A NOTE ON THE INTEGRAL FOR BIRTH-DEATH MARKOV PROCESSES

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Abstract

Let $X(t)$ be a birth-death Markov process. Here it is shown how the expectation of the time to absorption and of the integral under $X(t)$ up to absorption time can be found by substituting transitions to state 0 by transitions to the initial state of the process, provided the stationary distribution of the modified process exists. Examples of applications to some special cases of birth-death Markov processes are given.

STOCHASTIC INTEGRALS; BIRTH AND DEATH PROCESS; MARKOV PROCESSES

1. Introduction

Integrals of nonnegative stochastic processes arise naturally in engineering, biology and inventories. Functionals of this integral correspond to first emptiness problems in queuing, storage and traffic problems and to inventory systems with holding cost associated with the stock over a particular period of time (see for instance [3], [5] and [12]). In biology it has been associated with total food consumption and production of toxins of a bacteria ([15]) and total cost of epidemics ([2], [4], [6], [7]). Limiting

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properties for the integral have been studied in [1], [8], [9], [16]- [18], whereas integral functionals were studied in [11] and [13]. Here we deal with a different approach to evaluate the expectation of the integral for a birth-death Markov process as well as of the expected time to absorption.

2. Methodology

Let $X(t)$ be a birth-death process on a subset of $\mathcal{N}=\{0,1,2,...\}$, with birth and death rates λ_i and μ_i respectively for state i. Define

$$Z_k = \inf\{t\colon X(t) = 0 | X(0) = k\}$$

and

$$Y_k = \int_0^{Z_k} X(t)\, dt$$

thus Z_k is the time to absorption given $X(0) = k$ and Y_k is the area under $X(t)$ up to the time when the process vanishes.

If we substitute transitions to state 0 by transitions to the initial state k, providing that the resulting process is ergodic a stationary distribution exists. Call this distribution the Modified Stationary Distribution (MSD) and denote it by $\underline{\Pi}'=\{\pi_1,\pi_2,\pi_3...\}$. Let S_r be the random vector corresponding to the total amount of time spent in state r before absorption, $r = 1, 2, 3, ...$ Observe that

(1) $$Y_k = \int_0^{Z_k} X(t)\, dt = \sum_i i\ S_i$$

a result pointed out by Puri [14].

Let r be an arbitrary but fixed state, $r \neq 0$. If the modified process is ergodic then when $t \rightarrow \infty$ state r will be visited infinitely often. Assume a cycle has been completed every time a death occurs with the process being in state 1. Define also $E\{S_r\}$ as the the expected time spent in state r in a cycle. It is clear that $E\{S_r\}$ equals also the expected time spent in state r before the process goes to absorption in the original process.

Define now S_{rj} be the time spent in state r in the $j-th$ cycle in the modified process, $j = 1, 2, ..., n$. Thus we have:

$$E\{S_r\} = E\left\{n^{-1}\sum_{j=1}^{n} S_{rj}\right\} = \lim_{n\rightarrow\infty} \frac{E\left\{\sum\limits_{j=1}^{n} S_{rj}\right\}}{n}$$

Note that in the modified Markov Process

$$\lim_{n\rightarrow\infty} \frac{\sum\limits_{j=1}^{n} S_{rj}}{\sum\limits_{j=1}^{n}\sum\limits_{i} S_{ij}} = \pi_r$$

with π_r being the corresponding element of the modified stationary distribution $\underline{\Pi}$. The equality follows from the fact that the numerator is the total time spent in state r trough n cycles and the denominator is the equivalent through all states. Observe that

$$\pi_r = \lim_{n\rightarrow\infty} \frac{\sum\limits_{j=1}^{n} S_{rj}}{\sum\limits_{j=1}^{n}\sum\limits_{i} S_{ij}} = \frac{\lim\limits_{n\rightarrow\infty} n^{-1}\sum\limits_{j=1}^{n} S_{rj}}{\lim\limits_{k\rightarrow\infty} n^{-1}\sum\limits_{j=1}^{n}\sum\limits_{i} S_{ij}} = \frac{E\{S_r\}}{E\{\sum\limits_{i} S_i\}}$$

Note $E\{\sum\limits_{i} S_i\}$ is $E\{Z_k\}$, thus it follows that:

$$E\{S_r\} = \pi_r \, E\{Z_k\}$$

since $E\{S_1\} = 1/\mu$ we have:

(2) $$E\{Z_k\} = (\pi_1 \mu_1)^{-1}$$

and then from (1) and (2) it follows that:

(3) $$E\{Y_k\} = E\left\{\int_0^{Z_k} X(t)\, dt\right\} = \sum_i i \, E\{S_i\} = (\pi_1 \, \mu_1)^{-1} \sum_i i \, \pi_i$$

3. Examples

The following are applications of (3) to some birth-death Markov processes. In all examples the initial state is assumed to be $k = 1$, and thus the stationary distribution corresponds to the "reflecting state 0 approximation to the quasi-stationary distribution". (see [10] for details), which satisfies the following system of linear equations:

$$0 = \pi_2 \mu_2 - \pi_1 \lambda_1$$

$$0 = \pi_{n+1} \, \mu_{n+1} + \pi_{n-1} \, \lambda_{n-1} - \pi_n(\lambda_n + \mu_n), \quad n = 2,3,...$$

these can be solved recursively to give the well known solution:

$$\pi_n = \frac{\lambda_1 \, \lambda_2 \, \lambda_3 ... \lambda_{n-1}}{\mu_2 \, \mu_3 \, \mu_4 ... \mu_n} \pi_1$$

In all following cases assume $\lambda < \mu$ so that the stationary distribution exists. Observe that upon defining

(4) $$H(n) = \frac{\lambda_1 \, \lambda_2 \, \lambda_3 ... \lambda_{n-1}}{\mu_2 \, \mu_3 \, \mu_4 ... \mu_n}$$

then, $E\{Y_1\}$ in birth-death Markov processes with initial state 1 and 0 as an absorbing state can be simplified to:

(5) $$E\{Y_1\} = \frac{1}{\pi_1\mu_1}\sum_{i=1}^{\infty} i\,H(i)\pi_1 \;=\; \mu_1^{-1}\sum_{i=1}^{\infty} i\,H(i)$$

The last equality will be used in the following examples:

(a) $\lambda_n = \lambda,\ \mu_n = \mu\ , k = 1.$

This is the $M/M/1$ queue. Let $\rho = \lambda/\mu$. Y_1 of equation (1) corresponds to the total amount of time waited by all customers during the length of a busy period. From (4) we have $H(n) = \rho^{n-1}$. The expected time to absorption is $E\{Z_1\} = \lambda(1-\lambda/\mu)$, and applying (5) we have:

$$E\{Y_1\} = \mu^{-1}\sum_{i=1}^{\infty} i\,\rho^{i-1}$$

$$= \frac{1}{\mu\,(1-\lambda/\mu)^2}$$

(b) $\lambda_n = \lambda,\ \mu_n = \mu\,n, k = 1.$

This is the $M/M/\infty$ queue or immigration-death process. Y_1 is again the total amount of time waited by all customers during the length of a busy period. In this case we have $H(n) = \rho^{n-1}/n!$. $E\{Z_1\}$ can be shown to be $(e^{\rho}-1)/\rho\,\mu$, and

$$E\{Y_1\} = \mu^{-1}\sum_{i=1}^{\infty} i\,\rho^{i-1}/i! \;=\; \mu^{-1}\,e^{\lambda/\mu}$$

(c) $\lambda_n = \lambda n,\ \mu_n = \mu\,n, k = 1.$

This is the linear birth-death process (Yule process). We have $H(n) = \rho^{n-1}/n$. Here $E\{Z_1\} = -\,log(1-\lambda/\mu)\,\lambda^{-1}$ and

$$E\{Y_1\} = \mu^{-1} \sum_{i=1}^{\infty} i\, \rho^{\,i-1} / i$$

$$= \frac{1}{(\mu - \lambda)}$$

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