# "HETEROGENEOUS SEXUAL MIXING IN POPULATIONS WITH ARBITRARILY CONNECTED MULTIPLE GROUPS"

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## Abstract

An extension is presented of the social/sexual mixing formalism of Blythe/Castillo-Chavez/Busenberg, for incompletely connected activity groups. This is shown to include as special cases the one-sex and two-sex general solutions. A simple procedure for constructing mixing models for arbitrarily classified (e.g. by sex, age, geographical location, sexual preference) populations is outlined, including a scheme for finding the number of independent mixing parameters required, and a simple (linear algebra) means for finding the values of the dependent mixing parameters. Various worked examples are presented, including the two-sex problem and structured and selective mixing.



*Key Words: social/sexual mixing; sexual heterogeneity; mixing graphs; general solution; connectedness; HIV.*

## 1. INTRODUCTION

Interest in the sexual transmission of HIV has led to the rapid development in recent years of new mathematical descriptions of social/sexual mixing in heterogeneous populations (Nold 1980; Sattenspiel 1987a,b; Jacquez et al. 1988, 1989; Hyman and Stanley 1988, 1989; Blythe and Castillo-Chavez 1989; Castillo-Chavez and Blythe 1989; Busenberg and Castillo-Chavez, 1989, 1990). The solution due to Blythe/Castillo-Chavez/Busenberg has been shown to be the general solution for the one-sex mixing problem (Busenberg and Castillo-Chavez, 1990), and a related form has been derived for, and shown to be the general solution of the two-sex mixing problem (Castillo-Chavez and Busenberg, 1989, 1990).

All of these solutions are written in terms of contact distributions - individuals in each of the groups comprising the population all take the same number of partners (contacts) per unit of time, and this number varies between groups, and possibly with time. An alternative approach studied by Kendall (1949), Frederickson (1971), Dietz and Hadeler (1988), Dietz (1988), Hadeler (1989 and MS) and Waldštatter (1989), considers the processes of pair formation and dissolution, with heterogeneity arising because different individuals have different rates for these processes. Some recent results show that is may be possible to map one approach into the other in the two-sex case (Castillo-Chavez and Busenberg MS), and stochastic simulation of the pair formation/dissolution processes may be a useful aid in formulating and understanding deterministic models of both kinds (Blythe and Castillo-Chavez 1990, Blythe et al. in prep).

A major (if largely unrecognized) drawback of the contact distribution framework is that it is very difficult to deal with incompletely connected groups (i.e., not every group mixes with every other group). If any "social" aspect to mixing is to be incorporated into contact distribution models, this deficiency must be overcome (cf Sattenspiel 1987a,1988b; Sattenspiel and Simon 1988; Sattenspiel and Castillo-Chavez, 1990).

In this paper I extend the heterogeneous contact distribution formalism of Blythe/Castillo-Chavez/Busenberg (see also Castillo-Chavez et.al., 1990; Castillo-Chavez and Busenberg, 1990) to take account of incomplete connectance between groups and show thereby that one and two sex models, and models with multiple classes (e.g., male and female homosexuals, bisexuals and heterosexuals) are all special cases. Connectance does not change with time.

## 2. EXISTING MIXING SOLUTION

Consider an arbitrary population divided into N groups. At time t, the $i^{th}$ group contains $T_i(t)$ individuals, each of whom is defined as taking $c_i(t)$ sexual partners per unit time (e.g. month, year). If $p_{ij}(t)$ $(i, j = 1, 2, \cdots, N)$ is the fraction of the $c_i(t)$ partners taken from group j, then the mixing problem is defined by the set of four constraints:

(i) $0 < p_{ij}(t) < 1$ for all i, j and t

(ii) $\sum_{j=1}^{N} p_{ij}(t) = 1$ for all i.

(iii) $p_{ij}(t) = p_{ji}(t) = 0$ if $c_i(t)\, T_i(t) = 0$ or $c_j(t)\, T_j(t) = 0$ for i, j and t

(iv) $c_i(t)\, T_i(t)\, p_{ij}(t) = c_j(t)\, T_j(t)\, p_{ij}(t)$, all i, j and t.

A solution to the mixing problem is a set of parameterized functions exactly prescribing the $\{p_{ij}(t)\}$ in terms of the $\{c_i(t), T_i(t)\}$. It is now known that the Blythe/Castillo-Chavez/Busenberg solution is the general solution to the one-sex mixing problem (Blythe and Castillo-Chavez 1989; Castillo-Chavez and Blythe 1989; Busenberg and Castillo-Chavez, 1989, 1990; Blythe and Castillo-Chavez MS). The BCB solution is:

$$p_{ij}(t) = \bar{p}_j(t) \left[ \frac{R_i(t) R_j(t)}{V(t)} + \phi_{ij} \right], \quad \text{all i, j, t} \tag{1}$$

where

$$\bar{p}_i(t) = \frac{c_i(t) T_i(t)}{\sum_{k=1}^{N} c_k(t) T_k(t)}, \quad \text{all i} \tag{2}$$

with

$$R_i(t) = 1 - \sum_{k=1}^{N} \bar{p}_k(t)\, \phi_{ik}, \quad \text{all i} \tag{3}$$

and

$$V(t) = \sum_{k=1}^{N} \bar{p}_k(t)\, R_k(t), \quad \text{all i} \tag{4}$$

The $\{\phi_{ij}\}$ are a set of parameters known as the mixing parameters, and subject to

(v) $\phi_{ij} = \phi_{ji}$, all i, j and t

(vi) $0 \leq \phi_{ij} \leq U_{ij}$, (all i, j, t)

where the $\{U_{ij}\}$ are simply the largest values of the $\{\phi_{ij}\}$ such that all the $\{R_i(t)\}$ are non-negative. The $\{\phi_{ij}\}$ for all published particular solutions to the one-sex mixing problem have now been found (Blythe and Castillo-Chavez, MS; Blythe et.al., MS).

A mixing framework is said to be completely connected if every group may mix with every other group (subject to (iv)) and with itself (self-loops). With the exception of the trivial case of pure self-mixing within each group ($p_{ii}(t) = 1$, zero elsewhere, deriving from $\phi_{ii}(t) = 1/\bar{p}_i(t)$, zero elsewhere), all solutions generated by Equation (1) are completely connected.

## 3. MIXING WITH INCOMPLETE CONNECTION

An incompletely connected mixing framework is one where at least one $p_{ij}(t)$ is zero for all time t, regardless of the activity levels and population sizes of the groups: people in these groups do not mix.

In principle the original formulation (Equation (1)) can handle such cases, but there are problems. Say groups l and k do not mix. Then for $p_{lk}(t) = p_{kl}(t) = 0$ for all t we require $R_l(t) = 0$ all t, and $\phi_{lk} = \phi_{kl} = 0$. Thus all the elements in the $l^{th}$ row and $k^{th}$ column of p are either zeroes (as required) or else of the form $p_{ij}(t) = \bar{p}_j(t)\phi_{ij}$. As it stands, this latter form is inconvenient, as the relevant $\{\phi_{ij}\}$ need to be functions of time, such that some $R_i = 0$ while the others must satisfy $R_i \geq 0$ (essentially a linear programming problem). With just one missing connection per row, for example, all the $R_i = 0$, and we have no flexibility in the choice of the $\{\phi_{ij}\}$.

To avoid this, we re-formulate Equation (1), taking explicit account of inter-group connectedness. The re-formulated mixing framework must have the following characteristics. First, it should permit a general description of mixing where an arbitrary number of connections are missing, and do so regardless of <u>which</u> connections these are. The second condition may be considered a corollary of the first, but is sufficiently important to be worth stating separately. If the population is partitioned into two classes, such that every group in each class mixes with every group in the other class, but with no group in the home class, then we have a situation exactly equivalent to two-sex mixing. This case represents complete degeneracy in Equation (1). We require that our new solution be able to cope with this case, and further that it must thereby agree with the general solution to the two-sex mixing problem (for complete bipartite connectedness), recently found by Castillo-Chavez and Busenberg (1989, 1990). Of course the new solution must reduce to the BCB equations under conditions of complete connectedness.

The third condition is at once more subtle and more fundamental. It may readily be demonstrated that not every incompletely connected mixing framework with heterogeneous $c_i(t)\ T_i(t)$ is valid – there may not be a solution. We expect that some relatively straightforward characteristic of the new solution will reveal whether or not a solution set exists, and if so that we can converge to a member of it. Figure (1) shows a graphical representation of a simple four group population where for some $c_i(t)\ T_i(t)$ a solution exists, but not for others.

A new solution for incompletely connected mixing groups is now presented, and its ability to meet the above conditions evaluated. Again note that the pattern of connections between groups is assumed to hold for all time.

Borrowing from graph theory, let $\underset{\sim}{x}$ be the adjacency matrix for a mixing structure, with

$$x_{ij} = \begin{cases} 1 \text{ if group i and j directly linked} \\ 0 \text{ if not} \qquad i, j = 1, 2, \cdots, N \end{cases} \tag{5}$$

Now define

$$q_{ij}(t) = \frac{x_{ij} c_j(t)\ T_j(t)}{\sum_{k=1}^{N} x_{ik} c_k(t)\ T_k(t)} \tag{6}$$

i.e., $q_{ij}(t)$ is the activity of group j relative to that of all groups linked to group i, or just the <u>relative activity</u>. If groups i and j are not connected, they have relative activity of zero with respect to one another. Next, define

$$D_{ij}(t) = \frac{\sum_{k=1}^{N} x_{ik} c_k(t)\ T_k(t)}{\sum_{k=1}^{N} x_{jk} c_k(t)\ T_k(t)}\ , \qquad \text{all i and j} \tag{7}$$

as the ratio of relative activities between groups i and j. Note that for completeness we require

$$x_{ij}/x_{ji} = 0 \qquad \text{if } x_{ij} = x_{ji} = 0. \tag{8}$$

Redefining

$$R_i(t) = 1 - \sum_{k=1}^{N} q_{ik}(t)\ \phi_{ik}(t)\ , \qquad \text{all i} \tag{9}$$

and introducing the new quantities

$$V_i(t) = \sum_{k=1}^{N} q_{ik}(t)\ R_k(t)\ , \qquad \text{all i} \tag{10}$$

We may write the solution to the incompletely connected mixing problem as

$$p_{ij}(t) = q_{ij}(t)\left[\frac{R_i(t)\ R_j(t)}{V_i(t)} + \phi_{ij}(t)\right], \qquad \text{all i and j} \tag{11}$$

which strongly resembles Equation (1). However, now the $\{\phi_{ij}(t)\}$ are no longer symmetric, as we require

$$\phi_{ji}(t) = D_{ji}(t)\ \phi_{ij}(t)\ , \tag{12}$$

and $\phi_{ij}(t)$ is only defined where $x_{ij} = 1$. The time-dependence implicitly introduced into the $\{\phi_{ij}\}$ in Equation (10) is intended to take account of that explicitly appearing in Equation (13).

We obtain solutions by specifying a set of $\phi_{ij}$ for $j \geq i$ (i.e. the upper triangular matrix), and obtain the rest ($i < j$) using Equation (12) (note that this is arbitrary: we must specify the diagonal terms $\phi_{ii}$, and then half of the remainder). In the next section we consider the question of when valid solutions exist.

## 4. EXISTENCE OF VALID SOLUTIONS

A solution to the incompletely connected mixing problem for specified $\underset{\sim}{x}$ exists if a set of $\underset{\sim}{\phi}$ can be found such that conditions (i) to (iv) are satisfied for all non-negative $\underset{\sim}{c}$ and $\underset{\sim}{T}$. Consider each condition in turn:

I) For small enough non-negative $\{\phi_{ij}(t)\}$, the $R_i(t) \geq 0$, so that condition (i) is satisfied.

II) Summing over all j in Equation (11), the $p_{ij}(t)$ clearly sum to unity, for all i. Therefore (ii) is satisfied.

III) Condition (iii) must be imposed, just as in the BCB and two-sex problems.

IV) Constraint (iv) may be written

$$\frac{p_{ij}(t)}{p_{ji}(t)} = \frac{x_{ij}c_j(t)\ T_j(t)}{x_{ji}c_i(t)\ T_i(t)}$$

$$= \frac{q_{ij}(t)}{q_{ji}(t)\ D_{ji}(t)} \tag{13}$$

for all i and j where $x_{ij} = 1$. Substituting into Equation (11), this amounts to the requirement (for the non-degenerate case of all $R_i > 0$), that

$$V_{ij}(t) = D_{ji}(t)\ V_j(t)\ , \qquad \text{where } x_{ij} = 1. \tag{14}$$

As diagonal cases (i=j) are automatically satisfied, and as $D_{ji} = 1/D_{ij}$, we may dispose of all but M of these simultaneous equations at once (where M is the number of non-zero $x_{ij}$ above the diagonal, i.e., the number of links between groups). Further, these equations are not all independent, and in fact we require at most N-1 of them (if (14) holds for any i and all $j \neq i$, then it holds for all i).

Now let

$$\pi_i = \prod_{j=1}^{N} x_{ij}\ , \qquad \text{all i} \tag{15}$$

Then $\pi_i = 0$ if row i of $\underset{\sim}{x}$ contains any zeros, $\pi_i = 1$ if row i contains only ones. So

$$r = \sum_{i=1}^{N} \pi_i \tag{16}$$

is the number of rows of $\underset{\sim}{x}$ which do not contain zeros. For any pair of rows (say k and l) with $\pi_k = \pi_l = 1$, we have $D_{kj} = D_{lk} = 1$, and $q_{kj} = q_{lj}$ (all j), i.e. $V_k = V_l$. Hence (14) is satisfied for $l = k$, $j = l$. Further $D_{kj} = D_{lj}$ (all j), so that choosing $i = k$ (or l) in (14) reduces the number of simultaneous equations needed by two. This result readily extends to general r, and we may write the following:

(a) If $\exists$ i such that $\pi_i = 1$, then choose one of these (say i=k). Equation (14) then becomes the set of N−r independent equations

$$D_{ij}V_j(t) = V_k(t), \qquad \text{where } j \neq k \text{ and } \pi_j\pi_k = 0 \tag{17}$$

(b) Alternatively, if r=0, then choose any i, and there are then N−1 independent equations of the form (14) with $j \neq i$.

Thus, if r=N (completely connected), we need solve no equations, and if r=0 or 1 we need N−1 equations. Hence we may have at most $K = M+m_s-N+\max\{1,r\}$ arbitrary constant $\phi_{ij}$ $(j \geq i)$, where $m_s$ is the number of self-loops (where $x_{ii} = 1$, each i). So for the completely connected case ($r=N$, $m_s=N$, $M=N(N-1)/2$), we may have $N(N+1)/2$ arbitrary $\phi_{ij}$, i.e. all of them. For $r<N$, $K<M+m_s$, with $N-\max\{1,r\}$ of the $\phi_{ij}$ having to take on values at each time t such that (17) is satisfied. Now let $\chi$ be the set of all $\phi_{ij}$ for $j \geq i$, let $\chi_A$ be the subset which will be functions of time such that (17) is satisfied, and $\chi_C$ the complementary subset of K arbitrary constant $\phi_{ij}$ (i.e. $\chi = \chi_A \cup \chi_C$). Then a valid solution to the mixing problem is a partition of $\chi$ into $\chi_A$ and $\chi_C$ with non-negative constant $\phi_{ij} \in \chi_C$, such that (17) is satisfied, $\phi_{ij} \in \chi_A$ are non-negative and $R_i(t) \geq 0$ (all i, t) with at least one $R_i(t) > 0$. The conditions for existence of such solutions are not yet known for the incompletely connected problem, but the strong intuitive conjecture that $m_s=N$ (all groups have self-loops) is a sufficient condition has not yet found a counter example.

## 5. SOME EXAMPLES

We will look at a few examples of the general equation (11) for incompletely connected multi-group mixing, paying particular attention to how well it meets the requirements stated in Section 3, and to strategies for choosing the complementary set $\chi_C$.

Example 1: $X_{ij} = 1$, all i and j. This corresponds to complete connection (all vertices of the mixing graph are adjacent). Then

$$q_{ij}(t) = \frac{c_j(t)T_j(t)}{\sum_{k=1}^{N} c_k(t)T_k(t)} = \bar{p}_j(t) \tag{18}$$

and $D_{ij}(t) = 1$, implying $\phi_{ij}(t) = \phi_{ji}(t)$, and the time dependence may be dropped. Hence $V_i(t) = V(t)$ for all i, and the solution collapses back to the BCB equation (1), as required.

Example 2: $x_{1j} = x_{j1} = 1$, $x_{jj} = 1$ (all j), $x_{ij} = 0$ elsewhere.

This describes a homosexual population with a core group (i=1 w.l.o.g.), and N−1 peripheral groups. The peripheral groups mix in the home group and the core group, but no other (see Figure 2).

Here M=N−1, $m_s$=N and r=1, so we can have K=N arbitrary constants. Let us choose these as the mixing parameters of the core group, and denote the N−1 $\phi_{ij}$ to be calculated by $\theta_k$, k=1,2,⋯N−1. Then

$$\begin{aligned} \phi_{ij} &= \alpha_j \qquad (0 < \alpha_j < 1, \text{ all } j) \\ \phi_{jj} &= \theta_{j-1}, \qquad j \neq 1 \\ \phi_{j1} &= D_{j1}\alpha_j \qquad \text{all } j \end{aligned} \tag{19}$$

where $\alpha_j$ are arbitrary parameters.

Choosing our reference group in Equation (17) as group 1, we have N−1 equations

$$D_{jl}(t)\, V_j(t) = V_1(t), \qquad j = 2, 3, \cdots N \tag{20}$$

which may easily be cast in the form

$$\underset{\sim}{A}\, \underset{\sim}{\theta} = \underset{\sim}{B} \tag{21}$$

where $\underset{\sim}{A}$ is the (N−1) × (N−1) matrix

$$\underset{\sim}{A} = \begin{bmatrix} 0 & q_{13}q_{33} & q_{14}q_{44} & \cdots & q_{1N}q_{NN} \\ q_{12}q_{22} & 0 & q_{14}q_{44} & \cdots & q_{1N}q_{NN} \\ q_{12}q_{22} & q_{13}q_{33} & 0 & \cdots & q_{1N}q_{NN} \\ \vdots & \vdots & & \ddots & \vdots \\ q_{12}q_{22} & q_{13}q_{33} & \cdots & & 0 \end{bmatrix} \tag{22}$$ .

and $\underset{\sim}{B}$ is the vector

$$\underset{\sim}{B} = \begin{bmatrix} 1 - q_{11}\, L + q_{11}^2\, \alpha_1 - D_{21} + q_{11}\, q_{12}\, \alpha_2 \\ \vdots \\ 1 - q_{11}\, L + q_{11}^2\, \alpha_1 - D_{N1} + q_{11}\, q_{1N}\, \alpha_N \end{bmatrix} \tag{23}$$

where

$$L = \sum_{k=1}^{N} q_{1k}\, \alpha_k \;\leq\; 1. \tag{24}$$

Now

$$|\underset{\sim}{A}| = \frac{(-1)^N (N-1)}{\prod_{k=2}^{N} q_{1k} q_{kk}} \neq 0\ , \tag{25}$$

$\underset{\sim}{A}$ is non-negative, $\underset{\sim}{B}$ may be shown to be positive, so we are guaranteed a positive solution vector $\underset{\sim}{\theta}$. This gives a valid solution to the mixing problem if all $R_i \geq 0$. A sufficient condition ($\theta_k < 1$) is

$$B_k \leq q_{1k+1}\, q_{k+1\, k+1} \qquad \text{for } k = 1, \cdots, N-1 \tag{26}$$

if $0 < \alpha_j \leq 1$ for all j. For example, we can only use $\alpha_j = 1$, all j, if

$$q_{jj} \geq \frac{(1-q_{11})}{q_{1j}}\, [1 - (q_{11} + q_{1j})], \qquad \text{all } j > 1 \tag{27}$$

<u>Example 3</u>: Adjacency matrix $\underset{\sim}{x}$ bipartite. There are two cases of note. In the first, the "in-mixing" case, we may write $\underset{\sim}{x}$ as the partition (see Figure 3)

$$\underset{\sim}{x} = \begin{bmatrix} \underset{\sim}{J}_1 & \emptyset_2 \\ \emptyset_1 & \underset{\sim}{J}_2 \end{bmatrix} \tag{29}$$

where $\underset{\sim}{J}_1$ and $\underset{\sim}{J}_2$ are n × m and n × m matrices of ones respectively, and $\emptyset_1$ and $\emptyset_2$ are m × n and n × m matrices of zeroes respectively. Defining

$$\bar{p}_j^{(1)}(t) = \frac{c_j\, T_j}{\sum_{k=1}^{m} c_k T_k} = q_{ij} \qquad \text{for } i \leq n, j \leq m \tag{29}$$

and

$$\bar{p}_\ell^{(2)}(t) = \frac{c_{m+\ell}\, T_{m+\ell}}{\sum_{r=1}^{n} c_{m+r}\, T_{m+r}} = q_{ij} \text{ for } i > n, j > m \tag{30}$$

with

$$R_j^{(1)}(t) = 1 - \sum_{k=1}^{m} \bar{p}_k^{(1)}(t)\, \phi_{jk}(t) \tag{31}$$

$$R_\ell^{(2)}(t) = 1 - \sum_{r=1}^{n} \bar{p}_r^{(2)}(t)\, \phi_{m+\ell,\, n+\ell}(t) \tag{32}$$

$$V^{(1)}(t) = \sum_{k=1}^{m} \bar{p}_k^{(1)}\, R_k^{(1)} \tag{33}$$

$$V^{(2)}(t) = \sum_{\ell=1}^{n} \bar{p}_\ell^{(2)}\, R_\ell^{(2)} \tag{34}$$

we see that this partitioning results in two decoupled but internally completely connected subpopulations. We may of course then treat each as separate populations.

The second major case of interest is complete "out mixing" case (see Figure 4). Here,

$$\underset{\sim}{x} = \begin{bmatrix} \emptyset_1 & \underset{\sim}{J}_1 \\ & \\ \underset{\sim}{J}_2 & \emptyset_2 \end{bmatrix} \tag{36}$$

which is the adjacency matrix for a fully bipartite graph (two classes, with every element in one connected to all the elements in the other, but to none in the home class). Here $\emptyset_1$ is $n_1 \times n_2$. $\underset{\sim}{J}_2$ is $n_1 \times n_1$, $\underset{\sim}{J}_2$ is $n_2 \times n_2$ and $\emptyset_2$ is $n_2 \times n_1$, where $n_2 = n - n_1$.

This problem may be approached with the full equation for all i and j (Equation (11)), but as with the previous example, it is instructive to define new variables for the two classes

($i \leq n_1$ and $i > n_1$).

$$
\begin{aligned}
c_k^{(1)} T_k^{(1)} &= c_k T_k \qquad \text{for } k \leq n_1 \\
c_\ell^{(2)} T_\ell^{(2)} &= c_{n_1+\ell}\, T_{n_1+\ell} \qquad \text{for } \ell \leq n
\end{aligned}
\tag{36}
$$

Then

$$
q_\ell^{(1)} = \frac{c_\ell^{(2)} T_\ell^{(2)}}{\sum_{r=1}^{n_2} c_r^{(2)} T_r^{(2)}}, \qquad \ell \leq n_2 \tag{37}
$$

and

$$
q_k^{(2)} = \frac{c_k^{(2)} T_k^{(1)}}{\sum_{r=1}^{n_1} c_r^{(1)} T_r^{(1)}}, \qquad k \leq n_1 \tag{38}
$$

are the non-zero partitions of the $q_{ij}$ matrix. Next

$$
R_k^{(1)} = 1 - \sum_{r=1}^{n_2} q_r^{(1)} \phi_{kr}^{(1)}, \qquad k \leq n_1 \tag{39}
$$

and

$$
R_\ell^{(2)} = 1 - \sum_{r=1}^{n} q_r^{(2)} \phi_{\ell r}^{(2)}, \qquad \ell \leq n_2 \tag{40}
$$

where

$$
\begin{aligned}
\phi_{k\ell}^{(1)} &= \phi_{k,n_1+\ell} \\
\phi_{\ell k}^{(2)} &= \phi_{n_1+\ell_1\,,\,k}
\end{aligned}
\tag{41}
$$

is the partition of the $\phi$ matrix. Now let

$$
V^{(1)}(t) = \sum_{r=1}^{n_2} q_r^{(1)} R_r^{(2)} \tag{42}
$$

$$
V^{(2)}(t) = \sum_{r=1}^{n_1} q_r^{(2)} R_r^{(1)} \tag{43}
$$

so that we have mixing probabilities for the two subpopulations

$$p_{k\ell}^{(1)} = q_\ell^{(1)} \left[ \frac{R_k^{(1)}(t)\, R_\ell^{(2)}(t)}{V^{(1)}(t)} + \phi_{k\ell}^{(1)} \right] \tag{44}$$

and

$$p_{k\ell}^{(2)} = q_\ell^{(2)} \left[ \frac{R_k^{(2)}(t)\, R_\ell^{(1)}(t)}{V^{(2)}(t)} + \phi_{k\ell}^{(2)} \right] \tag{45}$$

with $k = 1,2,\cdots n_1$, and $\ell = 1,2,\cdots n_2$. Equations (44) and (45) are now exactly in the form derived by Busenberg and Castillo-Chavez (1989, 1990) as the general solution to the two-sex mixing problem (they also give the general solution for an age-structured population). Their analysis reveals the existence of many types of particular solution, with only one of them ("Ross's solution") being separable.

Where cross-connection between the two classes (sexes) is not complete, we must use the full model Equation (11) rather than the special case Equation (44, 45), but this need present no especial difficulties, as the problem is no different in kind from any other incompletely connected multi-group problem – we have merely classified the rows of the mixing matrix.

Example 4: (Structured mixing)

The "structured mixing" formalism of Jacquez et.al. (1989) is a complicated parameter-rich model designed to include a variety of mixing structures for AIDS transmission modelling. We shall use slightly different notation than that of Jacquez et.al. (1989) for the sake of consistency with the rest of this paper.

The population is divided along two dimensions. The first partition is into "population sub-groups," according to type of person (e.g. drug user, male homosexual) with characteristic levels of sexual activity $\{c_i(t)\}$. The second partition is into "behavior sub-groups" (called activity sub-groups by Jacquez et.al. (1989), according to location or practices. Structured mixing may best be understood as follows.

Let $N_r(t)$ ($r = 1, 2, \cdots, n$) be the population of population sub-group r, at time t, and let $c_r(t)$ be the sexual activity (partners per unit time) of individuals in the sub-group. Now partition the members of each population sub-group according to the discrete probability density function f, such that $f_{rs}N(t)$ is the number of r-type people in behavior sub-group s ($s = 1, 2, \cdots, m$). The $\{f_{rs}\}$ are specified, and may be functions of time. Mixing is specified across $R = 1, 2, \cdots, n$ within each behavior sub-group.

In order to phrase this model in the language of this paper, we must "unpack" the compact notation of Jacquez et.al. (1989). We regard the behavior groups as labelled blocks of an (n×m) × (n×m) mixing matrix. Each such block is comprised of n rows (one for each level of activity) and n columns (for those portions of mixing occurring within the behavior group). The $\{c_i(t)\}$ in the mixing matrix are reflected by

$$c_{\ell+km} = c_\ell \qquad \ell = 1, 2, \cdots, m \qquad k = 1, 2, \cdots, n \tag{46}$$

reflecting the fact that the same population sub-groups are represented in every behavior sub-group. The population associated with each class (i.e. each row of our block-composed mixing matrix) is given by

$$T_{r+(s-1)m} = f_{rs}N_r(t) \qquad r = 1, 2, \cdots, n \qquad s = 1, 2, \cdots, m \tag{47}$$

The adjacency matrix $\underset{\sim}{x}$ contains no zeroes: absence of contact between classes i and j (say) occurs because one of these classes is empty (some $f_{rs} = 0$_. This is covered by constraint (iv), so that structured mixing is <u>not</u> an example of incomplete connection, and is covered by Equation (1) (with one provision discussed in the next example).

It should be noted that Jacquez et.al. (1989) do not introduce the matrix $\underset{\sim}{f}$ with the interpretation (partition of r-people across the s-groups) used here. They interpret $f_{rs}$ as the fraction of the partners of a r-person who come from behavior group s. The two interpretations are equivalent, as $f_{rs}$ acts as a scale factor for $N_r$ in behavior groups (c.f. Jacquez et.al., 1989, p. 310); this may clearly be seen by writing the balance constraint (iii) in the manner of Jacquez et.al. (1989) for sexual contacts between group r and group r$'$ individuals in behavior group s:

$$c_r N_r f_{rs} \rho(s)_{rr'} = c_{r'} N_{r'} f_{r's} \rho(s)_{r'r} \tag{48}$$

<u>Example 5</u>: (Selective mixing)

Another paper from the Michigan group (Koopman et.al., 1989) considers sexual mixing, effectively within a single behavior class of the previous example, according to the following schema. For a population consisting of N groups, Koopman et.al. (1989) introduce a "precursor," social, mixing process. Here all the individuals in group i have $h_i$ social contacts per unit time, and there is some prescribed mixing pattern (Koopman et.al. (1989) use proportionate mixing). Conditional upon a social encounter between an i and a j individual, there is then a probability $d_{ij}$ that they find each other mutually acceptable, and that they then have sex. In Koopman et.al. (1989), a component of $d_{ij}$ is a time-dependent function which adjusts the probability of sex occurring according to the availability of prospective partners. Morris (pers. comm.) uses a simpler version of this model in her log-linear estimation scheme: all $\{h_i\}$ are assumed to be equal, and the $\{d_{ij}\}$ are not functions.

Some of the $\{d_{ij}\}$ may have zero values for all time, if one group never accepts people from another as sexual partners, so this formulation is inherently one with incomplete connectedness. It is convenient (but does not change the model in any way) to define the usual adjacency matrix $\underset{\sim}{x}$ by

$$x_{ij} = \begin{cases} 1 & \text{if } d_{ij}(t) > 0 \text{ for any } t \\ 0 & \text{if } d_{ij}(t) = 0 \text{ for all } t. \end{cases} \tag{49}$$

Then selective mixing may be written

$$p_{ij}(t) = \frac{x_{ij}\, d_{ij}\, h_j\, T_j\,(t)}{\sum_{k=1}^{N} x_{ik}\, d_{ik}\, h_k\, T_k\,(t)} \tag{50}$$

Not all social encounters lead to sex, and we may calculate $\{c_i(t)\}$, the rates of acquisition of partners, per unit time, for all groups:

$$c_i(t) = \frac{h_i}{\sum_{k=1}^{N} h_k T_k(t)} \sum_{k=1}^{N} x_{ik} d_{ik} h_k T_k(t) \tag{51}$$

$(i = 1, 2, \cdots, N)$. Hence we may write

$$\begin{aligned} p_{ij}(t) &= x_{ij}\, L_{ij}(t)\, c_j(t)\, T_j(t) \\ &= q_{ij}(t)\, L_{ij}(t) \sum_{k=1}^{N} x_{ik} c_k(t) T_k(t) \end{aligned} \tag{52}$$

(all i and j), where $q_{ij}(t)$ is given by Equation (6), and

$$L_{ij}(t) = L_{ji}(t) = \frac{d_{ij}\, h_i\, h_j}{c_i(t)\, c_j(t) \sum_{k=1}^{N} h_k\, T_k(t)} \tag{53}$$

Equation (52) is thus a particular case of Equation (11), where the $\{\phi_{ij}\}$ should be obtained from

$$\frac{R_i(t) R_j(t)}{V_i(t)} + \phi_{ij} = L_{ij} \sum_{k=1}^{N} x_{ik} c_k T_k \tag{54}$$

In the numerical example considered by Koopman et.al. (1989), N=9, M=27, $m_s$=9 and r=3, so that out of a total of 36 entries in the $\underset{\sim}{\phi}$ matrix, we have at most K=30 of them available for arbitrary assignment. In fact, in this case making the $\underset{\sim}{\phi}$ some function of the $\underset{\sim}{L}$ would probably be a better strategy.

It is thus clear that the general case of selective mixing falls within the circuit of Equation (11); of course, if the $\{d_{ij}\}$ are always positive, connectedness is complete and Equation (1) will suffice. Note that if an incompletely connected mixing model is used within one or more of the behavior groups of Jacquez et.al. (1989)--see previous example--then of course the structured mixing model is incompletely connected, and Equation (11) rather than Equation (1) must be used. The difference between zeroes in $\{d_{ij}\}$ and zeroes in $\{f_{rs}\}$ should be appreciated.

<u>Example 6</u>: (Costly search)

Kaplan et.al. (1989) suggest a model for the partner selection process in a male homosexual population with N groups. Each group has its distinct $c_i(t)$ as usual, and are ordered in such a way that increasing i means increasing risk. If individuals in group i refuse partners whose risk index is at most i + k (and hence presumably search for others), then one way to describe the mixing process is with the adjacency matrix

$$x_{ij} = \begin{cases} 1 & i - k \leq j \leq i + k \\ 0 & \text{elsewhere} \end{cases} \tag{55}$$

(the lower occurs because $x_{ji} = x_{ij}$; the i class is itself too risky for people in classes lower than i–k). There is thus a band running diagonally across the adjacency matrix. In this case, with N groups, we have

$$M = \frac{N(N-1)}{2} - \frac{(N-k)(N-k-1)}{2} \tag{56}$$

with $m_s = N$ and $r = 0$, then $K = M+1$. For example, with N=6 and K=1, K = 6; i.e., of the 11 $\{\phi_{ij}\}$ on or above the diagonal, all but 6 are required to maintain balance between groups and ensure a solution.

An alternative would be to use a like-with-like formalism (Blythe and Castillo-Chavez, 1989; Castillo-Chavez and Blythe, 1989), where all groups are connected but preference falls off away from the diagonal. Castillo-Chavez and Blythe (1989) looked at a preference function $\phi$ in the form of a narrow rectangle around the line i = j. As the original spirit of Kaplan et.al.'s (1989) paper really involves preference or selection rather than complete disconnection, the like-with-like approach may be more appropriate in this case.

## 6.. DISCUSSION AND CONCLUSIONS

We have seen that the incomplete-connection solution, Equation (11), encompasses both the general solution to the one-sex and to the two-sex problem. As Blythe and Castillo-Chavez (MS) and this paper have demonstrated, all published solutions to the one-sex problem can be derived from particular choices of $\{\phi_{ij}\}$.

A point worth commenting on is the calculation of the $\{\phi_{ij}\}$. It is not difficult to write down an algorithm for calculating the $\phi$ values needed to satisfy Equation (14), but to be completely general (allowing arbitrary choice of $\chi_C$ and its reference element k in Equation (17)) this involves an unwarranted proliferation of indices. It should be clear, however, that all that is involved is the solution of an equation of the form $\underset{\sim}{A}\ \underset{\sim}{\theta} = \underset{\sim}{B}$, where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are given, and $\underset{\sim}{\theta}$ contains the necessary $\phi_{ij}$ values. This is a simple operation in linear algebra that can be performed by standard software incorporated into the epidemic modelling package.

The only point of concern is whether a valid solution can be found at all for certain mixing systems, particularly if there are few or no self-loops. It must be accepted that a mixing problem, in epidemiology or otherwise, where no solution can be found for given population and sexual activity vectors, is fundamentally ill-posed. It may be that the most effective approach will be to sacrifice the independence of the levels of sexual activity in the various groups by making some at least of these vary with time, perhaps as in Koopman et.al. (1989), or as functions of the $\{\bar{p}_j(t)\}$, in order to satisfy the mixing constants. This should be seen as an attempt to keep the $\phi_{ij}$ constant, rather than the more rigid constraint of keeping all the $p_{ij}(t)$ constant (as in Anderson et.al. 1989, Gupta et.al. 1989).


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## FIGURE CAPTIONS

1. Not all mixing problems have a solution. Shown are two graphs representing N=4 mixing cases and their adjacency matrices. Each circle represents a group (number in parenthesis is group number) with a given population and characteristic activity level (lower number in each circle, $c_i$ $T_i$). Lines between groups indicate that two groups mix, and self-loops join a group to itself. (a) No solution possible (b) An infinite number of solutions possible.

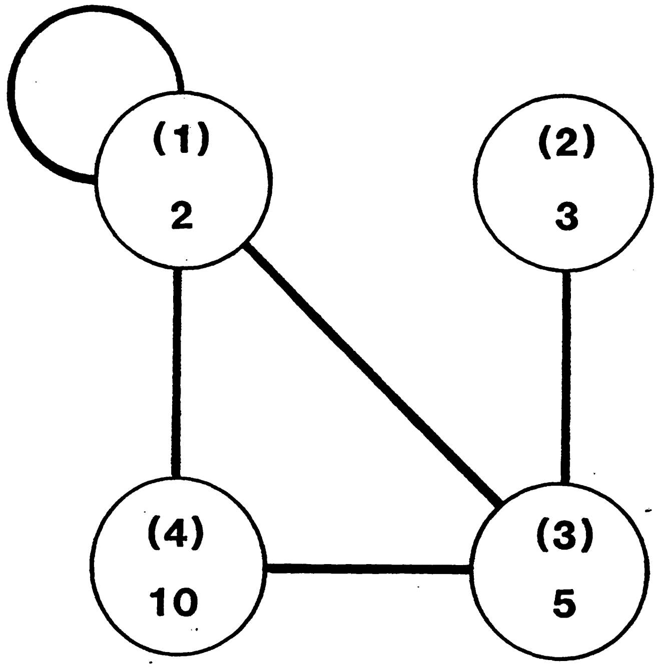
2. Mixing graph for multi-group mixing with a "core group" (group 1), with which all groups mix, and N−1 peripheral or satellite groups, which mix only with the home group and the core group, not with each other. The adjacency matrix has ones in the first row, the first column, and the main diagonal, with zero elsewhere.

3. Bipartite graph, in-mixing case. All groups in class (1) mix, and all groups in class (2) mix, with no cross-over between classes. See Equation (29) for the adjacency matrix. Problem collapses to two separate populations.

4. Bipartite graph, out-mixing or two-sex case. All groups in class (1) mix with all groups in class (2), with no group mixing with any group in the home class. See Equation (36) for the adjacency matrix.

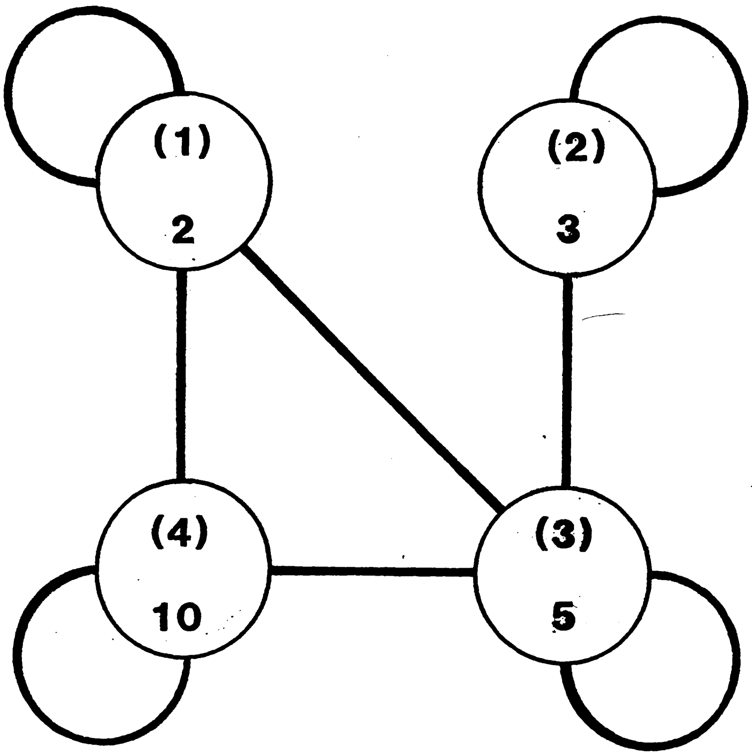
Blythe 1-10 1

(a)



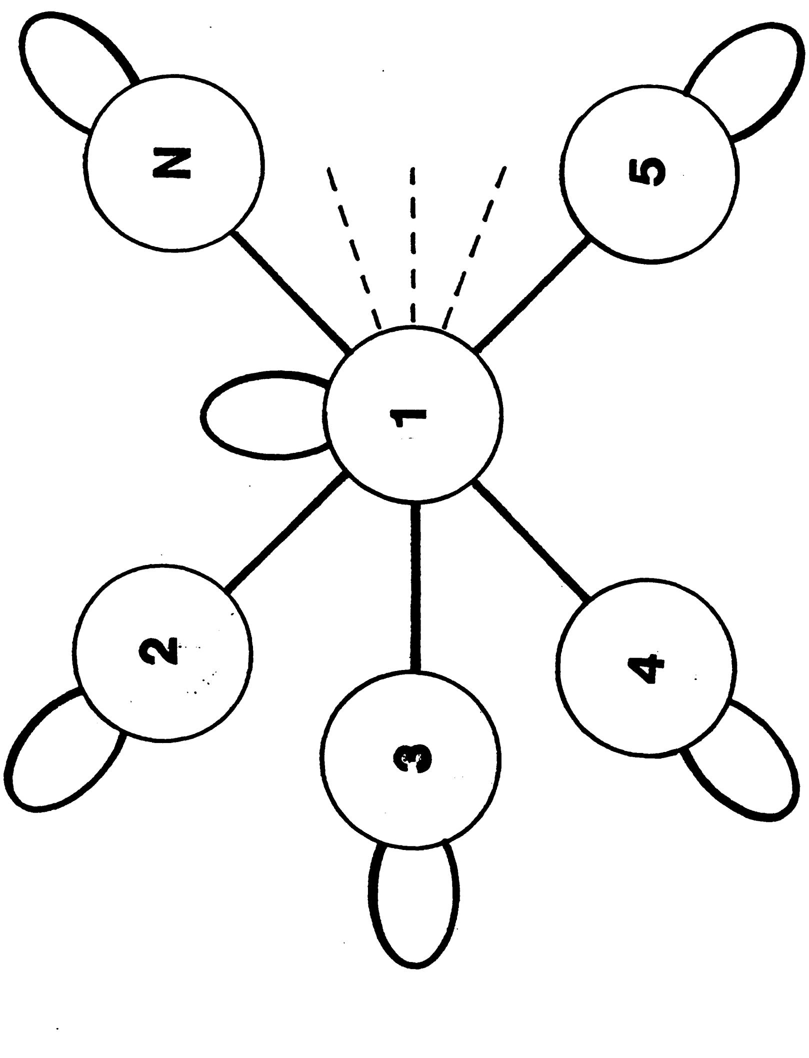

$$\underset{\sim}{x} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(b)

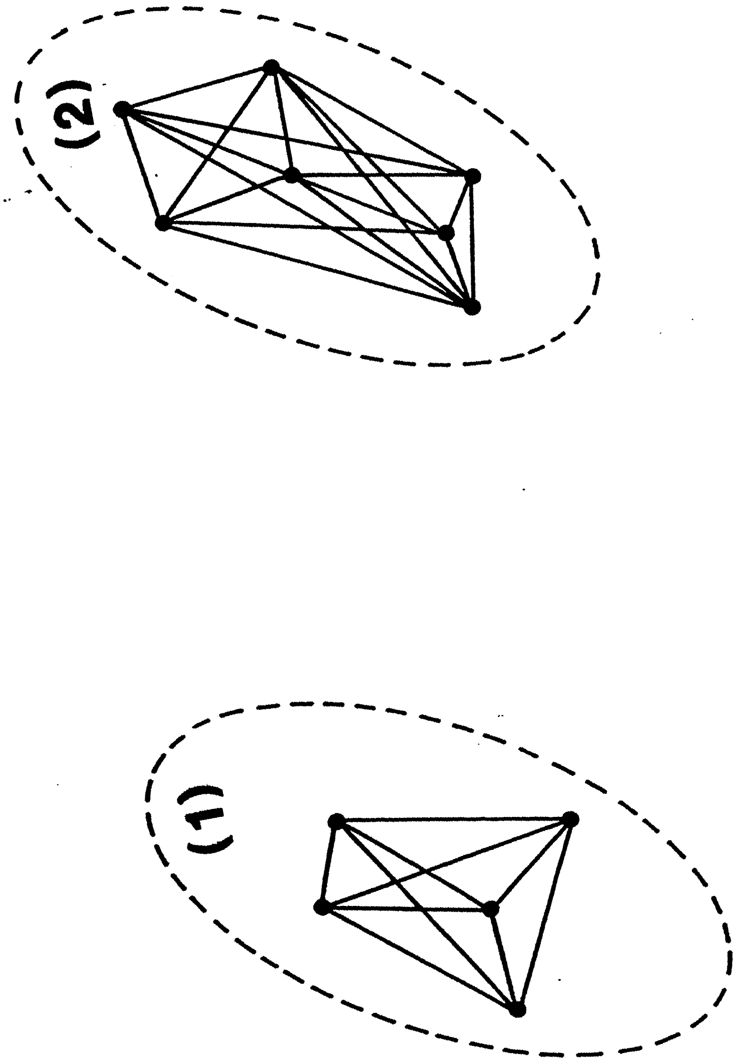



$$\underset{\sim}{x} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Blythe Fig 2



Blythe Fig 3



Blythe Fig 4