THE CONTRIBUTION OF TRADER INTERACTION TO MARKET NOISE

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Inspired by the Cucker-Smale flocking idea, we introduce a heterogeneous agent-based price model that captures explicitly the impact of trader interaction on asset price dynamics, in order to provide insights to a wide range of puzzling stylized facts observed in financial asset returns. Discrete-time models for communication among individual market participants are investigated in Chapter 3, while the role of an influential central authority, such as an equity analyst's report, is studied under a continuous-time setting in Chapter 4. In both cases, we provide limit theorems for normalized sums of dependent stochastic processes that allow us to study analytically the aggregated effect of micro-level communications among a large number of market participants. In addition, we demonstrate via numerical examples that our price model is capable of reproducing asset returns with statistical properties, such as heavy tails, aggregational Gaussianity and volatility clustering, that are in harmony with empirical observations.

BIOGRAPHICAL SKETCH

Xiaofei "Sophia" Liu was born in Jinan, China on November 2, 1982. After high school, she moved to Ontario, Canada to pursue undergraduate studies and obtained her Honors Bachelor of Science with High Distinction in Computer Science Joint Mathematics from the University of Toronto Scarborough Campus in 2007. That same year, she was admitted to the Ph.D. program in the field of Operations Research and Information Engineering at Cornell University. Her doctoral research in Financial Mathematics, under the guidance of Professor Philip Protter, focused on the contribution of trader interaction to market noise.

To my parents:

Jianzheng Liu and Lirong Zhao

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CHAPTER 1

INTRODUCTION

1.1 Motivation

The price of a risky asset is arguably one of the most important elements of many mathematical finance models. In 1900, Louis Bachelier [4] introduced Brownian motion as a model for stock prices in his Ph.D. thesis. At a first glance, this approach has an intuitive economic interpretation: suppose a large number of homogeneous agents trade independently in the market at any given time, where the actual price of the traded asset is determined by its instantaneous supply and demand. While each trade is small and insignificant on its own, their aggregate will generate enough force to push the asset price up or down. By the Central Limit Theorem, the resulting change of asset price will follow a normal distribution under mild conditions. One immediate problem, however, is that Brownian motion does not guarantee the positivity of asset prices at all times. To correct this, Paul Samuelson [51] proposed to use geometric Brownian motion (GBM) as a replacement in 1965. The idea was then championed by the famous Black-Scholes-Merton[10] formula and became a benchmark approach in the modeling of asset prices.

Although mathematically elegant, many empirical properties of stock prices observed in the actual market suggest that geometric Brownian motion is not an accurate representation of asset prices. For example, according to the GBM model, the logarithmic asset return over a certain time period Δt should follow a normal distribution, where Δt can range from a few seconds to several weeks. In reality however, empirical facts shared by price variations of a wide range of financial instruments across different markets show otherwise [15]. In particular, the following statistical properties are often observed in the logarithmic returns of financial assets[1]:

(1) *"Heavy tails"*: the distribution of logarithmic returns often displays a leptokurtic shape, with kurtosis greater than three and tail index between two to five (which excludes the normal distribution and stable laws with infinite variance).

(2) *"Aggregational Gaussianity"*: as we increase the time scale over which logarithmic returns are calculated, their distribution looks more and more similar to a normal distribution.

(3) *"Volatility clustering"*: of either sign, large (resp. small) logarithmic returns tend to be followed by large (resp. small) logarithmic returns. i.e. the magnitude of logarithmic return displays a significant positive autocorrelation.

Figure 1.1 and 1.2 demonstrate the above empirical findings for logrithmic returns of the S&P 500 Index between January 2000 and December 2010[2].

The incompetency of GBM in capturing the stylized facts of asset returns has detrimental impact in areas such as risk management and option pricing. For example, quantile-based portfolio risk measures, such as the industry standard Value-at-Risk (VaR), may be very different when calculated using a heavy-tailed return distribution rather than a Normal distribution. This is particularly true for the highest quantiles of the return distribution, which is associated to a portfolio's potential loss during rare but extremely adverse market movements.

[1] see Cont (2001) [15] for a more comprehensive list of stylized facts.

[2] historical price data obtained from Yahoo! Finance. All calculations, plots and statistical estimations are carried out using software package R, version 2.12.0.

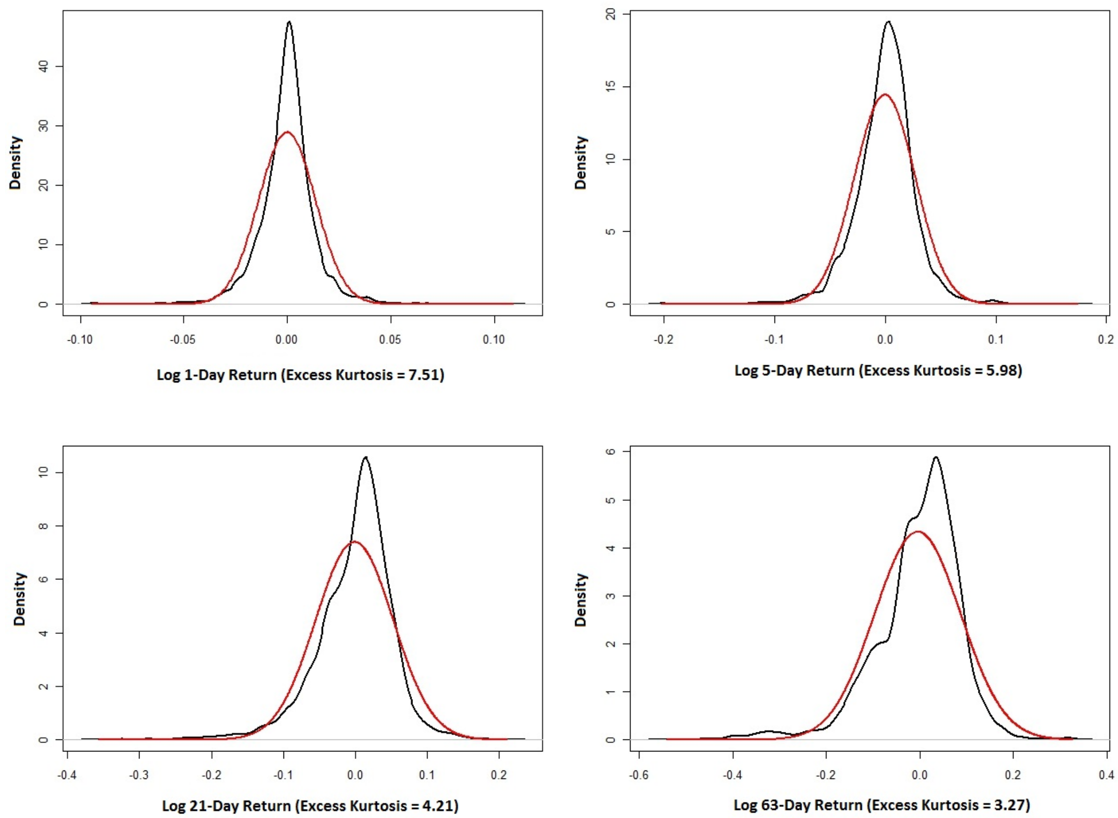

Figure 1.1: Kernel density estimations of 1, 5, 21 and 63-day logarithmic returns of the S&P 500 Index between January 1, 2000 and December 31, 2010, each compared with a Normal density (red) of matching mean and variance. Respective excess sample kurtosis are also reported.

In option pricing, a core premise of the most celebrated Black-Scholes-Merton (BSM) formula is that the underlying asset price follows a geometric Brownian motion. Since the constant volatility parameter σ in the BSM model is a property of the underlying asset, its value should not be dependent on a particular option's strike or maturity. In reality however, when equating an option's BSM model price with its actual market price, the resulting value of the volatility parameter, referred to as the *Implied Volatility*, clearly depends on the option's strike and maturity in a systematic and time-varying fashion. In particular, for a

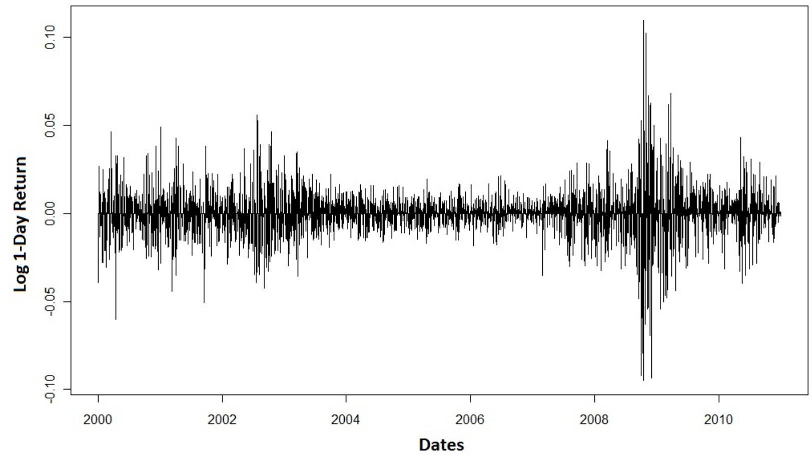

Figure 1.2: Time series of 1-day logarithmic return of the S&P 500 Index between January 1, 2000 and December 31, 2010.

fixed underlying asset, lower-strike options tend to have hight implied volatilities, which means that their actual prices in the market are higher than those predicted by the BSM model. Such deviations, known as the ”volatility skew”, have created many challenges for the pricing and hedging of exotic derivative securities.

In response to shortfalls of the geometric Brownian motion, many sophisticated mathematical models have been developed in attempts to provide a more adequate description of asset price fluctuations and to resolve conflicts arising from derivative securities pricing. Most of them fall under one of the following four categories:

1. *Lévy Jump-diffusion Models*

Since extremely large returns occur rather frequently in financial data, many suspected that asset prices not only move continuously, but jump from time to time. Merton (1976) proposed the first Jump-diffusion model for equity prices, which uses Brownian motion to capture small price movements and jumps for the larger ones. Under this model, option payoffs cannot be replicated by trading in the primitive assets, resulting in an incomplete market. Other models featuring jumps in asset returns include Madan et al. (1998) on the Variance Gamma model and Carr et al. (2002) on the CGMY model (named after the authors). Since the jump structures are usually specified to match the observed volatility skew, this class of models can generate implied volatility curves that are fairly consistent with market option prices for a single maturity. However, the implied volatility curve of the longer-dated options flattens out due to the models' i.i.d. return structure [6]. The models also fail to explain the volatility clustering and leverage effect observed in the historical realized volatility.

2. *Local Volatility Models*

Dupire (1994) and Derman and Kani (1994) note that under risk-neutrality, there exists a unique deterministic "local volatility function" of the underlying price S and calendar time t, which is consistent with and can be implied from the current European option prices. Such "local volatility" does not represent how volatilities evolve over time. Instead, it attempts to capture an "average" over all possible instantaneous volatilities under a stochastic setting [28]. Local volatility models retain a convenient hedging argument similar to that of the BSM model, and are widely used to price exotic options after proper calibration to market data. Nevertheless, the use of time-dependent instantaneous volatilities makes it difficult to implement empirical tests, which are based on stationarity assumptions with respect to calendar time.

3. *Stochastic Volatility Models*

Stochastic Volatility models, including Hull and White (1988) and Heston (1993), are largely inspired by the phenomenon of *Volatility Clustering*. They employ a separate Markovian stochastic process to capture the evolution of the instantaneous volatility, while assigning a non-zero correlation between the asset's log-returns and changes in its volatility process in order to reproduce the volatility skew observed in the market. Although it can be challenging to fit parameters to the current European option prices, these models are often qualitatively consistent with the stochastic, yet mean-reverting nature of implied volatilities. They are also capable of generating autocorrelations in absolute asset returns that are present in the empirical data. Unfortunately, the path continuity ties these models to Brownian motions and prevents the possibility of having very large variations in a short period of time. As a result, it is difficult for Stochastic Volatility models to replicate the steep skews typically observed for options with very short maturities.

4. *Affine Jump-diffusion Models with Stochastic Volatility*

This last class of price models are designed to combine the advantages of Lévy Jump-diffusion models and Stochastic Volatility models. Carr et al. (2003) employ an additional Markovian process to time-change the Lévy process that governs the underlying asset price dynamics. Bakshi and Madan (2000) and Duffie et al. (2000) provide general discussions of affine jump-diffusion models with stochastic volatility, which yield analytic solutions to derivative security pricing.

It is without a doubt that many of the above price models can successfully reproduce several empirical properties of asset returns. Nevertheless, they tend to be mathematically engineered and lack fundamental economic interpretation. This observation motivated us to search for a new asset price model, which not only captures the important stylized facts of asset returns observed in the market, but also is easy to understand from an economic point of view.

One of the most criticized assumptions embedded in the GBM model is that market participants act independently from one another. As we all know, an individual's thoughts and actions can be heavily influenced by others through observation and communication. The finance literature attributes a wide range of trading behaviors observed in security markets to interactions among market participants. One popular example, referred to as "herding", describes the situation where an agent imitates others' actions irrespective of his own private information [12]. Based on data from the Toronto Stock Exchange, Griffiths et al. (1998) find increased similarity in successive trades for securities that are exchanged in an open outcry market, which supports the hypothesis of traders engaging in imitating behavior when they can better identify each other [8]. Wermers (1999) analyze the trading activity of mutual funds from 1975 through 1994. He reports a higher level of herding among growth-oriented funds, and find that stocks the herd buy outperform stocks they sell during the following six months. Welch (2000) shows that the buy or sell recommendations of security analysts have a significant positive influence on the recommendations of the next two analysts and that analysts' choices are correlated with the prevailing consensus forecast even when it is subsequently proven inaccurate. Another well documented trading behavior, known as "contrarianism", is the intuitive counterpart to herding. In this case, agents choose to go against the direction of the crowd in stead of following. Griffin et al. (2003) study the trading of individual investors in NASDAQ 100 securities and provide evidence of contrarian behavior by traders who submit orders through retail brokers. Kaniel et al. (2008)

examine NYSE trading data from 2000 to 2003 and find that individuals buy stocks after prices decrease and sell stocks after prices increase.

While some believe that herding and contrarianism are results of investors' irrational "animal instinct", many economists support full or partial rationality behind these trading behaviors in financial markets. A seminal work is the famous *Keynesian Beauty Contest* analogy introduced by Keynes [38] in 1936, who argued that in order to maximize their profitability, rational investors would price an asset not based on their own information regarding its fundamental value, but on what they think other market participants might perceive that value to be. Scharfstein and Stein (1990), Graham (1999), Trueman (1994) and others[3] find that when the evaluation process is based on relative rather than absolute performance, reputation concerns will cause individual agents to mimic others' actions instead of following their private information. Welch (1992) proposes yet another explanation for rational herding, known as "informational cascade". When actions rather than private information are publicly visible, agents gain useful information from observing their predecessors' decisions, which ultimately lead to abandonment of their own private information. As a result, all subsequent agents will behave alike. Informational cascade can help explain phenomena such as massive herding on an inferior decision or sudden reverse of long-standing trends. References related to this notion can be found in [9].

In this dissertation, we use limit theorems and the Cucker-Smale flocking idea [19] to construct dynamic asset price models, which capture explicitly the impact of communications among market participants. We demonstrate, via simulation and statistical analysis, that certain rational behaviors of traders and

[3]For a list of references see Devenow and Welch (1996) and Hirshleifer and Teoh (2003).

specific dependence structures amongst them could provide a strong alternative explanation to many empirical properties of asset returns noted at the beginning of this section.

1.2 Agent-Based Price Models: A Review

During the past few decades, modeling economic markets from the bottom up with a large number of interacting heterogeneous agents has become a popular research field, especially in financial settings where information aggregation across the market is critical to the formation of asset prices. While some models provide rigorous theoretical analysis, others rely heavily on computational tools to break through the restrictions of analytic methods. In this section, we review a few important early papers from each category.

1.2.1 Analytic Models

There exist many analytically tractable heterogeneous agent-based models that focus on the stochastic interaction amongst market participants and its financial implications. Although most admittedly rely on unrealistic simplifications and assumptions, they provide valuable mathematical explanations to many puzzling stylized facts observed in empirical market data. As an early example, Föllmer (1974) analyzes an exchange economy with random preferences using results on interacting particle systems in physics, and shows that even short-ranged interaction among individual agents can propagate through the economy and cause significant impact on price dynamics in aggregation.

Kirman (1991) considers an exchange rate model consisting of two distinct types of traders - *fundamentalists* and *chartists* - choosing to invest in a risk free domestic currency or a risky foreign currency in order to maximize their expected utilities. While a fundamentalist believes that the exchange rate will always move towards a certain fundamental value and formulates her demand for the foreign currency accordingly, chartists base their exchange rate forecast on past rates observed in the market. Fractions of both types of agents evolve stochastically as agents try to assess the current majority type of the market and decide whether to act as a fundamentalist or a chartist. An equilibrium exchange rate is then obtained by equating the total supply of the foreign currency with the aggregated individual demands. Simulated time series of such an equilibrium rate captures several stylized facts observed in empirical market data [40]. In particular, when the two types of traders alternate to dominate the market, we observe clear volatility clustering in the corresponding exchange rate. Another agent-based model that's rather successful in explaining similar stylized facts was introduced in Lux (1998), where a fixed number of speculative traders are again divided into *fundamentalists* and *chartists*. Individuals from different groups randomly meet one another, compare their respective expected gains and losses, and possibly change to the opposite trading strategy afterwards. In addition, the *chartists* are further categorized as either *optimistic* or *pessimistic*, and may switch from one subgroup to another by following the predominant opinion in the current market as well as the actual price changes. Finally, a market maker absorbs the aggregated excess demand of individual participants while making price changes accordingly in any given period.

A major drawback shared by many analytic agent-based models is that market participants are usually divided into several distinct groups and are assumed to behave homogeneously within each group. As a result, both the interaction among agents and the aggregation of their excess demands are often formulated at the group-level, making these models only quasi heterogeneous. One of the few exceptions was Föllmer and Schweizer (1993) and Föllmer et al. (1994), who study the derivation of diffusion price models by combining the microeconomic point of view with an invariance principle. Their model motivates the equilibrium price process in terms of assumptions at the level of individual agents, who are not restricted to any specific strategy group and may have mixed trading behaviors. Nevertheless, as pointed out by Föllmer et al. (1994), this model focuses primarily on local interactions between the transition probability of each individual's behavior and the overall market environment, while the transitions themselves are made independently by different agents. If additional interactions appear directly among these transitions, the law of large numbers may no longer hold.

1.2.2 Computational Models

Analytic agent-based models gain their mathematical tractability at the price of deviating further away from the real world due to inevitable assumptions and simplifications. As abundant computational power and high quality financial data sets become widely available, computer simulated agent-based models attracted increasing attention from researchers. These models are able to incorporate complicated market dynamics and heterogeneity concerning information representation, preference types, the price formation mechanism, as well as individual learning and communication among market participants. Meanwhile, the complexity of the resulting artificial markets makes it almost impossible to study them analytically.

Using a Genetic Algorithm common in many computational learning models, Lettau (1997) implements a financial market model with a set of heterogeneous learning agents, whose only task is to decide how to divide their investment between a risky asset and a risk-free asset paying zero interest. Instead of looking at the impact of agents' excess demands on asset returns, the price of the risky asset is given exogenously, which allows the model to concentrate solely on agents' learning behavior. Results show that given simple agent preferences, the evolutionary Genetic Algorithm is able to discover the optimal portfolio weights, with a slight bias towards holding more risky asset.

Arifovic (1996) simulates a more complicated two-period, two-country equilibrium foreign exchange market, where agents have income and consumption in both periods, and may save their income from the first to the second period in either country's currency. Their goal is to maximize a two-period log utility function subject to certain budget constraints. Unlike Lettau (1997), agents' aggregated consumption and portfolio decisions endogenously determine the price levels in the market. Results from the Genetic Algorithm learning procedure show that the first period consumption level is rather stable but the exchange rate fails to settle to any constant value. Laboratory experiments on the same foreign exchange model with human subjects yield similar conclusions.

The Santa Fe Artificial Stock Market described in LeBaron et al. (1999) is one of the most complex agent-based computational models created to study the co-evolution of different types of strategies in a dynamic trading environment. Market participants have one-period myopic preferences of future wealth, and must allocate their investments between a risk free bond yielding constant interests and a risky stock paying stochastic dividends. A classifier system is used to form agents' individual expectations, which maps information regarding the current state of the economy into a future price and dividend forecast. In addition, traders have the flexibility of using or ignoring different pieces of information presented to them. They may also, with a certain probability, engage in learning processes to update their current forecasting rules at the end of each period. With its publicly available software, the Santa Fe model has served as a platform for many studies regarding interactions between different types of traders. Unfortunately, given the complexity of the model, it is ultimately difficult to make conclusions about how well the simulated market can reflect the real world.

1.3 A Note on Dependence and the Central Limit Theorem

From a microeconomic point of view, the geometric Brownian motion price model relies on the Central Limit Theorem (CLT) to describe the aggregated supply and demand of a large number of agents participating in the market. In its most basic form, the CLT states that if a sequence of random variables $X_1, X_2, \ldots$ are independent and identically distributed with zero mean and variance $0 < \sigma^2 < \infty$, the normalized partial sum $S_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i$ converges in law to a $N(0, \sigma^2)$ random variable as $N \to \infty$. If in addition $\mathbb{E}[|X_i|^3] = \rho < \infty$, the Berry-Esseen Theorem guarantees that the distance between the cumulative distribution functions of S_N and $N(0, \sigma^2)$ is at most $\frac{3\rho}{\sigma^3 \sqrt{N}}$.

As discussed in Section 1.1, it has become widely accepted that observation and communication may create a complicated dependence structure amongst participants in a financial market. As a result, newly proposed agent-based models that incorporate individual learning and interactions can no longer apply the traditional CLT when analyzing the aggregated macro-effect of micro-level trader behaviors. Much literature has been devoted to relaxing the CLT's independence assumption, including deep convergence results based on various strong mixing conditions proposed by Rosenblatt (1956) and others[4]. For example, Samur (1984) and Peligrad (1996) consider limit theorems for mixing triangular arrays. Unfortunately, strong mixing conditions can be quite difficult to check, and many classes of time series simply do not satisfy them [1]. On the other hand, most of these time series enter the scope of *Mixingales*, which is another popular concept developed to study the dependence among random variables. However, it is far more difficult to obtain limit theorems or even moment inequalities under the mixingale setting.

The limit theorems presented in Chapter 3 of this dissertation fall under the framework of *m-dependent random variables with unbounded m* given by Berk (1973). It is worth mentioning that limit theorems for mixing triangular arrays such as Samur (1984) and Peligrad (1996) do not imply our results, since in our model the dependence among random variables strengthens (as opposed to uniformly decay) with the row index. In Chapter 4 we provide a Central Limit Theorem for the normed sum of a sequence of dependent stochastic processes (instead of a sequence of dependent random variables). While literature in this area is rather scarce, Jacod and Shiryaev (1987) provides a set of results for i.i.d semimartingales that serve as a building block for the proof of our theorem.

[4]See Bradley (2005) for a survey.

CHAPTER 2

A FLOCKING-INSPIRED PRICE MODEL

2.1 The Cucker-Smale Flocking Model

The mathematical models we use to describe communication among market participants in Chapter 3 and Chapter 4 are inspired by Cucker and Smale's work on modeling emergent behavior in flocks [18] [19] [17].

It has been observed that under certain initial conditions, the state of a flock converges in time to one in which all birds fly with the same velocity. Cucker and Smale postulate a model for the evolution of a flock, where each bird adjusts its velocity by adding to it a weighted average of the difference between its own velocity and those of the other birds. The weights used to quantify how birds influence one another are assumed to be a function of the distance between each corresponding pair [18].

More specifically, consider a flock of k birds, where $x_i(t) \in \mathbb{R}^3$ and $v_i(t) \in \mathbb{R}^3$ represent the position and velocity of bird $i \in \{1, \ldots, k\}$ at time t, respectively. In a discrete-time setting, the Cucker-Smale flocking model is given by:

$$\begin{cases} x_i(t+h) &= x_i(t) + h v_i(t) \\ v_i(t+h) &= v_i(t) + h \sum_{j=1}^{k} a_{ij} \left(v_j(t) - v_i(t) \right) \end{cases} \qquad i \in \{1, \ldots, k\}. \tag{2.1}$$

Here $h > 0$ is the magnitude of the time step and the weights $\left\{ a_{ij} : i, j \in \{1, \ldots, k\} \right\}$ are defined as

$$a_{ij} = \frac{K}{\left(1 + \|x_i - x_j\|^2\right)^{\beta}}, \quad \text{for some fixed } K > 0 \text{ and } \beta \geq 0. \tag{2.2}$$

Intuitively, the communication strength between bird i and j decreases continuously as they separate in space, and the "rate of decay" is captured by the constant $\beta > 0$. It is worth mentioning that there exist other related models [55] where bird i only communicates with those birds that are "not too far away", i.e. $a_{ij} \neq 0$ if and only if $\|x_i - x_j\| \leq r$ for some constant $r > 0$. Our discrete-time model in Chapter 3 adopts the latter approach.

System (2.1) has the following continuous-time counterpart:

$$\begin{cases} x_i'(t) &= v_i(t) \\ v_i'(t) &= \sum_{j=1}^{k} a_{ij}\left(v_j(t) - v_i(t)\right) \end{cases} \qquad i \in \{1, \ldots, k\}. \tag{2.3}$$

In both discrete-time and continuous-time cases, under communication scheme (2.2), Cucker and Smale [18] provide a set of explicit conditions on the initial state of the flock $\{x_i(0), v_i(0) : i \in \{1, \ldots, k\}\}$ and the constants $\beta \geq 0$, $K > 0$, such that whenever these conditions are satisfied, the state of the flock is guaranteed to converge to one where all birds fly with the same velocity, i.e. there exists some $\hat{v} \in \mathbb{R}^3$ such that $v_i(t) \to \hat{v}$ as $t \to \infty$. They also extend the results to several different communication schema in [19], including

(1) *"k birds with sequential leadership"*, where bird i influences bird $(i+1)$ for all $i \in \{1, \ldots, k-1\}$ and no other influence between different birds occur. Our discrete-time model in Chapter 3 follows this scheme.

(2) *"k birds with a leader"*, where bird 1 influences birds $2, \ldots, k$ and no other

influence between different birds occur. The same communication structure is used in our continuous-time model in Chapter 4.

More recently, Cucker and Mordecki study a flock's emergent behavior in a noisy environment under a communication scheme very similar to (2.2). In particular, they perturb (2.1) and (2.3) with some additive random noises, $\{H_i(t) : i = 1, \ldots, k\}$, which result in the following systems:

In discrete-time:

$$\begin{cases} x_i(t+h) &= x_i(t) + hv_i(t) \\ v_i(t+h) &= v_i(t) + h\sum_{j=1}^{k} a_{ij}\left(v_j(t) - v_i(t)\right) + hH_i(t) \end{cases} \qquad i \in \{1, \ldots, k\}; \quad (2.4)$$

In continuous-time:

$$\begin{cases} x_i'(t) &= v_i(t) \\ v_i'(t) &= \sum_{j=1}^{k} a_{ij}\left(v_j(t) - v_i(t)\right) + H_i(t) \end{cases} \qquad i \in \{1, \ldots, k\}. \quad (2.5)$$

For the discrete-time model (2.4), the perturbation terms $\mathbf{H}(t) = (H_1(t), \ldots, H_k(t))$, $t \in \{0, h, 2h, \ldots\}$ form an i.i.d sequence of random variables, where $\mathbf{H}(t) \in \mathbb{R}^{3k}$ follows either a Uniform distribution on $\mathcal{B}(0, r) \subset \mathbb{R}^{3k}$ for some $r > 0$, or a centered Gaussian distribution with covariance matrix $\sigma^2\mathbf{Id}_{3k}$. The latter probability structure is sometimes referred to as a "Gaussian white noise sequence". We investigate a stochastic model involving a multiplicative (rather than additive) noise of the same type in Chapter 3 of this dissertation. For the continuous-time model (2.5), the perturbation terms are constructed by differentiating smooth approximations of Wiener processes with respect to time. The authors also remarked an alternative way of modeling $\mathbf{H}(t)$ using Itô Stochastic Calculus [17].

Due to the presence of noise, convergence of the birds' velocities to a common value cannot be expected as described in the deterministic case [19]. Indeed, when $\{v_1, \ldots, v_k\}$ are "similar enough" compared with the perturbation, the latter could outdo the contractive nature of the flock. Nevertheless, in both discrete-time and continuous-time cases, Cucker and Mordecki show that with high probability, a "nearly-alignment" of $\{v_1, \ldots, v_k\}$ occurs after a give time point, when certain restrictions on initial state of the flock and the model parameters are satisfied. In particular, if the perturbation remains small relative to the "dissimilarities" among $\{v_1, \ldots, v_k\}$ throughout the evolution of the flock, a perfect alignment of the birds' velocities will occur almost surely as $t \to \infty$.

2.2 A Mathematical Model for Asset Price

In Chapters 3 and 4 of this dissertation, we present two Flocking-inspired mathematical models that allow us to describe the (normalized) aggregated excess demand of all interactive agents in the entire market. To incorporate this information into asset price formation, we adopt the following slow price adjustment approach: Suppose the market-maker quotes a price that reflects the current fundamental value of the traded asset. Individual traders can then submit their buy or sell orders at this price. If an agent has private reasons to speculate that an asset will become more valuable in the near future, her interest in buying this asset will increase, i.e. the "positive" speculation will generate certain extra demand for the asset on top of its fundamental economic demand. Similarly, an agent's "negative" speculation will raise her interest in selling the asset and in turn lead to additional supply in the market. When aggregated over all market participants, such speculative demand or supply will also cause the asset price to fluctuate. In particular, if the sum of all market orders turns out to be an excess ”sell”, the price of the asset will decrease from its original quote. Otherwise, if the market orders total to an excess ”buy”, the price will increase instead. The magnitude of the price change is often assumed to be proportional to the amount of overall excess supply or demand.

To put the above idea in a mathematical context, let $S(t)$ be the logarithmic price of a risky asset and $V^*(t)$ be its normalized aggregated speculative demand at time t, as modeled in Chapter 3 and Chapter 4. Changes in $S(t)$ then satisfy the following equality:

$$\Delta S(t) = f(\Delta W(t)) + g(\Delta V^*(t))$$

where $f(\cdot)$ and $g(\cdot)$ are both deterministic, monotone increasing functions that vanish at 0. $W(t)$ is a standard Brownian motion and the term $f(\Delta W(t))$ represents fluctuation in the logarithmic asset price caused by changes in the fundamental value of the asset. Taking a first-order approximation, we get

$$\Delta S(t) \approx c \cdot \Delta W(t) + d \cdot \Delta V^*(t)$$

where $c, d > 0$ are some constants. Therefore, to study the process $S(\cdot)$, we need only to understand the process $V^*(\cdot)$.

Although our price formation method makes it easy to pin down the impact of agent communication through the aggregated speculative demand process $V^*(\cdot)$, it implies that the market is never really in equilibrium. One thing we should always keep in mind is that under such a setting, it’s often possible for the resulting asset price to spend a lot of time being far away from the value that actually clears the market. It is worth mentioning that there indeed exist price formation methods that clear the market in each time period. However, it is difficult to apply them to our problem, as agents’ speculative demands are affected solely by their interaction with one another, not the current asset price.

CHAPTER 3

DISCRETE-TIME INTERACTION AMONG MARKET PARTICIPANTS

3.1 Model Specification

Consider a finite time horizon $[0, T]$. For all $i \in \mathbb{N}^+$, V_t^i represents the speculative demand of agent i at time $t \in [0, T]$. For each fixed $K \in \mathbb{N}^+$, let

$$\Pi^K := \left\{0 = t_0^K < t_1^K < \cdots < t_{K-1}^K < t_K^K = T\right\}$$

be an equidistant partition of the interval $[0, T]$, where

$$t_k^K = k \cdot \frac{T}{K} =: k \cdot h^K \quad \text{for all } \; k \in \{1, \ldots, K\}.$$

From here onwards, we will suppress the superscript K whenever the context is clear.

At time $t_0 = 0$, agents form i.i.d. speculative demand ξ_i according to some common distribution, i.e.

$$V_0^i = \xi_i$$

where $\mathbb{E}[\xi_i] = 0$ and $\mathbb{E}[\xi_i^2] = \sigma^2 < \infty$ for all $i \in \mathbb{N}^+$. Subsequently, at times $\{t_k : k \in \mathbb{N}^+\}$, agent i modifies her speculative demand by adding to it a weighted average of the difference between several other agents' speculative demands and that of her own. More specifically, for all $i \in \mathbb{N}^+$ and $k \in \{1, \ldots, K\}$,

$$V_{t_k}^i = V_{t_{k-1}}^i + \sum_{j \in A_i} \alpha_{t_k}^{ij} \cdot h \cdot \left(V_{t_{k-1}}^j - V_{t_{k-1}}^i\right), \tag{3.1}$$

where the set $A_i \subset \mathbb{N}^+ \backslash \{i\}$ contains all market participants whom agent i actively communicates with. The stochastic process α^{ij} captures the directed instantaneous impact agent j has on agent i through their interactions, so α^{ij} and α^{ji} are different processes in general. By construction, the processes V^i depend on the total number of periods K we have in the time horizon [0,T] for all $i \in \mathbb{N}^+$.

Finally, we consider the process[1]

$$\overline{V}^{K,N} := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} V^{i,K}, \tag{3.2}$$

As $K, N \to \infty$, $\overline{V}^{K,N}$ captures the normalized aggregated speculative demand of all agents participating in the market as their interactions become more and more frequent.

3.2 Case I: Constant Sequential Communication

3.2.1 Assumptions and the Main Theorem

Assumption 3.2.1 In addition to Section 3.1, suppose the following hold:

(1) the initial speculative demand of agent i satisfies $\mathbb{E}[\xi_i^4] = \nu < \infty$ for all $i \in \mathbb{N}^+$ and some constant $\nu > 0$;

(2) $A_i = \{i+1\}$ for all $i \in \mathbb{N}^+$, i.e. at each time step, agent i adjusts her speculative demand only through communication with agent (i+1);

(3) for each fixed $K \in \mathbb{N}^+$, $\alpha^{i,i+1} \equiv \alpha \cdot \frac{K}{T} > 0$ for all $i \in \mathbb{N}^+$, where $0 < \alpha < 1$ is some constant. That is, agent i mimics the trading behavior of agent (i+1) at a constant rate that's inversely proportional to the length of the time steps

[1] We include the superscript K in this definition explicitly as a reminder that the process $\overline{V}^{K,N}$ depends not only on the number of agents in the summation, but also the number of time steps in the corresponding partition of $[0, T]$.

throughout the entire horizon $[0, T]$. As a result, equation (3.1) simplifies to

$$V_{t_k}^i = (1-\alpha)V_{t_{k-1}}^i + \alpha V_{t_{k-1}}^{i+1}. \tag{3.3}$$

Theorem 3.2.2 *Let $N = N(K)$. Suppose Assumption 3.2.1 holds, and*

$$\lim_{K\to\infty} \frac{K^3}{N} = 0.$$

Then

$$\overline{V}_t^{K,N} := \frac{1}{\sqrt{N(K)}} \sum_{i=1}^{N(K)} V_t^{i,K} \overset{\mathcal{L}}{\Longrightarrow} N(0, \sigma^2) \quad \text{as } K \longrightarrow \infty$$

for all $t \in [0, T]$, where $\sigma^2 = \mathbb{E}[\xi_i^2] < \infty$.

3.2.2 Proof of the Main Theorem

We first prove a few useful Lemmas.

Lemma 3.2.3 *For each $K \in \mathbb{N}^+$, the corresponding $\{V_{t_k}^i : i \in \mathbb{N}^+\}$ is a k-dependent sequence of random variables such that $\mathbb{E}[V_{t_k}^i] = 0$ and $\mathbb{E}[|V_{t_k}^i|^4] \le \nu + 6\sigma^4$ for all $k \in \{1, \ldots, K\}$, where $\nu = \mathbb{E}[\xi_i^4]$ and $\sigma^2 = \mathbb{E}[\xi_i^2]$.*

Proof By the model specification in Section 3.1 and Assumption 3.2.1, agent i's speculative demand satisfies the following system of difference equations for all $i \in \mathbb{N}^+$:

$$\begin{cases} V_{t_0}^i &= \xi_i \\ V_{t_k}^i &= (1-\alpha)V_{t_{k-1}}^i + \alpha V_{t_{k-1}}^{i+1} \qquad \text{for all } k = 1, \ldots, K. \end{cases}$$

Iterated computations yield

$$
\begin{aligned}
V^i_{t_0} &= \xi_i \\
V^i_{t_1} &= (1-\alpha)\xi_i + \alpha\xi_{i+1} \\
V^i_{t_2} &= (1-\alpha)^2\xi_i + 2\alpha(1-\alpha)\xi_{i+1} + \alpha^2\xi_{i+2} \\
& \quad \dots \\
V^i_{t_k} &= \sum_{j=0}^{k}\binom{k}{j}\alpha^j(1-\alpha)^{k-j}\xi_{i+j} =: \sum_{j=0}^{k}\theta_{k,j}\cdot\xi_{i+j}
\end{aligned}
$$

for all $k = 1, \dots, K$ and $i \in \mathbb{N}^+$. Since $0 < \alpha < 1$ by assumption and

$$
\sum_{j=0}^{k}\theta_{k,j} = \sum_{j=0}^{k}\binom{k}{j}\alpha^j(1-\alpha)^{k-j} = (\alpha + 1 - \alpha)^k = 1
$$

by the Binomial Theorem, we know that $0 < \theta_{k,j} < 1$ for all k and j. Moreover, $\{\xi_i : i = 1, 2, \dots\}$ are i.i.d. random variables with

$$
\mathbb{E}[\xi_i] = 0, \quad \mathbb{E}\left[\xi_i^2\right] = \sigma^2 < \infty \quad \text{and} \quad \mathbb{E}\left[\xi_i^4\right] = \nu < \infty
$$

Thus,

$$
\mathbb{E}\left[V^i_{t_k}\right] = \mathbb{E}\left[\sum_{j=0}^{k}\theta_{k,j}\cdot\xi_{i+j}\right] = \mathbb{E}[\xi_1]\cdot\sum_{j=0}^{k}\theta_{k,j} = 0,
$$

and the Multinomial Theorem implies that

$$
\begin{aligned}
\mathbb{E}\left[\left|V^i_{t_k}\right|^4\right] &= \mathbb{E}\left[\left(\sum_{j=0}^{k}\theta_{k,j}\cdot\xi_{i+j}\right)^4\right] \\
&= \mathbb{E}\left[\sum_{\substack{d_0,\dots,d_k\in\mathbb{N} \\ d_0+\dots+d_k=4}}\binom{4}{d_0,\dots,d_k}\prod_{j=0}^{k}\left(\theta_{k,j}\cdot\xi_{i+j}\right)^{d_j}\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{k}\left(\theta_{k,j}\cdot\xi_{i+j}\right)^4\right] + \mathbb{E}\left[\sum_{\substack{j,r=0 \\ j<r}}^{k} 6\left(\theta_{k,j}\cdot\xi_{i+j}\right)^2\left(\theta_{k,r}\cdot\xi_{i+r}\right)^2\right] \\
&= \mathbb{E}\left[\xi_1^4\right]\cdot\sum_{j=0}^{k}\theta_{k,j}^4 + 6\mathbb{E}\left[\xi_1^2\right]\cdot\mathbb{E}\left[\xi_1^2\right]\cdot\sum_{\substack{j,r=0 \\ j<r}}^{k}\theta_{k,j}^2\theta_{k,r}^2 \\
&= \nu\cdot\sum_{j=0}^{k}\theta_{k,j}^4 + 6\sigma^4\cdot\sum_{j=0}^{k}\left(\theta_{k,j}^2\cdot\sum_{r=j+1}^{k}\theta_{k,r}^2\right) \\
&\leq \nu + 6\sigma^4.
\end{aligned}
$$

The last inequality holds as $0 < \theta_{k,j} < 1$ for all k and j implies that

$$0 < \sum_{j=0}^{k} \theta_{k,j}^{4} < \sum_{j=0}^{k} \theta_{k,j}^{2} < \sum_{j=0}^{k} \theta_{k,j} = 1.$$

Finally, for any $r \in \mathbb{N}^+$,

$$\left\{V_{t_k}^i : i \leq r, i \in \mathbb{N}^+\right\} \;=\; \{g(\xi_i, \ldots, \xi_{i+k}) : i \leq r, i \in \mathbb{N}^+\} =: \mathcal{S}_1,$$

and

$$\begin{aligned}\left\{V_{t_k}^i : i > r + k, i \in \mathbb{N}^+\right\} &= \{g(\xi_i, \ldots, \xi_{i+k}) : i > r + k, i \in \mathbb{N}^+\} \\ &= \{g(\xi_{i+k}, \ldots, \xi_{i+2k}) : i > r, j \in \mathbb{N}^+\} =: \mathcal{S}_2.\end{aligned}$$

Since $\{\xi_i : i = 1, 2, \ldots\}$ are i.i.d. random variables and $g(\cdot)$ is a measurable function of its arguments, we know that $\mathcal{S}_1$ and $\mathcal{S}_2$ are also independent. By definition[2], $\{V_{t_k}^i : i \in \mathbb{N}^+\}$ is a k-dependent sequence of random variables. This completes the proof.

□

Lemma 3.2.4 *For each $K \in \mathbb{N}^+$, the corresponding sequence $\{V_{t_k}^i : i \in \mathbb{N}^+\}$ satisfies*

$$Var\left[V_{t_k}^{i+1} + \cdots + V_{t_k}^{i+r}\right] \leq r \cdot 2\sigma^2,$$

for all $k \in \{1, \ldots, K\}$ and $i, r \in \mathbb{N}^+$.

Proof As we've seen in the proof of Lemma 3.2.3,

$$V_{t_k}^i = \sum_{j=0}^{k} \theta_{k,j} \cdot \xi_{i+j} \qquad \text{for all } i \text{ and } k.$$

Since

$$Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var[X_i] + \sum_{i=1}^{n} \sum_{j=1}^{n} Cov\left[X_i, X_j\right]$$

[2]See Chapter 16 of Athreya and Lahiri (2006)

and

$$Cov\left[\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov\left[X_i, Y_j\right],$$

we have

$$\begin{aligned}
& Var\left[V_{t_k}^{i+1} + \cdots + V_{t_k}^{i+r}\right] \\
= \; & \sum_{l=1}^{r} Var\left[V_{t_k}^{i+l}\right] + \sum_{l=1}^{r}\sum_{s=1}^{r} Cov\left[V_{t_k}^{i+l}, V_{t_k}^{i+s}\right] \\
= \; & \sum_{l=1}^{r} Var\left[\sum_{j=0}^{k} \theta_{k,j} \cdot \xi_{i+l+j}\right] + \sum_{l=1}^{r}\sum_{s=1}^{r} Cov\left[\sum_{j=0}^{k} \theta_{k,j} \cdot \xi_{i+l+j}, \sum_{j'=0}^{k} \theta_{k,j'} \cdot \xi_{i+s+j'}\right] \\
= \; & \sum_{l=1}^{r} \left(Var\left[\xi_1\right]\left(\sum_{j=0}^{k} \theta_{k,j}^2\right)\right) + \sum_{l=1}^{r}\sum_{j=0}^{k} \theta_{k,j} \sum_{j'=0}^{k} \theta_{k,j'} Cov\left[\xi_{i+l+j}, \sum_{s=1}^{r} \xi_{i+s+j'}\right].
\end{aligned}$$

Note that $\{\xi_i : i = 1, 2, \ldots\}$ are i.i.d. random variables, so

$$Cov\left[\xi_i, \xi_{i'}\right] = \begin{cases} \sigma^2 & \text{if } i = i' \\ 0 & \text{if } i \neq i'. \end{cases}$$

Therefore, for any i, j, j', l and r,

$$Cov\left[\xi_{i+l+j}, \sum_{s=1}^{r} \xi_{i+s+j'}\right] \leq \sigma^2.$$

Moreover, $0 < \theta_{k,j} < 1$ for all k and j implies that

$$0 < \sum_{j=0}^{k} \theta_{k,j}^4 < \sum_{j=0}^{k} \theta_{k,j}^2 < \sum_{j=0}^{k} \theta_{k,j} = 1.$$

Thus,

$$\begin{aligned}
& Var\left[V_{t_k}^{i+1} + \cdots + V_{t_k}^{i+r}\right] \\
\leq \; & \sum_{l=1}^{r} Var\left[\xi_1\right] + \sum_{l=1}^{r}\sum_{j=0}^{k} \theta_{k,j}\left(\sum_{j'=0}^{k} \theta_{k,j'}\right)\sigma^2 \\
= \; & r\sigma^2 + r\sigma^2 \\
= \; & r \cdot 2\sigma^2.
\end{aligned}$$

□

We are now ready to prove Theorem 3.2.2.

Proof of Theorem 3.2.2 For each $K \in \mathbb{N}^+$, let $\Pi^K = \{0 = t_0 < t_1 < \cdots < t_K = T\}$ be the equidistant partition of $[0, T]$ as specified in Section 3.1. Given any $t \in [0, T]$, there exists a unique $k = k(K) = \lfloor \frac{tK}{T} \rfloor$ and a corresponding point $t_k = t_{k(K)}$ in the partition Π^K such that $t_{k(K)} \leq t < t_{k(K)+1}$. By construction of the model, we have $V_t^{i,K} = V_{t_{k(K)}}^i$ for all $i \in \mathbb{N}^+$. Moreover, since $0 \leq \left|t - t_{k(K)}\right| < \left|t_{k(K)+1} - t_{k(K)}\right| = \frac{T}{K} \to 0$ as $K \to 0$, we know that $t_{k(K)} \to t$ as $K \to 0$.

As shown in Lemma 3.2.3, for each $K \in \mathbb{N}^+$, $\{V_{t_{k(K)}}^i : i \in \mathbb{N}^+\}$ is a $k(K)$-dependent sequence of random variables such that

$$\mathbb{E}\left[V_{t_{k(K)}}^i\right] = 0 \quad \text{and} \quad \mathbb{E}\left[|V_{t_{k(K)}}^i|^4\right] \leq \nu + 6\sigma^4 \qquad \text{for all } i, K \in \mathbb{N}^+. \tag{3.4}$$

Moreover, by Lemma 3.2.4,

$$Var\left[V_{t_{k(K)}}^{i+1} + \cdots + V_{t_{k(K)}}^{i+r}\right] \leq r \cdot 2\sigma^2 \quad \text{for all } i, r \text{ and } K \in \mathbb{N}^+. \tag{3.5}$$

Since $\{\xi_i : i = 1, 2, \ldots\}$ are i.i.d. and $V_{t_{k(K)}}^i = \sum_{j=0}^{k(K)} \theta_{k(K),j} \cdot \xi_{i+j}$ for all $i \in \mathbb{N}^+$, we have

$$\begin{aligned}
& Var\left[\sum_{i=1}^{N} V_{t_{k(K)}}^i\right] \\
= \quad & Var\left[\sum_{i=1}^{N} \sum_{j=0}^{k(K)} \theta_{k(K),j} \cdot \xi_{i+j}\right] \\
= \quad & Var\left[\sum_{i=1}^{k(K)+1} \left(\sum_{j=0}^{i-1} \theta_{k(K),j}\right) \xi_i + \sum_{i=k(K)+2}^{N-1} \left(\sum_{j=0}^{k(K)} \theta_{k(K),j}\right) \xi_i + \sum_{i=N}^{k(K)+N} \left(\sum_{j=i-N}^{k(K)} \theta_{k(K),j}\right) \xi_i\right] \\
= \quad & Var\left[\xi_1\right] \cdot \left[\sum_{i=1}^{k(K)+1} \left(\sum_{j=0}^{i-1} \theta_{k(K),j}\right)^2 + \sum_{i=k(K)+2}^{N-1} \left(\sum_{j=0}^{k(K)} \theta_{k(K),j}\right)^2 + \sum_{i=N}^{k(K)+N} \left(\sum_{j=i-N}^{k(K)} \theta_{k(K),j}\right)^2\right].
\end{aligned}$$

In addition, we know that $0 < \theta_{k(K),j} < 1$ for all j and $\sum_{j=0}^{k(K)} \theta_{k(K),j} = 1$, so

$$0 < \sum_{i=1}^{k(K)+1} \left(\sum_{j=0}^{i-1} \theta_{k(K),j}\right)^2 < k(K) + 1,$$

$$0 < \sum_{i=N}^{k(K)+N} \left(\sum_{j=i-N}^{k(K)} \theta_{k(K),j} \right)^2 < k(K) + 1,$$

and for any $N >> K$,

$$\sum_{i=k(K)+2}^{N-1} \left(\sum_{j=0}^{k(K)} \theta_{k(K),j} \right)^2 = N - k(K) - 2.$$

By assumption, $N = N(K)$ satisfies $\lim_{K\to\infty} \frac{K^3}{N} = 0$, which implies that

$$\lim_{K\to\infty} \frac{k(K)^3}{N} = 0 \tag{3.6}$$

as $k(K) \in \{1, \dots, K\}$. Therefore,

$$\lim_{K\to\infty} \frac{1}{N} \sum_{i=1}^{k(K)+1} \left(\sum_{j=0}^{i-1} \theta_{k(K),j} \right)^2 = \lim_{K\to\infty} \frac{1}{N} \sum_{i=N}^{k(K)+N} \left(\sum_{j=i-N}^{k(K)} \theta_{k(K),j} \right)^2 = 0$$

and

$$\lim_{K\to\infty} \frac{1}{N} \sum_{i=k(K)+2}^{N-1} \left(\sum_{j=0}^{k(K)} \theta_{k(K),j} \right)^2 = 1.$$

As a result, we have

$$\lim_{K\to\infty} \frac{1}{N} Var\left[\sum_{i=1}^{N} V^i_{t_{k(K)}} \right] = \sigma^2 \cdot [0 + 1 + 0] = \sigma^2 > 0. \tag{3.7}$$

By Berk's Theorem in [7], (3.4), (3.5), (3.6), and (3.7) imply that

$$\frac{1}{\sqrt{N(K)}} \sum_{i=1}^{N(K)} V^i_{t_{k(K)}} \overset{\mathcal{L}}{\Longrightarrow} N(0, \sigma^2) \quad \text{as } K \longrightarrow \infty.$$

This completes the proof of Theorem 3.2.2.

□

3.3 Case II: Stochastic Communication with m Neighbors

3.3.1 Assumptions and the Main Theorem

Assumption 3.3.1 In addition to Section 3.1, suppose the following hold:

(1) there exists a constant $C_\xi > 0$ such that $|\xi_i| \leq C_\xi$ a.s. for all $i \in \mathbb{N}^+$;

(2) $A_i = \{j : 0 < j - i \leq m\} = \{i+1, \ldots, i+m\}$ for all $i \in \mathbb{N}^+$, where $m \in \mathbb{N}^+$ is some constant. That is, agent i can only consult with m individuals who are "sufficiently close" to her. Clearly, $|A_i| = m$ and $i \notin A_i$;

(3) $\alpha^{ij} \equiv \alpha^i$ for all $j \in A_i$, i.e. all agents in the set A_i communicate to agent i in the same fashion. In this case, equation (3.1) simplifies to

$$V^i_{t_k} = \left(1 - mh\alpha^i_{t_k}\right) V^i_{t_{k-1}} + h\alpha^i_{t_k} \sum_{j=i+1}^{i+m} V^j_{t_{k-1}}. \tag{3.8}$$

For any $i \in \mathbb{N}^+$ and $k \in \{1, \ldots, K\}$, assume that

$$\mathbb{E}[\alpha^i_{t_k}] = 0, \quad \mathbb{E}[(\alpha^i_{t_k})^2] = \gamma^2, \quad \text{and} \quad |\alpha^i_{t_k}| \leq C_\alpha \;\; a.s.$$

where $\gamma > 0$ and $C_\alpha > 0$ are some constants. In addition, let $\alpha^i_{t_k}$ and $\alpha^i_{t_{k'}}$ be independent for all $k \neq k'$. Such correlational structure of the process α^i is in fact similar to that of a "white noise" as described in Chapter I of Risken (1996). Finally, let α^i and α^j be identical and independent processes for all $i \neq j$, and take $\{\alpha^i : i \in \mathbb{N}^+\}$ to be independent from the set of initial speculative demands $\{\xi_i : i \in \mathbb{N}^+\}$.

Theorem 3.3.2 *Let $N = N(K)$. Supopose Assumption 3.3.1 holds, and*

$$\lim_{K \to \infty} \frac{K^{2+\epsilon}}{N} = 0 \quad \textit{for some small } \epsilon > 0.$$

Then

$$\frac{1}{\sqrt{N(K)}}\sum_{i=1}^{N(K)} V_t^{i,K} \stackrel{\mathcal{L}}{\Longrightarrow} N(0,\sigma^2) \quad as\ K \longrightarrow \infty$$

for all $t \in [0,T]$, *where* $\sigma^2 = \mathbb{E}[\xi_i^2] < \infty$.

3.3.2 Proof of the Main Theorem

We first prove a few useful Lemmas.

Lemma 3.3.3 *For each fixed* $K \in \mathbb{N}^+$, $\{V_{t_k}^i : i \in \mathbb{N}^+\}$ *is an mk-dependent sequence of random variables for each* $k \in \{1,\ldots,K\}$.

Proof For each fixed $K \in \mathbb{N}^+$, we prove the claim by induction on k. When $k = 1$,

$$V_{t_1}^i = (1 - mh\alpha_{t_1}^i)\xi_i + h\alpha_{t_1}^i \sum_{j=i+1}^{i+m} \xi_j = g\left(\alpha_{t_1}^i, \xi_i, \ldots, \xi_{i+m}\right) \quad \text{for all } \ i \in \mathbb{N}^+,$$

where $g(\cdot)$ is a real-valued polynomial function of its arguments, and thus continuous and Borel measurable. For any $r \in \mathbb{N}^+$, define

$$\begin{aligned}\left\{V_{t_1}^i : i \le r, i \in \mathbb{N}^+\right\} &= \left\{g\left(\alpha_{t_1}^i, \xi_i, \ldots, \xi_{i+m}\right) : i \le r, i \in \mathbb{N}^+\right\} \\ &= \left\{g\left(\alpha_{t_1}^{j-m}, \xi_{j-m}, \ldots, \xi_j\right) : m < j \le r+m, j \in \mathbb{N}^+\right\} \\ &= : \mathcal{S}_1,\end{aligned}$$

and

$$\begin{aligned}\left\{V_{t_1}^i : i > r+m, i \in \mathbb{N}^+\right\} &= \left\{g\left(\alpha_{t_1}^i, \xi_i, \ldots, \xi_{i+m}\right) : i > r+m, i \in \mathbb{N}^+\right\} \\ &= \left\{g\left(\alpha_{t_1}^j, \xi_j, \ldots, \xi_{j+m}\right) : j > r+m, j \in \mathbb{N}^+\right\} \\ &= : \mathcal{S}_2.\end{aligned}$$

By Assumption 3.3.1, the set $\left\{\alpha_{t_1}^{j-m}, \xi_{j-m}, \ldots, \xi_j : m < j \le r+m, j \in \mathbb{N}^+\right\}$ and the set $\left\{\alpha_{t_1}^{j}, \xi_j, \ldots, \xi_{j+m} : j > r+m, j \in \mathbb{N}^+\right\}$ are independent. Since $g(\cdot)$ is measurable, we know that $\mathcal{S}_1$ and $\mathcal{S}_2$ are also independent. Thus, by definition[3], the sequence $\{V_{t_1}^i : i \in \mathbb{N}^+\}$ is m-dependent.

Assume the lemma holds for every $k \in \{1, \ldots, k^*\}$, where $k^* \in \{1, \ldots, K\}$. For $k = k^* + 1$ and any $r \in \mathbb{N}^+$, define

$$
\begin{aligned}
& \left\{V_{t_{k^*+1}}^i : i \le r,\ i \in \mathbb{N}^+\right\} \\
= & \left\{g\left(\alpha_{t_{k^*+1}}^i, V_{t_{k^*}}^i, \ldots, V_{t_{k^*}}^{i+m}\right) : i \le r,\ i \in \mathbb{N}^+\right\} \\
= & \left\{g\left(\alpha_{t_{k^*+1}}^{j-m}, V_{t_{k^*}}^{j-m}, \ldots, V_{t_{k^*}}^{j}\right) : m < j \le r+m,\ j \in \mathbb{N}^+\right\} \\
= & : \mathcal{S}_1'
\end{aligned}
$$

and

$$
\begin{aligned}
& \left\{V_{t_{k^*+1}}^i : i > r + m(k^*+1),\ i \in \mathbb{N}^+\right\} \\
= & \left\{g\left(\alpha_{t_{k^*+1}}^i, V_{t_{k^*}}^i, \ldots, V_{t_{k^*}}^{i+m}\right) : i > r+m+mk^*,\ i \in \mathbb{N}^+\right\} \\
= & \left\{g\left(\alpha_{t_{k^*+1}}^{j}, V_{t_{k^*}}^{j}, \ldots, V_{t_{k^*}}^{j+m}\right) : j > r+m+mk^*,\ j \in \mathbb{N}^+\right\} \\
= & : \mathcal{S}_2'.
\end{aligned}
$$

Since $\{V_{t_{k^*}}^i : i \in \mathbb{N}^+\}$ is mk^*-dependent by the induction hypothesis and α^i is assumed to be independent of α^j for any $i \neq j$, we know that the set

$$\left\{\alpha_{t_{k^*+1}}^{j-m}, V_{t_{k^*}}^{j-m}, \ldots, V_{t_{k^*}}^{j} : m < j \le r+m,\ j \in \mathbb{N}^+\right\}$$

and the set

$$\left\{\alpha_{t_{k^*+1}}^{j}, V_{t_{k^*}}^{j}, \ldots, V_{t_{k^*}}^{j+m} : j > r+m+mk^*,\ j \in \mathbb{N}^+\right\}$$

are also independent. Thus, the measurability of $g(\cdot)$ implies that $\mathcal{S}_1'$ and $\mathcal{S}_2'$ are independent for any $r \in \mathbb{N}^+$. By definition, the sequence $\{V_{t_{k^*+1}}^i : i \in \mathbb{N}^+\}$ is $m(k^*+1)$-dependent.

[3]See Chapter 16 of Athreya and Lahiri (2006).

By induction, we conclude that for each fixed $K \in \mathbb{N}^+$, $\{V^i_{t_k} : i \in \mathbb{N}^+\}$ is an mk-dependent sequence of random variables for all $k \in \{1, \ldots, K\}$.

$\square$

Lemma 3.3.4 *For each fixed $K \in \mathbb{N}^+$, $\{V^i_{t_k} : i \in \mathbb{N}^+\}$ is a stationary sequence of random variables for all $k \in \{0, \ldots, K\}$.*

Proof For each fixed $K \in \mathbb{N}^+$, we prove the claim by induction on k. When $k = 0$, $\{V^i_{t_0} : i \in \mathbb{N}^+\} = \{\xi_i : i \in \mathbb{N}^+\}$ is an i.i.d. sequence, so

$$P(\xi_1 \le x_1, \ldots, \xi_l \le x_l) = P(\xi_{i+1} \le x_1, \ldots, \xi_{i+l} \le x_l) = \prod_{j=1}^{l} P(\xi_1 \le x_j)$$

for all $l \in \mathbb{N}^+$, $(x_1, \ldots, x_l)' \in \mathbb{R}^l$, and $i \in \mathbb{N}^+$. By definition[4], the sequence $\{V^i_{t_0} : i \in \mathbb{N}^+\}$ is stationary.

Now suppose the lemma holds for every $k \in \{0, \ldots, k^*\}$, where $k^* \in \{0, \ldots, K\}$. For $k = k^* + 1$, since

$$\begin{aligned} V^i_{t_{k^*+1}} &= (1 - mh\alpha^i_{t_{k^*+1}})V^i_{t_{k^*}} + h\alpha^i_{t_{k^*+1}} \sum_{j=i+1}^{i+m} V^j_{t_{k^*}} \\ &= g\left(\alpha^i_{t_{k^*+1}}, V^i_{t_{k^*}}, \ldots, V^{i+m}_{t_{k^*}}\right) \end{aligned}$$

where $g(\cdot)$ is measurable, we have

$$\begin{aligned} & P\left(V^1_{t_{k^*+1}} \le x_1, \ldots, V^l_{t_{k^*+1}} \le x_l\right) \\ = \ & P\left(g\left(\alpha^1_{t_{k^*+1}}, V^1_{t_{k^*}}, \ldots, V^{1+m}_{t_{k^*}}\right) \le x_1, \ldots, g\left(\alpha^l_{t_{k^*+1}}, V^l_{t_{k^*}}, \ldots, V^{l+m}_{t_{k^*}}\right) \le x_l\right) \\ = \ & \int P\left(g\left(z_1, V^1_{t_{k^*}}, \ldots, V^{1+m}_{t_{k^*}}\right) \le x_1, \right. \\ & \qquad \left. \ldots, g\left(z_l, V^l_{t_{k^*}}, \ldots, V^{l+m}_{t_{k^*}}\right) \le x_l\right) dF_{\alpha^1_{t_{k^*+1}} \cdots \alpha^l_{t_{k^*+1}}}(z_1, \ldots, z_l) \end{aligned}$$

[4]See Chapter 8 of Athreya and Lahiri (2006).

and

$$
\begin{aligned}
& P\left(V^{i+1}_{t_{k^*+1}} \leq x_1, \dots, V^{i+l}_{t_{k^*+1}} \leq x_l\right) \\
= \; & P\left(g(\alpha^{i+1}_{t_{k^*+1}}, V^{i+1}_{t_{k^*}}, \dots, V^{i+1+m}_{t_{k^*}}) \leq x_1, \dots, g(\alpha^{i+l}_{t_{k^*+1}}, V^{i+l}_{t_{k^*}}, \dots, V^{i+l+m}_{t_{k^*}}) \leq x_l\right) \\
= \; & \int P\left(g(z_1, V^{i+1}_{t_{k^*}}, \dots, V^{i+1+m}_{t_{k^*}}) \leq x_1,\right. \\
& \qquad \left. \dots, g(z_l, V^{i+l}_{t_{k^*}}, \dots, V^{i+l+m}_{t_{k^*}}) \leq x_l\right) dF_{\alpha^{i+1}_{t_{k^*+1}} \cdots \, \alpha^{i+l}_{t_{k^*+1}}}(z_1, \dots, z_l).
\end{aligned}
$$

By Assumption 3.3.1, we know that α^i and α^j are identical and independent processes for all $i \neq j$, so

$$
F_{\alpha^{1}_{t_{k^*+1}} \cdots \, \alpha^{l}_{t_{k^*+1}}}(z_1, \dots, z_l) = F_{\alpha^{i+1}_{t_{k^*+1}} \cdots \, \alpha^{i+l}_{t_{k^*+1}}}(z_1, \dots, z_l)
$$

for all $l \in \mathbb{N}^+$, $(z_1, \dots, z_l)' \in \mathbb{R}^l$ and $i \in \mathbb{N}^+$. Moreover, by the induction hypothesis, $\{V^i_{t_{k^*}} : i \in \mathbb{N}^+\}$ is a stationary sequence of random variables, so

$$
\{V^1_{t_{k^*}}, \dots, V^{l+m}_{t_{k^*}}\} \overset{\mathcal{D}}{=} \{V^{i+1}_{t_{k^*}}, \dots, V^{i+l+m}_{t_{k^*}}\}.
$$

Therefore, for all $l \in \mathbb{N}^+$, $x_1, \dots, x_l \in \mathbb{R}$ and $i \in \mathbb{N}^+$,

$$
P\left(V^1_{t_{k^*+1}} \leq x_1, \dots, V^l_{t_{k^*+1}} \leq x_l\right) = P\left(V^{i+1}_{t_{k^*+1}} \leq x_1, \dots, V^{i+l}_{t_{k^*+1}} \leq x_l\right)
$$

and the sequence $\{V^i_{t_{k^*+1}} : i \in \mathbb{N}^+\}$ is stationary.

By induction, we conclude that for each fixed $K \in \mathbb{N}^+$, the sequence $\{V^i_{t_k} : i \in \mathbb{N}^+\}$ is stationary for all $k \in \{0, \dots, K\}$.

$\square$

Lemma 3.3.5 *For each fixed $K \in \mathbb{N}^+$,*

(1) $\mathbb{E}\left[V^i_{t_k}\right] = 0$,

(2) $\mathbb{E}\left[\left(V^i_{t_k}\right)^2\right] = (1 + mh^2\gamma^2 + m^2h^2\gamma^2)^k\sigma^2$, *and*

(3) $\mathbb{E}\left[V^i_{t_k} V^d_{t_k}\right] = 0$ *if* $i \neq d$.

for all $i \in \mathbb{N}^+$ and $k \in \{1, \dots, K\}$.

Proof Recall that by Assumption 3.3.1, the set $\{\alpha^i : i \in \mathbb{N}^+\}$ is taken to be independent from the set of initial speculative demands $\{\xi_i : i \in \mathbb{N}^+\}$, and α^i is independent from α^j for all $i \neq j$. Moreover, for each $i \in \mathbb{N}^+$, $\alpha^i_{t_k}$ is independent from $\alpha^i_{t_{k'}}$ for all $k \neq k'$. Since

$$
\begin{aligned}
V^i_{t_k} &= (1 - mh\alpha^i_{t_k})V^i_{t_{k-1}} + h\alpha^i_{t_k} \sum_{j=i+1}^{i+m} V^j_{t_{k-1}} \\
&= g\left(\alpha^i_{t_k}, V^i_{t_{k-1}}, \ldots, V^{i+m}_{t_{k-1}}\right) \\
&= g\left(\alpha^i_{t_k}, g\left(\alpha^i_{t_{k-1}}, V^i_{t_{k-2}}, \ldots, V^{i+m}_{t_{k-2}}\right), \ldots, g\left(\alpha^{i+m}_{t_{k-1}}, V^{i+m}_{t_{k-2}}, \ldots, V^{i+2m}_{t_{k-2}}\right)\right) \\
&\vdots \\
&= \hat{g}\left(\left\{\alpha^j_t : t \in \{t_1, \ldots, t_k\}, j \in \{i, \ldots, i + (k-1)m\}\right\}, \left\{\xi^j : j \in \{i, \ldots, i + km\}\right\}\right),
\end{aligned}
$$

the previous assumptions imply that $\alpha^i_{t_{k+1}}$ is independent from $\{V^i_{t_k} : i \in \mathbb{N}^+\}$ for all $i \in \mathbb{N}^+$ and $k \in \{0, \ldots, K-1\}$.

We now prove each claim by induction on k. Throughout the proof, we rely on the above independence structures to factor expectations of products into products of expectations.

For $k = 1$, since $\mathbb{E}[\xi_i] = 0$ for all $i \in \mathbb{N}^+$, we have

$$
\begin{aligned}
\mathbb{E}\left[V^i_{t_1}\right] &= \mathbb{E}\left[(1 - mh\alpha^i_{t_1})\xi_i + h\alpha^i_{t_1} \sum_{j=i+1}^{i+m} \xi_j\right] \\
&= \mathbb{E}\left[(1 - mh\alpha^i_{t_1})\right] \cdot \mathbb{E}[\xi_i] + \mathbb{E}\left[h\alpha^i_{t_1}\right] \cdot \sum_{j=i+1}^{i+m} \mathbb{E}\left[\xi_j\right] \\
&= 0
\end{aligned}
$$

Assume $\mathbb{E}\left[V^i_{t_k}\right] = 0$ for all $i \in \mathbb{N}^+$, then

$$
\begin{aligned}
\mathbb{E}\left[V^i_{t_{k+1}}\right] &= \mathbb{E}\left[(1 - mh\alpha^i_{t_{k+1}})V^i_{t_k} + h\alpha^i_{t_{k+1}} \sum_{j=i+1}^{i+m} V^j_{t_k}\right] \\
&= \mathbb{E}\left[(1 - mh\alpha^i_{t_{k+1}})\right] \cdot \mathbb{E}\left[V^i_{t_k}\right] + \mathbb{E}\left[h\alpha^i_{t_{k+1}}\right] \cdot \sum_{j=i+1}^{i+m} \mathbb{E}\left[V^j_{t_k}\right] \\
&= 0
\end{aligned}
$$

By induction, this completes the proof for (1).

For (2) and (3), when $k = 1$,

$$\begin{aligned}
\mathbb{E}\left[\left(V^i_{t_1}\right)^2\right] &= \mathbb{E}\left[\left((1 - mh\alpha^i_{t_1})\xi_i + h\alpha^i_{t_1}\sum_{j=i+1}^{i+m}\xi_j\right)^2\right] \\
&= \mathbb{E}\left[1 - 2mh\alpha^i_{t_1} + m^2h^2(\alpha^i_{t_1})^2\right]\cdot\mathbb{E}\left[\xi_i^2\right] + \mathbb{E}\left[h^2(\alpha^i_{t_1})^2\right]\cdot\mathbb{E}\left[\left(\sum_{j=i+1}^{i+m}\xi_j\right)^2\right] \\
&\qquad + 2\mathbb{E}\left[(1 - mh\alpha^i_{t_1})\cdot h\alpha^i_{t_1}\right]\cdot\sum_{j=i+1}^{i+m}\mathbb{E}\left[\xi_i\xi_j\right]
\end{aligned}$$

for all $i \in \mathbb{N}^+$ and $d \neq i$,

$$\begin{aligned}
&\mathbb{E}\left[V^i_{t_1}V^d_{t_1}\right] \\
= {}& \mathbb{E}\left[\left((1 - mh\alpha^i_{t_1})\xi_i + h\alpha^i_{t_1}\sum_{j=i+1}^{i+m}\xi_j\right)\left((1 - mh\alpha^d_{t_1})\xi_d + h\alpha^d_{t_1}\sum_{j=d+1}^{d+m}\xi_j\right)\right] \\
= {}& \mathbb{E}\Big[(1 - mh\alpha^i_{t_1})(1 - mh\alpha^d_{t_1})\Big]\mathbb{E}\Big[\xi_i\xi_d\Big] + \mathbb{E}\Big[(1 - mh\alpha^i_{t_1})\Big]\mathbb{E}\Big[h\alpha^d_{t_1}\Big]\cdot\sum_{j=d+1}^{d+m}\mathbb{E}\Big[\xi_i\xi_j\Big] \\
&+ \mathbb{E}\Big[h\alpha^i_{t_1}\Big]\mathbb{E}\Big[(1 - mh\alpha^d_{t_1})\Big]\sum_{j=i+1}^{i+m}\mathbb{E}\Big[\xi_d\xi_j\Big] + \mathbb{E}\Big[h\alpha^i_{t_1}\Big]\mathbb{E}\Big[h\alpha^d_{t_1}\Big]\mathbb{E}\left[\left(\sum_{j=i+1}^{i+m}\xi_j\right)\left(\sum_{j=d+1}^{d+m}\xi_j\right)\right]
\end{aligned}$$

By model specifications in Section 3.1 and Assumption 3.3.1,

$$\mathbb{E}[\xi_i] = 0, \quad \mathbb{E}[\xi_i^2] = \sigma^2, \quad \mathbb{E}[\alpha^i_{t_1}] = 0, \quad \mathbb{E}[(\alpha^i_{t_1})^2] = \gamma^2,$$

and $\{\xi_i : i \in \mathbb{N}^+\}$ is an i.i.d. sequence of random variables. Thus,

$$\mathbb{E}[(V^i_{t_1})^2] = (1 + mh^2\gamma^2 + m^2h^2\gamma^2)\sigma^2 \quad \text{and} \quad \mathbb{E}[V^i_{t_1}V^d_{t_1}] = 0$$

for all $i \in \mathbb{N}^+$ and $d \neq i$.

Now suppose $\mathbb{E}[(V^i_{t_k})^2] = (1 + mh^2\gamma^2 + m^2h^2\gamma^2)^k\sigma^2$ and $\mathbb{E}[V^i_{t_k}V^d_{t_k}] = 0$ for all $i \in \mathbb{N}^+$ and $d \neq i$. Then,

$$
\begin{aligned}
& \mathbb{E}\left[\left(V_{t_{k+1}}^i\right)^2\right] \\
= \; & \mathbb{E}\left[\left((1-mh\alpha_{t_{k+1}}^i)V_{t_k}^i + h\alpha_{t_{k+1}}^i \sum_{j=i+1}^{i+m} V_{t_k}^j\right)^2\right] \\
= \; & \mathbb{E}\left[1-2mh\alpha_{t_{k+1}}^i + m^2h^2(\alpha_{t_{k+1}}^i)^2\right]\mathbb{E}\left[(V_{t_k}^i)^2\right] + \mathbb{E}\left[h^2(\alpha_{t_{k+1}}^i)^2\right]\mathbb{E}\left[\left(\sum_{j=i+1}^{i+m} V_{t_k}^j\right)^2\right] \\
& \quad + 2\mathbb{E}\left[(1-mh\alpha_{t_{k+1}}^i)\cdot h\alpha_{t_{k+1}}^i\right]\cdot \sum_{j=i+1}^{i+m} \mathbb{E}\left[V_{t_k}^i V_{t_k}^j\right] \\
= \; & (1+m^2h^2\gamma^2)(1+mh^2\gamma^2+m^2h^2\gamma^2)^k\sigma^2 + h^2\gamma^2\cdot m(1+mh^2\gamma^2+m^2h^2\gamma^2)^k\sigma^2 + 0 \\
= \; & (1+mh^2\gamma^2+m^2h^2\gamma^2)^{k+1}\sigma^2
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbb{E}\left[V_{t_{k+1}}^i V_{t_{k+1}}^d\right] \\
= \; & \mathbb{E}\left[\left((1-mh\alpha_{t_{k+1}}^i)V_{t_k}^i + h\alpha_{t_{k+1}}^i \sum_{j=i+1}^{i+m} V_{t_k}^j\right)\left((1-mh\alpha_{t_{k+1}}^d)V_{t_k}^d + h\alpha_{t_{k+1}}^d \sum_{j=d+1}^{d+m} V_{t_k}^j\right)\right] \\
= \; & \mathbb{E}\Big[(1-mh\alpha_{t_{k+1}}^i)(1-mh\alpha_{t_{k+1}}^d)\Big]\mathbb{E}\Big[V_{t_k}^i V_{t_k}^d\Big] \\
& \quad + \mathbb{E}\Big[(1-mh\alpha_{t_{k+1}}^i)\Big]\mathbb{E}\Big[h\alpha_{t_{k+1}}^d\Big]\cdot \sum_{j=d+1}^{d+m} \mathbb{E}\Big[V_{t_k}^i V_{t_k}^j\Big] \\
& \quad + \mathbb{E}\Big[h\alpha_{t_{k+1}}^i\Big]\mathbb{E}\Big[(1-mh\alpha_{t_{k+1}}^d)\Big] \sum_{j=i+1}^{i+m} \mathbb{E}\Big[V_{t_k}^d V_{t_k}^j\Big] \\
& \quad + \mathbb{E}\Big[h\alpha_{t_{k+1}}^i\Big]\mathbb{E}\Big[h\alpha_{t_{k+1}}^d\Big]\mathbb{E}\left[\left(\sum_{j=i+1}^{i+m} V_{t_k}^j\right)\left(\sum_{j=d+1}^{d+m} V_{t_k}^j\right)\right] \\
= \; & 0
\end{aligned}
$$

for all $i \in \mathbb{N}^+$ and any $d \neq i$. By induction, this completes the proof of (2) and (3).

□

Lemma 3.3.6 *Suppose there exist constants C_ξ, C_α such that $|\xi_i| \leq C_\xi$ and $|\alpha^i| \leq C_\alpha$ a.s. for all $i \in \mathbb{N}^+$. Then for each fixed $K \in \mathbb{N}^+$,*

$$
\left|V_{t_k}^i\right| \leq (1+2mhC_\alpha)^k C_\xi
$$

for all $i \in \mathbb{N}^+$ *and* $k \in \{1, \ldots, K\}$.

Proof For each fixed $K \in \mathbb{N}^+$, we prove the claim by induction on k. For $k = 1$, since $|\xi_i| \le C_\xi$ and $|\alpha^i_{t_1}| \le C_\alpha$ a.s. for all $i \in \mathbb{N}^+$, we know that

$$\begin{aligned}
\left|V^i_{t_1}\right| &= \left|(1 - mh\alpha^i_{t_1})\xi_i + h\alpha^i_{t_1} \sum_{j=i+1}^{i+m} \xi_j\right| \\
&\le (1 + mhC_\alpha)C_\xi + mhC_\alpha C_\xi \\
&= (1 + 2mhC_\alpha)C_\xi.
\end{aligned}$$

Now suppose $\left|V^i_{t_k}\right| \le (1 + 2mhC_\alpha)^k C_\xi$ for all $i \in \mathbb{N}^+$, then

$$\begin{aligned}
\left|V^i_{t_{k+1}}\right| &= \left|(1 - mh\alpha^i_{t_{k+1}})V^i_{t_k} + h\alpha^i_{t_{k+1}} \sum_{j=i+1}^{i+m} V^j_{t_k}\right| \\
&\le (1 + mhC_\alpha) \cdot (1 + 2mhC_\alpha)^k C_\xi + hC_\alpha \cdot m(1 + 2mhC_\alpha)^k C_\xi \\
&= (1 + 2mhC_\alpha)^{k+1} C_\xi
\end{aligned}$$

By induction, for each $K \in \mathbb{N}^+$, $\left|V^i_{t_k}\right| \le (1 + 2mhC_\alpha)^k C_\xi$ for all $i \in \mathbb{N}^+$, $k \in \{1, \ldots, K\}$.

□

We are now ready to prove Theorem 3.3.2.

Proof of Theorem 3.3.2 For each $K \in \mathbb{N}^+$, let $\Pi^K = \{0 = t_0 < t_1 < \cdots < t_K = T\}$ be the equidistant partition of the interval $[0, T]$ as specified in Section 3.1, where $h^K := |\Pi^K| = \frac{T}{K}$ and $t_k = k \cdot h^K$ for all $k \in \{1, \ldots, K\}$. Given any $t \in [0, T]$, there exists a unique $k = k(K) = \lfloor \frac{tK}{T} \rfloor$ and a corresponding point $t_k = t_{k(K)}$ in the partition Π^K such that $t_{k(K)} \le t < t_{k(K)+1}$. By construction of the model, we have $V^i_t = V^i_{t_{k(K)}}$ for all $i \in \mathbb{N}^+$. Moreover, since $0 \le \left|t - t_{k(K)}\right| < \left|t_{k(K)+1} - t_{k(K)}\right| = \frac{T}{K} \to 0$ as $K \to 0$, we know that $t_{k(K)} \to t$ as $K \to 0$.

By Lemma 3.3.3 and Lemma 3.3.5, $\{V^i_{t_{k(K)}} : i \in \mathbb{N}^+\}$ is an $m \cdot k(K)$-dependent sequence of random variables with zero means. In order to prove the convergence in Theorem 3.2.2, we check the criterion given in Berk (1973). As shown in Lemma 3.3.6, for each $K \in \mathbb{N}^+$,

$$\left|V^i_{t_{k(K)}}\right| \leq (1 + 2mh^K C_\alpha)^{k(K)} C_\xi \leq \left(1 + 2mC_\alpha \frac{T}{K}\right)^K$$

for all $i \in \mathbb{N}^+$ and $k(K) \in \{1, \ldots, K\}$. Since

$$\lim_{K\to\infty} (1 + \frac{2mC_\alpha T}{K})^K = e^{2mC_\alpha T},$$

there must exist some constant $\epsilon > 0$ and $n(\epsilon) \in \mathbb{N}^+$ such that

$$\left|\left(1 + \frac{2mC_\alpha T}{K}\right)^K - e^{2mC_\alpha T}\right| < \epsilon \quad \text{for all } K \geq n(\epsilon).$$

Thus,

$$\left(1 + \frac{2mC_\alpha T}{K}\right)^K \leq \max\left(\left\{(1 + \frac{2mC_\alpha T}{j})^j : j = 1, 2, \ldots, n(\epsilon)\right\},\ e^{2mC_\alpha T} + \epsilon\right) =: M_1$$

for all $K \in \mathbb{N}^+$, which implies that for any given $\delta > 0$,

$$\mathbb{E}\left[\left|V^i_{t_{k(K)}}\right|^{2+\delta}\right] \leq \mathbb{E}\left[M_1^{2+\delta}\right] = M_1^{2+\delta} \tag{3.9}$$

for all $K \in \mathbb{N}^+$, $i \in \mathbb{N}^+$, and $k(K) \in \{1, \ldots, K\}$.

On the other hand, for each $K \in \mathbb{N}^+$,

$$\mathbb{E}\left[(V^i_{t_{k(K)}})^2\right] = \sigma^2 \left(1 + m(\frac{T}{K})^2\gamma^2 + m^2(\frac{T}{K})^2\gamma^2\right)^{k(K)}$$

for all $i \in \mathbb{N}^+$ and $k(K) \in \{1, \ldots, K\}$ by Lemma 3.3.5. Therefore,

$$\begin{aligned}
\sigma^2 &\leq \lim_{K\to\infty} \mathbb{E}\left[\left(V^i_{t_{k(K)}}\right)^2\right] \\
&= \lim_{K\to\infty} \sigma^2 \left(1 + m(\frac{T}{K})^2\gamma^2 + m^2(\frac{T}{K})^2\gamma^2\right)^{\lfloor \frac{tK}{T} \rfloor} \\
&\leq \lim_{K\to\infty} \sigma^2 \left(1 + \frac{mt^2\gamma^2 + m^2t^2\gamma^2}{\frac{tK}{T}} \cdot \frac{T}{tK}\right)^{\frac{tK}{T}} \\
&< \lim_{K\to\infty} \sigma^2 \left(1 + \frac{(mt^2\gamma^2 + m^2t^2\gamma^2) \cdot \epsilon}{\frac{tK}{T}}\right)^{\frac{tK}{T}} \\
&= \sigma^2 \cdot e^{(mt^2\gamma^2 + m^2t^2\gamma^2)\epsilon}
\end{aligned}$$

for any $\epsilon > 0$. Since

$$e^{(mt^2\gamma^2+m^2t^2\gamma^2)\epsilon} \longrightarrow 1 \text{ as } \epsilon \longrightarrow 0,$$

we know that

$$\lim_{K\to\infty} \mathbb{E}\left[\left(V^i_{t_{k(K)}}\right)^2\right] = \sigma^2 > 0,$$

which in turn implies that

$$\mathbb{E}\left[\left(V^i_{t_{k(K)}}\right)^2\right] \le \max\left(\left\{(1+\frac{mT^2\gamma^2+m^2T^2\gamma^2}{j^2})^{\lfloor \frac{t}{T}j\rfloor} : j = 1,\ldots,n(\epsilon)\right\},\ \sigma^2+\epsilon\right) =: M_2$$

for all $i \in \mathbb{N}^+$, $K \in \mathbb{N}^+$ and $k(K) \in \{1,\ldots,K\}$, where $\epsilon > 0$ is some constant and $n(\epsilon) \in \mathbb{N}^+$.

Finally, by Lemma 3.3.5, $\mathbb{E}[V^i_{t_{k(K)}}] = 0$ and $\mathbb{E}[V^i_{t_{k(K)}}V^d_{t_{k(K)}}] = 0$ for all $i \in \mathbb{N}^+$ and $d \neq i$. Thus, for all $i,\ j$ and $K \in \mathbb{N}^+$, we have

$$\begin{aligned}
Var\left[V^{i+1}_{t_{k(K)}} + \cdots + V^{j}_{t_{k(K)}}\right] &= Var\left[V^{i+1}_{t_{k(K)}}\right] + \cdots + Var\left[V^{j}_{t_{k(K)}}\right] \\
&= \mathbb{E}\left[\left(V^{i+1}_{t_{k(K)}}\right)^2\right] + \cdots + \mathbb{E}\left[\left(V^{j}_{t_{k(K)}}\right)^2\right] \\
&= (j-i)\cdot \mathbb{E}\left[\left(V^{i}_{t_{k(K)}}\right)^2\right] \\
&\le (j-i)\cdot M_2, \qquad (3.10)
\end{aligned}$$

and for any $N = N(K)$,

$$\lim_{K\to\infty}\frac{1}{N(K)}Var\left[V^1_{t_{k(K)}} + \cdots + V^{N(K)}_{t_{k(K)}}\right] = \lim_{K\to\infty}\frac{N(K)}{N(K)}\cdot\mathbb{E}\left[\left(V^i_{t_{k(K)}}\right)^2\right] = \sigma^2 > 0. \qquad (3.11)$$

Moreover,

$$\lim_{K\to\infty}\frac{K^{2+\epsilon}}{N} = 0 \quad \text{for some small } \epsilon > 0 \qquad (3.12)$$

by assumption. Taking the constant δ in (3.9) to be equal to the constant ϵ in (3.12), we see that (3.9), (3.10), (3.11) and (3.12) together imply

$$\frac{1}{\sqrt{N(K)}}\sum_{i=1}^{N(K)} V^i_{t_{k(K)}} \overset{\mathcal{L}}{\Longrightarrow} N(0,\sigma^2) \quad \text{as } K \longrightarrow \infty$$

by Berk's Theorem in [7]. This completes the proof of Theorem 3.2.2.

□

3.4 Numerical Analysis

3.4.1 Statistical Properties of Simulated Data

To investigate statistical properties of the normalized total speculative demand $\overline{V}^{K,N}$ as defined in Section 3.1, we simulate[5] the stochastic communication model as specified in Section 3.3[6], where

- $T = 1$ is the length of the entire time horizon.
- $K = 200$ is the number of time steps in the equidistant partition.
- $N = 52132$ is the total number of agents participating in the market[7].
- $m = 10$ is the number of individuals each agent communicates with.
- All initial speculative demands ξ_i's follow a Uniform$[-2, 2]$ distribution.
- All communication rates $\alpha^i_{t_k}$'s follow a Uniform$[-5, 5]$ distribution.

Figure 3.1 shows a simulated sample path of $\overline{V}^{K,N}$, along with the path of a standard Brownian motion sampled over the same time grid. The former seems to have a higher volatility towards the end of the time horizon, while the latter displays a rather stable volatility throughout the entire period.

In order to test our model's ability to capture the empirically observed "stylized facts" of logarithmic asset returns as discussed in Section 1.1, we turn our

[5]All simulations, statistical tests and data plotting are carried out using software package R, Version 2.12.0.

[6]Simulation of the constant communication model as specified in Section 3.2 yields similar qualitative results.

[7]Note that $200^{2+0.05} \approx 52132$.

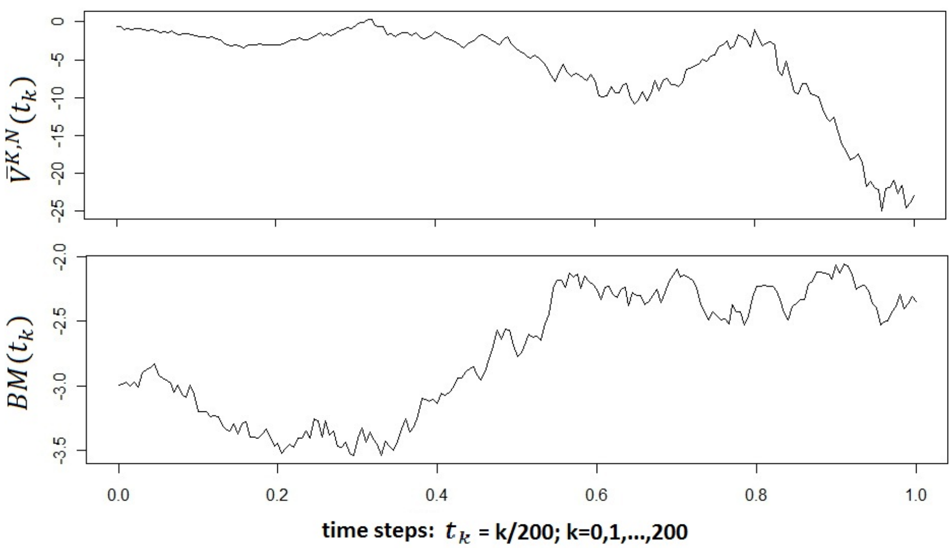

Figure 3.1: (Upper panel) Simulated sample paths of $\overline{V}^{K,N}$; (Lower panel) simulated sample paths of a standard Brownian motion over the same time grid.

attention to the set of values

$$\left\{(\overline{V}^{K,N}_{t_k} - \overline{V}^{K,N}_{t_{k-1}}) : k = 1, \dots, K\right\},$$

which can be viewed as the logarithmic asset returns generated by the normalized total speculative demand over each individual time step t_k, $k = 1, \dots, K$.

As shown in Figure 3.2, the "distribution" of 1-step returns generated by an arbitrary sample path of $\overline{V}^{K,N}$ displays a clear leptokurtic shape. That is, it seems to have a more acute peak around the mean and heavier tails than those of a Normal distribution. Some descriptive statistics[8] associated with the set of 1-step returns given by this particular sample are reported in Table 3.1. In addition, we simulate 500 additional sample paths of $\overline{V}^{K,N}$ and calculate the same statistics for the set of 1-step returns generated by each one of them. The corre-

[8]From here onward, "kurtosis" is always calculated as $(\mathbb{E}[X^4]/\mathbb{E}[X^2]^2) - 3$, i.e. the "excess kurtosis" relative to a normal distribution.

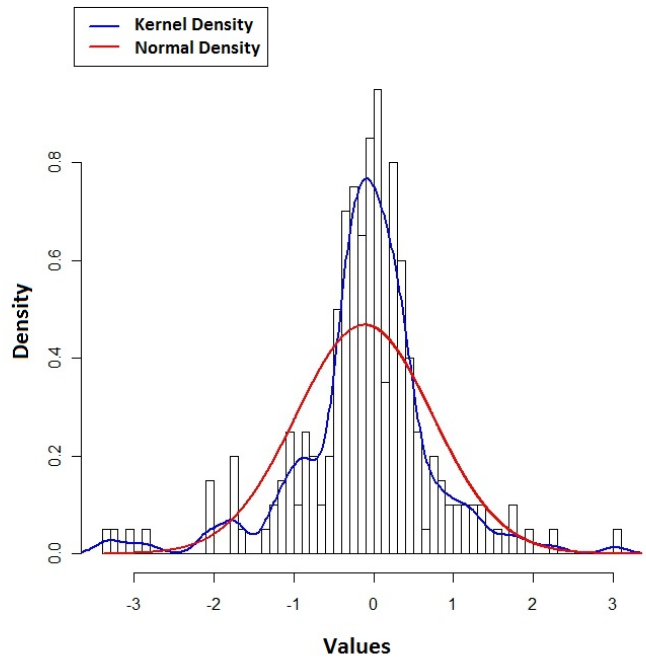

Figure 3.2: Histogram generated by the set of values $\{(\overline{V}^{K,N}_{t_k} - \overline{V}^{K,N}_{t_{k-1}}) : k = 1, \ldots, K\}$ given by the sample path of $\overline{V}^{K,N}$ shown in Figure 3.1. The Kernel density estimation (blue) and a Normal density (red) with mean and variance matching the simulated data are also plotted.

sponding results are summarized in Table 3.2. It's easy to spot that the "distribution" of the 1-step returns $\left\{(\overline{V}^{K,N}_{t_k} - \overline{V}^{K,N}_{t_{k-1}}) : k = 1, \ldots, K\right\}$ given by an arbitrary sample path of $\overline{V}^{K,N}$ is characterized by significant excess kurtosis.

Table 3.1: Descriptive statistics for the set of 1-step returns associated with the sample path of $\overline{V}^{K,N}$ shown in Figure 3.1. All statistics are calculated using the *fBasics* package in R 12.2.0.

Mean	Variance	Skewness	Kurtosis
-0.112	0.727	-0.574	3.16

Two normality tests are carried out on each of the 500 sets of 1-step returns associated with 500 individual sample paths of $\overline{V}^{K,N}$. The Shapiro–Wilk test

Table 3.2: Mean, Variance, Skewness and Kurtosis are repeatedly calculated for each of the 500 sets of 1-step returns associated respectively with 500 individual sample paths of $\overline{V}^{K,N}$. The resulting values are then averaged and reported along with the corresponding standard deviation.

	Mean	Variance	Skewness	Kurtosis
Average Value (over 500 sample sets)	-2.50×10^{-4}	0.618	9.95×10^{-3}	3.22
Standard Deviation (over 500 sample sets)	5.89×10^{-2}	0.533	0.520	2.15

is based on order statistics while the Jarque−Bera[9] test relies on sample kurtosis and skewness. Summary of the resulting test statistics and p−values are reported in Table 3.3, which lead to strong rejections of the null hypothesis of normality in both cases.

Table 3.3: Shapiro−Wilk and Jarque−Bera normality tests are performed repeatedly on each of the 500 sets of 1-step returns associated respectively with 500 simulated paths of $\overline{V}^{K,N}$. The resulting test statistics and p-values are then averaged and reported along with the corresponding standard deviations.

	Shapiro−Wilk		**Jarque−Bera**	
	Test Statistic	p−value	Test Statistic	p−value
Average Value (over 500 sample sets)	0.935	2.17×10^{-3}	138	2.14×10^{-3}
Standard Deviation (over 500 sample sets)	4.06×10^{-2}	1.67×10^{-2}	303	1.54×10^{-2}

Recall from Section 1.1 that *"Heavy tails"* is one of the most well-known statistical properties observed in the empirical logarithmic returns of financial assets. The Q-Q plot in Figure 3.3, together with the previously calculated sample

[9]The *stats* and *tseries* packages in R 12.2.0 are used to implement the Shapiro−Wilk test and the Jarque−Bera test, respectively.

kurtosis in Table 3.1, seem to indicate a similar behavior for the "distribution" of 1-step returns given by the simulated path of $\overline{V}^{K,N}$ as shown in Figure 3.1. To further investigate this observation, we turn our attention to *Hill Estimators* for the 10%-, 5%- and 2.5%- tails of the set of 1-step returns associated with sample paths of $\overline{V}^{K,N}$.

We calculate the Hill estimators for the 10%−, 5%− and 2.5%−tails of each of the 500 sets of 1-step returns given by individual simulated sample paths of $\overline{V}^{K,N}$. The corresponding results, summarized in Table 3.4, are in great harmony with various tail index estimates for empirical financial asset returns, which typically range from 2 to 5.

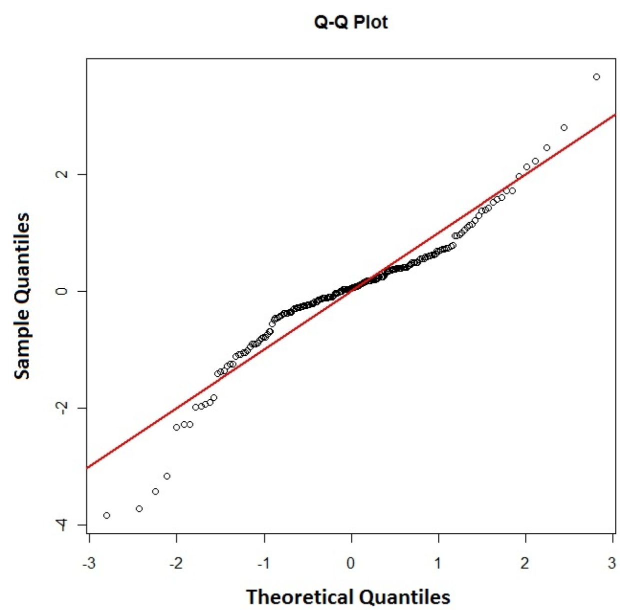

Figure 3.3: Q-Q plot of the set of values $\left\{\left(\overline{V}_{t_k}^{K,N} - \overline{V}_{t_{k-1}}^{K,N}\right) : k = 1, \ldots, K\right\}$ given by the simulated sample path of $\overline{V}^{K,N}$ shown in Figure 3.1.

Aggregational Gaussianity is also a common empirical property shared by the logarithmic returns of a wide set of financial assets. As we increase the length of the time interval over which the returns are calculated, their "distributions" ap-

Table 3.4: Hill Estimators for 10%−, 5%− and 2.5%−tails are repeatedly calculated for each of the 500 sets of 1-step returns associated with 500 individual sample paths of $\overline{V}^{K,N}$. The resulting values are then averaged and reported along with the corresponding standard deviations.

	10%	5%	2.5%
Average Value (over 500 sample sets)	2.15	2.97	4.13
Standard Deviation (over 500 sample sets)	0.469	0.877	2.18

pear to become more and more similar to a Normal distribution. In our model, the logarithmic return generated by the normalized total speculative demand, namely $\overline{V}^{K,N}$, seems to possess the same property, as illustrated graphically by Figure 3.4. To further support this finding, we calculate the sample kurtosis[10] for the sets of 1-, 7- and 14-step returns associated with each of the 500 individual sample paths of $\overline{V}^{K,N}$. The results are summarized in Table 3.5.

Table 3.5: The sample kurtosis is repeatedly calculated for the sets of 1-, 7- and 14-step returns associated respectively with 500 individual sample paths of $\overline{V}^{K,N}$. The resulting values are then averaged and reported along with the corresponding sample standard deviations.

	1-step	7-step	14-step
Average Value (over 500 sample sets)	3.22	2.39	1.64
Standard Deviation (over 500 sample sets)	2.15	2.20	1.93

Volatility clustering is yet another empirical property of logarithmic asset returns observed in the market. Large returns, positive or negative, tend to be

[10]Recall that in this dissertation, all kurtosis are calculated as $(\mathbb{E}[X^4]/\mathbb{E}[X^2]^2)-3$, i.e. the "excess kurtosis" relative to a normal distribution.

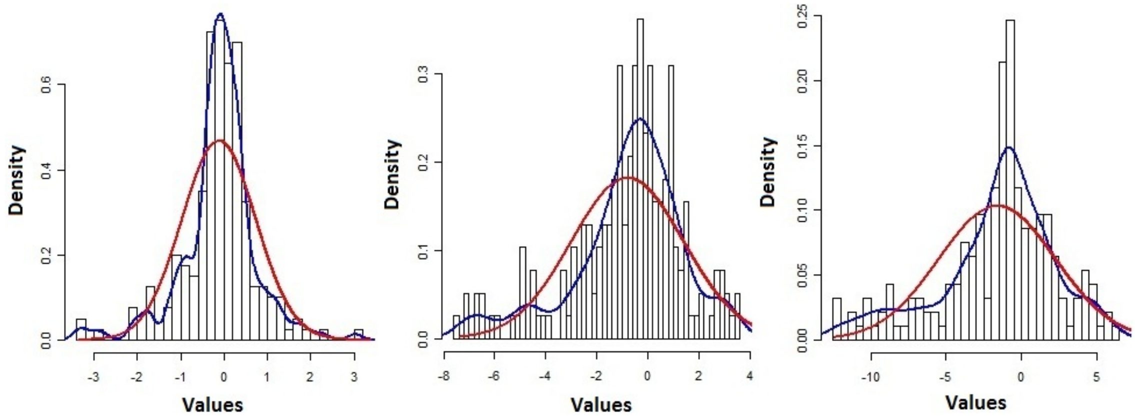

Figure 3.4: (Left to right) Histograms generated by the sets of 1-, 7- and 14-step returns given by the simulated sample path of $\overline{V}^{K,N}$ shown in Figure 3.1. The corresponding Kernel density estimation (blue) and a Normal density (red) with mean and variance matching the simulated data are also plotted.

followed by large returns, while small returns are often followed by small returns. In other words, the magnitude of logarithmic asset returns display a significant, positive autocorrelation [15]. The same phenomenon appears in Figure 3.5, when we plot the set of values $\left\{\left(\overline{V}^{K,N}_{t_k} - \overline{V}^{K,N}_{t_{k-1}}\right) : k = 1, \ldots, K\right\}$ given by an arbitrary simulated sample path of $\overline{V}^{K,N}$ against the time steps $\{t_k : k = 1, \ldots, K\}$.

A more quantitative manifestation of *Volatility Clustering* lies in the fact that, while the logarithmic asset returns are themselves uncorrelated, their absolute values usually display a significant, positive, and slow-decaying autocorrelation function [15]. In Figure 3.6, we plot the autocorrelation function of the set of 1-step returns associated with the simulated sample path of $\overline{V}^{K,N}$ in Figure 3.1, as well as the autocorrelation function of their absolute values. In contrast to the same plot created for a standard Brownian motion, our simulated data is again in qualitative agreement with empirical findings.

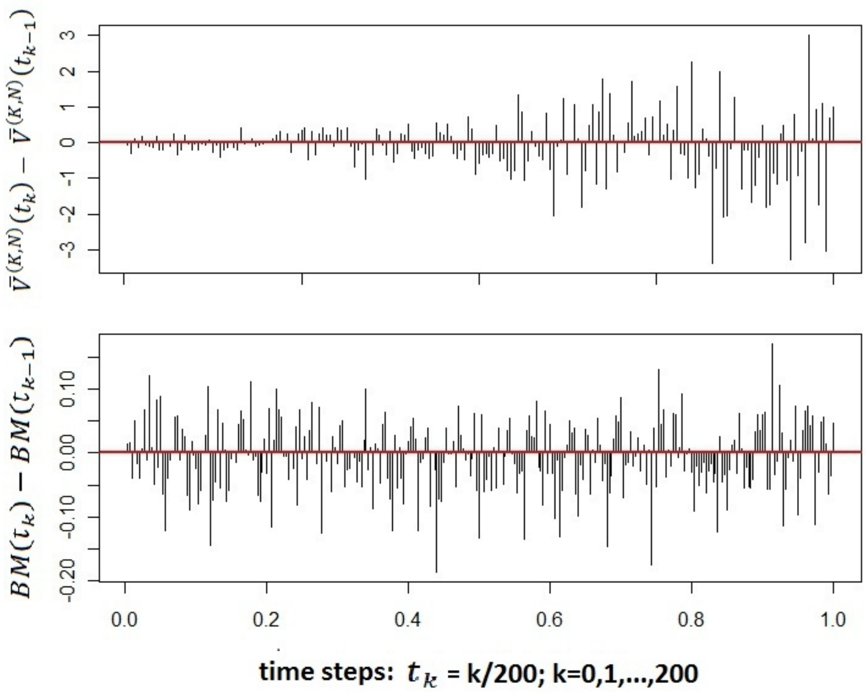

Figure 3.5: (Upper panel) The set of 1-step returns associated with the simulated sample path of $\overline{V}^{K,N}$ in Figure 3.1, plotted against the time grid $\{t_k : k = 1, \ldots, K\}$; (Lower panel) the same plot for a standard Brownian motion is presented for comparison.

A Multi-period Extension

As a direct extension to our 1-period model, consider an M-period communication model over a finite time horizon $[0, T]$, where the i-th period is denoted by $[T_{i-1}, T_i)$. Suppose $T_0 = 0$ and the length of each time period is $\frac{T}{M}$. Similar to the 1-period model, we partition every interval $[T_{i-1}, T_i)$ into K equidistant time steps for some fixed $K \in \mathbb{N}^+$. For each $i = 1, \ldots, M$, agents form i.i.d speculative demand at time T_{i-1} and subsequently update them throughout the interval $[T_{i-1}, T_i)$ according to the communication scheme specified in Section 3.3.

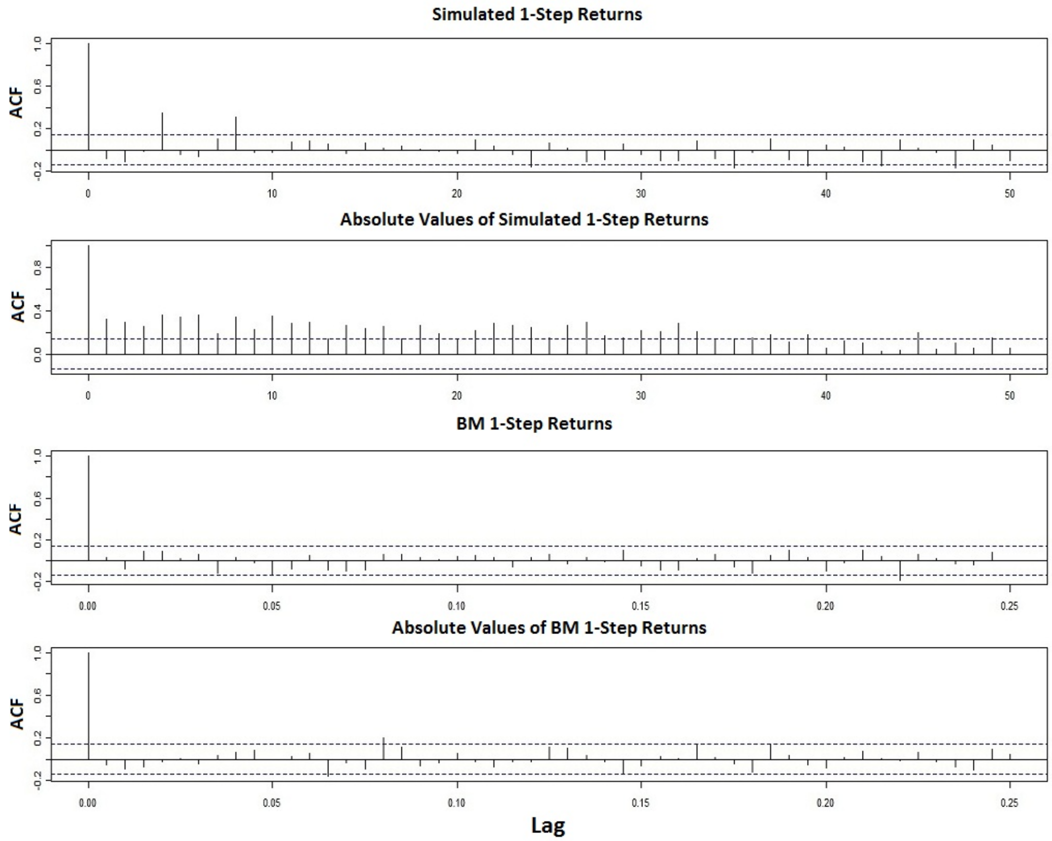

Figure 3.6: Autocorrelation functions of: (1) the 1-step returns given by the simulated sample path of $\overline{V}^{K,N}$ in Figure 3.1; (2) their corresponding absolute values; (3) the 1-step returns given by a standard Brownian motion sampled over the same time grid; (4) their absolute values.

Intuitively, at time T_{i-1}, investors are informed about an upcoming event i (e.g. a company's quarterly earnings release), which may seriously affect the value of a risky asset traded in the market. This information leads them to form independent initial speculative demands, which are subsequently adjusted over time via communication with other market participants. At time T_i, the outcome of event i is revealed and all related speculations vanish. In the meanwhile, agents begin to speculate about the next event $i + 1$ and same cycle repeats itself until we reach the end of the time horizon.

Figure 3.7 shows a simulated sample path of the normalized total speculative demand $\overline{V}^{M\cdot K,N}$ in our model[11], along with a path of the standard Brownian motion sampled over the same time grid. Note that in the case of $\overline{V}^{M\cdot K,N}$, jumps may occur naturally at mesh points of consecutive time periods.

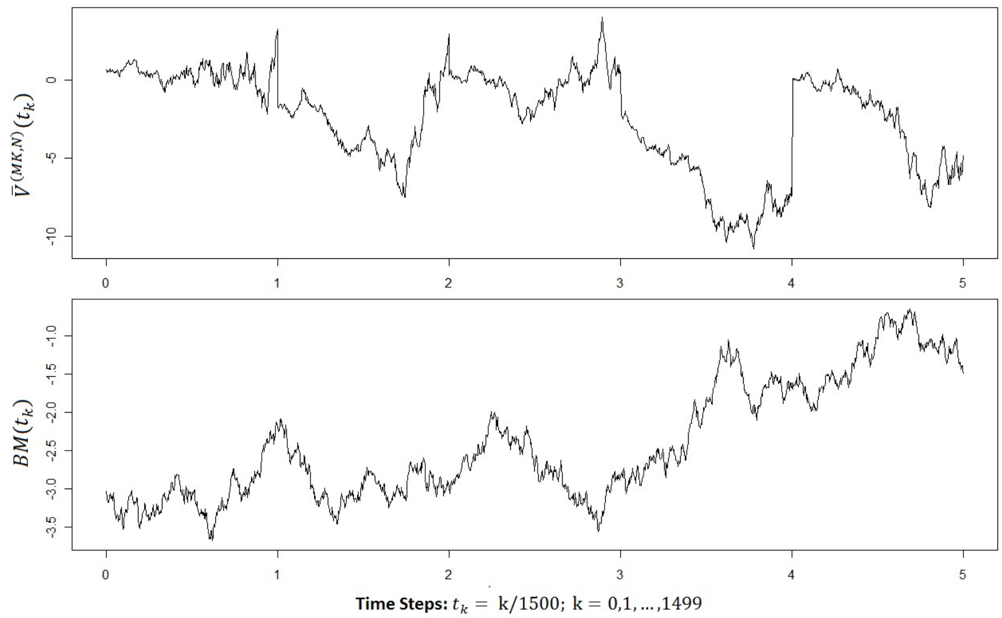

Figure 3.7: (Upper panel) Simulated sample path of $\overline{V}^{M\cdot K,N}$; (Lower panel) sample path of a standard Brownian motion simulated over the same time grid.

3.4.2 Rate of Convergence

By Theorem 3.3.2, as K and $N = N(K) \approx K^{2+\delta}$ ($\delta > 0$ is some small constant) tend to infinity, the distribution of $\overline{V}_{t_k}^{K,N}$, $k \in \{1, \dots, K\}$ converges to $N(0, \sigma^2)$, where σ^2 is the variance of the agents' initial speculative demands ξ_i, $i \in \{1, \dots, N\}$. To illustrate the convergence described in the theorem and

[11] $\overline{V}^{M\cdot K,N}$ is the M-period analogy to $\overline{V}^{K,N}$ defined in Section 3.1.

gain some insights regarding the corresponding rate of convergence, we simulate the system specified in Section 3.4.1 under different values of K and N(K), where $K = 50, 100, 200, 300, 400, 500, 600, 1000, 1500$ and $N(K) = [K^{2.05}]$. We then study in each case the empirical distribution of $\overline{V}_{t_K}^{K,N}$ given by 300 samples and compare their statistical properties with those of the $N(0, \sigma^2 = \frac{4}{3})$[12] distribution.

As shown in Figure 3.8, starting from $K = 300$, the shape of each kernel density estimator of $\overline{V}_{t_K}^{K,N}$ seems rather similar to that of the theoretical Normal density with matching mean and variance. To further test a more strict hypothesis that "the sample comes from a population with a $N(0, \frac{4}{3})$ distribution", we apply the Komogorov-Smirnov Goodness-of-Fit Test to each set of the 300 sampled values of $\overline{V}_{t_K}^{K,N}$. The resulting test statistics and p-values are presented in Table 3.6. Note that in the case of $K = 1500$, we no longer have evidence to reject the null hypothesis.

Table 3.6: Results of Komogorov-Smirnov Goodness-of-Fit Tests performed on samples of $\overline{V}_{t_K}^{K,N}$ with different values of K. Each sample is of size 300.

K =	Test Statistics	p-values
50	0.514	0
100	0.503	0
200	0.381	0
300	0.316	0
400	0.231	2.39×10^{-14}
500	0.208	1.05×10^{-11}
600	0.176	1.68×10^{-8}
1000	0.095	8.94×10^{-3}
1500	0.055	3.13×10^{-1}

[12]In Section 3.4.1, ξ_i is assumed to follow a Uniform$[-2, 2]$ distribution, so $\sigma^2 = \mathrm{Var}(\xi_i) = \frac{4}{3}$.

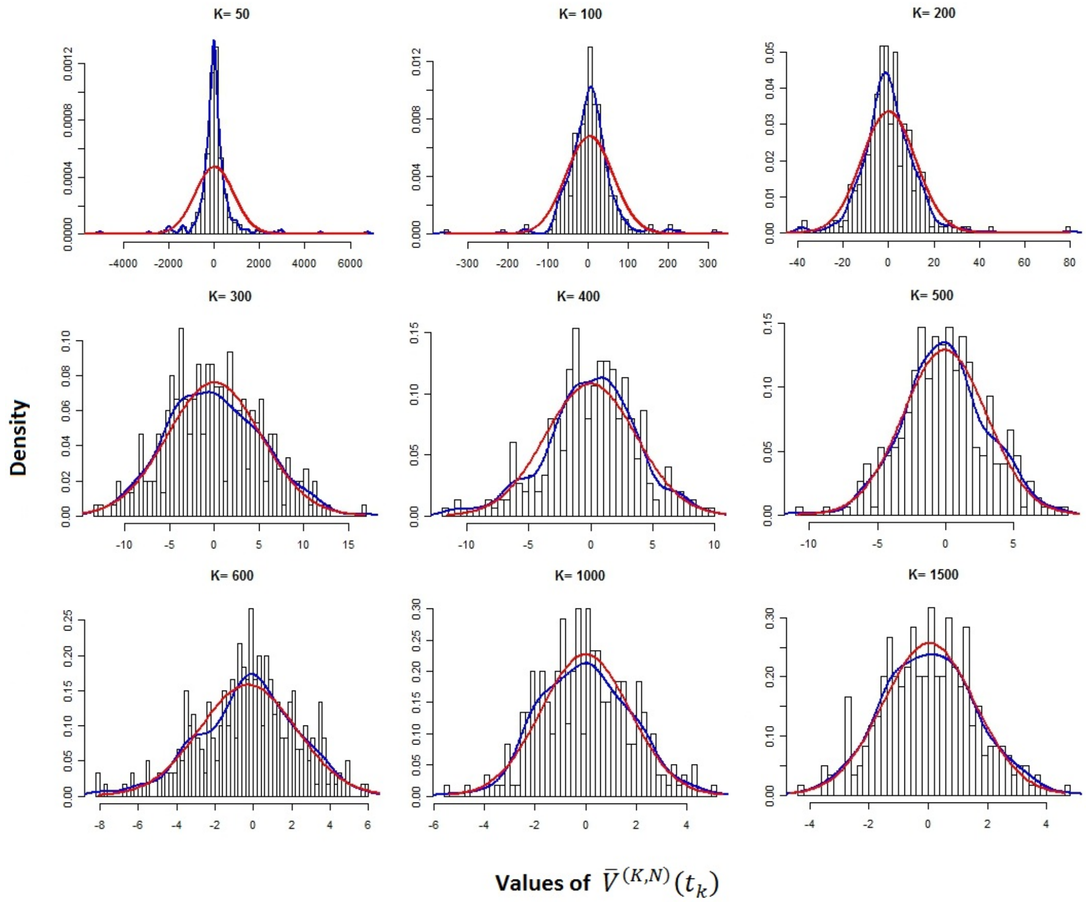

Figure 3.8: Histograms of $\overline{V}_{t_K}^{K,N}$ for different values of K, each generated by 300 samples. The corresponding Kernel density estimation (blue) and a Normal density (red) with mean and variance matching the simulated data are also plotted.

For each value of K, several basic statistics of $\overline{V}_{t_K}^{K,N}$ calculated from a sample of size 300 are presented in Table 3.7[13], along with the corresponding theoretical values of the $N(0, \frac{4}{3})$ distribution.

In Figure 3.9, we plot the logarithmic values of the sample "Absolute Mean" of $\overline{V}_{t_K}^{K,N}$ (reported in Column 3 of Table 3.7) against the corresponding values of K. The shape of the resulting curve indicates that $\mathbb{E}\left[\left|\overline{V}_{t_K}^{K,N}\right|\right]$ converges to $\mathbb{E}\left[|V^*|\right]$

[13]The "Absolute Mean" of a random variable X is calculated as $\mathbb{E}[|X|]$ and "Kurtosis" as usual refers to the "excess kurtosis".

Table 3.7: Basic statistics calculated from samples of $\overline{V}_{t_K}^{K,N}$ with different values of K. Each sample is of size 300. The corresponding theoretical values of the $N(0, \frac{4}{3})$ distribution is provided for comparison.

K =	Mean	Absolute Mean	Variance	Standard Deviation	Skewness	Kurtosis
50	32.2	434.8	7.14×10^5	845.2	1.57	20.6
100	3.26	38.6	3.47×10^3	58.9	5.8×10^{-2}	8.64
200	0.115	8.36	142.0	11.9	0.984	7.2
300	-0.01	4.24	27.6	5.25	0.183	-0.24
400	-6.63×10^{-3}	2.86	13.6	3.69	-0.312	0.431
500	-0.054	2.42	9.54	3.09	-0.013	0.209
600	-0.28	1.97	6.36	2.52	-0.307	0.118
1000	3.58×10^{-3}	1.42	3.1	1.76	0.105	-0.148
1500	0.038	1.25	2.41	1.55	0.097	-0.161
$N(0, \frac{4}{3})$	0	0.921	1.33	1.15	0	0

at polynomial rate $O\left[C_0 \cdot K^{-\rho}\right]$, where V^* is a $N(0, \frac{4}{3})$ random variable and $C_0, \rho > 0$ are some constants. Since

$$\left|\mathbb{E}\left[\left|\overline{V}_{t_K}^{K,N}\right|\right] - \mathbb{E}\left[|V^*|\right]\right| \sim C_0 \cdot K^{-\rho}$$

implies that

$$\log\left(\left|\mathbb{E}\left[\left|\overline{V}_{t_K}^{K,N}\right|\right] - \mathbb{E}\left[|V^*|\right]\right|\right) \sim \log C_0 + (-\rho) \cdot \log K,$$

we perform a linear regression of $y = \log\left(\left|\mathbb{E}\left[\left|\overline{V}_{t_K}^{K,N}\right|\right] - \mathbb{E}\left[|V^*|\right]\right|\right)$ on $x = \log K$ to help identify the two constants $C_0 > 0$ and $\rho > 0$. The estimated model parameters are given in Table 3.8 while the fitted line is plotted in Figure 3.10 .

Results from the linear regression suggest that

$$C_0 \approx e^{13.4} \approx 6.6 \times 10^5 \quad \text{and} \quad \rho \approx 2.06.$$

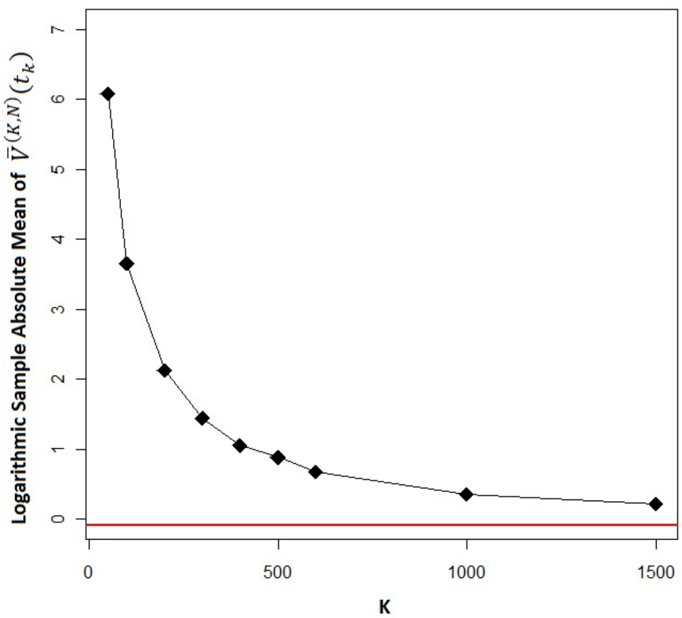

Figure 3.9: Plot of logarithmic values of sample "Absolute Mean" of $\overline{V}_{t_k}^{K,N}$ reported in Table 3.7 against the corresponding values of K. The horizontal line $y = \log(0.921)$ represents the logarithmic of the theoretical absolute mean of $N(0, \frac{4}{3})$.

Table 3.8: Results of linear regression using data in Columns 1 and 3 of Table 3.7, where $x = \log K$, $y = \log\left(\left|\text{Absolute Mean} - 0.921\right|\right)$.

Model: $\mathbf{y} = a + b\mathbf{x}$		$R^2 = 0.963$		
Coefficients	Estimate	Std. Error	t-value	$\Pr(>\|t\|)$
a	13.4	0.908	14.7	1.6×10^{-6}
b	-2.06	0.154	-13.4	2.98×10^{-6}

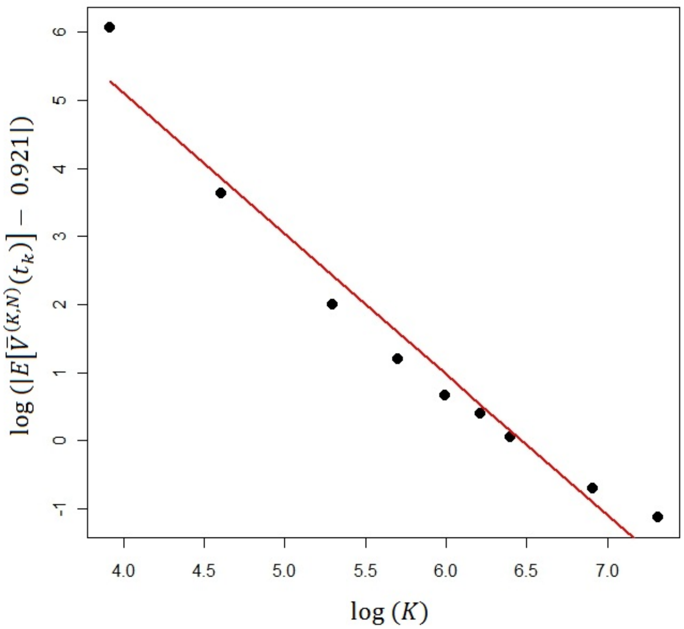

Figure 3.10: Plot of data points and the fitted line as specified in Table 3.8.

Thus,

$$\left|\mathbb{E}\left[\left|\overline{V}^{K,N}_{t_K}\right|\right] - \mathbb{E}\left[|V^*|\right]\right| \sim 6.6 \times 10^5 \cdot K^{-2.06}.$$

Such slow rate of convergence, compared to that of a typical Central Limit Theorem where the random variables are identically and independently distributed, is not surprising given the dependence structure of our model.

CHAPTER 4

CONTINUOUS-TIME INTERACTION WITH A CENTRAL AUTHORITY

4.1 Model Specification

Consider a finite time horizon $[0, T]$. For all $i \in \mathbb{N}^+$, V_t^i represents the speculative demand of agent i at time $t \in [0, T]$, with initial value $V_0^i = 0$. Over time, each agent modifies her speculative demand by continuously comparing it against signals given by an independent central authority $(Y_t)_{t\geq 0}$, e.g. some analyst report, and making adjustments accordingly. More specifically, for each $i \in \mathbb{N}^+$, the process V^i satisfies the following Stochastic Differential Equation:

$$dV_t^i = \alpha \left(Y_t - V_{t-}^i \right) dM_t^i$$

where Y is a continuous semimartingale, M^i is a càdlàg semimartingale independent from Y, and α is some constant. The term αdM_t^i captures the instantaneous stochastic rate of impact the central authority has on agent i at time t.

In the rest of the chapter, we investigate mainly two types of traders commonly observed in the financial market, namely the *Followers* and the *Non-Followers*. If agent i is a *Follower*, then $\alpha < 0$ and $M_t^i \equiv N_t^i - \lambda t$, where N^i is a Poisson process with arrival rate $a(t) = \lambda t$. Thus, the corresponding process V^i satisfies

$$dV_t^i = \alpha \left(Y_t - V_{t-}^i \right) d(N_t^i - \lambda t).$$

Intuitively, a *Follower* actively mimic the opinion of the central authority Y at all times, except when she receives certain private information that causes her to act differently. On the other hand, if agent i is a *Non-Follower*, then $\alpha > 0$ and

$M_t^i \equiv B_t^i$ is taken to be a standard Brownian motion, and the corresponding V^i satisfies

$$dV_t^i = \alpha\left(Y_t - V_t^i\right)dB_t^i.$$

In this case, agent i processes the received signal randomly, i.e. depending on the sign of dB_t^i, she either follows or acts opposite to the central authority.

Finally, we consider the process

$$\overline{V}^N := \frac{1}{\sqrt{N}}\sum_{i=1}^{N} V^i. \tag{4.1}$$

As $N \to \infty$, $\overline{V}^N$ captures the normalized aggregated speculative demand of all agents across the entire market.

4.2 A Central Limit Theorem for Processes

4.2.1 Mathematical Setup

Consider a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on which a càdlàg, finite activity Lévy processes M is defined. M has bounded jumps, i.e. $\sup_s |\Delta M_s^i| \le a$ a.s. for some constant $a > 0$, $M_0^i = 0$ and $\widetilde{\mathbb{E}}[M_t^i] = 0$ for all $t \in [0, T]$. Let $\left(\mathcal{B}^i, M^i\right)_{i\ge1}$ be a sequence of identical copies of the pair $(\mathcal{B}, M)$. Define the stochastic basis $\widehat{\mathcal{B}} = (\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{\mathbb{P}})$ as the tensor product of all $\mathcal{B}^i$'s. More specifically,

$$\widehat{\Omega} = \Omega^1 \times \Omega^2 \times \cdots \qquad \widehat{\mathcal{F}}_t = \cap_{s>t}\mathcal{F}_s^1 \otimes \mathcal{F}_s^2 \otimes \cdots$$
$$\widehat{\mathcal{F}} = \mathcal{F}^1 \otimes \mathcal{F}^2 \otimes \cdots \qquad \widehat{\mathbb{P}}(d\hat{\omega}) = \mathbb{P}^1(d\omega^1)\mathbb{P}^2(d\omega^2)\cdots$$

In addition, let $\mathcal{B}' = (\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$ be another stochastic basis and Y be a continuous square-integrable martingale on $\mathcal{B}'$ such that $Y_0 = 0$. We can then define the stochastic basis $\widetilde{\mathcal{B}} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, \widetilde{\mathbb{P}})$ as the tensor product of $\mathcal{B}'$ and $\widehat{\mathcal{B}}$ in the same fashion as before. Note that any process defined on Ω' or Ω^i's can also be considered as a process on $\widetilde{\Omega}$ and/or $\widehat{\Omega}$ by setting

$$X(\widetilde{\omega}) = X(\omega', \hat{\omega}) = X(\omega')$$

$$X(\widetilde{\omega}) = X(\omega', \hat{\omega}) = X(\omega', \omega^1, \ldots) = X(\omega^i)$$

$$X(\hat{\omega}) = X(\omega^1, \omega^2, \ldots) = X(\omega^i).$$

Moreover, since the extensions are *very good*, all martingales on the basis $\mathcal{B}'$ or $\mathcal{B}^i$ are also martingales on $\widetilde{\mathcal{B}}$ and/or $\widehat{\mathcal{B}}$[1].

4.2.2 Main Theorem

Theorem 4.2.1 *Let $Y = (Y_t)_{t \geq 0}$ be a continuous square-integrable martingale such that $Y_0 = 0$. For each $i \in \mathbb{N}^+$, let $M^i = (M^i_t)_{t \geq 0}$ be a càdlàg, finite activity Lévy processes with bounded jumps, i.e. $\sup_s |\Delta M^i_s| \leq a$ a.s. for some constant $a > 0$, such that $M^i_0 = 0$ and $\widetilde{\mathbb{E}}[M^i_t] = 0$ for all $t \in [0, T]$. Suppose the processes V^i, $i = 1, 2, \ldots$, satisfy*

$$dV^i_t = \alpha \left(Y_t - V^i_{t-} \right) dM^i_t, \quad V^i_0 = 0 \tag{4.2}$$

for some constant $\alpha \neq 0$. Then the sequence of processes

$$\overline{V}^N := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} V^i, \quad N = 1, 2, \ldots$$

converges in law to a process V^* as $N \to \infty$, where V^* is a stochastic time-changed Wiener process. Moreover,

(a) (Non-Followers) *If $M^i_t \equiv B^i_t$ is a standard Brownian motion, then*

$$[V^*, V^*]_t = \alpha^2 e^{\alpha^2 t} \int_0^t e^{-\alpha^2 s} Y_s^2 ds.$$

[1] See Chapter VI of Jacod and Shiryaev (1987)

(b) (Followers) If $M^i_t \equiv N^i_t - \lambda t$, where N^i is a Poisson process with arrival rate $a(t) = \lambda t$, then

$$[V^*, V^*]_t = \alpha^2 \lambda e^{\alpha^2 \lambda t} \int_0^t e^{-\alpha^2 \lambda s} Y_s^2 ds.$$

4.2.3 Proof of Main Theorem

We first prove a few useful Lemmas.

Lemma 4.2.2 *For each $i = 1, 2, \ldots$, the Stochastic Differential Equation (4.2) has a unique (strong) solution V^i, which is a semimartingale. Moreover,*

(a) (Non-Followers) If $M^i_t \equiv B^i_t$ is a standard Brownian motion, then

$$V^i_t = \mathcal{E}(-\alpha B^i)_t \int_0^t \mathcal{E}(-\alpha B^i)_s^{-1} \cdot \Big(\alpha Y_s dB^i_s + \alpha^2 Y_s ds\Big);$$

(b) (Followers) If $M^i_t \equiv N^i_t - \lambda t$, where N^i is a Poisson process with arrival rate $a(t) = \lambda t$, then

$$V^i_t = \int_0^t \exp\Big\{\alpha\lambda(t-s)\Big\}\Bigg(\prod_{s \le u \le t}\Big(1 - \Delta\alpha M^i_u\Big)\Bigg)\Big(\alpha^2 Y_s dN^i_s + \alpha Y_s dM^i_s\Big).$$

Proof Since $V^i_0 = 0$, we can rewrite the SDE (4.2) as

$$V^i_t = \int_0^t \alpha(Y_s - V^i_{s-}) dM^i_s = \int_0^t \alpha Y_s dM^i_s + \int_0^t -\alpha V^i_{s-} dM^i_s. \tag{4.3}$$

By assumption, M^i is a semimartingale with $M^i_0 = 0$ and Y is an adapted, continuous process. Thus, $J^i_t := \int_0^t \alpha Y_s dM^i_s$ is also a semimartingale and Theorem 7 in Chapter V of Protter (2004) implies that (4.3) has a unique solution V^i, which is again a semimartingale.

(a) If $M^i_t \equiv B^i_t$, the closed form of V^i can be obtained by directly applying Theorem 52 in Chapter V of Protter (2004).

(b) If $M_t^i \equiv N_t^i - \lambda t$, let $\Delta M_t^i := M_t^i - M_{t-}^i = M_t^i - \lim_{s\uparrow t} M_s^i$. To obtain the closed-form V^i, we first define a sequence of stopping times

$$\begin{aligned}
T_0 &= 0, \\
T_1 &= \inf\{t > 0 : 1 + \Delta(-\alpha M_t^i) = 0\}, \\
&\vdots \\
T_{n+1} &= \inf\{t > T_n : 1 + \Delta(-\alpha M_t^i) = 0\}
\end{aligned}$$

Since $M_t^i = N_t^i - \lambda t$ and $\alpha < 0$, we know that $\Delta(-\alpha M_t^i) = -\alpha \Delta N_t^i \geq 0$ a.s. for all $t \in [0, \infty)$. Thus, $T_n = +\infty$ for all $n \geq 1$. Moreover, $[M^i, M^i]^c \equiv 0$ and $M_0^i = 0$. By Exercise 27 in Chapter V of Protter (2004), we have

$$V_t^i = \sum_{n\geq 0} Z_t^n \mathbf{1}_{[T_n, T_{n+1})}(t) = Z_t^0 \quad \text{for all } t \in [0, T],$$

where

$$Z_t^0 = \left(\int_0^t \frac{1}{U_{s-}^0} \alpha Y_s dM_s^i - \int_0^t \frac{1}{U_{s-}^0} (\alpha Y_s)(-\alpha) d[M^i, M^i]_s \right) \cdot U_t^0$$

with

$$U_t^0 = \exp\left\{-\alpha M_t^i\right\} \cdot \prod_{0<s\leq t} \left(1 + \Delta(-\alpha M_s^i)\right) \exp\left\{\Delta \alpha M_s^i\right\}.$$

Since

$$\sum_{0<s\leq t} \Delta M_s^i = \sum_{0<s\leq t} \Delta N_s^i = N_t^i \quad \text{for all } \ t \in [0, T],$$

we know that

$$\begin{aligned}
U_t^0 &= \exp\left\{-\alpha M_t^i\right\} \cdot \left\{ \prod_{0<s\leq t} \left(1 - \Delta \alpha M_s^i\right) \right\} \cdot \left\{ \prod_{0<s\leq t} \exp\left\{\Delta \alpha M_s^i\right\} \right\} \\
&= \exp\left\{-\alpha M_t^i\right\} \cdot \left\{ \prod_{0<s\leq t} \left(1 - \Delta \alpha M_s^i\right) \right\} \cdot \exp\left\{\alpha \sum_{0<s\leq t} \Delta M_s^i\right\} \\
&= \exp\left\{-\alpha (M_t^i - N_t^i)\right\} \cdot \left\{ \prod_{0<s\leq t} \left(1 - \Delta \alpha M_s^i\right) \right\} \\
&= \exp\left\{\alpha \lambda t\right\} \cdot \left\{ \prod_{0<s\leq t} \left(1 - \Delta \alpha M_s^i\right) \right\}.
\end{aligned}$$

So for any $0 < s \le t$,

$$\frac{U_t^0}{U_{s-}^0} = \frac{\exp\{\alpha\lambda t\}\cdot\left\{\prod_{0<u\le t}\left(1-\Delta\alpha M_u^i\right)\right\}}{\exp\{\alpha\lambda s\}\cdot\left\{\prod_{0<u<s}\left(1-\Delta\alpha M_u^i\right)\right\}} = \exp\left\{\alpha\lambda(t-s)\right\}\left\{\prod_{s\le u\le t}\left(1-\Delta\alpha M_u^i\right)\right\}$$

Moreover, $d[M^i, M^i]_s = dN_s^i$. Therefore,

$$V_t^i = \int_0^t \exp\left\{\alpha\lambda(t-s)\right\}\left(\prod_{s\le u\le t}\left(1-\Delta\alpha M_u^i\right)\right)\left(\alpha^2 Y_s dN_s^i + \alpha Y_s dM_s^i\right).$$

$\square$

Lemma 4.2.3 *For each $i \in \mathbb{N}^+$, V^i is a square-integrable martingale on $\widetilde{\mathcal{B}}$, i.e. $C_t := \widetilde{\mathbb{E}}\left[(V_t^i)^2\right] < \infty$ for all $t \in [0, T]$. Moreover,*

(a) (Non-Followers) *If $M_t^i \equiv B_t^i$ is a standard Brownian motion, then*

$$C_t := \widetilde{\mathbb{E}}\left[(V_t^i)^2\right] = \alpha^2 e^{\alpha^2 t}\int_0^t e^{-\alpha^2 s}\widetilde{\mathbb{E}}[Y_s^2]ds \quad \forall\, t \in [0, T];$$

(b) (Followers) *If $M_t^i \equiv N_t^i - \lambda t$, where N^i is a Poisson process with arrival rate $a(t) = \lambda t$, then*

$$C_t := \widetilde{\mathbb{E}}\left[(V_t^i)^2\right] = \alpha^2\lambda e^{\alpha^2\lambda t}\int_0^t e^{-\alpha^2\lambda s}\widetilde{\mathbb{E}}\left[Y_s^2\right]ds \quad \forall\, t \in [0, T].$$

Proof By assumption, for each $i \in \mathbb{N}^+$, M_t^i is a finite activity Lévy process with bounded jumps, i.e. $\sup_s |\Delta M_s^i| \le a$ a.s. for some constant $a > 0$. Moreover, $\widetilde{\mathbb{E}}[M_t^i] = 0$ for all $t \in [0, T]$, so M^i is a square-integrable martingale by Theorem 34 and Theorem 41 in Chapter I of Protter (2004), and the Lévy-Khintchine triplet (b, c, F) of M^i must satisfy $b = 0$ and $\int_{\mathbb{R}} F(dx) = \lambda < \infty$ [24]. In addition, the Lévy Decomposition Theorem in Chapter I of Protter (2004) implies that $M_t^i = \sqrt{c}W_t^i + J_t^i$, where W_t^i is a standard Brownian motion and J_t^i is a purely discontinuous martingale independent of W_t. Denote by $\mu^{M^i}(\omega; dt, dx)$ the random measure of jumps associated with process M^i, then

$$J_t = \int_0^t \int_{\{|x|\le a\}} x\left(\mu^{M^i}(ds,dx) - dsF(dx)\right)$$

and we have

$$V_t^i = \int_0^t \sqrt{c}\alpha\left(Y_s - V_{s-}^i\right)dW_s^i + \int_0^t \alpha\left(Y_s - V_{s-}^i\right)dJ_s^i.$$

By Lemma 4.2.2, V^i is a semimartingale. Since Y is adapted and continuous, we know that $\alpha(Y_t - V_{t-}^i)$ is adapted and càglàd, thus predictable. Moreover, M^i is a square-integrable martingale, so Theorem 33 in Chapter III of Protter (2004) implies that V^i is a local martingale. Thus, by Corollary 3 in Chapter II of Protter (2004), in order to prove V^i is a square-integrable martingale on $\widetilde{\mathcal{B}}$, we are left to show that $\widetilde{\mathbb{E}}\left[[V^i,V^i]_t\right] < \infty$ for all $t \in [0,T]$.

For $n \ge 1$, define $\tau_n := \inf\left\{ t \ge 0 : |V_t^i| \ge n \right\}$. Since V^i is an adapted and càdlàg process with $V_0^i = 0$, we know that $(\tau_n)_{n\ge1}$ is a sequence of stopping times increasing to infinity almost surely as $n \uparrow \infty$, and $\left|V_-^i \mathbf{1}_{[0,\tau_n]}\right| \le n$ a.s., where V_-^i is the left continuous version of V^i. Let $V_t^{i,\tau_n} = V_{t\wedge\tau_n}^i$ be the stopped process, then

$$V_t^{i,\tau_n} = \underbrace{\int_0^t \sqrt{c}\alpha\left(Y_s - V_{s-}^i\right)\mathbf{1}_{\{0\le s\le\tau_n\}}dW_s^i}_{=:C_t^{i,\tau_n}} + \underbrace{\int_0^t \alpha\left(Y_s - V_{s-}^i\right)\mathbf{1}_{\{0\le s\le\tau_n\}}dJ_s^i}_{=:D_t^{i,\tau_n}}$$

by Theorem 12 in Chapter II of Protter (2004). Since $\alpha(Y_t - V_{t-}^i)\mathbf{1}_{\{0\le t\le\tau_n\}}(t)$ is an adapted, càglàd process (hence predictable) and W^i, J^i are both martingales, by Theorem 33 in Chapter III of Protter (2004), we know that C^{i,τ_n} and D^{i,τ_n} are both local martingales. Moreover,

$$
\begin{aligned}
\widetilde{\mathbb{E}}\left[[C^{i,\tau_n}, C^{i,\tau_n}]_t\right] &= \widetilde{\mathbb{E}}\left[\int_0^t c\alpha^2 \left(Y_s - V^i_{s-}\right)^2 \mathbf{1}_{\{0\le s\le \tau_n\}} ds\right] \\
&\le \widetilde{\mathbb{E}}\left[\int_0^t 2c\alpha^2 \left(Y_s^2 + (V^i_{s-})^2\right) \mathbf{1}_{\{0\le s\le \tau_n\}} ds\right] \\
&\le \int_0^t 2c\alpha^2 \widetilde{\mathbb{E}}\left[Y_s^2 \mathbf{1}_{\{0\le s\le \tau_n\}}\right] ds + \int_0^t 2c\alpha^2 \widetilde{\mathbb{E}}\left[(V^i_{s-})^2 \mathbf{1}_{\{0\le s\le \tau_n\}}\right] ds \\
&\le \int_0^t 2c\alpha^2 \widetilde{\mathbb{E}}\left[Y_s^2\right] ds + \int_0^t 2c\alpha^2 n^2 ds \\
&< \infty
\end{aligned}
$$

for all $t \in [0, T]$, where the interchanging of expectation and integral is justified by Fubini's Theorem and $\int_0^t \widetilde{\mathbb{E}}\left[Y_s^2\right] ds < \infty$ by assumption. Thus, C^{i,τ_n} is a square-integrable martingale. On the other hand,

$$
D_t^{i,\tau_n} = \int_0^t \int_{\{|x|\le a\}} x\alpha \left(Y_s - V^i_{s-}\right) \mathbf{1}_{\{0\le s\le \tau_n\}} \left(\mu^{M^i}(ds, dx) - dsF(dx)\right).
$$

Since $x\alpha\left(Y_s - V^i_{s-}\right)\mathbf{1}_{\{0\le s\le \tau_n\}}$ is an adapted, càglàd process (hence predictable) and

$$
\begin{aligned}
& \widetilde{\mathbb{E}}\left[\int_0^T \int_{\mathbb{R}} x^2\alpha^2 \left(Y_s - V^i_{s-}\right)^2 \mathbf{1}_{\{0\le s\le \tau_n\}} ds F(dx)\right] \\
=\ & \widetilde{\mathbb{E}}\left[\int_0^T \alpha^2 \left(Y_s - V^i_{s-}\right)^2 \mathbf{1}_{\{0\le s\le \tau_n\}} \left(\int_{\{|x|\le a\}} x^2 F(dx)\right) ds\right] \\
\le\ & \widetilde{\mathbb{E}}\left[\int_0^T 2\alpha^2 \left(Y_s^2 + (V^i_{s-})^2\right) \mathbf{1}_{\{0\le s\le \tau_n\}} a^2\lambda ds\right] \\
\le\ & \int_0^T 2a^2\lambda\alpha^2 \widetilde{\mathbb{E}}\left[Y_s^2 \mathbf{1}_{\{0\le s\le \tau_n\}}\right] ds + \int_0^T 2a^2\lambda\alpha^2 \widetilde{\mathbb{E}}\left[\left(V^i_{s-}\mathbf{1}_{\{0\le s\le \tau_n\}}\right)^2\right] ds \\
\le\ & \int_0^T 2a^2\lambda\alpha^2 \widetilde{\mathbb{E}}\left[Y_s^2\right] ds + \int_0^T 2a^2\lambda\alpha^2 n^2 ds \\
<\ & \infty,
\end{aligned}
$$

by Proposition 8.8 in Chapter 8 of Cont and Tankov(2004), D^{i,τ_n} is also a square-integrable martingale with

$$
\widetilde{\mathbb{E}}\left[(D_t^{i,\tau_n})^2\right] = \widetilde{\mathbb{E}}\left[\int_0^t \int_{\{|x|\le a\}} x^2\alpha^2 \left(Y_s - V^i_{s-}\right)^2 \mathbf{1}_{\{0\le s\le \tau_n\}} ds F(dx)\right].
$$

Combine the above arguments, we see that $V_t^{i,\tau_n} = C_t^{i,\tau_n} + D_t^{i,\tau_n}$ is itself a square-integrable martingale. Moreover, since W^i and J^i are independent, $[W^i, J^i]_t = 0$ for all $t \in [0,T]$. Thus,

$$\left[C^{i,\tau_n}, D^{i,\tau_n}\right]_t = \int_0^t \sqrt{c}\alpha^2 \left(Y_s - V_{s-}^i\right)^2 \mathbf{1}_{\{0\le s\le \tau_n\}} d[W^i, J^i]_s = 0.$$

By Proposition 4.50 in Chapter I of Jacod and Shiryaev (1987), $C^{i,\tau_n} D^{i,\tau_n}$ is a uniformly integrable martingale. Thus, $\widetilde{\mathbb{E}}\left[C_t^{i,\tau_n} D_t^{i,\tau_n}\right] = C_0^{i,\tau_n} D_0^{i,\tau_n} = 0$, which implies that

$$\widetilde{\mathbb{E}}\left[(V_t^{i,\tau_n})^2\right] = \widetilde{\mathbb{E}}\left[(C_t^{i,\tau_n})^2\right] + \widetilde{\mathbb{E}}\left[(D_t^{i,\tau_n})^2\right] \quad \text{for all } t \in [0,T].$$

Since $\left|V_-^i \mathbf{1}_{[0,\tau_n]}\right| \le \left|V_-^{i,\tau_n}\right|$ a.s. and $\widetilde{\mathbb{E}}\left[(V_t^{i,\tau_n})^2\right] = \widetilde{\mathbb{E}}\left[(V_{t-}^{i,\tau_n})^2\right]$, we see that

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[(C_t^{i,\tau_n})^2\right] &= \widetilde{\mathbb{E}}\left[[C^{i,\tau_n}, C^{i,\tau_n}]_t\right] \\
&\le \int_0^t 2c\alpha^2 \widetilde{\mathbb{E}}\left[Y_s^2 \mathbf{1}_{\{0\le s\le\tau_n\}}\right] ds + \int_0^t 2c\alpha^2 \widetilde{\mathbb{E}}\left[\left(V_{s-}^i \mathbf{1}_{\{0\le s\le\tau_n\}}\right)^2\right] ds \\
&\le \int_0^t 2c\alpha^2 \widetilde{\mathbb{E}}\left[Y_s^2\right] ds + \int_0^t 2c\alpha^2 \widetilde{\mathbb{E}}\left[(V_s^{i,\tau_n})^2\right] ds
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[(D_t^{i,\tau_n})^2\right] &= \widetilde{\mathbb{E}}\left[\int_0^t \int_{\{|x|\le a\}} x^2\alpha^2 \left(Y_s - V_{s-}^i\right)^2 \mathbf{1}_{\{0\le s\le\tau_n\}} ds F(dx)\right] \\
&\le \int_0^t 2a^2\lambda\alpha^2 \widetilde{\mathbb{E}}\left[Y_s^2 \mathbf{1}_{\{0\le s\le\tau_n\}}\right] ds + \int_0^t 2a^2\lambda\alpha^2 \widetilde{\mathbb{E}}\left[\left(V_{s-}^i \mathbf{1}_{\{0\le s\le\tau_n\}}\right)^2\right] ds \\
&\le \int_0^t 2a^2\lambda\alpha^2 \widetilde{\mathbb{E}}\left[Y_s^2\right] ds + \int_0^t 2a^2\lambda\alpha^2 \widetilde{\mathbb{E}}\left[(V_s^{i,\tau_n})^2\right] ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[(V_t^{i,\tau_n})^2\right] &\le \int_0^t 2c\alpha^2 \widetilde{\mathbb{E}}\left[Y_s^2\right] ds + \int_0^t 2c\alpha^2 \widetilde{\mathbb{E}}\left[(V_s^{i,\tau_n})^2\right] ds \\
&\qquad + \int_0^t 2a^2\lambda\alpha^2 \widetilde{\mathbb{E}}\left[Y_s^2\right] ds + \int_0^t 2a^2\lambda\alpha^2 \widetilde{\mathbb{E}}\left[(V_s^{i,\tau_n})^2\right] ds \\
&= \int_0^t 2\alpha^2(c + a^2\lambda) \widetilde{\mathbb{E}}\left[Y_s^2\right] ds + \int_0^t 2\alpha^2(c + a^2\lambda) \widetilde{\mathbb{E}}\left[(V_s^{i,\tau_n})^2\right] ds,
\end{aligned}$$

and the Gronwall's Inequality implies that

$$\widetilde{\mathbb{E}}\left[(V_t^{i,\tau_n})^2\right] \le e^{2\alpha^2(c+a^2\lambda)t} \cdot \int_0^t 2\alpha^2(c + a^2\lambda) \widetilde{\mathbb{E}}\left[Y_s^2\right] ds.$$

Therefore, we have

$$\widetilde{\mathbb{E}}\left[[V^i, V^i]_t\right] = \widetilde{\mathbb{E}}\left[\lim_{n\to\infty}[V^i, V^i]_{t\wedge\tau_n}\right] = \widetilde{\mathbb{E}}\left[\lim_{n\to\infty}[V^{i,\tau_n}, V^{i,\tau_n}]_t\right] = \lim_{n\to\infty}\widetilde{\mathbb{E}}\left[[V^{i,\tau_n}, V^{i,\tau_n}]_t\right]$$
$$= \lim_{n\to\infty}\widetilde{\mathbb{E}}\left[(V_t^{i,\tau_n})^2\right] \leq e^{2\alpha^2(c+a^2\lambda)t}\cdot\int_0^t 2\alpha^2(c+a^2\lambda)\widetilde{\mathbb{E}}\left[Y_s^2\right]ds < \infty,$$

where the interchanging of limit and expectation is justified by the Monotone Convergence Theorem. This in turn implies that V^i is a square-integrable martingale.

Finally, since Y is a square-integrable martingale independent from M^i,

$$\left[Y, V^i\right]_t = \int_0^t \alpha\left(Y_s - V_{s-}^i\right)d[Y, M^i]_s = 0 \quad \text{for all } t \in [0, T].$$

Again by Proposition 4.50 in Chapter I of Jacod and Shiryaev (1987), YV^i is a uniformly integrable martingale and $\widetilde{\mathbb{E}}\left[Y_t V_t^i\right] = Y_0 V_0^i = 0$ for all $t \in [0, T]$. Therefore,

(a) If $M_t^i \equiv B_t^i$, we have

$$\widetilde{\mathbb{E}}\left[(V_t^i)^2\right] = \widetilde{\mathbb{E}}\left[[V^i, V^i]_t\right] = \int_0^t \alpha^2\widetilde{\mathbb{E}}\left[\left(Y_s - V_s^i\right)^2\right]ds = \int_0^t \alpha^2\widetilde{\mathbb{E}}\left[Y_s^2\right]ds + \int_0^t \alpha^2\widetilde{\mathbb{E}}\left[(V_s^i)^2\right]ds$$

i.e.

$$d\widetilde{\mathbb{E}}\left[(V_t^i)^2\right] = \alpha^2\widetilde{\mathbb{E}}\left[Y_t^2\right]dt + \alpha^2\widetilde{\mathbb{E}}\left[(V_t^i)^2\right]dt, \quad \text{with } \widetilde{\mathbb{E}}\left[(V_0^i)^2\right] = 0$$

Solving the nonhomogeneous ODE, we get

$$\widetilde{\mathbb{E}}\left[(V_t^i)^2\right] = \alpha^2 e^{\alpha^2 t}\int_0^t e^{-\alpha^2 s}\widetilde{\mathbb{E}}\left[Y_s^2\right]ds \qquad \forall\, t > 0.$$

(b) If $M_t^i \equiv N_t^i - \lambda t$. Since $\alpha\left(Y_s - V_{s-}^i\right)$ is an adapted, càglàd process (hence predictable), and

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[\int_0^t \alpha^2 (Y_s - V^i_{s-})^2 d\langle M^i, M^i\rangle_s\right] &= \widetilde{\mathbb{E}}\left[\int_0^t \alpha^2 (Y_s - V^i_{s-})^2 \lambda ds\right] \\
&= \int_0^t \alpha^2 \lambda \widetilde{\mathbb{E}}\left[Y_s^2 - 2Y_s V^i_{s-} + (V^i_{s-})^2\right] ds \\
&= \int_0^t \alpha^2 \lambda \widetilde{\mathbb{E}}\left[Y_s^2\right] ds + \int_0^t \alpha^2 \lambda \widetilde{\mathbb{E}}\left[(V^i_s)^2\right] ds \\
&< \infty.
\end{aligned}$$

By Theorem 6.5.8 in Kuo (2006),

$$\widetilde{\mathbb{E}}\left[(V^i_s)^2\right] = \int_0^t \alpha^2 \lambda \widetilde{\mathbb{E}}\left[Y_s^2\right] ds + \int_0^t \alpha^2 \lambda \widetilde{\mathbb{E}}\left[(V^i_s)^2\right] ds$$

i.e.

$$d\widetilde{\mathbb{E}}\left[(V^i_s)^2\right] = \alpha^2 \lambda \widetilde{\mathbb{E}}\left[Y_t^2\right] dt + \alpha^2 \lambda \widetilde{\mathbb{E}}\left[(V^i_t)^2\right] dt, \quad \text{with } \widetilde{\mathbb{E}}\left[(V^i_0)^2\right] = 0$$

Solving the above ODE, we get

$$\widetilde{\mathbb{E}}\left[(V^i_t)^2\right] = \alpha^2 \lambda e^{\alpha^2 \lambda t} \int_0^t e^{-\alpha^2 \lambda s} \widetilde{\mathbb{E}}\left[Y_s^2\right] ds.$$

This completes the proof.

□

Recall that for each $N \in \mathbb{N}^+$, we have

$$\overline{V}^N = \frac{1}{\sqrt{N}} \sum_{i=1}^N V^i,$$

where $(\overline{V}^N)_{N\geq 1}$ can be considered as a sequence of $\mathbb{R}$-valued continuous processes on the stochastic basis $\widetilde{\mathcal{B}} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, \widetilde{\mathbb{P}})$.

Lemma 4.2.4 *The sequence $(\overline{V}^N)$ is tight*[2].

[2] See Chapter VI of Jacod and Shiryaev (1987) for definition.

Proof: To establish tightness for the sequence $(\overline{V}^N)$, we check the criteria given in Chapter VI of Jacod and Shiryaev (1987).

Since $\overline{V}_0^N = 0$ for all $N \in \mathbb{N}^+$, the sequence $(\overline{V}_0^N)$ is trivially tight in $\mathbb{R}$. Moreover, since V^i is a square-integrable martingale (see Lemma 4.2.3), we know that

1. $\widetilde{\mathbb{E}}\left[V_t^i\right] = V_0^i = 0,$
2. $\widetilde{\mathbb{E}}\left[V_t^i V_t^j\right] = \widetilde{\mathbb{E}}\left[\left[V^i, V^j\right]_t\right] = \widetilde{\mathbb{E}}\left[\int_0^t \alpha^2 (Y_s - V_{s-}^i)(Y_s - V_{s-}^j) d[M^i, M^j]_s\right] = 0,$
3. $\widetilde{\mathbb{E}}\left[(V_t^i)^2\right] = \widetilde{\mathbb{E}}\left[[V^i, V^i]_t\right] \leq e^{2\alpha^2(c+a^2\lambda)t} \cdot \int_0^t 2\alpha^2(c + a^2\lambda)\widetilde{\mathbb{E}}\left[Y_s^2\right] ds$

for all $t \in [0, T]$ and $i, j \in \mathbb{N}^+$, $i \neq j$. Therefore, the Markov inequality implies that $\forall\ \epsilon > 0$,

$$\begin{aligned}
& \lim_{\delta \downarrow 0} \limsup_N \widetilde{\mathbb{P}}\left[\left|\overline{V}_\delta^N - \overline{V}_0^N\right| > \epsilon\right] \\
\leq\ & \lim_{\delta \downarrow 0} \limsup_N \frac{1}{\epsilon^2} \cdot \widetilde{\mathbb{E}}\left[(\overline{V}_\delta^N)^2\right] \\
=\ & \lim_{\delta \downarrow 0} \limsup_N \frac{1}{\epsilon^2} \cdot \frac{1}{N} \sum_{i=1}^N \widetilde{\mathbb{E}}\left[(V_\delta^i)^2\right] \\
\leq\ & \lim_{\delta \downarrow 0} \frac{1}{\epsilon^2} \cdot e^{2\alpha^2(c+a^2\lambda)t} \int_0^\delta 2\alpha^2(c + a^2\lambda)\widetilde{\mathbb{E}}\left[Y_s^2\right] ds \\
=\ & 0.
\end{aligned}$$

In addition, for any $\xi > 0$, $s < r < t$ and $N \in \mathbb{N}^+$, we have

$$\begin{aligned}
& \widetilde{\mathbb{P}}\left[|\overline{V}_r^N - \overline{V}_s^N| \geq \xi,\ |\overline{V}_t^N - \overline{V}_r^N| \geq \xi\right] \\
\leq\ & \widetilde{\mathbb{P}}\left[(\overline{V}_r^N - \overline{V}_s^N)^2 + (\overline{V}_t^N - \overline{V}_r^N)^2 \geq 2\xi^2\right] \\
\leq\ & \frac{1}{2\xi^2} \cdot \widetilde{\mathbb{E}}\left[(\overline{V}_r^N - \overline{V}_s^N)^2 + (\overline{V}_t^N - \overline{V}_r^N)^2\right] \\
=\ & \frac{1}{2\xi^2} \cdot \frac{1}{N} \cdot \widetilde{\mathbb{E}}\left\{\left[\sum_{i=1}^N (V_r^i - V_s^i)\right]^2 + \left[\sum_{i=1}^N (V_t^i - V_r^i)\right]^2\right\}.
\end{aligned}$$

Note that for any $i, j \in \mathbb{N}^+$, $i \neq j$ and $s < r$,

$$\begin{aligned}
& \widetilde{\mathbb{E}}\left[(V_r^i - V_s^i)(V_r^j - V_s^j)\right] \\
= \; & \widetilde{\mathbb{E}}\left[V_r^i V_r^j\right] + \widetilde{\mathbb{E}}\left[V_s^i V_s^j\right] - \widetilde{\mathbb{E}}\left[V_r^i V_s^j\right] - \widetilde{\mathbb{E}}\left[V_s^i V_r^j\right] \\
= \; & 0 + 0 - \widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}\left[V_r^i V_s^j | \widetilde{\mathcal{F}}_s\right]\right] - \widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}\left[V_s^i V_r^j | \widetilde{\mathcal{F}}_s\right]\right] \\
= \; & -\widetilde{\mathbb{E}}\left[V_s^j \cdot \widetilde{\mathbb{E}}\left[V_r^i | \widetilde{\mathcal{F}}_s\right]\right] - \widetilde{\mathbb{E}}\left[V_s^i \cdot \widetilde{\mathbb{E}}\left[V_r^j | \widetilde{\mathcal{F}}_s\right]\right] \\
= \; & -\widetilde{\mathbb{E}}\left[V_s^j V_s^i\right] - \widetilde{\mathbb{E}}\left[V_s^i V_s^j\right] \\
= \; & 0,
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[(V_r^i - V_s^i)^2\right] &= \widetilde{\mathbb{E}}\left[(V_r^i)^2\right] + \widetilde{\mathbb{E}}\left[(V_s^i)^2\right] - 2\widetilde{\mathbb{E}}\left[V_r^i V_s^i\right] \\
&= \widetilde{\mathbb{E}}\left[(V_r^i)^2\right] + \widetilde{\mathbb{E}}\left[(V_s^i)^2\right] - 2\widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}\left[V_r^i V_s^i | \widetilde{\mathcal{F}}_s\right]\right] \\
&= \widetilde{\mathbb{E}}\left[(V_r^i)^2\right] + \widetilde{\mathbb{E}}\left[(V_s^i)^2\right] - 2\widetilde{\mathbb{E}}\left[V_s^i \cdot \widetilde{\mathbb{E}}\left[V_r^i | \widetilde{\mathcal{F}}_s\right]\right] \\
&= \widetilde{\mathbb{E}}\left[(V_r^i)^2\right] + \widetilde{\mathbb{E}}\left[(V_s^i)^2\right] - 2\widetilde{\mathbb{E}}\left[(V_s^i)^2\right] \\
&= \widetilde{\mathbb{E}}\left[(V_r^i)^2\right] - \widetilde{\mathbb{E}}\left[(V_s^i)^2\right].
\end{aligned}$$

So

$$\begin{aligned}
& \widetilde{\mathbb{P}}\left[|\overline{V}_r^N - \overline{V}_s^N| \geq \xi,\ |\overline{V}_t^N - \overline{V}_r^N| \geq \xi\right] \\
\leq \; & \frac{1}{2\xi^2} \cdot \frac{1}{N} \cdot \widetilde{\mathbb{E}}\left[\sum_{i=1}^N (V_r^i)^2 - (V_s^i)^2 + (V_t^i)^2 - (V_r^i)^2\right] \\
= \; & \frac{1}{2\xi^2} \cdot \frac{1}{N} \cdot \widetilde{\mathbb{E}}\left[\sum_{i=1}^N (V_t^i)^2 - (V_s^i)^2\right].
\end{aligned}$$

Finally, recall that

$$V_t^i = \underbrace{\int_0^t \sqrt{c}\alpha\left(Y_s - V_{s-}^i\right) dW_s^i}_{=:C_t^i} + \underbrace{\int_0^t \alpha\left(Y_s - V_{s-}^i\right) dJ_s^i}_{=:D_t^i}.$$

Following similar arguments as in the proof of Lemma 4.2.3, we see that

$$\begin{aligned}\widetilde{\mathbb{E}}\left[(V_t^i)^2\right] &= \widetilde{\mathbb{E}}\left[(C_t^i)^2\right] + \widetilde{\mathbb{E}}\left[(D_t^i)^2\right] \\ &= \widetilde{\mathbb{E}}\left[\int_0^t c\alpha^2 \left(Y_s - V_{s-}^i\right)^2 ds\right] + \widetilde{\mathbb{E}}\left[\int_0^t \int_{\{|x|\le a\}} x^2\alpha^2 \left(Y_s - V_{s-}^i\right)^2 dsF(dx)\right].\end{aligned}$$

Thus, for each $i \in \mathbb{N}^+$,

$$\begin{aligned}\widetilde{\mathbb{E}}\left[(V_t^i)^2 - (V_s^i)^2\right] &= \widetilde{\mathbb{E}}\left[\int_s^t c\alpha^2 \left(Y_u - V_{u-}^i\right)^2 du + \int_s^t \int_{\{|x|\le a\}} x^2\alpha^2 \left(Y_u - V_{u-}^i\right)^2 duF(dx)\right] \\ &\le \widetilde{\mathbb{E}}\left[\int_s^t c\alpha^2 \left(Y_u - V_{u-}^i\right)^2 du + \int_s^t a^2\lambda\alpha^2 \left(Y_u - V_{u-}^i\right)^2 du\right] \\ &\le \widetilde{\mathbb{E}}\left[\int_s^t \alpha^2(c + a^2\lambda)\left(2Y_u^2 + 2(V_{u-}^i)^2\right) du\right],\end{aligned}$$

and we have

$$\begin{aligned}&\widetilde{\mathbb{P}}\left[|\overline{V}_r^N - \overline{V}_s^N| \ge \xi,\ |\overline{V}_t^N - \overline{V}_r^N| \ge \xi\right] \\ \le\ & \frac{1}{2\xi^2}\cdot\frac{1}{N}\sum_{i=1}^N \widetilde{\mathbb{E}}\left[(V_t^i)^2 - (V_s^i)^2\right] \\ \le\ & \frac{1}{2\xi^2}\cdot\frac{1}{N}\sum_{i=1}^N \widetilde{\mathbb{E}}\left[\int_s^t \alpha^2(c + a^2\lambda)\left(2Y_u^2 + 2(V_{u-}^i)^2\right) du\right] \\ \le\ & \frac{1}{\xi^2}\cdot\alpha^2(c + a^2\lambda)\left(\widetilde{\mathbb{E}}\left[Y_t^2\right] + \widetilde{\mathbb{E}}\left[(V_t^i)^2\right]\right)(t - s) \\ \le\ & \xi^{-2}\cdot\left[\alpha^2(c + a^2\lambda)t\left(\widetilde{\mathbb{E}}\left[Y_t^2\right] + \widetilde{\mathbb{E}}\left[(V_t^i)^2\right]\right) - \alpha^2(c + a^2\lambda)s\left(\widetilde{\mathbb{E}}\left[Y_s^2\right] + \widetilde{\mathbb{E}}\left[(V_s^i)^2\right]\right)\right].\end{aligned}$$

The last two inequalities hold as both $\widetilde{\mathbb{E}}\left[Y_t^2\right]$ and $\widetilde{\mathbb{E}}\left[(V_t^i)^2\right]$ are continuous, increasing and finite-valued functions of $t \ge 0$, so $\alpha^2(c + a^2\lambda)t\left(\widetilde{\mathbb{E}}\left[Y_t^2\right] + \widetilde{\mathbb{E}}\left[(V_t^i)^2\right]\right)$ is also a continuous, increasing function of $t \ge 0$. By Theorem 4.1 in Chapter VI of Jacod and Shiryaev (1987), we conclude that the sequence $(\overline{V}^N)$ is tight.

□

As specified in Section 4.2.1, $\mathcal{B} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and $\mathcal{B}' = (\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$ are stochastic bases on which a finite activity Lévy process M with bounded jumps and a continuous square-integrable martingale Y are defined, respectively. For each fixed $\omega' \in \Omega'$, let $y_t := Y_t(\omega')$. Following similar arguments as in Lemma 4.2.2 and Lemma 4.2.3, we see that the SDE

$$dV_t(\omega', \cdot) = \alpha \left(y_t - V_{t-}(\omega', \cdot) \right) dM_t \tag{4.4}$$

has a unique strong solution $V(\omega', \cdot)$, which is a càdlàg square-integrable martingale on the stochastic basis $\mathcal{B}$, with $C_t^{\omega'} := \mathbb{E}\left[V_t(\omega', \cdot)^2\right] < \infty$ for all $t \in [0, T]$.

Lemma 4.2.5 *For fixed $\omega' \in \Omega'$ and $y_t = Y_t(\omega')$, the sequence of processes $\overline{V}^N(\omega', \cdot)$ converges in law to a Wiener process with characteristics $(0, C^{\omega'}, 0)$ as $N \to \infty$. In particular,*

(a) (Non-Followers) *If $M_t^i \equiv B_t^i$ is a standard Brownian motion, then*

$$C_t^{\omega'} = \alpha^2 e^{\alpha^2 t} \int_0^t e^{-\alpha^2 s} y_s^2 ds \quad \forall\, t \in [0, T];$$

(b) (Followers) *If $M_t^i \equiv N_t^i - \lambda t$, where N^i is a Poisson process with arrival rate $a(t) = \lambda t$, then*

$$C_t^{\omega'} = \alpha^2 \lambda e^{\alpha^2 \lambda t} \int_0^t e^{-\alpha^2 \lambda s} y_s^2 ds \quad \forall\, t \in [0, T].$$

Proof: By construction (see Section 4.2.1), $(\mathcal{B}^i, M^i)_{i \geq 1}$ is a sequence of identical copies of the pair $(\mathcal{B}, M)$. Moreover, for each fixed $\omega' \in \Omega'$ and $y_t = Y_t(\omega')$, the process $V^i(\omega', \cdot)$ for each $i \in \mathbb{N}^+$ is defined as the unique strong solution of

$$dV_t^i(\omega', \cdot) = \alpha \left(y_t - V_{t-}^i(\omega', \cdot) \right) dM_t^i$$

(see Lemma 4.2.2). As a result, we can view $\left(\mathcal{B}^i, V^i(\omega', \cdot)\right)_{i \geq 1}$ as a sequence of identical copies of the pair $(\mathcal{B}, V(\omega', \cdot))$, where $V(\omega', \cdot)$ is a càdlàg, square-integrable martingale. Since $C_t^{\omega'} = \mathbb{E}\left[V_t(\omega', \cdot)^2\right]$ is continuous and finite-valued for all $t \in [0, T]$, Theorem 3.46 in Chapter VIII of Jacod and Shiryaev (1987) applies, which gives the desired convergence. The specific forms of $C_t^{\omega'}$ in the *Non-Followers* and *Followers* cases follow directly from Lemma 4.2.3.

□

We are now ready to prove our Main Theorem in Section 4.2.2.

Proof of Theorem 4.2.1 Let $\check{\mathcal{B}} = (\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{F}}, \check{\mathbb{P}})$ be a stochastic basis endowed with a standard Wiener process W. Let $\mathcal{B}^*$ be the tensor product of $\mathcal{B}'$ and $\check{\mathcal{B}}$ as usual. In addition, define $V_t^* := W_{C_t}$, where C_t is a stochastic process on the basis $\mathcal{B}'$ such that for each $\omega' \in \Omega'$, $C_t(\omega') \equiv C_t^{\omega'}$ as defined in Lemma 4.2.5. Thus, V^* is a stochastic time-changed Wiener process on the basis $\mathcal{B}^*$, with $[V^*, V^*]_t = C_t$.

Recall that given a sequence of stochastic processes X^n and an additional process X, in order to show that $X^n \xrightarrow{\mathcal{L}} X$, it is necessary and sufficient to check the following two conditions[3]:

(1) $(X^n_{t_1}, \ldots, X^n_{t_k}) \xrightarrow{\mathcal{L}} (X_{t_1}, \ldots, X_{t_k})$ for all $t_i \in D, k \in \mathbb{N}^+$,

(2) (X^n) is tight.

By Lemma 4.2.4, the sequence $(\overline{V}^N)$ is tight. Thus, to prove $\overline{V}^N \xrightarrow{\mathcal{L}} V^*$, we must show that $\overline{V}^N$ converges in finite-dimensional distribution to V^* as $N \to \infty$. For any $t \in [0, T]$ and Borel set $A \subset \mathbb{R}$,

$$\begin{aligned} \widetilde{\mathbb{P}}(\overline{V}_t^N \in A) &= \int_{\Omega' \times \hat{\Omega}} \mathbf{1}_{\left\{\overline{V}_t^N(\omega', \hat{\omega}) \in A\right\}} \mathbb{P}'(d\omega') \hat{\mathbb{P}}(d\hat{\omega}) \\ &= \int_{\Omega'} \left(\int_{\hat{\Omega}} \mathbf{1}_{\left\{\overline{V}_t^N(\omega', \hat{\omega}) \in A\right\}} \hat{\mathbb{P}}(d\hat{\omega}) \right) \mathbb{P}'(d\omega') \\ &= \int_{\Omega'} \hat{\mathbb{P}}\left(\overline{V}_t^N(\omega', \cdot) \in A\right) \mathbb{P}'(d\omega'). \end{aligned}$$

[3]See Chapter VI 3.20 in Jacod and Shiryaev (1987).

Similarly,

$$\mathbb{P}^*(V_t^* \in A) = \int_{\Omega'} \check{\mathbb{P}}\,(V_t^*(\omega', \cdot) \in A)\,\mathbb{P}'(d\omega').$$

By construction of the process V^* and Lemma 4.2.5, we know that

$$\hat{\mathbb{P}}\left(\overline{V}_t^N(\omega', \cdot) \in A\right) \longrightarrow \check{\mathbb{P}}\,(V_t^*(\omega', \cdot) \in A) \quad \text{as } N \to \infty, \ \forall\ \omega' \in \Omega'.$$

Thus,

$$\begin{aligned}
\lim_{N\to\infty} \widetilde{\mathbb{P}}(\overline{V}_t^N \in A) &= \lim_{N\to\infty} \int_{\Omega'} \hat{\mathbb{P}}\left(\overline{V}_t^N(\omega', \cdot) \in A\right)\mathbb{P}'(d\omega') \\
&= \int_{\Omega'} \lim_{N\to\infty} \hat{\mathbb{P}}\left(\overline{V}_t^N(\omega', \cdot) \in A\right)\mathbb{P}'(d\omega') \\
&= \int_{\Omega'} \check{\mathbb{P}}\,(V_t^*(\omega', \cdot) \in A)\,\mathbb{P}'(d\omega') \\
&= \mathbb{P}^*(V_t^* \in A),
\end{aligned}$$

where the interchanging of limit and integration is justified by Dominated Convergence Theorem as $\hat{\mathbb{P}}\left(\overline{V}_t^N(\omega', \cdot) \in A\right) \in [0, 1]$. Finally, a direct extension of the above proof shows that for any $k \in \mathbb{N}^+$, $0 \leq t_1 < \ldots < t_k$ and $A_1, \ldots, A_k \in \mathbb{R}$, we have

$$\widetilde{\mathbb{P}}\left(\overline{V}_{t_1}^N \in A_1, \ldots, \overline{V}_{t_k}^N \in A_k\right) \longrightarrow_N \mathbb{P}^*\left(V_{t_1}^* \in A_1, \ldots, V_{t_k}^* \in A_k\right),$$

i.e. $\overline{V}^N$ converges to V^* in finite-dimensional distribution as $N \to \infty$. This completes the proof of Theorem 4.2.1.

□

4.3 Numerical Analysis

Let N be the total number of agents participating in the market throughout some finite time horizon $[0, T]$. For some $\rho \in [0, 1]$, let $N_f = [\rho N]$ traders be *Followers* and the rest $N_n = N - N_f$ traders be *Non-Followers*. Recall that the process Y represent signals given by an independent central authority. If agent i is a *Non-Follower*, her speculative demand process $V^{n,i}$ evolves according to the following SDE

$$dV_t^{n,i} = \alpha_n \left(Y_t - V_t^{n,i}\right) dB_t^i \quad V_0^{n,i} = 0, \tag{4.5}$$

where $\alpha_n > 0$ is some constant and B_t^i is a standard Brownian motion. On the other hand, for each *Follower* j, the corresponding process $V^{f,j}$ is governed by

$$dV_t^{f,j} = \alpha_f \left(Y_t - V_{t-}^{f,j}\right) d\left(N_t^j - \lambda t\right) \quad V_0^{f,j} = 0, \tag{4.6}$$

$\alpha_f < 0$ is again some constant while N_t^j is a homogeneous Poisson process with instantaneous arrival intensity $\lambda > 0$. As discussed in Section 4.2.1, all processes Y_t, $\{V_t^{n,i} : i = 1, \ldots, N_n\}$ and $\{V_t^{f,j} : j = 1, \ldots, N_f\}$ are assumed to be mutually independent.

In order to simulate the above model, we discretize equations (4.5) and (4.6) via the following simple Euler-type scheme, which allows the incorporation of jumps:

Step 1: Create a basic discretization time grid by partitioning the interval $[0, T]$ into $K \in \mathbb{N}^+$ equidistant steps $\{0 = t_0 < t_1 < \ldots < t_K = T\}$, where $t_k = k\frac{T}{K}$ for all $k = 0, \ldots, K$.

Step 2: Simulate the jump times of each Poisson process N^j, $j = 1, \ldots, N_f$ in the interval $[0, T]$, denoted by

$$\{\tau_1^j, \ldots, \tau_{N_T^j}^j\}, \quad j \in \{1, \ldots, N_f\}.$$

The superscripts serve as a reminder that the simulated number of jumps N_T^j as well as the set of corresponding jump times vary from agent to agent.

Step 3: Combine the basic time grid $\{t_k : k = 0, \ldots, K\}$ in Step 1 and the simulated jump times $\left\{\{\tau_1^j, \ldots, \tau_{N_T^j}^j\} : j = 1, \ldots, N_f\right\}$ in Step 2 to create a final discretization time grid $\{0 = t'_0 < t'_1, \ldots, < t'_M = T\}$, where $M = K + 1 + \sum_{j=1}^{N_f} N_T^j$.

Step 4: Simulate processes Y, $\{V^{n,i} : i = 1, \ldots, N_n\}$ and $\{V^{f,j} : j = 1, \ldots, N_f\}$ over the final time grid. More specifically, for each $k \in \{1, \ldots, M\}$,

- *Central Authority:*

$$Y_{t'_k} = Y_{t'_{k-1}} + \sqrt{\left(t'_k - t'_{k-1}\right)} \times Z,$$

  where $Z \sim N(0, 1)$. In this case, the corresponding continuous-time process Y is assume to be a standard Brownian motion.

- *Non-Followers:*

$$V_{t'_k}^{n,i} = V_{t'_{k-1}}^{n,i} + \alpha_n \left(Y_{t'_{k-1}} - V_{t'_{k-1}}^{n,i}\right) \cdot \sqrt{\left(t'_k - t'_{k-1}\right)} \times Z^i$$

  for all $i = 1, \ldots, N_n$, where Z^i are i.i.d. $N(0, 1)$ random variables.

- *Followers:*

  If t'_k is NOT a simulated jump time of the Poisson process N^j,

$$V_{t'_k}^{f,j} = V_{t'_{k-1}}^{f,j} + \alpha_f \left(Y_{t'_{k-1}} - V_{t'_{k-1}}^{f,j}\right) \cdot (-\lambda) \left(t'_k - t'_{k-1}\right).$$

  Otherwise, if t'_k is a simulated jump time of the Poisson process N^j,

$$V_{t'_k}^{f,j} = U_{t'_k}^{f,j} + \alpha_f \left(Y_{t'_k} - U_{t'_k}^{f,j}\right),$$

  where $U_{t'_k}^{f,j}$ is given by

$$U_{t'_k}^{f,j} = V_{t'_{k-1}}^{f,j} + \alpha_f \left(Y_{t'_{k-1}} - V_{t'_{k-1}}^{f,j}\right) \cdot (-\lambda) \left(t'_k - t'_{k-1}\right).$$

4.3.1 Statistical Properties of Simulated Data

To investigated statistical properties of the normalized total speculative demand $\overline{V}^N$ as defined in Section 4.1, we simulate the discretized model with the following parameter values:

- $T = 5$ is the length of the entire time horizon.
- $K = 500$ is the number of time steps in the equidistant partition of $[0, T]$.
- $N = 1000$ is the total number of agents participating in the market.
- $\rho = 0.5$, i.e. we have $N_f = 500$ *Followers* and $N_n = 500$ *Non-Followers*.
- $\alpha_n = 2$ and $\alpha_f = -2$.
- $\lambda = 1$ is the instantaneous arrival rate of a *Follower*'s private information.

As shown in Figure 4.1, the simulated sample path of $\overline{V}^N$ contains distinct "quiet" and "turbulent" periods, which are similar to those observed in the time series of the S&P 500 index level.

Next, we follow the same methodology as used in Section 3.4.1 to check the model's ability to capture empirically observed stylized facts of logarithmic asset returns. In particular, we examine the set of values

$$\left\{\left(\overline{V}^N_{t_k} - \overline{V}^N_{t_{k-1}}\right) : k = 1, \ldots, K\right\},$$

which can be viewed as the logarithmic asset returns generated by the normalized total speculative demand over each time step t_k, $k = 1, \ldots, K$. The histogram and kernel density estimation in Figure 4.2, as well as the Q-Q plot in Figure 4.3 show that the "distribution" of the set of 1-step logarithmic returns generated by an arbitrary sample path of $\overline{V}^N$ is indeed *Heavy-tailed*.

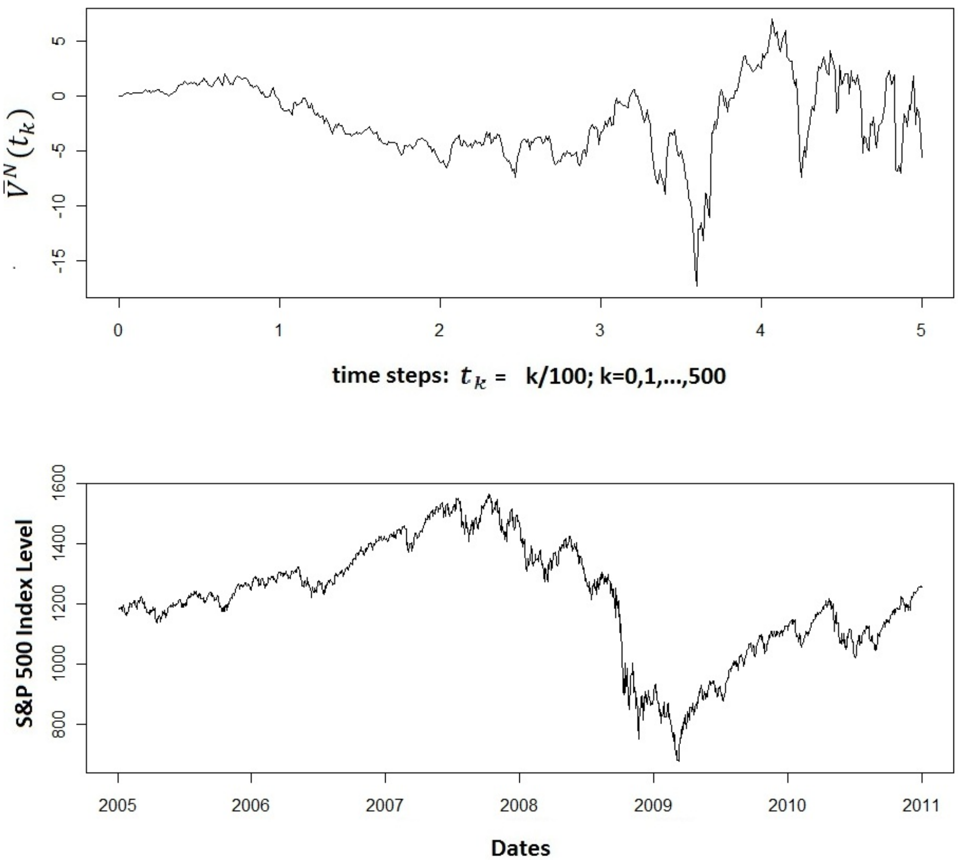

Figure 4.1: (Upper panel) Simulated sample paths of $\overline{V}^N$; (Lower panel) time series of the S&P 500 index level between January 1, 2005 and December 31, 2010.

In addition, various descriptive statistics and Hill estimators are calculated for the "distribution" of simulated 1-step returns, and compared with those computed for the "distribution" of daily logarithmic returns of the S&P 500 index between January 1, 2005 and December 31, 2010. The corresponding results, reported in Table 4.1 and Table 4.2, confirm that the tail behavior of the simulated data is in close accordance with empirical observations made in financial markets.

Aggregational Gaussianity, as discussed in Sections 1.1 and 3.4.1, is another stylized fact of empirical financial returns that is well captured by our simulated

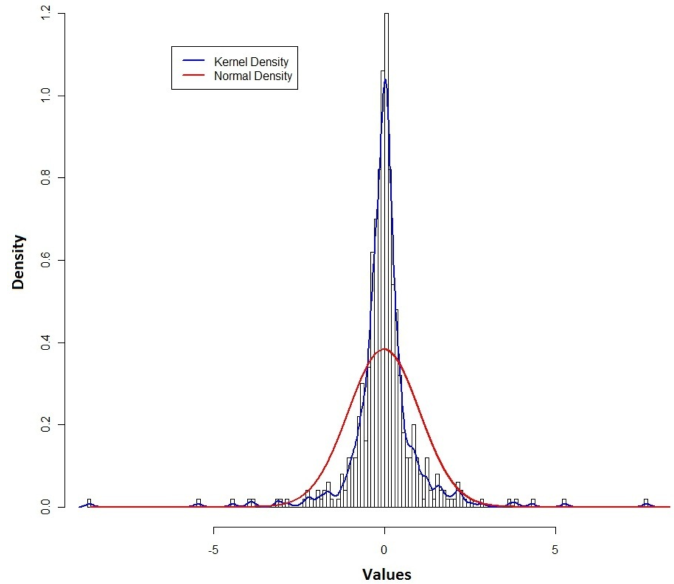

Figure 4.2: Histogram generated by the set of values $\{(\overline{V}^N_{t_k} - \overline{V}^N_{t_{k-1}}) : k = 1, \ldots, K\}$ associated with the simulated sample path of $\overline{V}^N$ shown in Figure 4.1. The Kernel density estimation (blue) and a Normal density (red) with mean and variance matching the simulated data are also plotted.

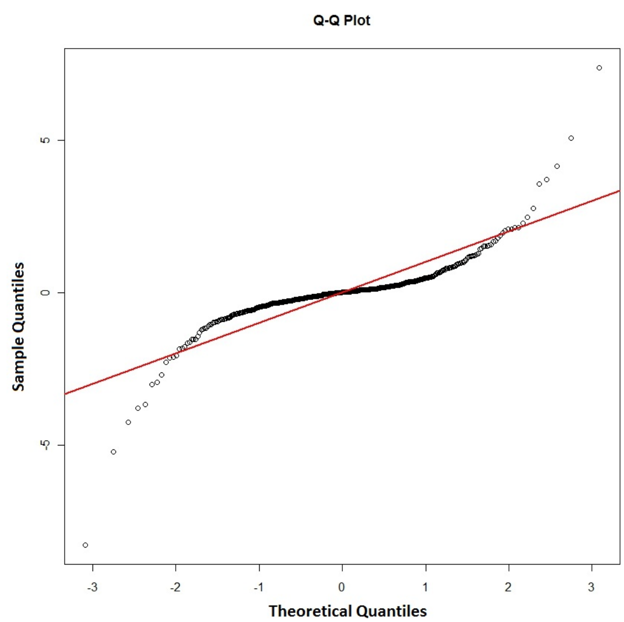

Figure 4.3: Q-Q plot of the set of values $\{(\overline{V}^N_{t_k} - \overline{V}^N_{t_{k-1}}) : k = 1, \ldots, K\}$ given by the simulated sample path of $\overline{V}^N$ shown in Figure 4.1.

Table 4.1: (Top) Mean, Variance, Skewness and Kurtosis are repeatedly calculated for each of the 300 sets of 1-step logarithmic returns associated with 300 simulated paths of $\overline{V}^N$. The resulting values are then averaged and reported along with the corresponding standard deviation; (Bottom) Mean, Variance, Skewness and Kurtosis are calculated for daily logarithmic returns of the S&P 500 index between January 1, 2005 and December 31, 2010.

	Mean	Variance	Skewness	Kurtosis
Average Value (over 300 sample sets)	-1.02×10^{-2}	31.5	8.14×10^{-2}	34.2
Standard Deviation (over 300 sample sets)	0.198	135	4.07	54.2

Mean	Variance	Skewness	Kurtosis
-3×10^{-5}	2.13×10^{-4}	0.247	9.99

Table 4.2: (Top) Hill Estimators for 10%−, 5%− and 2.5%−tails are repeatedly calculated for each of the 300 sets of 1-step returns associated with 300 simulated paths of $\overline{V}^N$. The resulting values are then averaged and reported along with the corresponding standard deviations; (Bottom) Hill Estimators for 10%−, 5%− and 2.5%−tails are calculated for daily logarithmic returns of the S&P 500 index between January 1, 2005 and December 31, 2010.

	10%	5%	2.5%
Average Value (over 300 sample sets)	1.69	2.16	3.25
Standard Deviation (over 300 sample sets)	0.533	0.736	1.31

10%	5%	2.5%
1.73	2.47	2.87

data. Not only can we easily visualized it in Figure 4.4, relevant sample statistics are calculated using both the simulated data and the S&P 500 index time series to provide additional quantitative support. The corresponding results can be found in Table 4.3.

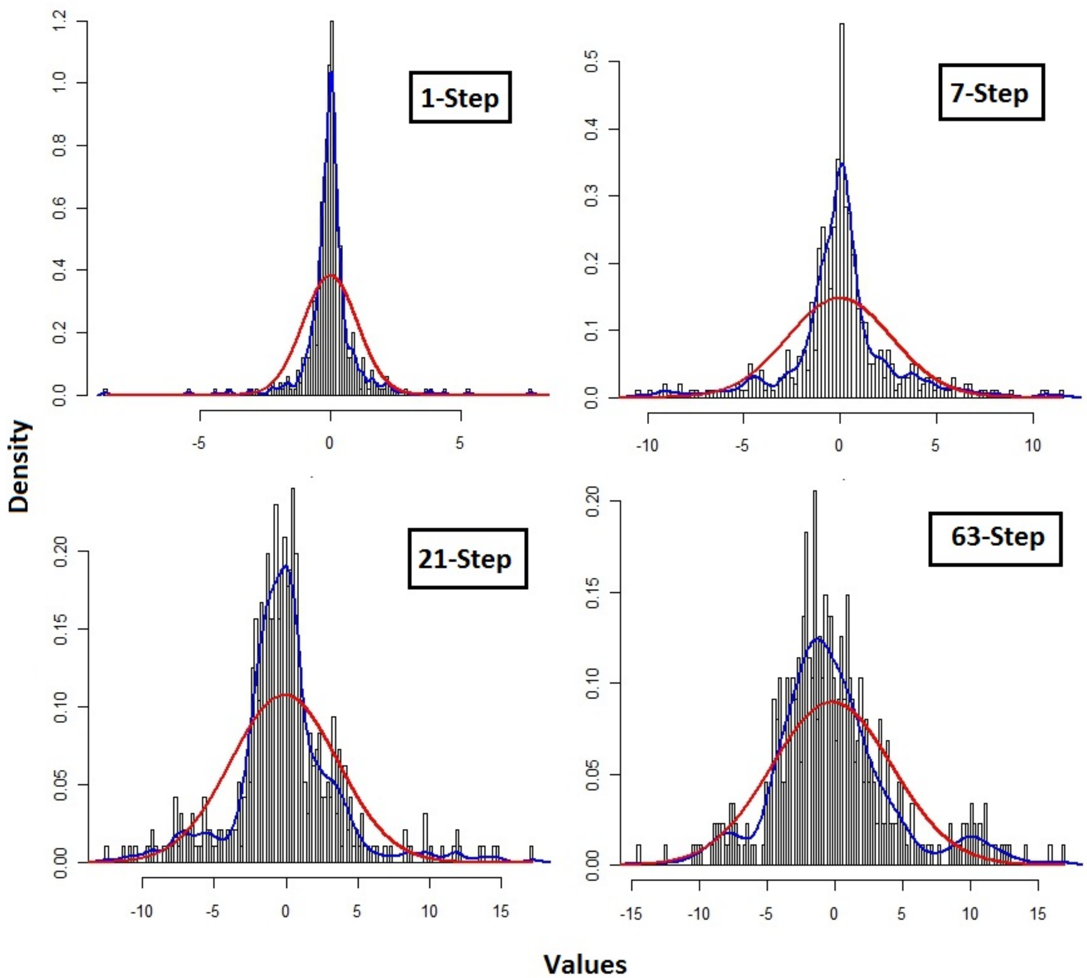

Figure 4.4: Histograms generated by 1-, 7-, 21-, and 63-step returns given by the simulated sample path of $\overline{V}^N$ shown in Figure 4.1. The corresponding Kernel density estimation (blue) and a Normal density (red) with mean and variance matching the simulated data are also plotted.

Finally, we point out that when the set of values $\left\{\left(\overline{V}_{t_k}^N - \overline{V}_{t_{k-1}}^N\right) : k = 1, \ldots, K\right\}$ given by a simulated path of $\overline{V}^N$ are plotted against the time steps $\{t_k$:

Table 4.3: (Top) The sample kurtosis is repeatedly calculated for each of the 300 sets of 1-, 7-, 21- and 63-step returns associated with 300 simulated paths of $\overline{V}^N$. The resulting values are then averaged and reported along with the corresponding standard deviations; (Bottom) sample kurtosis calculated for the 1-, 7-, 21- and 63-day logarithmic returns of the S&P 500 index between January 1, 2005 and December 31, 2010.

	1-step	7-step	21-step	63-step
Average Value (over 300 sample sets)	34.2	8.93	4.39	2.35
Standard Deviation (over 300 sample sets)	54.2	9.08	5.12	3.78

1-day	7-day	21-day	63-day
9.99	8.26	6.52	4.20

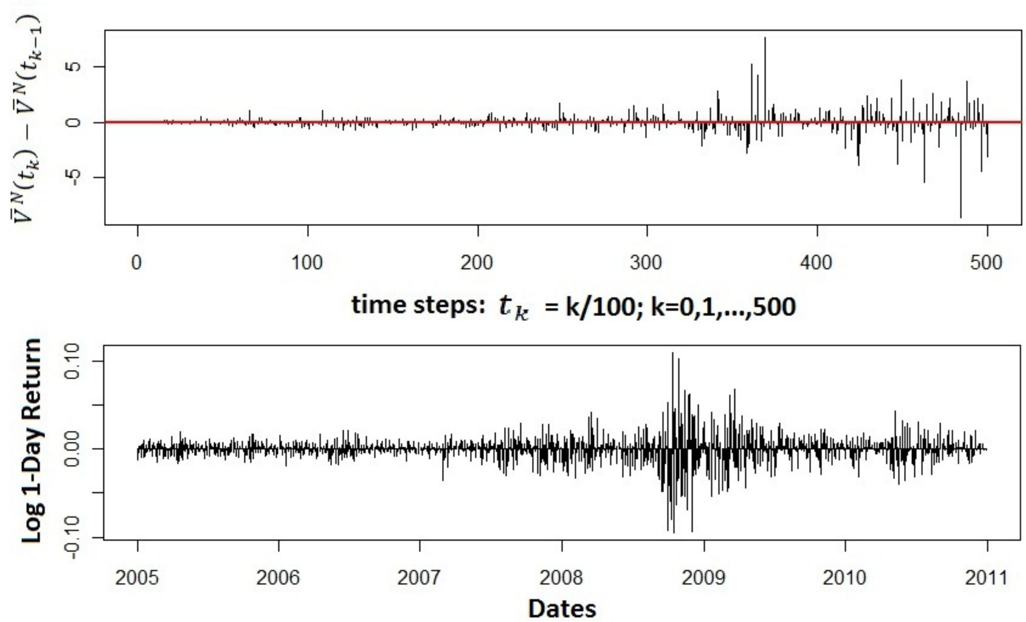

Figure 4.5: (Upper panel) The set of 1-step returns given by the simulated path of $\overline{V}^N$ shown in Figure 4.1 is plotted against the time grid $\{t_k : k = 1, \ldots, K\}$; (Lower panel) daily logarithmic returns of the S&P 500 index between January 1, 2005 and December 31, 2010 are plotted against time.

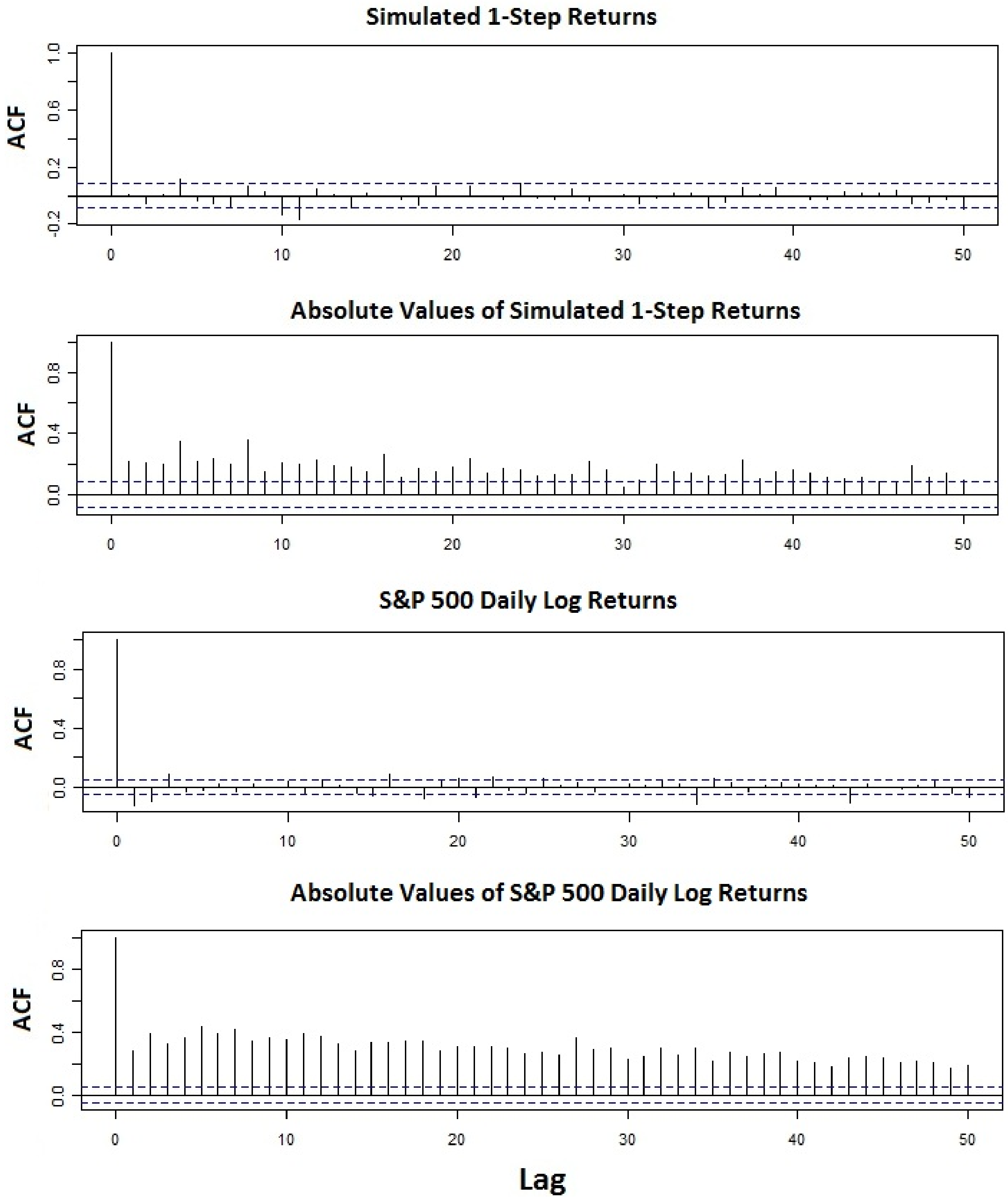

Figure 4.6: Autocorrelation functions of: (1) 1-step returns given by the simulated path of $\overline{V}^N$ shown in Figure 4.1; (2) their corresponding absolute values; (3) daily logarithmic returns of the S&P 500 index between January 1, 2005 and December 31, 2010; (4) their absolute values.

$k = 1, \ldots, K\}$ in Figure 4.5, we observe clear *Volatility Clustering*, which again matches empirical findings in financial asset returns. As discussed in Section 3.4.1, the autocorrelation function plots in Figure 4.6 provide a more quantitative manifestation of this phenomenon.

CHAPTER 5

SUMMARY AND FUTURE RESEARCH

The modeling of asset price dynamics has long been an integral part of mathematical finance literature. As a former benchmark, the Geometric Brownian motion possesses intuitive economic interpretation and great analytical tractability, but fails to capture many important stylized facts of asset returns observed across financial markets. Subsequent developments in this area include Lévy Jump-diffusions and Stochastic Volatility models, which are often mathematically engineered to encompass empirical properties of asset returns and lack motivation from a fundamental economic point of view. In this dissertation, we construct a heterogeneous agent-based price model that overcomes some of the aforementioned limitations. We show that one possible explanation for various stylized facts observed in financial asset returns, such as heavy tails, aggregational Gaussianity and volatility clustering, lies within the interaction among traders participating in the market[1]. In particular, such interaction generates certain speculative demands for the traded asset in addition to its fundamental economic demand, which in turn causes the price to fluctuate. All communication models presented in this dissertation are inspired by the Cucker-Smale flocking idea summarized in Section 2.1. A slow price adjustment approach is discussed in Section 2.2, which allows us to incorporate the total communication-caused speculative demand into the final asset price formation.

In Chapter 3, we study interaction among individual market participants under a discrete-time setting. The constant sequential communication scheme in

[1]Other possible explanations for various stylized facts of asset returns have also been proposed in the literature. For example, Grabchak and Samorodnitsky (2010) argue that financial returns may demonstrate heavy tails and aggregational Gaussianity when modeled as i.i.d. random variables with tempered heavy tails.

Section 3.2 describes a well-documented phenomenon known as "herding" in the literature, while the stochastic communication scheme introduced in Section 3.3 models interaction among noise traders. Both Theorem 3.2.2 and Theorem 3.3.2 fall under the framework of the Central Limit Theorem for m-dependent random variables with unbounded m given by Berk (1973). Numerical analysis in Section 3.4 demonstrates our model's ability to capture several stylized empirical facts of financial asset returns simultaneously and provides insights to the convergence rate of Theorem 3.3.2. In Chapter 4, we investigate under a continuous-time setting how the presence of a central authority, such as an equity analyst's report, may influence individual agents' trading behaviors and in turn the price dynamics of the traded asset. The model focuses on two distinct types of agents, namely the *Followers*, also known as *Fundamentalists*, and the *Non-Followers*, also known as *Noise Traders*. While a *Follower* forms her speculative demand by constantly mimicking the opinion of the central authority, except when she receives opposite private trading signals, a *Non-Follower* may choose to follow or act against the central authority's advice purely at random. We show in Theorem 4.2.1 that the normalized total speculative demand of a large number of market participants, as a result of their interaction with the central authority, turns out to be a stochastic time-changed Wiener process in both cases. Numerical analysis of the model is carried out in Section 4.3 by discretizing relevant stochastic differential equations using an Euler-type scheme.

To the best of our knowledge, agent-based models introduced in this dissertation are among the first in the literature to study the impact of trader interaction on price dynamics using limit theorems and the Cucker-Smale flocking idea. While initial results show great promise, many extensions to the models should be considered in future research. For example, the communication scheme described in Section 3.1 is far more general than those analyzed in Sections 3.2 and 3.3. Several assumptions made on communication patterns and communication rates will need to be relaxed before our model can fully capture the increasingly complex interaction among traders participating in today's financial markets. Also, since models introduced in this dissertation consider only one risky asset, a natural generalization would be to include multiple assets in the market so that individual traders can decide which asset they would like to invest in by communicating with others. Another research idea we would like to pursue is to calibrate our price model to different financial markets so it can be used in practice to capture the real dynamics of asset price revolution. Not only does our model possess great analytical tractability, it is also very natural to simulate, as the agent-based framework allows us to first motivate various assumptions at a micro-level and then examine the subsequent macro-effect via proper aggregation and normalization. This type of bottom-up approach, coupled with the increasingly available computational power, has become more and more popular among practitioners in fields such as option pricing and risk management.

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