

On “Optimal bidding in a uniform price auction with multi-unit demand”

Woonghee Tim Huh,^{*} Columbia University
Michael Freimer,[†] Pennsylvania State University
Robin Roundy,[‡] Cornell University
Amar Sapra,[§] Cornell University

November 26, 2003

Abstract

In a multiple-unit, uniform-price sealed-bid auction where price is the lowest accepted bid, we show that the ordering constraint between bids may become active. We point out that Draaisma and Noussair (1997)’s partial characterization of symmetric, strictly monotone equilibria neglects this constraint.

Keywords: Auction; Multi-unit auction; Sealed bid; Uniform Price; Constrained optimization

^{*}Department of Industrial Engineering and Operations Research, Columbia University, 500 W 120th Street, MC 4704, New York, NY 10027, USA. Tel: (212) 854-1802. Email: huh@ieor.columbia.edu.

[†]Smeal College of Business Administration, Pennsylvania State University, 509N Business Administration Building I, University Park, PA 16802, USA. Tel: (814) 865-5234. Email: mbf10@psu.edu.

[‡]School of Operations Research and Industrial Engineering, Cornell University, 216 Rhodes Hall, Ithaca, NY 14853, USA. Tel: (607) 255-9137. Email: robin@orie.cornell.edu.

[§]School of Operations Research and Industrial Engineering, Cornell University, 294 Rhodes Hall, Ithaca, NY 14853, USA. Tel: (607) 255-1270. Email: sapra@orie.cornell.edu.

1 Bid-Pooling

We study multiple-unit auctions in which $k \geq 2$ units are for sale, and bidders have symmetric independent private values. If each bidder has use for only one unit, the multiple-unit auction inherits most of the theoretical properties of the single-unit auction (?? (rjw); Branco (1996); Vickrey (1961)). papers study the bidding strategies of multiple-unit auctions when each bidder demands two units. Vickrey (1961) show the incentive-compatibility of the multiple-unit Vickrey auction. Engelbrecht-Wiggans and Kahn (1998) study the pay-your-bid discriminatory auction. Noussair (1995) and Engelbrecht-Wiggans (1999) study the uniform-price auction (UPFR auction) where price is the *first-rejected* (highest-rejected) bid. Draaisma and Noussair (1997) study another version of the uniform-price auction (UPLA auction) where price is the *last-accepted* (lowest-accepted) bid.

Engelbrecht-Wiggans and Kahn (1998) shows that in the discriminatory auction, the *pooling* of bids, or the submitting of identical bids when the valuation of two goods are different, can occur. However, in the UPFR auction, Engelbrecht-Wiggans (1999) shows that the first bid of each bidder is truth-telling and the second bid is shaded. Thus, pooling cannot occur in the UPFR auction. In the UPLA auction, we show by example that pooling can occur. It is contrary to the implicit assumption in Draaisma and Noussair (1997) that the bid-ordering constraint is inactive.

Draaisma and Noussair (1997) report a partial characterization of a class of symmetric, strictly monotonic equilibria to a multi-unit UPLA auction. Each bidder i simultaneously submits two bids, denoted by b_1^i and b_2^i , where $b_1^i \geq b_2^i$. Let v_1^i be the higher valued unit, and v_2^i be the lower valued unit. A pure strategy is a mapping from valuations into bids $B(v_1, v_2) = (B_1(v_1, v_2), B_2(v_1, v_2))$, where $B_1(v_1, v_2) \geq B_2(v_1, v_2)$. They present the first-order necessary conditions, from which it is shown that any strictly monotone symmetric equilibrium strategy $B^*(\cdot, \cdot)$ is separable, i.e., $B^*(v_1, v_2) = (B_1^*(v_1), B_2^*(v_2))$.

Let $E\pi^i(b_1^i, b_2^i)$ be the expected profit of bidder i with values (v_1^i, v_2^i) and bids (b_1^i, b_2^i) . Draaisma and Noussair (1997) correctly show that $\frac{\partial}{\partial b_j^i} E\pi^i(b_1^i, b_2^i)$ is independent of $b_{j'}^i$, where $j' \neq j$. However, they conclude without justification that $b_j^i = B_j^*(v_j^i)$ is independent of

$b_{j'}^i$. Such a conclusion is correct in the case of unconstrained optimization, but not in a general constrained optimization. They have neglected the inequality constraint $b_1^i \geq b_2^i$ in the derivation of the Kuhn-Tucker condition. (See Page 159, (2)). Consequently, their results (Lemma 1 and Proposition 1) are valid only when the constraint $b_1^i \geq b_2^i$ is redundant.

2 Example

In this section we present an example in which $b_1^i \geq b_2^i$ is not redundant, and a weakly monotone symmetric equilibrium strategy may *not* be separable. While this example does not disprove the Draaisma and Noussair (1997) result, we show non-separability of bids, and thus the pooling of bids can occur.

Suppose there are two symmetric bidders indexed by $i = 1, 2$. There are two units for sale. The values (V_1^i, V_2^i) of bidder i have continuous densities and compact supports $[\frac{1}{2}, 1]$ and $[0, \frac{1}{2}]$, respectively. Let $B^*(v_1, v_2)$ be a strictly monotone (i.e., increasing) symmetric equilibrium strategy. We first prove that in this example, there exists no strictly monotone bidding strategy that is separable.

Assume, by the way of contradiction, the separability of $B^*(v_1, v_2) = (B_1^*(v_1), B_2^*(v_2))$ as in Lemma 1 of Draaisma and Noussair (1997). Let $(D_1^i, D_2^i) = (B_1^*(V_1^i), B_2^*(V_2^i))$ be the *ex ante* distribution of bids for bidder i . Let H_1 be the cumulative density function of D_1^1 (and also of D_1^2 by symmetry). Similarly, let H_2 be the cumulative density function of D_2^1 (and also of D_2^2). Let h_1 and h_2 be the corresponding probability density functions. It is easy to verify $B_1^*(v_1) \leq v_1$ and $B_2^*(v_2) \leq v_2$ for all v_1 and v_2 , and it follows that both D_1^i and D_2^i have bounded supports. By the strict monotonicity of $B^*(\cdot, \cdot)$ and the continuity of (V_1^i, V_2^i) 's density, H_1 strictly increases from 0 to 1 in $[B_1^*(\frac{1}{2}), B_1^*(1)]$, while H_2 strictly increases from 0 to 1 in $[B_2^*(0), B_2^*(\frac{1}{2})]$.

Now we claim $B_1^*(1) = B_2^*(\frac{1}{2})$, i.e., the right endpoints of the distribution of bids H_1 and H_2 are the same. To see this, we recall the partial derivatives of bidder i 's expected profit

as given in expression (2) of Draaisma and Noussair (1997), for $b_1^1, b_2^1 > 0$:

$$\frac{\partial E\pi^i(b_1^i, b_2^i)}{\partial b_1^i} = (v_1^i - b_1^i)h_2(b_1^i) - (H_2(b_1^i) - H_1(b_1^i)), \text{ and} \quad (1)$$

$$\frac{\partial E\pi^i(b_1^1, b_2^i)}{\partial b_2^i} = (v_2^i - b_2^i)h_1(b_2^i) - 2(H_1(b_2^i)). \quad (2)$$

Let $t_1 = B_1^*(1)$ and $t_2 = B_2^*(\frac{1}{2})$. If $t_1 < t_2$, there is positive probability that $b_2 \in (t_1, t_2)$, violating the constraint $b_1 \geq b_2$ since $b_1 \leq t_1$. If $t_1 > t_2$, then for $b_1 \in (t_2, t_1)$, the first term in (1) is 0 since $h_2(b_1) = 0$. The second term is strictly negative, implying that perturbing b_1 downward improves the expected profit. Thus, we conclude $t_1 = t_2$. This result can be informally obtained by observing that one bidder's first bid competes with the other bidder's second bid.

Let $v_1 = \frac{1}{2}$ and $v_2 = \frac{1}{2}$. If we assume the strict monotonicity of B_1^* , we obtain

$$B_1^*(v_1) < B_1^*(1) = t_1 = t_2 = B_2^*(\frac{1}{2}),$$

violating the bid-ordering constraint $B_1^*(v_1) \geq B_2^*(v_2)$ for all $v_1 \geq v_2$. Thus, B^* is not a separable function. This completes the proof.

In particular, suppose $(V_1^1, V_2^1), (V_1^2, V_2^2) \sim (U[\frac{1}{2}, 1], U[0, \frac{1}{2}])$. We show that weakly monotone symmetric functions B_1^* and B_2^* satisfying the first-order conditions (1) and (2) may violate the inequality constraint $b_1 \geq b_2$. Let

$$B_1^*(v_1) = \begin{cases} \frac{2}{3}v_1 + \frac{1}{24} = \frac{2}{3}(v_1 - \frac{1}{2}) + \frac{3}{8} & \text{if } v_1 \in [\frac{1}{2}, \frac{9}{16}] \\ \frac{5}{12} & \text{if } v_1 \in [\frac{9}{16}, 1] \end{cases}$$

$$B_2^*(v_2) = \begin{cases} v_2 & \text{if } v_2 \in [0, \frac{3}{8}] \\ \frac{1}{3}v_2 + \frac{1}{4} = \frac{1}{3}(v_2 - \frac{3}{8}) + \frac{3}{8} & \text{if } v_2 \in [\frac{3}{8}, \frac{1}{2}]. \end{cases}$$

It follows

$$H_1(b_1) = \begin{cases} 0 & \text{if } b_1 \in [0, \frac{3}{8}] \\ 3(b_1 - \frac{3}{8}) & \text{if } b_1 \in [\frac{3}{8}, \frac{5}{12}) \\ 1 & \text{if } b_1 \in [\frac{5}{12}, 1] \end{cases}$$

$$H_2(b_2) = \begin{cases} 2b_2 & \text{if } b_2 \in [0, \frac{3}{8}] \\ 6(b_2 - \frac{3}{8}) + \frac{3}{4} & \text{if } b_2 \in [\frac{3}{8}, \frac{5}{12}] \\ 1 & \text{if } b_2 \in [\frac{5}{12}, 1]. \end{cases}$$

It is easy to verify the first-order conditions (i.e., setting (1) and (2) equal to zero) when $v_1 \in (\frac{1}{2}, \frac{9}{16})$ and $v_2 \in (\frac{3}{8}, \frac{1}{2})$ (corresponding to $b_1, b_2 \in (\frac{3}{8}, \frac{5}{12})$). Indeed, B_1^* and B_2^* maximize the unconstrained objective function $E\pi^i$. However, with positive probability, the constraint $b^1 \geq b^2$ is violated, and it follows that B_1^* and B_2^* are not separable.

We remark that B_1^* and B_2^* are neither strictly monotone nor feasible. Thus, this example is *not* a counter-example of the Draaisma and Noussair (1997) results. It points out that their proof is incomplete.

References

- Branco, F. 1996. Multiple Unit Auctions of an Indivisible Good. *Economic Theory* 8:77–101.
- Draaisma, T., and C. Noussair. 1997. Optimal Bidding in a Uniform Price Auction with Multi Unit Demand. *Economic Letters* 56:157–162.
- Engelbrecht-Wiggans, R. 1999. An Example of Multi Unit Auctions with Atypically Many Equilibria. *Economics Letters* 65:67–69.
- Engelbrecht-Wiggans, R., and C. M. Kahn. 1998. Multi-Unit Pay Your Bid Auctions with Variable Awards. *Games and Economic Behavior* 23:25–42.
- Noussair, C. 1995. Equilibria in a Multi-Object Uniform Price Sealed Bid Auction with Multi-Unit Demands. *Economic Theory* 5:337–351.
- Vickrey, W. 1961. Counterspeculation, Auctions and Competitive Sealed Tenders. *The Journal of Finance* XVII:8–7.