf{X};\widetilde{\mathbf{W}},\widetilde{\mathbf{V}}|\mathbf{Y_2}) + I(\mathbf{X};\widetilde{\mathbf{U}}|\widetilde{\mathbf{W}},\widetilde{\mathbf{V}},\mathbf{Y})\} \\
&\quad + (1-\lambda)\max\{I(\mathbf{X};\widehat{\mathbf{W}},\widehat{\mathbf{U}}|\mathbf{Y_1}) + I(\mathbf{X};\widehat{\mathbf{V}}|\widehat{\mathbf{W}},\widehat{\mathbf{U}},\mathbf{Y}), I(\mathbf{X};\widehat{\mathbf{W}},\widehat{\mathbf{V}}|\mathbf{Y_2}) + I(\mathbf{X};\widehat{\mathbf{U}}|\widehat{\mathbf{W}},\widehat{\mathbf{V}},\mathbf{Y})\} \\
&\leq \lambda R'_{lo}(\widetilde{\mathbf{D}},\mathbf{Y}) + (1-\lambda)R'_{lo}(\widehat{\mathbf{D}},\mathbf{Y}) + \epsilon.
\end{aligned}
$$

By letting $\epsilon \to 0$, we conclude that $R'_{lo}(\mathbf{D},\mathbf{Y})$ is a convex function of $\mathbf{D}$. □

6.3

Proof of Lemma 5. First we consider $A > 0$. Using the matrix inversion lemma, we can write

$$\begin{aligned}([M^{-1}+A]^{-1})_{diag} &= (A^{-1} - A^{-1}[M + A^{-1}]^{-1}A^{-1})_{diag}\\ &= A^{-1} - A^{-1}([M + A^{-1}]^{-1})_{diag}A^{-1}\\ &\le A^{-1} - A^{-1}[M_{diag} + A^{-1}]^{-1}A^{-1},\end{aligned}$$

since $(M_{diag})^{-1} \le (M^{-1})_{diag}$, [52, Theorem 7.7.8]. By the matrix inversion lemma, the right hand side of the last inequality is $[(M_{diag})^{-1} + A]^{-1}$.

Now, we consider $A \ge 0$. Without loss of generality we can assume that all positive diagonal entries are on the upper left corner of A. Hence we can write

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $A_1 > 0$, $m_1 \times m_1$ matrix, $m_1 \le m$. Also we can represent M in terms of block matrices,

$$M = \begin{pmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{pmatrix},$$

where $M_1 > 0$, $m_1 \times m_1$ matrix, and $M_3 > 0$, $(m - m_1) \times (m - m_1)$ matrix. Then we can write inverse of M as

$$M^{-1} = \begin{pmatrix} \bar{M}_1 & \bar{M}_2 \\ \bar{M}_2^T & \bar{M}_3 \end{pmatrix},$$

where

$$\begin{aligned}\bar{M}_1 &= (M_1 - M_2M_3^{-1}M_2^T)^{-1}\\ \bar{M}_3 &= (M_3 - M_2^TM_3^{-1}M_2)^{-1}\\ \bar{M}_2 &= -M_1^{-1}M_2(M_1 - M_2M_3^{-1}M_2^T)^{-1}.\end{aligned}$$

Also,

$$[M^{-1} + A] = \begin{pmatrix} \bar{M}_1 + A_1 & \bar{M}_2 \\ \bar{M}_2 & \bar{M}_3 \end{pmatrix}.$$

When we take the inverse of $[M^{-1} + A]$ we have,

$$[M^{-1} + A]^{-1} = \begin{pmatrix} \widetilde{M}_1 & \widetilde{M}_2 \\ \widetilde{M}_2 & \widetilde{M}_3 \end{pmatrix}.$$

where $\widetilde{M_2}$ is a matrix in terms of $\bar{M}_1$, $\bar{M}_2$, $\bar{M}_3$, A and

$$\widetilde{M}_1 = (\bar{M}_1 + A_1 - \bar{M}_2\bar{M}_3^{-1}\bar{M}_2^T)^{-1}$$
$$\widetilde{M}_3 = (\bar{M}_3 - \bar{M}_2^T(\bar{M}_1 + A_1)^{-1}\bar{M}_2)^{-1}.$$

Since, $M_1 = (\bar{M}_1 - \bar{M}_2\bar{M}_3^{-1}\bar{M}_2^T)^{-1}$ and $M_3 = (\bar{M}_3 - \bar{M}_2^T\bar{M}_1^{-1}\bar{M}_2)^{-1}$, we can write

$$\widetilde{M}_1 = [M_1^{-1} + A_1]^{-1}$$
$$\widetilde{M}_3 \preceq M_3.$$

Then utilizing the inequalities above we can write

$$\begin{aligned} ([M^{-1} + A]^{-1})_{diag} &= \begin{pmatrix} \widetilde{M}_1 & \widetilde{M}_2 \\ \widetilde{M}_2 & \widetilde{M}_3 \end{pmatrix}_{diag} \\ &\preceq \begin{pmatrix} ([M_1^{-1} + A_1]^{-1})_{diag} & 0 \\ 0 & (M_3)_{diag} \end{pmatrix} \\ &\preceq [(M_{diag})^{-1} + A]^{-1}. \end{aligned}$$

$\square$

CHAPTER 7

CHAPTER 4 OF APPENDIX

7.1

Proof of Theorem 9. Let $\epsilon > 0$, $v \in \mathscr{V}$ be given and joint distribution of $(\mathscr{U}, X, Y_1, \ldots, Y_m)$ in $C_{ach,v}(\mathbf{D})$, denoted by p, is fixed. The scheme consists of three main steps; namely, code construction, encoding and decoding. First we explain each step then show that the resulting rate is $\mathbf{D}$-achievable.

Code construction and encoding is similar to the proof of the achievable scheme in [2], which depends on ϵ-letter typicality [53] arguments. Here, we use lowercase letter z to denote a realization of a random variable Z.
Code Construction : A codebook, denoted as $\mathscr{C}^{\mathscr{S}_j}$, of size $2^{n(R_{\mathscr{S}_j}+R'_{\mathscr{S}_j})}$ is created for each set $\mathscr{S}_j \in v$ in the following way. Let $\mathbf{k}_{\mathscr{S}_j} = (k_{\mathscr{S}_j}, k'_{\mathscr{S}_j})$, where $k_{\mathscr{S}_j} \in [2^{nR_{\mathscr{S}_j}}]$ and $k'_{\mathscr{S}_j} \in [2^{nR'_{\mathscr{S}_j}}]$. A codeword $u_{\mathscr{S}_j}(\mathbf{k}_{\mathscr{S}_j}) \in \mathcal{U}^n_{\mathscr{S}_j}$ of length n is created by drawing each component from $\mathcal{U}_{\mathscr{S}_j}$ with respect to $p(u_{\mathscr{S}_j})$ in an i.i.d (independent and identically distributed) manner.
Encoding : Let $0 < \epsilon_0 < \cdots < \epsilon_{2^m+1}$ be sufficiently small and $x^n \in \mathcal{X}^n$ be given to the encoder. Then encoding is performed in $2^m - 1$ stages. Specifically, at stage j encoder picks $\mathscr{C}^{\mathscr{S}_j}$ and looks for an index $\mathbf{k}_{\mathscr{S}_j}$ such that $u_{\mathscr{S}_j}(\mathbf{k}_{\mathscr{S}_j})$ is ϵ_j-letter typical with x^n and

$$u^{-}_{\mathscr{S}_j} = \{u_{\mathscr{S}_i}(\mathbf{k}_{\mathscr{S}_i}) | i < j\}. \tag{7.1}$$

If such $\mathbf{k}_{\mathscr{S}_j}$ (or multiple) exists then encoder picks one of them arbitrarily and sends the bin index $k_{\mathscr{S}_j}$ to decoders. Otherwise encoder picks a codeword randomly and sends the corresponding bin index.

Decoding : We apply simultaneous decoding [4, Section 4]. Consider decoder l. It forms reconstructions of all its messages, $u_{\mathscr{D}_l}(\widehat{\mathbf{k}}_{\mathscr{D}_l}) = \{u_{\mathscr{S}_j}(\widehat{\mathbf{k}}_{\mathscr{S}_j})|\mathscr{S}_j \in \mathscr{D}_l\}$, where $\widehat{\mathbf{k}}_{\mathscr{D}_l} = \{\widehat{\mathbf{k}}_{\mathscr{S}_j}|\mathscr{S}_j \in \mathscr{D}_l\}$[1], in the following way. Decoder l takes the set of bin indices $k_{\mathscr{D}_l} = \{k_{\mathscr{S}_j}|\mathscr{S}_j \in \mathscr{D}_l\}$ then looks for a set of indices $\widetilde{\mathbf{k}}_{\mathscr{D}_l}$ such that

$$\widetilde{k}_{\mathscr{S}_j} = k_{\mathscr{S}_j} \text{ for all } \mathscr{S}_j \in \mathscr{D}_l \text{ and} \tag{7.2}$$

$$u_{\mathscr{D}_l}(\widetilde{\mathbf{k}}_{\mathscr{D}_l}) \text{ are } \epsilon_{l^*+1}\text{-letter typical with } y_l^n, \tag{7.3}$$

where $l^* = \max_{j:\mathscr{S}_j \in \mathscr{D}_l} j$. Note that if no error occurs at the encoder, $u_{\mathscr{D}_l}(\mathbf{k}_{\mathscr{D}_l})$ is ϵ_{l^*}-typical with x^n. If there is more than one set of codewords $u_{\mathscr{S}_j}(\widetilde{\mathbf{k}}_{\mathscr{S}_j})$, $\mathscr{S}_j \in \mathscr{D}_l$ whose indices, $\widetilde{\mathbf{k}}_{\mathscr{S}_j}$, satisfies (7.2) and (7.3), decoder l selects one arbitrarily and sets $\widehat{\mathbf{k}}_{\mathscr{S}_j} = \widetilde{\mathbf{k}}_{\mathscr{S}_j}$. If decoder l can not find any such set of indices, it sets $\widehat{\mathbf{k}}_{\mathscr{D}_l}$ to $\mathbf{1}$ (i.e., it declares an error). Since joint distribution of $(\mathscr{U}, X, Y_1, \ldots, Y_m)$ is in $C_{ach,v}(\mathbf{D})$, we can find a function $g_l(\cdot,\cdot)$ such that $g_l(u_{\mathscr{D}_l i}(\widehat{\mathbf{k}}_{\mathscr{D}_l}), y_{li}) = \hat{x}_{li}$, where $u_{\mathscr{D}_l i}(\widehat{\mathbf{k}}_{\mathscr{D}_l})$, y_{li} and $\hat{x}_{li}$ are i^{th} components of $u_{\mathscr{D}_l}(\widehat{\mathbf{k}}_{\mathscr{D}_l})$, y_l^n and $\hat{x}_l^n$ respectively.

Now we analyze the error probabilities at the encoding and decoding steps respectively.

Error Analysis for Encoder : Note that encoding process is correct if the following is satisfied:

1. At each encoding stage j, we can find $U_{\mathscr{S}_i}(\mathbf{k}_{\mathscr{S}_i})$ such that it is ϵ_j-jointly typical with $(U^-_{\mathscr{S}_j}, X^n)$ i.e.,

$$C_{\mathscr{S}_j} = \left\{\exists \mathbf{k}_{\mathscr{S}_j} \text{ such that } u_{\mathscr{S}_j}(\mathbf{k}_{\mathscr{S}_j}) \in \mathcal{T}^{(n)}_{\epsilon_j}(p|U^-_{\mathscr{S}_j}, X^n)\right\}. \tag{7.4}$$

[1]Since $v \in \mathscr{V}$ is an ordered list, it induces an order on sets $\mathscr{S}_j$. Hence we can take $\widehat{\mathbf{k}}_{\mathscr{D}_l}$ as an ordered set and consider ordered set structure.

Then probability of error at the encoder, $Pr(E)$ can be expressed as

$$\begin{aligned}
Pr(E) &= Pr((C_{\mathscr{S}_1} \cap \ldots \cap C_{\mathscr{S}_{2^m-1}})^c) \\
&= Pr(C^c_{\mathscr{S}_1} \cup \ldots \cup C^c_{\mathscr{S}_{2^m-1}}) \\
&= Pr((C^c_{\mathscr{S}_1} \cap \bar{C}^1) \cup \ldots \cup (C^c_{\mathscr{S}_{2^m-1}} \cap \bar{C}^{2^m-1})), \qquad (7.5)
\end{aligned}$$

where $\bar{C}^j$ is defined as $\bigcap_{i=1}^{i<j} C_{\mathscr{S}_i}$ for all $j \in [2^m - 1] \setminus \{1\}$ and $\bar{C}^1 = \emptyset$. Then from (7.5) and union bound, we can write

$$\begin{aligned}
Pr(E) &\le Pr(C^c_{\mathscr{S}_1} \cap \bar{C}^1) + \cdots + Pr(C^c_{\mathscr{S}_{2^m-1}} \cap \bar{C}^{2^m-1}) \\
&\le Pr(C^c_{\mathscr{S}_1} | \bar{C}^1) + \cdots + Pr(C^c_{\mathscr{S}_{2^m-1}} | \bar{C}^{2^m-1}) \qquad (7.6)
\end{aligned}$$

Note that $P(C^c_{\mathscr{S}_j} | \bar{C}^j)$, $j \in [2^m - 1]$ represents the probability of the event that there is no $U_{\mathscr{S}_j}(\mathbf{k}_{\mathscr{S}_j})$ ϵ_j-jointly typical with $(U^-_{\mathscr{S}_j}, X^n)$ given that for each $i < j$ we find $U_{\mathscr{S}_i}(\mathbf{k}_{\mathscr{S}_i})$ such that $U_{\mathscr{S}_i}(\mathbf{k}_{\mathscr{S}_i})$ is ϵ_i-jointly typical with $(U^-_{\mathscr{S}_i}, X^n)$, i.e.,

$$Pr(C^c_{\mathscr{S}_j} | \bar{C}^j) = Pr\left(\forall \mathbf{k}_{\mathscr{S}_j}, U_{\mathscr{S}_j}(\mathbf{k}_{\mathscr{S}_j}) \notin \mathcal{T}^{(n)}_{\epsilon_j}(p | U^-_{\mathscr{S}_j}, X^n) | (U^-_{\mathscr{S}_j}, X^n) \in \mathcal{T}^{(n)}_{\epsilon_{j-1}}(p)\right).$$

From Lemma 19 in Appendix 7.2 and since $(1-\alpha)^\beta < e^{-\alpha\beta}$ we can write

$$\begin{aligned}
Pr(C^c_{\mathscr{S}_j} | \bar{C}^j) &< e^{-\left[(1-\delta_{\epsilon_{j-1},\epsilon_j}(n)) 2^{-n\left(I(X, U^-_{\mathscr{S}_j}; U_{\mathscr{S}_j}) + 2\epsilon_j H(U_{\mathscr{S}_j})\right)} 2^{n(R_{\mathscr{S}_j} + R'_{\mathscr{S}_j})}\right]} \\
&= e^{-\left[(1-\delta_{\epsilon_{j-1},\epsilon_j}(n)) 2^{n\left((R_{\mathscr{S}_j} + R'_{\mathscr{S}_j}) - I(X, U^-_{\mathscr{S}_j}; U_{\mathscr{S}_j}) - 2\epsilon_j H(U_{\mathscr{S}_j})\right)}\right]}, \qquad (7.7)
\end{aligned}$$

where $\delta_{\epsilon_{j-1},\epsilon_j}(n) \to 0$ as $n \to \infty$. Note that when $H(U_{\mathscr{S}_j}) = 0$, $Pr(C^c_{\mathscr{S}_j} | \bar{C}^j)$ is equal to zero. Then $Pr(C^c_{\mathscr{S}_j} | \bar{C}^j) < \frac{\epsilon'}{2^m}$ if $n \ge n_1(\epsilon', \epsilon_j H(U_{\mathscr{S}_j}))$, and

$$R_{\mathscr{S}_j} + R'_{\mathscr{S}_j} \ge I(X, U^-_{\mathscr{S}_j}; U_{\mathscr{S}_j}) + 3\epsilon_j H(U_{\mathscr{S}_j}). \qquad (7.8)$$

Hence, if $(R_{\mathscr{S}_j}, R'_{\mathscr{S}_j})$ satisfy the condition in (7.8) for all $j \in [2^m - 1]$, from (7.6) we can conclude that probability of error at the encoder,

$$Pr(E) < \frac{2^m - 1}{2^m} \epsilon' \qquad (7.9)$$

when $n \geq N_1$ where $N_1 = \max_{j \in [2^m - 1]} n_1(\epsilon', \epsilon_j H(U_{\mathscr{S}_j}))$.

Error Analysis for Decoders : Let us focus on decoder l for some fixed $l \in [m]$. Decoding at this decoder is successful if the following conditions are satisfied:

1. There is no error at the encoder.

2. The source and the side information are ϵ_0-typical, i.e.,

$$D_0 = \left\{ (X^n, Y_1^n, \ldots, Y_m^n) \in \mathcal{T}_{\epsilon_0}^{(n)}(p) \right\}, \tag{7.10}$$

3. The set of codewords $U_{\mathscr{D}_l}(\mathbf{k}_{\mathscr{D}_l}) = \{U_{\mathscr{S}_j}(\mathbf{k}_{\mathscr{S}_j}) | \mathscr{S}_j \in \mathscr{D}_l\}$ chosen by the encoder are ϵ_{l^*+1}-letter typical with Y_l^n, i.e.,

$$D_{1,l} = \left\{ (U_{\mathscr{D}_l}(\mathbf{k}_{\mathscr{D}_l}), X^n, Y_l^n) \in \mathcal{T}_{\epsilon_{l^*+1}}^{(n)}(p) \right\}. \tag{7.11}$$

4. Within the received bins $k_{\mathscr{D}_l} = \{k_{\mathscr{S}_j} | \mathscr{S}_j \in \mathscr{D}_l\}$, decoder l can find a unique set of codewords, $U_{\mathscr{D}_l}(\widehat{\mathbf{k}}_{\mathscr{D}_l}) = \{U_{\mathscr{S}_j}(\widehat{\mathbf{k}}_{\mathscr{S}_j}) | \widehat{k}_{\mathscr{S}_j} = k_{\mathscr{S}_j}, \mathscr{S}_j \in \mathscr{D}_l\}$, such that $U_{\mathscr{D}_l}(\widehat{\mathbf{k}}_{\mathscr{D}_l})$ are ϵ_{l^*+1}-letter typical with Y_l^n, i.e.,

$$D_{2,l} = \left\{ \nexists \widetilde{\mathbf{k}}_{\mathscr{D}_l} \neq \mathbf{k}_{\mathscr{D}_l} \text{ such that } \widetilde{k}_{\mathscr{D}_l} = k_{\mathscr{D}_l}, \left(U_{\mathscr{D}_l}(\widetilde{\mathbf{k}}_{\mathscr{D}_l}), Y_l^n \right) \in \mathcal{T}_{\epsilon_{l^*+1}}^{(n)}(p) \right\}. \tag{7.12}$$

Then we can write the probability of error at decoder l, denoted by $Pr(D_{err,l})$, as

$$\begin{aligned}
Pr(D_{err,l}) &= Pr((E^c \cap D_0 \cap D_{1,l} \cap D_{2,l})^c) \\
&= Pr(E \cup D_0^c \cup D_{1,l}^c \cup D_{2,l}^c) \\
&= Pr(\bar{E} \cup (D_{1,l}^c \cap \bar{E}^c) \cup (D_{2,l}^c \cap \bar{E}^c \cap D_{1,l})), \text{ where } \bar{E} = E \cup D_0^c, \\
&\leq Pr(\bar{E}) + Pr(D_{1,l}^c \cap \bar{E}^c) + Pr(D_{2,l}^c \cap \bar{E}^c \cap D_{1,l}). \qquad (7.13)
\end{aligned}$$

First we analyze $Pr(\bar{E})$. By Lemma 18 in Appendix 7.2, $Pr(D_0^c) < \delta_{\epsilon_0}(n)$ where $\delta_{\epsilon_0}(n) \to 0$ as $n \to \infty$. Then we can find $n_2(\epsilon', \delta_{\epsilon_0})$, $\epsilon' > 0$ such that if $n \geq n_2(\epsilon', \delta_{\epsilon_0})$,

$Pr(D_0^c) < \frac{\epsilon'}{2^m}$. Hence, from (7.9) and the union bound, we have

$$Pr(\bar{E}) \leq Pr(E) + Pr(D_0^c) < \epsilon'. \tag{7.14}$$

when $n \geq \max\{n_2(\epsilon', \delta_{\epsilon_0}), N_1\}$.

Now we focus on $Pr(D_{1,l}^c \cap \bar{E}^c)$ and $Pr(D_{2,l}^c \cap \bar{E}^c \cap D_{1,l})$. We can upper bound $Pr(D_{1,l}^c \cap \bar{E}^c)$ by

$$Pr\left((U_{\mathscr{D}_l}(\mathbf{k}_{\mathscr{D}_l}), X^n, Y_l^n) \notin \mathcal{T}_{\epsilon_{l^*+1}}^n(p) \middle| (U_{\mathscr{D}_l}(\mathbf{k}_{\mathscr{D}_l}), X^n) \in \mathcal{T}_{\epsilon_{l^*}}^{(n)}(p)\right). \tag{7.15}$$

By Lemma 20 in Appendix 7.2, the probability in (7.15) is less than or equal to $\delta_{\epsilon_{l^*}\epsilon_{l^*+1}}(n)$ which goes to 0 as $n \to \infty$. Hence, $Pr(D_{1,l}^c \cap \bar{E}^c) < \epsilon'$ if $n \geq n_3(\epsilon', \delta_{\epsilon_{l^*}\epsilon_{l^*+1}})$.

Now we consider $Pr(D_{2,l}^c \cap \bar{E}^c \cap D_{1,l})$. Note that event $D_{2,l}^c$ can be rewritten as

$$D_{2,l}^c = \bigcup_{\mathscr{D}_l' : \mathscr{D}_l' \subseteq \mathscr{D}_l, \mathscr{D}_l' \neq \emptyset} F_{\mathscr{D}_l'}, \text{ where}$$

$$F_{\mathscr{D}_l'} = \Big\{\exists \widetilde{\mathbf{k}}_{\mathscr{D}_l} \text{ such that } \widetilde{\mathbf{k}}_{\mathscr{S}_j} \neq \mathbf{k}_{\mathscr{S}_j} \text{ for all } \mathscr{S}_j \in \mathscr{D}_l', \widetilde{k}_{\mathscr{D}_l'} = k_{\mathscr{D}_l'}, \widetilde{\mathbf{k}}_{\mathscr{S}_j} = \mathbf{k}_{\mathscr{S}_j}$$
$$\text{for all } \mathscr{S}_j \in \mathscr{D}_l \setminus \mathscr{D}_l' \text{ and } \left(U_{\mathscr{D}_l}(\widetilde{\mathbf{k}}_{\mathscr{D}_l}), Y_l^n\right) \in \mathcal{T}_{\epsilon_{l^*+1}}^{(n)}(p)\Big\}.$$

Using the union bound on the probabilities, we can write

$$Pr(D_{2,l}^c \cap \bar{E}^c \cap D_{1,l}) \leq \sum_{\mathscr{D}_l' : \mathscr{D}_l' \subseteq \mathscr{D}_l, \mathscr{D}_l' \neq \emptyset} Pr(F_{\mathscr{D}_l'} \cap \bar{E}^c \cap D_{1,l}). \tag{7.16}$$

Notice that $F_{\mathscr{D}_l'} \cap \bar{E}^c \cap D_{1,l}$ denotes the error event that there is no error at the encoder and source and side information are ϵ_0 typical (event $\bar{E}^c$), decoder l can find set of indices $\{\widehat{\mathbf{k}}_{\mathscr{S}_j} | \mathscr{S}_j \in \mathscr{D}_l\}$ such that $U_{\mathscr{D}_l}(\widehat{\mathbf{k}}_{\mathscr{D}_l})$ are ϵ_{l^*+1}-jointly typical with (X^n, Y_l^n) (event $D_{1,l}$); however, $\mathbf{k}_{\mathscr{D}_l'} = \{\mathbf{k}_{\mathscr{S}_j} | \mathscr{S}_j \in \mathscr{D}_l'\}$ subset of those indices are not unique (event $F_{\mathscr{D}_l'}$). Now we bound each term inside the summation in (7.16).

To do this, first we define an event $\bar{F}_{\mathscr{D}'_l}$ by replacing the typical set $\mathcal{T}^{(n)}_{\epsilon_{l^*+1}}(p)$ in event $F_{\mathscr{D}'_l}$ with $\mathcal{T}^{(n)}_{\epsilon_{l^*+2}}(p)$. In other words,

$$\bar{F}_{\mathscr{D}'_l} = \Big\{\exists \widetilde{\mathbf{k}}_{\mathscr{D}_l} \text{ such that } \widetilde{\mathbf{k}}_{\mathscr{S}_j} \neq \mathbf{k}_{\mathscr{S}_j} \text{ for all } \mathscr{S}_j \in \mathscr{D}'_l, \widetilde{k}_{\mathscr{D}'_l} = k_{\mathscr{D}'_l}, \widetilde{\mathbf{k}}_{\mathscr{S}_j} = \mathbf{k}_{\mathscr{S}_j}$$
$$\text{for all } \mathscr{S}_j \in \mathscr{D}_l \setminus \mathscr{D}'_l \text{ and } \left(U_{\mathscr{D}_l}(\widetilde{\mathbf{k}}_{\mathscr{D}_l}), Y^n_l\right) \in \mathcal{T}^{(n)}_{\epsilon_{l^*+2}}(p)\Big\},$$

giving $F_{\mathscr{D}'_l} \subseteq \bar{F}_{\mathscr{D}'_l}$. Let $S_1 = \{\widetilde{\mathbf{k}}_{\mathscr{D}'_l} | \widetilde{k}'_{\mathscr{S}_j} \neq k'_{\mathscr{S}_j}, \widetilde{k}_{\mathscr{D}'_l} = k_{\mathscr{D}'_l}, \forall \mathscr{S}_j \in \mathscr{D}'_l\}$ and $S_2 = \{\widetilde{\mathbf{k}}_{\mathscr{D}'_l} | \widetilde{k}_{\mathscr{S}_j} = 1, \forall \mathscr{S}_j \in \mathscr{D}'_l\}$. Then we can write,

$$\begin{aligned}
&Pr(\bar{F}_{\mathscr{D}'_l} \cap \bar{E}^c \cap D_{1,l}) \\
&\leq Pr\left(\bigcup_{S_1} U_{\mathscr{D}'_l}(\widetilde{\mathbf{k}}_{\mathscr{D}'_l}) \in \mathcal{T}^{(n)}_{\epsilon_{l^*+2}}(p|U_{\mathscr{D}_l\setminus\mathscr{D}'_l}(\mathbf{k}_{\mathscr{D}_l\setminus\mathscr{D}'_l}), Y^n_l)\Big|(U_{\mathscr{D}_l\setminus\mathscr{D}'_l}(\mathbf{k}_{\mathscr{D}_l\setminus\mathscr{D}'_l}), Y^n_l) \in \mathcal{T}^{(n)}_{\epsilon_{l^*+1}}(p)\right) \quad (7.17)\\
&\leq Pr\left(\bigcup_{S_2} U_{\mathscr{D}'_l}(\widetilde{\mathbf{k}}_{\mathscr{D}'_l}) \in \mathcal{T}^{(n)}_{\epsilon_{l^*+2}}(p|U_{\mathscr{D}_l\setminus\mathscr{D}'_l}(\mathbf{k}_{\mathscr{D}_l\setminus\mathscr{D}'_l}), Y^n_l)\Big|(U_{\mathscr{D}_l\setminus\mathscr{D}'_l}(\mathbf{k}_{\mathscr{D}_l\setminus\mathscr{D}'_l}), Y^n_l) \in \mathcal{T}^{(n)}_{\epsilon_{l^*+1}}(p)\right), \quad (7.18)
\end{aligned}$$

where (7.18) is obtained by using Lemma 21 in Appendix 7.2. Then due to the union bound of probabilities we can write

$$\begin{aligned}
&Pr(\bar{F}_{\mathscr{D}'_l} \cap \bar{E}^c \cap D^c_{1,l}) \\
&\leq \sum_{S_2} Pr\left(U_{\mathscr{D}'_l}(\widetilde{\mathbf{k}}_{\mathscr{D}'_l}) \in \mathcal{T}^{(n)}_{\epsilon_{l^*+2}}(p|U_{\mathscr{D}_l\setminus\mathscr{D}'_l}(\mathbf{k}_{\mathscr{D}_l\setminus\mathscr{D}'_l}), Y^n_l)\Big|(U_{\mathscr{D}_l\setminus\mathscr{D}'_l}(\mathbf{k}_{\mathscr{D}_l\setminus\mathscr{D}'_l}), Y^n_l) \in \mathcal{T}^{(n)}_{\epsilon_{l^*+1}}(p)\right) \\
&\leq 2^{n\sum_{\mathscr{S}_j\in\mathscr{D}'_l} R'_{\mathscr{S}_j}} 2^{-n\left(\sum_{\mathscr{S}_j\in\mathscr{D}'_l} H(U_{\mathscr{S}_j}) - H(U_{\mathscr{D}'_l}|U_{\mathscr{D}_l\setminus\mathscr{D}'_l}, Y_l) - 2\epsilon_{l^*+2}\left(\sum_{\mathscr{S}_j\in\mathscr{D}'_l} H(U_{\mathscr{S}_j})\right)\right)}, \text{ from Corollary 3.} \quad (7.19)
\end{aligned}$$

Note that $R'_{\mathscr{S}_j} \geq 0$, for all $j \in [2^m - 1]$ and when each $R'_{\mathscr{S}_j} = 0$, $\mathscr{S}_j \in \mathscr{D}'_l$, there is only one codeword $U(\mathbf{k}_{\mathscr{S}_j})$, $\mathscr{S}_j \in \mathscr{D}'_l$ in each bin. Then, from (7.17) $Pr(\bar{F}_{\mathscr{D}'_l} \cap \bar{E}^c \cap D_{1,l}) = 0$ in this case. Also, when each $H(U_{\mathscr{S}_j}) = 0$, $\mathscr{S}_j \in \mathscr{D}'_l$, $Pr(\bar{F}_{\mathscr{D}'_l} \cap \bar{E}^c \cap D_{1,l})$ is equal to 0.

Thus from (7.19), if

$$\sum_{\mathscr{S}_j \in \mathscr{D}'_l} R'_{\mathscr{S}_j} \leq \max\left\{\left(\sum_{\mathscr{S}_j \in \mathscr{D}'_l} H(U_{\mathscr{S}_j})\right) - H(U_{\mathscr{D}'_l} | U_{\mathscr{D}_l \setminus \mathscr{D}'_l}, Y_l) - 3\epsilon_{l^*+2}\left(\sum_{\mathscr{S}_j \in \mathscr{D}'_l} H(U_{\mathscr{S}_j})\right), 0\right\} \tag{7.20}$$

and $n \geq n_4(\epsilon', \epsilon_{l^*+2}, H(U_{\mathscr{S}_j}))$, $Pr(D^c_{2,l} \cap \bar{E}^c \cap D^c_{1,l}) < \frac{\epsilon'}{2^{|\mathscr{D}_l|}}$. Then from (7.13), if $R'_{\mathscr{S}_j}$ satisfies (7.20) for all $\mathscr{D}_l$, $l \in [m]$ and $n > N$, where $N = \max\{N_1, n_2(\epsilon', \delta_{\epsilon_0}), n_3(\epsilon', \delta_{\epsilon_{l^*}\epsilon_{l^*+1}}), \max_{l\in[m]}\{n_4(\epsilon', \epsilon_{l^*+2}, H(U_{\mathscr{S}_j}))\}\}$

$$Pr(D_{err,l}) < 3\epsilon'. \tag{7.21}$$

Let

$$D_{err} = \cup_{l\in[m]} D_{err,l}$$

denote the event that there is a decoding error at some decoder. By (7.21) and the union bound we have

$$Pr(D_{err}) < 3\epsilon' m. \tag{7.22}$$

Thus there must exists a single code in the ensemble for which (7.22) holds. Now we focus on the distortion constraints at decoders for this particular code. Assuming that there is no error occurring at the encoder and decoders (corresponding event is $E^c \cap D^c_{err,l}$), decoder l can find unique $u_{\mathscr{D}_l}(\mathbf{k}_{\mathscr{D}_l})$ such that $(u_{\mathscr{D}_l}(\mathbf{k}_{\mathscr{D}_l}), y^n_l, x^n)$ are ϵ_{l^*+1}-jointly typical and it can reconstruct $\hat{x}^n_l$ symbol by symbol through $\hat{x}_{li} = g_l(u_{\mathscr{D}_l i}, y_{li})$, $i \in [n]$. Then using the arguments in [53, page 57] we can bound the average distortion at decoder l as

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} d_l(x_i, \widehat{x}_{li}) &= \sum_{i=1}^{n} d_l(x_i, g_l(u_{\mathscr{D}_l i}, y_{li})) \\
&\leq E\left[d_l(X, g_l(U_{\mathscr{D}_l}, Y_l))\right] + \epsilon_{l^*+1} D_{l,max} \\
&\leq D_l + \epsilon_{l^*+1} D_{l,max},
\end{aligned} \tag{7.23}$$

where $D_{l,max}$ is the maximum distortion that $d_l(.,.)$ can give. Then the expected distortion at decoder l can be bounded by

$$\begin{aligned}
E\left[\frac{1}{n}\sum_{i=1}^{n} d_l(x_i,\widehat{x_{li}})\right] &\leq (D_l+\epsilon_{l^*+1}D_{l,max})Pr(E^c\cap D^c_{err,l})+D_{l,max}Pr(E\cup D_{err,l})\\
&\leq D_l+D_{l,max}(\epsilon_{l^*+1}+Pr(E\cup D_{err,l}))\\
&< D_l+D_{l,max}(\epsilon_{l^*+1}+4\epsilon' m), \qquad (7.24)
\end{aligned}$$

where (7.24) holds if $n > N$ and $(R_{\mathscr{S}_j}, R'_{\mathscr{S}_j})$, $\mathscr{S}_j \subseteq [m]$ satisfy the conditions in (7.8), (7.20), and the following non-negativity conditions.

$$R_{\mathscr{S}_j} \geq 0, \text{ for all } j \in [2^m-1] \qquad (7.25)$$

$$R'_{\mathscr{S}_j} \geq 0, \text{ for all } j \in [2^m-1]. \qquad (7.26)$$

Thus for all sufficiently large n, there exists a code whose expected distortion at decoder l satisfies (7.24) and whose rate does not exceed

$$\inf \sum_{j=1}^{2^m-1} R_{\mathscr{S}_j} \qquad (7.27)$$

$$\text{subject to}: R_{\mathscr{S}_j}, R'_{\mathscr{S}_j}, j\in[2^m-1] \text{ satisfying (7.8), (7.20), (7.25), and (7.26),}$$

Lemma 17. *Let $0 < \epsilon_0 < \epsilon_1 < \ldots < \epsilon_{2^m+1}$, and $U_{\mathscr{S}_j}$,$\mathscr{S}_j \in v$, be as in the proof of Theorem 9. For $\gamma \geq 0$, consider the following linear program:*

$$\widetilde{R}(\gamma) = \inf_{C^{LP}_{ach}(\gamma)} \sum_{j=1}^{2^m-1} R_{\mathscr{S}_j}, \qquad (7.28)$$

where $C^{LP}_{ach}(\gamma)$ denotes the set of $R_{\mathscr{S}_j}$ and $R'_{\mathscr{S}_j}$ such that

1) $R_{\mathscr{S}_j} \geq 0$ and $R'_{\mathscr{S}_j} \geq 0$, for all $j \in [2^m-1]$;

2) $R_{\mathscr{S}_j} + R'_{\mathscr{S}_j} \geq I(X, U^-_{\mathscr{S}_j}; U_{\mathscr{S}_j}) + 3\gamma$, for all $j \in [2^m-1]$;

3) For each decoder l, $l \in [m]$

$$\sum_{\mathscr{S}_j\in\mathscr{D}'_l} R'_{\mathscr{S}_j} \leq \max\left\{\left(\sum_{\mathscr{S}_j\in\mathscr{D}'_l} H(U_{\mathscr{S}_j})\right) - H(U_{\mathscr{D}'_l}|U_{\mathscr{D}_l\setminus\mathscr{D}'_l}, Y_l) - 3(2^m-1)\gamma, 0\right\}.$$

Then $\widetilde{R}(\gamma)$ is continuous at $\gamma = 0$ and is greater than or equal to the optimal value in (7.27) if

$$\gamma \geq \epsilon_{2^m+1} \max_{U_{\mathscr{S}_j}} H(U_{\mathscr{S}_j}). \tag{7.29}$$

Proof of Lemma 17. Note that when $\gamma = 0$, $C^{LP}_{ach}(\gamma)$ is equal to C^{LP}_{ach}. Also, since the alphabets are finite, $C^{LP}_{ach}(\gamma)$ is nonempty for any $\gamma \geq 0$. The continuity of $\widetilde{R}(\gamma)$ in γ then follows from standard results on the continuity of LPs [47]. The relation with (7.27) follows from noting that $C^{LP}_{ach}(\gamma)$ is contained in the set defined by the constraints (7.8), (7.20), (7.25), and (7.26), whenever (7.29) holds. □

Now given $\epsilon > 0$, choose $0 < \epsilon_0 < \epsilon_1 < \ldots < \epsilon_{2^m+1}$, ϵ' and γ such that

$$D_{l,max}(\epsilon_{l^*+1} + 4\epsilon' m) < \epsilon \text{ for all } l \in [m]$$
$$\gamma \geq \epsilon_{2^m+1} \max_{U_{\mathscr{S}_j}} H(U_{\mathscr{S}_j})$$

and $\widetilde{R}(\gamma) < \widetilde{R}(0) + \epsilon$. Then we have that for all sufficiently large n, there exists a code with rate at most $\widetilde{R}(0) + \epsilon$ whose expected distortion at decoder l is at most $D_l + \epsilon$. It follows that $\widetilde{R}(0)$ is **D**-achievable as desired. □

7.2

We first give the definition of ϵ-letter typical sequences and then provide theorems [53] that are useful to prove Theorem 9.

Definition 18. *Let $\epsilon > 0$ be given. $x^n \in \mathcal{X}^n$ is called ϵ-letter typical sequence with respect to p_X if*

$$\left|\frac{1}{n}N(a|x^n) - p_X(a)\right| \leq p_X(a)\epsilon, \textit{ for all } a \in \mathcal{X},$$

where $N(a|x^n)$ denotes the number of a occurring in x^n. Also $\mathcal{T}_{\epsilon}^{(n)}(p_X)$ denotes the set of all ϵ-letter typical sequences with respect to p_X.

Definition 19. *Let $\epsilon > 0$ be given. $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$ is called* jointly typical sequence *with respect to p_{XY} if*

$$\left|\frac{1}{n}N(a, b|x^n, y^n) - p_{XY}(a, b)\right| \le p_{XY}(a, b)\epsilon, \text{ for all } (a, b) \in \mathcal{X} \times \mathcal{Y},$$

Also $\mathcal{T}_{\epsilon}^{(n)}(p_{XY})$ denotes the set of all jointly typical sequences with respect to p_{XY}.

Definition 20. *Let $\epsilon > 0$ be given. The set of* conditionally typical sequence, *$\mathcal{T}_{\epsilon}^{(n)}(p_{XY}|x^n)$, is defined as*

$$\mathcal{T}_{\epsilon}^{(n)}(p_{XY}|x^n) = \{y^n|(x^n, y^n) \in \mathcal{T}_{\epsilon}^{(n)}(p_{XY})\}.$$

Lemma 18. *[53, Theorem 1.1] Let $0 < \epsilon \le \mu_X$ where $\mu_X = \min_{x \in support(p_X)} p(x)$ and $X^n \in \mathcal{X}^n$ drawn i.i.d with respect to p_X. Then*

$$1 - \delta_{\epsilon}(n) \le Pr[X^n \in \mathcal{T}_{\epsilon}^{(n)}(p_X)] \le 1,$$

where $\delta_{\epsilon}(n) = 2|\mathcal{X}|e^{-n\epsilon^2\mu_X}$.

Lemma 19. *[53, Theorem 1.3] Let $0 < \epsilon_1 < \epsilon_2 \le \mu_{XY}$ where $\mu_{XY} = \min_{(x,y) \in support(p_{XY})} p(x, y)$ and $Y^n \in \mathcal{Y}^n$ drawn i.i.d with respect to p_Y. If $x^n \in \mathcal{T}_{\epsilon_1}^{(n)}(p_X)$ then*

$$(1 - \delta_{\epsilon_1,\epsilon_2}(n))\, 2^{-n(I(X;Y)+2\epsilon_2 H(Y))} \le \Pr\left[Y^n \in \mathcal{T}_{\epsilon_2}^{(n)}(p_{XY} \mid x^n)\right] \le 2^{-n(I(X;Y)-2\epsilon_2 H(Y))},$$

where $\delta_{\epsilon_1,\epsilon_2}(n) = 2|\mathcal{X}||\mathcal{Y}| \cdot e^{-n\frac{(\epsilon_2-\epsilon_1)^2}{1+\epsilon_1}\mu_{XY}}$.

Corollary 3. *Let $0 < \epsilon_1 < \epsilon_2 \le \mu_{XYZ}$ where $\mu_{XYZ} = \min_{(x,y,z) \in support(p_{XYZ})} p(x, y, z)$. $Y^n \in \mathcal{Y}^n$ is drawn i.i.d with respect to p_Y and $Z^n \in \mathcal{Z}^n$ is drawn i.i.d with respect to p_Z. If $x^n \in \mathcal{T}_{\epsilon_1}^{n}(p_X)$ then*

$$\Pr\left[(Y^n, Z^n) \in \mathcal{T}_{\epsilon_2}^{(n)}(p_{XYZ} \mid x^n)\right] \le 2^{-n((H(Y)+H(Z)-H(Y,Z|X))-2\epsilon_2(H(Y)+H(Z)))},$$

Proof.

$$
\begin{aligned}
&\Pr\left[(Y^n, Z^n) \in \mathcal{T}_{\epsilon_2}^{(n)}(p_{XYZ} \mid x^n)\right] \\
&= \sum_{(y^n,z^n)\in\mathcal{T}_{\epsilon_2}^{(n)}(p_{XYZ}|x^n)} p_Y^n(y^n)p_Z^n(z^n) \\
&\leq 2^{-n(1-\epsilon_2)H(Y)}2^{-n(1-\epsilon_2)H(Z)}|\mathcal{T}_{\epsilon_2}^{(n)}(p_{XYZ} \mid x^n)|, \text{ by [53, Theorem 1.1]} \\
&\leq 2^{-n(1-\epsilon_2)H(Y)}2^{-n(1-\epsilon_2)H(Z)}2^{nH(Y,Z|X)(1+\epsilon_2)}, \text{ by [53, Theorem 1.2]} \\
&\leq 2^{-n((H(Y)+H(Z)-H(Y,Z|X))-2\epsilon_2(H(Y)+H(Z)))}
\end{aligned}
$$

□

Lemma 20. *[53, Markov Lemma] Let $0 < \epsilon_1 < \epsilon_2 \leq \mu_{XYZ}$ where $\mu_{XYZ} = \min_{(x,y,z)\in support(p_{XYZ})} p(x,y,z)$ and (X^n, Y^n, Z^n) drawn i.i.d with respect to p_{XYZ} such that $X \leftrightarrow Y \leftrightarrow Z$. If $(x^n, y^n) \in \mathcal{T}_{\epsilon_1}^n(p_{XY})$ then*

$$
\begin{aligned}
\Pr\left[Z^n \in \mathcal{T}_{\epsilon_2}^{(n)}(p_{XYZ} \mid x^n, y^n)\,|Y^n = y^n\right] &= \Pr\left[Z^n \in \mathcal{T}_{\epsilon_2}^{(n)}(p_{XYZ} \mid x^n, y^n)\,|Y^n = y^n, X^n = x^n\right] \\
&\geq 1 - \delta_{\epsilon_1,\epsilon_2}(n)
\end{aligned}
$$

where $\delta_{\epsilon_1,\epsilon_2}(n) = 2|\mathcal{X}||\mathcal{Y}||\mathcal{Z}| \cdot e^{-n\frac{(\epsilon_2-\epsilon_1)^2}{1+\epsilon_1}\mu_{XYZ}}$.

Lemma 21. *Let A, B and C denote the events*

$\{\exists\widetilde{\mathbf{k}}_{\mathscr{D}_l'}$ *such that* $\widetilde{\mathbf{k}}_{\mathscr{D}_l'} \neq \mathbf{k}_{\mathscr{D}_l'}, \widetilde{k}_{\mathscr{D}_l'} = k_{\mathscr{D}_l'}, U_{\mathscr{D}_l'}(\widetilde{\mathbf{k}}_{\mathscr{D}_l'}) \in \mathcal{T}_{\epsilon_{l^*+2}}^{(n)}(p|U_{\mathscr{D}_l\setminus\mathscr{D}_l'}(\mathbf{k}_{\mathscr{D}_l\setminus\mathscr{D}_l'}), Y_l^n)\}$ *and*

$\{\exists\widetilde{\mathbf{k}}_{\mathscr{D}_l'}$ *such that* $\widetilde{k}_{\mathscr{D}_l'} = \mathbf{1}, U_{\mathscr{D}_l'}(\widetilde{\mathbf{k}}_{\mathscr{D}_l'}) \in \mathcal{T}_{\epsilon_{l^*+2}}^{(n)}(p|U_{\mathscr{D}_l\setminus\mathscr{D}_l'}(\mathbf{k}_{\mathscr{D}_l\setminus\mathscr{D}_l'}), Y_l^n)\}$

$\{(U_{\mathscr{D}_l\setminus\mathscr{D}_l'}(\mathbf{k}_{\mathscr{D}_l\setminus\mathscr{D}_l'}), Y_l^n) \in \mathcal{T}_{\epsilon_{l^*+1}}^{(n)}(p)\}$

respectively. Then

$$Pr(A|C) \leq Pr(B|C).$$

Proof. The proof follows the steps in [4, Lemma 11.1]. We start with showing that for a particular set of bin indices $b_{\mathscr{D}'_l}$,

$$Pr\Big(A|C, k_{\mathscr{D}'_l} = b_{\mathscr{D}'_l} \text{ is chosen at the encoder }\Big)$$
$$\leq Pr\Big(B|C, k_{\mathscr{D}'_l} = b_{\mathscr{D}'_l} \text{ is chosen at the encoder }\Big). \tag{7.30}$$

We can write

$$\begin{aligned}
&Pr\Big(A|C, k_{\mathscr{D}'_l} = b_{\mathscr{D}'_l} \text{ is chosen at the encoder}\Big)\\
&= \sum_{b'_{\mathscr{D}'_l}} p(b'_{\mathscr{D}'_l}|b_{\mathscr{D}'_l}) Pr\Big(\exists \widetilde{\mathbf{k}}_{\mathscr{D}'_l} \text{ such that } \widetilde{k}_{\mathscr{D}'_l} = b_{\mathscr{D}'_l}, \widetilde{k}'_{\mathscr{D}'_l} \neq b'_{\mathscr{D}'_l}, U_{\mathscr{D}'_l}(\widetilde{\mathbf{k}}_{\mathscr{D}'_l})\\
&\qquad \text{is in } \mathcal{T}^{(n)}_{\epsilon_{l^*+2}}(p|U_{\mathscr{D}_l\setminus\mathscr{D}'_l}(\mathbf{k}_{\mathscr{D}_l\setminus\mathscr{D}'_l}), Y^n_l)\Big| C, \mathbf{k}_{\mathscr{D}'_l} = (b_{\mathscr{D}'_l}, \bar{b}'_{\mathscr{D}'_l}) \text{ is chosen at the encoder}\Big)\\
&\overset{a}{=} \sum_{b'_{\mathscr{D}'_l}} p(b'_{\mathscr{D}'_l}|b_{\mathscr{D}'_l}) Pr\Big(\exists \widetilde{\mathbf{k}}_{\mathscr{D}'_l} \text{ such that } \widetilde{k}_{\mathscr{S}_j} = 1, \widetilde{k}'_{\mathscr{S}_j} \in [2^{R'_{\mathscr{S}_j}} - 1] \forall \mathscr{S}_j \in \mathscr{D}'_l, U_{\mathscr{D}'_l}(\widetilde{\mathbf{k}}_{\mathscr{D}'_l})\\
&\qquad \text{is in } \mathcal{T}^{(n)}_{\epsilon_{l^*+2}}(p|U_{\mathscr{D}_l\setminus\mathscr{D}'_l}(\mathbf{k}_{\mathscr{D}_l\setminus\mathscr{D}'_l}), Y^n_l)\Big| C, \mathbf{k}_{\mathscr{D}'_l} = (b_{\mathscr{D}'_l}, \bar{b}'_{\mathscr{D}'_l}) \text{ is chosen at the encoder}\Big)\\
&\overset{b}{\leq} \sum_{b'_{\mathscr{D}'_l}} p(b'_{\mathscr{D}'_l}|b_{\mathscr{D}'_l}) Pr\Big(\exists \widetilde{\mathbf{k}}_{\mathscr{D}'_l} \text{ such that } \widetilde{k}_{\mathscr{S}_j} = 1 \text{ for all } \mathscr{S}_j \in \mathscr{D}'_l, U_{\mathscr{D}'_l}(\widetilde{\mathbf{k}}_{\mathscr{D}'_l})\\
&\qquad \text{is in } \mathcal{T}^{(n)}_{\epsilon_{l^*+2}}(p|U_{\mathscr{D}_l\setminus\mathscr{D}'_l}(\mathbf{k}_{\mathscr{D}_l\setminus\mathscr{D}'_l}), Y^n_l)\Big| C, \mathbf{k}_{\mathscr{D}'_l} = (b_{\mathscr{D}'_l}, \bar{b}'_{\mathscr{D}'_l}) \text{ is chosen at the encoder}\Big)\\
&= Pr\Big(\exists \widetilde{\mathbf{k}}_{\mathscr{D}'_l} \text{ such that } \widetilde{k}_{\mathscr{S}_j} = 1 \text{ for all } \mathscr{S}_j \in \mathscr{D}'_l, U_{\mathscr{D}'_l}(\widetilde{\mathbf{k}}_{\mathscr{D}'_l}) \in \mathcal{T}^{(n)}_{\epsilon_{l^*+2}}(p|U_{\mathscr{D}_l\setminus\mathscr{D}'_l}(\mathbf{k}_{\mathscr{D}_l\setminus\mathscr{D}'_l}), Y^n_l)\Big|\\
&\qquad\qquad C, k_{\mathscr{D}'_l} = b_{\mathscr{D}'_l} \text{ is chosen at the encoder}\Big)\\
&= P(B|C, k_{\mathscr{D}'_l} = b_{\mathscr{D}'_l} \text{ is chosen at the encoder}),
\end{aligned}$$

where

a : Given any set of codeword indices $\mathbf{b}_{\mathscr{D}'_l} = (b_{\mathscr{D}'_l}, \bar{b}'_{\mathscr{D}'_l})$ and event C, for each $\mathscr{S}_j \in \mathscr{D}'_l$, any collection of $[2^{R'_{\mathscr{S}_j}} - 1]$ number of codewords $u^n(\mathbf{k}_{\mathscr{S}_j})$ whose index $\mathbf{k}_{\mathscr{S}_j}$ is different from $\mathbf{b}_{\mathscr{S}_j}$ has the same distribution.

b: Each bin in codebook $C^{\mathscr{S}_j}$ has size $2^{R'_{\mathscr{S}_j}}$.

Multiplying both sides of (21) with $p(b_{\mathscr{D}_l'})$ and summing over all bin indices $b_{\mathscr{D}_l'}$ concludes the proof. $\square$

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