RIGIDITY ON EINSTEIN MANIFOLDS AND SHRINKING RICCI SOLITONS IN HIGH DIMENSIONS

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

Lihai Qian

May 2017

RIGIDITY ON EINSTEIN MANIFOLDS AND SHRINKING RICCI SOLITONS IN HIGH DIMENSIONS

Lihai Qian, Ph.D.
Cornell University 2017

This thesis consists of three parts. Each part solves a geometric problem in geometric analysis using differential equations.

The first part gives a rigidity result to high dimensional positive Einstein manifolds, by controlling the upper bound of the integration of Weyl tensor.

Part of the idea of the second part came from the new weighted Yamabe invariant from [4]. According to the definition, we can show a rigidity theorem to high-dimensional compact shrinking Ricci solitons.

The third part is an analytical result to 4-dimensional Ricci solitons. By the Weitzenbock for Ricci solitons introduced in [5], we proved an integral version of the Weitzenbock formula.

BIOGRAPHICAL SKETCH

Lihai Qian was born in April 27th, 1987 in Huai'an, a small and beautiful city in eastern China. He showed his love to math at an early age. Having solved a bunch of difficult Sudoku problems during primary school, he dreamed that one day he could be the quickest to solve Sudoku problems in the world. It quickly turned out that math has a lot more fun than that and he decided to learn more and maybe to contribute in math.

He attended Nanjing University in 2005 and earned a Bachelor of Science degree in mathematics in 2009. After that, he spent two years' graduate studying in Nanjing University and Beijing International Center For Mathematical Research. Then he came to Cornell University to pursue a PhD degree in mathematics in 2012. In 2017, he completed his thesis under the supervision of his advisor, Xiaodong Cao.

This work is dedicated to my mother, Jianchun Wang and my father, Shao Qian
for their endless love and unceasing support.

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my advisor, Professor Xiaodong Cao, for his great help about my thesis. Xiaodong is not only a well-known geometer but also an amazing person. I owe my gratitude to him for teaching me geometric analysis and making suggestions to research topics. He always listens to my immature ideas, gives helpful suggestions and corrects computational mistakes I made. Without his guidance, I would not have finished this thesis.

I would also like to thank Laurant Saloff-Coste and Leonard Gross, who are always ready to give their suggestions and share their insights. I want to thank Jeffrey Case for sharing some of his ideas to my thesis topics. I would not have been able to complete my thesis without their help.

I owe great gratitude to Jiaqiang Mei and Xiaochun Rong, who taught me Riemannian geometry and metric geometry when I was in China. I wouldn't have a chance to study geometry at Cornell without their help.

I want to thank many wonderful friends I met at Cornell. I want to thank Beihui Yuan, Shengmao Cao, Jie He and Yuexing Li for their companies and encouragements through my hard time. Their advice sustained my faith in time of difficulty and frustration.

I want to thank Ran Wei and Jun Lu for sharing their experiences and making suggestions.

I want to thank the department staff for making everything easy for us.

Cornell has been a great place to study and live and feels like a second hometown to me. Wherever I go in the future, I will always think of myself as a Cornellian.

And this is all because of you, the wonderful people I met during this time. Thank you all!

Last but not lease, I would like to thank my parents, for their truthful love and full support throughout my life. This thesis is dedicated to you.

TABLE OF CONTENTS

CHAPTER 1

BACKGROUND

In this paper, we are mainly going to discuss the geometry about two special types of manifolds, Einstein manifolds and Ricci solitons. We will list some of the known results and prove several new rigidity results. Throughout the entire thesis, we only consider compact manifolds without boundaries.

Let's begin with some geometric concepts and properties.

1.1 Curvature tensor

In Riemannian geometry, the Riemann curvature tensor is the most common method used to express the curvature of Riemannian manifolds. It associates a tensor to each point of a Riemannian manifold, that measures the extent to which the metric tensor is not locally isometric to that of Euclidean space.

For an $n-$ *dimensional* Riemannian manifold (M^n, g) we can define the curvature operator as $R(U,V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W$ where ∇ is the Levi-Civita connection and the coordinate components of the $(1,3)$-Riemann curvature tensor by

$$R^l_{ijk} = \frac{\partial}{\partial x^j}\Gamma^l_{ik} - \frac{\partial}{\partial x^k}\Gamma^l_{ij} + \Gamma^l_{js}\Gamma^s_{ik} - \Gamma^l_{ks}\Gamma^s_{ij} \tag{1.1}$$

where Γ is the Christoffel symbol.

Lowing indices with $R_{lijk} = g_{ls}R^s_{ijk}$ one gets the $(4,0)$-Riemann curvature tensor

$$R_{iklm} = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l}\right) + g_{np}(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}). \tag{1.2}$$

The symmetries of the tensor are

$$R_{iklm} = R_{lmik} \; and \; R_{iklm} = -R_{kilm} = -R_{ikml}.$$

That is, it is symmetric in the exchange of the first and last pair of indices, and antisymmetric in the flipping of a pair.

The first Bianchi identity is

$$R_{iklm} + R_{imkl} + R_{ilmk} = 0. \tag{1.3}$$

The second Bianchi identity is

$$\nabla_m R^n_{ikl} + \nabla_l R^n_{imk} + \nabla_k R^n_{ilm} = 0, \tag{1.4}$$

that is,

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0 \tag{1.5}$$

which amounts to a cyclic permutation sum of the last three indices, leaving the first two unchanged. Ricci and scalar curvatures are contractions of the Riemann tensor. They simplify the Riemann tensor, but contains less information.

The Ricci curvature tensor is essentially the unique nontrivial way of tracing the Riemann tensor

$$R_{ij} = R^l_{ilj} = g^{lm}R_{iljm} = g^{lm}R_{limj}. \tag{1.6}$$

The Ricci tensor R_{ij} is symmetric. By the contracting relations on the Chirstoffel symbols, we have

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \Gamma^m_{il}\Gamma^l_{km} - \nabla_k\Big(\frac{\partial}{\partial x^i}\Big(log\sqrt{|g|}\Big)\Big). \tag{1.7}$$

The scalar curvature is the trace of the Ricci curvature,

$$R = g^{ij}R_{ij} = g^{ij}g^{lm}R_{iljm}. \tag{1.8}$$

The gradient of the scalar curvature follows from the Bianchi identity

$$\nabla_l R^l_m = \frac{1}{2}\nabla_m R, \tag{1.9}$$

1.2 Kullkarni-Nomizu product

The Kullkarni-Nomizu product is an important tool for constructing new tensors from existing tensors on a Riemannian manifold. Let h and k be symmetric covariant 2-tensors. In coordinates,

$$h_{ij} = h_{ji} \qquad k_{ij} = k_{ji}.$$

Then we can define a new covariant 4-tensor by multiplying the two tensors, which is often denoted as $h \circ k$. The definition of the Kullkarni-Nomizu tensor is

$$(h \circ k)_{ijkl} = h_{ik}k_{jl} + h_{jl}k_{ik} - h_{il}k_{jk} - h_{jk}k_{il}. \tag{1.10}$$

Clearly, the product is symmetric, i.e.,

$$h \circ k = k \circ h.$$

1.3 Curvature decompositon

The Riemann $(0,4)$-curvature tensor can be viewed as a section of the vector bundle $\Lambda^2T^*M \otimes \Lambda^2T^*M$ where Λ^2T^*M denotes the vector bundle of 2-forms and $\otimes$ denotes the symmetric tensor product. From the first Bianchi identity we know that Rm is a section of $ker(b)$ where

$$b : \Lambda^2T^*M \otimes \Lambda^2T^*M \to T^*M \otimes \Lambda^3T^*M$$

is defined by

$$b(\Omega)(X,Y,Z,W) = \frac{1}{3}\Big(\Omega(X,Y,Z,W) + \Omega(X,Z,W,Y) + \Omega(X,W,Y,Z)\Big). \tag{1.11}$$

We will call $CM := ker(b)$ the bundles of curvature tensors. For every $x \in M$, the fiber C_xM has the structure of an $O(T_x^*M)$-module, given by

$$\times : O(T_x^*M) \times C_xM \to C_xM,$$

where

$$A \times (\alpha \otimes \beta \otimes \gamma \otimes \delta) := A\alpha \otimes A\beta \otimes A\gamma \otimes A\delta$$

for $A \in O(T_x^*M)$ and $\alpha,\ \beta,\ \gamma,\ \delta \in T_x^*M$. As an $O(T_x^*M)$ representation space, C_xM has a natural decomposition into its irreducible components. This yields a corresponding decomposition of the Riemann curvature tensor. To describe this, it will be convenient to consider the Kullkarni-Nomizu product

$$\circ : S^2M \times S^2M \to CM$$

defined in (1.10).

Here $S^2M = T^*M \otimes T^*M$ is the bundle of symmetric 2-tensors. The irreducible decomposition of C_xM as an $O(T_x^*M)$-module is given by

$$CM = \mathbb{R}g \circ g \oplus (S_0^2M \circ g) \oplus WM, \tag{1.12}$$

where S_0^2M is the bundle of symmetric, trace-free 2-tensors and

$$WM := Ker(b) \cap Ker(c)$$

is the bundle of Weyl curvature tensors. Here

$$c : \Lambda^2M \otimes \Lambda^2M \to S^2M$$

is the contraction map defined by

$$c\Omega)(X, Y) := \sum_{i=1}^{n} \Omega(e_i, X, e_i, Y).$$

Note also that $(g \circ g)_{ijkl} = 2(g_{ik}g_{jl} - g_{il}g_{jk})$.

The irreducible decomposition of CM yields the following irreducible decomposition of the Riemann curvature tensor:

$$Rm = fg \circ g + (h \circ g) + W,$$

where $f \in C^\infty(M)$, $h \in C^\infty(S_0^2M)$ and $W \in C^\infty(WM)$. Take the contraction c of this equation implies

$$R_{jk} = 2(n-1)fg_{jk} + (n-2)h_{jk}.$$

Taking two contraction, we find that

$$R = 2n(n-1)f.$$

Therefore we have for $n \geqslant 3$

$$Rm = -\frac{R}{2(n-1)(n-2)} g \circ g + \frac{1}{n-2} Rc \circ g + Weyl \tag{1.13}$$

$$= \frac{R}{2(n-1)n} g \circ g \oplus \frac{1}{n-2} E \circ g \oplus Weyl, \tag{1.14}$$

where $E := Rc - \frac{R}{n} g$ is the traceless Ricci tensor and Weyl is the Weyl tensor. The Weyl tensor has the same algebraic symmetries as the Riemann curvature tensor and in addition the Weyl tensor is totally trace-free, all of its traces are zero. Furthermore, the Weyl tensor is conformally invariant:

$$Weyl(e^{2f} g) = e^{2f} Weyl(g) \tag{1.15}$$

for any smooth function f on M.

In local coordinates, (1.13) says that for $n \geqslant 3$

$$R_{ijkl} = -\frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) + \frac{1}{n-2}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}) + W_{ijkl} \tag{1.16}$$

In particular, if $n \leqslant 3$ the the Weyl tensor vanishes. If $n = 2$, we have

$$R_{ijkl} = \frac{1}{2} R(g_{ik}g_{jl} - g_{il}g_{jk}),$$

and $R_{ij} = \frac{1}{2} R g_{ij}$. When $n = 3$,

$$R_{ijkl} = R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il} - \frac{1}{2} R(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{1.17}$$

1.4 Conformal metric

Definition 1.4.1. *Let (M^n, g) be a Riemannian manifold and f be a smooth function on M^n. Then $h = e^{2f} g$ is called a conformal metric of g.*

Proposition 1.4.1. *If $\tilde{g} = e^{2f} g$, then*

$$\tilde{R}^i_{jkl} = R^i_{jkl} - a^i_k g_{jl} - a_{jl}\delta^i_k + a^i_l g_{jk} + a_{jk}\delta^i_l, \tag{1.18}$$

where

$$a_{ij} := \nabla_i \nabla_j f - \nabla_i f \nabla_j f + \frac{1}{2}|\nabla f|^2 g_{ij}.$$

That is, as $(0, 4)$*-tensors,*

$$e^{-2f}\widetilde{Rm} = Rm - a \circ g. \tag{1.19}$$

By contracting the above formula we get

$$\tilde{R}_{ij} = R_{ij} - (n-2)a_{ij} - (\Delta f + \frac{n-2}{2}|\nabla f|^2)g_{ij} \tag{1.20}$$

and

$$e^{2f}\tilde{R} = R - 2(n-1)(\Delta f + \frac{n-2}{2}|\nabla f|^2)g_{ij}. \tag{1.21}$$

The Yamabe problem is to find a conformal metric $\tilde{g} \in [g]$, the conformal class of g, such that the scalar curvature of $\tilde{g}$ equals to a constant $c \in \mathbb{R}$. This problem is equivalent to solve the equation

$$ce^{2f} = R - 2(n-1)(\Delta f + \frac{n-2}{2}|\nabla f|^2).$$

We call

$$S := \frac{1}{n-2}\Big(Rc - \frac{1}{2(n-2)}Rg\Big)$$

the Schouten tensor. By (1.14) we can easily show that

$$Rm = Weyl \oplus S \circ g. \tag{1.22}$$

We may compute that

$$
\begin{aligned}
\tilde{S}_{ij} &= S_{ij} - \nabla_i \nabla_j f + \nabla_i f \nabla_j f - \frac{1}{2}|\nabla f|^2 g_{ij} \\
&= S_{ij} - a_{ij}. \qquad (1.23)
\end{aligned}
$$

By (1.22) and (1.23) we can conclude that

Proposition 1.4.2. *If $\tilde{g} = e^{2f} g$, then the $(1,3)$-Weyl tensor satisfies*

$$\widetilde{W}^i_{jkl} = W^i_{jkl} \qquad (1.24)$$

and $(0,4)$-Weyl tensor satisfies

$$\widetilde{W}_{ijkl} = e^{2f} W_{ijkl} \qquad (1.25)$$

Proposition 1.4.3. *If $n \geqslant 3$, then*

$$\nabla^l W_{lijk} = \frac{n-3}{n-2} C_{ijk}, \qquad (1.26)$$

where

$$C_{ijk} := \nabla_k S_{ij} - \nabla_j S_{ik}$$

is the Cotton tensor.

From Proposition 1.4.3, we know that for $n \geqslant 4$, if the Weyl tensor vanishes, then the Cotton tensor also vanishes. We also see that when $n = 3$, the Weyl tensor always vanishes but the Cotton tensor does not vanish in general.

Proposition 1.4.4. *When $n = 3$, if $\tilde{g} = e^{2f} g$, then*

$$\widetilde{C}_{ijk} = C_{ijk}. \qquad (1.27)$$

1.5 Locally conformally flat manifolds

We say a Riemannian manifold (M^n, g) is locally conformally flat if for every point $p \in M^n$, there exists a local coordinates $\{x_i\}$ in a neighborhood U of p such that

$$g_{ij} = v \cdot \delta_{ij}$$

for some function v defined on U, e.g., $v^{-1}g$ is a flat metric. When $n = 2$, every Riemannian manifold is locally conformally flat. Indeed, if (M^2, g) is a Riemannian surface and u is a function on M, then we have

$$\tilde{R}(e^u g) = e^{-u}(R(g) - \Delta_g u). \tag{1.28}$$

Thus to find u locally so that $\tilde{R}(e^u g) = 0$, we just need to solve the Poission equation

$$\Delta_g u = R(g)$$

which is definitely possible.

Theorem 1.5.1. *(Weyl, Schouten) A Riemannian manifold (M^n, g) is locally conformally flat if and only if*

(a) for $n \geqslant 4$ the Weyl tensor vanishes,

(b) for $n = 3$ the Cotton tensor vanishes.

Proof. Since the Weyl tensor is conformal invariant, it is clear that if (M^n, g) is locally conformally flat, then the Weyl tensor vanishes.

For $n = 3$, the Ricci tensor vanishes and therefore the Cotton tensor vanishes also. Conversely, if the Weyl tensor vanishes, then the equation that the metric $\tilde{g} = e^{2f}g$ is flat:

$$\widetilde{Rm} = 0$$

is equivalent to

$$\begin{aligned} 0 &= Rm - a \circ g \\ &= \Big(\frac{1}{n-2}(Rc - \frac{1}{2(n-2)}Rg) - a\Big) \circ g. \end{aligned} \tag{1.29}$$

Since the map $\circ : S^2M \to CM$ defined by $\circ(h) := h \circ g$ is injective, the above formula is equivalent to

$$\frac{1}{n-2}(Rc - \frac{1}{2(n-2)}Rg) = a.$$

That is,

$$\nabla_i \nabla_j f = S_{ij} + \nabla_i f \nabla_j f - \frac{1}{2}|\nabla f|^2 g_{ij}, \tag{1.30}$$

where

$$S_{ij} = \frac{1}{n-2}(R_{ij} - \frac{1}{2(n-2)}Rg_{ij})$$

Theorem 1.5.1 is now a consequence of the following, which gives the condition for when the flat metric equation for $\tilde{g}$ is locally solvable. □

Lemma 1.5.2. *If the Weyl tensor vanishes, equation* (1.30) *is locally solvable if and only if the following integrability condition is satisfied*

$$\nabla_k S_{ij} = \nabla_i S_{kj}, \tag{1.31}$$

that is, if and only if the Cotton tensor vanishes. Recall that when $n \geqslant 4$, (1.31) *follows from that the Weyl tensor vanishes. On the other hand, when $n = 3$, the Weyl tensor vanishes for any metric.*

Proof. To solve (1.30) it is necessary and sufficient to find a 1-form X locally such that

$$\nabla_i X_j = c_{ij} := S_{ij} + X_i X_j - \frac{1}{2}|X|^2 g_{ij}, \tag{1.32}$$

where $c = c(X, g)$ is a symmetric 2-tensor depending only on X and g. Clearly, if f is a solution of (1.30) then $X = df$ is a solution of (1.32). On the other hand, if X is a solution of (1.32), by the symmetry of the right hand side, we have

$$\nabla_i X_j = \nabla_j X_i,$$

which implies $dX = 0$. Thus locally X is the exterior derivative of some function f, which then solves (1.30). Now we rewrite (1.32) as

$$\frac{\partial}{\partial x^i} X_j = \tilde{c}_{ij}, \tag{1.33}$$

where

$$\begin{aligned} \tilde{c}_{ij} &= \tilde{c}(X, g)_{ij} := c(X, g)_{ij} + \Gamma^k_{ij} X_k \\ &= S_{ij} + X_i X_j - \frac{1}{2}|X|^2 g_{ij} + \Gamma^k_{ij} X_k. \end{aligned}$$

Suppose $p \in M$ and that the coordinates $\{x^i\}$ is defined in a neighborhood of p. The Frobenius theorem gives a necessary and sufficient condition to locally solve (1.33) with $X(p) = X_0$ for any $X_0 \in T_p M$ is the following integrability condition arising from $\frac{\partial^2}{\partial x^k \partial x^i} X_j = \frac{\partial^2}{\partial x^i \partial x^k} X_j$:

$$\frac{\partial}{\partial x^k} \tilde{c}_{ij} = \frac{\partial}{\partial x^j} \tilde{c}_{jk}.$$

More invariantly, the integrability condition arises from

$$\nabla_k \nabla_i X_j = \nabla_i \nabla_k X_j + R^i_{jkl} X_l$$

and

$$\nabla_k c_{ij} - \nabla_j c_{ik} = R^l_{jik} X_l = (S_i^{\ l} g_{jk} + S_{jk} \delta^l_i - S_k^{\ l} g_{ji} - S_{ji} \delta^l_k) X_l \tag{1.34}$$

where for the last equality we used $W^l_{jik} = 0$. From (1.32) we have

$$\nabla_k c_{ij} - \nabla_k S_{ij} + X_j \nabla_k X_i + X_i \nabla_k X_j - X^l \nabla_k X_l g_{ij}.$$

Therefore by (1.34) we have

$$C_{ijk} = \nabla_k S_{ij} - \nabla_j S_{ik} = 0.$$

□

Corollary 1.5.1. *If a Riemannian manifold (M^n, g) has constant sectional curvature, then (M^n, g) is locally conformally flat.*

Definition 1.5.1. *We say two Riemannian manifolds (M_1^n, g_1) and (M_2^n, g_2) are conformally equivalent if there exist a diffeomorphism $\phi : M_1 \to M_2$ and a function $f : M_1^n \to \mathbb{R}$ such that $g_1 = e^f \phi^* g_2$.*

Theorem 1.5.3. *(Kuiper) If (M^n, g) is a simply connected, locally conformally flat, closed Riemannian manifold, then (M^n, g) is conformally equivalent to the standard sphere S^n.*

A map ψ from one manifold (M_1^n, g_1) to another (M_2^n, g_2) is said to be conformal if there exists a function $f : M_1 \to M_2$ such that $g_1 = e^f \psi^* g_2$.

Theorem 1.5.4. *(Schoen and Yau) If (M^n, g) is a simply connected locally conformally flat, complete Riemannian manifold in the conformal class of a metric with nonnegative scalar curvature, then there exists a one-to-one conformal map of (M^n, g) into the standard sphere S^n.*

When M^n is not simply connected, it is useful to apply the above results to the universal cover $(\tilde{M}^n, \tilde{g})$.

1.6 Ricci flow

In this section, we are going to give a brief introduction to Hamilton's Ricci flow program, which is aimed at proving the Poincare Conjecture. In this section we will present the context for the Ricci flow: what is it, what are the problems that it is intended to solve, and why might it be expected to solve them. In the process we will also see some simple solutions to the Ricci flow.

The Poincare Conjecture was one of the iconic unsolved problems of 20th century mathematics. Around 1900, Poincare asked if a simply-connected closed 3-manifold is necessarily the 3-sphere S^3. This question remained open for a century and lots of ideas and techniques were introduced during that period of time. One of the main technique which was used to solve the Poincare conjecture was so called Hamilton's Ricci flow.

Richard Hamilton published a groundbreaking paper [42] in 1982, introducing the concept of the Ricci flow. If you have a Riemannian manifold M^n with initial metric g_0, the Ricci flow is a PDE that evolves the metric tensor

$$\frac{\partial}{\partial t} g(t) = -2Rc(g(t)) \tag{1.35}$$

$$g(0) = g_0 \tag{1.36}$$

where $Rc(g(t))$ denotes the Ricci curvature of the metric $g(t)$.

Before we can do anything with the Ricci flow, we must show that a solution exists for a short time. We would like to apply the short-time existence and uniqueness theorem for parabolic PDEs to the system.

Theorem 1.6.1. *Given a smooth Riemannian metric g_0 on a closed manifold M, there exists a maximal time interval $[0; T)$ such that a solution $g(t)$ to the Ricci flow, with $g(0) = g_0$, exists and is smooth on $[0; T)$, and this solution is unique.*

The idea is to try to evolve the metric in some way that will give the manifold a "better" metric or "shape". In choosing what should go on the right hand side of the equation of the Ricci flow, we know that it should be a rank-2 tensor, symmetric and it should involve the curvature somehow-the Ricci curvature tensor is the obvious choice. The minus sign makes the Ricci flow a heat-type equation, so it is expected to "average out" the curvature. This should make the metric rounder in the way that we want. The following theorem was proved by Hamilton in 1982 in [42].

Theorem 1.6.2. *Let M^3 be a closed 3-manifold which admits a Riemannian metric with strictly positive Ricci curvature. Then M^3 also admits a metric of constant positive curvature.*

In particular, any simply-connected closed 3-manifold which admits a metric of strictly positive Ricci curvature is diffeomorphic to the 3-sphere. We are starting to get into the Poincare Conjecture with this result.

More specifically, we will see that if the initial metric g_0 has strictly positive Ricci curvature then the manifold M^3 will shrink to a point in finite time under the Ricci flow. But if we dilate the metric by a time-dependent factor so that the volume remains constant, the problem of shrinking to a point is removed. Furthermore, we can show that the rescaled metric converges uniformly to the desired metric of constant positive curvature on M^3. This process of ”blowing up” the manifold when it is becoming singular is a crucial one in the Ricci flow program.

The maximum principle is the key tool in understanding many parabolic partial differential equations. It appears in many guises, but it always essentially expresses the fact that parabolic or heat-type PDEs will ”average out” the values of whatever quantity is evolving. It is crucial to understanding the Ricci flow. In some situations we will need more refined estimates than can be obtained by applying the maximum principle to scalar quantities related to curvature, so we must apply the maximum principle to tensor quantities like the curvature operator. The question of what it means for a tensor quantity to ”average out” naturally arises.

Theorem 1.6.3. *(Maximum Principle) Let $(M, g(t))$ be a closed manifold with a time-dependent Riemannian metric $g(t)$. Suppose that $u : M \times [0, T) \to \mathbb{R}$ satisfies*

$$\begin{aligned} \frac{\partial u}{\partial t} &\leqslant \Delta_{g(t)} u + \langle X(t), \nabla u \rangle + F(u) \\ u(x, 0) &\leqslant C \quad \forall x \in M, \end{aligned}$$

for some constant C, where $X(t)$ is a time-dependent vector field on M and $F : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz. Suppose that $\Phi : \mathbb{R} \to \mathbb{R}$ is the solution to the associated ODE, which is formed by neglecting the Laplacian and gradient terms

$$\begin{aligned} \frac{d\Phi}{dt} &= F(\Phi) \\ \Phi(0) &= C. \end{aligned} \tag{1.37}$$

Then

$$u(x, t) \leqslant \Phi(t)$$

for all $x \in M$ and $t \in [0, T)$ such that $\Phi(t)$ exists.

The theorem essentially tells us that our upper bound grows no faster than we would expect from the reaction term $F(u)$.

To apply maximum principle arguments to the curvature, we need to know what the equations describing the evolution of curvature quantities under the Ricci flow are. The evolution equations for the Ricci flow follow by the following proposition.

Proposition 1.6.1. *Suppose that $g(t)$ is a solution to the Ricci flow*

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

Then the various geometric quantities evolve according to the following equations:

(a) Metric inverse:

$$\frac{\partial}{\partial t} g^{ij} = 2R^{ij}. \tag{1.38}$$

(b) Christoffel symbols:

$$\frac{\partial}{\partial t} \Gamma_{ij}^{k} = -g^{kl}(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}). \tag{1.39}$$

(c) Riemannian curvature tensor:

$$\begin{aligned} \frac{\partial}{\partial t} R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ &- (R_i^p R_{pjkl} + R_j^p R_{ipkl} + R_k^p R_{ijpl} + R_l^p R_{ijkp}), \end{aligned} \tag{1.40}$$

where $B_{ijkl} := -R_{pij}^{q} R_{qlk}^{p}$.

(d) Ricci tensor:

$$\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} + 2R_{pijq}R^{pq} - 2R_i^p R_{pj}. \tag{1.41}$$

(e) Scalar curvature:

$$\frac{\partial}{\partial t} R = \Delta R + 2|Rc|^2. \tag{1.42}$$

(f) Einstein tensor on 3-manifolds:

$$\frac{\partial}{\partial t}|E|^2 = \Delta|E|^2 - 2|\nabla E|^2 - 8R_i^j R_j^k R_k^i + \frac{26}{3} R|Rc|^2 - 2R^3. \tag{1.43}$$

(g) Volume element:

$$\frac{\partial}{\partial t} d\mu = -R d\mu. \tag{1.44}$$

(h) Volume of a manifold:

$$\frac{\partial}{\partial t}\int_M d\mu = -\int_M R d\mu. \tag{1.45}$$

(i) Total scalar curvature on a closed manifold:

$$\frac{\partial}{\partial t}\int_M R d\mu = \int_M (-R^2 + 2|Rc|^2) d\mu. \tag{1.46}$$

Finally in this section we will show a special solution to the Ricci flow, which is called Ricci solitons. The first thing is to know what they do on the spaces of constant curvature.

On an $n-$*dimensional* sphere of radius r where $n > 1$, the metric is given by $g = r^2\bar{g}$ where $\bar{g}$ is the metric on the unit sphere. The sectional curvatures are all $\frac{1}{r^2}$. Thus for any unit vector v, we have $Rc(v, v) = \frac{n-1}{r^2}$. Therefore

$$Rc = \frac{n-1}{r^2} g = (n-1)\bar{g},$$

so the Ricci flow equation becomes an ODE:

$$\frac{\partial}{\partial t} g = -2Rc$$

$$\frac{\partial}{\partial t}(r^2\bar{g}) = -2(n-1)\bar{g}$$

$$\frac{d(r^2)}{dt} = -2(n-1).$$

Therefore we have the solution

$$r(t) = \sqrt{R_0^2 - 2(n-1)t},$$

where R_0 is the initial radius of the sphere. The manifold shrinks to a point as $t \to \frac{R_0^2}{2(n-1)}$.

Similarly, for a hyperbolic $n - space$ $\mathbb{H}$ where $n > 1$, the Ricci flow reduces to the ODE

$$\frac{d(r^2)}{dt} = 2(n-1)$$

which has the solution

$$r(t) = \sqrt{R_0^2 + 2(n-1)t}.$$

So the solution expands out to infinity.

Of course the at metric on $\mathbb{E}^n$ has zero Ricci curvature, so it does not evolve at all under the Ricci flow. There are other non-trivial Riemannian manifolds with vanishing Ricci curvature (the metric is flat, i.e. locally isometric to Euclidean space, if and only if the Riemann curvature tensor vanishes). These metrics can be regarded as the "fixed points" of the Ricci flow. However, we ought really to regard $\mathbb{S}^n$ and $\mathbb{H}^n$ as honorary fixed points of the flow even though the metric was changing under the flow, it only ever changed by a rescaling of the metric.

Even more generally, one can regard as "generalized fixed points" of the Ricci flow those manifolds which change only by a diffeomorphism and a rescaling under the Ricci flow. Let $(M^n, g(t))$ be a solution to the Ricci flow, and suppose that $\phi_t : M^n \to M^n$ is a time-dependent family of diffeomorphisms with $\phi_o = id$ and $\sigma(t)$ is a time-dependent scale factor with $\sigma(0) = 1$.

If we then have

$$g(t) = \sigma(t)\phi_t^* g(0)$$

then the solution $(M^n, g(t))$ is called a Ricci soliton, which we will discuss in Chapter 3. Therefore Einstein manifolds and Ricci solitons are special solutions to the Ricci flow.

1.7 Weighted Yamabe invariant

The Yamabe constant and Perelman's $\nu - entropy$ are two important geometric invariants in Riemannian geometry which share many similarities. Both constants are intimately related to sharp Sobolev-type inequalities on Euclidean space, with the Yamabe constant recovering the best constant for the Sobolev inequality and the $\nu - entropy$ recovering the best constant for the logarithmic Sobolev inequality. In the curved setting, these constants are defined as the infima of Sobolev-type quotients involving scalar curvature, and one can show that the infima are achieved by positive smooth functions through a two-step process.

First, one shows that minimizing sequences cannot concentrate provided the Yamabe constant (resp. $\nu - entropy$) is strictly less than the best constant for the Sobolev inequality (resp. logarithmic Sobolev inequality) on Euclidean space. Second one shows that strict inequality always holds on a compact manifold, except in the case of the Yamabe constant on the standard conformal sphere.

It turns out that there is a natural one-parameter family of geometric invariants which interpolate between the Yamabe constant and the $\nu - entropy$. These invariants, which is called weighted Yamabe constants, were introduced by the author [4] as curved analogues of the best constants in the family of Gagliardo-Nirenberg inequalities studied by Del Pino and Dolbeault.

The purpose of this section is to introduce to what extent these invariants interpolate between the Yamabe constant and the $\nu - entropy$, focusing on issues related to the problem of finding minimizers of the weighted Yamabe quotients.

In order to explain their results, we first recall the aforementioned result of Del Pino and Dolbeault [13].

Theorem 1.7.1. *(Del Pino-Dolbeault) Fix* $m \in [0, \infty)$*. Given any* $w \in W^{1,2}(\mathbb{R}) \cap L^{\frac{2(m+n)}{m+n-2}}(\mathbb{R})$ *it holds that*

$$\Lambda_{m,n}\left(\int_{\mathbb{R}^n} w^{\frac{2(m+n)}{m+n-2}}\right)^{\frac{2m+n-2}{n}} \leqslant \left(\int_{\mathbb{R}^n} |\nabla w|^2\right)\left(\int_{\mathbb{R}^n} w^{\frac{2(m+n-1)}{m+n-2}}\right)^{\frac{2m}{n}}, \tag{1.47}$$

where the constant $\Lambda_{m,n}$ *is given by*

$$\Lambda_{m,n} = \frac{n\pi(m+n-2)^2}{2m+n-2}\left(\frac{2(m+n-1)}{2m+n-2}\right)^{\frac{2m}{n}}\left(\frac{\Gamma(\frac{2m+n}{2})}{\Gamma(m+n)}\right)^{\frac{2}{n}}. \tag{1.48}$$

Moreover, equality holds in (1.47) *if and only if there exists a constant* $\epsilon > 0$ *and a fixed point* $x_0 \in \mathbb{R}^n$ *such that* w *is a constant multiple of the function*

$$w_{\epsilon,x_0}(x) := \left(\frac{2\epsilon}{\epsilon^2 + |x - x_0|^2}\right)^{\frac{m+n-2}{2}}. \tag{1.49}$$

There are four features of Theorem 1.7.1 which should be emphasized. First, Theorem 1.1 recovers the sharp Sobolev inequality [22] in the case $m = 0$ and the sharp logarithmic Sobolev inequality in the case $m = \infty$. Second, the extremal functions (1.49) are all the same, except for the dependence of the exponent on the parameter m. Third, the functions w_{ϵ,x_0} concentrate at x_0 as $\epsilon \to 0$. Fourth, the family (1.47) of Gagliardo-Nirenberg (GN) inequalities is, in a certain sense, the only such family with geometrically significant extremal functions. This last point requires further explanation.

Given constants $2 \leqslant p \leqslant q \leqslant \frac{2n}{n-2}$, the sharp Sobolev inequality and Holder's inequality yields a positive constant $C_{p,q}$ such that the GN inequality

$$\|w\|_q \leqslant C_{p,q}\, \|\nabla w\|_2^{\theta}\, \|w\|_p^{1-\theta} \tag{1.50}$$

holds for all $w \in C_0^\infty(\mathbb{R}^n)$. Now, only when $2p = q + 2$, corresponding to the family (1.47), is the best constant $C_{p,q}$ known (there are other cases known in the range $1 \leqslant p \leqslant q \leqslant \frac{2n}{n-2}$). This yields one to wonder if there is some geometric property distinguishing this family. One such property was previously described in [10]: The formalism of smooth metric measure spaces allows one to define conformal invariants which give a curved analogue of the sharp constant $C_{p,q}$ in (1.50) as the infimum of the total weighted scalar curvature subject to certain volume constraints. In this framework, the family (1.47) has the property that it is the only family of GN inequalities (1.50) for which the extremal functions on Euclidean space are also critical points of the constrained total weighted scalar curvature functional through variations of the metric or the measure. This generalizes the fact that extremal functions of the Sobolev inequality (resp. logarithmic Sobolev inequality) give rise to conformally flat Einstein metrics on $\mathbb{R}^n$ (resp. Gaussian measures on $\mathbb{R}^n$).

To explain the results of this section requires some terminology. A smooth metric measure space is a four-tuple $(M^n, g, e^{-\phi}dvol, m)$ of a Riemannian manifold (M^n, g), a smooth measure $e^{-\phi}dvol$ determined by a function $\phi \in C^\infty(M)$ and the Riemannian volume element of g, and a dimensional parameter $m \in [0, \infty]$. The weighted scalar curvature R_ϕ^m of a smooth metric measure space is

$$R_\phi^m := R + 2\Delta\phi - \frac{m+1}{m}|\nabla\phi|^2, \tag{1.51}$$

where R and Δ are the scalar curvature and Laplacian associated to the metric g, respectively. The weighted Yamabe quotient is the functional

$$Q(w) := \frac{\left(\int |\nabla w|^2 + \frac{m+n-2}{4(m+n-1)} R^m_\phi w^2\right)\left(\int |w|^{\frac{2(m+n-1)}{m+n-2}} e^{\frac{\phi}{m}}\right)^{\frac{2m}{n}}}{\left(\int |w|^{\frac{2(m+n)}{m+n-2}}\right)^{\frac{2m+n-2}{n}}}, \tag{1.52}$$

where all integrals are taken with respect to $e^{-\phi}dvol$; in the limit $m = \infty$, this is

$$Q(w) := \frac{\int |\nabla w|^2 + \frac{1}{4} R^\infty_\phi w^2}{\int w^2} exp\left(-\frac{2}{n}\int_M \frac{w^2}{\|w\|_2^2} log \frac{w^2 e^{-\phi}}{\|w\|_2^2}\right). \tag{1.53}$$

The weighted Yamabe quotient is conformally invariant in the sense that if

$$\left(M^n, \hat{g}, e^{-\hat{\phi}} dvol_{\hat{g}}, m\right) = \left(M^n, e^{\frac{2\sigma}{m+n-2}} g, e^{\frac{(m+n)\sigma}{m+n-2}} e^{-\phi} dvol_g\right)$$

for some $\sigma \in C^\infty(M)$, then $Q(w) = Q(we^{\sigma/2})$. There are similar conformally invariant functionals on smooth metric measure spaces generalizing (1.50) for $2 \leqslant p \leqslant \frac{2(m+n)}{m+n-2} = q$, and it is through these functionals that one obtains the characterization described in the previous paragraph.

The weighted Yamabe constant of a compact smooth metric measure space is

$$\Lambda[g, e^{-\phi} dvol, m] := inf\{Q(w) : 0 < w \in C^\infty(M)\}. \tag{1.54}$$

When $m = 0$, this is the Yamabe constant. When $m = \infty$ and $\Lambda > 0$, this is equivalent to Perelman's $\nu - entropy$ [19]. Thus the weighted Yamabe constant interpolates between the Yamabe constant and Perelman's $\nu - entropy$. In this section we list some results about the weighted Yamabe problem, which asks about the existence of functions which minimize the weighted Yamabe quotient. Their results illustrate the interpolatory nature of the weighted Yamabe constants, though, as described below, there are some surprises.

Their approach to these problems is similar to approaches to the Yamabe problem [18, 20, 23, 25] and to Perelman's $\nu - entropy$ [19]. Much of the analysis is based on the Euler-Lagrange equation

$$-\Delta_\phi w + \frac{m+n-2}{4(m+n-1)} R_\phi^m w + c_1 w^{\frac{(m+n)}{m+n-2}} e^{\frac{\phi}{m}} = c_2 w^{\frac{(m+n+2)}{m+n-2}} \tag{1.55}$$

for critical points of the functional Q. When $m > 0$, the equation (1.55) has a subcritical nonlinearity. The main difficulty is instead that minimizing sequences for the weighted Yamabe constant need not be uniformly bounded in $W^{1,2}(M)$. They overcome this difficulty by introducing a generalization of Perelman's W-functional. Using this functional, they obtain an Aubin-type criterion for the existence of minimizers of the weighted Yamabe constant.

Theorem 1.7.2. *Let $(M^n, g, e^{-\phi} dvol, m)$ be a compact smooth metric measure space, then*

$$\Lambda[g, e^{-\phi} dvol, m] \leqslant \Lambda[\mathbb{R}^n, dx^2, dvol, m] = \Lambda_{m,n}. \tag{1.56}$$

Moreover, if the inequality (1.56) *is strict, then there exists a positive function $w \in C^\infty(M)$ such that*

$$Q(w) = \Lambda[g, e^{-\phi} dvol, m].$$

Theorem 1.7.2 implies that the weighted Yamabe constant of Euclidean space $\Lambda[\mathbb{R}^n, dx^2, dvol, m]$ is $\Lambda_{m,n}$.

They solved the weighted Yamabe problem when $m \in \mathbb{N} \cup \{0, \infty\}$ using the following necessary condition for equality to hold in (1.56).

Theorem 1.7.3. *Let $(M^n, g, e^{-\phi}dvol, m)$ be a compact smooth metric measure space such that $m \in \mathbb{N} \cup \{0, \infty\}$. If*

$$\Lambda[g, e^{-\phi}dvol, m] = \Lambda[\mathbb{R}^n, dx^2, dvol, m]$$

then $m \in \{0, 1\}$ and $(M^n, g, e^{-\phi}dvol, m)$ is conformally equivalent to $(S^n, g_0, dvol, m)$ for g_0 a metric of constant sectional curvature. In particular, there exists a positive function $w \in C^\infty(M)$ such that

$$Q(w) = \Lambda[g, e^{-\phi}dvol, m].$$

Theorem 1.7.4. *There does not exist a minimizer for the weighted Yamabe constant of $(S^n, g_0, dvol)$.*

We expect that the weighted Yamabe problem is always solvable for $m \in \{0\} \cup [1, \infty)$, but not for $m \in (0, 1)$.

CHAPTER 2

EINSTEIN MANIFOLD

In differential geometry and mathematical physics, an Einstein manifold is a differentiable Riemannian manifold whose Ricci tensor is proportional to the metric. They are named after Albert Einstein because this condition is equivalent to saying that the metric is a solution of the vacuum Einstein field equations, which bring interests to a number of mathematicians and physicians. In this chapter we are going to describe the curvature decomposition of Einstein manifolds, list some well-known rigidity results and prove the main result in section 2.6.

2.1 Introduction

The purpose of modern Riemannian geometry is to understand the relation between topology and geometry. One question is the existence of an Einstein metric, that is, a Riemannian metric g such that

$$Rc = \lambda g, \tag{2.1}$$

where Rc is the Ricci curvature tensor of the metric g and λ is some constant on a given smooth compact n-manifold M. When such a metric exists, moreover, it is natural to ask to what extent it is unique; in other words, one would like to understand the Einstein moduli space of M, i.e. the set of unit-volume Einstein metrics on M, modulo the action of the diffeomorphism group.

In dimension 2, there is a complete classification of compact oriented 2-manifolds by the Euler characteristic, originally obtained by purely topological methods through work of Mobius, Dehn, Heegard and Rado. In dimension 3, Perelman proved the Poincare conjecture, which claimed that every simply connected, closed 3-manifold is homeomorphic to the 3-sphere fifteen years ago.

The existence and uniqueness problems are commensurately harder when $n \geqslant 4$. Indeed, there are , to date, no non-existence or uniqueness results known when $n > 4$. Fortunately, however, a constellation of low-dimensional accidents makes the borderline case of $n = 4$ comparatively tractable. The aim in this chapter is to introduce some of our knowledge of the rigidity results when $n \geqslant 4$.

2.2 Positive isotropic curvature

Definition 2.2.1. *A Riemannian manifold with dimension at least 4 is said to have positive (respectively, non-negative) isotropic curvature if for every orthonormal 4-frame we have*

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0 \ \ (respectively, \ \geqslant 0). \tag{2.2}$$

2.3 Curvature decomposition for Einstein 4-manifolds

In this and the next sections, we restrict our discussion on dimension 4. We will talk about two curvature decompositions, the duality decomposition and Berger decomposition. Let's start from duality decomposition

The Hodge star operator $\star : \Lambda^2 TM \to \Lambda^2 TM$ induces a natural decomposition of the vector bundle of 2-forms $\Lambda^2 TM$ on an oriented 4-manifold (M^4, g),

$$\Lambda^2 TM = \Lambda^+ TM \oplus \Lambda^- TM,$$

where $\Lambda^\pm M$ are eigenspaces of ± 1 respectively, sections of which are called self-dual and anti-self-dual 2-forms respectively. It further induces a decomposition for the curvature operator $Rm : \Lambda^2 TM \to \Lambda^2 TM$,

$$Rm = \begin{pmatrix} \frac{R}{12} Id + W^+ & E \\ E & \frac{R}{12} Id + W^- \end{pmatrix} \tag{2.3}$$

where E is the traceless part of Ricci curvature, and R is the scalar curvature. If (M^4, g) is Einstein, then the traceless Ricci tensor vanishes, we get

$$Rm = \begin{pmatrix} Rm^+ & 0 \\ 0 & Rm^- \end{pmatrix} = \begin{pmatrix} \frac{R}{12} Id + W^+ & 0 \\ 0 & \frac{R}{12} Id + W^- \end{pmatrix} \tag{2.4}$$

The duality decomposition for Einstein 4-manifolds implies that Rm, $Rm^\pm$, W, $W^\pm$ are all harmonic, using the Weitzenbock formula which we discuss in Proposition 2.3.3

Definition 2.3.1. *A curvature operator is said to be k-positive (k-nonnegative), if the sum of any k eigenvalues of the curvature operator is positive (nonnegative).*

Berger in [31] discussed the following beautiful curvature decomposition for Einstein 4-manifolds (see also Singer and Thorpe [30]).

Proposition 2.3.1. *Let (M, g) be an Einstein 4-manifold with $Rc = \lambda g$. For any $p \in M$, there exists an orthonormal basis $\{e_i\}_{1 \leqslant i \leqslant 4}$ of T_pM, such that relative to the corresponding basis $\{e_i \wedge e_j\}_{1 \leqslant i \leqslant 4}$ of $\Lambda^2 T_pM$, Rm takes the form*

$$Rm = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \tag{2.5}$$

where $A = diag\{a_1, a_2, a_3\}$, $B = diag\{b_1, b_2, b_3\}$ satisfying the following properties,

(1) $a_1 = K(e_1, e_2) = K(e_3, e_4) = min\{K(\sigma) : \sigma \in \Lambda^2 T_pM, \|\sigma\| = 1\}$,

(2) $a_3 = K(e_1, e_4) = K(e_2, e_3) = max\{K(\sigma) : \sigma \in \Lambda^2 T_pM, \|\sigma\| = 1\}$,

(3) $a_2 = K(e_1, e_3) = K(e_2, e_4)$, *and* $a_1 + a_2 + a_3 = \lambda$;

(4) $b_1 = R_{1234}$, $b_2 = R_{1342}$, $b_3 = R_{1423}$;

(5) $|b_2 - b_1| \leqslant a_2 - a_1$, $|b_3 - b_1| \leqslant a_3 - a_1$, $|b_3 - b_2| \leqslant a_3 - a_2$.

Diagonalizing the matrix in Berger's decomposition, we get eigenvalues of the curvature operator in order (see [33]),

$$a_1 + b_1 \leqslant a_2 + b_2 \leqslant a_3 + b_3, \tag{2.6}$$

$$a_1 - b_1 \leqslant a_2 - b_2 \leqslant a_3 - b_3. \tag{2.7}$$

Therefore, 2-positive curvature operator is equivalent to $(a_1 + a_2) \pm (b_1 + b_2) > 0$ and $a_1 > 0$; positive isotropic curvature implies $a_1 + a_2) \pm (b_1 + b_2) > 0$; 3-positive curvature operator is equivalent to $2a_1 + a_2 \pm b_2 > 0$; and 4-positive curvature operator is equivalent to $a_1 + a_2 > 0$ and $1 + (a_1 \pm b_1) > 0$.

The relationship between Berger's curvature decomposition and special duality decomposition can be described as follows:

$$
\begin{aligned}
W^+(\omega_i^+, \omega_j^+) &= [(a_i + b_i) - \frac{R}{12}]\delta_{ij}, \\
W^-(\omega_i^-, \omega_j^-) &= [(a_i - b_i) - \frac{R}{12}]\delta_{ij},
\end{aligned}
\tag{2.8}
$$

where $\{\omega_i^+\}_{1\leqslant i\leqslant 3}$ and $\{\omega_i^-\}_{1\leqslant i\leqslant 3}$ are the corresponding orthonormal bases of Λ^+M and Λ^-M in Berger curvature decomposition.

In [12] X. Cao and P. Wu proved the following result.

Proposition 2.3.2. *Let (M, g) be an Einstein 4-manifold with $Rc = g$, then*

(1) Rm is 2-positive if and only if the isotropic curvature is positive.

(2) If Rm is 3-positive, then $K > \frac{1}{30}$.

(3) If $K > \frac{1}{12}$, then Rm is 3-positive.

(4) Rm is 4-positive if and only if $K < 1$, which implies $K > 4 - \sqrt{17}$.

(5) Rm is 6-positive if and only if $R > 0$.

(6) Rm is 2-nonnegative if and only if the isotropic curvature is nonnegative.

(7) If Rm is 3-nonnegative, then either min K=0 or $K \geqslant \frac{1}{30}$.

(8) If $K \geqslant \frac{1}{12}$, then Rm is 3-nonnegative.

(9) Rm is 4-nonnegative if and only if $K \leqslant 1$, which implies $K \geqslant 4 - \sqrt{17}$.

(10) Rm is 6-nonnegative if and only if $R \geqslant 0$.

We will end this section by introducing the famous Weitzenbock formula and Kato's inequality.

Proposition 2.3.3. *(Weitzenbock formula* [75]*) Let* (M^4, g) *be an Einstein 4-manifold, then*

$$\Delta|W^{\pm}|^2 = 2|\nabla W^{\pm}|^2 + R|W^{\pm}|^2 - 36 det\ W^{\pm}. \tag{2.9}$$

Gursky and LeBrun [32] and later Yang [36] proved a refined Kato's inequality, which was proved to be optimal by Branson [37] and Calderbank, Gauduchon and Herzlich [39].

Proposition 2.3.4. *Let* (M, g) *be an Einstein 4-manifold, then*

$$|\nabla W^{\pm}|^2 \geqslant \frac{5}{3}|\nabla|W^{\pm}||^2. \tag{2.10}$$

2.4 Hitchin-Thorpe inequality

For closed 4-manifolds, the topology of the manifolds are influenced by their curvatures. By the Gauss-Bonnet theorem we can write explicitly the Euler characteristic and Hirzebruch signature as linear combinations of integrals of R, $|E|$, W^+ and W^-.

Proposition 2.4.1. *For compact oriented Riemannian 4-manifolds, we have*

$$\chi(M) = \frac{1}{8\pi^2}\int_M \Big[|W^+|^2 + |W^-|^2 + \frac{R^2}{24} - \frac{|E|^2}{2}\Big]d\mu \tag{2.11}$$

and

$$\tau(M) = \frac{1}{12\pi^2}\int_M \Big[|W^+|^2 - |W^-|^2\Big] d\mu. \tag{2.12}$$

Since E vanishes for any Einstein metric, we have the following celebrated result of Hitchin and Thorpe.

Theorem 2.4.1. *If the smooth compact oriented Riemannian 4-manifold M admits an Einstein metric g, then*

$$2\chi(M) \geqslant 3|\tau(M)|,$$

with equality if and only if the connection on one of the bundles $\Lambda^{\pm}$ is flat.

2.5 Some known results

In this section we will present some known results on the rigidity of Einstein manifolds. First we list a few results regarding Einstein 4-manifolds.
Berger in [31] proved the following theorem in 1961.

Theorem 2.5.1. *(Berger) Let (M, g) be Einstein 4-manifolds with $\frac{1}{4}$-pinched sectional curvature. Then (M, g) is isometric to (S^4, g_0) where g_0 is the round metric on S^4.*

Later in 1974, Hitchin proved the following in [34].

Theorem 2.5.2. *(Hitchin) Suppose (M, g) is a half conformally flat Einstein 4-manifold with positive scalar curvature, then (M, g) isometric to either (S^4, g_0) or $(\mathbb{CP}^2, g_{FS})$ up to rescaling.*

In 1999, Gursky and LeBrun were able to prove

Theorem 2.5.3. *(Gursky-LeBrun) Let (M, g) be a compact Einstein 4-manifold with nonnegative sectional curvature and positive intersection form, then (M, g) isometric to $(\mathbb{CP}^2, g_{FS})$ up to rescaling.*

Dagang Yang in [36] proved the following result in 2000.

Theorem 2.5.4. *(D. Yang) Let (M, g) be an oriented Einstein 4-manifold with $Rc = g$ and sectional curvature $K \geqslant \frac{\sqrt{1249}-23}{120} \approx 0.102$, then (M, g) is isometric to (S^4, g_0) or $(\mathbb{CP}^2, g_{FS})$ up to rescaling.*

Under the same assumption of Theorem 2.5.4, Costa was able to improve the result by reducing the lower bound of K.

Theorem 2.5.5. *(Costa) Let (M, g) be an oriented Einstein 4-manifold with $Rc = g$ and sectional curvature $K \geqslant \frac{2-\sqrt{2}}{6} \approx 0.097$, then (M, g) is isometric to (S^4, g_0) or $(\mathbb{CP}^2, g_{FS})$ up to rescaling.*

In 2008 Bohm and Wilking proved the following result in [35].

Theorem 2.5.6. *(Bohm and Wilking) On a compact manifold the normalized Ricci flow evolves a Riemannian metric with 2-positive curvature operator to a limit metric with constant sectional curvature.*

As an application from the above theorem to Einstein 4-manifold, we get the following result.

Corollary 2.5.1. *Let (M, g) be an oriented Einstein 4-manifold with 2-positive curvature operator, then (M, g) is isometric to (S^4, g_0) up to rescaling.*

In 2013, X. Cao and P. Wu improved Corollary 2.5.1 by the following result in [12].

Theorem 2.5.7. *(Cao-Wu) Let (M, g) be an oriented Einstein 4-manifold with 3-positive curvature operator, then (M, g) is isometric to (S^4, g_0) or $(\mathbb{CP}^2, g_{FS})$ up to rescaling.*

Later they showed

Theorem 2.5.8. *Let (M, g) be an Einstein 4-manifold with 4-nonnegative curvature operator and positive intersection form, then (M, g) is isometric to $(S^2 \times S^2, g_0 \oplus g_0)$ up to rescaling.*

For n-dimensional Einstein manifolds ($n \geqslant 4$), we have the following results.

Theorem 2.5.9. *(Tachibana) Let (M, g) be an Einstein n-manifold with positive curvature operator, then (M, g) is isometric to space forms.*

In 2010, Brendle improved Tachibana's result by the following result.

Theorem 2.5.10. *(Brendle) Let (M, g) be an Einstein n-manifold with positive isotropic curvature, then (M, g) is isometric to space forms.*

2.6 Weighted Yamabe invariant for Einstein manifolds

In this section, we are going to introduce the main rigidity results for this chapter. Recall the weighted Yamabe invariant defined in Section 1.7, for Einstein manifolds, we have the simple forms.

Definition 2.6.1. *For n-dimensional Einstein manifolds, the weighted Yamabe invariant defined in* (1.54)*. We take* $m = 1$ *and* ϕ *to be constant we have the simplified weighted Yamabe invariant*

$$Y := inf \frac{(\int |\nabla u|^2 + \frac{n-1}{4n} R u^2)(\int u^{\frac{2n}{n-1}})^{\frac{2}{n}}}{\int u^{\frac{2(n+1)}{n-1}}}. \tag{2.13}$$

We are going to prove the following computational lemma first.

Lemma 2.6.1. *Under the assumption of Theorem 2.6.2,*

$$W_{ijkl}(2W_{ihlm}W_{hjkm} + \frac{1}{2}W_{ijhm}W_{klhm}) \leqslant C(n)|W|^3. \tag{2.14}$$

Proof. First, by $Cauchy - Schwarz$ inequality,

$$W_{ijkl}(2W_{ihlm}W_{hjkm} + \frac{1}{2}W_{ijhm}W_{klhm}) \leqslant \frac{5}{2}|W|^3$$

in any dimension.

In dimension $n = 4$, $C(4) = \frac{\sqrt{6}}{4}$. See [1, $Lemma$3.5].

In dimension $n = 5$, $W_{ijhm}W_{klhm} = 4W_{ihlm}W_{hjkm}$, see [61] so

$$W_{ijkl}(2W_{ihlm}W_{hjkm} + \frac{1}{2}W_{ijhm}W_{klhm}) = W_{ijkl}W_{ijhm}W_{klhm} \leqslant |W|^3.$$

□

Now we can prove the main theorem for this chapter.

Theorem 2.6.2. *Let (M^n, g) be an Einstein manifold with $Rc = g$ where $4 \leqslant n \leqslant 8$ and Y be the weighted Yamabe invariant defined as in* (2.14)*. If*

$$Y > 2C(n)\left\| |W| - RD(n) \right\|_{\frac{n+1}{2}} \left(vol(M) \right)^{\frac{2}{n(n+1)}},$$

where R is the scalar curvature, $C(4) = \frac{\sqrt{6}}{4}$, $C(5) = 1$, $C(n) = \frac{5}{2}$ for $6 \leqslant n \leqslant 8$ and $D(n) = \frac{9-n}{8nC(n)}$. Then (M^n, g) is isometric to S^n with round metric.

Proof.

$$\begin{aligned}
\frac{1}{2}\Delta|W|^2 &= |\nabla W|^2 + < W, \Delta W > \\
&= |\nabla W|^2 + W_{ijkl}W_{ijkl,mm} \\
&= |\nabla W|^2 + W_{ijkl}(W_{ijkm,lm} + W_{ijml,km}) \\
&= |\nabla W|^2 + 2W_{ijkl}W_{ijkm,lm} \\
&= |\nabla W|^2 + 2W_{ijkl}(W_{ijkm,ml} + W_{hjkm}R_{hilm} + W_{ihkm}R_{hjlm} + W_{ijhm}R_{hklm} + W_{ijkh}R_{hmlm}) \\
&= |\nabla W|^2 + 2W_{ijkl}(W_{hjkm}R_{hilm} + W_{ihkm}R_{hjlm} + W_{ijhm}R_{hklm} + W_{ijkh}R_{hmlm}) \\
&= |\nabla W|^2 + 2W_{ijkl}(W_{hjkm}W_{hilm} + W_{ihkm}W_{hjlm} + W_{ijhm}W_{hklm} + W_{ijkh}W_{hmlm}) \\
&\quad + \frac{2R}{n(n-1)}W_{ijkl}(W_{ljki} + W_{ilkj} + W_{ijlk}) + \frac{2R}{n}|W|^2 \\
&= |\nabla W|^2 + \frac{2R}{n}|W|^2 - 2W_{ijkl}(2W_{ihlm}W_{hjkm} + \frac{1}{2}W_{ijhm}W_{klhm}).
\end{aligned}$$

By *Kato's Inequality* and Lemma 2.2.1, we integrate over M and get

$$0 \geqslant \int |\nabla|W||^2 + \frac{2}{n}\int R|W|^2 - 2C(n)\int |W|^3$$

$$= \int |\nabla|W||^2 + \frac{n-1}{4n}\int R|W|^2 - 2C(n)\int |W|^3 + \frac{9-n}{4n}\int R|W|^2$$
$$= \int |\nabla|W||^2 + \frac{n-1}{4n}\int R|W|^2 - 2C(n)\int |W|^2(|W| - RD(n)).$$

Taking $u = |W|$ in the weighted Yamabe invariant, then

$$\begin{aligned}
0 &\geqslant (\int |\nabla|W||^2 + \frac{n-1}{4n}\int R|W|^2)(\int |W|^{\frac{2n}{n-1}})^{\frac{2}{n}}) \\
&\quad -2C(n)(\int |W|^2(|W| - RD(n)))(\int |W|^{\frac{2n}{n-1}})^{\frac{2}{n}}) \\
&\geqslant Y\int |W|^{\frac{2(n+1)}{n-1}} - 2C(n)(\int |W|^2(|W| - RD(n)))(\int |W|^{\frac{2n}{n-1}})^{\frac{2}{n}}). \qquad (1)
\end{aligned}$$

As long as we can show the right hand side of (1) > 0, we can conclude $|W| \equiv 0$.

By Holder inequality,

$$(\int |W|^{\frac{2n}{n-1}})^{\frac{2}{n}} \leqslant (\int |W|^{\frac{2(n+1)}{n-1}})^{\frac{2}{n+1}} vol(M)^{\frac{2}{n(n+1)}},$$

it suffices to show

$$Y vol(M)^{-\frac{2}{n(n+1)}}(\int |W|^{\frac{2(n+1)}{n-1}})^{\frac{n-1}{n+1}} > 2C(n)\int |W|^2(|W| - RD(n)). \qquad (2)$$

By Holder inequality again, we have

$$\int |W|^2(|W| - RD(n)) \leqslant (\int |W|^{\frac{2(n+1)}{n-1}})^{\frac{n-1}{n+1}} \| |W| - RD(n)\|_{\frac{n+1}{2}}.$$

By assumption, (2) is true.

Therefore, (M^n, g) is isometric to S^n with round metric. □

CHAPTER 3

RICCI SOLITON

The concept of Ricci soliton was introduced by Hamilton in mid 80's. Ricci solitons are natural generalizations of Einstein metrics. They also correspond to special solutions to Hamilton's Ricci flow and often arise as limits of dilations of singularities in the Ricci flow. They can be viewed as fixed points of the Ricci flow, as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms and scalings. Ricci solitons are of interests to physicists as well and are called quasi-Einstein metrics in physics literature (see, e.g., [40]). In this chapter we talk about some of the recent progress on Ricci solitons as well as the role they play in the singularity study of the Ricci flow. Also we will present a rigidity result to compact shrinking Ricci solitons.

3.1 Introduction

Recall that a Riemannian metric g_{ij} is Einstein if $R_{ij} = \lambda g_{ij}$ for some constant λ. Ricci soliton, introduced by Hamilton, are natural generalizations of Einstein metrics.

Definition 3.1.1. *A Riemannian metric g_{ij} on a smooth manifold M^n is called a Ricci soliton if there exists a smooth vector field $V = (V^i)$ such that the Ricci tensor R_{ij} of the* *metric g_{ij} satisfies the equation*

$$R_{ij} + \frac{1}{2}(\nabla_i V^j + \nabla_j V^i) = \lambda g_{ij}, \tag{3.1}$$

for some constant λ. Moreover, if V is a gradient vector field, then we have a gradient Ricci soliton satisfying the following equation

$$R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}, \tag{3.2}$$

for some smooth function f on M.

If $\lambda > 0$, the soliton is shrinking, if $\lambda < 0$ the soliton is expanding and if $\lambda = 0$ the soliton is steady. The function f is called a potential function of the Ricci soliton.

Since $\nabla_i V^j + \nabla_j V^i$ is the Lie derivative $L_V g_{ij}$ of the metric g in the direction of V, we also write the Ricci soliton equation (3.1) and (3.2) as

$$Rc + \frac{1}{2} L_V g = \lambda g \ \ \text{and} \ \ Rc + \nabla^2 f = \lambda g \tag{3.3}$$

respectively.

When the underlying manifold is a Kahler manifold, we have the corresponding notion of Kahler-Ricci solitons.

Definition 3.1.2. *A complete Kahler metric $g_{\alpha\bar{\beta}}$ on a Kahler manifold X^n of complex dimension n is called a Kahler-Ricci soliton if there exists a holomorphic vector field $V = (V^\alpha)$ on X such that the Ricci tensor $R_{\alpha\bar{\beta}}$ of the metric $g_{\alpha\bar{\beta}}$ satisfies the equation*

$$R_{\alpha\bar{\beta}} + \frac{1}{2}(\nabla_{\bar{\beta}} V_\alpha + \nabla_\alpha V_{\bar{\beta}}) = \lambda g_{\alpha\bar{\beta}} \tag{3.4}$$

for some real constant λ. It is called a gradient Kahler-Ricci soliton if the holomorphic vector field V comes from the gradient vector field of a real-valued function f on X^n so that

$$R_{\alpha\bar{\beta}} + \nabla_\alpha \nabla_{\bar{\beta}} f = \lambda g_{\alpha\bar{\beta}} \;\; \textit{and} \;\; \nabla_\alpha \nabla_\beta f = 0. \tag{3.5}$$

Again, if $\lambda > 0$, the soliton is shrinking, if $\lambda < 0$ the soliton is expanding and if $\lambda = 0$ the soliton is steady.

Note that the case $V = 0$ (i.e., f being a constant function) is an Einstein (or Kahler-Einstein) metric. Thus Ricci solitons are natural extensions of Einstein metrics. In fact, we will see in the next section that there are no non-Einstein compact steady or expanding Ricci solitons. Also, by a suitable scaling of the metric g, we can normalize $\lambda = 0, 1$ or -1.

3.2 Notation and Preliminaries

Before we continue on soliton equations, let's fix the notation.
Given an orthonormal basis $\{E_i\}_{i=1}^n$ of T_pM, we can construct an orthonormal frame about p such that $e_i(p) = E_i$ and $\nabla e_i|_p = 0$. Such a frame is called normal at p. Also e_{12} is the short notation for $e_1 \wedge e_2 \in \Lambda^2$, the space of two-forms.
The modified Laplacian is defined as

$$\Delta_f = \Delta - \nabla_{\nabla f}. \tag{3.6}$$

For any $(m,0)$-tensor T, its divergence operator is defined as

$$(\delta T)_{p_2\dots p_m} = \sum_i \nabla_i T_{ip_2\dots p_m}, \tag{3.7}$$

while its interior product by a vector field X is defined as

$$(i_X T)_{p_2\dots p_m} = T_{X_{p_2\dots p_m}}, \tag{3.8}$$

Furthermore, we will interchange the perspective of a vector and a covector freely, i.e., a $(2,0)$-tensor can also be considered as a $(1,1)$-tensor. Similarly, a $(4,0)$-tensor such as Rm, W can be seen as an operator on bi-vectors, that is, a map from $\Lambda^2(TM) \to \Lambda^2(TM)$. Therefore the norm of these operators is agreed to be sum of all eigenvalues squared. More precisely,

$$|W|^2 = \sum_{i<j;k<l} W_{ijkl}^2. \tag{3.9}$$

In addition, the norm of covariant derivative and divergence on these tensors can be defined as follows

$$|\nabla W|^2 = \sum_i \sum_{a<b;c<d} (\nabla_i W_{abcd})^2, \tag{3.10}$$

$$|\delta W|^2 = \sum_i \sum_{a<b} ((\delta W)_{iab})^2. \tag{3.11}$$

For a tensor $T : \Lambda^2(TM) \otimes \Lambda^2(TM) \to \mathbb{R}$, we define

$$\langle T, \delta W\rangle = \sum_{i<j;k} T_{ijk}(\delta W)_{kij}, \tag{3.12}$$

$$\langle T, i_X W\rangle = \sum_{i<j;k} T_{ijk}(i_X W)_{kij} \tag{3.13}$$

Now we are ready to introduce some of the computational formulas from the soliton equation, listed as the following lemma.

Lemma 3.2.1. *Let (M, g) be a gradient Ricci soliton with soliton equation $Rc + \nabla^2 f = \lambda g$, we have*

$$
\begin{aligned}
R + \Delta f &= n\lambda, && (3.14)\\
\frac{1}{2}\nabla_i R &= \nabla^j R_{ij} = R_{ij}\nabla^j f, && (3.15)\\
Rc(\nabla f) &= \frac{1}{2}\nabla R, && (3.16)\\
R + |\nabla f|^2 - 2\lambda f &= constant, && (3.17)\\
\Delta R + 2|Rc|^2 &= \langle \nabla f, \nabla R\rangle + 2\lambda R. && (3.18)
\end{aligned}
$$

Remark 3.2.1. *If $\lambda \geqslant 0$, then $R \geqslant 0$ by the maximum principle and equation* (3.18). *Moreover, a complete gradient Ricci soliton has positive scalar curvature unless it is isometric to the flat Euclidean space.*

Proof. (3.14)-(3.16) are just straight forward calculation from the soliton equation. For (3.17), suppose that $R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}$. Taking the covariant derivatives and using the commutating formula for covariant derivatives, we obtain

$$\nabla_i R_{jk} - \nabla_j R_{ik} + R_{ijkl}\nabla_l f = 0. \tag{3.19}$$

Taking the trace on j and k, and using the contracted second Bianchi identity (3.15) we get

$$\nabla_i(R + |\nabla f|^2 - 2\lambda f) = 2(R_{ij} + \nabla_i\nabla_j f - \lambda g_{ij})\nabla_j f = 0.$$

Therefore

$$R + |\nabla f|^2 - 2\lambda f = const$$

for some constant.

For (3.18) we just take the Laplacian to both sides of (3.17). □

Before we move further to the geometry of Ricci solitons, we'd like to take a look at a few examples. The following two propositions state that there are no non-Einstein compact steady or expanding Ricci solitons.

Proposition 3.2.1. *(Hamilton, Ivey) On a compact manifold M^n, a gradient steady or expanding Ricci soliton is Einstein.*

Proof. Taking the difference of (3.14) and (3.17) we get

$$\Delta f - |\nabla f|^2 + 2\lambda f = n\lambda - C.$$

When M is compact and $\lambda \leqslant 0$, it follows from the maximum principle that f must be a constant and hence (M, g) is Einstein. □

More generally, we have

Proposition 3.2.2. *On a compact manifold M^n, a steady or expanding Ricci soliton is Einstein.*

Proof. It follows from the above proposition and Perelman's result in [19] that any compact Ricci soliton is necessarily a gradient soliton. □

For compact shrinking Ricci solitons in low dimensions we have

Proposition 3.2.3. *(Hamilton, Ivey) In dimension $n \leqslant 3$, there are no compact shrinking Ricci solitons other than those of constant positive curvature.*

These propositions give us a guideline of what types of non-Einstein Ricci soliton we may look for.

When $n \geqslant 4$, there exist nontrivial compact gradient shrinking solitons. Also, there exist complete non-compact Ricci solitons (steady, shrinking and expanding) that are not Einstein. Below we list a number of such examples. It turns out that most of the examples are gradient, and all the known examples of nontrivial shrinking solitons so far are Kahler.

Example 3.2.1. *(Compact shrinking solitons) For real dimension 4, the first example of a compact shrinking soliton was constructed by Koiso* [62] *on compact complex surface* $\mathbb{CP}^2\#(-\mathbb{CP}^2)$ *where* $-\mathbb{CP}^2$ *means* $\mathbb{CP}^2$ *with the opposite orientation. This is a gradient Kahler-Ricci soliton with* $U(2)$*-symmetry and positive Ricci curvature. More generally, they found* $U(n)$*-invariant Kahler-Ricci solitons on twisted projective line bundle over* $\mathbb{CP}^{n-1}$ *for* $n \geqslant 2$*. Moreover, in* [66] *Wang-Zhu found a gradient Kahler-Ricci soliton on* $\mathbb{CP}^2\#(-2\mathbb{CP}^2)$ *with* $U(1) \times U(1)$ *symmetry.*

Example 3.2.2. *(Non-compact shrinking solitons) In* [67] *Feldman-Ilmanen-Knopf found the first complete non-compact* $U(n)$*-invariant gradient shrinking Kahler-Ricci solitons, which are cone-like at infinity. It has positive scalar curvature but the Ricci curvature doesn't have a fixed sign.*

Example 3.2.3. *(Non-compact steady solitons in dimension 2) In dimension two, Hamilton in* [44] *discovered the first example of a complete non-compact steady Ricci soliton on* $\mathbb{R}^2$*, called the cigar soliton, where the metric is given by*

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

with potential function

$$f = -log(1 + x^2 + y^2).$$

The cigar soliton has positive Gaussian curvature and linear volume growth, and is asymptotic to a cylinder of finite circumference at infinity.

Example 3.2.4. *(Non-compact steady solitons in high dimensions) Higher dimensional examples of non-compact gradient steady solitons were found by Robert Bryant in* [65] *on* $\mathbb{R}^n$*. They are rotationally symmetric and have positive sectional curvature. Furthermore, the volume of geodesic balls* $B_r(0)$ *grow on the order of* $r^{\frac{n+1}{2}}$*.*

Example 3.2.5. *(Non-compact expanding solitons) In* [71]*, H-D Cao constructed a one-parameter family of complete non-compact expanding solitons on* $\mathbb{C}^n$*. These expanding solitons all have* $U(n)$*-symmetry and positive sectional curvature, and are cone-like at infinity.*
More examples of complete non-compact expanding solitons were found by Feldman-Ilmanen-Knopf [67] *on* $\mathbb{C}^n/\mathbb{Z}_k$*,* $k = n + 1,\ n + 2,\ ...$

Example 3.2.6. *(Warped products) Using doubly warped product and multiple warped product constructions, Ivey and Dancer-Wang* [63] *produced non-compact gradient steady solitons, which generalize the construction of Bryant's soliton Also, Gastel-Kronz* [64] *produced a two-parameter family of gradient expanding solitons on* $\mathbb{R}^{m+1} \times N$ *where* N^n $(n \geqslant 2)$ *is an Einstein manifold with positive scalar curvature.*

We conclude examples of Ricci solitons with

Example 3.2.7. *(Gaussian solitons)* $(\mathbb{R}^n, g_0)$ *with the flat Euclidean metric can be also* *equipped with both shrinking and expanding gradient Ricci solitons, called the Gaussian shrinker or expander.*

(a) $(\mathbb{R}^n, g_0, \frac{|x|^2}{4})$ *is a gradient shrinker with potential function* $f = \frac{|x|^2}{4}$*:*

$$Rc + \nabla^2 f = \frac{1}{2} g_0.$$

(b) $(\mathbb{R}^n, g_0, -\frac{|x|^2}{4})$ *is a gradient shrinker with potential function* $f = -\frac{|x|^2}{4}$*:*

$$Rc + \nabla^2 f = -\frac{1}{2} g_0.$$

3.3 Variational structures

In this section we focus on Perelman's $\mathcal{F}$-functional and $\mathcal{W}$-functional and the associated λ-energy and ν-energy respectively. The critical points of the λ-energy (respectively ν-energy) are precisely given by compact gradient steady (respectively, shrinking) solitons.

Definition 3.3.1. *($\mathcal{F}$-functional and λ-energy) In* [19] *Perelman considered the functional*

$$\mathcal{F}(g_{ij}, f) = \int_M (R + |\nabla f|^2) e^{-f} dvol \tag{3.20}$$

defined on the space of Riemannian metrics and smooth functions on M. Here again R is the scalar curvature and f is a smooth function on M^n. Note that when $f = 0$, $\mathcal{F}$ is simply a total scalar curvature of g.

In [19] Perelman introduced the following variation formulas.

Lemma 3.3.1. *(First variation formula of $\mathcal{F}$-functional) If $v_{ij} = \delta g_{ij}$ and $\phi = \delta f$ are variations of g_{ij} and f respectively, then the first variation of $\mathcal{F}$ is given by*

$$\delta\mathcal{F}(v_{ij}, \phi) = \int_M \Big[- v_{ij}(R_{ij} + \nabla_i\nabla_j f) + (\frac{v}{2} - \phi)(2\Delta f - |\nabla f|^2 + R)\Big] e^{-f} dvol \qquad (3.21)$$

where $v = g^{ij} v_{ij}$.

Next we consider the associated energy

$$\lambda(g_{ij}) = inf\{\mathcal{F}(g_{ij}, f) : f \in C^\infty(M), \int_M e^{-f} dvol = 1\}. \qquad (3.22)$$

Clearly, $\lambda(g_{ij})$ is invariant under diffeomorphisms. If we set $u = e^{-f/2}$, then the functional $\mathcal{F}$ can be written as

$$\mathcal{F} = \int_M (Ru^2 + 4|\nabla u|^2) dvol. \qquad (3.23)$$

Thus

$$\lambda(g_{ij}) = inf\{\int_M (Ru^2 + 4|\nabla u|^2) dvol : \int_M u^2 = 1\}, \qquad (3.24)$$

the first eigenvalue of the operator $-4\Delta + R$. Let $u_0 > 0$ be a first eigenfunction of the operator $-4\Delta + R$ so that

$$-4\Delta u_0 + Ru_0 = \lambda(g_{ij}) u_0.$$

Then $f_0 = -2log\ u_0$ is a minimizer of $\lambda(g_{ij})$:

$$\lambda(g_{ij}) = \mathcal{F}(g_{ij}, f_0).$$

Note that f_0 satisfies the equation

$$-2\Delta f_0 + |\nabla f_0|^2 - R = \lambda(g_{ij}). \tag{3.25}$$

For any symmetric 2-tensor $h = h_{ij}$, consider the variation $g_{ij}(s) = g_{ij} + sh_{ij}$. It is an easy consequence of Lemma 3.3.1 and (3.25) that the variation $D_g\lambda(h)$ of $\lambda(g_{ij})$ is given by

$$\frac{d}{ds}\bigg|_{s=0} \lambda(g_{ij}(s)) = \int -h_{ij}(R_{ij} + \nabla_i\nabla_j f)e^{-f}dvol, \tag{3.26}$$

where f is the minimizer of $\lambda(g_{ij})$. In particular, the critical points of λ are steady gradient Ricci solitons.

Note that by diffeomorphism invariance of λ, $D_g\lambda$ vanishes on any Lie derivative $h_{ij} = \frac{1}{2}L_V g_{ij}$. By inserting $h = -2(Rc + \nabla^2 f)$ in (3.26) we have the following results from Perelman.

Proposition 3.3.1. *Suppose that $g_{ij}(t)$ is a solution to the Ricci flow on a compact manifold M^n. Then $\lambda(g_{ij}(t))$ is non-decreasing in t and the monotonicity is strict unless we are on a steady gradient soliton. In particular, a compact steady Ricci soliton is necessarily a gradient soliton.*

We remark that by considering the quantity

$$\bar{\lambda}(g_{ij}) = \lambda(g_{ij})(vol(g_{ij}))^{\frac{2}{n}},$$

which is a scale invariant version of $\lambda(g_{ij})$. Perelman also showed the following proposition.

Proposition 3.3.2. *$\bar{\lambda}(g_{ij})$ is non-decreasing along the Ricci flow whenever it is non-positive; moreover, the monotonicity is strict unless we are on a gradient expanding soliton. In particular, any compact expanding Ricci soliton is necessarily a gradient soliton.*

Definition 3.3.2. *($\mathcal{W}$-functional and ν-energy) To study the shrinking Ricci solitons, Perelman introduced the $\mathcal{W}$-functional*

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M \Big[\tau(R + |\nabla f|^2) + f - n\Big](4\pi\tau)^{\frac{n}{2}} e^{-f} dvol, \tag{3.27}$$

where g_{ij} is a Riemannian metric, f a smooth function on M^n, and τ a positive scale parameter. Clearly the functional $\mathcal{W}$ is invariant under simultaneous scaling of τ and g_{ij}, and invariant under diffeomorphisms.

In the same paper Perelman also derived the following variation formula.

Lemma 3.3.2. *(First variation of $\mathcal{W}$-functional) If $v_{ij} = \delta g_{ij}$, $\phi = \delta f$ and $\eta = \delta\tau$, then*

$$\begin{aligned} \delta\mathcal{W}(v_{ij}, \phi, \eta) &= \int_M -\tau(R_{ij} + \nabla_i f \nabla_j f - \frac{1}{2\tau} g_{ij})(4\pi\tau)^{-\frac{n}{2}} e^{-f} dvol \\ &+ \int_M (\frac{v}{2} - \phi - \frac{n}{2\tau}\eta)\Big[\tau(R + 2\Delta f - |\nabla f|^2) + f - n - 1\Big](4\pi\tau)^{-\frac{n}{2}} e^{-f} dvol \\ &+ \int_M \eta(R + |\nabla f|^2 - \frac{n}{2\tau}\eta)(4\pi\tau)^{-\frac{n}{2}} e^{-f} dvol \end{aligned} \tag{3.28}$$

Similar to the λ-entropy, we can consider

$$\nu(g_{ij}, \tau) = inf\{\mathcal{W}(g_{ij}, f, \tau) : f \in C^\infty(M), \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dvol = 1\}. \tag{3.29}$$

Note that if we let $u = e^{-f/2}$, then the functional $\mathcal{W}$ can be expressed as

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M \Big[\tau(Ru^2 + 4|\nabla u|^2) - u^2 logu^2 - nu^2\Big](4\pi\tau)^{-\frac{n}{2}} dvol, \tag{3.30}$$

and the constraint $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dvol = 1$ becomes $\int_M u^2 (4\pi\tau)^{-\frac{n}{2}} dvol = 1$. Therefore $\nu(g_{ij}, \tau)$ corresponds to the best constant of a logarithmic Sobolev inequality.

Since the nonquadratic term is subcritical (in view of Sobolev exponent), it is rather straightforward to show that $\nu(g_{ij}, \tau)$ is achieved by some nonnegative function $u \in H^1(M)$ which satisfies the Euler-Lagrange equation

$$\tau(-4\Delta u + Ru) - 2ulog\, u - nu = \nu(g_{ij}, \tau)u. \tag{3.31}$$

One can further show that the minimizer ν is positive and smooth. This is equivalent to say that $\nu(g_{ij}, \tau)$ is achieved by some minimizer f satisfying the nonlinear equation

$$\tau(2\Delta f - |\nabla f|^2 + R) + f - n = \nu(g_{ij}, \tau). \tag{3.32}$$

Proposition 3.3.3. *(Perelman* [19]*) Suppose $g_{ij}(t)$, $0 \leqslant t < T$ is a solution to the Ricci flow on a compact manifold M^n. Then $\nu(g_{ij}(t), T - t)$ is nondecreasing in t; moreover, the monotonicity is strict unless we are on a shrinking gradient soliton. In particular, any compact shrinking Ricci soliton is necessarily a gradient soliton.*

Remark 3.3.1. *Nader in* [58] *showed that if (M^n, g) is a complete non-compact shrinking Ricci soliton with bounded curvature $|Rm| < C$ with respect to some smooth vector field V, then there exists a smooth function f on M such that (M^n, g) is a gradient soliton with f as its potential function.*

3.4 Ricci solitons and Ricci flow

In Section 1.7 we knew that Ricci soliton was a special solution to the Ricci flow. To better understand the Ricci flow, we introduce the singularity models to the Ricci flow.

Definition 3.4.1. *The maximal solution to the Ricci flow is defines as the solution to the Ricci flow that exists on a maximal time interval* $[0, T)$ *where* $T \leqslant \infty$

Definition 3.4.2. *Under the Ricci flow* (1.35), *Hamilton defined*

$$K_{max}(t) = \sup_{x \in M} |Rm(x,t)|_{g(t)} \tag{3.33}$$

According to Hamilton, one can classify maximal solutions into three types; every maximal solution is clearly of one and only one of the following three types:

$$\begin{aligned}
&\textbf{\textit{Type I:}} && T < +\infty \quad \textit{and} \quad sup\,(T-t)K_{max}(t) < +\infty; \\
&\textbf{\textit{Type II(a):}} && T < +\infty \quad \textit{but} \quad sup\,(T-t)K_{max}(t) = +\infty; \\
&\textbf{\textit{Type II(b):}} && T = +\infty \quad \textit{but} \quad sup\; tK_{max}(t) = +\infty; \\
&\textbf{\textit{Type III:}} && T = +\infty \quad \textit{and} \quad sup\; tK_{max}(t) < +\infty
\end{aligned} \tag{3.34}$$

For each type of the maximal solution, Hamilton defined a corresponding type of limiting singularity model.

Definition 3.4.3. *A solution* $g_{ij}(x,t)$ *to the Ricci flow on the manifold* M, *where either* M *is compact or at each time* t *the metric* $g_{ij}(\cdot,t)$ *is complete and has bounded curvature, is* *called a singularity model if it is not flat and of one of the following three types:*

Type I: *The solution exists for $t \in (-\infty, \Omega)$ for some Ω with $0 < \Omega < \infty$ and*

$$|Rm| \leqslant \frac{\Omega}{\Omega - t}$$

everywhere with equality somewhere at $t = 0$;

Type II: *The solution exists for $t \in (-\infty, +\infty)$ and*

$$|Rm| \leqslant 1$$

everywhere with equality somewhere at $t = 0$;

Type III: *The solution exists for $t \in (-A, +\infty)$ for some constant A with $0 < A < \infty$ and*

$$|Rm| \leqslant \frac{A}{A + t}$$

everywhere with equality somewhere at $t = 0$.

There are a few known rigidity results about singularity model worth listing.

Theorem 3.4.1. *(Hamilton [68]) For any complete maximal solution to the Ricci flow with bounded and non-negative curvature operator on a Riemannian manifold, or on a Kahler manifold with bounded and nonnegative holomorphic bisectional curvature, there exists a sequence of dilations which converges to a singular model.*
For $Type\ I$ solutions: the limit model exists for $t \in (-\infty, \Omega)$ with $0 < \Omega < +\infty$ and has

$$R \leqslant \frac{\Omega}{\Omega - t}$$

everywhere with equality somewhere at $t = 0$;
For $Type\ II$ solutions: the limit model exists for $t \in (-\infty, +\infty)$ and has

$$R \leqslant 1$$

everywhere with equality somewhere at $t = 0$*;*

For $Type\ III$ *solutions: the limit model exists for* $t \in (-A, +\infty)$ *with* $0 < A < +\infty$ *and has*

$$R \leqslant \frac{A}{A + t}$$

everywhere with equality somewhere at $t = 0$.

For $Type\ II$ or $Type\ III$ singularities with non-negative curvature we have the following results.

Theorem 3.4.2. *(Hamilton* [46]*) Any* $Type\ II$ *singularity model of the Ricci flow with nonnegative curvature operator and positive Ricci curvature must be a steady Ricci soliton.*

Theorem 3.4.3. *(H-D Cao* [71]*)*

(a) Any $Type\ II$ *singularity model on a Kahler manifold with non-negative holomorphic bisectional curvature and positive Ricci curvature must be a steady Kahler-Ricci soliton;*

(b) Any $Type\ III$ *singularity model on a Kahler manifold with non-negative holomorphic bisectional curvature and positive Ricci curvature must be a shrinking Kahler-Ricci soliton.*

Theorem 3.4.4. *(Chen-Zhu* [45]*) Any* $Type\ III$ *singularity model of the Ricci flow with non-negative curvature operator and positive Ricci curvature must be an expanding Ricci soliton.*

In [51] N. Sesum was able to show the following result.

Theorem 3.4.5. *Let M be a smooth, compact n-dimensional Riemannian manifold* $(n \geqslant 3)$ *and $g(\cdot, t)$ be a solution to the Ricci flow. Assume there is a constant C so that* $\sup_M |R(\cdot, t)| \leqslant C$ *for all $t \in [0, T)$ and $T < \infty$. Assume that at T we have a Type II singularity and the norm of the curvature operator blows up. Then by suitable rescalings of our metrics, we get a Gaussian shrinker in the limit.*

This result is later improved by X. Cao in [14].

Theorem 3.4.6. *(X. Cao) Let $(M, g(t))$, $t \in [0, T)$ be a maximal solution to the Ricci flow with positive scalar curvature. Then we have one of the following:*

(a) either $\limsup_{[0,T)} R = \infty$,

(b) or $\limsup_{[0,T)} R < \infty$, *then* $\limsup_{[0,T)} \frac{|W|}{R} = \infty$. *This must be a Type II maximal solution, furthermore, the dilation limit must be a complete Ricci-flat solution with* $\max |W| = 1$.

3.5 Geometry of gradient Ricci solitons

In this section we introduce some known rigidity results to gradient Ricci solitons. We start from gradient steady and expanding solitons.

Definition 3.5.1. *Given any positive constant $\kappa > 0$ and $r > 0$, we say a solution to the Ricci flow is κ-noncollapsed at (x_0, t_0) on the scale r if it satisfies the following property: if $|Rm|(x,t) \leqslant r^{-2}$ for all $(x,t) \in B_{t_0}(x_0, r) \times [t_0 - r^2, t_0]$, then*

$$vol_{t_0}(B_{t_0}(x_0, r)) \geqslant \kappa r^n. \tag{3.35}$$

Theorem 3.5.1. *(Hamilton [68]) Suppose we have a complete non-compact gradient steady Ricci soliton (M^n, g_{ij}) so that*

$$R_{ij} = \nabla_i \nabla_j f$$

for some potential function f on M. Assume the Ricci curvature operator is positive, and the scalar curvature R attains its maximum R_{max} at a point $x_0 \in M^n$. Then

$$|\nabla f|^2 + R = R_{max}$$

everywhere on M^n. Furthermore, the function f is convex and attains its minimum at x_0.

In case of a complete gradient expanding soliton with nonnegative Ricci curvature, the potential function f is a convex exhaustion function of quadratic growth. Hence we have

Theorem 3.5.2. *(Hamilton [68]) Let (M^n, g, f) be a gradient expanding soliton with $Rc \geqslant 0$. Then M^n is diffeomorphic to $\mathbb{R}^n$.*

In the Kahler setting we have the following result.

Theorem 3.5.3. *(Bryant [69] and Chau-Tam [70]) Suppose we have a complete non-compact gradient steady Kahler-Ricci soliton $(X^n, g_{i\bar{j}})$. Assume Ricci curvature is positive*

$Rc > 0$, and the scalar curvature R attains its maximum R_{max} at a point $x_0 \in X^n$. Then X^n is biholomorphic to the complex Euclidean space $\mathbb{C}^n$.

Theorem 3.5.4. *(Chau-Tam [70]) Let (X^n, g_{ij}) be a complete noncompact gradient expanding Kahler-Ricci soliton with non-negative Ricci curvature, then X^n is biholomorphic to $\mathbb{C}^n$.*

Theorem 3.5.5. *(Hamilton [44]) The only complete steady Ricci soliton on a two-dimensional manifold with bounded (scalar) curvature R which attains its maximum $R_{max} = 1$ at an origin is the cigar soliton on the plane $\mathbb{R}^2$ with the metric*

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

For $n \geqslant 3$, Perelman claimed that any complete non-compact κ-noncollapsed gradient steady soliton with bounded positive curvature must be the Bryant soliton. He also conjectured that any complete non-compact three-dimensional κ-noncollapsed ancient solution with bounded positive curvature is necessarily a Bryant soliton.

Now we are going to describe recent progress on gradient shrinking Ricci solitons.

Theorem 3.5.6. *(Hamilton [43]) A complete shrinking Ricci soliton with bounded and non-negative curvature operator either has positive curvature operator everywhere or its universal cover splits as a product $N \times \mathbb{R}^k$, where $k \geqslant 1$ and N is a shrinking soliton with positive curvature operator.*

On the other hand, Hamilton ($n = 3, 4$) and Bohm-Wilking ($n \geqslant 5$) have the following result.

Theorem 3.5.7. *(Hamilton* [42], [43] *and Bohm-Wilking* [35]*) Compact shrinking solitons with positive curvature operator are isometric to finite quotients of round spheres.*

For dimension $n = 3$, Perelman proved the following

Theorem 3.5.8. *(Perelman* [19]*) There does not exist a three-dimensional complete non-compact κ-noncollapsed gradient shrinking soliton with bounded and positive sectional curvature.*

In other words, a three-dimensional complete κ-noncollapsed gradient shrinking soliton with bounded and positive sectional curvature must be compact.
Based on the above proposition, Perelman obtained the following important classification result.

Theorem 3.5.9. *(Perelman* [19]*) Let $g(t)$ be a non-flat gradient shrinking soliton to the Ricci flow on a three-manifold M^3. Suppose $g(t)$ has bounded and nonnegative sectional curvature and is κ-noncollapsed on all scales for some $\kappa > 0$. Then $(M, g(t))$ is one of the following:*

(a) the round three-sphere $\mathbb{S}^3$, or one of its metric quotients;

(b) the round infinite cylinder $\mathbb{S}^2 \times \mathbb{R}$, or its $\mathbb{Z}^2$ quotient.

Therefore the only three-dimensional complete non-compact κ-noncollapsed gradient shrinking soliton with bounded and nonnegative sectional curvature are either $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{R}^3$.

In the past decades of years, there has been a lot of attempts to improve and generalize the above results of Perelman. We list the following results.

Theorem 3.5.10. *(Ni-Wallach for dimension 3* [72]*) Any 3-dimensional complete noncompact non-flat gradient shrinking soliton with nonnegative Ricci curvature $Rc \geqslant 0$ and with $|Rm|(x) \leqslant Ce^{ar(x)}$ must be a quotient of the round cylinder $\mathbb{S}^2 \times \mathbb{R}$.*

This result is improved by the following results.

Theorem 3.5.11. *(Cao-Chen-Zhu* [73]*) Let (M^3, g) be a 3-dimensional complete noncompact non-flat shrinking gradient soliton. Then (M^3, g) is a quotient of the round cylinder $\mathbb{S}^2 \times \mathbb{R}$.*

Theorem 3.5.12. *(Naber* [58]*) Any 4-dimensional complete noncompact shrinking Ricci soliton with bounded and nonnegative curvature operator is isometric to either $\mathbb{R}^4$, or a finite quotient of $\mathbb{S}^3 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}^2$.*

For high dimensions, we have the following results.

Theorem 3.5.13. *(Ni-Wallach in high dimensions* [72]*) Let (M^n, g) be a complete, locally conformally flat gradient shrinking soliton with nonnegative Ricci curvature. Assume that*

$$|Rm|(x) \leqslant Ce^{a(r(x)+1)}$$

for some constant $a > 0$, where $r(x)$ is the distance function to some origin. Then its universal cover is $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{R}$.

Theorem 3.5.14. *(Petersen-Wylie) Let (M^n, g) be a complete gradient shrinking Ricci soliton with potential function f. Assume the Weyl tensor $W = 0$ and*

$$\int_M |Rc|^2 e^{-f} dvol < \infty,$$

then (M^n, g) is a finite quotient of $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{R}$.

In the meanwhile, Z-H. Zhang improved Theorem 3.5.13 and 3.5.14 by removing all curvature bound assumptions.

Theorem 3.5.15. *(Z-H Zhang) Any complete gradient shrinking soliton with vanishing Weyl tensor must be a finite quotients of $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{R}$.*

Finally we'd like to introduce the Weitzenbock formula by Cao and Tran in [5], which is a new but very powerful tool to study the geometry of high dimensional Ricci solitons.

Theorem 3.5.16. *Let (M, g, f, λ) be a four-dimensional gradient Ricci soliton, then we have the following Bochner-Weitzenbock formula:*

$$\begin{aligned} \Delta_f |W^+|^2 &= 2|\nabla W^+|^2 + 4\lambda |W^+|^2 - 36 det\ W^+ - \langle Rc \circ Rc, W^+ \rangle \\ &= 2|\nabla W^+|^2 + 4\lambda |W^+|^2 - 36 det\ W^+ - \langle Hessf \circ Hessf, W^+ \rangle \end{aligned} \tag{3.36}$$

In [9], J-Y Wu, P. Wu and W. Wylie was able to improve Z-H Zhang's result via using Theorem 3.5.16.

Theorem 3.5.17. *A four-dimensional gradient shrinking Ricci soliton with $\delta W^{\pm} = 0$ is either Einstein, or a finite quotient of $\mathbb{R}^4$, $\mathbb{S}^3 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}^2$.*

3.6 Main results

In this section we are going to present the main result for this chapter. Recall the definition of weighted Yamabe invariant defined in Section 1.7. For this section we take $m = 1$ and $\phi = f$ which is the potential function of the soliton equation in (1.54).

Definition 3.6.1. *Let $\widetilde{R} = R + 2\Delta f - 2|\nabla f|^2$ be the weighted scalar curvature and*

$$Y = \inf_{u>0} \frac{[\int |\nabla u|^2 e^{-f} + \frac{n-1}{4n}\int (R + 2\Delta f - 2|\nabla f|^2)u^2 e^{-f}](\int u^{\frac{2n}{n-1}})^{\frac{2}{n}}}{\int u^{\frac{2(n+1)}{n-1}} e^{-f}}$$

be the weighted Yamabe invariant for the n-dimensional gradient shrinking Ricci soliton (M^n, g) with soliton equation $Rc + \nabla^2 f = g$. The weighted conformal Laplacian is defined as $L_f^m := -\Delta_f + \frac{m+n-2}{4(m+n-1)} R_f^m$ where R_f^m is defined in (1.51).

Theorem 3.6.1. *For n-dimensional compact gradient shrinking Ricci solitons (M^n, g) where n =4, 5, 7, 8 with soliton equation $Rc + \nabla^2 f = g$, if*

$$Y > \sqrt{\frac{(n-2)(n-1)^3}{32}} \left\| |W| + \sqrt{\frac{8}{n(n-2)}}|E| - k(n)|\nabla f|^2 \right\|_{\frac{n+1}{2}} \left(\int e^{(n+1)f} \right)^{\frac{2}{n(n+1)}},$$

where $k(n) = \frac{2(n^3-4n^2+5n-18)\sqrt{n-1}}{n^2(n^2-2n-7)\sqrt{2(n-2)}}$ and all integrals are with respect to the measure $d\widetilde{\mu} = e^{-f}d\mu$. Then (M^n, g) is isometric to the round sphere (S^n, g_{round}).

To prove this theorem, we need the following lemmas.

Lemma 3.6.2. *Let (M^n, g) be a gradient Ricci soliton with soliton equation $Rc + \nabla^2 f = g$, then*

$$\frac{1}{2}\Delta_f|E|^2 = |\nabla E|^2 + 2|E|^2 - 2W_{ijkl}E_{ik}E_{jl} + \frac{4}{n-2}E_{ij}E_{jk}E_{ik} - \frac{2(n-2)}{n(n-1)}R|E|^2$$

where E denotes the traceless Ricci tensor.

Proof. See computation from [2]. □

Lemma 3.6.3. *For every $n-$dimensional Riemannian manifold,*

$$\left| - W_{ijkl}E_{ik}E_{jl} + \frac{2}{n-2}E_{ij}E_{jk}E_{ik}\right| \leqslant \sqrt{\frac{n-2}{2(n-1)}}\left(|W|^2 + \frac{8}{n(n-2)}|E|^2\right)^{\frac{1}{2}}|E|^2.$$

Proof. First of all we have

$$(E \circ g)_{ijkl} = E_{ik}g_{jl} - E_{il}g_{jk} + E_{jl}g_{ik} - E_{jk}g_{il}$$

$$(E \circ E)_{ijkl} = 2(E_{ik}E_{jl} - E_{il}E_{jk}).$$

By some simple computation we have

$$W_{ijkl}E_{ik}E_{jl} = \frac{1}{4}W_{ijkl}(E \circ E)_{ijkl}$$

$$E_{ij}E_{jk}E_{ik} = -\frac{1}{8}(E \circ g)_{ijkl}(E \circ E)_{ijkl}.$$

Therefore we get the following identity

$$-W_{ijkl}E_{ik}E_{jl} + \frac{2}{n-2}E_{ij}E_{jk}E_{ik} = -\frac{1}{4}\left(W + \frac{1}{n-2}E \circ g\right)_{ijkl}(E \circ E)_{ijkl}.$$

Now that $E \circ E$ has the same symmetries as the Riemann tensor, it can be decomposed orthogonally as

$$E \circ E = T \oplus V \oplus U$$

where T is trace-free and

$$V_{ijkl} = -\frac{2}{n-2}(E^2 \circ g)_{ijkl} + \frac{2}{n(n-2)}|E|^2(E \circ E)_{ijkl}$$

$$U_{ijkl} = -\frac{1}{n(n-1)}|E|^2(g \circ g)_{ijkl},$$

where $(E^2)_{ij} = E_{ip}E_{jp}$. Taking the squared norm we obtain

$$|E \circ E|^2 = 8|E|^4 - 8|E^2|^2$$

$$|V|^2 = \frac{16}{n-2}|E^2|^2 - \frac{16}{n(n-2)}|E|^4$$

$$|U|^2 = \frac{8}{n(n-1)}|E|^4.$$

In particular, we have

$$|T|^2 + \frac{n}{2}|V|^2 = |E \circ E|^2 + \frac{n-2}{2}|V|^2 - |U|^2 = \frac{8(n-2)}{n-1}|E|^4.$$

Now use the fact that W and T are trace-free and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&\left|\left(W + \frac{1}{n-2}E \circ g\right)_{ijkl}(E \circ E)_{ijkl}\right|^2 \\
&= \left|\left(W + \frac{1}{n-2}E \circ g\right)_{ijkl}(T + V)_{ijkl}\right|^2 \\
&= \left|\left(W + \frac{\sqrt{2}}{\sqrt{n}(n-2)}E \circ g\right)_{ijkl}(T + \sqrt{\frac{n}{2}}V)_{ijkl}\right|^2 \\
&\leqslant \left|\left(W + \frac{\sqrt{2}}{\sqrt{n}(n-2)}E \circ g\right)\right|^2\left(|T|^2 + \frac{n}{2}|V|^2\right) \\
&= \frac{8(n-2)}{n-1}\left(|W|^2 + \frac{8}{n(n-2)}|E|^2\right)|E|^4.
\end{aligned}$$

This concludes the proof. □

To make further computation, we need that the weighted Yamabe invariant is positive, which is guaranteed by the follow lemma.

Lemma 3.6.4. *The Yamabe invariant is always positive if* (M^n, g) *is a compact shrinking Ricci soliton.*

Proof. Since M^n is a compact shrinking Ricci soliton, we have $R > 0$.
Therefore the first eigenvalue of $-\Delta + \frac{R}{4}$ is positive.
Hence the first eigenvalue of the weighted Laplacian when $m = \infty$ is positive.
Suppose to the contrary that the $m = 1$ weighted Yamabe constant is nonpositive.
By [4. Proposition 3.5], the first eigenvalue of $m = 1$ weighted conformal Laplacian $\lambda_1(L_f^1)$ has the same sign as the weighted Yamabe invariant, therefore it is nonpositive.
By considering [4. (3.4)] that

$$\frac{(m+k+n-1)(m+n-2)}{(m+k+n-2)(m+n-1)}(L_f^{m+k}w, w) \leqslant (L_f^m w, w),$$

for all $w \in W^{1,2}(M)$. We can conclude that the first eigenvalue of the $m = k$ weighted conformal Laplacian is nonpositive, $\forall k \in N$.
Thus, there is a smooth function u such that

$$(L_f^m u, u) < 0$$

for all $m > 1$, where the left-hand side denotes the L^2-inner product with respect to the measure $v^m dvol = e^{-\phi} dvol$.
Taking the limit as m tends to infinity yields $(L_f^\infty u, u) < 0$, contradicting the fact that the first eigenvalue of the $m = \infty$ weighted conformal Laplacian is positive. □

Now we are going to introduce the last computational lemma before we prove the main theorem.

Lemma 3.6.5. *Under the assumption of Theorem 3.6.1, (M^n, g) is Einstein.*

Proof. Since M is compact, we have

$$\int \Delta_f |E|^2 e^{-f} = 0.$$

By Lemma 3.6.2 and Lemma 3.6.3, we have

$$0 \geqslant \int |\nabla E|^2 e^{-f} + 2\int |E|^2 e^{-f} - \sqrt{\frac{2(n-2)}{n-1}} \int \left(|W|^2 + \frac{8}{n(n-2)}|E|^2\right)^{\frac{1}{2}} |E|^2 e^{-f}$$
$$-\frac{2(n-2)}{n(n-1)} \int R|E|^2 e^{-f}$$

By the soliton equation we have

$$R + \Delta f = n,$$

which implies

$$2 = \frac{2}{n}R + \frac{2}{n}\Delta f.$$

Therefore we get

$$2\int |E|^2 e^{-f} = \frac{2}{n}\int R|E|^2 e^{-f} + \frac{2}{n}\int \Delta f |E|^2 e^{-f}.$$

By Kato's inequality

$$|\nabla E|^2 \geqslant |\nabla |E||^2$$

we have

$$0 \geqslant \int |\nabla |E||^2 e^{-f} + 2\int |E|^2 e^{-f} - \sqrt{\frac{2(n-2)}{n-1}} \int \left(|W|^2 + \frac{8}{n(n-2)}|E|^2\right)^{\frac{1}{2}} |E|^2 e^{-f}$$

$$
\begin{aligned}
&-\frac{2(n-2)}{n(n-1)}\int R|E|^2e^{-f}\\
=&\int|\nabla|E||^2e^{-f}+\frac{2}{n(n-1)}\int R|E|^2e^{-f}+\frac{2}{n}\int\Delta f|E|^2e^{-f}\\
&-\sqrt{\frac{2(n-2)}{n-1}}\int\Big(|W|^2+\frac{8}{n(n-2)}|E|^2\Big)^{\frac{1}{2}}|E|^2e^{-f}\\
=&\frac{8}{(n-1)^2}\Big[\int|\nabla|E||^2e^{-f}+\frac{n-1}{4n}\int\Big(R+2\Delta f-2|\nabla f|^2|E|^2e^{-f}\Big)\Big]\\
&+\Big(1-\frac{8}{(n-1)^2}\Big)\int|\nabla|E||^2e^{-f}+\frac{2(n-3)}{n(n-1)}\int\Delta f|E|^2e^{-f}+\frac{4}{n(n-1)}\int|\nabla f|^2|E|^2e^{-f}\\
&-\sqrt{\frac{2(n-2)}{n-1}}\int\Big(|W|^2+\frac{8}{n(n-2)}|E|^2\Big)^{\frac{1}{2}}|E|^2e^{-f}.
\end{aligned}
$$

Let $A=\frac{8}{(n-1)^2}\Big[\int|\nabla|E||^2e^{-f}+\frac{n-1}{4n}\int\Big(R+2\Delta f-2|\nabla f|^2|E|^2e^{-f}\Big)\Big]$, we have

$$
\begin{aligned}
0\geqslant A+\Big(1-\frac{8}{(n-1)^2}\Big)\int|\nabla|E||^2e^{-f}+\frac{2(n-3)}{n(n-1)}\int\Delta f|E|^2e^{-f}+\frac{4}{n(n-1)}\int|\nabla f|^2|E|^2e^{-f}\\
-\sqrt{\frac{2(n-2)}{n-1}}\int\Big(|W|^2+\frac{8}{n(n-2)}|E|^2\Big)^{\frac{1}{2}}|E|^2e^{-f}.
\end{aligned}
$$

Using integration by parts we know

$$
\int\Delta f|E|^2e^{-f}=-2\int\nabla f\nabla|E||E|e^{-f}+\int|\nabla f|^2|E|^2e^{-f}.
$$

Therefore

$$
0\geqslant A+B-\sqrt{\frac{2(n-2)}{n-1}}\int\Big(|W|^2+\frac{8}{n(n-2)}|E|^2\Big)^{\frac{1}{2}}|E|^2e^{-f}
$$

where

$$
\begin{aligned}
B&=\Big(1-\frac{8}{(n-1)^2}\Big)\int|\nabla|E||^2e^{-f}-\frac{4(n-3)}{n(n-1)}\int\nabla f\nabla|E||E|e^{-f}+\frac{2}{n}\int|\nabla f|^2|E|^2e^{-f}\\
&=D+\frac{2(n^3-4n^2+5n-18)}{n^2(n^2-2n-7)}\int|\nabla f|^2|E|^2e^{-f}
\end{aligned}
$$

where

$$D=\Big(1-\frac{8}{(n-1)^2}\Big)\int|\nabla|E||^2e^{-f}-\frac{4(n-3)}{n(n-1)}\int\nabla f\nabla|E||E|e^{-f}+\frac{4(n-3)^2}{n^2(n^2-2n-7)}\int|\nabla f|^2|E|^2e^{-f}.$$

By Holder's inequality we know $D\geqslant 0$, therefore $B\geqslant\frac{2(n^3-4n^2+5n-18)}{n^2(n^2-2n-7)}\int|\nabla f|^2|E|^2e^{-f}$.

Hence

$$0\geqslant A+\frac{2(n^3-4n^2+5n-18)}{n^2(n^2-2n-7)}\int|\nabla f|^2|E|^2e^{-f}-\sqrt{\frac{2(n-2)}{n-1}}\int\Big(|W|+\sqrt{\frac{8}{n(n-2)}}|E|\Big)|E|^2e^{-f}$$

$$=A-\sqrt{\frac{2(n-2)}{n-1}}\int\Big(|W|+\sqrt{\frac{8}{n(n-2)}}|E|-k(n)|\nabla f|^2\Big)|E|^2e^{-f}.$$

Let $C=\int\Big(|W|+\sqrt{\frac{8}{n(n-2)}}|E|-k(n)|\nabla f|^2\Big)|E|^2e^{-f}$, so till now we've shown

$$0\geqslant A-\sqrt{\frac{2(n-2)}{n-1}}C.$$

Now we are going to show that under the assumption of Theorem 3.6.1 we actually have $A-\sqrt{\frac{2(n-2)}{n-1}}C>0$.

Since $0\geqslant A-\sqrt{\frac{2(n-2)}{n-1}}C$, multiply by $\Big(\int|E|^{\frac{2n}{n-1}}\Big)^{\frac{2}{n}}$ to both sides we have

$$\begin{aligned}0\geq A\Big(\int|E|^{\frac{2n}{n-1}}\Big)^{\frac{2}{n}}&-\sqrt{\frac{2(n-2)}{n-1}}C\Big(\int|E|^{\frac{2n}{n-1}}\Big)^{\frac{2}{n}}\\ \geqslant\frac{8}{(n-1)^2}Y\int|E|^{\frac{2(n+1)}{n-1}}e^{-f}&-\sqrt{\frac{2(n-2)}{n-1}}C\Big(\int|E|^{\frac{2n}{n-1}}\Big)^{\frac{2}{n}}\end{aligned}$$

by the definition of Y.

By Holder's inequality, we have

$$\int|E|^{\frac{2n}{n-1}}\leqslant\Big(\int|E|^{\frac{2(n+1)}{n-1}}e^{-f}\Big)^{\frac{n}{n+1}}\Big(\int e^{nf}\Big)^{\frac{1}{n+1}},$$

which implies

$$\left(\int |E|^{\frac{2n}{n-1}}\right)^{\frac{2}{n}} \leqslant \left(\int |E|^{\frac{2(n+1)}{n-1}} e^{-f}\right)^{\frac{2}{n+1}} \left(\int e^{nf}\right)^{\frac{2}{n(n+1)}}$$

Therefore

$$0 \geqslant \frac{8}{(n-1)^2} Y\left(\int |E|^{\frac{2(n+1)}{n-1}} e^{-f}\right)^{\frac{n-1}{n+1}} \left(\int e^{nf}\right)^{-\frac{2}{n(n+1)}} - \sqrt{\frac{2(n-2)}{n-1}} C.$$

It suffices to show

$$\frac{8}{(n-1)^2} Y\left(\int |E|^{\frac{2(n+1)}{n-1}} e^{-f}\right)^{\frac{n-1}{n+1}} \left(\int e^{nf}\right)^{-\frac{2}{n(n+1)}} - \sqrt{\frac{2(n-2)}{n-1}} C > 0.$$

Let $g = |W| + \sqrt{\frac{8}{n(n-2)}}|E| - k(n)|\nabla f|^2$.

By Holder's inequality we have

$$\begin{aligned} C &= \int g|E|^2 e^{-f} \\ &\leqslant \left(\int g^{\frac{n+1}{2}} e^{-f}\right)^{\frac{2}{n+1}} \left(|E|^{\frac{2(n+1)}{n-1}} e^{-f}\right)^{\frac{n-1}{n+1}}. \end{aligned}$$

It suffices to show

$$\begin{aligned} Y &> \sqrt{\frac{(n-2)(n-1)^3}{32}} \left(\int g^{\frac{n+1}{2}} e^{-f}\right)^{\frac{2}{n+1}} \left(\int e^{nf}\right)^{\frac{2}{n(n+1)}}, \\ &= \sqrt{\frac{(n-2)(n-1)^3}{32}}\, \|g\|_{\frac{n+1}{2}} \left(\int e^{(n+1)f d\widetilde{\mu}}\right)^{\frac{2}{n(n+1)}}, \end{aligned}$$

which by the assumption is true.

Therefore $|E| \equiv 0$, (M^n, g) is Einstein. $\square$

Now we are going to prove Theorem 3.6.1.

Proof. By Lemma 3.6.5 we know that (M^n, g) is Einstein under the assumption. Now we are going to show it's actually a round sphere.

Since (M^n, g) is Einstein, the assumption of Theorem 1 changes to

$$Y > \sqrt{\frac{(n-2)(n-1)^3}{32}} \, \||W|\|_{\frac{n+1}{2}} \, vol^{\frac{2}{n+1}}.$$

Compare the coefficients of Y with Theorem 2.6.2, we have

$$\sqrt{\frac{(n-2)(n-1)^3}{32}} > 2C(n)$$

when n =4, 5, 7, 8.

This means when n =4, 5, 7, 8 we have

$$Y > 2C(n)\||W| - RD(n)\|_{\frac{n+1}{2}} vol^{\frac{2}{n(n+1)}}.$$

So by Theorem 2, (M^n, g) is isometric to (S^n, g_{round}). This completes the proof of Theorem 3.6.1. □

CHAPTER 4

INTEGRAL WEITZENBOCK FORMULA

We've introduced the Weitzenbock formulas for 4-dimensional Einstein manifolds and Ricci solitons in Chapter 2 and Chapter 3 respectively. In this chapter we are going to give an integral version of Weitzenbock formula for compact shrinking Ricci solitons.

4.1 Framework approach

Since we have the curvature decomposition from Section 1.3, in this chapter we will express the interior product $i_{\nabla f}$ and the divergence on each curvature component as linear combinations of four operators P, Q, M, N. The geometry of each operator gives us some information about the original manifold.

Let (M^n, g) be an n-dimensional oriented Riemannian manifold. Using the pointwisely induced inner product, any anti-symmetric $(2,0)$-tensor α can be written as an operator on the tangent space by

$$\alpha(X, Y) = \langle -\alpha(X), Y \rangle = \langle X, \alpha(Y) \rangle = \langle \alpha, X \wedge Y \rangle.$$

In particular, a bi-vector acts on a vector X as follows

$$(U \wedge V)X = \langle V, X \rangle U - \langle U, X \rangle V.$$

Similarly, any symmetric $(2,0)$-tensor b can be written as an operator on the tangent space

$$b(X,Y) = \langle b(X), Y\rangle = \langle X, b(Y)\rangle = \langle b, X \wedge Y\rangle.$$

Therefore, when b is viewed as a 1-form, $d_\nabla b$ denotes the exterior derivative. Then

$$(d_\nabla b)(X,Y,Z) = \nabla_X b(Y,Z) - \nabla_Y b(X,Z).$$

Now we can define the fundamental tensors of our interest here, via a local frame and then using the operator language. We now assume $\alpha \in \Lambda^2(TM)$, $X,\ Y,\ Z \in TM$, and $\{e_i\}_{i=1}^n$ to be a local orthonormal frame on a gradient Ricci soliton (M^n, g, f, λ).

4.2 Integral Weitzenbock formula

In Theorem 3.5.16 we introduced the Weitzenbock formula for Ricci solitons. Now we are going to prove the integral Weitzenbock formula. To show this, we need the following definitions and lemmas.

Definition 4.2.1. *The tensors P, Q, M, N: $\Lambda^2 TM \otimes TM \to \mathbb{R}$ are defined as:*

$$P_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} = \nabla_j f_{ik} - \nabla_i f_{jk} = R_{jikp}\nabla^p f, \tag{4.1}$$

$$P(X \wedge Y, Z) = -R(X,Y,Z,\nabla f) = (d_\nabla Rc)(X,Y,Z) = \delta Rm(Z,X,Y),$$

$$P(\alpha, Z) = R(\alpha, \nabla f \wedge Z) = \delta Rm(Z, \alpha);$$

$$Q_{ijk} = g_{ki}\nabla_j R - g_{kj}\nabla_i R = 2(g_{ik}R_{jp} - g_{jk}R_{ip})\nabla^p f, \tag{4.2}$$

$$Q(X \wedge Y, Z) = 2g(X, Z)Rc(Y, \nabla f) - 2g(Y, Z)Rc(X, \nabla f),$$

$$Q(\alpha, Z) = -2Rc(\alpha(Z), \nabla f) = -2g(\alpha(Z), Rc(\nabla f));$$

$$M_{ijk} = R_{kj}\nabla_i f - R_{ki}\nabla_j f, \tag{4.3}$$

$$M(X \wedge Y, Z) = Rc(Y, Z)\nabla_X f - Rc(X, Z)\nabla_Y f = -Rc((X \wedge Y)\nabla f, Z),$$

$$M(\alpha, Z) = -Rc(\alpha(\nabla f), Z) = -g(\alpha\nabla f, Rc(Z));$$

$$N_{ijk} = g_{kj}\nabla_i f - g_{ki}\nabla_j f \tag{4.4}$$

$$N(X \wedge Y, Z) = g(Y, Z)\nabla_X f - g(X, Z)\nabla_Y f = g((X \wedge Y)Z, \nabla f),$$

$$N(\alpha, Z) = g(\alpha Z, \nabla f) = -\alpha(Z, \nabla f).$$

Remark 4.2.1. *The tensors $P^{\pm}$, $Q^{\pm}$, $M^{\pm}$, $N^{\pm} : \Lambda^{2\pm}TM \otimes TM \to R$ are defined by restricting $\alpha \in \Lambda^{2\pm}TM$. They can be seen as operators on Λ^2 by standard projection.*

Remark 4.2.2. *Before further computation, let us remark on the essence of these operators. $P \equiv 0$ if and only if the curvature is harmonic; $Q \equiv 0$ if and only if the scalar curvature is constant; $M \equiv 0$ if and only if either $\nabla f = 0$ or Rc vanishes on the orthogonal complement of ∇f; finally $N \equiv 0$ if and only if the potential function f is constant.*

Using the framework above, the interior product $i_{\nabla f}$ on components of the curvature tensor can be represented as follows. The computational lemmas can be found in [5]. Again the Einstein summation convention is used.

Lemma 4.2.1. *Let (M, g, f, λ) be a gradient Ricci soliton, P, Q, M, N as in Definition 4.2.1. In a local orthonormal frame, we have*

$$R_{ijkp}\nabla^p f = R(e_i, e_j, e_k, \nabla f) = -P_{ijk} = \nabla^p R_{ijkp} = \delta Rm(e_k, e_i, e_j), \tag{4.5}$$

$$(g \circ g)_{ijkp}\nabla^p f = (g \circ g)(e_i, e_j, e_k, \nabla f) = -2N_{ijk}, \tag{4.6}$$

$$(Rc \circ g)_{ijkp}\nabla^p f = (Rc \circ g)(e_i, e_j, e_k, \nabla f) = \frac{1}{2}Q_{ijk} - M_{ijk}, \tag{4.7}$$

$$H_{ijkp}\nabla^p f = H(e_i, e_j, e_k, \nabla f) = M_{ijk} - \frac{1}{2}Q_{ijk} - 2\lambda N_{ijk}, \tag{4.8}$$

$$W_{ijkp}\nabla^p f = W(e_i, e_j, e_k, \nabla f) = -P_{ijk} - \frac{Q_{ijk}}{2(n-2)} + \frac{M_{ijk}}{n-2} - \frac{RN_{ijk}}{(n-1)(n-2)} \tag{4.9}$$

where $H = Hessf \circ g$.

Proof. For the first formula, we can just apply the soliton equation and Bianchi identities.

For the second formula, we have

$$(g \circ g)_{ijkp}\nabla^p f = 2(g_{ik}g_{jp} - g_{ip}g_{jk})\nabla^p f$$

$$= 2g_{ik}\nabla_j f - 2g_{jk}\nabla_i f = -2N_{ijk}.$$

For the third formula, we again use the definition of Kulkarni-Nomizu product to compute

$$\begin{aligned}(Rc \circ g)_{ijkp}\nabla^p f &= (R_{ik}g_{jp} + R_{jp}g_{ik} - R_{ip}g_{jk} - R_{jk}g_{ip})\nabla^p f \\ &= R_{ik}\nabla_j f + \frac{1}{2}(g_{ik}\nabla_j R - g_{jk}\nabla_i R) - R_{jk}\nabla_i f \\ &= \frac{1}{2}Q_{ijk} - M_{ijk}.\end{aligned}$$

For the fourth formula, we need to use the soliton equation $R_{ij} + (Hessf)_{ij} = \lambda g_{ij}$. Then the result is a combination of the second and third formula.

For the last formula, it comes from the curvature decomposition (1.14). □

We need to use another computational lemma from [5] to derive our main result.

Lemma 4.2.2. *Let (M, g, f, λ) be a gradient Ricci soliton, for P, Q, M, N as above, in a local orthonormal frame we have*

$$\nabla^p R_{ijkp} = -P_{ijk}, \tag{4.10}$$

$$\nabla^p (Rg \circ g)_{ijkp} = 2Q_{ijk}, \tag{4.11}$$

$$\nabla^p (Rc \circ g)_{ijkp} = -\nabla^p H_{ijkp} = -P_{ijk} + \frac{1}{2}Q_{ijk}, \tag{4.12}$$

$$\nabla^p W_{ijkp} = -\frac{n-3}{n-2}P_{ijk} - \frac{n-3}{2(n-1)(n-2)}Q_{ijk} := -\frac{n-3}{n-2}C_{ijk}. \tag{4.13}$$

Proof. The first formula comes from the second Bianchi identity.

For the second formula, we have

$$\begin{aligned}\nabla^p (Rg \circ g) &= 2\nabla^p (Rg_{ik}g_{jp} - Rg_{ip}g_{jk}) \\ &= 2g_{ik}g_{jp}\nabla^p R - 2g_{jk}g_{ip}\nabla^p R \\ &= 2g_{ik}\nabla_j R - 2g_{jk}\nabla_i R = 2Q_{ijk}.\end{aligned}$$

For the third formula, we have

$$\begin{aligned}\nabla^p (Rc \circ g)_{ijkp} &= \nabla^p (R_{ik}g_{jp} + R_{jp}g_{ik} - R_{ip}g_{jk} - R_{jk}g_{ip}) \\ &= g_{jp}\nabla^p R_{ik} + g_{ik}\nabla^p R_{jp} - g_{jk}\nabla^p R_{ip} - g_{ip}\nabla^p R_{jk} \\ &= \nabla_j R_{ik} + \frac{1}{2}(g_{ik}\nabla_j R - g_{jk}\nabla_i R) - \nabla_i R_{jk} \\ &= -P_{ijk} + \frac{1}{2}Q_{ijk}.\end{aligned}$$

For the last one formula, we again use formula (1.14) to get the result. □

Remark 4.2.3. *By the standard projection, we have*

$$(\delta W)^{\pm} = \delta(W^{\pm}),$$

$$(i_{\nabla f} W)^{\pm} = i_{\nabla f} W^{\pm},$$

the identities above hold if we replace $W,\ P,\ Q,\ M,\ N$ *by* $W^{\pm}$, $P^{\pm}$, $Q^{\pm}$, $M^{\pm}$, $N^{\pm}$ *respectively.*

The last computational lemma we need before deriving the main result is the following lemma in [5].

Lemma 4.2.3. *Let* (M, g, f, λ) *be a gradient shrinking Ricci soliton, then we have:*

$$2\langle P, Q\rangle = -|\nabla R|^2, \tag{4.14}$$

$$2\langle P, N\rangle = \langle \nabla f, \nabla R\rangle, \tag{4.15}$$

$$2\langle Q, Q\rangle = 2(n-1)|\nabla R|^2, \tag{4.16}$$

$$2\langle M, M\rangle = 2|Rc|^2|\nabla f|^2 - \frac{1}{2}|\nabla R|^2, \tag{4.17}$$

$$2\langle N, N\rangle = 2(n-1)|\nabla f|^2, \tag{4.18}$$

$$2\langle Q, M\rangle = |\nabla R|^2 - 2R\langle \nabla f, \nabla R\rangle, \tag{4.19}$$

$$2\langle Q, N\rangle = -2(n-1)\langle \nabla f, \nabla R\rangle, \tag{4.20}$$

$$2\langle M, N\rangle = 2R|\nabla f|^2 - \langle \nabla f, \nabla R\rangle, \tag{4.21}$$

Furthermore, if M *is closed, then*

$$\int_M 2\langle P, P\rangle e^{-f} = \int_M |\nabla Rc|^2 e^{-f}. \tag{4.22}$$

Proof. For the first formula,

$$2\langle P, Q\rangle = P_{ijk}Q_{ijk}$$

$$= (\nabla_i R_{jk} - \nabla_j R_{ik})(g_{ik}\nabla_j R - g_{jk}\nabla_i R)$$

$$= \nabla_i R_{jk} g_{ik} \nabla_j R - \nabla_j R_{ik} g_{ik} \nabla_j R$$

$$-g_{kj}\nabla_i R_{jk}\nabla_i R + \nabla_j R_{ik} g_{kj} \nabla_i R$$

$$= 2(\nabla_i R_{jk} - \nabla_j R_{ik}) g_{ik} \nabla_j R$$

$$= |\nabla R|^2 - 2|\nabla R|^2$$

$$= -|\nabla R|^2.$$

For the second formula,

$$2\langle P, N\rangle = P_{ijk}N_{ijk}$$

$$= (\nabla_i R_{jk} - \nabla_j R_{ik})(g_{jk}\nabla_i f - g_{ik}\nabla_j f)$$

$$= \nabla_i R_{jk} g_{jk} \nabla_i f + \nabla_j R_{ik} g_{ik} \nabla_j f$$

$$-\nabla_j R_{ik} g_{jk} \nabla_i f - \nabla_i R_{jk} g_{ik} \nabla_j f$$

$$= 2\langle \nabla f, \nabla R\rangle - \langle \nabla f, \nabla R\rangle$$

$$= \langle \nabla f, \nabla R\rangle.$$

For the third formula,

$$2\langle Q, Q\rangle = Q_{ijk}Q_{ijk}$$

$$= (g_{ik}\nabla_j R - g_{jk}\nabla_i R)^2$$

$$= (g_{ik}\nabla_j R)^2 + (g_{jk}\nabla_i R)^2$$

$$-2g_{ik}\nabla_j R g_{jk}\nabla_i R$$

$$= 2n|\nabla R|^2 - 2|\nabla R|^2$$

$$= 2(n-1)|\nabla R|^2.$$

For the fourth formula,

$$2\langle M, M\rangle = M_{ijk}M_{ijk}$$

$$= (R_{jk}\nabla_i f - R_{ik}\nabla_j f)^2$$

$$= (R_{jk}\nabla_i f)^2 + (R_{ik}\nabla_j f)^2$$

$$-2R_{jk}\nabla_i f R_{ik}\nabla_j f$$

$$= 2|Rc|^2|\nabla f|^2 - 2\times\frac{1}{2}\times\frac{1}{2}|\nabla R|^2$$

$$= 2|Rc|^2|\nabla f|^2 - \frac{1}{2}|\nabla R|^2.$$

For the fifth formula,

$$2\langle N, N\rangle = N_{ijk}N_{ijk}$$

$$= (g_{jk}\nabla_i f - g_{ik}\nabla_j f)^2$$

$$= 2n|\nabla f|^2 - 2g_{jk}\nabla_i f g_{ik}\nabla_j f$$

$$= 2n|\nabla f|^2 - 2|\nabla f|^2$$

$$= 2(n-1)|\nabla f|^2.$$

For the sixth formula,

$$2\langle Q, M\rangle = Q_{ijk}M_{ijk}$$

$$= (g_{ik}\nabla_j R - g_{jk}\nabla_i R)(R_{jk}\nabla_i f - R_{ik}\nabla_j f)$$

$$= g_{ik}\nabla_j R R_{jk}\nabla_i f + g_{jk}\nabla_i R R_{ik}\nabla_j f$$

$$-g_{jk}\nabla_i R R_{jk}\nabla_i f - g_{ik}\nabla_j R R_{ik}\nabla_j f$$

$$= 2 \times \frac{1}{2}|\nabla R|^2 - 2R\langle\nabla R, \nabla f\rangle$$

$$= |\nabla R|^2 - 2R\langle\nabla R, \nabla f\rangle.$$

For the seventh formula,

$$2\langle Q, N\rangle = Q_{ijk}N_{ijk}$$

$$= (g_{ik}\nabla_j R - g_{jk}\nabla_i R)(g_{jk}\nabla_i f - g_{ik}\nabla_j f)$$

$$= g_{ik}\nabla_j R g_{jk}\nabla_i f + g_{jk}\nabla_i R g_{ik}\nabla_j f$$

$$-g_{jk}\nabla_i R g_{jk}\nabla_i f - g_{ik}\nabla_j R g_{ik}\nabla_j f$$

$$= 2\langle\nabla f, \nabla R\rangle - 2n\langle\nabla f, \nabla R\rangle$$

$$= -2(n-1)\langle\nabla f, \nabla R\rangle.$$

For the eighth formula,

$$2\langle M, N\rangle = M_{ijk}N_{ijk}$$

$$= (R_{jk}\nabla_i f - R_{ik}\nabla_j f)(g_{jk}\nabla_i f - g_{ik}\nabla_j f)$$

$$= R_{jk}g_{jk}\nabla_i f\nabla_i f + R_{ik}g_{ik}\nabla_j f\nabla_j f$$

$$= R_{jk}g_{ik}\nabla_i f\nabla_j f - R_{ik}g_{jk}\nabla_j f\nabla_i f$$

$$= 2R|\nabla f|^2 - 2 \times \frac{1}{2}\langle\nabla f, \nabla R\rangle$$

$$= 2R|\nabla f|^2 - \langle\nabla f, \nabla R\rangle$$

□

From Lemma 4.2.3, we have the following two corollaries in [5].

Corollary 4.2.1. *Let (M, g, f, λ) be a gradient Ricci soliton, at each point, we have*

$$0 = \langle Q, i_{\nabla f} W\rangle = \langle N, i_{\nabla f} W\rangle = \langle Q, \delta W\rangle = \langle N, \delta W\rangle. \tag{4.23}$$

Proof. For the first equality,

$$\begin{aligned}
&\langle Q, i_{\nabla f} W\rangle \\
&= \sum_{i<j} Q_{ijk}(i_{\nabla f} W)_{kij} \\
&= \sum_{i<j} Q_{ijk}\nabla^p f W_{pkij} \\
&= -\sum_{i<j} Q_{ijk} W_{ijkp}\nabla^p f \\
&= \langle Q, P + \frac{Q}{2(n-2)} - \frac{M}{n-2} + \frac{SN}{(n-1)(n-2)}\rangle \\
&= -\frac{1}{2}|\nabla R|^2 + \frac{n-1}{2(n-2)}|\nabla R|^2 \\
&\quad -\frac{1}{2(n-2)}|\nabla R|^2 + \frac{1}{n-2}R\langle\nabla f, \nabla R\rangle \\
&\quad -\frac{n-1}{(n-1)(n-2)}R\langle\nabla f, \nabla R\rangle \\
&= 0.
\end{aligned}$$

For the second equality, we use similar argument for N.

For the third equality,

$$\begin{aligned}
&\langle Q, \delta W\rangle \\
&= \sum_{i<j} Q_{ijk}(\delta W)_{kij}
\end{aligned}$$

$$= \sum_{i<j} Q_{ijk} \nabla^p W_{pkij}$$

$$= -\sum_{i<j} Q_{ijk} \nabla^p W_{ijkp}$$

$$\langle Q, \frac{n-3}{n-2} P + \frac{n-3}{2(n-1)(n-2)} Q \rangle$$

$$= -\frac{n-3}{2(n-2)} |\nabla R|^2 + \frac{n-3}{2(n-2)} |\nabla R|^2$$

$$= 0.$$

For the fourth equality, we use similar argument for N.

□

Corollary 4.2.2. *Let (M, g, f, λ) be a closed gradient Ricci soliton, we have*

$$\int_M 2|\delta W|^2 e^{-f} = (\frac{n-3}{n-2})^2 \int_M (|\nabla Rc|^2 - \frac{1}{(n-1)} |\nabla R|^2) e^{-f}. \tag{4.24}$$

Proof. Since

$$\delta W = -\nabla^p W_{ijkp}$$

$$= \frac{n-3}{n-2} P + \frac{n-3}{2(n-1)(n-2)} Q.$$

Therefore

$$|\delta W|^2$$

$$= (\frac{n-3}{n-2})^2 \langle P + \frac{Q}{2(n-1)}, P + \frac{Q}{2(n-1)} \rangle$$

$$(\frac{n-3}{n-2})^2 \langle P + \frac{Q}{2(n-1)}, P \rangle.$$

Hence

$$\int_M 2|\delta W|^2 e^{-f} = (\frac{n-3}{n-2})^2 \int_M (|\nabla Rc|^2 - \frac{1}{(n-1)} |\nabla R|^2) e^{-f}$$

□

Now we can prove our main theorem, which is an integral version of the Weitzenbock formula for 4-dimensional Ricci solitons.

Theorem 4.2.4. *For 4-dimensional compact shrinking Ricci solitons, we have*

$$\int_M \langle W, Hessf \circ Hessf\rangle e^{-f} = 4\int_M |\delta W|^2 e^{-f} - \int \langle i_{\nabla f} W, M\rangle e^{-f}. \tag{4.25}$$

Proof. Since M is compact, we have

$$\begin{aligned} &\int_M W_{ijkl} f_{ik} f_{jl} e^{-f} \\ &= -\int_M \nabla_i W_{ijkl} f_k f_{jl} e^{-f} - \int_M W_{ijkl} f_k \nabla_i f_{jl} e^{-f} \\ &\quad + \int_M W_{ijkl} f_k f_{jl} f_i e^{-f} \\ &:= A + B + C \end{aligned}$$

where $A = \int_M \nabla_i W_{ijkl} f_k f_{jl} e^{-f}$, $B = \int_M W_{ijkl} f_k \nabla_i f_{jl} e^{-f}$ and $C = \int_M W_{ijkl} f_k f_{jl} f_i e^{-f}$.

$$\begin{aligned} A &= \int_M \nabla_i W_{ijkl} f_k (\lambda g_{jl} - R_{jl}) e^{-f} \\ &= -\int_M \nabla_i W_{ijkl} f_k R_{jl} e^{-f} \\ &= -\frac{1}{2}\int_M (\delta W)_{jkl} M_{klj} e^{-f} \\ &= -\int_M \langle \delta W, M\rangle e^{-f}; \\ B &= \int_M W_{ijkl} f_k \nabla_i (\lambda g_{jl} - R_{jl}) e^{-f} \\ &= \frac{1}{2}\int_M W_{ijkl} f_k P_{ijl} e^{-f} \end{aligned}$$

$$
\begin{aligned}
&= -\int_M \langle i_{\nabla f} W, P + \frac{Q}{2(n-1)} e^{-f} \rangle \\
&= \frac{n-2}{n-3} \int_M \langle \delta W, i_{\nabla f} W \rangle e^{-f} \\
&= \frac{n-2}{n-3} \int_M \langle \delta W, -P + \frac{M}{n-2} \rangle e^{-f}; \\
C &= \int_M W_{ijkl} f_k f_i (\lambda g_{jl} - R_{jl}) e^{-f} \\
&= -\int_M W_{ijkl} f_k f_i R_{jl} e^{-f} \\
&= \int_M W_{ijlk} f_k f_i R_{jl} e^{-f} \\
&= -\int_M \langle i_{\nabla f} W, M \rangle e^{-f}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_M \langle W, Hessf \circ Hessf \rangle e^{-f} \\
&= \frac{1}{n-3} \int_M \langle \delta W, (n-2)P + (n-4)M \rangle e^{-f} \\
&\quad - - \int_M \langle i_{\nabla f} W, M \rangle e^{-f}.
\end{aligned}
$$

In particular, when $n = 4$ we have

$$
\int_M \langle W, Hessf \circ Hessf \rangle e^{-f} = 4 \int_M |\delta W|^2 e^{-f} - \int \langle i_{\nabla f} W, M \rangle e^{-f}.
$$

□

By similar computation, we have the following corollary.

Corollary 4.2.3. *For 4-dimensional compact shrinking Ricci solitons, we have*

$$
\int_M 2\langle P, M \rangle e^{-f} = 2 \int_M (\lambda |Rc|^2 - Rc^3) e^{-f} + \int_M \langle \nabla f, \nabla |Rc|^2 \rangle e^{-f}. \tag{4.26}
$$

Proof.

$$\begin{aligned}
&\int_M 2\langle P, M\rangle e^{-f}\\
&= 2\int_M (\nabla_i R_{jk} - \nabla_j R_{ik}) R_{jk} \nabla_i f e^{-f}\\
&= \int_M \langle \nabla f, \nabla |Rc|^2\rangle e^{-f} - 2A
\end{aligned}$$

where $A := \int_M \nabla_j R_{ik} R_{jk} \nabla_i f e^{-f}$.

$$\begin{aligned}
A = &-\int_M R_{ik}\nabla_j R_{ik}\nabla_i f e^{-f} - \int_M R_{ik}R_{jk} f_{ij} e^{-f}\\
&+ \int_M R_{ik}R_{jk} f_i f_j e^{-f}\\
= &-\frac{1}{4}\int_M |\nabla R|^2 e^{-f} - \int_M R_{ik}R_{jk}\lambda g_{ij} e^{-f}\\
&+ \int_M R_{ik}R_{jk}R_{ij} e^{-f} + \frac{1}{4}\int_M |\nabla R|^2 e^{-f}\\
&\int_M (Rc^3 - \lambda |Rc|^2) e^{-f}.
\end{aligned}$$

Therefore

$$\int_M 2\langle P, M\rangle e^{-f} = 2\int_M (\lambda |Rc|^2 - Rc^3) e^{-f} + \int_M \langle \nabla f, \nabla |Rc|^2\rangle e^{-f}.$$

□

BIBLIOGRAPHY

[1] G. Huisken, *Ricci deformation of the metric on a Riemannian manifold*, J. Differential Geom. 21 (1985), 47-62.

[2] M. Eminenti, G. La Nave, and C. Mantegazza, *Ricci solitons: the equation point of view*, Manuscripta Math. 127 (2008), no. 3, 345-367.

[3] V. Bour, *Fourth order curvature flows and geometric applications*, ArXiv Preprint Server - http://arxiv.org, 2010.

[4] J. S. Case, *A Yamabe-type problem on smooth metric measure spaces*, J. Differential Geom. 101 (2015), 467-505.

[5] X. Cao, H. Tran, *The Weyl tensor of gradient Ricci solitons*, Geometry and Topology. 20 (2016), 389-436.

[6] H-D. Cao, *Recent progress on Ricci solitons*, ArXiv Preprint Server - http://arxiv.org, 2009.

[7] E. A. Carlen, M. Loss, *Sharp constant in Nash's inequality*, Internat. Math. Res. Notices (1993), no. 7, 213-215.

[8] G. Catino, *Integral pinched shrinking Ricci solitons*, ArXiv Preprint Server - http://arxiv.org, 2015.

[9] J-Y. Wu, P. Wu, W. Wylie, *Gradient shrinking Ricci solitons of half harmonic Weyl curvature*, ArXiv Preprint Server - http://arxiv.org, 2014.

[10] J. S. Case, *Conformal invariants measuring the best constants for Gagliardo-Nirenberg-Sobolev inequalities*, Calc. Var. Partial Differential Equations 48 (2013), no. 3-4, 507-526.

[11] A. Derdzinski, *Self-dual Kahler manifolds and Einstein manifolds of dimension four*, Com. Math. 49 (1983), 405-433.

[12] X. Cao, P. Wu, *Einstein four-manifolds of three-nonnegative curvature operator*, preprint, 2013.

[13] M. Del Pino, J. Dolbeault, *Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions*, J. Math. Pures Appl. (9) 81 (2002), no. 9, 847-875.

[14] X. Cao, *Curvature pinching estimate and singularities of the Ricci flow*, Comm. Anal. Geom., Vol. 19 (5): 975-990, 2011.

[15] E. Hebey, *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, Courant Lecture Notes in Mathematics, vol. 5, New York University Courant Institute of Mathematical Sciences, New York, 1999.

[16] D-S. Kim, Y. Ho Kim, *Compact Einstein warped product spaces with nonpositive scalar curvature*, Proc. Amer. Math. Soc. 131 (2003), no. 8, 2573-2576.

[17] J. M. Lee, T. H. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. (N.S.) 17 (1987), no. 1, 37-91.

[18] M. Obata, *The conjectures on conformal transformations of Riemannian manifolds*, J. Differential Geometry 6 (1971/72), 247-258.

[19] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:0211159, preprint.

[20] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. 20 (1984), no. 2, 479-495.

[21] R. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, Topics in calculus of variations (Montecatini Terme, 1987), Lecture Notes in Math., vol. 1365, Springer, Berlin, 1989, pp. 120-154.

[22] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) 110 (1976), 353-372.

[23] N. S. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa (3) 22 (1968), 265-274.

[24] J. A. Viaclovsky, *Conformally invariant Monge-Ampere equations: global solutions*, Trans. Amer. Math. Soc. 352 (2000), no. 9, 4371-4379.

[25] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. 12 (1960), 21-37.

[26] A. Besse, *Einstein manifolds*, Berlin-Heidelberg, Springer-Verlag, 1987.

[27] M. Cai, *On shrinking gradient Ricci soliton with nonnegative sectional curvature*, arXiv:math.DG/1303.2728.

[28] S. Calamai, D. Petrecca, *On Calabi extremal Kahler-Ricci solitons*, arXiv: math.DG/1401.6293, 2014.

[29] X. Cao, *Closed gradient shrinking Ricci solitons with positive curvature operator*, J. Geom. Anal. 17 (2007), 451-459.

[30] M. Singer, A. Thorpe, *The curvature of 4-dimensional Einstein spaces*, In Global Analysis (Papers in Honor of K. Kodaira), 355-365. Univ. Tokyo Press, Tokyo, 1969.

[31] M. Berger, *Sur quelques varietes d'Einstein compactes*, Ann. Mat. Pura Appl. 53 (1961), 89-95.

[32] M. Gursky, C. LeBrun, *On Einstein manifolds of positive sectional curvature*, Ann. Glob. An. Geom. 17 (1999), 315-328.

[33] P. Wu, *Studies on Einstein manifolds and gradient Ricci solitons*, Ph. D. dissertation, University of California, Santa Barbara, 2012.

[34] N. Hitchin, *Compact four-dimensional Einstein manifolds*, J. Differential Geometry 9 (1974), 435-441.

[35] C. Bohm, B. Wilking, *Manifolds with positive curvature operators are space forms*, Ann. of Math. (2) 167 (2008), 1079-1097.

[36] D. Yang, *Rigidity of Einstein 4-manifolds with positive curvature*, Invent. Math. 142 (2000), 435-450.

[37] T. Branson, *Kato constants in Riemannian geometry*, Math. Res. Let. 7 (2000), 245-261.

[38] S. Brendle, *Einstein manifolds with nonnegative isotropic curvature are locally symmetric*, Duke Math. Journal 151 (2010), 1-21.

[39] D. Calderbank, P. Gauduchon, M. Herzlich,*Refined Kato inequalities and conformal weights in Riemannian geometry*, J. of Functional Analysis 173 (2000), 214-255.

[40] H-D. Cao, *Existence of gradient Kahler-Ricci solitons*, Elliptic and Parabolic Methods in Geometry (Minneapolis, MN, 1994), A K Peters, Wellesley, MA, (1996) 1-16.

[41] S. Tachibana, *A theorem of Riemannian manifolds of positive curvature operator*, Proc. Japan Acad. 50 (1974), 301-302.

[42] R. Hamilton, *Three manifolds with positive Ricci curvature*, J. Diff. Geom. 17 (1982), 255-306.

[43] R. Hamilton, *Four?manifolds with positive curvature operator*, J. Differential Geom. 24 (1986), 153-179.

[44] R. Hamilton, *The Ricci flow on surfaces*, Contemporary Mathematics 71 (1988) 237-261.

[45] B.L. Chen, X.P. Zhu, *Complete Riemannian manifolds with pointwise pinched curvature*, Invent. Math. 140 (2000), no. 2, 423-452.

[46] R. Hamilton, *Eternal solutions to the Ricci flow*, J. Diff. Geom. 38 (1993), 1-11.

[47] B.L. Chen, X.P. Zhu, *Volume growth and curvature decay of positively curved Kahler manifolds*, Quarterly Journal of Pure and Applied Mathematics Vol.1. no.1 68-108(2005).

[48] R. Greene, H. Wu, *Gap theorems for noncompact Riemannian manifolds*, Duke Math. J., 49 (1982), 731-756.

[49] C. Guenther, J. Isenberg, D. Knopf, *Stability of the Ricci flow at Ricci-flat metrics*, Comm. Anal. Geom. 10 (2002) 741-777.

[50] T. Ivey, *Ricci solitons on compact three-manifolds*, Diff. Geom. Appl. 3(1993), 301-307.

[51] N. Sesum, *Limiting behaviour of the Ricci flow*, arxiv: DG.math.DG/0402194.

[52] N. Sesum, *Linear and dynamical stability of Ricci flat metrics*, Duke Math. J., 133 (2006), 1-26.

[53] T. Aubin, *Equations diffrentielles non linaires et problme de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl., 55:269-296, 1976.

[54] X. Chen, Y. Wang, *On four-dimensional anti-self-dual gradient Ricci solitons*, preprint, 2011.

[55] B. Chow, P. Lu, L. Ni, *Hamilton's Ricci flow*, volume 77 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2006.

[56] J. Lee, T. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. (N.S.), 17(1):37-91, 1987.

[57] O. Munteanu, N. Sesum, *On gradient Ricci solitons*, J. Geom. Anal., 23(2):539-561, 2013.

[58] A. Naber, *Noncompact shrinking four solitons with nonnegative curvature*, J. Reine Angew. Math., 645:125-153, 2010.

[59] P. Petersen, W. Wylie, *Rigidity of gradient Ricci solitons*, Pacific journal of mathematics, 241(2):329-345, 2009.

[60] Z-H. Zhang, *Gradient shrinking solitons with vanishing Weyl tensor*, Pacific J. Math., 242(1):189-200, 2009.

[61] I. Jack, L. Parker, *Linear independence of renormalization countqerterms in curved space-times of arbitrary dimensionality*, J. Math. Phys. 28 (1987), 1137-1139.

[62] N. Koiso, *On rotationally symmmetric Hamilton's equation for Kahler-Einstein metrics*, Recent Topics in Diff. Anal. Geom., Adv. Studies Pure Math.,18-I,Academic Press, Boston, MA (1990), 327-337.

[63] A.S. Dancer, M.Y. Wang, *Some New Examples of Non-Kahler Ricci Solitons*, arXiv:0806.4408.

[64] A. Gastel, M. Kronz, *A family of expanding Ricci solitons*, Variational problems in Riemannian geometry, Progr. Nonlinear Differential Equations Appl., 59, Birkhuser, Basel (2004), 81-93.

[65] R. Bryant, *Local existence of gradient Ricci solitons*, unpublished.

[66] X.J. Wang, X.H. Zhu, *Kahler-Ricci solitons on toric manifolds with positive first Chern class*, Adv. Math. 188 (2004), no. 1, 87-103.

[67] M. Feldman, T. Ilmanen, D. Knopf, *Rotationally symmetric shrinking and expanding gradient Kahler-Ricci solitons*, J. Diff. Geom., 65 (2003), 169-209.

[68] R. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Differential Geometry (Cambridge, MA, 1993), 2, 7-136, International Press, Combridge, MA,1995.

[69] R. Bryant, *Gradient Kahler Ricci solitons*, arXiv.org/abs/math.DG/0407453.

[70] A. Chau, L. Tam, *A note on the uniformization of gradient Kahler Ricci solitons*, Math. Res. Lett., 12 (2005), 19-21.

[71] H-D. Cao, *Limits of Solutions to the Kahler-Ricci flow*, J. Differential Geom., 45 (1997), 257-272.

[72] L. Ni, N. Wallach, *On a classification of the gradient shrinking solitons*, preprint, arXiv:0710.3194.

[73] H-D. Cao, B-L. Chen, X-P. Zhu, *Recent developments on Hamilton's Ricci flow*, Surveys in differential geometry, Vol. XII, 47-112, Surv. Differ. Geom., XII, Int. Press, Somerville, MA, 2008.

[74] P. Petersen, P. Wylie, *On the classification of gradient Ricci solitons*, arXiv:0712.1298.

[75] A. Derdzinski. *Self-dual Kahler manifolds and Einstein manifolds of dimension four*, Composition Math., 49(3):405-433, 1983.