SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NY 14853-3801

TECHNICAL REPORT NO. 1047

January 1993

LEVY MEASURES OF INFINITELY DIVISIBLE RANDOM VECTORS AND SLEPIAN INEQUALITIES[1]

by

Gennady Samorodnitsky
and Murad S. Taqqu[2]

[1]This research was supported by the ONR grant N00014-90-J-1287 and United States–Israel Binational Science Foundation.

[2]Department of Mathematics, Boston University, Boston, MA 02215.

Lévy measures of infinitely divisible random vectors and Slepian inequalities *†‡

Gennady Samorodnitsky
Cornell University

Murad S. Taqqu
Boston University

January 28, 1993

Abstract

We study Slepian inequalities for general non-Gaussian infinitely divisible random vectors. Conditions for such inequalities are expressed in terms of the corresponding Lévy measures of these vectors. These conditions are shown to be nearly best possible, and for a large subfamily of infinitely divisible random vectors these conditions are necessary and sufficient for Slepian inequalities. As an application we consider SαS Ornstein-Uhlenbeck processes and a family of infinitely divisible random vectors introduced by Brown and Rinott.

1 Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors in R^d. If for any $\boldsymbol{\lambda} \in R^d$,

$$P(\mathbf{X} > \boldsymbol{\lambda}) \geq P(\mathbf{Y} > \boldsymbol{\lambda}), \tag{1.1}$$

then the random vectors $\mathbf{X}$ and $\mathbf{Y}$ are said to satisfy the *right* Slepian inequality. If for any $\boldsymbol{\lambda} \in R^d$,

$$P(\mathbf{X} < \boldsymbol{\lambda}) \geq P(\mathbf{Y} < \boldsymbol{\lambda}), \tag{1.2}$$

then the random vectors $\mathbf{X}$ and $\mathbf{Y}$ are said to satisfy the *left* Slepian inequality. (Throughout this paper the notation $\mathbf{x} > \mathbf{y}$ for $\mathbf{x} = (x_1, \ldots, x_d)$, $\mathbf{y} = (y_1, \ldots, y_d)$ means that $x_i > y_i$ for every $i = 1, \ldots, d$, while the notation $\mathbf{x} \geq \mathbf{y}$ means that $x_i \geq y_i$ for every $i = 1, \ldots, d$, and similarily with reverse inequalities.) If $\mathbf{X}$ and $\mathbf{Y}$ satisfy both (1.1) and (1.2) then we say that these two vectors satisfy the two-sided Slepian inequality.

Since probability measures are continuous from above and from below, it is clear that $\mathbf{X}$ and $\mathbf{Y}$ satisfy the right Slepian inequality if and only if for any $\boldsymbol{\lambda} \in R^d$,

$$P(\mathbf{X} \geq \boldsymbol{\lambda}) \geq P(\mathbf{Y} \geq \boldsymbol{\lambda}), \tag{1.3}$$

*This research was supported by the ONR grant N00014-90-J-1287 and United States - Israel Binational Science Foundation

†*AMS 1980 subject classification.* Primary 60E07, 60E15.

‡*Keywords and phrases:* Slepian inequality, infinitely divisible random vectors, stable random vectors.

and they satisfy the left Slepian inequality if and only if for any $\boldsymbol{\lambda} \in R^d$,

$$P(\mathbf{X} \le \boldsymbol{\lambda}) \ge P(\mathbf{Y} \le \boldsymbol{\lambda}). \tag{1.4}$$

As a result, we may switch from one version of the Slepian inequalities to the other depending on the situation.

Our terminology is (only) slightly unorthodox. Firstly, the original Slepian inequality [15] has been proved for centered Gaussian vectors. Secondly, (1.1) and (1.2) are equivalent if the random vectors $\mathbf{X}$ and $\mathbf{Y}$ are symmetric. But since (1.1) and (1.2) are in general not always equivalent, it is necessary to consider both inequalities. The right (left) Slepian inequality has the following interpretation: the probability that the components are $\mathbf{X}$ are all very large (very small) is greater than the corresponding probability for $\mathbf{Y}$. Hence the components of $\mathbf{X}$ are "more positively dependent" than those of $\mathbf{Y}$.

The original Slepian result for centered Gaussian vectors can be formulated as follows. If $\mathbf{X}$ and $\mathbf{Y}$ are two zero mean normal random vectors then (1.1) and (1.2) hold if and only if

$$EX_i^2 = EY_i^2 \text{ for each } i = 1, \dots, d, \tag{1.5}$$

$$E(X_iX_j) \ge E(Y_iY_j) \text{ for every } i, j = 1, \dots, d.$$

This result has been used extensively for studying Gaussian processes, especially their sample paths (see, e.g., Fernique [7], Adler [1] and Ledoux and Talagrand [10]). Its importance has generated a lot of interest in extensions to as wide a class of (non-Gaussian) stochastic processes as possible. The task, however, has turned out not to be easy. The main difficulty seems to be that natural extensions of the conditions (1.5) do not in general suffice for Slepian inequalities. This is not surprising if one remembers that "a few numbers" like those appearing in (1.5) do characterize a Gaussian law, but this is not usually the case for many other laws of interest.

A weaker version of Slepain ineqaulity, the *Sudakov minorization* for Gaussian random vectors (Fernique [7]) was extended to symmetric α-stable random vectors with $\alpha > 1$ by Marcus and Pisier [11] and to general *type G* infinitely divisible random vectors by Samorodnitsky and Taqqu [13]. However, these extensions are not entirely satisfactory, in the sense that the result of Marcus and Pisier involves a dimension-dependent numerical constant that blows up as the dimension increases, and the conditions under which the result of Samorodnitsky and Taqqu holds are often difficult to verify.

The only instance known to the authors of a succesful nontrivial extension of the full Slepian inequality to non-Gaussian situations is due to Brown and Rinott [3]. It deals with an especially simple family of infinitely divisible random vectors. Our wish to understand the basic features that make the Slepian inequality work in this case has led us to the present research. We discuss the Brown and Rinott family later in the paper.

Our goal is to extend the Slepian inequality, that is the conclusions (1.2) and (1.1) to the class of all infinitely divisible random vectors. Our conditions involve comparison, not of covariances as in (1.5), but of Lévy measures. This is, of course, quite natural for these type of random vectors.

Our overall approach is akin to that used in Pitt [12] and Joag-dev et al. [9] under different circumstances. Professor Pitt kindly pointed to us the potential usefulness of his approach to our problem.

The paper is organized as follows. In Section 2 we introduce some notation and terminology. In Section 3 the main results are stated and proved, and Sections 4 and 5 contain examples: in Section 4 we specialize our results to the symmetric α-stable case and treat, in particular, symmetric α-stable Ornstein-Uhlenbeck processes, while in Section 5 we discuss the Brown and Rinott [3] family of infinitely divisible random vectors.

2 Infinitely divisible random vectors and semi-groups

A d-dimensional infinitely divisible random vector $\mathbf{X} = (X_1, \ldots, X_d)$ without a Gaussian component is usually characterized by its *Lévy measure* and *shift vector* through Lévy-Khinchine representation of its characteristic function $\phi_{\mathbf{X}}(\boldsymbol{\theta}) = E \exp i(\mathbf{X}, \boldsymbol{\theta})$:

$$\phi_{\mathbf{X}}(\boldsymbol{\theta}) = \exp\{\int_{R^d - \{\mathbf{0}\}} (e^{i(\boldsymbol{\theta}, \mathbf{x})} - 1 - i\mathbf{1}(||\mathbf{x}|| \leq 1)(\boldsymbol{\theta}, \mathbf{x}))\nu(d\mathbf{x}) + i(\boldsymbol{\theta}, \mathbf{b})\}. \tag{2.1}$$

The Lévy measure of $\mathbf{X}$, ν, is a σ-finite measure on the Borel subsets of $R^d - \{\mathbf{0}\}$ such that $\int_{R^d - \{\mathbf{0}\}} (1 \wedge ||\mathbf{x}||^2)\nu(d\mathbf{x}) < \infty$, and the shift vector of $\mathbf{X}$, $\mathbf{b}$, is a constant vector.

We will also consider the important subclass of infinitely divisible random vectors whose Lévy measure satisfies the additional relation

$$\int_{R^d - \{\mathbf{0}\}} (1 \wedge ||\mathbf{x}||)\nu(d\mathbf{x}) < \infty. \tag{2.2}$$

For such random vectors one can integrate out the third term of the integrand in the right hand side of (2.1) and absorb the result in the shift term, which results in

$$\phi_{\mathbf{X}}(\boldsymbol{\theta}) = \exp\{\int_{R^d - \{\mathbf{0}\}} (e^{i(\boldsymbol{\theta}, \mathbf{x})} - 1)\nu(d\mathbf{x}) + i(\boldsymbol{\theta}, \mathbf{c})\}. \tag{2.3}$$

With some abuse of terminology we will also call the vector $\mathbf{c}$ in (2.3) (different, in general, from the vector $\mathbf{b}$ in (2.1)) a shift vector. Incidentally, this name is more appropriate for the vector $\mathbf{c}$ because the vector $\mathbf{b}$ in (2.1) depends on the somewhat arbitrary compensator $\mathbf{1}(||\mathbf{x}|| \leq 1)(\boldsymbol{\theta}, \mathbf{x})$.

Infinitely divisible random vectors satisfying (2.2) have the nice property that if K is a closed convex cone in R^d such that the Lévy measure ν of $\mathbf{X}$ is supported by K and the shift vector $\mathbf{c} \in K$ then $P(\mathbf{X} \in K) = 1$. Examples include compound Poisson random vectors and α-stable random vectors with $0 < \alpha < 1$.

For a given infinitely divisible random vector $\mathbf{X}$ with characteristic function $\phi_{\mathbf{X}}(\boldsymbol{\theta})$ given by (2.1) and $t \geq 0$, let P^t denote the distribution of an infinitely divisible random vector with characterisitic function $(\phi_{\mathbf{X}}(\boldsymbol{\theta}))^t$. Then $P^t * P^s = P^{s+t}$ for any $t, s \geq 0$, where $*$ denotes convolution of probability measures. Moreover, if $\mathbf{X}^{*t} \sim P^t$, then $\{\mathbf{X}^{*t},\ t \geq 0\}$ is a process with stationary independent increments satisfying $\mathbf{X}^{*0} = 0$ and $\mathbf{X}^{*t} \stackrel{d}{=} \mathbf{X}$. We refer to $\{P^t,\ t \geq 0\}$ as the convolution semigroup generated by the infinitely divisible random vector $\mathbf{X}$, and these semigroups play an important role in our arguments. Recall that the generator G of the convolution semigroup $\{P^t,\ t \geq 0\}$ generated by $\mathbf{X}$ can be written in the form

$$Gg(\mathbf{y}) = \int_{R^d - \{\mathbf{0}\}} (g(\mathbf{x} + \mathbf{y}) - g(\mathbf{y}) - 1(||\mathbf{x}|| \leq 1)(\mathbf{x}, \triangle g(\mathbf{y}))\nu(d\mathbf{x}) + (\mathbf{b}, \triangle g(\mathbf{y})), \tag{2.4}$$

for $\mathbf{y} \in R^d$, where $g : R^d \to R$ is in the domain $\mathcal{D}$ of the generator G. Recall further that any $g \in C_b^\infty$ (the space of all infinitely differentiable functions $R^d \to R$ with bounded derivatives) is in the domain $\mathcal{D}$ (see, e,g, Fristedt [8]).

If the Lévy measure of an infinitely divisible random vector $\mathbf{X}$ satisfies (2.2) and its characteristic function is given by (2.3) then the generator of the convolution semigroup generated by $\mathbf{X}$ can be written in the form

$$Gg(\mathbf{y}) = \int_{R^d-\{\mathbf{0}\}} (g(\mathbf{x}+\mathbf{y}) - g(\mathbf{y}))\nu(d\mathbf{x}) + (\mathbf{c}, \triangle g(\mathbf{y})), \tag{2.5}$$

for $\mathbf{y} \in R^d$, $g \in \mathcal{D}$.

Since the Lévy measure ν of an infinitely divisible random vector is finite outside of a neighbourhood of the origin, one can transform any infinitely divisible random vector into a one satisfying (2.2) by restricting its Lévy measure to the complement of such a neighbourhood - a procedure used repeatedly in this paper. Specifically, given a vector $\mathbf{X}$ with characteristic function (2.1) and a Borel set $A \subset R^d$ such that $B_\delta := \{\mathbf{x} \in R^d : ||\mathbf{x}|| < \delta\} \subset A$ for some $\delta > 0$, we define an infinitely divisible random vector $\mathbf{X}^A$ as having Lévy measure $\nu^A = \mathbf{I}_{A^c}\nu$ and shift vector $\mathbf{b}^A = \mathbf{b}$. Then its characteristic function can be written in the form (2.3) with the shift vector $\mathbf{c}^A$ given by

$$c_i^A = b_i - \int_{A^c} \mathbf{1}(||\mathbf{x}|| \leq 1) x_i \nu(d\mathbf{x}), \ i = 1, \ldots, d. \tag{2.6}$$

3 Main results

In order to maintain the view of the forest beyond the trees, we describe the basic ideas behind our results. Looking back at the Gaussian conditions (1.5) one observes that they imply that the components of the random vector $\mathbf{X}$ "cluster together" more than the components of the random vector $\mathbf{Y}$ do. The Slepian inequalities (1.1) and (1.2) may then be regarded as an expression of that "clustering". In the non-Gaussian infinitely divisible case, criteria of "clustering" are naturally related to Lévy measures. Specifically, let $\mathbf{X}$ and $\mathbf{Y}$ be two infinitely divisible random vectors with Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ and shift vectors $\mathbf{b}_{\mathbf{X}}$ and $\mathbf{b}_{\mathbf{Y}}$ accordingly. If one is interested, say, in the *right* Slepian inequality (1.1), it is intuitive then that the appropriate "clustering" requirement on the Lévy measures should be

$$\nu_{\mathbf{X}}\{\mathbf{x} \in R^d : \mathbf{x} > \boldsymbol{\lambda}\} \geq \nu_{\mathbf{Y}}\{\mathbf{x} \in R^d : \mathbf{x} > \boldsymbol{\lambda}\} \tag{3.1}$$

for every $\boldsymbol{\lambda} \in R^d$, or a version of this condition with non-strict inequalities. The condition (3.1), however, can be awkward when $\boldsymbol{\lambda} \in R^d_- := \{\mathbf{x} \in R^d - \{\mathbf{0}\} : \mathbf{x} \leq \mathbf{0}\}$ because the origin, which plays a special role for Lévy measures, belongs to the set $\{\mathbf{x} \in R^d\text{: } \mathbf{x} > \boldsymbol{\lambda}\}$. For example, ν_X and ν_Y may have an infinite mass in a neighbourhood of the origin. We will therefore suppose

$$\nu_X\{\mathbf{x} \in R^d\text{: } \mathbf{x} > \boldsymbol{\lambda}\} \geq \nu_Y\{x \in R^d\text{: } \mathbf{x} > \boldsymbol{\lambda}\} \text{ for every } \boldsymbol{\lambda} \in R^d - R^d_-, \tag{3.2}$$

to which we add the "complementary" condition

$$\nu_{\mathbf{X}}\{\mathbf{x} \in R^d : \mathbf{x} \not\geq \boldsymbol{\lambda}\} \leq \nu_{\mathbf{Y}}\{\mathbf{x} \in R^d : \mathbf{x} \not\geq \boldsymbol{\lambda}\} \text{ for every } \boldsymbol{\lambda} \in R^d_-, \tag{3.3}$$

where $\mathbf{x} \not\geq \mathbf{y}$ for $\mathbf{x}$, $\mathbf{y}$ in R^d means that $x_i < y_i$ for at least one $i = 1, \ldots, d$, and similarily with $\mathbf{x} \not\leq \mathbf{y}$.

One also has to take into account the effect of the shift vectors. The vectors $\mathbf{b}_{\mathbf{X}}$ and $\mathbf{b}_{\mathbf{Y}}$ are ill suited for this purpose because, as we have mentioned above, they depend on a somewhat arbitrary centering. One should compare instead, whenever possible, the shift vectors in the representation (2.3). We will see in the sequel how to do this in general.

In fact, further reflection on the conditions (3.2) - (3.3) tells us that these conditions should properly be regarded as corresponding not to (1.1) *alone* but rather to the *right* Slepian inequality for the whole families of infinitely divisible random vectors arising from the corresponding convolution semigroups. Specifically, let $P^t_{\mathbf{X}}$ and $P^t_{\mathbf{Y}}$ be the convolution semigroups generated by the infinitely divisible random vectors $\mathbf{X}$ and $\mathbf{Y}$ respectively, and let $\mathbf{X}^{*t} \sim P^t_{\mathbf{X}}$ and $\mathbf{Y}^{*t} \sim P^t_{\mathbf{Y}}$ for $t > 0$ ($\mathbf{X} \stackrel{d}{=} \mathbf{X}^{*1}$). Then (3.2) - (3.3) should be regarded as corresponding to the family of *right* Slepian inequalities

$$P(\mathbf{X}^{*t} > \boldsymbol{\lambda}) \geq P(\mathbf{Y}^{*t} > \boldsymbol{\lambda}) \text{ for every } \boldsymbol{\lambda} \in R^d \tag{3.4}$$

for *all* $t > 0$. The distinction between (1.1) and (3.4) is a critical one, and we will have more to say about this point in the sequel.

For technical reasons, conditions (3.2) and (3.3) will have to be somewhat modified (strengthened, in fact), except when the infinitely divisible random vectors satisfy the condition (2.2). We do not know, at this point, whether this is intrinsic to the problem or stems from our approach only.

DEFINITION 3.1 A sequence $\{A_n,\ n \geq 1\}$ of Borel sets in R^d is said to be *deflating to the origin* if

(i) $A_1 \supset A_2 \supset \ldots$,

(ii) $\cap_{n\geq 1} A_n = \{\mathbf{0}\}$,

(iii) For every $n \geq 1$ there is a $\delta > 0$ such that $B_\delta = \{\mathbf{x} \in R^d : ||\mathbf{x}|| < \delta\} \subset A_n$.

Remark. A natural way to produce sequences of sets deflating to the origin is to choose sequences of balls of positive radius decreasing to zero in some norm, not necessarily the Euclidian norm $||\cdot||$. All the applications of our results considered in this paper use only sets of this kind. It is conceivable, however, that the greater generality may turn out to be useful in future applications.

Given an infinitely divisible random vector $\mathbf{X}$ with Lévy measure ν and shift vector $\mathbf{b}$ and a sequence of sets $\{A_n,\ n \geq 1\}$ deflating to the origin, we define a sequence of infinitely divisible random vectors $\{\mathbf{X}^{A_n},\ n \geq 1\}$ as described above, by restricting the Lévy measure to the complement of the corresponding set A_n. We record at this point the obvious observation that $\mathbf{X}^{A_n} \Rightarrow \mathbf{X}$ as $n \to \infty$. The strengthening of the assumptions (3.2) and (3.3) mentioned above amounts to assuming the following: that these assumptions hold for $\mathbf{X}$ and $\mathbf{Y}$ restricted as above to the complements of two (not necessarily identical) sequences of sets deflating to the origin.

THEOREM 3.1 *Let $\mathbf{X}$ and $\mathbf{Y}$ be two infinitely divisible random vectors in R^d with characteristic functions (2.1) with Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ and shift vectors $\mathbf{b}_{\mathbf{X}}$ and $\mathbf{b}_{\mathbf{Y}}$ accordingly.*

(i) Suppose there are two sequences of sets $\{A_n,\ n \geq 1\}$ and $\{B_n,\ n \geq 1\}$ deflating to the origin such that for every $n \geq 1$ the following three conditions hold:

$$\nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c : \mathbf{x} > \boldsymbol{\lambda}\} \geq \nu_{\mathbf{Y}}\{\mathbf{x} \in B_n^c : \mathbf{x} > \boldsymbol{\lambda}\} \text{ for every } \boldsymbol{\lambda} \in R^d - R^d_-; \tag{3.5}$$

$$\nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c : \mathbf{x} \not\geq \boldsymbol{\lambda}\} \leq \nu_{\mathbf{Y}}\{\mathbf{x} \in B_n^c : \mathbf{x} \not\geq \boldsymbol{\lambda}\} \text{ for every } \boldsymbol{\lambda} \in R_-^d; \tag{3.6}$$

and

$$\mathbf{c}_{\mathbf{X}}^{A_n} \geq \mathbf{c}_{\mathbf{Y}}^{B_n} \tag{3.7}$$

(cf. (2.6).*) Then for every* $\boldsymbol{\lambda} \in R^d$*, the right Slepian inequalities* (3.4) *hold for all* $t > 0$.

(ii) Suppose there are two sequences of sets $\{A_n,\ n \geq 1\}$ *and* $\{B_n,\ n \geq 1\}$ *deflating to the origin such that for every* $n \geq 1$ *the following three conditions hold:*

$$\nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c : \mathbf{x} < \boldsymbol{\lambda}\} \geq \nu_{\mathbf{Y}}\{\mathbf{x} \in B_n^c : \mathbf{x} < \boldsymbol{\lambda}\} \text{ for every } \boldsymbol{\lambda} \in R^d - R_+^d, \tag{3.8}$$

(where $R_+^d := \{\mathbf{x} \in R^d - \{\mathbf{0}\} : \mathbf{x} \geq \mathbf{0}\}$*);*

$$\nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c : \mathbf{x} \not\leq \boldsymbol{\lambda}\} \leq \nu_{\mathbf{Y}}\{\mathbf{x} \in B_n^c : \mathbf{x} \not\leq \boldsymbol{\lambda}\} \text{ for every } \boldsymbol{\lambda} \in R_+^d; \tag{3.9}$$

and

$$\mathbf{c}_{\mathbf{X}}^{A_n} \leq \mathbf{c}_{\mathbf{Y}}^{B_n}. \tag{3.10}$$

Then for every $\boldsymbol{\lambda} \in R^d$*, the left Slepian inequalities*

$$P(\mathbf{X}^{*t} < \boldsymbol{\lambda}) \geq P(\mathbf{Y}^{*t} < \boldsymbol{\lambda}) \text{ for every } \boldsymbol{\lambda} \in R^d \tag{3.11}$$

hold for all $t > 0$.

Theorem 3.1 is a simple consequence of the next theorem below. But first some remarks:

Remarks

1. Assuming (3.5) - (3.6) *for every* $n \geq 1$ is, clearly, a stronger assumption than just (3.2) - (3.3) in the sense that the former imply the latter.

2. For every $n \geq 1$ the pair of assumptions (3.5) - (3.6) is equivalent to the following assumption, which is in certain circumstances more tractable than the former: for every random vector $\mathbf{W} \in R^d$ whose coordinates $W_1, \ldots, W_d$ are atomless,

$$\int_{A_n^c} \Big(P(\mathbf{W} < \mathbf{x}) - P(\mathbf{W} < \mathbf{0})\Big)\nu_{\mathbf{X}}(d\mathbf{x}) \geq \int_{B_n^c} \Big(P(\mathbf{W} < \mathbf{x}) - P(\mathbf{W} < \mathbf{0})\Big)\nu_{\mathbf{Y}}(d\mathbf{x}). \tag{3.12}$$

To see the equivalence, observe that for a fixed $n \geq 1$

$$\int_{\mathbf{x} \in A_n^c} \Big(P(\mathbf{W} < \mathbf{x}) - P(\mathbf{W} < \mathbf{0})\Big)\nu_{\mathbf{X}}(d\mathbf{x}) \tag{3.13}$$

$$= \int_{\mathbf{x} \in A_n^c} \Big[\int_{R^d} \Big(\mathbf{1}(\mathbf{w} < \mathbf{x})\mathbf{1}(\mathbf{w} \not< \mathbf{0}) - \mathbf{1}(\mathbf{w} \not\leq \mathbf{x})\mathbf{1}(\mathbf{w} < \mathbf{0})\Big) F_{\mathbf{W}}(d\mathbf{w})\Big]\nu_{\mathbf{X}}(d\mathbf{x})$$

$$= \int_{R^d - R_-^d} \nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c : \mathbf{x} > \mathbf{w}\} F_{\mathbf{W}}(d\mathbf{w}) - \int_{R_-^d} \nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c : \mathbf{x} \not\geq \mathbf{w}\} F_{\mathbf{W}}(d\mathbf{w}),$$

where $F_{\mathbf{W}}$ is the probability law of $\mathbf{W}$. The implication (3.5) - (3.6) $\Rightarrow$ (3.12) is now obvious. Let us check the converse implication. To establish (3.6) (say), pick first a $\boldsymbol{\lambda}$ in R_-^d such that both

$\boldsymbol{\lambda} < \mathbf{0}$ and $\boldsymbol{\lambda}$ is continuity point of both $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$. That is, $\nu_{\mathbf{X}}\{\mathbf{x} : x_i = \lambda_i \text{ for some } i\} = 0$, and similarily for $\nu_{\mathbf{Y}}$. (Let us agree to call such points "nice".) For all $\epsilon > 0$ small enough the cube $C(\epsilon) = \prod_1^d(\lambda_i - \epsilon, \lambda_i + \epsilon)$ is entirely in R^d_-. Let $\mathbf{W}$ have the uniform distribution over $C(\epsilon)$. Then (3.12) implies

$$\int_{[-1,1]^d} \nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c : \mathbf{x} \not\geq \boldsymbol{\lambda} + \epsilon\mathbf{y}\} d\mathbf{y} \leq \int_{[-1,1]^d} \nu_{\mathbf{Y}}\{\mathbf{x} \in B_n^c : \mathbf{x} \not\geq \boldsymbol{\lambda} + \epsilon\mathbf{y}\} d\mathbf{y}.$$

Since $\boldsymbol{\lambda}$ is "nice", we recover (3.6) for all "nice" $\lambda \in R^d_-$ by letting ϵ go to zero. Observe now that "nice" points are dense in R^d_-. For *any* $\boldsymbol{\lambda} \in R^d_-$ choose a sequence of "nice" points $\boldsymbol{\lambda}_i$, $i = 1, 2, \ldots$ converging to $\boldsymbol{\lambda}$ *from below*. Then

$$\nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c : \mathbf{x} \not\geq \boldsymbol{\lambda}_i\} \uparrow \nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c : \mathbf{x} \not\geq \boldsymbol{\lambda}\},$$

and we obtain (3.6) for all $\boldsymbol{\lambda} \in R^d_-$. One can show in the same manner that (3.12) imply (3.5) for all $\boldsymbol{\lambda} \in R^d - R^d_-$.

Similarily, for every fixed $n \geq 1$ the pair of assumptions (3.8) - (3.9) is equivalent to the following assumption: for every random vector $\mathbf{W} \in R^d$ with atomless coordinates

$$\int_{A_n^c} \Big(P(\mathbf{W} > \mathbf{x}) - P(\mathbf{W} > \mathbf{0})\Big) \nu_{\mathbf{X}}(d\mathbf{x}) \geq \int_{B_n^c} \Big(P(\mathbf{W} > \mathbf{x}) - P(\mathbf{W} > \mathbf{0})\Big) \nu_{\mathbf{Y}}(d\mathbf{x}). \tag{3.14}$$

We have mentioned above that when the Lévy measures of the infinitely divisible random vectors $\mathbf{X}$ and $\mathbf{Y}$ satisfy (2.2), the assumptions (3.2) - (3.3) are adequate. This is made formal in the following theorem.

THEOREM 3.2 *Let $\mathbf{X}$ and $\mathbf{Y}$ be two infinitely divisible random vectors in R^d with Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ satisfying* (2.2) *with characteristic functions given in the form* (2.3) *and shift vectors $\mathbf{c}_{\mathbf{X}}$ and $\mathbf{c}_{\mathbf{Y}}$ respectively.*

(i) Suppose that conditions (3.2) *and* (3.3) *holds and*

$$\mathbf{c}_{\mathbf{X}} \geq \mathbf{c}_{\mathbf{Y}}. \tag{3.15}$$

Then for every $\boldsymbol{\lambda} \in R^d$, the right Slepian inequalities (3.4) *hold for all $t > 0$.*

(ii) Suppose the following conditions hold:

$$\nu_{\mathbf{X}}\{\mathbf{x} \in R^d : \mathbf{x} < \boldsymbol{\lambda}\} \geq \nu_{\mathbf{Y}}\{\mathbf{x} \in R^d : \mathbf{x} < \boldsymbol{\lambda}\} \text{ for every } \boldsymbol{\lambda} \in R^d - R^d_+; \tag{3.16}$$

$$\nu_{\mathbf{X}}\{\mathbf{x} \in R^d : \mathbf{x} \not\leq \boldsymbol{\lambda}\} \leq \nu_{\mathbf{Y}}\{\mathbf{x} \in R^d : \mathbf{x} \not\leq \boldsymbol{\lambda}\} \text{ for every } \boldsymbol{\lambda} \in R^d_+; \tag{3.17}$$

and

$$\mathbf{c}_{\mathbf{X}} \leq \mathbf{c}_{\mathbf{Y}}. \tag{3.18}$$

Then for every $\boldsymbol{\lambda} \in R^d$, the left Slepian inequalities (3.11) *hold for all $t > 0$.*

Remark. The argument of a previous remark shows, of course, that the pair of assumptions (3.2) - (3.3) is equivalent to the following assumption: when (2.2) holds,

$$\int_{R^d-\{\mathbf{0}\}}\Big(P(\mathbf{W}<\mathbf{x})-P(\mathbf{W}<\mathbf{0})\Big)\nu_{\mathbf{X}}(d\mathbf{x})\geq\int_{R^d-\{\mathbf{0}\}}\Big(P(\mathbf{W}<\mathbf{x})-P(\mathbf{W}<\mathbf{0})\Big)\nu_{\mathbf{Y}}(d\mathbf{x}),\tag{3.19}$$

for every $\mathbf{W}\in R^d$ whose distribution satisfies the following Lipschitz condition: for every $\mathbf{x},\mathbf{y}\in R^d$, there is a constant C such that

$$|P(\mathbf{W}<\mathbf{x}+\mathbf{y})-P(\mathbf{W}<\mathbf{y})|\leq C\|\mathbf{x}\|.\tag{3.20}$$

This condition is satisfied for example when $\mathbf{W}$ has all bounded marginal densities. It ensures, together with condition (2.2) that the right and left hand sides of (3.19) are well defined.

Similarly, the pair of assumptions (3.16) - (3.17) is equivalent to the following assumption: for every random vector $\mathbf{W}\in R^d$ satisfying (3.20),

$$\int_{R^d-\{\mathbf{0}\}}\Big(P(\mathbf{W}>\mathbf{x})-P(\mathbf{W}>\mathbf{0})\Big)\nu_{\mathbf{X}}(d\mathbf{x})\geq\int_{R^d-\{\mathbf{0}\}}\Big(P(\mathbf{W}>\mathbf{x})-P(\mathbf{W}>\mathbf{0})\Big)\nu_{\mathbf{Y}}(d\mathbf{x}).\tag{3.21}$$

We now prove Theorem 3.2.

PROOF: We may and will assume that $\mathbf{c}_{\mathbf{X}}=\mathbf{c}_{\mathbf{Y}}=\mathbf{0}$.

(i) It is obviously enough to prove (3.4) for $t=1$ (that is, to prove (1.1)).

Fix a $\boldsymbol{\lambda}\in R^d$ and let $f(\mathbf{x})=\prod_{i=1}^d\mathbf{1}(x_i>\lambda_i)$, $\mathbf{x}\in R^d$. Fix an $\epsilon>0$. For each $i=1,\ldots,d$ there clearly is a nondecreasing function $f_{i,\epsilon}:R\to R$, in C_b^∞ (of R), such that $\lim_{x\to-\infty}f_{i,\epsilon}(x)=0$, $\lim_{x\to\infty}f_{i,\epsilon}(x)=1$, and such that

$$|f_{i,\epsilon}(x)-\mathbf{1}(x>\lambda_i)|\leq\epsilon$$

for every $x\notin E_{i,\epsilon}$, where $E_{i,\epsilon}$ is a Borel set such that

$$P(X_i\in E_{i,\epsilon})\leq\epsilon\text{ and }P(Y_i\in E_{i,\epsilon})\leq\epsilon.$$

Further, let $f_\epsilon(\mathbf{x})=\prod_{i=1}^d f_{i,\epsilon}(x_i)$, and observe that $f_\epsilon\in C_b^\infty$ of R^d, that $|f_\epsilon(\mathbf{x})|\leq 1$ for every $\mathbf{x}\in R^d$, and moreover

$$|f(\mathbf{x})-f_\epsilon(\mathbf{x})|\leq\epsilon d.\tag{3.22}$$

for every $\mathbf{x}\notin E_\epsilon:=\{\mathbf{x}\in R^d:x_i\in E_{i,\epsilon}\text{ for some }i=1,\ldots,d\}$. We remark further that

$$P(\mathbf{X}\in E_\epsilon)\leq\epsilon d\text{ and }P(\mathbf{Y}\in E_\epsilon)\leq\epsilon d.\tag{3.23}$$

Write $f_{i,\epsilon}(x)=H_i((-\infty,x])$, where for $i=1,\ldots,d$, H_i is a probability measure on R with a bounded density with respect to Lebesgue measure. Then $H=H_1\times\ldots\times H_d$ is a probability measure on R^d. Let $P_{\mathbf{X}}^t$ and $P_{\mathbf{Y}}^t$ be the convolution semigroups generated by $\mathbf{X}$ and $\mathbf{Y}$ accordingly, and let $G_{\mathbf{X}}$ and $G_{\mathbf{Y}}$ be the corresponding generators. Note that for every $t>0$ and $\mathbf{x}\in R^d$,

$$P_{\mathbf{Y}}^t f_\epsilon(\mathbf{x}):=Ef_\epsilon(\mathbf{Y}^{*t}+\mathbf{x})=E\int_{R^d}\mathbf{1}(\mathbf{Z}\leq\mathbf{Y}^{*t}+\mathbf{x})H(d\mathbf{z})=P(\mathbf{W}\leq\mathbf{x}),\tag{3.24}$$

where $\mathbf{W} = \mathbf{Z} - \mathbf{Y}^{*t}$, and $\mathbf{Z}$ is an $\mathbf{Y}^{*t}$ R^d-valued random vector with the law H independent of $\mathbf{Y}^{*t}$. We conclude by (2.5) and (3.24) that for every $\mathbf{y} \in R^d$,

$$G_{\mathbf{X}} P_{\mathbf{Y}}^t f_\epsilon(\mathbf{y}) = \int_{R^d - \{\mathbf{0}\}} \Big(P_{\mathbf{Y}}^t f_\epsilon(\mathbf{x}+\mathbf{y}) - P_{\mathbf{Y}}^t f_\epsilon(\mathbf{y})\Big)\nu_{\mathbf{X}}(d\mathbf{x}) \tag{3.25}$$

$$= \int_{R^d - \{\mathbf{0}\}} \Big(P(\mathbf{W} \le \mathbf{x}+\mathbf{y}) - P(\mathbf{W} \le \mathbf{y})\Big)\nu_{\mathbf{X}}(d\mathbf{x}),$$

and similarily

$$G_{\mathbf{Y}} P_{\mathbf{Y}}^t f_\epsilon(\mathbf{y}) = \int_{R^d - \{\mathbf{0}\}} \Big(P(\mathbf{W} \le \mathbf{x}+\mathbf{y}) - P(\mathbf{W} \le \mathbf{y})\Big)\nu_{\mathbf{Y}}(d\mathbf{x}). \tag{3.26}$$

The distribution of $\mathbf{W}$ has bounded marginals because it is obtained by convoluting the distribution of $\mathbf{Z}$ which has bounded marginals. As noted in a previous remark, this implies that $\mathbf{W}$ satisfies (3.20), and hence (3.2) - (3.3) are equivalent to (3.19). It then follows from (3.25) and (3.26) that for every $t > 0$ and $\mathbf{y} \in R^d$,

$$G_{\mathbf{X}} P_{\mathbf{Y}}^t f_\epsilon(\mathbf{y}) \ge G_{\mathbf{Y}} P_{\mathbf{Y}}^t f_\epsilon(\mathbf{y}). \tag{3.27}$$

Define $h_\epsilon(t) = P_{\mathbf{X}}^t P_{\mathbf{Y}}^{1-t} f_\epsilon(\mathbf{0})$, $0 \le t \le 1$. Then for every $0 \le t \le 1$,

$$h'_\epsilon(t) = P_{\mathbf{X}}^t (G_{\mathbf{X}} - G_{\mathbf{Y}}) P_{\mathbf{Y}}^{1-t} f_\epsilon(\mathbf{0}).$$

An immediate conclusion of (3.27) is that $h'_\epsilon(t) \ge 0$ for every $0 < t < 1$, and so, in particular,

$$E f_\epsilon(\mathbf{X}^{*1}) = h_\epsilon(1) \ge h_\epsilon(0) = E f_\epsilon(\mathbf{Y}^{*1}). \tag{3.28}$$

Now (3.22) and (3.23) imply $Ef(\mathbf{X}^{*1}) \ge Ef(\mathbf{Y}^{*1}) - 4\epsilon d$. That is, for any $\boldsymbol{\lambda} \in R^d$,

$$P(\mathbf{X} > \boldsymbol{\lambda}) \ge P(\mathbf{Y} > \boldsymbol{\lambda}) - 4\epsilon d.$$

Since this is true for every $\epsilon > 0$, (1.1) follows.

The proof of the second part of the theorem is identical. ∎

Remarks

1. It is, of course, obvious how Theorem 3.1 follows from Theorem 3.2. Indeed, under the assumptions of, say, part (i) of Theorem 3.1 we get from part (i) of Theorem 3.2 that for every $n \ge 1$ and $\boldsymbol{\lambda} \in R^d$,

$$P((\mathbf{X}^{A_n})^{*t} > \boldsymbol{\lambda}) \ge P((\mathbf{Y}^{B_n})^{*t} > \boldsymbol{\lambda})$$

for all $t > 0$. Since $(\mathbf{X}^{A_n})^{*t} \Rightarrow \mathbf{X}^{*t}$ as $n \to \infty$ and $(\mathbf{Y}^{B_n})^{*t} \Rightarrow \mathbf{Y}^{*t}$ as $n \to \infty$, it follows that (3.4) holds for every $t > 0$.

2. In the important particular case when the infinitely divisible random vectors $\mathbf{X}$ and $\mathbf{Y}$ are *symmetric*, it is trivial (but useful) to note in the context of Theorem 3.1 that any choice of the sequences $\{A_n,\ n \ge 1\}$ and $\{B_n,\ n \ge 1\}$ such that the sets A_n's and B_n's are symmetric around the origin, makes the conditions (3.7) and (3.10) unnecessary. Moreover, in the symmetric case the two parts in each of Theorems 3.1 and 3.2 coalesce, in the sense that the assumptions of either part of the former imply the conclusions of both parts, and the same is true for the latter theorem with the difference that, in this case, even the assumptions of the two parts become identical.

We now turn to *necessary* conditions for Slepian inequalities.

THEOREM 3.3 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two infinitely divisible random vectors with Lévy measures* $\nu_{\mathbf{X}}$ *and* $\nu_{\mathbf{Y}}$ *correspondingly.*

(i) Suppose that for every $\boldsymbol{\lambda} \in R^d$*, the right Slepian inequalities* (3.4) *hold for every* $t > 0$. *Then* (3.2) *and* (3.3) *hold. Moreover, if* $\nu_{\mathbf{X}}$ *and* $\nu_{\mathbf{Y}}$ *satisfy* (2.2) *and* $\mathbf{c}_{\mathbf{X}}$ *and* $\mathbf{c}_{\mathbf{Y}}$ *are the corresponding shift vectors in the representation* (2.3) *of the characteristic functions, then* (3.15) *holds as well.*

(ii) Suppose that for every $\boldsymbol{\lambda} \in R^d$*, the left Slepian inequalities* (3.11) *hold for every* $t > 0$. *Then* (3.16) *and* (3.17) *hold. Further, if* $\nu_{\mathbf{X}}$ *and* $\nu_{\mathbf{Y}}$ *satisfy* (2.2) *and* $\mathbf{c}_{\mathbf{X}}$ *and* $\mathbf{c}_{\mathbf{Y}}$ *are the corresponding shift vectors in* (2.3)*, then* (3.18) *holds as well.*

PROOF: (i) Let $\mathbf{W}$ be any random vector in R^d. Define $f(\mathbf{x}) = P(\mathbf{W} < \mathbf{x})$, $\mathbf{x} \in R^d$. Since for every $t > 0$,

$$P^t_{\mathbf{X}} f(\mathbf{0}) = Ef(\mathbf{X}^{*t}) = EE\mathbf{1}(\mathbf{W} < \mathbf{X}^{*t}) = \int_{R^d} P(\mathbf{X}^{*t} > \mathbf{w}) F_{\mathbf{W}}(d\mathbf{w})$$

and

$$P^t_{\mathbf{Y}} f(\mathbf{0}) = \int_{R^d} P(\mathbf{Y}^{*t} > \mathbf{w}) F_{\mathbf{W}}(d\mathbf{w}),$$

we conclude by (3.4) that

$$P^t_{\mathbf{X}} f(\mathbf{0}) \geq P^t_{\mathbf{Y}} f(\mathbf{0}) \text{ for every } t > 0. \tag{3.29}$$

Suppose now that $\mathbf{W}$ has a C_b^∞ density with respect to the d-dimensional Lebesgue measure. Then $f \in C_b^\infty$, and so it is in the domains of both generators $G_{\mathbf{X}}$ and $G_{\mathbf{Y}}$. It follows from (3.29) and $P^0_{\mathbf{X}} f(\mathbf{0}) = f(\mathbf{0}) = P^0_Y f(\mathbf{0})$ that there is a sequence $t_n \downarrow 0$ such that

$$\left(P^t_{\mathbf{X}} f(\mathbf{0})\right)'\Big|_{t=t_n} \geq \left(P^t_{\mathbf{Y}} f(\mathbf{0})\right)'\Big|_{t=t_n}, \quad n = 1, 2, \ldots.$$

That is,

$$P^{t_n}_{\mathbf{X}} G_{\mathbf{X}} f(\mathbf{0}) \geq P^{t_n}_{\mathbf{Y}} G_{\mathbf{Y}} f(\mathbf{0})$$

for every $n = 1, 2, \ldots$. Letting $n \to \infty$, we obtain

$$G_{\mathbf{X}} f(\mathbf{0}) \geq G_{\mathbf{Y}} f(\mathbf{0}). \tag{3.30}$$

Using the representation (2.4) of the generators, we conclude that for every random vector $\mathbf{W} \in R^d$ with a C_b^∞ density

$$\int_{R^d} \Big(P(\mathbf{W} < \mathbf{x}) - P(\mathbf{W} < \mathbf{0}) - \mathbf{1}(\|\mathbf{x}\| \leq 1)(\mathbf{x}, \triangle F_{\mathbf{W}}(\mathbf{0})\Big)\nu_{\mathbf{X}}(d\mathbf{x}) + \Big(b_{\mathbf{X}}, \triangle F_{\mathbf{W}}(\mathbf{0})\Big) \tag{3.31}$$

$$\geq \int_{R^d} \Big(P(\mathbf{W} < \mathbf{x}) - P(\mathbf{W} < \mathbf{0}) - \mathbf{1}(\|\mathbf{x}\| \leq 1)(\mathbf{x}, \triangle F_{\mathbf{W}}(\mathbf{0})\Big)\nu_{\mathbf{Y}}(d\mathbf{x}) + \Big(b_{\mathbf{Y}}, \triangle F_{\mathbf{W}}(\mathbf{0})\Big).$$

We first prove (3.2). An earlier argument shows that it is enough to prove it for $\boldsymbol{\lambda} \in R^d - R^d_-$ which is a continuity point of both $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$. Let $\lambda_1 = \epsilon > 0$ (say).

Define $\psi_{\mathbf{X}}(\mathbf{y}) = \nu_{\mathbf{X}}\{\mathbf{x} : \mathbf{x} > \mathbf{y}\}$ and $\psi_{\mathbf{Y}}(\mathbf{y}) = \nu_{\mathbf{Y}}\{\mathbf{x} : \mathbf{x} > \mathbf{y}\}$, $\mathbf{y} \in R^d$. Observe that $\psi_{\mathbf{X}}$ and $\psi_{\mathbf{Y}}$ are bounded on $[\epsilon/2, \infty) \times R^{d-1}$ and continuous at $\mathbf{y} = \boldsymbol{\lambda}$.

Let $\mathbf{W}^* = \boldsymbol{\lambda}$ a.s., and choose a sequence $\{\mathbf{W}^n,\ n \geq 1\}$ of random vectors in R^d satisfying

$$\mathbf{W}^n \Rightarrow \mathbf{W}^* \text{ as } n \to \infty, \tag{3.32}$$

$$\mathbf{W}^n \text{ has a } C_b^\infty \text{ density}, \tag{3.33}$$

$$W_1^n \geq \epsilon/2 \text{ a.s. } n = 1, 2, \ldots. \tag{3.34}$$

Then it follows from (3.34) that $\triangle F_{\mathbf{W}^n}(\mathbf{0}) = \mathbf{0},\ n = 1, 2, \ldots$, and so (3.31) takes in this case the form

$$\int_{R^d} P(\mathbf{W}^n < \mathbf{x})\nu_{\mathbf{X}}(d\mathbf{x}) \geq \int_{R^d} P(\mathbf{W}^n < \mathbf{x})\nu_{\mathbf{Y}}(d\mathbf{x}),\ n = 1, 2, \ldots.$$

Equivalently,

$$E\psi_{\mathbf{X}}(\mathbf{W}^n) \geq E\psi_{\mathbf{Y}}(\mathbf{W}^n),\ n = 1, 2, \ldots.$$

Taking the limits as $n \to \infty$ we conclude that

$$E\psi_{\mathbf{X}}(\mathbf{W}^*) \geq E\psi_{\mathbf{Y}}(\mathbf{W}^*),$$

which is exactly (3.2).

We now turn to verification of (3.3). Of course, it is enough to check it for $\boldsymbol{\lambda} < \mathbf{0}$ which is, in addition, a continuity point of both $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$. We pursue a strategy similar to that used in the proof of (3.2). Define $\phi_{\mathbf{X}}(\mathbf{y}) = \nu_{\mathbf{X}}\{\mathbf{x} : \mathbf{x} \not\geq \mathbf{y}\}$ and $\phi_{\mathbf{Y}}(\mathbf{y}) = \nu_{\mathbf{Y}}\{\mathbf{x} : \mathbf{x} \not\geq \mathbf{y}\}$, $\mathbf{y} \in R^d$. Let $\epsilon = \min_{i=1,\ldots,d}(-\lambda_i) > 0$. Observe that $\phi_{\mathbf{X}}$ and $\phi_{\mathbf{Y}}$ are bounded on $(-\infty, -\epsilon/2]^d$ and are continuous at $\mathbf{y} = \boldsymbol{\lambda}$.

Let again $\mathbf{W}^* = \boldsymbol{\lambda}$ a.s., and choose a sequence $\{\mathbf{W}^n,\ n \geq 1\}$ satisfying (3.32), (3.33), and

$$\mathbf{W}_i^n \leq -\epsilon/2 \text{ for every } i = 1, \ldots, d \text{ and } n = 1, 2, \ldots. \tag{3.35}$$

Then again $\triangle F_{\mathbf{W}^n}(\mathbf{0}) = \mathbf{0},\ n = 1, 2, \ldots$, and so we immediately obtain from (3.31) that

$$\int_{R^d} P(\mathbf{W}^n \not< \mathbf{x})\nu_{\mathbf{X}}(d\mathbf{x}) \leq \int_{R^d} P(\mathbf{W}^n \not< \mathbf{x})\nu_{\mathbf{Y}}(d\mathbf{x}),\ n = 1, 2, \ldots,$$

which is equivalent to

$$E\phi_{\mathbf{X}}(\mathbf{W}^n) \leq E\phi_{\mathbf{Y}}(\mathbf{W}^n),\ n = 1, 2, \ldots.$$

Now take the limits as $n \to \infty$; we obtain

$$E\phi_{\mathbf{X}}(\mathbf{W}^*) \leq E\phi_{\mathbf{Y}}(\mathbf{W}^*),$$

thus proving (3.3).

It remains, therefore, to prove (3.15) under the assumption that the Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ satisfy (2.2).

Since we need to compare the vectors $\mathbf{c}_{\mathbf{X}}$ and $\mathbf{c}_{\mathbf{Y}}$ componentwise, we may as well restrict ourselves to the case $d = 1$. We have by (3.30) and (2.5)

$$\int_{-\infty}^{\infty} \Big(P(W < x) - P(W < 0)\Big)\nu_X(dx) + (c_X - c_Y)f_W(0) \tag{3.36}$$

$$\geq \int_{-\infty}^{\infty} \Big(P(W < x) - P(W < 0)\Big)\nu_Y(dx)$$

for every random variable W with a C_b^∞ density f_W.

Let $W \sim N(0, \sigma^2)$. Observe that

$$\int_0^\infty \Big(P(W < x) - P(W < 0)\Big)\nu_X(dx) = \int_0^\infty P(0 < W < x)\nu_X(dx) \tag{3.37}$$

$$= \int_0^\infty \Big(\Phi(\frac{x}{\sigma}) - \frac{1}{2}\Big)\nu_X(dx) \leq \int_0^1 \Big(\Phi(\frac{x}{\sigma}) - \frac{1}{2}\Big)\nu_X(dx) + \nu_X([1,\infty))$$

$$= o(\sigma^{-1}) \text{ as } \sigma \to 0$$

by the boundedness of the density and (2.2). Applying the same argument to the other half of the integral in the left hand side of (3.36) we conclude that

$$\int_{-\infty}^{\infty} \Big(P(W < x) - P(W < 0)\Big)\nu_X(dx) = o(\sigma^{-1}) \text{ as } \sigma \to 0. \tag{3.38}$$

Similarily,

$$\int_{-\infty}^{\infty} \Big(P(W < x) - P(W < 0)\Big)\nu_Y(dx) = o(\sigma^{-1}) \text{ as } \sigma \to 0. \tag{3.39}$$

Recalling that $f_W(0) = 1/\sigma\sqrt{2\pi}$, we conclude immediately from (3.36), (3.38) and (3.39) that $c_X \geq c_Y$.

This completes the proof of part (i) of the theorem, and the proof of part (ii) is identical. ■

When the Lévy measures of the infinitely divisible random vectors $\mathbf{X}$ and $\mathbf{Y}$ satisfy (2.2), Theorems 3.2 and 3.3 imply that (3.2), (3.3) and (3.15) are the necessary and sufficient conditions for the right Slepian inequality (3.4) and (3.16), (3.17) and (3.18) are the necessary and sufficient conditions for the left Slepian inequality (3.11)

Assuming only (1.1) instead of assuming (3.4) for *all* $t > 0$ is not, in general, sufficient for the conclusions of Theorem 3.3, even under the assumption (2.2). We show this through the following two examples.

EXAMPLE 3.1 ((1.1) *implies neither* (3.2) *nor* (3.3))

We modify an example of Samorodnitsky and Taqqu [13] as follows. Let $d = 1$, and let Y be a mean 1 Poisson random variable. Let X be a (nonnegative) infinitely divisible random variable such that $X \stackrel{d}{=} X_1 + X_2$, where X_1 and X_2 are independent infinitely divisible random variables with Lévy measures

$$\nu_{X_1}(dx) = n\delta_{1/2}(dx), \tag{3.40}$$

$$\nu_{X_2}(dx) = c\mathbf{1}(x \geq 1)\exp(-x(\log x)^{1/2})dx,$$

where n is a positive integer to be specified later, and $c > 0$ is chosen in such a way that $\nu_{X_2}([1,\infty)) = 1/2$. (The two random variables have zero shifts c_{X_1} and c_{X_2}.)

Observe that as $\lambda \to \infty$ (through integer values)

$$P(Y \geq \lambda) \sim P(Y = \lambda) = (e\lambda!)^{-1} = 0(e^{-\lambda(\log\lambda)^{1/2}}).$$

On the other hand, $\int_1^\infty \nu_{X_2}(dx) < \infty$ and hence X_2 is compound Poisson, that is, $X_2 = Z_1, + \ldots + Z_N$ where N is a Poisson random variable and the Z_i's are i.i.d. random variables independent of N. Since the distribution of Z_i is $(\nu_{X_2}(R))^{-1}\nu_{X_2}$, there is a constant C such that, as $\lambda \to \infty$,

$$P(X_2 \geq \lambda) > P(Z_1 \geq \lambda) \geq Ce^{-\lambda(\log \lambda)^{1/2}}.$$

Therefore

$$P(Y > \lambda) = o(P(X_2 > \lambda) \tag{3.41}$$

as $\lambda \to \infty$. Hence there is a $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$,

$$P(X > \lambda) \geq P(X_2 > \lambda) \geq P(Y > \lambda).$$

On the other hand one can choose n so large that

$$P(X > \lambda) \geq P(X_1 > \lambda) \geq P(Y > 0) \geq P(Y > \lambda)$$

for every $0 < \lambda < \lambda_0$, implying (1.1). However,

$$\nu_X((3/4, \infty)) = \nu_{X_2}((3/4, \infty)) = 1/2 < 1 = \nu_Y((3/4, \infty)),$$

and so (3.2) fails.

By taking $\tilde{X} = -Y$, $\tilde{Y} = -X$, we have an example of a situation where (1.1) holds in the absence of (3.3).

EXAMPLE 3.2 ((1.1) *does not imply* (3.15))

We take once again $d = 1$. Let E be a mean 1 exponential random variable. Let $X = E$, $Y = -E + \log 2$. Then the Lévy measures of X and Y, are $\nu_X(dx) = x^{-1}e^{-x}1(x > 0)dx$ and $\nu_Y(dx) = |x|^{-1}e^{x}1(x < 0)dx$ (Feller [6], XVII.3(d)) They satisfy (2.2), and moreover, $c_X = 0 < \log 2 = c_Y$, which means that (3.15) fails. However, it is elementary to verify that (1.1) holds in this case.

Remarks

1. Example 3.2 notwithstanding, (1.1) DOES imply (3.15), in the case (2.2) under the additional assumption that the Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ are *concentrated on the same quadrant of* R^d.

   To see this, suppose, for example, that $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ are concentrated on R^d_+. Then $X_1 = U_1 + (\mathbf{c_X})_1$ and $Y_1 = U_2 + (\mathbf{c_Y})_2$ where $U_1 \geq 0$ and $U_2 \geq 0$. Since (2.2) holds, we have $P(U_2 > \delta) < 1$ because U_2 is a limit in distribution of compound Poisson variables. If $(\mathbf{c_X})_1 < (\mathbf{c_Y})_1$ (say), then $P(Y_1 \geq (\mathbf{c_Y})_1) = 1$, but $P(X_1 \geq (\mathbf{c_Y})_1) < 1$, contradicting (1.1).

2. In view of Theorem 3.3, Examples 3.1 and 3.2 illustrate situations where the right (say) Slepian inequality (3.4) holds for some $t > 0$ and fails for other $t > 0$. In the strictly α-stable case with $0 < \alpha \leq 2$ (including centered Gaussian case $\alpha = 2$) because of the relation $\mathbf{X}^{*t} \stackrel{d}{=} t^{1/\alpha}\mathbf{X}$, all the distinction between (1.1) and (3.4) disappears. In particular, strictly α-stable random vectors $\mathbf{X}$ and $\mathbf{Y}$ with $0 < \alpha < 1$ satisfy the right Slepian inequality (1.1) if and only if (3.2), (3.3) and (3.15) hold, and similarily with the left Slepian inequality (1.2).

4 Symmetric α-stable case; Ornstein-Uhlenbeck processes

In this section we specialize the results of Section 2 to the *symmetric α-stable* (SαS) case, $0 < \alpha < 2$. In this case the Lévy measure ν of a random vector $\mathbf{X}$ is given in the form

$$\nu(A) = \int_{S^d} \int_0^\infty \mathbf{1}(r\mathbf{s} \in A) r^{-(1+\alpha)} dr \Gamma(d\mathbf{s}), \tag{4.1}$$

where Γ is a finite symmetric measure on Borel subsets of the unit sphere $S^d = \{\mathbf{x} \in R^d : ||\mathbf{x}|| = 1\}$ in *some* norm $||\cdot||$ on R^d. The measure Γ is commonly called *the spectral measure of* $\mathbf{X}$.

Unless stated otherwise we will use in this section the maximum norm $||\mathbf{x}|| = \max_{i=1,\ldots,d} |x_i|$, as this norm is the most natural to use in our context.

Let, therefore, $\mathbf{X}$ and $\mathbf{Y}$ be two SαS random vectors, $0 < \alpha < 2$, with spectral measures $\Gamma_{\mathbf{X}}$ and $\Gamma_{\mathbf{Y}}$ accordingly. We will choose the following sequences $\{A_n,\ n \geq 1\}$ and $\{B,\ n \geq 1\}$ of sets deflating to the origin:

$$A_n = \prod_{i=1}^d (-\delta_{n,i}, \delta_{n,i}) \ \text{ and } \ B_n = \prod_{i=1}^d (-\theta_{n,i}, \theta_{n,i}), \tag{4.2}$$

where for a fixed $i \in \{1, \ldots, d\}$, $\{\delta_{n,i}\}_{n=1}^\infty$ and $\{\theta_{n,i}\}_{n=1}^\infty$ are two sequences of positive numbers decreasing to zero (refer to the remark following Definition 3.1). Since these sets are symmetric around the origin, conditions (3.7) and (3.10) hold automatically (see Remark 2 following the proof of Theorem 3.2). We will now obtain a more explicit form of the conditions (3.5) and (3.6).

For every $n = 1, 2, \ldots$ and $\boldsymbol{\lambda} \in R^d - R^d_-$,

$$\nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c : \mathbf{x} > \boldsymbol{\lambda}\} \tag{4.3}$$

$$= \int_{S^d} \int_0^\infty \mathbf{1}\left((r|s_i| \geq \delta_{n,i} \text{ for some } i \in \{1, \ldots, d\}) \text{ and } rs_i > \lambda_i \text{ for all } i \in \{1, \ldots, d\}\right) r^{-(1+\alpha)} dr \Gamma_{\mathbf{X}}(d\mathbf{s}).$$

For an $\mathbf{x} \in R^d$ let

$$\mathcal{P}_{\mathbf{x}} = \{i : x_i \geq 0\}.$$

Then the right hand side of (4.3) becomes

$$\int_{S^d} \mathbf{1}(\mathcal{P}_\lambda \subseteq \mathcal{P}_{\mathbf{s}}) \int_0^\infty \mathbf{1}(r \geq \min_{i=1,\ldots,d} \frac{\delta_{n,i}}{|s_i|},\ r > \max_{i \in \mathcal{P}_\lambda} \frac{\lambda_i}{s_i},\ r < \min_{i \notin \mathcal{P}_{\mathbf{s}}} \frac{\lambda_i}{s_i}) r^{-(1+\alpha)} dr \Gamma_{\mathbf{X}}(d\mathbf{s})$$

$$= \alpha^{-1} \int_{S^d} \mathbf{1}(\mathcal{P}_\lambda \subseteq \mathcal{P}_{\mathbf{s}}) \left[\left(\max_{i \in \mathcal{P}_\lambda} \frac{\lambda_i}{s_i} \vee \min_{i \in \{1,\ldots,d\}} \frac{\delta_{n,i}}{|s_i|} \right)^{-\alpha} - \left(\min_{i \notin \mathcal{P}_{\mathbf{s}}} \frac{\lambda_i}{s_i} \right)^{-\alpha} \right]_+ \Gamma_{\mathbf{X}}(d\mathbf{s}).$$

(In order to prevent the occurence of expressions of the type 0/0 we will consider, unless mentioned otherwise, only $\boldsymbol{\lambda} \in R^d$ with non-zero coordinates.) Similarily, for every $\boldsymbol{\lambda} \in R^d_-$,

$$\nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c : \mathbf{x} \not\geq \boldsymbol{\lambda}\}$$

$$= \int_{S^d} \int_0^\infty \mathbf{1}(r \geq \min_{i=1,\ldots,d} \frac{\delta_{n,i}}{|s_i|},\ r \geq \min_{i \notin \mathcal{P}_{\mathbf{s}}} \frac{\lambda_i}{s_i}) r^{-(1+\alpha)} dr \Gamma_{\mathbf{X}}(d\mathbf{s})$$

$$= \alpha^{-1} \int_{S^d} \left(\min_{i \notin \mathcal{P}_{\mathbf{s}}} \frac{\lambda_i}{s_i} \vee \min_{i \in \{1,\ldots,d\}} \frac{\delta_{n,i}}{|s_i|} \right)^{-\alpha} \Gamma_{\mathbf{X}}(d\mathbf{s}).$$

This proves

PROPOSITION 4.1 *In the $S\alpha S$ case, conditions* (3.5) *and* (3.6) *take respectively the following forms. For every $\boldsymbol{\lambda} \in R^d - R^d_-$,*

$$\int_{S^d} \mathbf{1}(\mathcal{P}_\lambda \subseteq \mathcal{P}_\mathbf{s}) \left[\left(\max_{i \in \mathcal{P}_\lambda} \frac{\lambda_i}{s_i} \vee \min_{i \in \{1,\dots,d\}} \frac{\delta_{n,i}}{|s_i|} \right)^{-\alpha} - \left(\min_{i \notin \mathcal{P}_\mathbf{s}} \frac{\lambda_i}{s_i} \right)^{-\alpha} \right]_+ \Gamma_\mathbf{X}(d\mathbf{s}) \tag{4.4}$$

$$\geq \int_{S^d} \mathbf{1}(\mathcal{P}_\lambda \subseteq \mathcal{P}_\mathbf{s}) \left[\left(\max_{i \in \mathcal{P}_\lambda} \frac{\lambda_i}{s_i} \vee \min_{i \in \{1,\dots,d\}} \frac{\theta_{n,i}}{|s_i|} \right)^{-\alpha} - \left(\min_{i \notin \mathcal{P}_\mathbf{s}} \frac{\lambda_i}{s_i} \right)^{-\alpha} \right]_+ \Gamma_\mathbf{Y}(d\mathbf{s}),$$

and for every $\boldsymbol{\lambda} \in R^d_-$,

$$\int_{S^d} \left(\min_{i \notin \mathcal{P}_\mathbf{s}} \frac{\lambda_i}{s_i} \vee \min_{i \in \{1,\dots,d\}} \frac{\delta_{n,i}}{|s_i|} \right)^{-\alpha} \Gamma_\mathbf{X}(d\mathbf{s}) \leq \int_{S^d} \left(\min_{i \notin \mathcal{P}_\mathbf{s}} \frac{\lambda_i}{s_i} \vee \min_{i \in \{1,\dots,d\}} \frac{\theta_{n,i}}{|s_i|} \right)^{-\alpha} \Gamma_\mathbf{Y}(d\mathbf{s}). \tag{4.5}$$

As a particular case consider the situation where the spectral measures $\Gamma_\mathbf{X}$ and $\Gamma_\mathbf{Y}$ are concentrated on $R^d_- \cup R^d_+$. In this case (4.4) reduces to

$$\int_{S^d_+} \left(\max_{i \in \{1,\dots,d\}} \frac{\lambda_i}{s_i} \vee \min_{i \in \{1,\dots,d\}} \frac{\delta_{n,i}}{s_i} \right)^{-\alpha} \Gamma_\mathbf{X}(d\mathbf{s}) \geq \int_{S^d_+} \left(\max_{i \in \{1,\dots,d\}} \frac{\lambda_i}{s_i} \vee \min_{i \in \{1,\dots,d\}} \frac{\theta_{n,i}}{s_i} \right)^{-\alpha} \Gamma_\mathbf{Y}(d\mathbf{s}) \tag{4.6}$$

for every $\boldsymbol{\lambda} \in R^d_+$, where $S^d_+ = S^d \cap R^d_+$, while (4.5) reduces to

$$\int_{S^d_+} \left(\min_{i \in \{1,\dots,d\}} \frac{\lambda_i}{s_i} \vee \min_{i \in \{1,\dots,d\}} \frac{\delta_{n,i}}{s_i} \right)^{-\alpha} \Gamma_\mathbf{X}(d\mathbf{s}) \leq \int_{S^d_+} \left(\min_{i \in \{1,\dots,d\}} \frac{\lambda_i}{s_i} \vee \min_{i \in \{1,\dots,d\}} \frac{\theta_{n,i}}{s_i} \right)^{-\alpha} \Gamma_\mathbf{Y}(d\mathbf{s}) \tag{4.7}$$

for every $\boldsymbol{\lambda} \in R^d_+$.

Stationary $S\alpha S$ moving averages are stationary SαS stochastic processes that are important in applications (e.g Cambanis [4], see also Surgailis et al. [16] for a recent study). They can be represented as

$$X(t) = \int_{-\infty}^{\infty} f(t+x) M(dx), \ -\infty < t < \infty, \tag{4.8}$$

where $f \in L^\alpha(-\infty, \infty)$, and M is an independently scattered SαS random measure on $(-\infty, \infty)$ with Lebesgue control measure. The d-dimensional random vector $(X(t_1), \dots, X(t_d))$, $t_1, \dots, t_d \in R$, is SαS with spectral measure given by

$$\Gamma(A) = (\frac{1}{2}) m_1(T^{-1}(A)) + (\frac{1}{2}) m_1(T^{-1}(-A)), \tag{4.9}$$

where m_1 is a finite measure on Borel subsets of R defined by

$$m_1(dx) = ||f(\cdot + x)||^\alpha dx,$$

$||f(\cdot + x)|| = \max_{i=1,\dots,d} |f(t_i + x)|$ and $T : R \to S^d$ is given by

$$T(x) = \begin{cases} (f(t_1+x)/||f(\cdot+x)||, \dots, f(t_d+x)/||f(\cdot+x)||) & \text{if } ||f(\cdot+x)|| \neq 0, \\ (1, 0, \dots, 0) & \text{if } ||f(\cdot+x)|| = 0. \end{cases}$$

See, e.g. Samorodnitsky and Taqqu [14]. In particular, if $f(x) \geq 0$ for all $x \in R$ then the spectral measure given by (4.9) is concentrated on $R^d_- \cup R^d_+$. Substituting (4.9) into (4.6) and (4.7) , we get:

COROLLARY 4.1 *Let $\{X(t),\ t \in R\}$ and $\{Y(t),\ t \in R\}$ be two $S\alpha S$ moving averages as in* (4.8), *defined by two nonnegative functions f and g in $L^\alpha(-\infty,\infty)$. For given $t_1,\dots,t_d \in R$ consider the two $S\alpha S$ random vectors $\mathbf{X} = (X(t_1),\dots,X(t_d))$ and $\mathbf{Y} = (Y(t_1),\dots,Y(t_d))$. Then the Slepian inequalities* (1.1) *and* (1.2) *hold for these vectors, if for every $\boldsymbol{\lambda} > \mathbf{0}$ and $n = 1,2\dots$*

$$\int_{-\infty}^{\infty}\Big(\max_{i\in\{1,\dots,d\}}\frac{\lambda_i}{f(t_i+x)}\vee\min_{i\in\{1,\dots,d\}}\frac{\delta_{n,i}}{f(t_i+x)}\Big)^{-\alpha}dx \geq \int_{-\infty}^{\infty}\Big(\max_{i\in\{1,\dots,d\}}\frac{\lambda_i}{g(t_i+x)}\vee\min_{i\in\{1,\dots,d\}}\frac{\theta_{n,i}}{g(t_i+x)}\Big)^{-\alpha}dx \tag{4.10}$$

and

$$\int_{-\infty}^{\infty}\Big(\min_{i\in\{1,\dots,d\}}\frac{\lambda_i}{f(t_i+x)}\vee\min_{i\in\{1,\dots,d\}}\frac{\delta_{n,i}}{f(t_i+x)}\Big)^{-\alpha}dx \leq \int_{-\infty}^{\infty}\Big(\min_{i\in\{1,\dots,d\}}\frac{\lambda_i}{g(t_i+x)}\vee\min_{i\in\{1,\dots,d\}}\frac{\theta_{n,i}}{g(t_i+x)}\Big)^{-\alpha}dx. \tag{4.11}$$

EXAMPLE 4.1 *$S\alpha S$ Ornstein-Uhlenbeck processes.* The conditions of the corollary are relatively easy to verify for $S\alpha S$ *Ornstein-Uhlenbeck processes.* These are stationary $S\alpha S$ moving averages of the type (4.8), with

$$f(x) = \gamma^{1/\alpha}e^{-\gamma x}\mathbf{1}(x>0) \quad\text{and}\quad g(x) = \mu^{1/\alpha}e^{-\mu x}\mathbf{1}(x>0), \tag{4.12}$$

with $\gamma,\mu > 0$. (The purpose of the normalization $\gamma^{1/\alpha}$ and $\mu^{1/\alpha}$ is to give the two processes the same scale.) The Ornstein-Uhlenbeck processes are also Markov; (see Adler, Cambanis and Samorodnitsky [2]).

We will see that if $0 < \gamma < \mu$, then the two processes satisfy Slepian inequalities in the sense that for every $t_1,\dots,t_d \in R$ and all real numbers $\lambda_1,\dots,\lambda_d$,

$$P(X(t_1) > \lambda_1,\dots,X(t_d) > \lambda_d) \geq P(Y(t_1) > \lambda_1)\dots,Y(t_d) > \lambda_d), \tag{4.13}$$

(and by symmetry the left Slepian inequality as well). This result is known in the Gaussian case because for Gaussian Ornstein-Uhlenbeck processes,

$$E(X(t)X(s)) = e^{-\gamma|t-s|} \geq e^{-\mu|t-s|} = E(Y(t)Y(s))$$

for all $t,s \in R$, and so (1.5) are trivially verified. Intuitively, a large μ causes $Y(t)$ to be affected mainly by the increments of the random measure M "near t". Thus, independence of the increments of M leads us to suspect that the components of the process Y "cluster together" less than the components of X do.

For a rigorous verification, assume that $-\infty = t_0 < t_1 < \dots < t_d < t_{d+1} = \infty$ and choose

$$\delta_{n,i} = \delta_n e^{-\gamma(t_i-t_1)},$$

$$\theta_{n,i} = \delta_n e^{-\mu(t_i-t_1)},$$

$\delta_n \downarrow 0$ and $i = 1,\dots,d$. In the subsequent computations we will drop the subscript n.

In our case the left hand side of (4.10) is

$$\int_{-t_1}^{\infty}\Big(\max_{i\in\{1,\dots,d\}}\frac{\lambda_i}{\gamma^{1/\alpha}e^{-\gamma(t_i+x)}}\vee\min_{i\in\{1,\dots,d\}}\frac{\delta e^{-\gamma(t_i-t_1)}}{\gamma^{1/\alpha}e^{-\gamma(t_i+x)}}\Big)^{-\alpha}dx$$

$$= \int_{-t_1}^{\infty} \gamma e^{-\gamma\alpha x} \Big(\max_{i\in\{1,\dots,d\}} \lambda_i e^{\gamma t_i} \vee \delta e^{\gamma t_1} \Big)^{-\alpha} dx$$

$$= \alpha^{-1} \Big(\max_{i\in\{1,\dots,d\}} \lambda_i e^{\gamma(t_i - t_1)} \vee \delta \Big)^{-\alpha}.$$

Since the last expression is clearly non-increasing in γ, (4.10) follows.

We now check (4.11). Since $t_0 = -\infty$, the left hand side of the latter equals

$$\sum_{j=0}^{d-1} \int_{-t_{j+1}}^{-t_j} \Big(\min_{i\in\{j+1,\dots,d\}} \frac{\lambda_i}{\gamma^{1/\alpha} e^{-\gamma(t_i+x)}} \vee \min_{i\in\{j+1,\dots,d\}} \frac{\delta_n e^{-\gamma(t_i - t_1)}}{\gamma^{1/\alpha} e^{-\gamma(t_i+x)}} \Big)^{-\alpha} dx \tag{4.14}$$

$$= \sum_{j=0}^{d-1} \int_{-t_{j+1}}^{-t_j} \gamma e^{-\gamma\alpha x} \Big(\min_{i\in\{j+1,\dots,d\}} \lambda_i e^{\gamma t_i} \vee \delta e^{\gamma t_1} \Big)^{-\alpha} dx$$

$$= \alpha^{-1} \sum_{j=0}^{d-1} \Big(e^{\gamma\alpha t_{j+1}} - e^{\gamma\alpha t_j} \Big) \Big(\min_{i\in\{j+1,\dots,d\}} \lambda_i e^{\gamma t_i} \vee \delta e^{\gamma t_1} \Big)^{-\alpha}.$$

Now (4.11) follows from (4.14) and the following lemma.

LEMMA 4.1 *Let $-\infty < t_1 < \dots < t_d < \infty$, λ_i, $i = 1,\dots,d$ positive numbers. Then for every $\delta > 0$ and $u, v \le t_1$ the function*

$$k(\gamma) = \sum_{j=0}^{d-1} \Big(e^{\gamma\alpha t_{j+1}} - e^{\gamma\alpha t_j} \Big) \Big(\min_{i\in\{j+1,\dots,d\}} \lambda_i e^{\gamma t_i} \vee \delta e^{\gamma u} \Big)^{-\alpha} \tag{4.15}$$

with $t_0 = v$ is non-decreasing in $\gamma > 0$.

PROOF: The proof is by induction in d. For $d = 1$ we have

$$\begin{aligned} k(\gamma) &= \Big(e^{\gamma\alpha t_1} - e^{\gamma\alpha v} \Big) \Big(\lambda_1 e^{\gamma t_1} \vee \delta e^{\gamma u} \Big)^{-\alpha} \\ &= \Big(1 - e^{-\gamma\alpha(t_1 - v)} \Big) \Big(\lambda_1 \vee \delta e^{-\gamma(t_1 - u)} \Big)^{-\alpha}, \end{aligned}$$

and this is non-decreasing in γ because $t_1 \ge u \vee v$.

Suppose now that the statement of the lemma is true for a $d \ge 1$. and let us prove it for $d+1$. We have now

$$k(\gamma) = \sum_{j=0}^{d} \Big(e^{\gamma\alpha t_{j+1}} - e^{\gamma\alpha t_j} \Big) \Big(\min_{i\in\{j+1,\dots,d\}} \lambda_i e^{\gamma t_i} \vee \delta e^{\gamma u} \Big)^{-\alpha}.$$

Consider two cases.

Case 1. $\lambda_1 e^{\gamma t_1} \le \min_{i\in\{2,\dots,d\}} \lambda_i e^{\gamma t_i}$.

Then

$$k(\gamma) = \Big(e^{\gamma\alpha t_1} - e^{\gamma\alpha v} \Big) \Big(\lambda_1 e^{\gamma t_1} \vee \delta e^{\gamma u} \Big)^{-\alpha} + \sum_{j=1}^{d} \Big(e^{\gamma\alpha t_{j+1}} - e^{\gamma\alpha t_j} \Big) \Big(\min_{i\in\{j+1,\dots,d\}} \lambda_i e^{\gamma t_i} \vee \delta e^{\gamma u} \Big)^{-\alpha}.$$

The fact that the first term in the sum above is non-decreasing has been proved when we considered the case $d = 1$, while the second term is non-decreasing by the assumption of the induction with $v = t_1$.

Case 2. $\lambda_1 e^{\gamma t_1} > \min_{i \in \{2,\ldots,d\}} \lambda_i e^{\gamma t_i}$.

Then combining together the first two terms in the sum we obtain

$$k(\gamma) = \left(e^{\gamma\alpha t_2} - e^{\gamma\alpha v}\right)\left(\min_{i\in\{2,\ldots,d\}} \lambda_i e^{\gamma t_i} \vee \delta e^{\gamma u}\right)^{-\alpha} + \sum_{j=2}^{d}\left(e^{\gamma\alpha t_{j+1}} - e^{\gamma\alpha t_j}\right)\left(\min_{i\in\{j+1,\ldots,d\}} \lambda_i e^{\gamma t_i} \vee \delta e^{\gamma u}\right)^{-\alpha},$$

and this is non-decreasing once again by the assumption of the induction (we preserve the same u and v, delete t_1 and let t_2 play the role of t_1).

This completes the proof of the lemma. ■

5 The Brown-Rinott family

In 1988, Brown and Rinott [3] obtained Slepian inequalities for a particular family of infinitely divisible random vectors. The authors of the present paper have wanted to understand for some time how to place their result in a general theory of Slepian inequalities for infinitely divisible random vectors. We show in this section where the results of Brown and Rinott fit in the general theory, and we also provide some extensions.

Brown and Rinott consider a particular subclass of the infinitely divisible distributions, defined by $2^d - 1$ numbers, labeled t_A, where $A \neq \emptyset$ runs through all subsets of the set $\{1, 2, \ldots, d\}$.

Let Q be an infinitely divisible probability law on R, not necessarily symmetric, and let $\{Q^t,\ t \geq 0\}$ be the corresponding convolution semigroup. The Brown-Rinott family $\mathcal{BR}_Q$ of infinitely divisible random vectors in R^d is constructed as follows. Let $\mathcal{A} = \{A : A \subseteq \{1, \ldots, d\}\}$. Choose a vector of nonnegative numbers $\mathbf{t} = \{t_A,\ A \in \mathcal{A},\ A \neq \emptyset\}$, and let $\{Z_A,\ A \in \mathcal{A},\ A \neq \emptyset\}$ be independent (real valued) random variables, with $Z_A \sim Q^{t(A)}$, $A \in \mathcal{A}$, $A \neq \emptyset$. Then define an infinitely divisible random vector $\mathbf{X} = (X_1, \ldots, X_d)$ by

$$X_i = \sum_{A: i \in A} Z_A,\ i = 1, \ldots, d. \tag{5.1}$$

The family $\mathcal{BR}_Q$ is obtained by allowing $\mathbf{t}$ to vary. It is indeed, a family of infinitely divisible random vectors because any linear combination of components of members of the family can be expressed as a linear combination of independent infinitely divisible random variables. It is easy, moreover, to identify the parameters of the random vector $\mathbf{X}$ in $\mathcal{BR}_Q$ for fixed $\mathbf{t}$. Namely, suppose that the infinitely divisible law Q has (in the representation (2.1)) one-dimensional Lévy measure μ and shift a. Then the d-dimensional Lévy measure ν of $\mathbf{X}$ is given by

$$\nu = \sum_{\substack{A \in \mathcal{A} \\ A \neq \emptyset}} t_A \mu^{(A)}, \tag{5.2}$$

where for an $A \in \mathcal{A}$, $A \neq \emptyset$ $\mu^{(A)}$ is the measure μ placed on the line $l_A = \{\mathbf{x} \in R^d : x_i = 0\ \forall i \notin$

$A,\ x_{i_1} = x_{i_2}\ \forall i_1, i_2 \in A\}$. That is, for every nonnegative measurable function $h : R^d \to R$

$$\int_{R^d - \{\mathbf{0}\}} h(\mathbf{x})\nu(d\mathbf{x}) = \sum_{\substack{A \in \mathcal{A} \\ A \neq \emptyset}} t_A \int_{-\infty}^{\infty} h(\ldots, x, \ldots, 0, \ldots, x, \ldots, 0, \ldots)\mu(dx),$$

when the arguments of h in the integral under the sum are equal to x for all i's in A and to 0 for all i's not in A. Further, by (5.1), the shift vector $\mathbf{b}$ of $\mathbf{X}$ is given by

$$b_i = a \sum_{A : i \in A} t_A \tag{5.3}$$

(if the norm $||\cdot||$ in (2.1) is chosen to be the maximum norm as well).

Brown and Rinott give as example the multivariate Poisson distribution $(X_1, \ldots, X_d)$ which is defined by (5.1) with Z_A Poisson with mean t_A. They also consider the $M/G/\infty$ queue (Poisson arrivals, general service time, infinite number of servers). If $X(\tau)$ denotes the number of customers in the queue at time τ, then $(X(\tau_1), \ldots, X(\tau_d))$, $0 < \tau_1 < \ldots < \tau_d$ is multivariate Poisson with $t_A \neq 0$ if A consists of consecutive numbers and $t_A = 0$ otherwise. (They give an explicit expression for t_A that involves the rate of the Poisson arrival process and the distribution of the service times.)

Let $\mathbf{X}$ and $\mathbf{Y}$ be two infinitely divisible random vectors in $\mathcal{BR}_Q$ defined by vectors $\mathbf{t}$ and $\mathbf{t}^*$ accordingly. We want to use our results to derive Slepian inequalities for $\mathbf{X}$ and $\mathbf{Y}$ based on a proper comparison of the vectors $\mathbf{t}$ and $\mathbf{t}^*$.

We start by obtaining an explicit form of (3.5) and (3.6) in this case. Choose $A_n = B_n = \{\mathbf{x} \in R^d : ||\mathbf{x}|| \leq \delta\}$ with $\delta = \delta_n > 0$ and $||\cdot||$ our usual maximum norm. It turns out to be somewhat more convenient to work with the equivalent condition (3.12) here. For every random vector $\mathbf{W} \in R^d$ we have by (5.2)

$$\int_{A_n^c} \Big(P(\mathbf{W} < \mathbf{x}) - P(\mathbf{W} < \mathbf{0})\Big)\nu_{\mathbf{X}}(d\mathbf{x}) \tag{5.4}$$

$$= \sum_{\substack{A \in \mathcal{A} \\ A \neq \emptyset}} t_A \int_{|x| > \delta} \Big[P\Big((\bigcap_{i \in A}\{W_i < x\})\bigcap(\bigcap_{i \notin A}\{W_i < 0\})\Big) - P\Big(\bigcap_{i=1}^{d}\{W_i < 0\}\Big)\Big]\mu(dx)$$

$$= \sum_{\substack{A \in \mathcal{A} \\ A \neq \emptyset}} t_A \int_{\delta}^{\infty} P\Big[\Big((\bigcap_{i \in A}\{W_i < x\})\bigcap(\bigcap_{i \notin A}\{W_i < 0\})\Big) - \Big(\bigcap_{i=1}^{d}\{W_i < 0\}\Big)\Big]\mu(dx)$$

$$- \sum_{\substack{A \in \mathcal{A} \\ A \neq \emptyset}} t_A \int_{-\infty}^{\delta} P\Big[\Big(\bigcap_{i=1}^{d}\{W_i < 0\}\Big) - \Big((\bigcap_{i \in A}\{W_i < x\})\bigcap(\bigcap_{i \notin A}\{W_i < 0\})\Big)\Big]\mu(dx)$$

$$= \sum_{\substack{B \in \mathcal{A} \\ B \neq \emptyset}} \int_{\delta}^{\infty} P\Big((\bigcap_{i \in B}\{0 \leq W_i < x\})\bigcap(\bigcap_{i \notin B}\{W_i < 0\})\Big)\mu(dx) \sum_{A : A \supseteq B} t_A$$

$$- \sum_{\substack{B \in \mathcal{A} \\ B \neq \emptyset}} \int_{-\infty}^{-\delta} P\Big((\bigcap_{i \in B}\{x \leq W_i < 0\})\bigcap(\bigcap_{i \notin B}\{W_i < x\})\Big)\mu(dx) \sum_{A : A \cap B \neq \emptyset} t_A.$$

We immediately conclude that if for all $B \in \mathcal{A}$ and $B \neq \emptyset$, the following two conditions

$$\sum_{A:A\supseteq B} t_A \geq \sum_{A:A\supseteq B} t^*_A \tag{5.5}$$

and

$$\sum_{A:A\cap B\neq\emptyset} t_A \leq \sum_{A:A\cap B\neq\emptyset} t^*_A, \tag{5.6}$$

hold, then (3.12) holds. Moreover, use (5.5) and (5.6) with B running over singletons to conclude, using (5.2) and (5.3), that (3.7) holds (with $\geq$ replaced by an equality). In the same way one can easily check that for any random vector $\mathbf{W} \in R^d$ we have

$$\int_{A_n^c} \Big(P(\mathbf{W} > \mathbf{x}) - P(\mathbf{W} > \mathbf{0})\Big)\nu_{\mathbf{X}}(d\mathbf{x}) \tag{5.7}$$

$$= \sum_{\substack{B\in\mathcal{A}\\ B\neq\emptyset}} \int_{-\infty}^{-\delta} P\Big(\big(\bigcap_{i\in B}\{x < W_i \leq 0\}\big)\bigcap\big(\bigcap_{i\notin B}\{W_i > 0\}\big)\Big)\mu(dx) \sum_{A:A\supseteq B} t_A$$

$$- \sum_{\substack{B\in\mathcal{A}\\ B\neq\emptyset}} \int_{\delta}^{\infty} P\Big(\big(\bigcap_{i\in B}\{0 < W_i \leq x\}\big)\bigcap\big(\bigcap_{i\notin B}\{W_i > x\}\big)\Big)\mu(dx) \sum_{A:A\cap B\neq\emptyset} t_A.$$

We conclude that that (5.5) and (5.6) imply (3.14) and (3.10) as well. Applying Theorem 3.1, we recover Theorem 1.1 of Brown and Rinott:

PROPOSITION 5.1 *Relations* (5.5) *and* (5.6) *imply the right and left Slepian inequalities* (3.4) *and* (3.11) *for all* $t > 0$.

Furthermore, suppose that the distribution Q is supported by $[0, \infty)$. This is well known to be equivalent to: (i) μ satisfies (2.2) and is supported by $(0, \infty)$ and (ii) $a \geq 0$ (Feller [6], XV11.3(f)). By using (5.4) and (5.3) (with B once again running over the singletons), we recover the direct part of Theorem 1.2 of Brown and Rinott:

PROPOSITION 5.2 *If the distribution Q is supported by* $[0, \infty)$, *then*
(5.5) $\Rightarrow$ (3.4) *for all* $t > 0$;
(5.6) $\Rightarrow$ (3.11) *for all* $t > 0$.

Turning to converse statements and using our Theorem 3.3 we obtain the following refinement of the results of Brown and Rinott:

PROPOSITION 5.3 *Suppose that the right Slepian inequalities* (3.4) *hold for all* $t > 0$. *Then*
$\mu((0, \infty)) > 0 \Rightarrow$ *condition* (5.5),
$\mu((-\infty, 0)) > 0 \Rightarrow$ *condition* (5.6).
Suppose that the left Slepian inequalities (3.11) *hold for all* $t > 0$. *Then*
$\mu((0, \infty)) > 0 \Rightarrow$ *condition* (5.6),
$\mu((-\infty, 0)) > 0 \Rightarrow$ *condition* (5.5).

PROOF: Setting $\mathbf{W} = \boldsymbol{\lambda}$ in the identity (5.4), we get: for any $\boldsymbol{\lambda} \in R^d - R^d_-$,

$$\nu_{\mathbf{X}}\{\mathbf{x} \in R^d : \mathbf{x} > \boldsymbol{\lambda}\} = \sum_{A:A \supseteq \mathcal{P}_\lambda} t_A \mu\{x > 0 : \ x > \max_{i \in \mathcal{P}_\lambda} \lambda_i\} \tag{5.8}$$

and for any $\boldsymbol{\lambda} < \mathbf{0}$,

$$\nu_{\mathbf{X}}\{\mathbf{x} \in R^d : \mathbf{x} \not> \boldsymbol{\lambda}\} = \sum_{\substack{B \in \mathcal{A} \\ B \neq \emptyset}} \sum_{A:A \cap B \neq \emptyset} t_A \mu\{x < 0 : \ x \leq \lambda_i \ \forall i \in B, \ x > \lambda_i \ \forall i \notin B\}. \tag{5.9}$$

We can now apply Theorem 3.3. Because of (5.8), the right Slepian inequalities (3.4) for all $t > 0$ implies (5.5) provided $\mu((0, \infty)) > 0$. It is almost as easy to see that because of (5.9), they also imply (5.6) provided $\mu((-\infty, 0)) > 0$: for a fixed $B \in \mathcal{A}$, $B \neq \emptyset$ use (5.9) with $\lambda_i \downarrow -\infty \ \forall i \notin B$ and $\lambda_i \uparrow 0 \ \forall i \in B$.

The other statements follow in a similar way if we start with (5.7) and set $\mathbf{W} = \boldsymbol{\lambda}$. ∎

Brown and Rinott [3] also discuss the extent to which (1.1) alone implies (5.5), and (1.2) alone implies (5.6), when Q is supported by $[0, \infty)$. Although we do not have a complete answer to this problem, we are able to shed some additional light. The following proposition generalizes Proposition 1.3 of Brown and Rinott and Theorem 1 of Ellis [5]: we remove the compound Poisson assumption (while retaining, in the first part, the assumption of existence of exponential moments of the Lévy measure).

PROPOSITION 5.4 *Suppose that Q is supported by $[0, \infty)$, and that $\mu((0, \infty)) > 0$.*

(i) Assume that for all $\theta > 0$

$$\int_1^\infty e^{\theta x} \mu(dx) < \infty. \tag{5.10}$$

Then the right Slepian inequality (1.1) implies (5.5).

(ii) Assume that the shift $a = 0$, and that the Lévy measure μ has slowly varying tails at 0, that is for any $r > 0$

$$\lim_{x \to 0} \frac{\mu\Big((rx, \infty)\Big)}{\mu\Big((x, \infty)\Big)} = 1. \tag{5.11}$$

Then the left Slepian inequality (1.2) implies (5.6).

PROOF: Since μ satisfies (2.2), so do $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$. We will assume, therefore, that all relevant characteristic functions are given in the form (2.3). In particular, we assume that a is the shift corresponding to the representation (2.3). Then by analogy with (5.3) we conclude that the shift vector $\mathbf{c}_{\mathbf{X}}$ is given by

$$(\mathbf{c}_{\mathbf{X}})_i = a \sum_{A:i \in A} t_A, \tag{5.12}$$

and similarily with $\mathbf{c}_{\mathbf{Y}}$. Now, choose $\boldsymbol{\theta} \geq \mathbf{0}$. For part (i), observe that (5.10) implies

$$Ee^{(\boldsymbol{\theta}, \mathbf{X})} = \exp\Big(\int_{R^d_+ - \{\mathbf{0}\}} \Big(e^{(\boldsymbol{\theta}, \mathbf{x})} - 1\Big) \nu_{\mathbf{X}}(d\mathbf{x}) + (\boldsymbol{\theta}, \mathbf{c}_{\mathbf{X}})\Big) \tag{5.13}$$

$$= \exp\Big\{ \sum_{\substack{A\in\mathcal{A}\\ A\neq\emptyset}} t_A \Big[\int_0^\infty \Big(e^{x\sum_{i\in A}\theta_i} - 1\Big)\mu(dx) + a\sum_{i\in A}\theta_i\Big)\Big]\Big\} < \infty.$$

Since (1.1) implies

$$Ee^{(\boldsymbol{\theta},\mathbf{X})} = \prod_{i=1}^d \theta_i \int_{R^d} e^{(\boldsymbol{\theta},\mathbf{x})} P(\mathbf{X} > \mathbf{x})d\mathbf{x} \geq \prod_{i=1}^d \theta_i \int_{R^d} e^{(\boldsymbol{\theta},\mathbf{x})} P(\mathbf{Y} > \mathbf{x})d\mathbf{x} = Ee^{(\boldsymbol{\theta},\mathbf{Y})},$$

we conclude by (5.13) that

$$\sum_{\substack{A\in\mathcal{A}\\ A\neq\emptyset}} t_A \Big[\int_0^\infty \Big(e^{x\sum_{i\in A}\theta_i} - 1\Big)\mu(dx) + a\sum_{i\in A}\theta_i\Big)\Big] \geq \sum_{\substack{A\in\mathcal{A}\\ A\neq\emptyset}} t_A^* \Big[\int_0^\infty \Big(e^{x\sum_{i\in A}\theta_i} - 1\Big)\mu(dx) + a\sum_{i\in A}\theta_i\Big)\Big]. \tag{5.14}$$

Take now any $B \in \mathcal{A},\ B \neq \emptyset$, and choose

$$\theta_i = \begin{cases} \theta & \text{if } i \in B \\ 0 & \text{if } i \notin B, \end{cases}$$

$\theta > 0$. Then (5.14) reduces to

$$\Big(\sum_{A\supseteq B} t_A\Big)\Big[\int_0^\infty \Big(e^{x\theta|B|} - 1\Big)\mu(dx) + a\theta|B|\Big] + \sum_{\substack{A\not\supseteq B\\ A\neq\emptyset}} t_A \Big[\int_0^\infty \Big(e^{x\theta|A\cap B|} - 1\Big)\mu(dx) + a\theta|A\cap B|\Big] \tag{5.15}$$

$$\geq \Big(\sum_{A\supseteq B} t_A^*\Big)\Big[\int_0^\infty \Big(e^{x\theta|B|} - 1\Big)\mu(dx) + a\theta|B|\Big] + \sum_{\substack{A\not\supseteq B\\ A\neq\emptyset}} t_A^* \Big[\int_0^\infty \Big(e^{x\theta|A\cap B|} - 1\Big)\mu(dx) + a\theta|A\cap B|\Big],$$

where $|A|$ stands for the cardinality of A. Setting

$$c_n(\theta) = \int_0^\infty \Big(e^{x\theta n} - 1\Big)\mu(dx) + a\theta n,\ \theta > 0,\ n = 0, 1, 2, \ldots,$$

we can rewrite (5.15) as

$$c_{|B|}(\theta)\sum_{A\supseteq B} t_A + \sum_{\substack{A\not\supseteq B\\ A\neq\emptyset}} c_{|A\cap B|}(\theta)t_A \geq c_{|B|}(\theta)\sum_{A\supseteq B} t_A^* + \sum_{\substack{A\not\supseteq B\\ A\neq\emptyset}} c_{|A\cap B|}(\theta)t_A^*.$$

Now (5.5) follows from the easily verifiable fact

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \infty,$$

by letting $\theta \uparrow \infty$.

For part (ii) we have $Ee^{-(\boldsymbol{\theta},\mathbf{X})} \geq Ee^{-(\boldsymbol{\theta},\mathbf{Y})}$, and hence an argument identical to that leading to (5.15) shows that (1.2) implies in our case that for every $\theta > 0$,

$$\sum_{A\cap B\neq\emptyset} t_A \Big[\int_0^\infty \Big(1 - e^{-x\theta|A\cap B|}\Big)\mu(dx)\Big] \leq \sum_{A\cap B\neq\emptyset} t_A^* \Big[\int_0^\infty \Big(1 - e^{-x\theta|A\cap B|}\Big)\mu(dx)\Big]. \tag{5.16}$$

Denote $b(\theta) = \int_0^\infty (1 - e^{-\theta x})\mu(dx)$, $\theta > 0$. Using (5.11) one can easily check that for every $c > 0$

$$\lim_{\theta\to\infty} \frac{b(c\theta)}{b(\theta)} = \lim_{\theta\to\infty} \frac{\int_0^\infty e^{-t}\mu\Big(\big(\frac{t}{c\theta},\infty\big)\Big)dt}{\int_0^\infty e^{-t}\mu\Big(\big(\frac{t}{\theta},\infty\big)\Big)dt} = 1.$$

Now (5.6) follows from (5.16) upon letting $\theta \uparrow \infty$. ∎

Remarks

1. If Q is supported on $[0,\infty)$ and is *compound Poisson*, then it is the distribution of the random variable $V = \sum_{i=1}^N U_i$, where the $\{U_i\}$ are i.i.d. non-negative with distribution $\mu/\mu(R_+)$ (μ is the Lévy measure of V), and N is a Poisson random variable independent of the U_i's. Then (5.10) reduces to $Ee^{\theta U_1} < \infty$ which is the assumption of Proposition 1.3 of Brown and Rinott [3] and Theorem 1 of Ellis [5].

2. For the $\mathcal{BR}_Q$ family,

$$\begin{aligned}(1.1) \;&\Rightarrow\; (5.5)\ \text{[Proposition 5.4(i)]}\\ &\Rightarrow\; (3.4)\ \text{ for all } t > 0\ \text{[Theorem 3.3]}\\ &\Rightarrow\; (3.2)\text{ and }(3.3).\end{aligned}$$

   In general, however, (1.1) does not imply (3.2) or (3.3) *even in the presence of all exponential moments*, as Example 3.1 demonstrates.

In their Theorem 1.2 (ii), Brown and Rinott [3] state that for the $\mathcal{BR}_Q$ family, conditions (1.2) and (5.6) are equivalent when Q is supported by $[0,\infty)$ without any additional assumptions. We do not find their proof convincing, and the following example seems to provide a counterexample to that statement.

EXAMPLE 5.1 Let $d = 2$ and $\mu(dx) = x^{-1}e^{-x}dx$, $x > 0$, with shift $a \geq 0$ to be chosen later. (μ is the Lévy measure of a unit mass exponential random variable.) Let

$$t_1 = t_2 = 3, \quad t_{12} = 0 \tag{5.17}$$

$$t_1^* = t_2^* = 0, \quad t_{12}^* = 5.$$

Observe that (5.6) fails for $B = \{1,2\}$. Let $\Gamma(5)$, $\Gamma_1(3)$ and $\Gamma_2(3)$ be independent random variables with Gamma (5), Gamma (3) and Gamma (3) distributions accordingly (all with the scale parameter equal to 1). Then we can represent the vectors $\mathbf{X}$ and $\mathbf{Y}$ (in law) as follows:

$$X_1 = \Gamma_1(3) + 3a, \quad X_2 = \Gamma_2(3) + 3a, \quad Y_1 = Y_2 = \Gamma(5) + 5a.$$

We claim that one can choose $a > 0$ so large that for every $\boldsymbol{\lambda} \in R^2$ (1.2) holds.

Since $Y_1 = Y_2$ a.s., the "worst case" for (1.2) is the case $\lambda_1 = \lambda_2 = \lambda$. In that case (1.2) reduces to $P(X_1 < \lambda, X_2 < \lambda) \geq P(Y_1 < \lambda)$ or, equivalently, to

$$\Big(P(\Gamma_1(3) + 3a < \lambda)\Big)^2 \geq P(\Gamma(5) + 5a < \lambda). \tag{5.18}$$

Observe that (5.18) holds trivially for every $\lambda \leq 5a$. We need, therefore, only to consider the case $\lambda > 5a$. Letting $x = \lambda - 5a > 0$ we see that we only need to exhibit an $a > 0$ for which

$$\Big(P(\Gamma_1(3) < x + 2a)\Big)^2 \geq P(\Gamma(5) < x) \tag{5.19}$$

for every $x > 0$. Choose an $x_0 > 0$ such that for every $x \geq x_0$

$$P(\Gamma(5) > x) \geq \frac{1}{48} e^{-x} x^4 \text{ and } P(\Gamma_1(3) > x) \leq e^{-x} x^2. \tag{5.20}$$

Let $x_1 = x_0 \vee \sqrt{96}$. Choose now a so large that

$$\Big(P(\Gamma_1(3) \leq 2a)\Big)^2 \geq P(\Gamma(5) \leq x_1).$$

Then (5.19) holds trivially for all $0 < x \leq x_1$, while its truth for $x > x_1$ is a simple consequence of (5.20).

References

[1] R.J. Adler. *An Introduction to Continuity, Extrema and Related Topics for General Gaussian Processes.* IMS Lecture Notes 12, Hayward, 1990.

[2] R.J. Adler, S. Cambanis and G. Samorodnitsky. Stable Markov processes. *Stoch. Proc. Appl.*, 34:1–17,1990.

[3] L.D. Brown and Y. Rinott. Inequalities for multivariate infinitely divisible processes. *Ann. Probab.*, 16:642–657, 1988.

[4] S. Cambanis. Similarities and contrasts between Gaussian and other stable signals. *Proceedings of Fifth Aachen Colloquium on Mathematical Methods in Signal Processing* (P.L. Butzer, ed.), 113-120, 1984.

[5] R. Ellis. Inequalities for multivariate compound Poisson distributions. *Ann. Probab.*, 16:658-661, 1988.

[6] W. Feller. *An Introduction to Probability Theory and Its Applications*, Vol. 2. Wiley, New York, 1966.

[7] X. Fernique. Regularité des trajectoires des fonctions aléatoires gaussiennes. In *Lecture Notes in Mathematics, Vol. 480*, pages 1–96, New York, 1975. Springer Verlag.

[8] B. Fristedt. Sample functions of stochastic processes with stationary independent increments. *Advances in Probability* 3, Marcel Dekker, New York, 1974.

[9] K. Joag-dev, M.D. Perlman, and L.D. Pitt. Association of normal random variables and Slepian's inequality. *Ann. Probab.*, 11:451–455, 1983.

[10] M. Ledoux and M. Talagrand. *Isoperimetry and Processes in Probability in Banach Spaces.* Springer-Verlag, Berlin, 1992.

[11] M.B. Marcus and G. Pisier. Characterization of almost surely continuous p-stable random Fourier series and strongly stationary processes. *Acta Math.*, 152:245–301, 1984.

[12] L.D. Pitt. Positively correlated normal variables are associated. *Ann. Probab.*, 10:496–499, 1982.

[13] G. Samorodnitsky and M.S. Taqqu. Stochastic monotonicity and Slepian-type inequalities for infinitely divisible and stable random vectors. *Ann. Probab.*, to appear.

[14] G. Samorodnitsky and M.S. Taqqu. *Stable Non-Gaussian Processes.* To appear, 1993.

[15] D. Slepian. The one-sided barrier problem for Gaussian noise. *Bell System Tech. J.*, 41:463–501, 1962.

[16] D. Surgailis, J. Rosinski, V. Mandrekar and S. Cambanis. Stable generalized moving averages. Preprint, 1991.

Gennady Samorodnitsky
School of ORIE
Cornell University
Ithaca, NY 14853
Email: gennady@orie.cornell.edu

Murad S. Taqqu
Boston University
111 Cummington Street
Boston, MA 02215-2411, USA
Email: murad@math.bu.edu