RESEARCH ON A NEW KIND OF MAGIC SQUARE

BU-405-M

January, 1972

1. A very curious question which has taxed the wisdom of many people for some time led me to make the following investigation which seems to open a new Field of Analysis and in particular of the theory of combinations. This question concerns a group of 36 officers, of six different ranks and drawn from six different regiments, whom it was a question of arranging in a square in such a way that on each line, horizontal as well as vertical, there would be found six officers different from each other in both rank and regiment. However, in spite of the trouble taken to resolve this problem, one is obliged to admit that such an arrangement is absolutely impossible, although one is not able to give a rigorous proof.

2. To explain more clearly the question mentioned, I will mark the six different regiments by the Latin letters

$$a, b, c, d, e, f,$$

and the six different grades by the Greek letters

$$\alpha, \beta, \gamma, \delta, \epsilon, \xi;$$

and it is clear that the characteristics of each officer are determined by two letters, one Latin and the other Greek, of which the first indicated his regiment and the other his rank, and that there will be actually 36 combinations of two of these letters, as follows:

$a\alpha$	$a\beta$	$a\gamma$	$a\delta$	$a\epsilon$	$a\xi$
$b\alpha$	$b\beta$	$b\gamma$	$b\delta$	$b\epsilon$	$b\xi$
$c\alpha$	$c\beta$	$c\gamma$	$c\delta$	$c\epsilon$	$c\xi$
$d\alpha$	$d\beta$	$d\gamma$	$d\delta$	$d\epsilon$	$d\xi$
$e\alpha$	$e\beta$	$e\gamma$	$e\delta$	$e\epsilon$	$e\xi$
$f\alpha$	$f\beta$	$f\gamma$	$f\delta$	$f\epsilon$	$f\xi$

each one of which expresses the characteristic of an officer. It is a question of writing these 36 terms in the 36 divisions of a square so that on each line, horizontal as well as vertical, one finds the six Latin and the six Greek letters.

3. One will then have three conditions to fulfill; first, that on each row one finds the six letters, Latin as well as Greek; second, that the same be true in all the columns; and lastly, that all the 36 terms above be found actually inscribed in the square, or that no symbol be found twice, which comes to the same thing. For, if it were only a matter of satisfying the first two conditions, it would not be difficult to find several solutions; here is one

$$\begin{array}{cccccc}
a\alpha & b\xi & c\delta & d\epsilon & e\gamma & f\beta \\
b\beta & c\alpha & f\epsilon & e\delta & a\xi & d\gamma \\
c\gamma & d\epsilon & a\beta & b\xi & f\delta & e\alpha \\
d\delta & f\gamma & e\xi & c\beta & b\alpha & a\epsilon \\
e\epsilon & a\delta & b\gamma & f\alpha & d\beta & c\xi \\
f\xi & e\beta & d\alpha & a\gamma & c\epsilon & b\delta
\end{array}$$

but this arrangement has the fault that the terms $b\xi$ and $d\epsilon$ are found twice and that the terms $b\epsilon$ and $d\xi$ are lacking entirely.

4. Then after all the care that has been used for the construction of such a square of thirty-six entries has proved useless, to give wider generality to my research, in place of six regiments and six different ranks, I will put an arbitrary number n, in such a way that there will be n Latin letters

a b c d etc.

and as many Greek letters

$\alpha\ \beta\ \gamma\ \delta$ etc.

to combine in n^2 different ways and to arrange in an $n \times n$ square array in such a way that each row and each column contains all the Latin and Greek letters and that no term is found twice in the square.

5. Since each line of the square contains all these different letters and consequently the sum is everywhere the same, it is clear that such an arrangement will satisfy the condition of the ordinary magic squares. For, to produce all the numbers in the natural order, one has only to give to the Latin letters, a, b, c, d, e, etc., the values o, n, 2n, 3n, 4n, ... $(n-1)n$, and to the Greek letters α, β, γ, δ, ϵ, etc., the values 1, 2, 3, 4, 5, ... n. But since in these squares it is a question only of the sum of all the numbers which are found in each line, horizontal as well as vertical, it is not at all necessary that all the numbers be found on each line provided that the sum be everywhere the same; which is also the reason that one can construct ordinary magic squares of 36 boxes.

6. To make easier the operations which I will have to perform eventually, I will put in place of the Latin and Greek letters the natural numbers 1, 2, 3, 4, 5, etc., where in order to distinguish between them I will call the ones Latin numbers and the others Greek numbers; and finally, so as never to confuse them, I will join the Greek numbers to the Latin numbers in the form of superscripts, in the way that will be seen in the following square of 49 boxes,

$$\begin{array}{ccccccc}
1^1 & 2^6 & 3^4 & 4^3 & 5^7 & 6^5 & 7^2 \\
2^2 & 3^7 & 1^5 & 5^4 & 4^1 & 7^6 & 6^3 \\
3^3 & 6^1 & 5^6 & 7^5 & 1^2 & 4^7 & 2^4 \\
4^4 & 5^2 & 6^7 & 1^6 & 7^3 & 2^1 & 3^5 \\
5^5 & 1^3 & 7^1 & 2^7 & 6^4 & 3^2 & 4^6 \\
6^6 & 7^4 & 4^2 & 3^1 & 2^5 & 5^3 & 1^7 \\
7^7 & 4^5 & 2^3 & 6^2 & 3^6 & 1^4 & 5^1
\end{array}$$

in which I have arranged the Latin numbers following their natural order, in the first row as well as the first column, in such a way that these numbers represent simultaneously the indices of these two lines and those of their companions. I have also made the Greek numbers, or superscripts, equal to the Latin numbers in the first vertical line, as I will do everywhere below, since the significance of these numbers is completely arbitrary.

7. Since it is easy to convince oneself that all the terms written in the preceding square satisfy perfectly the three conditions required and indicated above; to bring the reader closer to the point of view from which one must picture most of the methods which have brought us to the following research, we are going to begin by the analysis of the construction of the square mentioned above. For this purpose, we take once more the fundamental Latin square which, omitting the superscripts, will have the following form:

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	6	5	7	1	4	2
4	5	6	1	7	2	3
5	1	7	2	6	3	4
6	7	4	3	2	5	1
7	4	2	6	3	1	5

where each one of the seven lines, horizontal as well as vertical, contains all the seven numbers, 1, 2, 3, 4, 5, 6, 7.

8. Having thus established this Latin square, everything comes back to finding a sure method of joining Greek numbers, or superscripts, to each Latin number of this square; and first, in order to begin with the superscript 1, since it is necessary that it recur in each line, horizontal as well as vertical, it is a matter of taking from the vertical columns seven numbers such that they are different from each other and that they are related at the same time to different horizontal rows or rather: the numbers which one takes from

each vertical column should all be taken from different levels, which must be done similarly in relation to the other exponents, 2, 3, 4, 5, etc. At this point, it must once more be noted that since we suppose the exponents of the first column to be known, and since we always make them equal to the Latin numbers of this column, the first terms of these functions which we are going to describe will always follow the order of the natural numbers 1, 2, 3, 4, 5, 6, 7.

9. Since, then, in the following investigation everything depends on these functions which serve to regulate the writing of the superscripts, or to determine the ranks of the officers arranged, I will call them below square-forming functions; one must have one for each superscript. Thus, in the square of 49 entries listed above in the 6th paragraph, the square-forming functions are:

for the	superscript		1	this:	1 6 7 3 4 2 5,
"	"	"	2	"	2 5 4 6 1 3 7,
"	"	"	3	"	3 1 2 4 7 5 6,
"	"	"	4	"	4 7 3 5 6 1 2,
"	"	"	5	"	5 4 1 7 2 6 3,
"	"	"	6	"	6 2 5 1 3 7 4,
"	"	"	7	"	7 3 6 2 5 4 1.

This then is what one must understand by the term square-forming functions, which we will make use of throughout in the following; and it is first of all evident that, in order to construct a complete square, it is necessary to have such a function for each Greek number or superscript. To follow, it is necessary that all the functions should agree in such a way among themselves that in writing them one on top of the other, one will find in each column all the different numbers, because otherwise the same number of the Latin (or base) square should receive two different exponents.

10. Having therefore established for an arbitrary case a square of Latin numbers, the first step consists of finding the square-forming functions for each superscript, and if it happens that for a single one of these numbers one is unable to find any such a function one can boldly state that the Latin square is incapable of providing a complete square. And even if one has found functions for all the superscripts, if it is impossible to choose them in such a way that they agree among themselves in the way in which I have just described, since that has succeeded in the example above, it is once more a sure sign that the Latin square is not capable of furnishing a solution to the problem. But one must be careful not to come to this conclusion except after being fully convinced that one has found and studied all the square-forming functions which are valid for the proposed square.

11. The formation of the square-forming functions is therefore the first and the main object of this paper; but I must admit that up to this time I have not had any sure method by which I can conduct this investigation. It even seems that one should be content with a sort of simple process of trial and error that I am going to explain for the Latin square of 49 boxes set forth above.

For example, to find the characteristic function of the superscript 4 of this square let us choose arbitrarily the first four entries which I will take as they have been marked

$$4 \quad 7 \quad 3 \quad 5$$

and which are taken from the first four columns and from the four rows which correspond to the indices 4, 6, 1, 2; and it is clear that the last three values of our function,

$$1 \quad 2 \quad 6,$$

should be drawn from the last three columns and from the three rows which correspond to the indices 3, 5, 7. Therefore, the remaining pieces of the 3rd, 5th, and 7th rows furnish us the following table (or array)

1	4	2
6	3	4
3	1	5

from which obviously result the last three terms of our functions in the order 6, 1, 2 as we have shown them above. If the first four terms had not been known to us, one sees by what we have just said that it would have been necessary to examine in the same way all the possible combinations.

12. After having shown in general the operations which one must undertake in order to construct such complete squares, I go on to more particular investigations which naturally will vary in accordance with the nature of the Latin squares, which can be formed in as many more different ways as the number of entries of which it is made up is large; and one easily perceives that soon the number of all the possible methods of constructing it becomes so great that one no longer knows how to make a count. This is the reason that I will be content here to run through some simple and regular kinds of Latin squares, which will not fail to lead us to some much more complicated types.

13. First, the simplest Latin square is without doubt the one where all the numbers 1, 2, 3, 4, ... n progress cyclically in each row and column. The squares of this first kind, of a classification which so to speak arose naturally, will have in general for any number n^2 of entries, the following form:

1	2	3	4	5	6	. . .		n
2	3	4	5	6	. . .		n	1
3	4	5	6	. . .		n	1	2
4	5	6	. . .		n	1	2	3
5	6	. . .		n	1	2	3	4
6	. . .		n	1	2	3	4	5

etc.

The squares of this first type which occur for all n by n arrays will hereafter be named Latin squares <u>à simple marche</u>.

14. Following this classification, the second kind will contain the Latin squares <u>à double marche</u>, which are formed by taking the numbers of the first line, arranged in their natural order, two by two and transposing them in the second line, which will consequently be:

2 1 4 3 7 6 5 8 7 etc.

From this and from the first row, one then constructs the third and the fourth by adding 2 to each of their terms, the fifth and sixth by adding 2 to the terms of the third and fourth, and so forth. The squares of the second rank thus formed will have in general the following form:

1	2	3	4	5	6	7	8	etc.
2	1	4	3	6	5	8	7	etc.
3	4	5	6	7	8	9	10	etc.
4	3	6	5	8	7	10	9	etc.
5	6	7	8	9	10	11	12	etc.
6	5	8	7	10	9	12	11	etc.

etc.,

by which one can easily see that this second kind could not occur except for the squares where the number of boxes in each line is even.

15. For the third class, I refer to the Latin squares <u>à triple marche</u>, where in the first line one considers three numbers jointly, in order to vary them in three different ways, before forming the subsequent lines, which one obtains three by three by adding 3 to the terms of the three preceding, as one can see in the general form which follows:

1	2	3	4	5	6	7	8	9	etc.
2	3	1	5	6	4	8	9	7	etc.
3	1	2	6	4	5	9	7	8	etc.
4	5	6	7	8	9	10	11	12	etc.
5	6	4	8	9	7	11	12	10	etc.
6	4	5	9	7	8	12	10	11	etc.
7	8	9	10	11	12	13	14	15	etc.

etc.,

which shows us that this construction is valid only if the numbers of the boxes contained in a line is divisible by 3.

16. In the same way, one can form squares of the fourth kind proceeding <u>à quadruple marche</u> by taking separately four by four the entries of the first horizontal line and passing through all the transpositions which are possible and which form the four first horizontal lines, from which one derives the four following by adding 4 to each entry, and so on with the others. But since the first four entries,

1 2 3 4,

allow several different transpositions, we will have several general forms for the squares of this kind, of which it will be sufficient to cite the first member (I call "member of a square" any one of these parts which form a separate square) since it is easy to deduce from it the general form, the transpositions being the same in all the other members or simple squares from which is formed the large Latin square which, in this case, should always have a number of boxes divisible by $4^2=16$. Here are 4 similar transpositions

I				II				III				IV			
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
2	1	4	3	2	1	4	3	2	3	4	1	2	4	1	3
3	4	1	2	3	4	2	1	3	4	1	2	3	1	4	2
4	3	2	1	4	3	1	2	4	1	2	3	4	3	2	1

since it would be superfluous to form or to cite the general forms for the squares composed of similar members. One perceives easily that following the same laws one has only to vary the following Quaternaries of the first row.

One sees also that this classification could guide us to many other regular squares; but we stop here, to develop more carefully in the following sections the four kinds which we have just established and to deduce from them some complete squares.

First Section

SOME LATIN SQUARES A SIMPLE MARCHE OF THE GENERAL FORM

1	2	3	4	5	6	. . .		n
2	3	4	5	6	. . .		n	1
3	4	5	6	. . .		n	1	2
4	5	6	. . .		n	1	2	3
5	6	. . .		n	1	2	3	4
6	. . .		n	1	2	3	4	5

etc.

CASE OF n = 2

17. Let us begin by the simplest case, where n = 2 and the Latin square is

$$\begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix}$$

from which one is not able to take any square-forming function, and consequently this case is impossible, since it is not possible to deduce any other square. And in fact, if one satisfies the first two conditions of the question, cited in Section 3, one comes to the square

$$\begin{matrix} 1^1 & 2^2 \\ 2^2 & 1^1 \end{matrix}$$

where the two terms 1^1 and 2^2 are found twice, while the two others, 2^1 and 1^2, are missing entirely. Thus, if the question concerns a group of 4 officers of two different ranks and regiments, one sees first that it will be impossible to arrange them in a square in the manner prescribed.

CASE OF n = 3

18. Let us go on to the case of n = 3, and our Latin square will be

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix}$$

where the diagonal with different entries, 1 3 2, furnishes first a characteristic function for the superscript 1; and since all the numbers increase, while descending, by one, it is clear that the characteristic functions will follow the same order and that consequently they will be

for the exponent	1 - 1	3	2,
" " "	2 - 2	1	3,
" " "	3 - 3	2	1.

So in inserting the exponents following this system of functions one will obtain the following complete square;

$$\begin{matrix} 1^1 & 2^3 & 3^2 \\ 2^2 & 3^1 & 1^3 \\ 3^3 & 1^2 & 2^1 \end{matrix}$$

which is the only solution which can take place for the squares à simple marche of nine entries since function 1, 3, 2 is the only square-forming function for the exponent 1 and since the fundamental or Latin square proposed is the only one for the case cited.

CASE OF n = 4

19. Let us consider the case where n = 4, which brings us to the following Latin square:

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{matrix}$$

but here, one sees first that it is impossible to find any function for the superscript 1, and, on examining the square according to the prescribed rules, we will see that it is the same for all other superscripts; from which one must conclude that this Latin square cannot furnish any complete square for the value n = 4. But one must notice carefully that this Latin square is not the only one which can occur for the value cited, considering that one can form three others, among which will be found one which will lead us to three beautiful solutions, and so it is only the square of 16 entries à simple marche which fails to meet the required conditions.

20. The same inconvenience is found in all the values where the number n is even, and this observation leads us to the following theorem:

For all the cases where the number n is even the Latin square à simple marche can never furnish a solution to the question proposed.

To prove this, one has only to show that it is impossible to find any functions for the superscript 1 of any square à simple marche where the number of horizontal or vertical entries is even. Let us suppose for this purpose that such a function might be

$$1 \quad a \quad b \quad c \quad d \quad e \quad \text{etc.}$$

where the letters a, b, c, d, etc., of which the number is $n - 1$, indicate the numbers 2, 3, 4 ... n, in a given order, which is determined by the (horizontal) rows corresponding to the values α β γ δ ϵ etc., which indicate also the numbers 2, 3, 4, 5 etc. in such a way that the sum of all the numbers α, β, γ, δ, etc. must be equal to the sum of a, b, c, d, etc.

Then since, in our Latin square, all the numbers of the (horizontal) rows increase in arithmetical progression where the difference is 1, noticing that in passing on to the numbers beyond n it is necessary to begin again with one, it follows that, because the second number, a, of the assumed function is drawn from the second (vertical) column and from the (horizontal) row which corresponds to the value α, one will have

$$a = \alpha + 1.$$

In the same way, since the third entry, b, of this function is drawn from the third (vertical) column and from the (horizontal) row corresponding to the value β there will be

$$b = \beta + 2.$$

In following this reasoning, one finds that there will be for the other entries

$$c = \gamma + 3, \; d = \delta + 4, \; e = \epsilon + 5, \; f = \xi + 6 \quad \text{etc.},$$

noticing always that having arrived at a number larger than n one will put in its place the excess above n. Now let the sum of all the letters be

$$\alpha + \beta + \gamma + \delta + \text{etc.} = S,$$

and the sum of the letters

$$a + b + c + d + \text{etc. will be} = S + 1 + 2 + 3 + \ldots + (n - 1),$$

or rather there will be

$$a + b + c + d + \text{etc.} = S + n(n - 1)/2 .$$

Now, the sum of the Latin letters, $a + b + c + d +$ etc., and that of the Greek letters $\alpha + \beta + \gamma + \delta +$ etc., as we have seen above, should be equal to each other or, which comes to the same thing, the difference should be a multiple of the number n, which being put $= \lambda n$ brings us to this equation

$$n(n - 1)/2 = \lambda n,$$

which gives

$$\lambda = (n - 1)/2.$$

Consequently, since λ is a whole number, this equality could not exist unless $n - 1$ was an even number or n an odd number. In this way, the truth of our theorem is rigorously proven, and it would be useless to wish to apply the Latin squares to any case where n is an even number.

CASE OF n = 5

21. Let us go back to our squares, and the case of $n = 5$ leads us to the following Latin square <u>à simple marche</u>

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

from which one can without difficulty derive the following three functions for the exponent 1:

1	3	5	2	4
1	4	2	5	3
1	5	4	3	2

By adding one to each of the terms of these functions, one will obtain those for the exponent 2, which, by adding one to it again, will give those for the exponent 3, and so on for the others. In this way, one will be able to construct the following three squares capable of determining the writing of the superscripts

I					II					III				
1	3	5	2	4	1	4	2	5	3	1	5	4	3	2
2	4	1	3	5	2	5	3	1	4	2	1	5	4	3
3	5	2	4	1	3	1	4	2	5	3	2	1	5	4
4	1	3	5	2	4	2	5	3	1	4	3	2	1	5
5	2	4	1	3	5	3	1	4	2	5	4	3	2	1

22. By means of these three complete systems of functions, we will be able to make three complete squares of 25 entries and consequently as many solutions, if the problem concerns a group of 25 officers of five different ranks from five different regiments. Here are the three complete squares: *

*To establish the relation between these three complete squares and the functions at the end of Section 21, one must invert the order of squares II and III. L.G.D.

I				
1^1	2^5	3^4	4^3	5^2
2^2	3^1	4^5	5^4	1^3
3^3	4^2	5^1	1^5	2^4
4^4	5^3	1^2	2^1	3^5
5^5	1^4	2^3	3^2	4^1

II				
1^1	2^3	3^5	4^2	5^4
2^2	3^4	4^1	5^3	1^5
3^3	4^5	5^2	1^4	2^1
4^4	5^1	1^3	2^5	3^2
5^5	1^2	2^4	3^1	4^3

III				
1^1	2^4	3^2	4^5	5^3
2^2	3^5	4^3	5^1	1^4
3^3	4^1	5^4	1^2	2^5
4^4	5^2	1^5	2^3	3^1
5^5	1^3	2^1	3^4	4^2

The construction of these three squares is all the easier since after one has written the superscript 1, the others follow in their natural order, descending by the (vertical) columns.

23. It remains for us to remark again regarding the square-forming functions, that their entries go in arithmetical progression, by increasing in the first by 2, in the second by 3, in the third by 4, in the 4th by 5 and so on with the others. Next, that the superscripts of the first rows of the three complete squares are

in the first	1	5	4	3	2
in the second	1	3	5	2	4
in the third	1	4	2	5	3,

which agree with the three square-forming functions. Finally, the first of these three kinds of squares, in changing the order of the (horizontal) rows, furnishes the following very remarkable square

1^1	2^5	3^4	4^3	5^2
3^3	4^2	5^1	1^5	2^4
5^5	1^4	2^3	3^2	4^1
2^2	3^1	4^5	5^4	1^3
4^4	5^3	1^2	2^1	3^5

in which not only do the (vertical) columns and (horizontal) rows contain the different Greek and Latin letters, but where even the diagonals and their completed parallels* as

$$3^3 \quad 1^4 \quad 4^5 \quad 2^1 \quad 5^2 \ ,$$

satisfy all the prescribed conditions.

CASE OF n = 7

24. The case of n = 7 gives us the following Latin square à simple marche of 49 entries.

1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3
5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6

where the consideration of the functions increasing in arithmetical progression (section 20) furnishes us first the following functions for the superscript 1

1	3	5	7	2	4	6
1	4	7	3	6	2	5
1	5	2	6	3	7	4
1	6	4	2	7	5	3
1	7	6	5	4	3	2

where the first increases by 2, the second by 3, the third by 4, the fourth by 5, and the fifth by 6. But one must not think that here are all the functions for

* This is what is called broken diagonals. L.G.D.

the superscript 1, because in examining the square more carefully one in addition finds the following 14:

1	3	6	2	7	5	4
1	3	7	6	4	2	5
1	4	6	3	2	7	5
1	4	7	5	3	2	6
1	4	7	2	6	5	3
1	4	2	7	6	3	5
1	5	4	2	7	3	6
1	5	7	3	6	4	2
1	6	4	7	3	5	2
1	6	4	3	7	2	5
1	6	5	2	4	7	3
1	6	2	5	7	4	3
1	7	4	6	2	5	3
1	7	5	3	6	2	4

25. All these square-forming functions have been found by the very laborious method explained above (section 8); but the beautiful order which prevails in the squares _à simple marche_ gives us very easy methods of finding many such functions, as soon as one has found only one, which will be the subject of the following problem:

Having found a square-forming function for some square à simple marche of which the number n is odd, find the sure rules by means of which one can find several other square-forming functions.

26. Let

1 a b c d e etc.

be the characteristic function which has been found, which refers to the superscript 1, and from which the entry which corresponds to the undetermined index t be $= x$, so that taking $t = 1$, x also becomes $= 1$. It is necessary to note 1st) that giving to t all the values possible from 1 to n, the entry x should also receive all these different values; 2nd) that, since t is the index of the (vertical) column from which the number x is taken, the index of the (horizontal) row will be, as one sees from the construction of the square, $= x - t + 1$, which corresponds also to the entry x. Then since the numbers a, b, c, d, etc. must be taken from different (horizontal) rows, it follows that this formula $x - t + 1$, and consequently also $x - t$, should include all the different values, in the same manner as the numbers t and x.

27. That noted, let

$$1 \quad A \quad B \quad C \quad D \quad E \quad \text{etc.},$$

be a new square-forming function which one wishes to derive from the one given and where the value of any entry X will be $= T$; and one understands, because we have said so above, that in giving to T all the values possible, not only the entry X, but also the difference $X - T$ should likewise receive all these same different values. These conditions will be obviously filled in taking

$$T = x \text{ and } X = t$$

and consequently one will always obtain a new function by exchanging the two numbers t and x (between themselves) from which comes this rule for the formation of a new function:

<u>Take x for the index and t for the entry which corresponds to it.</u>

This new function will then be formed <u>per inversionem</u>, by reversal.

28. One can make another new function by taking

$$T = t \text{ and } X = \alpha + t - x \, .$$

For, if one varies the value of t through all the values from 1 to n, it is evident that the entry $\alpha + t - x$ will also receive all these different values, no matter what may be the number α. Therefore, since $X - T = \alpha - x$, this equation will also receive all the possible values. But, for this found function to correspond to the superscript 1, it is necessary that, putting $t = 1$ and $x = 1$, there also ensue $X = 1$, which gives us $\alpha = 1$.

<u>Therefore one will always obtain a new function by taking</u>

$$T = t \text{ and } X = 1 + t - x \; ,$$

and it is in this that the second rule which I have proposed to give consists.

29. In combining the two rules which I have just explained, it will be easy to derive from only one given function a number of new functions which one can represent in the following way:

	I	II	III	IV	V	VI
T = t	t	x	1 + t - x	x	1 + t - x	1 + x - t
X = x	1 + t - x	t	t	1 + x - t	2 - x	x
	VII	VIII	IX	X	XI	XII
	2 - x	1 + x - t	2 - x	2 - t	2 - t	2 - t
	1 + t - x	2 - t	2 - t	1 + x - t	2 - x	2 - x

Here then are eleven different rules by the use of which one can derive eleven new functions, all different (from each other), from one single proposed square-forming function.

30. To clarify the two principal rules and those of the preceding section, which have been derived from them, by an example, let us take at random one of the functions cited above (section 24), for example this one

$$1 \quad 4 \quad 2 \quad 7 \quad 6 \quad 3 \quad 5 \; ,$$

to which we can apply alternately the first and second rule, or else the second and the first. The two sets of functions which they give rise to are as follows:

given function	1 4 2 7 6 3 5	given function	1 4 2 7 6 3 5
1st rule	1 3 6 2 7 5 4	2nd rule	1 6 2 5 7 4 3
2nd rule	1 7 5 3 6 2 4	1st rule	1 3 7 6 4 2 5
1st rule	1 6 4 7 3 5 2	2nd rule	1 7 4 6 2 5 3
2nd rule	1 4 7 5 3 2 6	1st rule	1 5 7 3 6 4 2
1st rule	1 6 5 2 4 7 3	2nd rule	1 5 4 2 7 3 6
2nd rule	1 4 6 3 2 7 5	1st rule	1 4 6 3 2 7 5
1st rule	1 5 4 2 7 3 6	2nd rule	1 6 5 2 4 7 3
2nd rule	1 5 7 3 6 4 2	1st rule	1 4 7 5 3 2 6
1st rule	1 7 4 6 2 5 3	2nd rule	1 6 4 7 3 5 2
2nd rule	1 3 7 6 4 2 5	1st rule	1 7 5 3 6 2 4
1st rule	1 6 2 5 7 4 3	2nd rule	1 3 6 2 7 5 4
2nd rule	1 4 2 7 6 3 5	1st rule	1 4 2 7 6 3 5

perfectly equal, with the only difference being that in beginning with the second rule, the order of the functions is reversed.

31. Here then are eleven new functions which all derive their origin from only one, and even from any one among them. Also all these new functions are found among the 14 cited above (section 24) and there are only two which failed to appear in the course of this operation, namely

1 4 7 2 6 5 3 and 1 6 4 3 7 2 5

both of an unusual kind, since each one reproduces itself by the first rule, while, by the second rule, one is reproduced by the other.

32. After having found for the value $n = 7$ nineteen different functions, we could derive from each one a complete square, and consequently nineteen different kinds. For in taking any one and letting it be 1, a, b, c, d, e, f, and continuing these numbers according to their natural order, one will have the functions for the following exponents, 2, 3, 4, 5, 6 and 7, and in this way, one will obtain the following square of functions:

1	a	b	c	d	e	f
2	a + 1	b + 1	c + 1	d + 1	e + 1	f + 1
3	a + 2	b + 2	c + 2	d + 2	e + 2	f + 2
4	a + 3	b + 3	c + 3	d + 3	e + 3	f + 3
5	a + 4	b + 4	c + 4	d + 4	e + 4	f + 4
6	a + 5	b + 5	c + 5	d + 5	e + 5	f + 5
7	a + 6	b + 6	c + 6	d + 6	e + 6	f + 6

where it is evident that each (vertical) column as well as each row, contains all the (various) numbers from 1 to 7, no matter what may be the order of the numbers a, b, c, d, e and f.

33. To facilitate the construction of the complete square being sought, it will be well to assign the superscripts which agree with the first row, which is always the natural series of (the) numbers 1, 2, 3, 4, 5, 6, 7. For this purpose, in the proposed function

$$1 \quad a \quad b \quad c \quad d \quad e \quad f$$

let the entry which corresponds to the index $t = x$; and to this entry x in the square one should add the superscript 1. Then, as the superscripts increase, in descending through each (vertical) column, following their natural order, the following entry, $x + 1$, will have the superscript 2 and, in general, the entry $x + \lambda$ will have the superscript $\lambda + 1$. Let us take then λ in such a way that it becomes $x + \lambda = t$, from which one derives $\lambda = t - x$; and consequently, the number t in the first horizontal line will have the superscript

$$\lambda + 1 = t + 1 - x .$$

Then let us give to t successively the values 1, 2, 3, 4 etc., and the superscripts of the first row will be the following:

$$1,\ 3 - a,\ 4 - b,\ 5 - c,\ 6 - d,\ 7 - e,\ 8 - f .$$

34. We have seen above that this function is also a square-forming function which results from the first by use of the second rule. This is why, to construct the complete square, one should first take each square-forming function, to show the superscripts which should be given to the numbers of the first (horizontal) row; then, going down the (vertical) columns, one has only to increase the higher superscripts following their natural order. In this way, if the proposed function 1, a, b, c, d, e, f is at the same time the sequence of the superscripts of the first row, the complete square which is derived will have the following form:

$$\begin{array}{ccccccc}
1^1 & 2^a & 3^b & 4^c & 5^d & 6^e & 7^f \\
2^2 & 3^{a+1} & 4^{b+1} & 5^{c+1} & 6^{d+1} & 7^{e+1} & 1^{f+1} \\
3^3 & 4^{a+2} & 5^{b+2} & 6^{c+2} & 7^{d+2} & 1^{e+2} & 2^{f+2} \\
4^4 & 5^{a+3} & 6^{b+3} & 7^{c+3} & 1^{d+3} & 2^{e+3} & 3^{f+3} \\
5^5 & 6^{a+4} & 7^{b+4} & 1^{c+4} & 2^{d+4} & 3^{e+4} & 4^{f+4} \\
6^6 & 7^{a+5} & 1^{b+5} & 2^{c+5} & 3^{d+5} & 4^{e+5} & 5^{f+5} \\
7^7 & 1^{a+6} & 2^{b+6} & 3^{c+6} & 4^{d+6} & 5^{e+6} & 6^{f+6}
\end{array}$$

35. Having then found, in all, nineteen functions for the value $n = 7$, one could make from them as many complete squares; so that, if the question concerns 49 officers of seven different ranks and taken from seven different regiments, one can derive a large number of different solutions, all derived from a single Latin square <u>à simple marche</u>. One can even draw from the same

source several other solutions; since the number of functions is so considerable, having taken one at random for the superscript 1, one could take the functions for the following superscripts from other kinds, so that always in arranging these different functions in a square, the numbers in the (vertical) columns all differ among themselves. One sees then, in this way, that one will obtain a much larger number of new kinds of complete squares, mixed with several square-forming functions joined together. It will suffice to clarify this mixture of functions by a single example.

Functions of superscripts							Types of square-forming functions						
1	4	7	2	6	5	3	1	4	7	2	6	5	3
2	7	5	4	1	3	6	1	6	4	3	7	2	5
3	6	1	5	4	2	7	1	4	6	3	2	7	5
4	1	3	6	2	7	5	1	5	7	3	6	4	2
5	2	6	3	7	4	1	1	5	2	6	3	7	4
6	3	2	7	5	1	4	1	5	4	2	7	3	6
7	5	4	1	3	6	2	1	6	5	2	4	7	3

One easily understands, from this single example, that one can find many other equally suitable combinations whose number it would even be very difficult to determine.

36. If one inserts the exponents in conformity with these functions, the complete square which results therefrom will have this form:

1^1	2^5	3^4	4^2	5^6	6^7	7^3
2^2	3^6	4^7	5^3	6^1	7^4	1^5
3^3	4^1	5^2	6^4	7^5	1^6	2^7
4^4	5^7	6^5	7^6	1^2	2^3	3^1
5^5	6^3	7^1	1^7	2^4	3^2	4^6
6^6	7^2	1^3	2^1	3^7	4^5	5^4
7^7	1^4	2^6	3^5	4^3	5^1	6^2

Here, one sees first that the superscripts of the rows are no longer square-forming functions, as in the nineteen preceding kinds, and that one is not able to discover any order, since one finds there a mixture of seven different kinds. This observation is of the utmost importance, because the consideration of the regular squares might delude us into believing that the superscripts of the first (horizontal) rows should generally have the properties of square-forming functions.

Moreover, it is without doubt very surprising that, while the case of $n = 7$ furnishes us such a prodigious number of solutions, which will be further augmented later, the case of $n = 6$ is not able to furnish even one, even though the case which precedes it, $n = 5$, has led us to three different solutions.

CASE OF $n = 9$

37. Now let $n = 9$; and the Latin square à simple marche to which the following applies, will have this form:

1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	1
3	4	5	6	7	8	9	1	2
4	5	6	7	8	9	1	2	3
5	6	7	8	9	1	2	3	4
6	7	8	9	1	2	3	4	5
7	8	9	1	2	3	4	5	6
8	9	1	2	3	4	5	6	7
9	1	2	3	4	5	6	7	8

38. Since it would be very difficult to look for all the functions which might be found in this Latin square and since the number undoubtedly will be enormous, I will be satisfied to consider only those which go in arithmetical progression, excluding those where the difference would be 3 or 6, since it would not be prime to the number n = 9. For generally, it is always necessary that the difference between the terms of these progressions, as well as the difference between the number x and t, or rather x - t, have no common divisor with the number n, because a function chosen without regard for this rule would not contain all the values from 1 to n, or could not be arranged in the class of square-forming functions. So when these two cases are excluded, the functions which go in arithmetical progression are these:

1	3	5	7	9	2	4	6	8
1	6	2	7	3	8	4	9	5
1	9	8	7	6	5	4	3	2

from which one can make three complete squares of 81 entries by taking the functions for the following superscripts of the same kind, considering that we are excepting ourselves from making an enumeration of the others.

For, in taking one of these three functions, and letting it be

1 a b c d etc.,

for the superscript 1, one will see from what we have said above, (sections 23 and 33), that the superscripts of the first row which are 1, 3 - a, 4 - b, 5 - c, 6 - d, etc., also comprise a square-forming function and that, consequently, one can take first each of these three functions which we have found for the superscripts of the first (horizontal) row, which gives us these three complete squares:

I.

1^1	2^3	3^5	4^7	5^9	6^2	7^4	8^6	9^8
2^2	3^4	4^6	5^8	6^1	7^3	8^5	9^7	1^9
3^3	4^5	5^7	6^9	7^2	8^4	9^6	1^8	2^1
4^4	5^6	6^8	7^1	8^3	9^5	1^7	2^9	3^2
5^5	6^7	7^9	8^2	9^4	1^6	2^8	3^1	4^3
6^6	7^8	8^1	9^3	1^5	2^7	3^9	4^2	5^4
7^7	8^9	9^2	1^4	2^6	3^8	4^1	5^3	6^5
8^8	9^1	1^3	2^5	3^7	4^9	5^2	6^4	7^6
9^9	1^2	2^4	3^6	4^8	5^1	6^3	7^5	8^7

II.

1^1	2^6	3^2	4^7	5^3	6^8	7^4	8^9	9^5
2^2	3^7	4^3	5^8	6^4	7^9	8^5	9^1	1^6
3^3	4^8	5^4	6^9	7^5	8^1	9^6	1^2	2^7
4^4	5^9	6^5	7^1	8^6	9^2	1^7	2^3	3^8
5^5	6^1	7^6	8^2	9^7	1^3	2^8	3^4	4^9
6^6	7^2	8^7	9^3	1^8	2^4	3^9	4^5	5^1
7^7	8^3	9^8	1^4	2^9	3^5	4^1	5^6	6^2
8^8	9^4	1^9	2^5	3^1	4^6	5^2	6^7	7^3
9^9	1^5	2^1	3^6	4^2	5^7	6^3	7^8	8^4

III.

1^1	2^9	3^8	4^7	5^6	6^5	7^4	8^3	9^2
2^2	3^1	4^9	5^8	6^7	7^6	8^5	9^4	1^3
3^3	4^2	5^1	6^9	7^8	8^7	9^6	1^5	2^4
4^4	5^3	6^2	7^1	8^9	9^8	1^7	2^6	3^5
5^5	6^4	7^3	8^2	9^1	1^9	2^8	3^7	4^6
6^6	7^5	8^4	9^3	1^2	2^1	3^9	4^8	5^7
7^7	8^6	9^5	1^4	2^3	3^2	4^1	5^9	6^8
8^8	9^7	1^6	2^5	3^4	4^3	5^2	6^1	7^9
9^9	1^8	2^7	3^6	4^5	5^4	6^3	7^2	8^1

40. Here then are three complete squares, derived from the three regular (isometrical?) functions which we intended to examine. To clarify better the use of the rules cited above (sections 26, 27 and 28) for the formation of functions and finally to be able to judge more easily their number, we are going to choose one of the functions at random; and by applying to it successively the two rules, we will obtain the following twelve functions:

	characteristic adopted	1	6	5	9	2	4	8	7	3
	of which the reverse is	1	5	9	6	3	2	8	7	4
from which one derives	by the 2nd rule	1	6	8	5	4	3	9	2	7
		1	7	4	8	3	5	9	2	6
	by the 1st rule	1	8	6	5	4	2	9	3	7
		1	8	5	3	6	9	2	4	7
	by the 2nd rule	1	4	7	9	2	5	8	6	3
		1	4	8	2	9	7	6	5	3
	by the 1st rule	1	5	9	2	6	8	3	7	4
		1	4	9	2	8	7	6	3	5
	by the 2nd rule	1	7	4	3	9	8	5	2	6
		1	8	4	3	7	9	2	6	5

where one must stop at the sixth pair, since, if one wished to apply to it again the first rule, one would obtain the same functions where one reproduces the other by reversal; thus, our two rules have given us all together eleven new functions.

41. One very important observation, which is still to be made, is that in making use of the third rule, which we have been able to dispense with in the preceding article about $n = 7$, since it would not have been of any help to us, one could find another dozen new functions. This rule can be stated in the following way:

<u>Given for the proposed characteristic the index = t and the entry which corresponds to it = x, one can take, for the new function, the index $T = 2t - 1$ and the term itself $X = 2x - 1$;</u> for which the reason is clear, 1st) because in taking $t = 1$ and $x = 1$, there results

$$T = 1 \text{ and } x = 1 \;;$$

2nd) because if the x's vary through all the values, also the 2x's and consequently the $2x - 1$'s will also pass through (satisfy) all the different values; and 3rd) since, if $x - t$ contains all the values from 1 to 9, likewise

$$X - T = 2(x - t)$$

will pass through (satisfy) all the proper variations.

42. It will be well to clarify by an example this new rule, so fruitful in functions jointly with the two preceding ones; and for this result, we are going to take the function chosen above; which will give us the following dozen:

The function adopted furnishes		1	6	5	9	2	4	8	7	3
by the 3rd rule from which one		1	7	2	6	9	4	8	5	3
derives by the 1st rule		1	3	9	6	8	4	2	7	5
and then	by the 2nd rule	1	5	2	8	6	3	9	4	7
		1	9	4	8	7	3	6	2	5
	by the 1st rule	1	3	6	8	2	5	9	4	7
		1	8	6	3	9	7	5	4	2
	by the 2nd rule	1	9	7	6	4	2	8	5	3
		1	4	7	2	6	9	3	5	8
	by the 1st rule	1	6	9	5	8	4	3	7	2
		1	4	7	2	8	5	3	9	6
	by the 2nd rule	1	6	4	9	7	3	5	2	8
		1	8	6	3	7	2	5	9	4

43. Let us apply again this third rule to the first of the new dozen functions which we have just found and we will obtain, with the help of the two preceding rules, the following new dozen:

From the function adopted one		1	7	2	6	9	4	8	5	3
obtains by the 3rd rule and		1	7	4	6	3	9	2	5	8
from that one by the 1st rule		1	7	5	3	8	4	2	9	6
	by the 2nd rule	1	5	9	8	3	7	6	4	2
		1	5	8	2	7	3	6	9	4
	by the 1st rule	1	9	5	8	2	7	6	4	3
		1	4	6	9	2	7	5	3	8
which gives us	by the 2nd rule	1	3	8	6	4	9	2	5	7
		1	8	7	5	4	9	3	6	2
	by the 1st rule	1	7	2	5	8	4	9	3	6
		1	9	7	5	4	8	3	2	6
	by the 2nd rule	1	5	2	9	7	3	8	6	4
		1	3	6	9	2	8	5	7	4

Here, (as everywhere else) one has continued as far as the reproduction of the original functions, which has occurred until now at the sixth pair.

44. If one wishes to apply the third rule to the first of these functions, that is

1 7 4 6 3 9 2 5 8

one would derive this one

1 8 4 3 7 9 2 6 5

which is already found in the first dozen; so that our three rules have furnished us only three dozen functions, even though there certainly are for this case a much larger number, considering that, among all those we have just found, there is none which is in accord with its reversal. Nevertheless, one should find several for this case, since in the preceding case, where n = 7, there were at least two similar functions.

45. To convince ourselves entirely, let us look for a function which has the property of reproducing itself by reversal.

Such is this one - - - - - - - - - -	1	8	5	9	3	7	6	2	4
which reproduces itself by the 1st rule	1	8	5	9	3	7	6	2	4
we will have then by the 2nd rule	1	4	8	5	3	9	2	7	6
by the 1st rule	1	7	5	2	4	9	8	3	6
by the 2nd rule	1	5	8	3	2	7	9	6	4
by the 1st rule	1	5	4	9	2	8	6	3	7
by the 2nd rule	1	7	9	5	4	8	2	6	3

The first function adopted as reversible has thus brought us five other new functions; from which one sees that there are also functions not reversible which are found in a close union with those which are reversible and which are not found at all in the dozen preceding.

46. In examining the first of the characteristics cited, we shall see that they can give us another half-dozen altogether new functions. For this function of the preceding order

1 8 5 9 3 7 6 2 4

by rule 3 gives us this

1 4 6 2 9 3 8 7 5

which, being reversible, will give us the following characteristics

The reversible function		1	4	6	2	9	3	8	7	5
furnished by the	2nd rule	1	8	7	3	6	4	9	2	5
	1st rule	1	8	4	6	9	5	3	2	7
	2nd rule	1	4	9	8	6	2	5	7	3
	1st rule	1	6	9	2	7	5	8	4	3
	2nd rule	1	6	4	3	8	2	9	5	7

where we have continued the processes, as before, as far as the reproduction of a reversible function.

47. In the same way, in applying the third rule to the first function of the preceding order, one gets the following reversible function.

1 5 7 6 2 4 3 9 8

from which the following functions are derived by the alternate application of the first and second rule

Reversible function	1	5	7	6	2	4	3	9	8
2nd rule	1	7	6	8	4	3	5	9	2
1st rule	1	9	6	5	7	3	2	4	8
2nd rule	1	3	7	9	8	4	6	5	2
1st rule	1	9	2	6	8	7	3	5	4
2nd rule	1	3	2	8	7	9	5	4	6

48. If we should wish to repeat these operations, by applying again the third rule to the first function of this new order, we will come again to the first half dozen and then later to the others, so that this source of characteristics seems to have been used up by the three rules used. Having then found up till now three classes of twelve functions and three others of six functions, we have altogether 54, and with the first three which proceed in arithmetical progression, 57 different functions each one of which can give a complete square; and in mixing them together, as one can do in the manner shown above (sections 35 and 36) one easily understands that the number of all the possible solutions must become incomparably larger.

49. The 57 functions that we have found do not even come close to including all the possible functions; granted that by using the first direct method, which I have set forth above (sections 8, 9, and 10), one can easily find the 8 following functions which are not included in any of the orders(?) cited:

1	3	5	8	2	9	6	4	7
1	3	5	9	8	4	2	7	6
1	3	6	8	2	4	9	7	5
1	3	6	8	4	2	9	5	7
1	3	6	9	4	8	2	5	7
1	3	6	2	9	8	4	7	5
1	3	6	9	7	4	2	5	8
1	3	7	6	2	9	5	4	8

from which one can conclude that the total number of functions will be at least four times larger.

SOME ODD MAGIC SQUARES OF WHICH THE DIAGONALS AND THEIR PARALLELS ARE ALSO ENDOWED WITH THE PRESCRIBED CONDITIONS

50. Let n be some odd number and d the difference between terms of a function which proceeds in arithmetical progression

$$1,\ 1 + d,\ 1 + 2d,\ 1 + 3d \text{ etc.}$$

and of which the terms, if one subtracts the number n from all those which exceed this number, should produce all the different values from 1 to n, after having continued as far as the entry $1 + (n - 1)d$.

That being granted, it is clear that the difference d should be a prime number to n and that consequently, when n is a prime number, one can give to d all the values below n; whereas, if n has a factor p, one must exclude all the progressions where the difference d is p, 2p, 3p, 4p, etc. This essential condition is not even sufficient to give to this progression the property of a square-forming function; for since to the index $t = 1 + \lambda$ there corresponds the term $x = 1 + \lambda d$, as we have shown in another part (section 26) the function $x - t = \lambda(d - 1)$ must also produce all the different numbers. From this, it is evident that the number $d - 1$ must be prime to n; that consequently one must always exclude the value $d = 1$, and the values $d = p + 1$, $d = 2p + 1$, $d = 3p + 1$ etc., every time that n includes a factor p.

51. Now, it is not difficult to determine in general, for each number n, the number of values that the difference d can receive. For, if n is a prime number, the number of the values of d, which one takes always smaller than n, will be $n - 2$, and the number of the functions in progression which will occur will also be $n - 2$. If n is a product of two factors different from each other, as $n = pq$, the number of all the values of d will be

$$(p - 2)(q - 2) .$$

And in general, if n is a product of several different factors, $p\ q\ r\ s$ etc., the values of d will be to the number of

$$(p - 2)(q - 2)(r - 2)(s - 2) \text{ etc.}$$

But when n has two or more factors equal to each other, the form of the equation for the number of values of d will be a little different. For if $n = p^{\alpha}q^{\beta}r^{\gamma}s^{\delta}$ etc. the number of values one can give to d will be

$$p^{\alpha-1}q^{\beta-1}r^{\gamma-1}s^{\delta-1} \text{ etc.} \quad (p - 2)(q - 2)(r - 2)(s - 2) \text{ etc.}$$

52. After these remarks, it will be easy to construct in general a magic square in such a way that not only the rows and columns, but even the two diagonals and all their parallels (each completed by its corresponding one from the other side [section 23]) are made up of terms all of which are different from each other. For this purpose, I should first note that, whatever may be the form of such a square represented by Greek and Latin letters, as we did in the beginning, one can always reduce it to numbers and in such a way that the first (vertical) column contains all the entries in their natural order, as we have assumed thus far; and the problem will come down to seeing in what way one must transpose the

other (vertical) columns of the complete squares, so that the required property extends to the diagonals and to all their parallels.

53. Since we will consider here only the functions which go in arithmetic progression, it is evident that, in the (horizontal) rows, the Latin as well as the Greek numbers or superscripts will appear in arithmetical progression, and by using d for the difference in the progression of the Latin numbers and δ for that in the progression of Greek numbers, the first horizontal row will be

$$1^1 \quad (1+d)^{1+\delta} \quad (1+2d)^{1+2\delta} \quad \text{etc.}$$

Thus here, since for the following rows one has only to add one to the Latin as well as the Greek numbers, the complete square will have the following form:

$$\begin{array}{lllll}
1^1 & (1+d)^{1+\delta} & (1+2d)^{1+2\delta} & (1+3d)^{1+3\delta} & \text{etc.} \\
2^2 & (2+d)^{2+\delta} & (2+2d)^{2+2\delta} & (2+3d)^{2+3\delta} & \text{etc.} \\
3^3 & (3+d)^{3+\delta} & (3+2d)^{3+2\delta} & (3+3d)^{3+3\delta} & \text{etc.} \\
4^4 & (4+d)^{4+\delta} & (4+2d)^{4+2\delta} & (4+3d)^{4+3\delta} & \text{etc.} \\
 & & \text{etc.} & &
\end{array}$$

54. Now, since the Latin numbers of each (horizontal) row should include all the possible numbers, it follows that the difference d should be valued as we have shown above, that is to say in such a way that neither d nor d - 1 has any common divisor with the number n; and this particular condition extends also to the difference in the progression of the exponents δ and requires that both δ and $\delta - 1$ be prime to the number n. Then, it is evident that the two differences d and δ must not be equal, for if they were equal, all the entries would already have been found in the first (vertical) column; and this second condition suffices when the number n is prime; but if it is not prime, in addition to that the number $d - \delta$ must be prime to n.

55. These three conditions fulfilled, one will have satisfied the first of the principal conditions prescribed for the construction of the squares with different diagonals and parallels, that is to say one will obtain a square where the (horizontal) rows and (vertical) columns contain all the different numbers, such as we have constructed in the 1st part of this section. There remains only to see in what way one will be able to fulfill the other condition, of the diagonals and their parallels.

56. Let us consider for this purpose the first diagonal, which descends from left to right, and since the Latin numbers which form it make this progression

$$1,\ 2 + d,\ 3 + 2d,\ 4 + 3d,\ 5 + 4d,\ 6 + 5d \quad \text{etc.}$$

where the difference is $d + 1$, one sees that this diagonal will include all the different numbers every time that $d + 1$ is a number prime to n; and since all the parallels of this diagonal cross with the same difference $d + 1$, the required property (?) will extend itself also to the parallels. It is the same for the Greek numbers or exponents, which also receive all the values possible, provided the difference of their progressions, $\delta + 1$, is prime to the number n.

57. Let us consider also the second diagonal, which goes from left to right; and we see first that the Latin and Greek numbers of this diagonal as well as of their parallels form the arithmetical progressions where the difference of some is $d - 1$ and of the others $\delta - 1$. Then provided that both $d - 1$ and $\delta - 1$ are numbers prime to n, all the entries which are found in this diagonal and in all these parallels will also be different from each other. Besides, this last condition is already included in the nature of square-forming functions.

58. Here then are all the conditions required for the construction of the squares which are the object of this 2nd part. They are reduced to the three following; 1st that the numbers d, $d + 1$ and $d - 1$ be prime to the number n;

2nd) that the numbers δ, $\delta + 1$ and $\delta - 1$ also be prime to the number n and 3rd) that the number $d - \delta$ likewise not have any common divisor with n.

59. Let us suppose then that p is a divisor or any factor of the number n; and it will be necessary to exclude from the values of d these

$$d = \lambda p,\ d = \lambda p + 1,\ d = \lambda p - 1,$$

and from the values of the letter δ the following

$$\delta = \lambda p,\ \delta = \lambda p - 1,\ \delta = \lambda p + 1 .$$

Let $p = 3$; and it will be necessary to exclude from the values of d and of δ all the numbers possible; from which one sees that, in every case where the number n is divisible by 3, it will be impossible to construct a square where the diagonals and the parallels satisfy the required conditions.

60. Now, when the number n is prime, the number of all the different values which can be given to the differences d and δ will be

$$= n - 3$$

Then, if n is a product of two prime numbers unequal to each other, as $n = pq$, the number of the values of d and δ will be

$$(p - 3)(q - 3) .$$

And in general, if $n = p^{\alpha}q^{\beta}r^{\gamma}$ etc., the same number will be expressed by this formula

$$p^{\alpha-1}q^{\beta-1}r^{\gamma-1} \text{ etc. } (p-3)(q-3)(r-3) \text{ etc.}$$

61. After these general remarks, let us develop some particular cases, and since we have just excluded from the values of n the multiples of 3, let us take $n = 5$, where the suitable values for d and δ will be 2 and 3, one of which can be taken for d and the other for δ. Then let $d = 2$ and $\delta = 3$; and the square which results from them will have this form:

1^1	3^4	5^2	2^5	4^3
2^2	4^5	1^3	3^1	5^4
3^3	5^1	2^4	4^2	1^5
4^4	1^2	3^5	5^3	2^1
5^5	2^3	4^1	1^4	3^2

and it is evident that in changing the values of d and of δ, that is to say in putting $d = 3$ and $\delta = 2$, one can form another square; but it is not worth the trouble to distinguish it from this one.

62. Let $n = 7$: and the proper values of d and δ will be 2, 3, 4, 5, which, including two values which are unequal, give six different combinations, namely

$d = 2$ and $\delta = 3$, $d = 3$ and $\delta = 4$,

$d = 2$ and $\delta = 4$, $d = 3$ and $\delta = 5$,

$d = 2$ and $\delta = 5$, $d = 4$ and $\delta = 5$,

and the squares which result are the following:

I. If $d = 2$ and $\delta = 3$

1^1	3^4	5^7	7^3	2^6	4^2	6^5
2^2	4^5	6^1	1^4	3^7	5^3	7^6
3^3	5^6	7^2	2^5	4^1	6^4	1^7
4^4	6^7	1^3	3^6	5^2	7^5	2^1
5^5	7^1	2^4	4^7	6^3	1^6	3^2
6^6	1^2	3^5	5^1	7^4	2^7	4^3
7^7	2^3	4^6	6^2	1^5	3^1	5^4

II. If $d = 2$ and $\delta = 4$

1^1	3^5	5^2	7^6	2^3	4^7	6^4
2^2	4^6	6^3	1^7	3^4	5^1	7^5
3^3	5^7	7^4	2^1	4^5	6^2	1^6
4^4	6^1	1^5	3^2	5^6	7^3	2^7
5^5	7^2	2^6	4^3	6^7	1^4	3^1
6^6	1^3	3^7	5^4	7^1	2^5	4^2
7^7	2^4	4^1	6^5	1^2	3^6	5^3

III. If $\partial = 2$ and $\delta = 5$

1^1	3^6	5^4	7^2	2^7	4^5	6^3
2^2	4^7	6^5	1^3	3^1	5^6	7^4
3^3	5^1	7^6	2^4	4^2	6^7	1^5
4^4	6^2	1^7	3^5	5^3	7^1	2^6
5^5	7^3	2^1	4^6	6^4	1^2	3^7
6^6	1^4	3^2	5^7	7^5	2^3	4^1
7^7	2^5	4^3	6^1	1^6	3^4	5^2

IV. If $\partial = 3$ and $\delta = 4$

1^1	4^5	7^2	3^6	6^3	2^7	5^4
2^2	5^6	1^3	4^7	7^4	3^1	6^5
3^3	6^7	2^4	5^1	1^5	4^2	7^6
4^4	7^1	3^5	6^2	2^6	5^3	1^7
5^5	1^2	4^6	7^3	3^7	6^4	2^1
6^6	2^3	5^7	1^4	4^1	7^5	3^2
7^7	3^4	6^1	2^5	5^2	1^6	4^3

V. If $\partial = 3$ and $\delta = 5$

1^1	4^6	7^4	3^2	6^7	2^5	5^3
2^2	5^7	1^5	4^3	7^1	3^6	6^4
3^3	6^1	2^6	5^4	1^2	4^7	7^5
4^4	7^2	3^7	6^5	2^3	5^1	1^6
5^5	1^3	4^1	7^6	3^4	6^2	2^7
6^6	2^4	5^2	1^7	4^5	7^3	3^1
7^7	3^5	6^3	2^1	5^6	1^4	4^2

VI. If $\partial = 4$ and $\delta = 5$

1^1	5^6	2^4	6^2	3^7	7^5	4^3
2^2	6^7	3^5	7^3	4^1	1^6	5^4
3^3	7^1	4^6	1^4	5^2	2^7	6^5
4^4	1^2	5^7	2^5	6^3	3^1	7^6
5^5	2^3	6^1	3^6	7^4	4^2	1^7
6^6	3^4	7^2	4^7	1^5	5^3	2^1
7^7	4^5	1^3	5^1	2^6	6^4	3^2

63. The nature of these squares gives us also this advantage, that one can begin the inscription of his entries by any box of the square which one wishes. To show the multiplicity of the forms which are derived, let us take the first of the six squares which we have just constructed and let us fill the boxes in the following way:

4^7	6^3	1^6	3^2	5^5	7^1	2^4
5^1	7^4	2^7	4^3	6^6	1^2	3^5
6^2	1^5	3^1	5^4	7^7	2^3	4^6
7^3	2^6	4^2	6^5	1^1	3^4	5^7
1^4	3^7	5^3	7^6	2^2	4^5	6^1
2^5	4^1	6^4	1^7	3^3	5^6	7^2
3^6	5^2	7^5	2^1	4^4	6^7	1^3

64. If one wishes to apply all this to ordinary magic squares, one has only to put in place of the Latin numbers these values:

0, 7, 14, 21, 28, 35, 42

and in place of the Greek numbers the following:

1, 2, 3, 4, 5, 6, 7,

in some order, and then to put in place of each entry of the preceding square the sum of the two Latin and Greek numbers changed in this way. Thus, in the complete square we have just found let us put

in place of the Latin numbers	1	2	3	4	5	6	7
the following value	14	42	0	35	21	7	28
and in place of the Greek numbers	1	2	3	4	5	6	7
substitute these	5	4	1	7	2	3	6

and we will obtain the following ordinary magic square:

41	8	17	4	23	33	49
26	35	48	36	10	18	2
11	16	5	28	34	43	38
29	45	39	9	19	7	27
21	6	22	31	46	37	12
44	40	14	20	1	24	32
3	25	30	47	42	13	15

In this square, not only all the (horizontal) rows and (vertical) columns, but also all the diagonals and their corresponding and completed parallels, as for example:

$$8 \quad 26 \quad 38 \quad 7 \quad 46 \quad 20 \quad 30\ ,$$

will produce the same sum, namely 175.

65. To give still another idea of cases where the number n is not prime, but indivisible by 3, let us consider that of

$$n = 35 = 5 \cdot 7$$

in which the number of all the values which can be given to the letters d and δ will be 8. For since here, in putting $n = pq$, there is $p = 5$ and $q = 7$, the formula which expresses the number of the values is

$$(p - 3)(q - 3) = 2 \cdot 4 = 8,$$

which fits very well; for the values which the letters d and δ can receive are actually the 8 following:

$$2,\ 3,\ 12,\ 17,\ 18,\ 23,\ 32,\ 33\ .$$

Next, in excluding from d and δ the numbers whose difference $d - \delta$ is divisible by 5 or by 7, the allowable combinations will be

$$d = 2 \text{ and } \delta = 3, \quad d = 12 \text{ and } \delta = 23,$$
$$d = 2 \text{ and } \delta = 18, \quad d = 17 \text{ and } \delta = 18,$$
$$d = 2 \text{ and } \delta = 33, \quad d = 17 \text{ and } \delta = 23,$$
$$d = 3 \text{ and } \delta = 12, \quad d = 17 \text{ and } \delta = 33,$$
$$d = 3 \text{ and } \delta = 32, \quad d = 23 \text{ and } \delta = 32,$$
$$d = 12 \text{ and } \delta = 18, \quad d = 32 \text{ and } \delta = 33,$$

from which can be formed twelve different squares of 1225 divisions in which all the prescribed conditions would be fulfilled; but the reader will willingly excuse us from the actual construction of even one of them.

SECOND SECTION

LATIN SQUARE A DOUBLE MARCHE OF THE GENERAL FORM

1	2	3	4	5	6 ...	n-3	n-2	n-1	n
2	1	4	3	6	5 ...	n-2	n-3	n	n-1
3	4	5	6	7	8 ...	n-1	n	1	2
4	3	6	5	8	7 ...	n	n-1	2	1
5	6	7	8	9	10 ...	1	2	3	4

etc.

66. We have already noted in the preceding section, while establishing the classes of regular squares, that this type excludes completely the odd numbers n; and we shall see below that the values of n must be not only even numbers, but in addition evenly even numbers, or rather that the number of entries in a square à double marche must be divisible by 4. But before coming to the demonstration of this truth, it will be necessary to determine in general the relationship existing between the various numbers of the square and their position. In order to do this, I observe first that because the terms of the first row are at the same time the indices of the columns which correspond to them, as those of the first column are the indices of the corresponding rows, each entry of the square will be determined by two indices, one vertical and the other horizontal. Then let t in general be the vertical index of some term x, and u be its horizontal index; the problem will be to find the relation between the three letters, t, u, and x. To do this it is necessary to distinguish carefully the case where one or the other of the two numbers t and u is odd from that in which both t and u are even; and we will see right away that the first case gives

$$x = t + u - 1$$

and the second,

$$x = t + u - 3 ,$$

which shows at the same time that the two indices t and u can be exchanged without the term x changing value, since it depends only on the sum of these two letters. After this observation, we will be able to propose our above-mentioned theorem, stated in the following manner:

No square <u>à double marche</u> can give rise to a square-forming function (<u>formule directrice</u>) unless the number of horizontal or vertical terms is divisible by 4.

67. To demonstrate this theorem, let the series

$$a \quad b \quad c \quad d \quad e \quad \text{etc.}$$

be a function of the arbitrary index a; let

$$\alpha \quad \beta \quad \gamma \quad \delta \quad \epsilon \quad \text{etc.}$$

be the series of horizontal indices indicated by the letter u*, those of the vertical indices marked by t*, which always follow the series of the natural numbers, being

$$1, \quad 2, \quad 3, \quad 4 \quad \text{etc.};$$

and it will be necessary, by virtue of the nature of the functions, which was shown in the preceding section, for both of these series to include all the numbers from 1 to n. Having the two indices, the vertical one t and the horizontal one u, we can by the preceding rules easily deduce from them the value of each term of our function.

68. First, it is clear that for the first term one will always have $a = \alpha$. For the second term, b, there are $t = 2$ and $u = \beta$, from which, by distinguishing between the two possible values of β, which can be even or odd, we will have for the first $b = \beta + 1$, and for the other $b = \beta - 1$. For the third term, c, because $t = 3$, which is odd and $u = \gamma$, there will always be $c = \gamma + 2$. For the fourth

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*In the original edition, as the result of an error, the letters t and u are reversed here. We have made them consistent with subsequent notation. -Ed.

term, d, where $t = 4$, which is even, and $u = \delta$, it is necessary to distinguish again between the two possible values of d; if it is odd, there will be $d = \delta + 3$, and if it is even, $d = \delta + 1$; and thus with the others. One will then have for the function

$$a \quad b \quad c \quad d \quad e \quad f \quad g \quad \text{etc.}$$

of a square whose horizontal indices are

$$\alpha, \quad \beta, \quad \gamma, \quad \delta, \quad \epsilon, \quad \zeta, \quad \eta \quad \text{etc.}$$

and whose vertical indices are

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7 \quad \text{etc.,}$$

the following terms:

$$a = \alpha$$

$$b = \begin{cases} \beta + 1 \ (\beta \text{ odd}) \\ \beta - 1 \ (\beta \text{ even}) \end{cases}$$

$$c = \gamma + 2$$

$$d = \begin{cases} \delta + 3 \ (\delta \text{ odd}) \\ \delta + 1 \ (\delta \text{ even}) \end{cases}$$

$$e = \epsilon + 4$$

$$f = \begin{cases} \zeta + 5 \ (\zeta \text{ odd}) \\ \zeta + 3 \ (\zeta \text{ even}) \end{cases}$$

$$g = \eta + 6$$

$$h = \begin{cases} \theta + 7 \ (\theta \text{ odd}) \\ \theta + 5 \ (\theta \text{ even}) \end{cases}$$

$$i = \iota + 8$$

etc.

69. We see, then, that the determination of the letters a, b, c, d etc. by the indices α, β, γ, δ etc. would be absolutely regular if none of the alternate letters β, δ, ζ, etc. were even; and that we would then have $a = \alpha$, $b = \beta + 1$, $c = \gamma + 2$, $d = \delta + 3$, $e = \epsilon + 4$, $f = \zeta + 5$, etc., the number of these terms being always equal to n. Let us suppose for a moment that all these alternate letters are odd. Let the sum of the series of horizontal indices be

$$\alpha + \beta + \gamma + \delta + \epsilon + \text{etc.} = \sum ,$$

and the sum of the terms of the function be

$$a + b + c + d + e + \text{etc.} = S ;$$

and by adding all these terms, we will have this equation:

$$S = \sum + 1 + 2 + 3 + 4 + 5 + \ldots + (n-1) = \sum + \frac{1}{2} n(n-1) .$$

Now since both of our series must include all the numbers from 1 to n, it follows that the two sums S and $\sum$ must equal each other, or else that their difference must be a multiple of n, λn, from which is gotten

$$S = \sum + \lambda n$$

and consequently, it will have to be in this case

$$\frac{1}{2} n(n-1) = \lambda n .$$

But we have already said above that squares <u>à double marche</u> completely exclude odd values of n; whence, supposing n (an even number) = 2k, k being some integer, we will have

$$k(2k - 1) = 2\lambda k$$

or rather $\lambda = k - \frac{1}{2}$ or $k = \lambda + \frac{1}{2}$, which is impossible.

70. But this conclusion originates in the supposition that all the alternate letters β, δ, ζ, etc. are odd and it is only for this case that the functions become completely impossible, whatever value is assigned to n. In order that

there exist functions which generate the square à double marche, it is absolutely necessary that at least one of the letters β, δ, ζ, θ etc. denote an even number; and to see what will result from this, we will suppose first that there is only one, which will decrease the sum of the series of horizontal indices by 2, and we will have *

$$\frac{1}{2}\,n(n-1) - 2 = \lambda n;$$

or rather, putting $n = 2k$, it will have to be

$$k(2k - 1) - 2 = 2\lambda k,$$

whence it is evident that k must be an even number. Then let $k = 2m$ and consequently $n = 4m$; and our equation will become

$$m(4m - 1) - 1 = 2\lambda m,$$

or rather

$$1 = m(4m - 1) - 2\lambda m = m(4m - 2\lambda - 1)\ .$$

Now since this equation could not occur unless $m = 1$ and $\lambda = 1$, it is clear that this case can exist only when $n = 4$.

71. Let us suppose in general that among the alternate numbers β, δ, ζ, θ, etc. there are π even numbers; and since the total number of these letters is $\frac{1}{2}\,n$, it is clear that π cannot be greater than $\frac{1}{2}\,n$. Then, since each even value of these letters produces, in the sum $\frac{1}{2}\,n(n-1)$ a decrease of two, our equation will be

$$\frac{1}{2}\,n(n-1) - 2\pi = \lambda n$$

or rather, taking $n = 2k$, we will have the following:

$$k(2k - 1) - 2\pi = 2\lambda k,$$

which can occur only when k is an even number, $= 2m$, and consequently $n = 4m$. Then our equation will be

* The original edition has here, incorrectly, $n(2n - 1) - 2 = \lambda n$, and four lines further, n in place of k. -Ed.

$$m(4m - 1) - \pi = 2\lambda m,$$

from which is obtained the numbers of the alternate letters which are even, namely

$$\pi = m(4m - 2\lambda - 1),$$

that is, equal to a product of two factors, one m and the other $4m - 2\lambda - 1$. Now since π cannot be greater than $\frac{1}{2} n = 2m$, and since the coefficient of m, $4m - 2\lambda - 1$, is an odd number, it is absolutely necessary that it be

$$4m - 2\lambda - 1 = 1$$

from which is obtained

$$\lambda = 2m - 1$$

and

$$\pi = m.$$

Then it is necessary for half of the letters β, δ, ζ, θ etc. to be even and for the number n to be divisible by 4; in consequence, oddly even numbers, 2, 6, 10, 14, etc., will be completely excluded from this section, considering that they could never give rise to square-forming functions, which was to be proved (QED).

72. Thus we will establish throughout this entire section that the number n be divisible by 4, by making $n = 4m$, and in all these cases, the preceding demonstration enables us to see the possibility of the functions being square-forming. Let us then consider principally the functions which correspond to the first superscript, 1, and which, because of $a = 1$, will have in general this form:

$$1 \quad b \quad c \quad d \quad e \quad f \quad g \quad \text{etc.},$$

to which corresponds this series of horizontal indices

$$1, \ \beta, \ \gamma, \ \delta, \ \epsilon, \ \zeta, \ \text{etc.},$$

the series of vertical indices being that of the natural numbers

$$1, \ 2, \ 3, \ 4, \ 5, \ 6 \ \text{etc.}$$

Granting this, we have seen that if t marks the vertical index and u the horizontal index, the term of the function will be

$$x = t + u - 1$$

except for the single case where the numbers t and u are even, where it will be

$$x = t + u - 3;$$

so that in both cases x is an odd number.

73. We have shown that for the number $n = 4m$, the case where t and u are even must always occur m times, from which it follows that there are also m cases where odd values for u correspond to the even numbers of t; and for the same reason, for the case of odd t there will be m even numbers and m odd numbers for u. Let us clarify this by the following example, where $m = 2$ and $n = 8$:

Vertical indices	$t = 1, 2, 3, 4, 5, 6, 7, 8.$
Horizontal indices	$u = 1, 6, 2, 5, 7, 4, 8, 3.$

Here the even indices $u = 6$ and 4 correspond to the even indices $t = 2$ and 6. The odd indices $t = 3$ and 7 correspond to the even indices $u = 2$ and 8. Next, the odd indices $u = 1$ and 7 correspond to the odd indices $t = 1$ and 5, and the odd indices $u = 3$ and 5 correspond to the even indices $t = 8$ and 4. Now from these two series can be formed, by the formulas $x = t + u - 1$ and $x = t + u - 3$, the following function,

$$1 \quad 5 \quad 4 \quad 8 \quad 3 \quad 7 \quad 6 \quad 2$$

in which all of the terms are different.

74. It is as easy to examine each proposed function to see whether or not it is square-forming. For when one has the numbers x and the horizontal indices t, one has only to find the indices u, considering one or the other of the formulas given for x, among which the last, $x = t + u - 3$ or $u = x - t + 3$, occurs only when t is even and x is odd; and when all the numbers found in this

way for u are different, the proposed function will always be square-forming. Here are some examples:

I.

Indices	$t = 1\ 2\ 3\ 4$
Function	$\underline{x = 1\ 3\ 4\ 2}$
Indices	$u = 1\ 4\ 2\ 3$

II.

Indices	$t = 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8$
Function	$\underline{x = 1\ 3\ 5\ 7\ 4\ 2\ 8\ 6}$
Indices	$u = 1\ 4\ 3\ 6\ 8\ 5\ 2\ 7$

III.

Indices	$t = 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12$
Function	$\underline{x = 1\ 4\ 6\ 8\ 10\ 12\ 2\ 3\ 5\ 7\ 9\ 11}$
Indices	$u = 1\ 3\ 4\ 5\ 6\ 7\ 8\ 10\ 9\ 12\ 11\ 2$

where it is observed that the series u include all the different values, so that the functions proposed for x are truly square-forming.

75. Considering carefully the last two examples, one will observe that it is easy to examine in general the functions for all numbers divisible by 4. To do this one need only divide them into two equal parts, each of which contains 2m terms, and one will see, by the prescribed rules, if the series found for u includes all the different values. Here is the example of two general square-forming functions for all numbers $n = 4m$:

First function.

$$\text{First half}\begin{cases} t = 1\quad 2\quad 3\quad 4\quad 5\quad 6\ldots 2m \\ \underline{x = 1\quad 4\quad 6\quad 8\quad 10\quad 12\ldots 4m} \end{cases}$$

$$u = 1\quad 3\quad 4\quad 5\quad 6\quad 7\ldots 2m-1$$

$$\text{Second half}\left\{\begin{array}{llllll} t = & 2m+1 & 2m+2 & 2m+3 & 2m+4 & \ldots 4m \\ x = & 2 & 3 & 5 & 7 & \ldots 4m-1 \end{array}\right.$$
$$\begin{array}{llllll} \hline u = & 2m+2 & 2m+4 & 2m+3 & 2m+6 & \ldots 2 \end{array}$$

where it is easily verified that, in the two parts found for u, all the different numbers actually appear.

Second function.

$$\text{First half}\left\{\begin{array}{lllllll} t = & 1 & 2 & 3 & 4 & 5 & 6 \ldots 2m \\ x = & 1 & 3 & 5 & 7 & 9 & 11 \ldots 4m-1 \end{array}\right.$$
$$\begin{array}{lllllll} \hline u = & 1 & 4 & 3 & 6 & 5 & 8 \ldots 2m+2 \end{array}$$

$$\text{Second half}\left\{\begin{array}{llllll} t = & 2m+1 & 2m+2 & 2m+3 & 2m+4 & \ldots 4m \\ x = & 4 & 2 & 8 & 6 & \ldots 4m-2 \end{array}\right.$$
$$\begin{array}{llllll} \hline u = & 2m+4 & 2m+1 & 2m+6 & 2m+3 & \ldots 4m-1 \end{array}$$

In the last part the next-to-last term of u is 4m+2 or rather 2, from which it is seen that among the values of u are found all the numbers from 1 to 4m.

76. Now, having found a single square-forming function, one can obtain from it several others by rules which are similar to those which we used in the preceding section. To clarify this completely, we will consider an arbitrary function

$$1 \quad a \quad b \quad c \quad d \quad e \quad f \quad \text{etc.},$$

in which the term which corresponds to the index t is =x, and we have seen that, taking u as the horizontal index, two cases must be distinguished; one where t is even and x odd, which produces $u = 3 + x - t$; and the other where $u = 1 + x - t$, which includes all the other values. To make it a little easier, one may represent both of these cases by this ambiguous formula

$$u = x - t + 2 \pm 1,$$

where the plus occurs when x is odd and t even; in all other cases the minus must be used.

77. Now, the nature of all square-forming functions includes the two following properties:

1) that, while the letter t varies over all values from one to $n = 4m$, the letter x must also range over all these different values;

2) that, while the two letters t and x are varied over all the values, the formula $u = x - t + 2 \pm 1$ will also range over all the possible values. From these emerge naturally the following third property, that while the letters x and t vary from 1 to n, the formula $t - x + 2 \pm 1$ will likewise furnish all the different values, provided that one is aware of the ambiguity of the signs, the upper of which occurs only when the number t is odd and x even; above all, since these two letters are so closely linked that, while one varies over all the values, the other also shows the same variations, they can be considered interchangeable, at least in this respect.

78. Let us now see how one can deduce the new square-forming function from the one which we have supposed to be known. For this purpose let

$$1 \quad A \quad B \quad C \quad D \quad \text{etc.}$$

be such a function, of which the term corresponding to the index T is X; and it will be necessary, while T varies over all the values, for X also to undergo the same variations, the same being true for $X - T + 2 \pm 1$ and $T - X + 2 \pm 1$, provided that the stated rules regarding the ambiguity of the signs are observed. Now since it has been noted on the other hand that the letters t and u may be interchanged, it follows that a new square-forming function can be found by fitting the same term x into the index $u = x - t + 2 \pm 1$, that is by taking

$$T = x - t + 2 \pm 1 \quad \text{and} \quad X = x.$$

Thus, having for the case n = 8 this function

$$1 \quad 3 \quad 5 \quad 8 \quad 2 \quad 4 \quad 6 \quad 7,$$

to which corresponds, for u,

$$1 \quad 4 \quad 3 \quad 5 \quad 6 \quad 7 \quad 8 \quad 2,$$

one will deduce a new function by putting the second term, 3, in the fourth place, assigned by the number u written under it; the third term, 5, in the third place, and so on for the rest, which will give the new function

$$1 \quad 7 \quad 5 \quad 3 \quad 8 \quad 2 \quad 4 \quad 6 .$$

79. Next, one will always find a new square-forming function by interchanging the letters t and x and taking

$$T = x \quad \text{and} \quad X = t;$$

in this way, the first property is already fulfilled by itself, and the other, which concerns the formula $X - T + 2 \pm 1$, will also be perfectly fulfilled; for this formula, at present $t - x + 2 \pm 1$, will take on all the values, provided that it is observed that the upper (plus) sign occurs only when t is even and x odd. It is easy to see that this rule agrees with the first of those which we gave in Section 27 of the preceding section and which we characterized by the term "reversal"; so that the same rule can always be used without any alteration in this section. The square-forming function of the preceding example, that is

$$1 \quad 3 \quad 5 \quad 8 \quad 2 \quad 4 \quad 6 \quad 7,$$

will thus produce by reversal the following

$$1 \quad 5 \quad 2 \quad 6 \quad 3 \quad 7 \quad 8 \quad 4 .$$

80. Another rule may be deduced from the first case by taking

$$T = t \quad \text{and} \quad X = t - x + 2 \pm 1,$$

since from this results $U = X - T + 2 \pm 1$, where the ambiguity of the signs works in the opposite way from the preceding, so that one obtains, by substituting, in place of T and X, their values,

$$U = -x + 4,$$

a formula which, without ambiguity, will receive all the possible variations while t and therefore also x are taking on all the values. This rule is analogous to the second one in the preceding section, where we also had

$$T = t \quad \text{and} \quad X = t - x + 1,$$

which occurs in all cases except those where t is odd and x even, which obliges us to use the value

$$X = t - x + 3 .$$

If we let, for example, the

$$t\text{'s} = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8,$$

and the

$$x\text{'s} = 1 \quad 3 \quad 5 \quad 8 \quad 2 \quad 4 \quad 6 \quad 7,$$

we will have

$$X = 1 \quad 8 \quad 7 \quad 5 \quad 6 \quad 3 \quad 4 \quad 2^{*} .$$

81. By means of these two rules, one can deduce from each known function several others and almost always a dozen new ones, as happened in the preceding section (compare the example of section 42 and what follows it), provided however, that the second rule is used with the indicated rectification.

To illuminate all this by an example, let us take once more the function which we have used up to this point, and write under it its reverse, applying then the second and first rules alternately; one will obtain a total of a dozen new functions, including the proposed one, as may be seen from the following.

* The original edition erroneously has here 1 8 7 5 6 3 2 4. -Ed.

The proposed function	1	3	5	8	2	4	6	7	I
gives by reversal	1	5	2	6	3	7	8	4	
and then*									
by the second rule	1	8	7	5	6	3	4	2	II
[applied to I]	1	6	4	7	3	8	2	5	
by the first rule	1	7	5	3	8	2	4	6	III
[applied to I]	1	8	6	7	4	5	3	2	
by the second rule	1	4	7	2	8	5	6	3	IV
[applied to III]	1	3	8	6	4	2	5	7	
by the first rule	1	6	2	5	7	4	8	3	V
[applied to II]	1	4	8	2	6	7	3	5	
by the second rule	1	5	4	8	7	3	2	6	VI
[applied to V]	1	7	6	3	2	8	5	4	

where we have continued these operations until the reproduction of the last functions, which occurs at the sixth pair.

82. Let us apply the same operations to a function of twelve terms, adopting one formed from those which proceed in arithmetic progression, and the complete dozen which are obtained by these two rules will be:

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* In the original edition, the order of succession of the functions is interchanged for groups III and IV. The numbering of the pairs of functions (Roman numerals) is not found in the original edition.

The same remark about order of succession applies to section 82 as well as to each of the four dozens of section 94. -Ed.

Proposed	1	3	5	7	9	11	4	2	8	6	12	10	I
Reversed	1	8	2	7	3	10	4	9	5	12	6	11	
by the second rule	1	12	11	10	9	8	6	7	4	5	2	3	II
[applied to I]	1	7	4	10	3	9	6	12	5	11	8	2	
by the first rule	1	12	5	3	9	7	2	11	6	4	10	8	III
[applied to I]	1	11	12	9	10	7	8	6	5	4	3	2	
by the second rule	1	3	11	2	9	12	8	10	6	7	4	5	IV
[applied to III]	1	4	6	8	10	12	2	3	5	7	9	11	
by the first rule	1	7	8	2	9	3	10	4	11	5	12	6	V
[applied to II]	1	4	2	11	12	9	10	7	5	8	3	6	
by the second rule	1	8	10	3	9	4	12	5	11	6	2	7	VI
[applied to V]	1	11	4	6	8	10	12	2	5	3	9	7	

of which the last pair are reproduced by the first rule.

83. Now each of the square-forming functions for the superscript 1 furnishes, as we have shown above, appropriate functions for all the other superscripts and even such that they present different terms for all the columns. For it is clear, by the construction of the Latin square, that by increasing by a value of 2 the terms of the function for the superscript 1, one will obtain another function for the superscript 3, and by increasing the terms of the latter by 2, another function for the superscript 5. And in general, for a function for the superscript a, letting it be

a b c d e etc.,

a function will be deduced for the superscript a + 2 by adding 2 to each term of the preceding function. Thus for the case of $n = 8$, each function for the superscript 1, of which we have set forth a dozen, will furnish appropriate functions for the odd superscripts; for example

for the superscript 1	1	3	5	7	4	2	8	6
for the superscript 3	3	5	7	1	6	4	2	8
for the superscript 5	5	7	1	3	8	6	4	2
for the superscript 7	7	1	3	5	2	8	6	4

where each vertical column contains different numbers, even or odd, separately.

84. The formation of functions for the superscript 2 and the other even numbers is not so obvious; nevertheless, as in the Latin square the second row is deduced from the first by adding one to all the odd terms and subtracting one from the even ones, one can suppose that, in doing the same thing in regard to the proposed function, one will get the function for the superscript 2, because in effect all the odd terms produce in this way all the even ones, and reciprocally all the even terms, when one is subtracted from them, produce the odd ones. But it must again be demonstrated that the function which results from this is effectively a square-forming function.

85. For this purpose, in the function for the superscript 1, let the term which corresponds to the index $t = x$, and let x' be the one which corresponds to the same index in the function for the superscript 2. In the same way let u be the horizontal index of the same term x of the first function, and u' that of the term x' in the other; and one will have, by observing the prescribed rules about the ambiguity of the signs,

$$u = x - t + 2 \pm 1 \quad \text{and} \quad u' = x' - t + 2 \pm 1.$$

Thus there will be four cases to consider, according to whether the two numbers t and x are even or odd; and the values of u and u' will be for each case expressed in the following manner:

I	II	III	IV
$t = 2i$	$t = 2i$	$t = 2i+1$	$t = 2i+1$
$x = 2k$ $u = 2k-2i+1$	$x = 2k+1$ $u = 2k-2i+4$	$x = 2k$ $u = 2k-2i$	$x = 2k+1$ $u = 2k-2i+1$
$x' = 2k-1$ $u' = 2k-2i+2$	$x' = 2k+2$ $u' = 2k-2i+3$	$x' = 2k-1$ $u' = 2k-2i+1$	$x' = 2k+2$ $u' = 2k-2i+2$

86. From this it is seen that the second and third cases give even values for u and that the values of u' are less than one, from which it is evident that all the even values of u produce for u' all the odd values.

Next, the first and the fourth case, where the values of u are odd, furnish for u' values greater than one, and thus all the odd values of u produce for u' all the even values; so that all the values of u, different one from another, produce also for u' all the possible values, and the function is unquestionably square-forming, since it has all the necessary characteristics.

87. Having thus found the function for the superscript 2 in the way which we have just taught, one will form from it, by the first rule, functions for all the other even superscripts, and by this means one will easily construct, from each function proposed for the superscript 1, a complete system of functions similar to the one which we give here for the function

$$1 \quad 3 \quad 5 \quad 7 \quad 4 \quad 2 \quad 8 \quad 6.$$

For the superscript 1	1	3	5	7	4	2	8	6
For the superscript 2	2	4	6	8	3	1	7	5
For the superscript 3	3	5	7	1	6	4	2	8
For the superscript 4	4	6	8	2	5	3	1	7
For the superscript 5	5	7	1	3	8	6	4	2
For the superscript 6	6	8	2	4	7	5	3	1
For the superscript 7	7	1	3	5	2	8	6	4
For the superscript 8	8	2	4	6	1	7	5	3

where one sees that in each row the terms are all different one from another and that consequently when the superscripts are joined in the manner which has been explained to all the numbers in the proposed Latin square, no term can appear more than once, and the square will be complete.

88. In considering more attentively the complete system of functions which we have formed one will see first that all the rows fit in perfectly with those of the Latin square "à double marche" and that there is no difference except in their order, which is changed, that is to say the horizontal indices, which in the square appear in the natural order 1 2 3 4 5 6 7 8, are here 1 3 5 7 4 2 8 6. In considering then in general any row, and letting its index be t and its highest (supreme) term be x expressed with the superscript 1, if we express the terms which follow the x, in descending order, by

$$x', \quad x'', \quad x''', \quad \text{etc.}$$

and give them the

$$2, \quad 3, \quad 4, \quad \text{etc.}$$

the term $x^{(\varphi)}$ will have the exponent $\varphi + 1$; and taking φ so that it becomes

$$x^{(\varphi)} = t,$$

which is the term which corresponds to the same index in the first row of the Latin square, it will be necessary to give to that term the superscript $\varphi+1$. Or, whenever the values $x, x', x'', x''', \ldots x^{(\varphi)}$ take the same order as in the Latin square, one will always have $t = x + \varphi - \begin{cases}0\\2\end{cases}$ or rather $\varphi = t - x + \begin{cases}0\\2\end{cases}$, and therefore

$$\varphi + 1 = t - x + \begin{cases}1\\3\end{cases} = t - x + 2 \pm 1,$$

where the ambiguity of the signs follows the same laws that we have stated above.

89. From this it is clear that the superscripts of the first row of our Latin square also form a square-forming function, derived from the proposed function by the second rule, and that in order to construct a complete square one can begin with the first row, assigning to it superscripts according to any function and continuing to assign the others by descending according to that column of the square which begins with the same number. Thus, since one derives from the proposed function,

$$1 \quad 3 \quad 5 \quad 7 \quad 4 \quad 2 \quad 8 \quad 6,$$

by the second rule the function

$$1 \quad 8 \quad 7 \quad 6 \quad 4 \quad 5 \quad 2 \quad 3,$$

one can begin with this function combining it with the first row of the original (<u>simple</u>) square, for whose terms it will serve as superscripts; and the others will be inserted in the way which we have just explained and which we will make clearer by the example of the following square:

$$\begin{array}{cccccccc}
1^1 & 2^8 & 3^7 & 4^6 & 5^4 & 6^5 & 7^2 & 8^3 \\
2^2 & 1^7 & 4^8 & 3^5 & 6^3 & 5^6 & 8^1 & 7^4 \\
3^3 & 4^2 & 5^1 & 6^8 & 7^6 & 8^7 & 1^4 & 2^5 \\
4^4 & 3^1 & 6^2 & 5^7 & 8^5 & 7^8 & 2^3 & 1^6 \\
5^5 & 6^4 & 7^3 & 8^2 & 1^8 & 2^1 & 3^6 & 4^7 \\
6^6 & 5^3 & 8^4 & 7^1 & 2^7 & 1^2 & 4^5 & 3^8 \\
7^7 & 8^6 & 1^5 & 2^4 & 3^2 & 4^3 & 5^8 & 6^1 \\
8^8 & 7^5 & 2^6 & 1^3 & 4^1 & 3^4 & 6^7 & 5^2
\end{array}$$

But it must be noted that this beautiful property of the superscripts of the first row can occur only when the system of functions is formed from a single proposed function.

90. However, it is easy to combine several square-forming functions together to form such a complete system, as we have shown in section 35 of the preceding section. I add also, in regard to that section, that after having obtained the functions for the odd superscripts of some function of the superscript 1, one can deduce the functions for the even superscripts of another function, provided that its terms follow the same order as far as even and odd are concerned. Thus for the preceding example, after deducing the odd square-forming functions of the function 1 3 5 7 4 2 8 6, one will be able to obtain those which determine the formation of the even superscripts of that function: 1 5 7 3 8 4 6 2, which is also square-forming and whose terms, as concerns even and odd, follow the same order. Here is the complete system:

For the superscript 1	1	3	5	7	4	2	8	6
For the superscript 2	2	6	8	4	7	3	5	1
For the superscript 3	3	5	7	1	6	4	2	8
For the superscript 4	4	8	2	6	1	5	7	3
For the superscript 5	5	7	1	3	8	6	4	2
For the superscript 6	6	2	4	8	3	7	1	5
For the superscript 7	7	1	3	5	2	8	6	4
For the superscript 8	8	4	6	2	5	1	3	7

which, used as instructed, will give the following complete square*

1^1	2^6	3^7	4^2	5^8	6^5	7^4	8^3
2^2	1^7	4^6	3^5	6^3	5^4	8^1	7^8
3^3	4^8	5^1	6^4	7^2	8^7	1^6	2^5
4^4	3^1	6^8	5^7	8^5	7^6	2^3	1^2
5^5	6^2	7^3	8^6	1^4	2^1	3^8	4^7
6^6	5^3	8^2	7^1	2^7	1^8	4^5	3^4
7^7	8^4	1^5	2^8	3^6	4^3	5^2	6^1
8^8	7^5	2^4	1^3	4^1	3^2	6^7	5^6

where the superscripts of the first row have this order:

$$1 \quad 6 \quad 7 \quad 2 \quad 8 \quad 5 \quad 4 \quad 3,$$

which plainly is not square-forming, since the values of u would be

$$1 \quad 5 \quad 5 \quad 7 \quad 4 \quad 2 \quad 6 \quad 6,$$

and therefore far from being different one from another.

91. After these general thoughts, which can be applied to all Latin squares <u>à double marche</u>, no matter how large n is, as long as it is divisible by 4, we will develop some particular cases where n = 4 and n = 8, but omitting larger ones, which would lead us too far; and since, for the case of n = 8, we have already given several examples, we will limit ourselves to finding all its square-forming functions; having shown that each of them can furnish a complete system and that two different square-forming functions can also lead to a complete system, as long as the terms maintain the same order as far as even and odd are concerned. Once these systems have been formed, whose number obviously is much larger than that of the first square-forming functions, the construction of the squares offers not the slightest difficulty.

* The original edition erroneously gives 8^2 instead of 8^6 and 7^6 instead of 7^1 in the fourth column. -Ed.

THE CASE OF n = 4

92. The Latin square <u>à double marche</u> is in this case

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

which gives for the superscript 1 only the two following functions:

1 4 2 3 and 1 3 4 2,

of which one, if one applies the two prescribed rules, produces the other. From these two functions can be formed the two complete systems which follow.

	I				II			
For the superscript 1	1	4	2	3	1	3	4	2
For the superscript 2	2	3	1	4	2	4	3	1
For the superscript 3	3	2	4	1	3	1	2	4
For the superscript 4	4	1	3	2	4	2	1	3

and by writing out the superscripts according to these functions, one obtains the two complete squares below:

I				II			
1^1	2^3	3^4	4^2	1^1	2^4	3^2	4^3
2^2	1^4	4^3	3^1	2^2	1^3	4^1	3^4
3^3	4^1	1^2	2^4	3^3	4^2	1^4	2^1
4^4	3^2	2^1	1^3	4^4	3^1	2^3	1^2

One will easily be convinced that, whatever other Latin square one wants to construct, one will never be able to obtain from it other complete squares which satisfy the prescribed conditions. However, both of the squares which we have just formed also admit of transpositions of the columns such that the prescribed properties appear even in the diagonals. Here are two examples:

I				II			
1^1	3^4	4^2	2^3	1^1	4^3	2^4	3^2
2^2	4^3	3^1	1^4	2^2	3^4	1^3	4^1
3^3	1^2	2^4	4^1	3^3	2^1	4^2	1^4
4^4	2^1	1^3	3^2	4^4	1^2	3^1	2^3

CASE OF n = 8

93. The fundamental Latin square is:

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	5	6	7	8	1	2
4	3	6	5	8	7	2	1
5	6	7	8	1	2	3	4
6	5	8	7	2	1	4	3
7	8	1	2	3	4	5	6
8	7	2	1	4	3	6	5

for which the two general formulae furnish first the two following square-forming functions

1	3	5	7	4	2	8	6
1	4	8	6	2	3	5	7

of which the first begins with the four odd numbers; and it is not hard to find all the functions whose even and odd terms preserve the same order; they are the four which follow:

1	3	5	7	4	2	8	6
1	5	7	3	4	8	2	6
1	5	7	3	8	4	6	2
1	7	5	3	8	2	4	6

94. Let us then apply successively the two rules presented and demonstrated above (section 78 ff.) to these four functions, which will give us the four following groups of twelve.

First group of twelve*

fundamental	1	3	5	7	4	2	8	6	I
reversed	1	6	2	5	3	8	4	7	
second rule	1	8	7	6	4	5	2	3	II
(applied to I)	1	5	4	8	3	7	6	2	
first rule	1	8	5	3	2	7	6	4	III
(applied to I)	1	7	8	5	6	4	3	2	
second rule	1	3	7	2	6	8	4	5	IV
(applied to III)	1	4	6	8	2	3	5	7	
first rule	1	5	6	2	7	3	8	4	V
(applied to II)	1	4	2	7	8	5	3	6	
second rule	1	6	8	3	7	4	2	5	VI
(applied to V)	1	7	4	6	8	2	5	3	

Second group of twelve

fundamental	1	5	7	3	4	8	2	6	I
reversed	1	7	4	5	2	8	3	6	
second rule	1	6	5	2	4	7	8	3	II
(applied to I)	1	4	2	8	6	7	5	3	
first rule	1	3	8	2	7	5	6	4	III
(applied to I)	1	4	8	5	3	2	6	7	
second rule	1	8	6	3	7	2	4	5	IV
(applied to III)	1	7	6	8	3	5	4	3	
first rule	1	8	5	7	6	3	2	4	V
(applied to II)	1	6	4	7	8	3	5	2	
second rule	1	3	7	6	2	4	8	5	VI
(applied to V)	1	5	2	6	8	4	3	7	

* In the original edition, the order of succession of the square-forming functions is reversed in groups III and IV of each group of twelve. See note 1, p. 340. - Ed.

Third group of twelve

fundamental	1	5	7	3	8	4	6	2	I
reversed	1	8	4	6	2	7	3	5	
second rule	1	6	5	2	8	3	4	7	II
(applied to I)	1	3	2	7	6	8	5	4	
first rule	1	3	2	8	7	5	4	6	III
(applied to I)	1	4	6	7	3	2	8	5	
second rule	1	8	4	5	7	2	6	3	IV
(applied to III)	1	7	8	6	3	5	2	4	
first rule	1	7	5	8	6	4	2	3	V
(applied to II)	1	6	8	3	4	7	5	2	
second rule	1	4	7	5	2	3	8	6	VI
(applied to V)	1	5	6	2	4	8	3	7	

Fourth group of twelve

fundamental	1	7	5	3	8	2	4	6	I
reversed	1	6	4	7	3	8	2	5	
second rule	1	4	7	2	8	5	6	3	II
(applied to I)	1	5	2	6	3	7	8	4	
first rule	1	3	5	8	2	4	6	7	III
(applied to I)	1	4	8	2	6	7	3	5	
second rule	1	8	7	5	6	3	4	2	IV
(applied to III)	1	7	6	3	2	8	5	4	
first rule	1	5	4	8	7	3	2	6	V
(applied to II)	1	8	6	7	4	5	3	2	
second rule	1	6	2	5	7	4	8	3	VI
(applied to V)	1	3	8	6	4	2	5	7	

95. Thus here are forty-eight square-forming functions, which exhaust our whole Latin square; for all the functions which can be obtained from it by the

ordinary method are found in the four preceding groups of twelve. Thus by using only one of these functions, one can construct from it a complete square and consequently forty-eight different solutions, without counting those which spring from the combination of several of these functions whose even and odd terms preserve the same order and whose number is probably very considerable. To facilitate such combinations and to be able at the same time to judge of the number of all the different solutions, we are going to distribute these forty-eight functions in different classes, according to the order that is observed as far as even and odd are concerned, and we will designate even numbers by the letter e and odd ones by the letter o; we will obtain the following types:

I.

o	o	o	o	e	e	e	e
1	3	5	7	4	2	8	6
1	5	7	3	4	8	2	6
1	5	7	3	8	4	6	2
1	7	5	3	8	2	4	6

II.

o	o	o	e	e	e	e	o
1	3	5	8	2	4	6	7
1	3	7	2	6	8	4	5
1	3	7	6	2	4	8	5
1	7	5	8	6	4	2	3

III.

o	o	e	e	e	e	o	o
1	3	8	6	4	2	5	7
1	5	2	6	8	4	3	7
1	5	6	2	4	8	3	7
1	7	4	6	8	2	5	3

IV.

o	e	e	e	e	o	o	o
1	4	2	8	6	7	5	3
1	4	6	8	2	3	5	7
1	4	8	2	6	7	3	5
1	8	4	6	2	7	3	5

V.

o	o	e	o	e	e	o	e
1	3	2	7	6	8	5	4
1	7	4	5	2	8	3	6
1	7	6	3	2	8	5	4
1	7	8	5	6	4	3	2

VI.

o	e	e	o	e	o	o	e
1	4	2	7	8	5	3	6
1	6	4	7	8	3	5	2
1	6	8	3	4	7	5	2
1	8	6	7	4	5	3	2

VII.

o	e	o	o	e	o	e	e
1	4	7	5	2	3	8	6
1	8	5	3	2	7	6	4
1	8	5	7	6	3	2	4
1	8	7	5	6	3	4	2*

VIII.

o	e	o	e	e	o	e	o
1	4	7	2	8	5	6	3
1	6	5	2	4	7	8	3
1	6	5	2	8	3	4	7
1	8	7	6	4	5	2	3

* The original edition has, erroneously, 1 8 7 5 6 3 2 4. -Ed.

IX.	o	o	e	e	o	o	e	e
	1	3	2	8	7	5	4	6
	1	3	8	2	7	5	6	4
	1	5	2	6	3	7	8	4
	1	5	4	8	3	7	6	2
	1	5	4	8	7	3	2	6
	1	5	6	2	7	3	8	4
	1	7	6	8	3	5	4	2
	1	7	8	6	3	5	2	4

X.	o	e	e	o	o	e	e	o
	1	4	6	7	3	2	8	5
	1	4	8	5	3	2	6	7
	1	6	2	5	3	8	4	7
	1	6	2	5	7	4	8	3
	1	6	4	7	3	8	2	5
	1	6	8	3	7	4	2	5
	1	8	4	5	7	2	6	3
	1	8	6	3	7	2	4	5

96. In considering some class of square-forming functions containing λ functions, it is clear that, since one can combine each of them with each of the functions of another class, one will obtain λ^2 different solutions. Thus, since we have in all eight classes each of which contains four functions, of which each can be combined with one or the other of the same class*, one can deduce sixteen solutions from each class and consequently 128 solutions from the eight classes; and by adding to them the two classes of eight functions, each of which furnishes 64 solutions, the number of all the possible solutions will be 256, all of which will equally satisfy the problem. But it must be noted that the Latin squares <u>à quadruple marche</u> will give a still greater number of them, without counting those which can be obtained by several transformations which are explained above and which will be even more clearly explained in what follows. This, added to the different solutions for the cases of n = 3, n = 4, n = 5, and n = 7, ought to increase our surprise in regard to the case of n = 6, the impossibility of which appears to be more and more confirmed.

End of second section

* The original edition erroneously has: "...combined with one or another of the other classes...". -Ed.

THIRD SECTION

ON LATIN SQUARES À TRIPLE MARCHE OF THE GENERAL FORM

1	2	3	4	5	6	7	8	9	etc.
2	3	1	5	6	4	8	9	7	etc.
3	1	2	6	4	5	9	7	8	etc.
4	5	6	7	8	9	10	11	12	etc.

etc.

96. [a]* Here, it is evident that the number n must necessarily be divisible by 3; we will thus establish throughout $n = 3m$, where m will indicate the number "members" (groups of 3) of which each row and column is composed. Thus, the simplest case will be the one where $m = 1$ or rather $n = 3$, and the Latin square comprising a single member of the general square <u>à triple marche</u> will be:

1	2	3
2	3	1
3	1	2

the construction of which has been sufficiently explained in section 18 of the first section.

97. The first question which presents itself here is that of knowing whether or not all the cases of this square <u>à triple marche</u> always admit of square-forming functions. Now I should first take note of the fact that when the square is made up of two members, it can never admit of square-forming functions, so that the case of $n = 6$ must again be excluded from this type of simple squares. One can convince oneself of the truth of this by the ordinary

* The original edition erroneously has section 96 twice. -Ed.

method of looking for functions, but this truth will acquire that much higher a degree of certainty since one can give a very rigorous demonstration of it, drawn from principles which are completely different from those by which we have proved the impossibility of the preceding cases, where the number n was oddly even, and which could not be applied in this section because of the multiplicity of different cases which one would be obliged to consider.

98. In order to make this demonstration clearer and easier, I will indicate the first member of the proposed square à triple marche, which is

1	2	3
2	3	1
3	1	2

by the letter A, which will thus include three rows and three columns; and the letter a will indicate each number contained in this small square, that is to say, 1, 2, or 3. In the same way, I will express the second member of the square, which is

4	5	6
5	6	4
6	4	5

by the letter B and each of the numbers which it contains by b. Granting this, we can represent the Latin square with two members, that is to say for the case where n = 6, in this way:

A	B
B	A

where each row and column includes six* terms.

* Original edition:"...includes three terms." -Ed.

99. Now I observe that, if this square admitted of a square-forming function, it should contain three a's and three b's, some of which would be taken from the first column AB, and others of which would be taken from the second column BA. Now since all the terms of such a function should be taken from different rows and columns, each term which one puts in the function excludes one row and one column. Thus, when one wishes to take all three a's from the first column, since they would be taken from the letter A, the first row will at once be excluded, and so will the first column, and consequently the three b's should be taken from the second part of the second column, that is from the member A, the only one remaining which contains no b's at all.

Let us suppose then, that one takes from the first column two a's and one b, that is, three terms; and it will be necessary for the other to provide as many, that is, one a and two b's. Now since the two a's are taken from the member A of the first row and the b from the member B of the second row, it is clear that the remaining term of the first row can be only b, and that of the second row a a, since the first column is excluded. Instead of the missing terms a b b, we obtain a a b. From this one already sees fairly clearly that while taking one a and two b's from the first column, it would be similarly impossible to derive from the second column the remaining terms a a b. Consequently, it is demonstrated that the case of $n = 6$ admits of no square-forming function.

100. But if for the case of $n = 9$ or $m = 3$, we mark the third member of the general square, that is

$$\begin{matrix} 7 & 8 & 9 \\ 8 & 9 & 7 \\ 9 & 7 & 8 \end{matrix}$$

by the letter C and the three numbers, 7, 8, 9 which it contains by the letter c, we shall have the square

A	B	C
B	C	A
C	A	B

to examine, and the square-forming function, if there is one, will include three a's, three b's, and three c's. In taking the three a's from the first column, the first row will be excluded and in consequence one will be able to take from the second column only the three c's, which will exclude the second row; and because there are still the three b's remaining in the third column, one easily sees that this case furnishes square-forming functions; one will even be able to deduce some in other ways.

101. In examining in the same way the case of $n = 12$ or $m = 4$ and designating the fourth member of the general square

10	11	12
11	12	10
12	10	11

by the letter D and the terms which it contains, 10, 11, 12 by d, so that the square to be examined is

A	B	C	D
B	C	D	A
C	D	A	B
D	A	B	C

one will see that, no matter in what way one takes the small letters from the rows and columns of this square, it will never furnish square-forming functions; and it seems that one can dare to draw the same conclusion for all cases where n is an even number, so that this section applies only to odd multiples of 3, like 3, 9, 15, 21 etc.

102. The beauty of the demonstration for the case of $n = 6$, presented in sections 98 and 99, leads me to digress to Latin squares à quintuple marche, or à septuple marche or that of any other odd number, for which one can demonstrate with the same ease that any of them which comprise only two members can never admit of square-forming functions. For designating, for the case of $n = 10 = 2 \times 5$, the two members of which it is composed by A and B, and the five terms which they each contain by a and b, it will be a matter of deducing from the square

A B

B A

or

a a a a a b b b b b

. . . .

a a a a a b b b b b

b b b b b a a a a a

. . . .

b b b b b a a a a a

a function which contains, in some order, five a's and five b's.

103. Thus, if we wanted to take all five a's from the first column, the first row would be excluded and there would remain in the second column only the term A, which includes no b's. If we took from the first column four a's and one b, the second column could furnish only one b and four a's, while we would need one a and four b's in order to complete the function. The same problem occurs when three a's and two b's are taken from the first column, for, instead of the two a's and three b's that we still would need the second column would furnish only two b's and three a's. From this one sees that there are

no square-forming functions to be expected from it; and the reason is frankly the fact that the number of small letters is odd, and it seems that one can maintain that the same impossibility exists in all cases where the number of members A, B etc. is even.

104. But in all cases where the number of small letters is even, this impossibility ceases completely. For let us suppose that it is a question of a square <u>à quadruple marche</u> which includes two members, A and B, each of which contains its small letter, a or b, four times, which would be the case for $n = 8$; it will be necessary to obtain from the square $\begin{matrix}AB\\BA\end{matrix}$ a function which includes, in some order, four a's and four b's, which presents not the slightest difficulty. One has only to take two a's and two b's from the first column; and since in the second column the first member, B, provides two more b's and the other member, A, provides two more a's, the square-forming function will be complete. From this one sees at the same time that in all these cases it is always necessary to take two a's and two b's from each column; and this reasoning holds good for all even numbers.

105. Let us return to our square <u>à triple marche</u>; and in order to find its square-forming functions, let us consider some terms, x, which corresponds to the vertical index t and to the horizontal index u; and by comparing this term to the sum of its indices, t and u, one will soon observe that there is a double relation between them; one of which is

$$x = t + u - 1$$

and the other

$$x = t + u - 4$$

the difference between them depending on the divisibility of the numbers t and u by 3. Now these numbers reduce to three types, which we can represent by

$3\lambda + 1$, $3\lambda + 2$, $3\lambda + 3$, or simply by 1, 2, 3, which can equally well designate the three types. Next, because of the ambiguity of the numbers 1 and 4 in the two expressions for x, we will put

$$x = t + u - w \quad .$$

Granting this, the following table will serve to determine the relation between x and its indices and the values of w, for all types of values of t and u.

$$\text{If } \begin{cases} t = \\ u = \end{cases} \begin{array}{|c|c|c||c|c|c||c|c|c} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \end{array}$$

$$\text{one will have } \begin{cases} w = \\ x = \end{cases} \begin{array}{|c|c|c||c|c|c||c|c|c} 1 & 1 & 1 & 1 & 1 & 4 & 1 & 4 & 4 \\ 1 & 2 & 3 & 2 & 3 & 1 & 3 & 1 & 2 \end{array}$$

from which one sees that there is $w = 4$ when one or the other of the indices t and u is $= 3$ and neither one $= 1$.

106. Having thus found $x = t + u - w$, one obtains reciprocally

$$u = x - t + w,$$

from which one can find the horizontal index u of each term x and the corresponding vertical index; and from there, one can assign the true value of w for all values t and x, as one can see from this table:

$$\text{If } \begin{cases} x = \\ t = \end{cases} \begin{array}{|c|c|c||c|c|c||c|c|c} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \end{array}$$

$$\text{one will have } w = \begin{array}{|c|c|c||c|c|c||c|c|c} 1 & 4 & 4 & 1 & 1 & 4 & 1 & 1 & 1 \end{array} \quad .$$

There are consequently three cases where $w = 4$, which we will represent separately thus:

$$w = 4, \text{ if } \begin{cases} x = \\ t = \end{cases} \begin{array}{c|c|c|} 1 & 1 & 2 \\ 2 & 3 & 3 \end{array}$$

107. This last table will be a great help in examining if a proposed function is square-forming or not. For one need only write this formula or the series of x and that of t one under the other, and deduce from it, according to

this table, the values of u, and when one finds that they are all different, it is a sure sign that the proposed function is in effect square-forming. To illustrate this with an example, let us take for the case of $n = 9$

this progression for x: 1 3 5 7 9 2 4 6 8

and writing under it the series of t: 1 2 3 4 5 6 7 8 9

by means of the stated rule,one will have u = 1 2 6 4 5 9 7 8 3

which, in including all the different values, shows that the arithmetic progression

1 3 5 7 9 2 4 6 8

is in effect square-forming.

108. Now, having found one square-forming function, one can, by methods similar to those which we have used in the preceding sections, deduce from it many other functions which are also square-forming. For, granting that for a new function the term X corresponds to the vertical index T and the horizontal index U, since we had a while ago $x = t + u - w$, one sees that the two indices t and u are permutable, so that, taking

$$T = u \quad \text{and} \quad U = t,$$

one will have

$$X = x.$$

Thus in the preceding example, having before one's eyes the values of u, one has only to arrange them in their natural order and to write under each one its number x, in the following manner:

T = 1	2	3	4	5	6	7	8	9
X = 1	3	8	7	9	5	4	6	2

and this function will surely be a new square-forming function, since all the U's, being the same as the t's, have different values.

109. One will thus be able, as in the preceding sections, to exchange the two letters t and x, taking

$$T = x \quad \text{and} \quad X = t,$$

from which one will get, as above, a new function, the inverse. Thus, the function proposed above

1 3 5 7 9 2 4 6 8

will furnish by inversion this new function

1 6 2 7 3 8 4 9 5,

and the function which we have gotten from the proposed one by the other rule,

1 3 8 7 9 5 4 6 2,

leads, when inverted, to the following:

1 9 2 7 6 8 4 3 5.

110. Having for U by virtue of this rule, where $T = x$ and $X = t$, the formula

$$U = X - T + w = t - x + w,$$

since these expressions range over all the values while t and x undergo the necessary variations, it follows that, taking

$$T = t,$$

one can put

$$X = t - x + w,$$

and that is the essence of the second rule which differs from those of the preceding sections only with respect to the meaning of w, which here will always be = 1, except for the three cases mentioned in section 106, that is*

* The permutation of x and t with respect to the end of section 106 comes from the fact that one supposed, at the beginning of section 110, $T = x$ and $X = t$. -Ed.

$$\begin{array}{r|c|c|c} t = & 1 & 1 & 2 \\ x = & 2 & 3 & 3 \end{array}$$

for which it is necessary to take $w = 4$. By means of these two rules, as soon as one has found some functions by the ordinary methods, one can deduce from them several others.

111. But here, one will soon discover great variety in the functions which one wants to transform by these rules. There are some which remain unalterable by both rules. Such a one is

1 3 2 7 9 8 4 6 5

which is the diagonal of the proposed square; it is reproduced by both the first and the second of our rules. Next, there are also functions which by the use of both of these rules produce only one new function. Such a one, for example, is the arithmetic progression decreasing by 1,

1 9 8 7 6 5 4 3 2,

which reproduces itself by the first rule, while the second rule furnishes this function:

1 6 5 7 3 2 4 9 8,

which reproduces itself by reversal.

112. Let us develop the proposed arithmetic progression

1 3 5 7 9 2 4 6 8,

which with the help of our two rules [sections 109 and 110] furnishes, as one will see, four new functions.

Proposed function	1	3	5	7	9	2	4	6	8
Reversed function	1	6	2	7	3	8	4	9	5
Second rule	1	3	8	7	9	5	4	6	2
	1	9	2	7	6	8	4	3	5

There are, thus, with the preceding ones, seven square-forming functions for the case of n = 9, which all have the excellent property that their terms follow the same order with respect to their divisibility by three. Now it is easy to find still others which in this regard follow the same laws, which we will set forth all together.

1	3	2	7	9	8	4	6	5
1	3	5	7	9	2	4	6	8
1	3	8	7	9	5	4	6	2
1	6	2	7	3	8	4	9	5
1	6	5	7	3	2	4	9	8
1	6	8	7	3	5	4	9	2
1	9	2	7	6	8	4	3	5
1	9	5	7	6	2	4	3	8*
1	9	8	7	6	5	4	3	2

all of which we have found, except the following two:

1	6	8	7	3	5	4	9	2
1	9	5	7	6	2	4	3	8*

which reproduce each other by both the first and second rules.

It is important to have set forth these 9 functions which keep the same order with respect to the terms which are divisible by three. For we will see in the following section that, in order to form a complete magic square, one can use 2 and even 3 similar functions for the different superscripts in regard to our three types of numbers; from this one sees that these nine functions are capable of producing a prodigious number of different squares.

* The original edition erroneously has 1 9 5 7 6 2 4 9 8. -Ed.

113. But there is also a quantity of square-forming functions which in this way furnish as many as a dozen new ones, as one can see from the following one, chosen at random.

Proposed function	1	3	8	6	7	9	2	5	4
Reversed function	1	7	2	9	8	4	5	3	6

from which one gets by the

second rule	1	3	5	2	8	7	9	4	6
	1	5	2	8	7	3	6	9	4
first rule*	1	4	2	8	3	9	6	5	7
	1	3	6	9	2	7	5	4	8
second rule	1	8	2	9	6	7	5	4	3
	1	3	7	8	4	9	6	5	2
first rule*	1	3	9	8	7	5	6	2	4
	1	9	2	5	8	7	3	4	6
second rule	1	3	4	9	8	2	5	7	6
	1	6	2	3	7	9	8	5	4

and consequently twelve, none of which was known to us before.

114. After these rules for the invention of square-forming functions for the superscript 1, it still remains to see what means are necessary to deduce functions for the other superscripts, or rather in what way it is necessary to construct the complete system. In order to do this, I observe in general that having found, for some superscript a, the function

a b c d e etc.,

* Here, the first rule consists in forming the reversed function. In this table, a pair of functions is always deduced from the one which precedes it, which is not the case in sections 81 and 94. See the note on sections 81 and 82. -Ed.

one will derive from it, by virtue of the form of the Latin square, the function for the superscript a + 3, by adding 3 to each term of the first function. And considering attentively the form of the proposed square, one may even suspect that, if some term of the proposed function is of the form $3\alpha + 1$, the term for the superscript 2 will be of the form $3\alpha + 2$ and the one for the superscript 3 of the form $3\alpha + 3$. Next, if a term of the proposed function for the superscript 1 has the form $3\alpha + 2$, the corresponding one for the function for the superscript 2 will have the form $3\alpha + 3$ and the one for the superscript 3 will have the form $3\alpha + 1$. Finally, if the term of the proposed function is of the form $3\alpha + 3$, the one for the superscript 2 will be of the form $3\alpha + 1$ and the one for the superscript 3 will be of the form $3\alpha + 2$. This conjecture, which it will be well to set forth for greater clarity in the following table:

form of the	for the superscripts	
term	2	3
$3\alpha + 1$	$3\alpha + 2$	$3\alpha + 3$
$3\alpha + 2$	$3\alpha + 3$	$3\alpha + 1$
$3\alpha + 3$	$3\alpha + 1$	$3\alpha + 2$

can even be demonstrated rigorously in the following way.

115. Let there be, in the proposed square-forming function for the superscript 1, some term x which corresponds to the vertical index t and the horizontal index u, so that

$$u = x - t + w \quad .$$

Next let there be, in the function for the superscript 2, a term x which corresponds to the same vertical index t, but to the horizontal index u', so that

$$u' = x' - t + w \quad .$$

Finally, let there be, in the function for the superscript 3, a term x'' corresponding to the same vertical index t and to the horizontal index

$$u'' = x'' - t + w \;.$$

It should be noted that the horizontal indices u, u', u'' must be taken from the same function explained above. Granting this, it is necessary to show that, while the index u ranges through all the values (and that is the essence of the nature of square-forming functions), the two other indices u' and u'' also range through all the values. Now this will seem clear in the table below, which represents all the possible cases with respect to the two given values t and x, where we have put, in order to shorten this, $\alpha - \beta = \gamma$.

$t = 3\beta +$	1	2	3	1	2	3	1	2	3
$x = 3\alpha +$	1	2	3	2	3	1	3	1	2
$u = 3\gamma +$	1	1	1	2	2	2	3	3	3
$x' = 3\alpha +$	2	3	1	3	1	2	1	2	3
$u' = 3\gamma +$	2	2	2	3	3	3	1	1	1
$x'' = 3\alpha +$	3	1	2	1	2	3	2	3	1
$u'' = 3\gamma +$	3	3	3	1	1	1	2	2	2

From this table, it is evident that, every time that $u = 3\gamma + 1$, one will have

$$u' = 3\gamma + 2 \quad \text{and} \quad u'' = 3\gamma + 3 \;.$$

Similarly, when $u = 3\gamma + 2$, one will have

$$u' = 3\gamma + 3 \quad \text{and} \quad u'' = 3\gamma + 1 \;.$$

Finally, when $u = 3\gamma + 3$, one will have

$$u' = 3\gamma + 1 \quad \text{and} \quad u'' = 3\gamma + 2 \;.$$

From this one sees that, since u varies through all the values, both the u' and the u'' must also vary through all the values, and consequently the rule given above gives us for each function for the superscript 1 two other functions for the superscripts 2 and 3, from which one can form the functions for the superscripts 4, 5, 6, by adding 3 to each term of the first three; and those for the superscripts 7, 8, 9, by doing the same thing to the preceding three.

116. In this way, the formation of a complete system of square-forming functions from a single proposed function for the superscript 1 of the fundamental Latin square will not present the slightest difficulty. Let us take once more, in order to give an example for the case of n = 9, the function which proceeds in arithmetic progression

1 3 5 7 9 2 4 6 8;

and the complete system will be

1	3	5	7	9	2	4	6	8
2	1	6	8	7	3	5	4	9
3	2	4	9	8	1	6	5	7
4	6	8	1	3	5	7	9	2
5	4	9	2	1	6	8	7	3
6	5	7	3	2	4	9	8	1
7	9	2	4	6	8	1	3	5
8	7	3	5	4	9	2	1	6
9	8	1	6	5	7	3	2	4

and the complete square which results from this system will have the following form:*

1^1	2^3	3^8	4^7	5^9	6^5	7^4	8^6	9^2
2^2	3^1	1^9	5^8	6^7	4^6	8^5	9^4	7^3
3^3	1^2	2^7	6^9	4^8	5^4	9^6	7^5	8^1
4^4	5^6	6^2	7^1	8^3	9^8	1^7	2^9	3^5
5^5	6^4	4^3	8^2	9^1	7^9	2^8	3^7	1^6
6^6	4^5	5^1	9^3	7^2	8^7	3^9	1^8	2^4
7^7	8^9	9^5	1^4	2^6	3^2	4^1	5^3	6^8
8^8	9^7	7^6	2^5	3^4	1^3	5^2	6^1	4^9
9^9	7^8	8^4	3^6	1^5	2^1	6^3	4^2	5^7

* The original edition has erroneously, in the third column, 4^9 instead of 1^9.-Ed.

117. In this square, we have taken the first three square-forming functions, for the superscripts 1, 2, 3, from the same function. But one might have used different functions, provided that their terms followed the same order with respect to their divisibility by 3. Having thus cited above nine different square-forming functions which all follow the same law, one can form from them 729 complete squares, all different. To illustrate this by an example, let us take once more

for the superscript 1 the function 1 3 5 7 9 2 4 6 8

for the superscript 2 the function 1 3 8 7 9 5 4 6 2

for the superscript 3 the function 1 6 8 7 3 5 4 9 2

and the complete system of square-forming functions will be:

1	3	5	7	9	2	4	6	8
2	1	9	8	7	6	5	4	3
3	5	7	9	2	4	6	8	1
4	6	8	1	3	5	7	9	2
5	4	3	2	1	9	8	7	6
6	8	1	3	5	7	9	2	4
7	9	2	4	6	8	1	3	5
8	7	6	5	4	3	2	1	9
9	2	4	6	8	1	3	5	7

from which one constructs the following complete square:

1^1	2^9	3^5	4^7	5^6	6^2	7^4	8^3	9^8
2^2	3^1	1^6	5^8	6^7	4^3	8^5	9^4	7^9
3^3	1^2	2^7	6^9	4^8	5^4	9^6	7^5	8^1
4^4	5^3	6^8	7^1	8^9	9^5	1^7	2^6	3^2
5^5	6^4	4^9	8^2	9^1	7^6	2^8	3^7	1^3
6^6	4^5	5^1	9^3	7^2	8^7	3^9	1^8	2^4
7^7	8^6	9^2	1^4	2^3	3^8	4^1	5^9	6^5
8^8	9^7	7^3	2^5	3^4	1^9	5^2	6^1	4^6
9^9	7^8	8^4	3^6	1^5	2^1	6^3	4^2	5^7

118. Here, we have profited from the fine link which exists among the nine functions cited above; but even while using some other square-forming function, it is not difficult to discover all the other functions which have the same property with respect to their divisibility by 3. Let us take for example the following function, chosen at random:

$$1 \quad 3 \quad 8 \quad 6 \quad 7 \quad 9 \quad 2 \quad 5 \quad 4$$

and let us write, for each term, in the form of a superscript, the value for $u = x - t + w$ as well as the others of the same type, in this way:

$$\begin{array}{ccccccccc}
t = 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
x = 1 & 3 & 8 & 6 & 7 & 9 & 2 & 5 & 4 \\
u = 1 & 2 & 9 & 3 & 6 & 4 & 5 & 7 & 8 \\
\hline
1^1 & 3^2 & 8^9 & 6^3 & 7^6 & 9^4 & 2^5 & 5^7 & 4^8 \\
1^1 & 6^5 & 2^3 & 3^9 & 4^3 & & 5^8 & & 7^2 \\
 & 9^8 & 5^6 & 9^6 & & 3^7 & 8^2 & 2^4 &
\end{array}$$

and now it comes down to taking from this simple function where not only all the terms themselves but also their superscripts are different: conditions fulfilled by:

$$1^1 \quad 6^5 \quad 2^3 \quad 3^9 \quad 7^6 \quad 9^4 \quad 8^2 \quad 5^7 \quad 4^8$$
$$1^1 \quad 6^5 \quad 8^9 \quad 4^3 \quad 3^7 \quad 5^8 \quad 2^4 \quad 7^2$$

from which one can deduce new functions of the same type, which, joined to the proposed function, can be used to construct 27 complete new squares.

119. Before finishing this section, I will add still one more proof of the first rule of reversal, supposed until now to be true without having been proved. This proof is all the more necessary since there is a large number of Latin squares where this reversal is in reality unable to produce square-forming functions. It is thus a matter of showing that, when the number u,

which is = x - t + w, varies through all the values, while t and x undergo their appropriate variations, the formula t - x + w, which I will call v, will also receive all the different values.

In order to do this, it is necessary to take into account all the different types which the two numbers t and x can include, as we have shown in the proof of the preceding theorem (sections 114 and 115), with regard to the functions which correspond to the superscripts 2 and 3 and as this table explains:

$t = 3\beta +$	1	2	3	1	2	3	1	2	3
$x = 3\alpha +$	1	2	3	2	3	1	3	1	2
$u = 3\gamma +$	1	1	1	2	2	2	3	3	3
$v = 3\gamma +$	1	1	1	3	3	3	2	2	2

from which it is clear that, when u has the form $3\gamma + 1$, v will have the form $-3\gamma + 1$, and consequently the sum will be 2; that is, in the case of $u = 3\gamma + 1$, the number v will be the difference between u and 2 or rather between u and n + 2, n being the root of the square in question. Now in the two other cases, $u = 3\gamma + 2$ or $u = 3\gamma + 3$, one will have $v = - 3\gamma + 3$ or $v = - 3\gamma + 2$, and consequently in either one u + v = 5 or rather = n + 5; that is, in these two cases v is the difference between u and 5 or rather between u and n + 5. It is thus decided that when u is varied, the number v will also pass through all the values.

For the case where n = 9, let us write the u's in their natural order, namely

$$u = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$$

and the v's will be according to the rules

$$v = 1 \quad 3 \quad 2 \quad 7 \quad 9 \quad 8 \quad 4 \quad 6 \quad 5$$

from which one sees even more obviously how all the values of v vary according to the variations of the letter u.

End of third section.

FOURTH SECTION

ON LATIN SQUARES À QUADRUPLE MARCHE OF THE GENERAL FORM

$$\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{etc.} \\
2 & - & - & - & 6 & - & - & - & \\
3 & - & - & - & 7 & - & - & - & \\
4 & - & - & - & 8 & - & - & - & \\
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \text{etc.} \\
6 & - & - & - & 10 & - & - & - & \\
7 & - & - & - & 11 & - & - & - & \\
8 & - & - & - & 12 & - & - & - & \\
\text{etc.} & & & & \text{etc.} & & & & \text{etc.}
\end{array}$$

120. Since, as it is evident by the general form, this section can pertain only to squares whose root n is divisible by 4, we will put n = 4m, and m will indicate the number of "members" of which the square is composed, which will contain four terms in each row and column or rather a total of 16 terms. Then if we represent, in the manner introduced at the beginning of the preceding section, these members by the letters A, B, C etc., so that

$$A = \left\{\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & - & - & - \\
3 & - & - & - \\
4 & - & - & -
\end{array}\right. \qquad
B = \left\{\begin{array}{cccc}
5 & 6 & 7 & 8 \\
6 & - & - & - \\
7 & - & - & - \\
8 & - & - & -
\end{array}\right. \qquad
C = \left\{\begin{array}{cccc}
9 & 10 & 11 & 12 \\
10 & - & - & - \\
11 & - & - & - \\
12 & - & - & -
\end{array}\right. \\ \text{etc.}$$

the different cases which we will have to consider will be included in the following forms:

$$A \qquad \begin{array}{cc} A & B \\ B & A \end{array} \qquad \begin{array}{ccc} A & B & C \\ B & C & A \\ C & A & B \end{array} \qquad \begin{array}{cccc} A & B & C & D \\ B & C & D & A \\ C & D & A & B \\ D & A & B & C \end{array}$$

etc.

121. If we were to treat these squares on the same footing as in the preceding sections, we would fall into some very laborious calculations. It will thus be necessary to use another method, which will also be able to be used when the proposed squares are <u>de tout autre marche au - dela de la quadruple</u>.* It is for this reason that I will propose here a method which will make these investigations considerably easier, and by which all the objects will be represented in a manner which is as clear as it is easy:

122. First, in considering some term of the proposed square, which we will indicate by the letter x, it is a question of discovering the relation which this term has with the indices, the vertical = t and the horizontal = u; from which it is clear that one must take into account four terms, for which the formulas will be $4\lambda + 1$, $4\lambda + 2$, $4\lambda + 3$, $4\lambda + 4$. In conformity with these four types, we will always put

$$t = 4p + f, \ u = 4q + g, \ x = 4s + h,$$

where the numbers p, q, and s will always be smaller than m and the other letters, f, g, and h, will always represent one of the four numbers 1, 2, 3, 4. Besides that, in considering the proposed square, it will be easily perceived that one will always have

$$s = p + q,$$

* That is to say, when the proposed squares permute their rows and columns by groups, or "members," larger than four terms. -M. L. Barr

by observing that, when the number x becomes larger than n = 4m, one must subtract from it the number n, and the remainder will indicate the correct value for the letter s.

123. We have already noted above that, in this case of squares <u>à quadruple marche</u>, the first member A can be of four different forms (see section 16) which it will be good to set forth here.

I				II				III				IV			
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
2	3	4	1	2	1	4	3	2	1	4	3	2	4	1	3
3	4	1	2	3	4	1	2	3	4	2	1	3	1	4	2
4	1	2	3	4	3	2	1	4	3	1	2	4	3	2	1

From these forms for the first member, it will be easy to obtain those for the following members B, C, and D, etc., by increasing all the terms; for the second, B, by 4; for the third, C, by 8; for the fourth, D, by 12; and so forth.

124. Let us begin with the first form whose first row will represent the values of f for the form t = 4p + f, while the first column gives the values of g for the form u = 4q + g; and the terms of this form themselves will represent the values of the letter h for the form x = 4s + h, if it is observed that s = p + q. This can be represented in the following way:

$$
g\left\{\begin{array}{c|cccc}
 & \multicolumn{4}{c}{\overbrace{\hphantom{1\quad 2\quad 3\quad 4}}^{f}} \\
1 & 1 & 2 & 3 & 4 \\
\hline
2 & 2 & 3 & 4 & 1 \\
3 & 3 & 4 & 1 & 2 \\
4 & 4 & 1 & 2 & 3
\end{array}\right.
$$

where the terms of the square indicate the numbers h for all the values of f and g.

125. From that, one will be able easily to construct another square which will represent the values of the letter g which correspond to the values of g and h.

$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{f}} \\ & & 1 & 2 & 3 & 4 \\ \hline & 1 & 1 & 4 & 3 & 2 \\ h & 2 & 2 & 1 & 4 & 3 \\ & 3 & 3 & 2 & 1 & 4 \\ & 4 & 4 & 3 & 2 & 1 \end{array} \qquad \begin{array}{cc|cccc} & & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{g^{*}}} \\ & & 1 & 2 & 3 & 4 \\ \hline & 1 & 1 & 4 & 3 & 2 \\ h^{*} & 2 & 2 & 1 & 4 & 3 \\ & 3 & 3 & 2 & 1 & 4 \\ & 4 & 4 & 3 & 2 & 1 \end{array} .$$

These diagrams can very conveniently be applied to judge square-forming functions, whose nature requires that there correspond to all the values of $t = 4p + f$ an equal number of different values for the letter $x = 4s + h$; and second, that the values of the index $u = 4q + g$ also all differ.

126. After expressing the values of the numbers t, u, x by two members, it will be good to note, for the sake of convenience in the following explanation, that the first is so to speak the "characteristic," which indicates the closest smaller multiple of four, and the other is the "mantissa," which indicates the remainder of a proposed number with respect to divisibility by 4.** Thus for the numbers of the first member, A, which are 1, 2, 3, 4, the "characteristic" will be 0; for those of the second member, B, which are 5, 6, 7, 8, the "characteristic" will be 4; for those of the third, C, namely 9, 10, 11, 12, it will be 8; and so forth. Moreover, it is evident that the "characteristic" of x is always equal to the sum of the "characteristics" of t and u, so that, if one takes into account only the "characteristics," one will always have $x = t + u$ and consequently

$$u = x - t.$$

---

* In the original edition, the letters g and h are erroneously interchanged.-Ed.

**In other words, this "mantissa" is the residue modulo 4. M. L. B.

127. Then, since in all cases the "characteristics" are subject to no difficulty, we can dispense with them entirely, and consequently we need only look at the residues of f, g, h, which form the remainders of t, u, x, that is, what remains after division by four; and because of this, we can also dispense with the letters f, g, h, in place of which we will use only t, u, and x, as we have done in the preceding sections, which will make our investigation considerably easier.

However, we will add to these three letters t, u, x, a fourth, v, which is related in the same way to the letters x and t, as u is related to the letters t and x, so that looking only at the "characteristics," one will have

$$v = t - x,$$

in the place where we had $u = x - t$, from which one sees that the "characteristic" of v will always be the negative of the "characteristic" of u or rather its difference from the number $n = 4m$; and the sum of the "characteristics" of these two letters will always be either 0 or n.

128. Now, it will be easy to represent by suitable diagrams how each of these four letters is determined by two others. For first, if one regards the letters t and u as known, the form of the number x will be determined by the first diagram, from which one can easily form the second, for the values of u when t and x are known.

Diagram 1
for the values of x

$$
\begin{array}{cc|cccc}
 & & \multicolumn{4}{c}{\overbrace{\hphantom{1\quad 2\quad 3\quad 4}}^{t}} \\
 & & 1 & 2 & 3 & 4 \\
\hline
\multirow{4}{*}{$u\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right.$} & 1 & 1 & 2 & 3 & 4 \\
 & 2 & 2 & 3 & 4 & 1 \\
 & 3 & 3 & 4 & 1 & 2 \\
 & 4 & 4 & 1 & 2 & 3
\end{array}
$$

Diagram 2
for the values of u

$$
\begin{array}{cc|cccc}
 & & \multicolumn{4}{c}{\overbrace{\hphantom{1\quad 2\quad 3\quad 4}}^{t}} \\
 & & 1 & 2 & 3 & 4 \\
\hline
\multirow{4}{*}{$x\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right.$} & 1 & 1 & 4 & 3 & 2 \\
 & 2 & 2 & 1 & 4 & 3 \\
 & 3 & 3 & 2 & 1 & 4 \\
 & 4 & 4 & 3 & 2 & 1
\end{array}
$$

From this diagram one next easily obtains the third, for the values of v by t and x; since one need only exchange the indices t and x; or rather, leaving them, one need only exchange the rows and columns, as one can see from this diagram:

Diagram 3
for the values of v

$$
\begin{array}{cc|cccc}
 & & \multicolumn{4}{c}{\overbrace{\hphantom{1\quad 2\quad 3\quad 4}}^{t}} \\
 & & 1 & 2 & 3 & 4 \\
\hline
\multirow{4}{*}{$x\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right.$} & 1 & 1 & 2 & 3 & 4 \\
 & 2 & 4 & 1 & 2 & 3 \\
 & 3 & 3 & 4 & 1 & 2 \\
 & 4 & 2 & 3 & 4 & 1
\end{array}
$$

129. From the first of these three diagrams, it is immediately clear that when the letters t and u are transposed, the figure stays the same. Thus, when one has found some square-forming function, in which the term x corresponds to the vertical index t, one can immediately deduce from it another in which, indicating by X the term which corresponds to the vertical index T, one has only to take T = u and X = x, and then, calling the horizontal index of this new function U, one will have U = t. For it is clear that, while the two letters T and X vary through all the values, the letter U will also range over the same variations. It thus is only a matter of arranging the different values of u = T according to their natural order.

130. Second, it will not be difficult to prove that having found a square-forming function from the letters t and x, one can always deduce another from T and X, by taking

$$T = x \quad \text{and} \quad X = t.$$

For one sees from the third diagram that the horizontal index u will be in this case = v, and consequently it is a question only of showing that, while the values of u pass through all the numbers from 1 to 4, those of v also undergo the same changes.

Let us take for this purpose a new diagram, which indicates to us the sum of the two letters u and v, t and x being given.

$$\begin{array}{cc|cccc}
 & & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{t}} \\
 & & 1 & 2 & 3 & 4 \\
\hline
\multirow{4}{*}{$x\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right.$} & 1 & 2 & 6 & 6 & 6 \\
 & 2 & 6 & 2 & 6 & 6 \\
 & 3 & 6 & 6 & 2 & 6 \\
 & 4 & 6 & 6 & 6 & 2
\end{array}$$

from which it is clear that, since the "characteristics" of u and v cancel each other out, one will always have $u + v = 2$ or $u + v = 6$; the first will occur whenever $u = 1$ or $u = 4\lambda + 1$; in all the other cases, there will be $u + v = 6$ or $u + v = n + 6$.

131. Let us develop these different cases. First, taking $u = 4\lambda + 1$, one will have $v = -4\lambda + 1$ or, after adding n to it, one will have $v = 4(m-\lambda) + 1$; from this it is seen that, while the letter u assumes all the values of type 1, the letter v will also take on all these values. Second, taking $u = 4\lambda + 2$, one will have $v = 4(m-\lambda) + 4$; or rather v will be the difference between u and 6 or between u and $n + 6$; thus, while u varies over all the values of type 2, the

letter v will range over those of type 4. And if u ranges over those of type 3, v will take on the values of the same type. Finally, while u assumes all the values of type 4, v will take on those of type 2. From this it is clear that in general, if u ranges over all the variations, $v = U$ undergoes them also; consequently "reversal" of the square-forming functions takes place in all cases without the least restriction.

132. From this double transformation of each square-forming function, one can deduce several others. For having arrived at the values

$$T = x, \quad X = t, \quad U = v,$$

one will have, by exchanging the letters T and U by following the first transformation, this new transformed formula for a function

$$T = v, \quad X = t, \quad U = x,$$

and from that, by exchanging the letters T and X by following the other transformation, one will have this new one

$$T = t, \quad X = v, \quad U = u,$$

which corresponds to the one which we have found in the preceding sections by our second rule.

133. Although we have found still other transformations, it will suffice to use the two which correspond to those in the other sections, in view of the fact that by the combination of these two rules, one can deduce as many as twelve new functions from each proposed square-forming function. This is why we will set them forth here:

<u>If one has some square-forming function, in which the term x corresponds to the vertical index t, and one calls the vertical index for the new function T, and the term which corresponds to it X, one will always have</u>

by the first rule $\quad T = x$ and $X = t$,

by the second rule $\quad T = t$ and $X = v$,

where the number v must be determined by the third diagram given above, which we will repeat here, since all the transformations which one will wish to perform depend on this form alone.

Diagram 3
for the values of v

$$x\left\{\begin{array}{c|cccc} & \multicolumn{4}{c}{\overbrace{1 \quad 2 \quad 3 \quad 4}^{t}} \\ \hline 1 & 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 2 & 3 \\ 3 & 3 & 4 & 1 & 2 \\ 4 & 2 & 3 & 4 & 1 \end{array}\right.$$

134. After finding all the functions for the superscript 1, or at least a large part of them, it is clear that by adding to each term of a given function either 4, 8, or 12, etc., one will have the functions for the superscripts 5, 9, 13, etc.; and consequently, there remains only to show how one can find functions for the superscripts 2, 3, 4, 6, 7, etc., so that one can extract a complete system of functions; after which, as has been already seen, it is no longer difficult to construct the complete square.

135. For the term x in the function for the superscript 1, let the horizontal index = u. In the function for the superscript 2, let the horizontal index for the term $x' = u'$; in the function for the superscript 3, let the term be x'' and the index $= u''$, and thus for the others, x''' and u''', x'''' and u'''' etc. This being granted, the first "member," A, shows us the following relationships among these different values for x:

$$\begin{array}{llllll} x & = 1, & 2, & 3, & 4 \\ x' & = 2, & 3, & 4, & 1 \\ x'' & = 3, & 4, & 1, & 2 \\ x''' & = 4, & 1, & 2, & 3\,. \end{array}$$

It will then be necessary to prove that, while the letter u varies through all the values, the letters u', u'', u''' will also undergo the same variations.

136. For this purpose let us consider the diagrams taken from the second one given above, which express the values of u by t and x; they will be represented in the following manner:

for u

$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{t}} \\ & & 1 & 2 & 3 & 4 \\ \hline \multirow{4}{*}{$x\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right.$} & 1 & 1 & 4 & 3 & 2 \\ & 2 & 2 & 1 & 4 & 3 \\ & 3 & 3 & 2 & 1 & 4 \\ & 4 & 4 & 3 & 2 & 1 \end{array}$$

for u' *

$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{t}} \\ & & 1 & 2 & 3 & 4 \\ \hline \multirow{4}{*}{$x'\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right.$} & 1 & 2 & 1 & 4 & 3 \\ & 2 & 3 & 2 & 1 & 4 \\ & 3 & 4 & 3 & 2 & 1 \\ & 4 & 1 & 4 & 3 & 2 \end{array}$$

for u''

$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{t}} \\ & & 1 & 2 & 3 & 4 \\ \hline \multirow{4}{*}{$x''\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right.$} & 1 & 3 & 2 & 1 & 4 \\ & 2 & 4 & 3 & 2 & 1 \\ & 3 & 1 & 4 & 3 & 2 \\ & 4 & 2 & 1 & 4 & 3 \end{array}$$

for u'''

$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{t}} \\ & & 1 & 2 & 3 & 4 \\ \hline \multirow{4}{*}{$x'''\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right.$} & 1 & 4 & 3 & 2 & 1 \\ & 2 & 1 & 4 & 3 & 2 \\ & 3 & 2 & 1 & 4 & 3 \\ & 4 & 3 & 2 & 1 & 4 \end{array}$$

* In the diagram for u', the original edition has $x'\left\{\begin{array}{l}2\\3\\4\\1\end{array}\right.$ instead of $x'\left\{\begin{array}{l}1\\2\\3\\4\end{array}\right.$ -Ed.

Comparing the second of these diagrams with the first, one sees that throughout there is $u' = u + 1$ or $u' = u - 3$, the second of which occurs when $u = 4$; in all the other cases, there is $u' = u + 1$. Next, comparing the third diagram with the first, one will have either $u'' = u + 2$, or $u'' = u - 2$, where the second case occurs when $u' = 3$ or $= 4$. Finally, the comparison of the fourth diagram asserts that one will have $u''' = u - 1$ in all cases except that of $u = 1$, for which it becomes $u''' = u + 3$. It is thus decided that while u receives all the suitable values, the letters u', u'', u''' will pass through the same variations.

137. Thus one clearly sees in what way, from some square-forming function for the superscript 1, one can form a complete system of functions and a complete square. But from what we have said in the preceding sections, one easily understands that, to form the functions for the superscripts 2, 3, 4, one can use different functions for the superscript 1, provided that their terms follow the same order with respect to divisibility by 4, which is a very fertile source which multiplies considerably the number of all the complete squares in comparison with all the different functions found for the superscript 1.

138. After these general investigations for all squares divisible by four, we will consider some particular cases. Now first, when the proposed square contains only one member, A, which is a square à simple marche, we have shown, in the first section, that it can have no square-forming functions. For this reason we will confine ourselves to citing the case of $n = 8$, where the square contains two members, A and B, whose form is:

1	2	3	4	5	6	7	8
2	3	4	1	6	7	8	5
3	4	1	2	7	8	5	6
4	1	2	3	8	5	6	7
5	6	7	8	1	2	3	4
6	7	8	5	2	3	4	1
7	8	5	6	3	4	1	2
8	5	6	7	4	1	2	3

This square apparently furnishes 48 functions for the superscript 1, when one examines it according to the rules given above; I will cite those of them which I have found by the first method, shown in section 11 etc., which are:

1	3	7	5	8	4	2	6	1	4	7	6	8	4	2	5
1	3	7	5	4	8	6	2	1	4	7	6	2	5	8	3
1	3	8	6	4	2	5	7	1	4	8	7	3	5	6	2
1	3	8	6	7	5	2	4	1	4	8	7	3	2	6	5
1	3	5	7	8	4	2	6	1	4	5	8	6	3	2	7
1	3	5	7	2	8	6	4	1	4	5	8	2	7	6	3
1	3	6	8	7	5	4	2	1	4	6	5	8	7	3	2
1	3	6	8	2	4	5	7	1	4	6	5	3	2	8	3

from which one can easily find the rest, by applying the rules which have been so often repeated.

139. We will not stop to develop the magic squares which this case can furnish since all the principles have been sufficiently explained and proved;

and since the three other cases of the form of the first member, A, give not the least difficulty when treated in the same way as the first form, it would be superfluous to carry this investigation any further. We will thus end this section with the remark that the case which we have just examined can not occur when the number of members A, B, C etc. is 3 or 5 or perhaps any other odd number.

End of fourth section.

FIFTH SECTION

ON THE TRANSFORMATION OF BOTH SIMPLE AND COMPLETE SQUARES

140. after seeing that all the methods which we have presented up to this point can not furnish any magic square for the case of $n = 6$ and that the same conclusion seems to hold true for all oddly even values of n, one might think that, if such squares are possible, the Latin squares which serve as their bases, since they do not follow any of the orders which we have just considered, would be totally irregular. It would then be necessary to examine all the possible cases of such Latin squares for the case of $n = 6$, whose number is doubtless very large. And since besides that the formation of irregular squares is not so easy, I am going to state a method by means of which one can easily transform, in several different forms, all the regular squares and then examine whether they admit of square-forming functions or not.

141. This method depends on this principle: that if, in a proposed Latin square, two numbers a and b are found in the corners of a rectangle, in the way shown by the following diagram

```
a . . . . . b
.           .
.           .
.           .
b . . . . . a
```

one can exchange these two letters, writing a in place of b and b in place of a; the reason for this is obvious, for it is easily seen that, notwithstanding this transposition, all the rows and columns will still include the same numbers. It is thus evident that by this principle one will be able to transform each proposed square into several other different forms which will have, with regard to the square-forming functions, quite special properties.

142. Let us consider for example the following Latin square <u>à simple marche</u> of 36 entries

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & \underline{3} & 4 & 5 & \underline{6} & 1 \\
3 & 4 & 5 & 6 & 1 & 2 \\
4 & 5 & 6 & 1 & 2 & 3 \\
5 & \underline{6} & 1 & 2 & \underline{3} & 4 \\
6 & 1 & 2 & 3 & 4 & 5
\end{array}$$

which, as we have demonstrated in section I, section 20, admits of no square-forming function. Let us transpose in the way which we have stated the two indicated numbers, 3 and 6, which are arranged in a rectangle, and we will obtain the following square

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 6 & 4 & 5 & 3 & 1 \\
3 & 4 & 5 & 6 & 1 & 2 \\
4 & 5 & 6 & 1 & 2 & 3 \\
5 & 3 & 1 & 2 & 6 & 4 \\
6 & 1 & 2 & 3 & 4 & 5
\end{array}$$

which, despite its apparent likeness, differs so essentially from the proposed square that one can deduce from it a large number of square-forming functions for all six superscripts, although the other didn't furnish any at all. Here they are:

1	6	5	2	4	3		4	3	2	5	1	6
1	6	5	3	2	4		4	3	2	6	5	1
1	4	6	2	3	5		4	1	3	5	6	2
1	4	2	5	6	3		4	1	5	2	3	6
1	5	4	3	6	2		4	2	1	6	3	5
1	5	2	3	6	4		4	2	5	3	6	1
1	3	4	6	2	5		4	6	1	3	5	2
1	3	6	5	4	2		4	6	3	2	1	5
2	4	3	1	6	5		5	1*	6	4	3	2
2	3	5	1	4	6		5	6	2	4	1	3
2	3	6	4	1	5		5	6	3	1	4	2
2	1	5	4	6	3		5	4	2	1	3	6
3	2	4	1	6	5		6	5	1	4	3	2
3	6	2	1	5	4		6	3	5	4	2	1
3	6	1	4	2	5		6	3	4	1	5	2
3	5	2	4	6	1		6	2	5	1	3	4

143. After finding all these functions, there remains only to examine if one can form a complete system from them, by means of which one can complete the simple square proposed. Now, considering carefully the functions for the superscripts 2, 3, 5, 6, one will see that, no matter in what way one wants to combine them, they furnish in the fourth column only the two numbers 1 and 4 so that these two numbers would necessarily be found twice in the same column of the complete system, whose absolute impossibility leaps to the eye. We can thus boldly assure that the simple square proposed cannot furnish a solution to the problem.

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* The Comm. Alg. here erroneously has 7. M. L. B.

144. I have examined by this method a large number of similar transformed squares without encountering a single one which didn't present the same difficulty of furnishing no system of functions in which some column didn't include one number twice, and I have not hesitated to conclude that one can not produce a complete square of 36 entries, and that the same impossibility extends to the cases of $n = 10$, $n = 14$, and in general to all oddly even numbers. For, once a method is found of transforming some magic square into several (as many as 24) different forms, if there existed a single complete square for the case of $n = 6$, there would certainly be several others whose fundamental Latin squares would all be different. Now since I have examined a very considerable number of such squares, it seems impossible to me that all the possible cases have eluded me.

145. This reasoning can be carried to a much greater degree of certainty by the general transformation which we are going to present, by means of which each proposed Latin square can be transformed into several others which all have the same property with respect to the square-forming functions, so that, if the proposed square admits of no square-forming functions, all the transformed squares will also be of the same nature, and in the case where the proposed square admits of a complete system, all those which have been derived from it will also furnish complete magic squares.

146. For this general transformation, one need only change the value of the numbers of which the Latin square is composed, by substituting in their place other numbers in some order and by then reducing the new square according to the order which we have observed up to this point, that is, that the numbers of the first row and the first column follow in their natural order. In this way one will always obtain a new square which has the same properties with respect to the square-forming functions, because one has only to transfer the same changes to the

functions of the proposed square. By this one sees that this method ought to be all the more fertile in the production of new squares in proportion as the number n is large. For, for the cases of n = 2, 3, 4, no change can be expected. For the case of n = 5, the number of variations can go as high as 3 and for the case of n = 6 the number must be that much more considerable in that the order of six numbers can receive up to 720 variations, of which several, however, will come down to the same form.

147. In order better to clarify the manner and use of these transformations, we are going to take as an example the last square of 6, which was so fertile in functions; from this, by exchanging the numbers at will in some way, for example by writing

4 6 1 3 2 5

instead of

1 2 3 4 5 6,

we will obtain the following square

4	6	1	3	2	5
6	5	3	2	1	4
1	3	2	5	4	6
3	2	5	4	6	1
2	1	4	6	5	3
5	4	6	1	3	2

which, when the rows and columns are reduced in order, will receive this ordinary form:

1	2	3	4	5	6
2	4	1	5	6	3
3	5	2	6	4	1
4	1	6	2	3	5
5	6	4	3	1	2
6	3	5	1	2	4

If we treated in the same way all the Latin squares of 36 entries, à simple, double or triple marche, which, as we have shown, do not admit of any square-forming function, we will obtain a great number of other similar squares which will not admit of such functions either; so that it will suffice to have examined a single one in order to pass judgment on all the others.

148. From that it is clear that, if there existed a single complete magic square of 36 entries, one could deduce from it several others by means of these transformations, which would equally well satisfy the conditions of the problem. Now, having examined a large number of such squares without having encountered even one, I find it more than probable that there are none. For the number of Latin squares could not be so enormous that the quantity of those which I have examined should not have furnished one which admits of square-forming functions, if any existed; in view of the fact that the cases of n = 2 and n = 3 furnish only one; the case of n = 4, four, the case of n = 5 fifty-six, by exact count, one can see that the number of variations for the case of n = 6 could not be so prodigious that the number of 50 or 60 which I might have examined would be only a small part of them. I observe again, on this occasion, that the exact count of all the possible cases of similar variations would be an object worthy of the attention of geometers, all the more since all the principles which are known in the theory of combinations cannot lend the slightest help.

149. While examining several such squares formed at random, I noticed an astounding difference with respect to square-forming functions; I encountered some which didn't furnish any, some which gave none for two superscripts but two for each of the others. Among other things, I happened also upon a square which seems to me to deserve particular attention, since it gave me four functions for each superscript, and even some which seemed to give promise of a complete system; for this reason I am going to set down here the square which produced them

Square

1	2	3	4	5	6
2	1	5	6	3	4
3	4	1	2	6	5
4	5	6	1	2	3
5	6	4	3	1	2
6	3	2	5	4	1

Square-forming functions

1	4	6	5	3	2	3	2	6	5	1	4	5	1	2	4	6	3
1	5	2	3	6	4	3	1	4	5	2	6	5	2	1	6	4	3
1	6	5	2	4	3	3	6	5	4	2	1	5	4	2	1	3	6
1	3	4	6	2	5	3	6	2	1	5	4	5	4	3	6	2	1
2	4	6	3	5	1	4	1	3	5	6	2	6	1	4	2	5	3
2	5	1	3	4	6	4	3	1	6	5	2	6	5	1	4	3	2
2	6	3	1	4	5	4	2	5	3	6	1	6	2	4	1	3	5
2	3	6	4	1	5	4	3	5	2	1	6	6	5	3	2	1	4

All of these functions have the nice property that each of them has its "reversal" among the others. But in order to form a complete system, one could combine only four of them, and those in the two following ways

1	5	2	3	6	4	1	3	4	6	2	5
2	6	3	1	4	5	2	5	1	3	4	6
3	1	4	5	2	6	3	6	2	1	5	4
4	3	1	6	5	2	4	1	3	5	6	2

and it is clear that as far as functions for the superscripts 5 and 6 are concerned, there are none which fit in to complete the system.

150. One could apply similar transformations to true magic or complete squares; but it would be superfluous to construct others by exchanging numbers. There is, on the other hand, another type of transformation which is peculiar to them, since in any magic square the Latin and Greek numbers can be exchanged, and from this one always obtains a new square which is entirely different. Thus, taking as an example the following complete square of 25 entries

$$\begin{array}{ccccc} 1^1 & 2^5 & 3^4 & 4^3 & 5^2 \\ 2^2 & 3^1 & 4^5 & 5^4 & 1^3 \\ 3^3 & 4^2 & 5^1 & 1^5 & 2^4 \\ 4^4 & 5^3 & 1^2 & 2^1 & 3^5 \\ 5^5 & 1^4 & 2^3 & 3^2 & 4^1 \end{array}$$

one will get, by the exchanging of numbers which we have mentioned, the following square

$$\begin{array}{ccccc} 1^1 & 5^2 & 4^3 & 3^4 & 2^5 \\ 2^2 & 1^3 & 5^4 & 4^5 & 3^1 \\ 3^3 & 2^4 & 1^5 & 5^1 & 4^2 \\ 4^4 & 3^5 & 2^1 & 1^2 & 5^3 \\ 5^5 & 4^1 & 3^2 & 2^3 & 1^4 \end{array}$$

which, being put in order, will reassume its original form; but this change is thus only a very particular case of the general transformation which we are going to propose.

151. Let us note that, as each term of a complete square contains two numbers, one of which has been called the Latin number, and the other the Greek number, the position which this term occupies is also determined by two numbers, one of which is the horizontal index and the other the vertical index. Each term with the position that it occupies is thus determined by four numbers, a, b, c, d: the first of these, a, is the horizontal index; b, the vertical index; c, the Latin number, and d, the Greek number; and all of these four numbers, a, b, c, d will be permutable. In this way, the terms of the last square of 25 entries can be represented as follows:

1 1 1 1	1 2 2 5	1 3 3 4	1 4 4 3	1 5 5 2
2 1 2 2	2 2 3 1	2 3 4 5	2 4 5 4	2 5 1 3
3 1 3 3	3 2 4 2	3 3 5 1	3 4 1 5	3 5 2 4
4 1 4 4	4 2 5 3	4 3 1 2	4 4 2 1	4 5 3 5
5 1 5 5	5 2 1 4	5 3 2 3	5 4 3 2	5 5 4 1

If one thinks at all about these quaternaries, one will easily perceive that all four numbers can be interchanged in all possible manners, and I need not add that the number of variations is 24, which in truth will not all produce new squares, but nonetheless a fair quantity, all the larger in proportion as n is large.

152. I had observed above [section 148] that an exact count of all the possible variations of the Latin squares would be a very important question, but which appeared to me to be extremely difficult and almost impossible once the number n exceeded 5. To approach this count, it is necessary to begin by this question:

<u>In how many different ways, if the first row is given, can one vary the second row for each proposed number n?</u>

The solution is contained in the following table:

n	number of variations	
1	0	
2	1	
3	1 = 1(1)	+ 0(0)
4	3 = 2(1)	+ 1(1)
5	11 = 3(3)	+ 2(1)
6	53 = 4(11)	+ 3(3)
7	309 = 5(53)	+ 4(11)
8	2119 = 6(309)	+ 5(53)
9	16687 = 7(2119)	+ 6(309)
10	148329 = 8(16687)	+ 7(2119)
etc.	etc.	

From that it is clear that these numbers make up a logical progression or a sort of recurrent series in which each term is determined by the two preceding ones, but whose scale of relation is variable. Thus if one calls P, Q, R, S the numbers of the variations which correspond to the numbers n, n + 1, n + 2, n + 3, one will always have

$$R = nQ + (n-1)\ P$$

and

$$S = (n-1)\ R + nQ\ .$$

One can find from that an independent formula for n, by which each term S can be expressed by the three preceding ones, P, Q, R. For, the next to the last equation giving

$$R - Q = (n-1)\ (Q + P),$$

there will be

$$n-1 = \frac{R - Q}{P + Q}\ ;$$

from which one sees that R - Q is always divisible by P + Q. In the same way one will have

$$S - R = n\ (Q + R)$$

and consequently

$$n = \frac{S - R}{Q + R}\ .$$

Then subtracting the preceding equation from this one, one will have

$$1 = \frac{S - R}{Q + R} - \frac{R - Q}{P + Q}\ ,$$

from which one gets

$$PS - PR + QS - QR - RR = PQ + PR + QR$$

and consequently

$$S = \frac{PQ - 2PR + 2QR + RR}{P + Q}$$

or rather

$$S = 2R + Q + \frac{RR + PQ}{P + Q} - Q$$

$$= 2R + Q + \frac{(R + Q)(R - Q)}{P + Q}\ .$$

Thus, taking

$$P = 53, \quad Q = 309, \quad R = 2119$$

one will have

$$2R + Q = 4547,\ R - Q = 1810,\ R + Q = 2428,\ P + Q = 362;$$

from that

$$\frac{R - Q}{P + Q} = 5$$

and consequently

$$S = 4547 + 5(2428) = 16687.$$

Or else, taking

$$P = 309, \quad Q = 2119, \quad R = 16687,$$

there will be

$$2R + Q = 35493,\ R - Q = 14568,\ R + Q = 18806,\ P + Q = 2428;$$

from that

$$\frac{R - Q}{P + Q} = 6$$

and consequently

$$S = 35493 + 6(18806) = 148329.$$

The series of the numbers of variations has again a very nice property, whose truth is nothing less than evident: it is <u>that one can even determine each term by the one which precedes it</u>. Thus, when the number of variations for the number of the terms of the second row, n, is $= P$ and for the number $n + 1 = Q$, there will always be*

$$Q = nP + \frac{-P \pm 1}{n},$$

* Original edition (three times): $Q = \frac{P \pm 1}{n}$. See the editor's preface. -Ed.

where the + occurs if n is an odd number and the - if n is even. Besides that, taking R as the number of variations for the case of n = 2, since we have found

$$R = nQ + (n-1)\ P,$$

if we put in place of Q the value which we have found, $Q = nP + \frac{-P \pm 1}{n}$, we will have a formula which determines the term R by the term before the preceding one, P, alone, that is

$$R = n^2P - P + 1 + (n-1)\ P = (n-1)\ (n+2)\ P \pm 1.$$

Thus, taking

$$n = 6 \quad \text{and} \quad P = 53,$$

one will have

$$R = 5(8)(53)\ -1 = 2119;$$

and taking

$$n = 7 \text{ where } P = 309,$$

there will be

$$R = 6(9)(309) + 1 = 16687.$$

But I must admit that I have not found the property of determining each number by the preceding number alone by anything except pure induction, and I don't see particularly well how one could deduce it from the nature of the series.

However, there is a means of deducing it immediately from the series, at least the following thoughts bring us closer still to the truth of the assertion that $Q = nP + \frac{-P \pm 1}{n}$. For, if Q is the number of variations for some case n, either odd or even, and R the number of variations for the following case, where the number of terms is n + 1, there would be, by virtue of the equation cited,

$$nQ = (n^2-1)\ P \pm 1$$

and

$$(n+1)\ R = (n^2 + 2n)\ Q \mp 1,$$

where the upper sign occurs if n is odd, the lower if n is even. Now the sum of these two equations gives this equation

$$(n+1)\ R = (n^2+n)\ Q + (n^2-1)\ P,$$

from which one gets, by dividing by n + 1, the value

$$R = nQ + (n-1)\ P,$$

which agrees perfectly with the one which we deduced above from the nature of the series.

That is what I have thought that I should add with respect to counting the variations which can occur in the simple fundamental squares, leaving to the Geometers to see if there are means of achieving the enumeration of all the possible cases, which appears to furnish a vast field for new and interesting investigations. I end mine here, on a question which, although it is in itself not very useful, has led us to rather important observations for both the theory of combinations and for the general theory of magic squares.

End