ALIASING CONTRASTS FOR A GIVEN FRACTIONAL REPLICATE

by

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ABSTRACT

Experimenters frequently select a particular fractional replicate from a complete factorial treatment design and then need to determine which effects are partially or completely confoiunded (aliased) with each other. The fractional replicate may have arisen by chance, by accident, or by design. A method for determining the aliasing structure for a given fractional replicate is addressed herein. Two examples with 3^{5-2} and with 3^{7-4} combinations, respectively, are used to demonstrate the procedure for symmetrical prime power factorials and orthogonal fractional replicates. The general case for non-orthogonal fractional replicates and for any symmetrical or assymmetrical factorial is discussed. No simple procedure was found for constructing the aliasing contrast and the aliasing structure for the general case.

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INTRODUCTION

It is not an uncommon occurrence in statistical consulting to find that an experimenter has already conducted an investigation using a specified fractional replicate. The particular set of combinations forming the fractional replicate may have been obtained by chance, by design, or by accident, and it is desired to know which effects are partially or completely confounded, aliased, with each other. This can be determined for any fraction by setting up the full design matrix for the full factorial as follows. Let $X_{N \times N}$ be the incidence matrix for the full factorial with N combinations, let $\beta_{N \times 1}$ be the vector of single degree of freedom parameters, and let $Y_{N \times 1}$ be the vector of observations (or means). Then for any fractional replicate X, β, and Y may be partitioned as follows:

$$X\,\beta = Y = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} Y1 \\ Y2 \end{bmatrix}, \tag{1}$$

where Y1 is the observation vector for the fractional replicate. Then,

$$[\,X_{11} \quad X_{12}] \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = X_{11}\,\beta_1 + X_{12}\,\beta_2 = Y1\,, \tag{2}$$

and

$$\beta_1 + [\,X_{11}'\,X_{11}]^{-1}\,X_{11}'\,X_{12}\,\beta_2 = [X_{11}'\,X_{11}]^{-1}\,X_{11}'\,Y1\,, \tag{3}$$

given that $X_{11}'\,X_{11}$ has an inverse. $A = [\,X_{11}'\,X_{11}\,]^{-1} X_{11}'\,X_{12}$ is called the aliasing matrix. It is useful to define another matrix which is the matrix of the combinations for the identifying subscripts of the responses in the Y1 vector. Denote this matrix as S for the Y vector and as S1 for the Y1 vector. There is a one to one correspondence between the S and X matrices for a specified parametric vector β_1. If the first parameter in β_1 is the mean, then the first row of A gives the defining contrast for the fractional replicate S1. Algebraically this is all very simple, but when n and/or s for s^n factorials become large (relatively), the implementation of this algebra can become very tedious and time-consuming and is prone to errors (for most people). β_2 can become quite large as will be illustrated below for the second example.

When s is a prime (or prime power) and S1 is an orthogonal fraction, computer aids are available to help with the process of determining the aliasing structure. Computer programs like ALIAS.PRG (written by C. E. McCulloch, 1991), WYLIE (written by Steven Wang, 1992), and GAUSS are extremely helpful and time-saving. Two examples are presented below to demonstrate this.

EXAMPLE ONE

An experimenter used the following 3^{-2} of a 3^5 factorial to obtain 3^{5-2} runs or combinations of a 3^5 factorial:

$$S1' = [\ 00000\ \ 00111\ \ 00222\ \ 01012\ \ 01120\ \ 01201\ \ 02021\ \ 02102\ \ 02210\ \ 10000\ \ 10111\ \ 10222\ \ 11012\ \ 11120\ \ 11201\ \ 12021\ \ 12102\ \ 12210\ \ 20000\ \ 20111\ \ 20222\ \ 21012\ \ 21120\ \ 21201\ \ 22021\ \ 22102\ \ 22210\ ]. \quad (4)$$

The matrix S1 has 27 rows and n = 5 columns. This experimenter wished to estimate main effects for the five factors (say A, B, C, D, and E) and the four two factor interactions of one of the factors with the other four factors (say $A \times B$, $A \times C$, $A \times D$, and $A \times E$). The particular fraction S1 that he used was to take an orthogonal main effects plan for four factors (say B, C, D, and E) in nine runs and repeat these combinations for each level of the fifth factor, say A. An ANOVA table for these 27 combinations is:

Source of variation	Degrees of freedom	Geometrical interactions
Total	27	
Mean	1	
A	2	
B	2	
C	2	
D	2	
E	2	
$A \times B$	4	$AB + AB^2$
$A \times C$	4	$AC + AC^2$
$A \times D$	4	$AD + AD^2$
$A \times E$	4	$AE + AE^2$

The question now is which effects are aliased with the effects in the above ANOVA table. Since this is an orthogonal fraction and a prime factorial, we may use the geometrical components of the various interaction terms. To determine the generators for this fraction, contruct a matrix $D_{5 \times k}$ where k is the number of generators to be tried to ascertain the defining contrast and then the aliasing structure; the entries in a column will be the exponents of the geometrical interaction letters. To start, select k equal to 4 and the terms CDE, BCD^2, BD^2E, and BC^2E, resulting in the matrix

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 2 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} . \tag{5}$$

The product of the matrices S1 and D using GAUSS, resulted in a 27 $\times$ 4 matrix which had all zeros, modulo 3, in the first two columns. This means that these two interaction terms, CDE and BCD^2, are completely aliased with the mean and hence can be used as the two generators for this fractional replicate. Then using these two generators and their generalized interactions BC^2E and BDE^2, the complete aliasing structure may easily be obtained using WYLIE. For example, the aliases of the mean, the defining contrast, and of A, say, are

$$I = CDE = BCD^2 = BC^2E = BDE^2 \tag{6}$$

$$A = ACDE = AC^2D^2E^2 = ABCD^2 = AB^2C^2D = ABC^2E = AB^2CE^2 = ABDE^2 = AB^2D^2E \tag{7}$$

where = means aliased with.

In the matrix D, it happened that the two interactions selected were the desired ones but this need not be the case. For example, the third and fourth generators could have been selected. Then, using ALIAS or WYLIE, it would be noted that B is an alias of BD^2E. This means that BDE^2 could be a generator and we may note that it appears in I as an interaction, i.e., any two of these four effects could be the generators.

EXAMPLE TWO

The experimentor wished to obtain estimates of the main effects for seven factors, say A, B, C, D, E, F, and G, at three levels each and the two factor interactions for three of the factors, say A $\times$ B, A $\times$ C, and B $\times$ C . The fractional replicate was constructed as follows. The 27 combinations for a 3^3 factorial were selected; then, the levels of ABC were equated to the levels of factor D, levels of ABC^2 to levels of factor E, levels of AB^2C to levels of factor F, and levels of AB^2C^2 to levels of factor G. This means that we know one alias of each of the factors D, E, F, and G. Thus, either the factor or the factor squared times the aliases will give four generators which is what is required for the $3^{7-4} = 3^3$ combinations. The 27 combinations selected for this particular fractional replicate are:

$$S1' = [0000000\ \ 0120210\ \ 0210120\ \ 1020202\ \ 1110112\ \ 1200022\ \ 2010101\ \ 2100011\ \ 2220221$$
$$0011212\ \ 0101122\ \ 0221002\ \ 1001111\ \ 1121021\ \ 1211201\ \ 2021010\ \ 2111220\ \ 2201100 \tag{8}$$
$$0022121\ \ 0112001\ \ 0202211\ \ 1012020\ \ 1102200\ \ 1222110\ \ 2002222\ \ 2122102\ \ 2212012] .$$

An ANOVA table for this example is:

Source of variation	Degrees of freedom	Geometrical interactions
Total	27	
Mean	1	
A	2	
B	2	
C	2	
D	2	
E	2	
F	2	
G	2	
A × B	4	$AB + AB^2$
A × C	4	$AC + AC^2$
B × C	4	$BC + BC^2$

The D matrix tried was

$$D = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 2 & 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 0 & 0 & 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 & 2 & 2 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 1 & 0 & 2 & 0 \\ 2 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 2 \end{bmatrix} . \tag{9}$$

The entries, modulo three, in the first, second, third, seventh, eighth, and ninth columns of the product of the two matrices S1 and D were all zero, modulo 3, meaning that these were all aliases of the mean. This means that the four generators $ABCD^2$, ABC^2E^2, AB^2CF^2, and $AB^2C^2G^2$ and their generalized interactions are sufficient to construct the defining contrast and consequently all the aliases of the main effects and the three two factor interactions listed above. The 40 aliases in the defining contrast with the three factor interactions listed first, the four factor interactions listed second, the five factor interactions listed third, the six factor interactions listed fourth, and the seven factor interactions listed last are:

$I = CDE^2 = AEF = BDF^2 = ADG = BEG^2 = CFG^2$

$= \underline{ABCD}^2 = \underline{ABC}^2\underline{E}^2 = ACDF = ABDE = \underline{AB}^2\underline{CF}^2 = BC^2EF^2 = \underline{AB}^2\underline{C}^2\underline{G}^2 =$
$AC^2EG = AB^2FG = DE^2F^2G = BCDG^2$

$= BCD^2E^2F^2 = AB^2D^2EF^2 = AC^2D^2E^2F = BC^2D^2E^2G^2 = BC^2DFG = BCEFG = AD^2E^2F^2G^2 = ABE^2FG^2 = ACD^2E^2G = AB^2DE^2G^2 = ABD^2F^2G = CD^2EF^2G = AC^2DF^2G^2 = ACEF^2G^2 = BD^2E^2FG$

$= AB^2C^2DE^2F^2 = AB^2CD^2EG^2 = ABC^2D^2FG^2 = ABCE^2F^2G$

$$= AB^2CDE^2FG = ABC^2DEF^2G = ABCDEFG^2 = AB^2C^2D^2EFG. \quad (10)$$

The four factor generators used for WYLIE are underlined. Other generators from the above set of aliases could just as well have been used.

Note that there are 40 effects aliased with the mean and 80 effects aliased with each of the desired effects. The vector β has 2,187 single degree of freedom parameters aassociated with the mean and the $(3^7 - 1) / (3 - 1) = 1{,}093$ geometrical components. The vector β_1 contains 27 parameters and the vector β_2 contains 2,160 parameters. Thus, the A matrix has only 27 rows but it has 2,160 columns. Writing down these aliases manually would be extremely tedious and time-consuming. However, with a program like WYLIE this becomes a simple computer task. In fact, WYLIE allows a listing of all aliases up to a specified number of letters and if only aliases up to three factor, say, interactions are desired, this is easily specified.

A COMMENT

The procedure described above works for prime powered factorials and orthogonal fractions. Whenever these conditions are not satisfied, resort must be made to determining the X_{11} and X_{12} matrices and then finding the aliasing matrix A. Various types of contrasts among the levels of an effect may be used such as, e.g., Helmert, orthogonal polynomial, Hadamard in certain cases, or the Anderson-Federer zero-one formulation. Regardless of the nature of the contrast, the construction of the design matrix will be an arduous task unless a computer program is written to aid in the process. Computer programs to aid in the use of fractional replicates for large factorials are extremely valuable aids and more program like the above are necessary.