ON THE EXISTENCE AND CONSTRUCTION OF ORTHOGONAL F-SQUARES OF ORDER $n = 2s^p$, s A PRIME NUMBER

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John P. Mandeli and Walter T. Federer

Abstract

The existence of a set of $(s - 1)[2(s^p - 1)/(s - 1) - 1]^2$ orthogonal F-squares of order n equal to $2s^p$, s a prime number and p a positive integer, with s symbols plus one additional orthogonal F-square of order n with two symbols has been proved by construction. The method of construction utilizes orthogonal arrays with $2s^p$ assemblies or columns, $2(s^p - 1)/(s - 1) - 1$ constraints or rows, s symbols, and of strength two, together with the addition tables, mod s, provided by a complete set of orthogonal Latin squares of order s. From the set of orthogonal F-squares, we show how to construct orthogonal arrays of the form $(4s^{2p},2,2s^p,2)$ + $(4s^{2p},(s - 1)[2(s^p - 1)/(s - 1) - 1]^2, s, 2)$ + $(4s^{2p},1,2,2)$. Thus, the method of constructing orthogonal F-squares is also a method for constructing orthogonal arrays. We further demonstrate that for $p \geq 2$, a complete set of orthogonal F-squares is approached as $s \to \infty$, and for s constant a complete set is approached as $p \to \infty$.

Key phrases: Complete sets of orthogonal F-squares; orthogonal arrays; asymptotically complete sets of orthogonal F-squares; Kronecker product construction of orthogonal F-squares.

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1. Introduction and Summary

In an effort to conserve space, the reader is referred to Hedayat and Seiden (1970), Hedayat, Raghavarao, and Seiden (1975), and to Federer (1977) for details and definitions concerning F-squares, orthogonality of F-squares design, and of a complete set of orthogonal F-squares design. The symbol OL(n,t) has been used frequently to denote a set of t mutually orthogonal Latin squares of order n; when $t = n - 1$, the set of orthogonal Latin squares of order n is complete. A set of t mutually orthogonal F-squares of order n and of the form $F(n;\lambda_1,\lambda_2,\cdots,\lambda_m)$, has been denoted as $OF(n;\lambda_1,\lambda_2,\cdots,\lambda_m;t)$, where m is a constant and represents the number of symbols in each F-square of the set. When $t = (n - 1)^2/(m - 1)$, an integer, the set is said to be complete. For $m = s$ and $n = s^p$, s a prime power, a complete set has been given by Hedayat, Raghavarao, and Seiden (1975); when $m = 2$ and $n = 4t$, a complete set has been given by Federer (1977). The definition of a complete set of orthogonal F-squares given by Federer (1977) needs to be extended as follows for i symbols, $i = 2, 3, \cdots, n$, in an $F(n;\lambda_1,\lambda_2,\cdots,\lambda_i)$-square design:

DEFINITION 1.1. The number N_i of orthogonal F-squares designs with i distinct symbols is denoted as $OF(n;\lambda_1,\lambda_2,\cdots,\lambda_i;N_i)$, where $\sum_{h=1}^{i} \lambda_h = n$. For i a variable ranging over the values $i = 2, 3, \cdots, n$, the total set of $t = \sum_{i=2}^{n} N_i$ orthogonal

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F-squares with i distinct elements is denoted by $\sum_{i=2}^{n} OF(n;\lambda_1,\lambda_2,\cdots,\lambda_i;N_i)$. When $\sum_{i=2}^{n} N_i(i - 1) = (n - 1)^2$, the set of $t = \sum_{i=2}^{n} N_i$ orthogonal F-squares designs is said to be complete.

In the spirit of Definition 1.1, Mandeli (1975) has constructed complete sets of $F(s^p;\lambda_1,\lambda_2,\cdots,\lambda_{m_i})$-squares where $m_i = s^k$, k = 1, 2, $\cdots$, p. Thus, the number of symbols in the complete set of orthogonal F-squares can vary from s, s^2, $\cdots$, s^k, $\cdots$, to s^p.

Note the fact that $\sum_{i=2}^{n} N_i(i - 1) = (n - 1)^2$ in the complete set of orthogonal F-squares designs (CSOFSD) follows directly from factorial design and analysis of variance theory. There are $(n - 1)^2$ row by column interaction degrees of freedom and these are the only ones available for constructing F-squares. When these $(n - 1)^2$ interaction degrees of freedom are completely utilized in the construction of orthogonal F-squares, the CSOFSD has been constructed. Also, when i = m, $\sum_{i=2}^{n} N_i = (n - 1)^2/(m - 1)$.

In the present paper, we show how to construct an orthogonal set of $t = (s - 1)[2(s^p - 1)/(s - 1) - 1]^2$ $OF(n = 2s^p;\lambda_1,\lambda_2,\cdots,\lambda_s)$-squares plus one of the form $F(n;s^p,s^p)$ for s a prime number, and thereby prove their existence. To date, work on this class for n = 2(3) = 6 has been reported for an OF(6;2,2,2;4)-squares set [Hedayat et al. (1975)] and for OF(6;2,2,2;8) + OF(6;3,3;1) and OF(6;1,1,1,1,1,1;1) + OF(6;2,2,2;7) sets [D. A. Anderson, W. T. Federer, and F-C. H. Lee (unpublished reports)]. To illustrate the number N_s of orthogonal $F(n;\lambda_1,\lambda_2,\cdots,\lambda_s)$-squares for all $n = 2s^p < 100$, s an odd prime number and p a positive integer, and the proportion of the degrees of freedom accounted for by the N_s $F(n;\lambda_1,\lambda_2,\cdots,\lambda_s)$ plus one $F(n;s^p,s^p)$-squares, the following table was prepared:

$n = 2s^p$	s	N_s	$(N_s(s - 1) + 1)/(n - 1)^2$
6	3	2	5/25 = 0.200
10	5	4	17/81 = 0.210
14	7	6	37/169 = 0.219
18	3	98	197/289 = 0.682
22	11	10	101/441 = 0.229
26	13	12	145/625 = 0.232
34	17	16	257/1089 = 0.236
38	19	18	325/1369 = 0.237
46	23	22	485/2025 = 0.240
50	5	484	1937/2401 = 0.807
54	3	1250	2501/2809 = 0.890
58	29	28	785/3249 = 0.242
62	31	30	901/3721 = 0.242
74	37	36	1297/5329 = 0.243
82	41	40	1601/6561 = 0.244
86	43	42	1765/7225 = 0.244
94	47	46	2117/8649 = 0.245
98	7	1350	8101/9409 = 0.861

The method of construction we use is that of orthogonal arrays (see e.g., Raghavarao (1971), Chapter 2, for definition and discussion). [Our method of constructing orthogonal F-squares differs from that used in the above cited papers.] Since it is known that an orthogonal array forming a k row × c column matrix with s distinct elements of strength 2 exists for all $n = 2s^p$, s a prime number, our method of construction is then quite general. Such orthogonal arrays have been denoted as (n,k,s,2) where n is the number of assemblies (columns), k is the number of constraints (rows), s is the number of levels (elements or symbols), and 2 is the strength of the array.

In addition, we demonstrate that the proportion of the $(n - 1)^2$ degrees of freedom for the row $\times$ column interaction accounted for by the $(s - 1)^2 \times (2(s^p - 1)/(s - 1) - 1)^2 + 1$ degrees of freedom for the set of orthogonal F-squares constructed by the above method, approaches one as $p \rightarrow \infty$ for s constant. We further show that this proportion approaches 1/4 for $p = 1$ as $s \rightarrow \infty$ and that it approaches one for $p \geq 2$ as $s \rightarrow \infty$. This means that as p becomes large the CSOFSD is approached for any s and that for any $p \geq 2$ the method of construction produces a set which becomes closer and closer to the CSOFSD as $s \rightarrow \infty$.

2. Construction and Existence of a Set of Orthogonal F-squares of Order $n = 2s^p$

Theorem 2.1. There exists a set of $(s - 1)[2(s^p - 1)/(s - 1) - 1]^2$ $F(2s^p;2s^{p-1}, 2s^{p-1},\cdots,2s^{p-1})$ plus one $F(2s^p;s^p,s^p)$ orthogonal F-squares of order $n = 2s^p$, s a prime number and p a positive integer.

Proof: The proof is by construction, using the orthogonal arrays ($n = 2s^p$, $k = (2(s^p - 1)/(s - 1) - 1)$, s, 2). These orthogonal arrays (OA) are known to exist and Addelman and Kempthorne (1961) give a method for constructing them. Making use of the arrays thus constructed, we shall show how to construct $(s - 1)(2(s^p - 1)/(s - 1) - 1)^2$ $F(n;2s^{p-1},2s^{p-1},\cdots,2s^{p-1})$-squares. Then, we shall show how to construct the additional $F(n;s^p,s^p)$-square.

Without loss in generality we may write the first two rows (constraints) of an OA in the order given in the first two columns of Table 2.1. Then, denoting the k constraints as y_1, y_2, $\cdots$, y_k in the (n,k,s,2) OA, we construct an $F(n;2s^{p-1}, 2s^{p-1},\cdots,2s^{p-1})$-square by using constraint 1 as the rows and constraint y_1 as the columns of an addition table mod s. The n-row by n-column addition table is an $F(n;2s^{p-1},2s^{p-1},\cdots,2s^{p-1})$-square for the following two reasons:

(i) Each row of the square is an integer $y_1 + c$, $0 \leq c \leq s - 1$; y_1 has precisely s elements, 0, 1, $\cdots$, s - 1, each replicated $2s^{p-1}$ times. Adding a value of c, mod s, does not alter this property. Therefore, each row of the $n \times n$ square has s elements each replicated $2s^{p-1}$ times.

(ii) Let z be an element, $0 \leq z \leq s - 1$, from y_1. That is, let z be the element in the i^{th} position in constraint y_1. Then, the element in the i^{th} position in $y_1 + 1$ is simply $z + 1$, $\cdots$, and the i^{th} position in constraint $y_1 + s - 1$ is $z + (s - 1)$. Thus, the i^{th} column, $i = 1, 2, \cdots, 2s^p$, of the $n \times n$ square consists of $2s^{p-1}$ values of z, $2s^{p-1}$ values of $z + 1$, $\cdots$, and $2s^{p-1}$ values of $z + s - 1$. By the properties of modulo arithmetic, z, $z + 1$, $\cdots$, $z + (s - 1)$, are merely some cyclic permutation of the elements 0, 1, $\cdots$, s - 1. Hence, the i^{th} column of the square has s elements each replicated $2s^{p-1}$ times.

Therefore, putting (i) and (ii) together, results in each of the s elements being replicated $2s^{p-1}$ times in each row and each column. Consequently, an $F(2s^p;2s^{p-1}, 2s^{p-1},\cdots,2s^{p-1})$-square is obtained.

In a similar manner we can construct a second F-square of this type by forming the addition table of constraint one as the rows and any other constraint, say y_2, as the columns (see fourth column of Table 2.1.). This second F-square will be orthogonal to the first F-square constructed above because of the following: Superimpose the first F-square upon the second, which in effect superimposes y_1 on y_2. That is, $y_1 + 1$ is superimposed upon $y_2 + 1$, $\cdots$, $y_1 + (s - 1)$ on $y_2 + (s - 1)$. By the definition of OA, each element of y_1 appears an equal number of times, $\lambda = 2s^{p-1}/s = 2s^{p-2}$, with each element of y_2. Since the addition of a constant c, $0 \leq c \leq s - 1$, to y_1 or y_2, does not alter this property, each element of $y_1 + c$ appears an equal number of times, $\lambda = 2s^{p-2}$, with each element of $y_2 + c$. Since there are $2s^p$ rows in the F-square, each element of the first F-square appears an equal number of times, $2s^p\lambda = 4s^{2p-2}$, with each element of the second F-square. This is precisely the definition of orthogonality for two F-squares.

Note that since y_1 and y_2 were any two arbitrary constraints of the $(n,k,s,2)$ OA and that there are $k = 2(s^p - 1)/(s - 1) - 1$ such constraints, $2(s^p - 1)/(s - 1) - 1$ orthogonal F-squares can be constructed in the above manner. Now, let us construct an additional orthogonal set of $[2(s^p - 1)/(s - 1) - 1]F(n; 2s^{p-1}, 2s^{p-1}, \cdots, 2s^{p-1})$-squares using the constraint two (second column of Table 2.1) with each of the constraints $y_1, y_2, \cdots, y_k$. Any of these F-squares will be orthogonal to those constructed above for the following reasons. For example, construct an F-square using constraint two as the rows and constraint y_3 as the columns of the addition table. Now superimposing the first square constructed on this square in effect means superimposing y_1 on each of $y_3, y_3 + 1, \cdots, y_3 + (s - 1)$; $y_1 + 1$ on each of $y_3, y_3 + 1, \cdots, y_3 + (s - 1)$; $\cdots$, $y_1 + (s - 1)$ on each of $y_3, y_3 + 1, \cdots, y_3 + (s - 1)$. Following the previous discussion and using the definition of OA, each element of y_1 appears with each element of y_3 an equal number λ of times, where $\lambda = 2s^{p-2}$. Again, the addition of a constant c_1, $0 \le c_1 \le s - 1$, to y_1 and a constant c_3 to y_3, $0 \le c_3 \le s - 1$, does not alter this property; each element of $y_1 + c_1$ appears $\lambda = 2s^{p-2}$ times with each element of $y_3 + c_3$. Since there are $2s^p$ rows in the F-squares, each element of the first square appears $2s^p\lambda = 4s^{2p-2}$ times with each element of the square just constructed. Hence, the two F-squares are orthogonal. In a similar manner, it can be shown that this square is orthogonal to any of the other F-squares constructed above. Since constraints 1 and 2 in Table 2.1 were chosen without any loss in generality from the set $y_1, y_2, \cdots, y_k$, $[2(s^p - 1)/(s - 1) - 1]^2$ orthogonal F-squares can be constructed as described above.

Up to this point we have used addition mod s in constructing the $F(n;2s^{p-1}, 2s^{p-1},\cdots,2s^{p-1})$-squares, which consisted of the following rule:

$$\begin{array}{r} 0\;0\;\cdots\;0 \\ +\;0\;1\;\cdots\;s\text{-}1 \\ \hline 0\;1\;\cdots\;s\text{-}1 \end{array} \quad \begin{array}{r} 1\;1\;\cdots\;1 \\ +\;0\;1\;\cdots\;s\text{-}1 \\ \hline 1\;2\;\cdots\;0 \end{array} \quad \cdots \quad \begin{array}{r} s\text{-}1\;\;s\text{-}1\;\cdots\;s\text{-}1 \\ +\;0\;\;1\;\cdots\;s\text{-}1 \\ \hline s\text{-}1\;\;0\;\cdots\;s\text{-}2 \end{array}$$

Note that the + rule is nothing but 0, 1, $\cdots$, s - 1 and all of its cyclic permutations. This, then, is an array of s^2 assemblies (columns), with s symbols each replicated s times. The OA $(s^2, s + 1, s, 2)$ can be constructed following Rao (1946). The above can be used as the first three rows (constraints) of the $(s^2, s + 1, s, 2)$ OA and constraints y_4, y_5, $\cdots$, y_{s+1} can be added.

Alternatively, one could represent the addition mod s by the following table:

+	0	1	2		s-1
0	0	1	2	$\cdots$	s-1
1	1	2	3	$\cdots$	0
2	2	3	4	$\cdots$	1
$\vdots$	$\vdots$				$\vdots$
s-1	s-1	0	1	$\cdots$	s-2

The above is a cyclic Latin square of order s. A set of s - 1 orthogonal Latin squares exists for all prime numbers with the above cyclic Latin square being a member of the set. Using each of the s - 1 Latin squares from the set, we have s - 1 addition tables. Using all of the s - 1 addition tables instead of just the single one used so far, we can construct $(s - 1)[2(s^p - 1)/(s - 1) - 1]^2$ orthogonal $F(2s^p;2s^{p-1},2s^{p-1},\cdots,2s^{p-1})$-squares.

We now show how to construct the additional $F(2s^p;s^p,s^p)$-square using the additional constraint having zero s^p times and one s^p times, which can be added to the

$(2s^p, 2(s^p - 1)/(s - 1), s, 2)$ OA. The F-square that this constraint forms is:

+	0	0	⋯	0	1	1	⋯	1
0	0	0	⋯	0	1	1	⋯	1
0	0	0		0	1	1		1
⋮	⋮							⋮
0	0	0		0	1	1		1
1	1	1		1	0	0		0
1	1	1		1	0	0		0
⋮	⋮							⋮
1	1	1	⋯	1	0	0	⋯	0

using addition mod 2. This is an $F(2s^p; s^p, s^p)$-square, and it can be shown to be orthogonal to all of the above constructed F-squares. Thus, we have obtained the following set of F-squares:

$$OF(2s^p; 2s^{p-1}, 2s^{p-1}, \cdots, 2s^{p-1}; (s - 1)[2(s^p - 1)/(s - 1) - 1]^2)$$
$$+ \; OF(2s^p; s^p, s^p; 1),$$

and the theorem is proved.

Constraint (row of $(n,k,s,2)$)

1	2	y_1	y_2	y_3	...	y_k
0	0	y_1	y_2	y_3		
0	1	y_1	y_2	$y_3 + 1$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
0	s-1	y_1	y_2	$y_3 + s-1$		
1	0	$y_1 + 1$	$y_2 + 1$	y_3		
1	1	$y_1 + 1$	$y_2 + 1$	$y_3 + 1$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
1	s-1	$y_1 + 1$	$y_2 + 1$	$y_3 + s-1$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
s-1	0	$y_1 + s-1$	$y_2 + s-1$	y_3		
s-1	1	$y_1 + s-1$	$y_2 + s-1$	$y_3 + 1$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
s-1	s-1	$y_1 + s-1$	$y_2 + s-1$	$y_3 + s-1$		
0	0	y_1	y_2	y_3		
0	1	y_1	y_2	$y_3 + 1$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
0	s-1	y_1	y_2	$y_3 + s-1$		
1	0	$y_1 + 1$	$y_2 + 1$	y_3		
1	1	$y_1 + 1$	$y_2 + 1$	$y_3 + 1$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
1	s-1	$y_1 + 1$	$y_2 + 1$	$y_3 + s-1$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
s-1	0	$y_1 + s-1$	$y_2 + s-1$	y_3		
s-1	1	$y_1 + s-1$	$y_2 + s-1$	$y_3 + 1$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
s-1	s-1	$y_1 + s-1$	$y_2 + s-1$	$y_3 + s-1$		

Table 2.1: Constraints of an $(n = 2s^p,\ k = 2(s^p - 1)/(s - 1) - 1,\ s,\ 2)$ orthogonal array.

3. Example Demonstrating the Construction of 99 Orthogonal F-squares of Order n = 18

The number 18 is the smallest nontrivial case of (n,k,s,2) OA's since when p = 1 a single constraint represents this OA. We shall first demonstrate this point by constructing the OF(6;2,2,2;2) + OF(6;3,3;1) and the OA(6,1,3,2).

We know that there exists an (2(3), 2(3 - 1)/(3 - 1) - 1, 3, 2) OA, that is, an (6,1,3,2) orthogonal array, it being simply

$$0\ 1\ 2\ 0\ 1\ 2\ .$$

We may append the constraint 0 0 0 1 1 1, to obtain the generalized orthogonal array

$$A = \begin{matrix} 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{matrix}\ .$$

We can check to see that A indeed has the property of an orthogonal array, namely that the 2 x 6 matrix A contains all possible 2 x 1 column vectors with the same frequency. Now our first F-square of order 6 can be obtained by forming the addition table of constraint 0 0 0 1 1 1 with itself, mod 2:

+	0	0	0	1	1	1
0	0	0	0	1	1	1
0	0	0	0	1	1	1
0	0	0	0	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

The above is an F(6;3,3)-square which we shall call F_1. Our second F-square can be obtained by forming the addition table of constraint 0 1 2 0 1 2 with itself, mod 3. Mod 3 consists of:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

,

and so the addition of 0 1 2 0 1 2 with itself is:

+	0	1	2	0	1	2
0	0	1	2	0	1	2
1	1	2	0	1	2	0
2	2	0	1	2	0	1
0	0	1	2	0	1	2
1	1	2	0	1	2	0
2	2	0	1	2	0	1

.

This is an $F(6;2,2,2)$-square, which we will call F_2. One can check to see that F_2 is orthogonal to F_1. Now addition, mod 3, is defined by the above 3×3 table which is really a Latin square of order 3,

$$\begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{matrix} .$$

But, there exists a second 3×3 Latin square orthogonal to this, namely

$$\begin{matrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{matrix} .$$

We thus have the following "addition" table, orthogonal to the preceding one:

"+"	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

.

A third F-square can be obtained by forming the "addition" table of constraint 0 1 2 0 1 2 with itself, by using the above "addition" table,

"+"	0	1	2	0	1	2
0	0	1	2	0	1	2
1	2	0	1	2	0	1
2	1	2	0	1	2	0
0	0	1	2	0	1	2
1	2	0	1	2	0	1
2	1	2	0	1	2	0 .

This is another $F(6;2,2,2)$-square, which we will call F_3. One can check to see that F_3 is orthogonal to F_1 and F_2. Thus the set $\{F_1, F_2, F_3\}$ forms an orthogonal set of F-squares of order 6.

We now use Theorem 2.1 and its proof to construct 98 $F(18;6,6,6)$ and 1 $F(18;9,9)$ orthogonal F-squares of order $n = 18$. Since $n = 18 = 2(3^2)$ we have, by Theorem 2.1, that there exists

$$(3 - 1)[2(3^2 - 1)/(3 - 1) - 1]^2 F(2(3^2);2(3),2(3),2(3))$$

and 1 $F(2(3^2);3^2,3^2)$ orthogonal F-squares; i.e., 98 $F(18;6,6,6)$ + 1 $F(18;9,9)$ orthogonal F-squares of order 18. To construct them we use the procedure given in the proof of Theorem 2.1. We show the simplicity of the method of construction by making use of the analysis of variance in Table 3.1. For our construction we need the $(2(3^2), 2(3^2 - 1)/(3 - 1) - 1, 3, 2)$; i.e., the $(18,7,3,2)$ orthogonal array. This OA was first constructed by Bose and Bush (1952), and later by Addelman and Kempthorne (1961), and was reproduced in Raghavarao (1971). It is:

0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2
0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
0	1	2	1	2	0	2	0	1	2	0	1	0	1	2	1	2	0
0	2	1	1	0	2	2	1	0	1	0	2	2	1	0	0	2	1
0	1	2	1	2	0	1	2	0	0	1	2	2	0	1	2	0	1
0	1	2	2	0	1	0	1	2	1	2	0	2	0	1	1	2	0
0	1	2	0	1	2	2	0	1	1	2	0	1	2	0	2	0	1 .

Let constraints 1 through 7 be labeled R_1, R_2, $\cdots$, R_7 respectively and C_1, C_2, $\cdots$, C_7 respectively. We use these constraints in our analysis of variance table. The $(18)^2$ observations forming an 18 row by 18 column square may be partitioned in such a manner as to have orthogonal sums of squares with one and two degrees of freedom. The orthogonal sums of squares with two degrees of freedom are precisely those sums of squares attributed from constraints R_1, $\cdots$, R_7, C_1, $\cdots$, C_7, and their interactions from the orthogonal array (18,7,3,2). The orthogonal sums of squares with one degree of freedom are those sums of squares attributed from constraints $R_8 = C_8 = 000000000111111111$ and their interaction. We may now construct the analysis of variance in Table 3.1 for the $n^2 = (18)^2$ observations. Now by the proof of the theorem each of the R_iC_j interactions for $i,j = 1, 2, \cdots, 7$, constructs two (= s - 1) orthogonal F(18;6,6,6) squares and the R_8C_8 interaction constructs an F(18;9,9) square. This gives us a total of $[(7)^2(2) = 98)F(18;6,6,6)]$ + 1 F(18;9,9) orthogonal F-squares of order 18 as we expect. For illustration let us construct the two orthogonal F(18;6,6,6) squares corresponding to the R_1C_2 interaction. The first F-square is simply obtained by forming the addition table mod 3 of R_1 with C_2. We have as follows:

C_2

	+	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	0	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	0	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	0	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	1	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
	1	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
	1	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
	2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
	2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
	2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
R_1	0	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	0	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	0	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	1	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
	1	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
	1	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
	2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
	2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
	2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1

This is clearly an $F(18;6,6,6)$-square. To construct it we used the addition mod 3 table:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1 .

Since this is a 3×3 Latin square, we can find a second 3×3 Latin square orthogonal to it, namely

$$\begin{array}{ccc} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{array} .$$

Hence, we have the following "addition" table which is orthogonal to addition mod 3:

"+"	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0 .

Using this "addition" table on R_1 and C_2 instead of the ordinary addition table mod 3, we get our second F(18;6,6,6)-square:

C_2

	"+"	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	0	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	0	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	0	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
	2	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
	2	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
	2	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
R_1	0	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	0	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	0	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
	2	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
	2	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
	2	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0

One can check to see that this F(18;6,6,6)-square is indeed orthogonal to the previous F(18; 6,6,6)-square. We therefore have obtained the two orthogonal F(18; 6,6,6)-squares corresponding to the R_1C_2 interaction. Similarly, we can obtain two orthogonal F(18; 6,6,6)-squares from each of the R_iC_j interactions for i,j = 1, 2, $\cdots$, 7 and so we have 49(2) = 98 F(18; 6,6,6)-squares, since there are (7)(7) = 49 interactions. By the proof of Theorem 2.1 these 98 F-squares are mutually orthogonal, so we see how to construct 98 orthogonal F(18; 6,6,6)-squares. To obtain the F(18; 9,9)-square from the R_8C_8 interaction, we form the addition table mod 2 of R_8 with C_8 as follows:

+	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0

We have therefore shown how to construct the set of 98 F(18;6,6,6) + 1 F(18;9,9) orthogonal F-squares of order 18.

Note that the interactions of R_8 with C_1, C_2, $\cdots$, C_7 and of C_8 with R_1, R_2, $\cdots$, R_7 yields 28 degrees of freedom; the interaction sum of squares of the row and column remainders with each other and with R_i and C_j, $i,j = 1, 2, \cdots, 8$, is associated with $4 + (2)(15 + 15) = 64$ degrees. Thus, $28 + 64 = 92$ which is the remainder of the row by column interaction degrees of freedom not used to construct F-squares. It is not yet known how to construct F-squares from these remaining interaction contrasts.

Source of Variation	Degrees of Freedom		
Correction for the mean	1		
Rows	17		
R_1		2	
R_2		2	
$\vdots$		$\vdots$	
R_7		2	
R_8		1	
Remainder		2	
Columns	17		
C_1		2	
C_2		2	
$\vdots$		$\vdots$	
C_7		2	
C_8		1	
Remainder		2	
Row x Column interaction	$(17)^2 = 289$		
R_1C_1		4	197
R_1C_2		4	
$\vdots$		$\vdots$	
R_1C_7		4	
R_2C_1		4	
$\vdots$		$\vdots$	
R_7C_7		4	
R_8C_8		1	
Remainder		92	
Total	$(18)^2 = 324$		

Table 3.1: Analysis of variance for 99 orthogonal F-squares of order 18.

4. The Number of Orthogonal F-squares When $p \to \infty$ and $s \to \infty$

From Definition 1.1 and the discussion after it, we see that we have a complete set of orthogonal F-squares of order n if we have enough F-squares to account for the $(n - 1)^2$ degrees of freedom in the row $\times$ column interaction in the analysis of variance table. In section 3 we have $(17)^2 = 289$ degrees of freedom in the row $\times$ column interaction; hence to have a complete set of orthogonal F-squares of order 18 we would need a set of orthogonal F-squares that accounts for the entire 289 degrees of freedom. We were able to obtain 98 $F(18;6,6,6) + 1$ $F(18;9,9)$ orthogonal F-squares of order 18. This accounts for $98(3 - 1) + 1(2 - 1)$ $= 98(2) + 1(1) = 197$ degrees of freedom in the row $\times$ column interaction. We therefore have $197/289 \times 100\% = 68.2\%$ of a complete set of orthogonal F-squares of order 18.

From the table in the introduction we note that holding s constant and letting p become larger increases the proportion of the degrees of freedom associated with the set of orthogonal F-squares obtained by this method. Likewise, setting $p = 1$ and letting s increase, the proportion of the interaction degrees of freedom associated with F-squares appears to approach $1/4$ and for $p \geq 2$ and letting s increase, the proportion appears to approach unity. The following theorem is in this spirit:

Theorem 4.1.
$$\lim_{\substack{p \to \infty \\ s \text{ a constant}}} \frac{(s - 1)^2[2(s^p - 1)/(s - 1) - 1]^2 + 1}{(n - 1)^2 = (2s^p - 1)^2} = 1 ,$$

$$\lim_{\substack{p = 1 \\ s \to \infty}} \frac{(s - 1)^2[2(s^p - 1)/(s - 1) - 1]^2 + 1}{(2s^p - 1)^2} = \frac{1}{4} ,$$

and

$$\lim_{\substack{p \geq 2 \\ s \to \infty}} \frac{(s - 1)^2[2(s^p - 1)/(s - 1) - 1]^2 + 1}{(2s^p - 1)^2} = 1 .$$

The proof of the theorem is straightforward. Thus, we have shown how to construct an asymptotically complete set of orthogonal F-squares.

5. Alternative Construction Method for Orthogonal F-squares of Order $n = 2s$

In constructing orthogonal F-squares of order $n = 2s^p$, we have a simpler method than that used previously for the special case when $p = 1$, i.e. $n = 2s$, namely the Kronecker product method.

Construction Method: Let $n = 2s$, where s is a prime number. Let $L_1(s)$, $L_2(s)$, $\cdots$, $L_{(s-1)}(s)$ be a complete set of orthogonal Latin squares of order s. We may construct $(s - 1)$ orthogonal F-squares of order n as follows:

$$\sum_{i=1}^{s-1} L_i(s) \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = L_1(s) \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + L_2(s) \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$+ \cdots + L_{(s-1)}(s) \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

where $\otimes$ denotes Kronecker product. In addition, the F-square obtained by

$$J_{s\times s} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where $J_{s\times s}$ is the $s \times s$ matrix whose elements are all ones, is orthogonal to the above $(s - 1)$ F-squares.

Example 5.1: Let $n = 10 = 2(5)$. Thus $s = 5$. We may construct 5 orthogonal F-squares of order 10. Using the above we have:

$$\sum_{i=1}^{4} L_i(5) \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \\ 5 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} .$$

One can check to see that the above form 4 orthogonal $F(10;2,2,2,2,2)$-squares; the following F-square with two symbols, i.e., $F(10;5,5)$-square, is orthogonal to the above 4 F-squares,

$$F(10;5,5) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

The above procedure of constructing F-squares may be extended to the case where $n = qk$. If t orthogonal Latin squares of order k exist, then a set of t orthogonal F-squares of order n may be constructed as follows:

$$\sum_{i=1}^{t} L_i(k) \otimes J_{q \times q} \,.$$

Further, suppose that ℓ Latin squares of order q exist. Then, the following F-squares are orthogonal to the above F-squares and to each other:

$$\sum_{i=1}^{\ell} L_i(q) \otimes J_{k \times k} \,.$$

6. Discussion

The method of constructing orthogonal F-squares for order $n = 2s^p$ can be used in general for any order n provided an $(n,k,t,2)$ orthogonal array exists for some k and some t a power of a prime. These orthogonal arrays are known to exist for $n = s^p$ and $n = 2s^p$. The case $n = 2s^p$ has just been considered. For $n = s^p$, complete sets of orthogonal F-squares of order n have been constructed previously by Hedayat, Raghavarao, and Seiden (1975). The following then presents an alternative, perhaps easier, method of constructing these F-squares. We first give the theorem due to Bose and Bush (1952) on the existence of $(n = s^p,k,t,2)$ orthogonal arrays:

Theorem 6.1. Given $t = s^v$, $\lambda = s^u$ (where s is a prime), then we can construct an orthogonal array $(\lambda t^2,k,t,2)$ of strength 2, in which the number of constraints k is given by

$$k = \frac{\lambda(t^{c+1} - 1)}{t^c - t^{c-1}} + 1 \,,$$

where $c = [u/v]$.

We therefore have the following theorem:

<u>Theorem 6.2.</u> <u>We can use the</u> $(\lambda t^2, k, t, 2)$ <u>orthogonal array, with</u> λ, t, <u>and</u> k <u>defined above, to construct a set, and when</u> $[u/v] = u/v$ <u>a complete set, of</u> $(t - 1)k^2$ <u>orthogonal</u> $F(\lambda t^2; \lambda t, \lambda t, \cdots, \lambda t)$-<u>squares</u>.

The proof is similar to the proof of Theorem 2.1 and so is omitted here.

Since there is a one-to-one correspondence between orthogonal F-squares and orthogonal arrays, we may restate Theorem 2.1 in terms of orthogonal arrays. First note that a set of r $F(n; \gamma, \gamma, \cdots, \gamma)$-squares corresponds to an orthogonal array $(n^2, r, n/\gamma, 2)$. Hence the set of $(s - 1)[2(s^p - 1)/(s - 1) - 1]^2$ orthogonal $F(2s^p; 2s^{p-1}, 2s^{p-1}, \cdots, 2s^{p-1})$-squares in Theorem 2.1 corresponds to an orthogonal array

$$(4s^{2p}, (s - 1)[2(s^p - 1)/(s - 1) - 1]^2, s, 2).$$

The $F(2s^p; s^p, s^p)$-square also in Theorem 2.1 can be written as the orthogonal array $(4s^{2p}, 1, 2, 2)$. Since the $F(2s^p; s^p, s^p)$-square is orthogonal to the $F(2s^p; 2s^{p-1}, 2s^{p-1}, \cdots, 2s^{p-1})$-squares we have that the orthogonal array $(4s^{2p}, 1, 2, 2)$ is orthogonal to the orthogonal array $(4s^{2p}, (s - 1)[2(s^p - 1)/(s - 1) - 1]^2, s, 2)$ and so we may put both arrays together and denote it as a single orthogonal array $(4s^{2p}, (s - 1)[2(s^p - 1)/(s - 1) - 1]^2, s, 2) + (4s^{2p}, 1, 2, 2)$.

Finally, consider the following orthogonal array $(4s^{2p}, 2, 2s^p, 2)$:

$$\begin{array}{cccccccccccccc} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots\cdots & 2s^p & 2s^p & \cdots & 2s^p \\ 0 & 1 & \cdots & 2s^p & 0 & 1 & \cdots & 2s^p & \cdots\cdots & 0 & 1 & \cdots & 2s^p \end{array}.$$

This orthogonal array corresponds to the sources of variation due to rows and columns in the analysis of variance; when written in square form the two constraints look like:

$$\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \\ 2s^p & 2s^p & \cdots & 2s^p \end{array} \quad \text{and} \quad \begin{array}{cccc} 0 & 1 & \cdots & 2s^p \\ 0 & 1 & \cdots & 2s^p \\ \vdots & & & \\ 0 & 1 & \cdots & 2s^p \end{array} .$$

One can check to see that the above orthogonal array $(4s^{2p},2,2s^p,2)$ is orthogonal to the afore-mentioned orthogonal array. We may thus write the orthogonal arrays as a single orthogonal array denoted as $(4s^{2p},2,2s^p,2) + (4s^{2p},(s-1)[2(s^p-1)/(s-1)-1]^2, s, 2) + (4s^{2p},1,2,2)$.

We thus see that our method of constructing F-squares of order $n = 2s^p$, is also a method for constructing orthogonal arrays of size $n = (2s^p)^2 = 4s^{2p}$ with 2, s, and $2s^p$ elements.

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