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DISTRIBUTIONS OF SUBADDITIVE FUNCTIONALS OF SAMPLE PATHS OF INFINITELY DIVISIBLE PROCESSES

by

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Distributions of subadditive functionals of sample paths of infinitely divisible processes *†‡

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Abstract

Subadditive functionals on the space of sample paths include suprema, integrals of paths, oscillation on sets, and many others. In this paper we find an optimal condition which ensures that the distribution of a subadditive functional of sample paths of an infinitely divisible process belongs to the subexponential class of distributions. Further, we give exact tail behavior for the distributions of such functionals, thus improving many recent results obtained for particular forms of subadditive functionals and for particular infinitely divisible processes.

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1 Introduction

In this paper we investigate the tail behavior of subadditive functionals of paths of infinitely divisible (i.d.) processes. We recall that a stochastic process $\mathbf{X} = \{X(t) : t \in T\}$, where T is an arbitrary index set, is said to be i.d. if all its finite dimensional distributions are i.d. The class of i.d. processes includes such processes as Gaussian, stable, Lévy and additive random fields. Important examples of i.d. processes are harmonizable, moving average, shot noise, fractional processes and others. Since the influence of the Gaussian component on the tail behavior of subadditive functionals of i.d. processes is asymptotically negligible in all the cases we consider, we restrict our study, to i.d. processes with no Gaussian component. The characteristic function of such a process $\mathbf{X}$ can be written in Lévy's form:

$$\begin{aligned} E \ \exp \ &\{i < \boldsymbol{\beta}, \mathbf{X} >\} = \\ \exp \ &\{i < \boldsymbol{\beta}, \mathbf{b} > + \int_{\mathbf{R}^T} [e^{i<\boldsymbol{\beta},\boldsymbol{\alpha}>} - 1 - i < \boldsymbol{\beta}, \boldsymbol{\tau}(\boldsymbol{\alpha}) >] \nu(d\boldsymbol{\alpha})\}, \boldsymbol{\beta} \in \mathbf{R}^{(T)}, \end{aligned} \tag{1.1}$$

where $\mathbf{b} \in \mathbf{R}^T$ and ν is the projective limit of the Lévy measures corresponding to the finite dimensional distributions of $\mathbf{X}$ (see Maruyama (1970)). Here $\mathbf{R}^{(T)}$ denotes the space of real functions $\boldsymbol{\beta}$ defined on T such that $\beta(t) = 0$ for all but finitely many $t, < \boldsymbol{\beta}, \boldsymbol{\alpha} >= \Sigma_{t \in T} \beta(t) \alpha(t)$, and $\tau(\boldsymbol{\alpha})(t) = \alpha(t)/(\alpha^2(t) + 1)$.

Our goal is to show that for a large class of i.d. processes $\mathbf{X}$ and subadditive measurable real functions ϕ defined on paths of $\mathbf{X}$, the distribution of $\phi^+(X(\cdot)) = max\{\phi(X(\cdot)), 0\}$ belongs to the class $\mathcal{S}$ of subexponential distributions, and also, to give an asymptotic evaluation for the tail of $\phi(X(\cdot))$. Natural examples of ϕ include supremum of the path, supremum of the absolute value of the path, L^p-norm (F-norm if $p < 1$) of the path, etc., therefore our results characterize the distributions of such nonlinear functionals of i.d. stochastic processes.

In Section 2 we prove that

$$P\{\phi(X(\cdot)) > x\} \sim \nu \circ \phi^{-1}((x, \infty)), \quad \text{as } x \to \infty, \tag{1.2}$$

provided $\nu \circ \phi^{-1}((x, \infty))$ is asymptotically equivalent to the tail of a distribution in the class $\mathcal{S}$. We recall that a distribution F on $[0, \infty)$ is said to belong to the subexponential class $\mathcal{S}$ if $F(x) < 1$ for every x and

$$\frac{1 - F * F(x)}{1 - F(x)} \to 2, \quad \text{as } x \to \infty. \tag{1.3}$$

The class $\mathcal{S}$ contains the distributions with regularly and slowly varying tails, log-normal distributions (see Embrechts et al. (1979)); other examples of $F \in \mathcal{S}$ can be obtained from a theorem of Pitman (1980), e.g. $F(x) \sim 1 - exp\{x(logx)^{-m}\}, m > 0, x \to \infty$, belongs to $\mathcal{S}$. Our result (1.2) can also be viewed as a generalization to the multidimensional (in fact, infinite-dimensional) case of a result of Embrechts et al. (1979) given for a positive random variable and ϕ being the identity mapping.

Most of the examples of i.d. processes can be obtained by the means of a stochastic integral

$$X(t) = \int_S f_t(s)M(ds), \, t \in T, \tag{1.4}$$

where M is an independently scattered i.d. random measure on a certain set S and, for each t, f_t is a deterministic function on S. In this case the measure ν and the shift $\mathbf{b}$ in (1.1) can be given explicitly in the terms of $\{f_t\}_{t\in T}$ and the parameters of M. This fact enables us to apply result (1.2) to processes represented by (1.4) to get some more explicit results in the cases of stable processes, ξ - radial processes and others (see Section 3).

In Section 4 we address the question when our assumption on the asymptotic behavior of $\nu \circ \phi^{-1}((x,\infty))$ can be easily verified. We show that in certain interesting cases of processes given by (1.4) this question reduces to another one: when does the product of two independent random variables have the distribution in the class $\mathcal{S}$? We conclude this paper quoting a result from a forthcoming paper by Cline and Samorodnitsky that provides a partial answer to the later question.

Finally, we should mention something about the methods in the paper. They combine certain techniques developed in the study of probabilities in Banach spaces with some standard methods in the study of subexponential distributions.

2 The main result.

Let $\mathbf{X} = \{X(t) : t \in T\}$ be an i.d. process determined by (1.1). In this section we shall assume that T is a countable set, therefore ν in (1.1) is a σ-finite Borel measure on $\mathbf{R}^T$. Let $\phi : \mathbf{R}^T \to (-\infty, \infty]$ be a measurable subadditive function, i.e.

$$\phi(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2) \le \phi(\boldsymbol{\alpha}_1) + \phi(\boldsymbol{\alpha}_2) \;\text{ for every } \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathbf{R}^T. \tag{2.1}$$

Further we shall assume that there exists a lower-semicontinuous pseudonorm $q : \mathbf{R}^T \to [0,\infty]$ such that

$$|\phi(\boldsymbol{\alpha})| \le q(\boldsymbol{\alpha}) \;\text{ for every } \boldsymbol{\alpha} \in \mathbf{R}^T \tag{2.2}$$

(recall that a function $q : \mathbf{F} \to [0,\infty]$, where $\mathbf{F}$ is a linear space, is said to be a pseudonorm if $q(x+y) \le q(x)+q(y), q(0) = 0$, and $q(cx) \le q(x)$ for all $x, y \in \mathbf{F}, |c| \le 1$). Define $H : (0,\infty) \to [0,\infty]$ by

$$H(x) = \nu(\{\boldsymbol{\alpha} \in \mathbf{R}^T : \phi(\boldsymbol{\alpha}) > x\}).$$

THEOREM 2.1 *Let* $\mathbf{X}$ *and* ϕ *satisfy (1.1), (2.1) and (2.2). Assume* $P\{q(X(\cdot)) < \infty\} = 1$ *and suppose that the distribution function* $F(x) = 1 - min\{H(x), 1\}$ *belongs to the subexponential class* $\mathcal{S}$ *(recall (1.3)). Then the distribution of* $\phi^+(X(\cdot)) = max\{\phi(X(\cdot)), 0\}$ *belongs to* $\mathcal{S}$ *and*

$$\lim_{x\to\infty} \frac{P\{\phi(X(\cdot)) > x\}}{H(x)} = 1.$$

Remark Under the conditions of Theorem 2.1 F is a nondefective distribution; see Lemma 2.1

The proof of Theorem 2.1 is preceded by a proposition and two lemmas. In the proposition we state several properties of the distributions from $\mathcal{S}$ for the sake of convenient reference in the sequel; we refer the reader to Embrechts et al. (1979) for the proofs of these properties.

PROPOSITION 2.1 *Let* $F \in \mathcal{S}$ *and put* $\overline{F}(x) = 1 - F(x)$. *Then*

(i) $\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1$ *uniformly in* y *over compact sets;*

(ii) $\lim_{x\to\infty} e^{\epsilon x}\overline{F}(x) = \infty$, *for each* $\epsilon > 0$;

(iii) If $\lim_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} = c \in (0, \infty)$, *where* G *is a distribution function on* $[0, \infty)$, *then* $G \in \mathcal{S}$;

(iv) If $\overline{G}(x) = o(\overline{F}(x))$ *as* $x \to \infty$, *where* G *is a distribution function on* $[0, \infty)$, *then* $F * G \in \mathcal{S}$ *and* $\lim_{x\to\infty} \frac{\overline{F*G}(x)}{\overline{F}(x)} = 1$;

(v) If $G(x) = e^{-\lambda}\Sigma_{n=0}^{\infty} \frac{\lambda^n}{n!} F^{*n}(x)$, *then* $\lim_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} = \lambda$.

The next two lemmas are well-known in the case when q is a norm on a Banach space. Since, in our case, q is only semicontinuous and non-necessarily homogeneous pseudonorm on $\mathbf{R}^T$, we provide complete proofs of these lemmas. Notice that the conclusion of the first lemma is true only for some $r_0 > 0$; an example showing that this is not true for all $r_0 > 0$ can easily be constructed.

LEMMA 2.1 *Let* $\mathbf{X}$ *be given by (1.1) and suppose that* $P\{q(X(\cdot)) < \infty\} = 1$. *Then there exists* $r_0 > 0$ *such that* $\nu(\{\boldsymbol{\alpha} \in \mathbf{R}^T : q(\boldsymbol{\alpha}) > r_0\}) < \infty$.

Proof. Since $\{X(t)\}_{t\in T}$ is a countable sequence of random variables, there exists $\boldsymbol{a} : T \to (0, \infty)$ such that $\Sigma_{t\in T}|a(t)X(t)|^2 < \infty$ a.s.. Therefore $\boldsymbol{a}\mathbf{X} = (a(t)X(t))_{t\in T}$ is an $\ell^2(T)$-valued i.d. random variable. The $\ell^2(T)$-valued random variable $\boldsymbol{a}(\mathbf{X} - \mathbf{X}')$, where $\mathbf{X}'$ is an independent copy of $\mathbf{X}$, is i.d. symmetric with the Lévy measure given by

$$\mu(A) = \nu(\{\boldsymbol{\alpha} \in \mathbf{R}^T : \boldsymbol{a\alpha} \in A\}) + \nu(\{\boldsymbol{\alpha} \in \mathbf{R}^T : \boldsymbol{a\alpha} \in -A\}), \tag{2.3}$$

for every Borel set A in $\ell^2(T)$. Let now $\{\mathbf{Z}(u) : u \geq 0\}$ be an $\ell^2(T)$-valued stationary independent increment process with $\mathbf{Z}(1) \stackrel{d}{=} \boldsymbol{a}(\mathbf{X} - \mathbf{X}')$. Let $p(\mathbf{h}) = q(\boldsymbol{a}^{-1}\mathbf{h}), \mathbf{h} \in \ell^2(T)$; p is a lower-semicontinuous pseudonorm on $\ell^2(T)$ and $P\{p(\mathbf{Z}(1)) < \infty\} = 1$. Choose now a decreasing to zero sequence $m_1 > m_2 > \ldots$ such that

$$\mu\{\mathbf{h} : \|\mathbf{h}\|_{\ell^2(T)} = m_i\} = 0 \text{ for every } i = 1, 2, \ldots. \tag{2.4}$$

By the lower-semicontinuity of p, the set

$$A_i = \{\mathbf{h} \in \ell^2(T) : m_i < \|\mathbf{h}\|_{\ell^2(T)} < m_{i-1}, p(\mathbf{h}) > r\}$$

is open, for every i ($m_0 = \infty$), and contained in $\{\mathbf{h} : \|\mathbf{h}\|_{\ell^2(T)} \geq m_i\}$. Thus, for each $i \geq 1$,

$$\underline{\lim}_{n\to\infty} nP\{\mathbf{Z}(n^{-1}) \in A_i\} \geq \mu(A_i).$$

By (2.4) and Fatou's lemma we now get

$$\begin{aligned} \mu(\{\mathbf{h} \in \ell^2(T) : p(\mathbf{h}) > r\}) &= \Sigma_{i=1}^{\infty} \mu(A_i) \\ &\leq \underline{\lim}_{n\to\infty} n \Sigma_{i=1}^{\infty} P\{\mathbf{Z}(n^{-1}) \in A_i\} \\ &= \underline{\lim}_{n\to\infty} nP\{p(\mathbf{Z}(n^{-1})) > r\}. \end{aligned} \tag{2.5}$$

On the other hand, using Lévy inequality adopted for pseudonorms, we obtain

$$\begin{aligned} 1 - [1 - P\{p(\mathbf{Z}(n^{-1})) > r\}]^n &= P\{\sup_{1\leq j\leq n} p(\mathbf{Z}(jn^{-1}) - \mathbf{Z}((j-1)n^{-1})) > r\} \\ &\leq P\{\sup_{1\leq j\leq n} p(\mathbf{Z}(jn^{-1})) > r/2\} \\ &\leq 2P\{p(\mathbf{Z}(1)/2) > r/4\} \\ &\leq 2P\{p(\mathbf{Z}(1)) > r/4\}. \end{aligned}$$

This implies that if $r_0 > 0$ is such that $P\{p(\mathbf{Z}(1)) > r_0/4\} < 1/8$, then we have

$$nP\{p(\mathbf{Z}(n^{-1})) > r_0\} \leq \log(8/7), \text{ for all} n = 1, 2, \ldots.$$

Combining the above bound with (2.5) yields $\mu(\{\mathbf{h} \in \ell^2(T) : p(\mathbf{h}) > r_0\}) \leq \log(8/7)$. In view of definitions of μ and p, the proof is complete. □

In the proof of the next lemma we adapt a technique of deAcosta (1980) who has proven a stronger result for Banach space valued random variables (with q being a norm). Due to the non-homogeneity of q, which invalidates a standard use Jensen's inequality (cf. the proof of Lemma 2.2 in deAcosta (1980)), and because q is only semicontinuous, this adaptation is not immediate.

LEMMA 2.2 *Let* $\mathbf{X}$ *be given by (1.1), and suppose that* $P\{q(X(\cdot)) < \infty\} = 1$ *and* $\nu(\{\boldsymbol{\alpha} \in \mathbf{R}^T : q(\boldsymbol{\alpha}) > r_0\}) = 0$ *for some* $\tau_0 > 0$. *Then* $E\exp(\epsilon q(X(\cdot))) < \infty$ *for some* $\epsilon > 0$.

Proof. We transform the problem to $\ell^2(T)$-valued random variables in the same way as in the proof of Lemma 2.1. Having μ, p, and the process $\{\mathbf{Z}(u) : u \geq 0\}$ already defined, let μ_δ denote the restriction of μ to the set $\{\mathbf{h} \in \ell^2(T) : \|\mathbf{h}\|_{\ell^2(T)} > \delta\}, \delta > 0$, and let $\{\mathbf{Z}_\delta(u) : u \geq 0\}$ be an $\ell^2(T)$-valued stationary independent increment process such that $\mathbf{Z}_\delta(1)$ has symmetric i.d. distribution with Lévy measure μ_δ and with no Gaussian component. Since $\mathcal{L}(\mathbf{Z}_\delta(1)) \Rightarrow \mathcal{L}(\mathbf{Z}(1))$, as $\delta \downarrow 0$, and p is lower semicontinuous, we have

$$E\exp(\epsilon p(\mathbf{Z}(1))) \leq \underline{\lim}_{\delta\to 0} E\exp(\epsilon p(\mathbf{Z}_\delta(1))), \tag{2.6}$$

for each $\epsilon > 0$. Fix now $\delta > 0$, and define

$$\mathbf{Y}_{nj} = \begin{cases} \mathbf{Z}_\delta(jn^{-1}) - \mathbf{Z}_\delta((j-1)n^{-1}) & \text{if } p(\mathbf{Z}_\delta(jn^{-1}) - \mathbf{Z}_\delta((j-1)n^{-1})) < 2r_0 \\ 0 & \text{otherwise.} \end{cases}$$

Put $\mathbf{S}_{nm} = \Sigma_{j=1}^m \mathbf{Y}_{nj}$, $m = 1, \ldots, n$. We claim that

$$\mathbf{S}_{nn} \to \mathbf{Z}_\delta(1) \text{ in probability, as } n \to \infty. \tag{2.7}$$

Indeed, for every $\theta > 0$,

$$\begin{aligned} P\{\|\mathbf{S}_{nn} - \mathbf{Z}_\delta(1)\| > \theta\} &\leq P\{\mathbf{S}_{nn} \neq \mathbf{Z}_\delta(1)\} \\ &\leq 1 - P\{p(\mathbf{Z}_\delta(n^{-1})) < 2r_0\}^n, \end{aligned} \tag{2.8}$$

and, setting $\lambda_n = n^{-1}\mu_\delta(\ell^2(T))$, we get

$$\begin{aligned} P\{p(\mathbf{Z}_\delta(n^{-1}) \geq 2r_0\} &= e^{-\lambda_n}\Sigma_{k=0}^\infty \frac{1}{k!n^k}\mu_\delta^{*k}\{p(\mathbf{h}) \geq 2r_0\} \\ &= e^{-\lambda_n}\Sigma_{k=0}^\infty \frac{1}{k!n^k}\underbrace{\mu_\delta \times \ldots \times \mu_\delta}_{k}\{p(\mathbf{h}_1 + \ldots + \mathbf{h}_k) \geq 2r_0\} \\ &\leq e^{-\lambda_n}\Sigma_{k=0}^\infty \frac{1}{k!n^k}\underbrace{\mu_\delta \times \ldots \times \mu_\delta}_{k}\{p(\mathbf{h}_1) + \ldots + p(\mathbf{h}_k) \geq 2r_0\} \\ &\leq e^{-\lambda_n}\Sigma_{k=2}^\infty \frac{1}{k!n^k}[\mu_\delta(\ell^2(T)\}]^k, \end{aligned}$$

because $\mu_\delta\{p(\mathbf{h}) > r_0\} = 0$. We infer from the above bound that

$$\begin{aligned} nP\{p(\mathbf{Z}_\delta(n^{-1})) \geq 2r_0\} &\leq ne^{-\lambda_n}[e^{\lambda_n} - 1 - \lambda_n] \\ &= n[1 - e^{-\lambda_n} - \lambda_n e^{-\lambda_n}] \to 0, \end{aligned}$$

as $n \to \infty$. This implies that the last term in (2.8) converges to 0, proving (2.7). By the lower semicontinuity of p we get, for each $\epsilon > 0$,

$$E \exp(\epsilon p(\mathbf{Z}_\delta(1))) \leq \underline{\lim}_{n\to\infty} E \exp(\epsilon p(\mathbf{S}_{nn})). \tag{2.9}$$

Now fix $n \geq 1$ (δ is fixed as well) and define

$$\begin{aligned} A_j &= \{p(\mathbf{S}_{ni}) \leq r, i = 1, \ldots, j-1, p(\mathbf{S}_{nj}) > r\},\ j = 1, \ldots, n; \\ B_m &= \cup_{j=1}^m A_j, m = 1, \ldots, n; \\ M &= \sup_{1\leq m\leq n} E \exp(\epsilon p(\mathbf{S}_{nm})), \end{aligned}$$

where $r, \epsilon > 0$ will be specified later. We have

$$\begin{aligned} E[\exp(\epsilon p(\mathbf{S}_{nm}))1_{B_m}] &= \Sigma_{j=1}^m E[\exp(\epsilon p(\mathbf{S}_{nm}))1_{A_j}] \\ &\leq \Sigma_{j=1}^m E[\exp(\epsilon p(\mathbf{S}_{nj}))1_{A_j}] E \exp(\epsilon p(\mathbf{S}_{nm} - \mathbf{S}_{nj})), \end{aligned}$$

by the subadditivity of p and independence, and since the terms in $\mathbf{S}_{nm}$ are i.i.d., the last expression is

$$\leq \Sigma_{j=1}^m E[\exp(\epsilon p(\mathbf{S}_{nj}))1_{A_j}]M.$$

Since on A_j we have $p(\mathbf{S}_{nj}) \leq p(\mathbf{S}_{nj-1}) + p(\mathbf{Y}_{nj}) \leq r + 2r_0$, we obtain the bound $E[\exp(\epsilon p(\mathbf{S}_{nm}))1_{B_m}] \leq \exp(\epsilon(r + 2r_0))P(B_m)M$, and because $p(\mathbf{S}_{nm}) \leq r$ on B_m^c, we conclude that

$$E \exp(\epsilon p(\mathbf{S}_{nm})) \leq \exp(\epsilon(r + 2r_0))P(B_m)M + \exp(\epsilon r),\ m = 1, \ldots, n.$$

Using the definition of M, we deduce from the above inequality that

$$M \leq \exp(\epsilon(r + 2r_0))P(B_n)M + \exp(\epsilon r). \tag{2.10}$$

Define now $\mathbf{V}_{nj}, j = 1, \ldots, n$, such that

$$\mathbf{Y}_{nj} + \mathbf{V}_{nj} = \mathbf{Z}_\delta(jn^{-1}) - \mathbf{Z}_\delta((j-1)n^{-1}).$$

Using the symmetry of the set $\{\mathbf{h} : p(\mathbf{h}) < 2r_0\}$ we infer that

$$\mathbf{Y}_{nj} - \mathbf{V}_{nj} \stackrel{d}{=} \mathbf{Y}_{nj} + \mathbf{V}_{nj}.$$

Hence we get

$$\begin{aligned} P(B_n) &= P\{\sup_{1\leq k\leq n} p(\Sigma_{j=1}^k \mathbf{Y}_{nj}) > r\} \\ &\leq 2P\{\sup_{1\leq k\leq n} p(\Sigma_{j=1}^k (\mathbf{Y}_{nj} + \mathbf{V}_{nj})) > r/2\} \\ &\leq 4P\{p(\mathbf{Z}_\delta(1)) > r/4\}, \end{aligned}$$

by Lévy's inequality. Using Lévy's inequality again we have

$$P\{p(\mathbf{Z}_\delta(1)) > r/4\} \le 2P\{p(\mathbf{Z}(1)) > r/8\}$$

for all $\delta > 0$. Choose now $r > 0$ such that $P\{p(\mathbf{Z}(1)) > r/8\} \le 1/32$. Then

$$P(B_n) \le \frac{1}{4} \text{ for all } n \in \mathbf{N},$$

which, combined with (2.10), yields

$$M \le 4^{-1} \exp(\epsilon(r + 2r_0))M + \exp(\epsilon r). \tag{2.11}$$

Choose now $\epsilon > 0$ such that $\exp(\epsilon(r + 2r_0)) \le 2$. We get by (2.11)

$$M \le 2^{-1}M + \exp(\epsilon r) \le 2^{-1}M + 2,$$

therefore

$$E \exp(\epsilon p(\mathbf{S}_{nn})) \le M \le 4 \text{ for every } n \in N.$$

In view of (2.9),

$$E \exp(\epsilon p(\mathbf{Z}_\delta(1))) \le 4 \text{ for every } \delta > 0,$$

and by (2.6) and the definitions of p and $\mathbf{Z}(1)$,

$$E \exp\{\epsilon q(X(\cdot) - X'(\cdot))\} = E \exp\{\epsilon p(\mathbf{Z}(1))\} \le 4.$$

Now a standard use of Fubini's theorem completes the proof of the lemma.

Proof of Theorem 2.1. By Lemma 2.1 there exists $r_0 > 0$ such that $\nu(\{\boldsymbol{\alpha} : q(\boldsymbol{\alpha}) > r_0\}) < \infty$. Hence by (2.2)

$$H(x) \le \nu(\{\boldsymbol{\alpha} : q(\boldsymbol{\alpha}) > x\}) < \infty, \quad \text{for every } x \ge r_0.$$

Consider now the following decomposition of ν :

$$\nu = \nu_1 + \nu_2 + \nu_3,$$

where

$$\begin{aligned}
\nu_1(A) &= \nu(A \cap \{\boldsymbol{\alpha} : \phi(\boldsymbol{\alpha}) > r_0\}), \\
\nu_2(A) &= \nu(A \cap \{\boldsymbol{\alpha} : q(\boldsymbol{\alpha}) > r_0, \phi(\boldsymbol{\alpha}) \le r_0\}), \\
\nu_3(A) &= \nu(A \cap \{\boldsymbol{\alpha} : q(\boldsymbol{\alpha}) \le r_0\}).
\end{aligned}$$

Notice that ν_1 and ν_2 are finite measures. Let $\mathbf{X}_j = \{X_j(t) : t \in T\}, j = 1, 2, 3$ be independent stochastic processes such that, for $j = 1, 2$,

$$E\exp\{i < \boldsymbol{\beta}, \mathbf{X}_j >\} = \exp\{\int_{\mathbf{R}^T}[e^{i<\boldsymbol{\beta},\boldsymbol{\alpha}>} - 1]\nu_j(d\boldsymbol{\alpha})\} \tag{2.12}$$

and

$$E\exp\{i < \boldsymbol{\beta}, \mathbf{X}_3 >\} = \exp\{i < \boldsymbol{\beta}, \mathbf{b}_1 > + \int_{\mathbf{R}^T}[e^{i<\boldsymbol{\beta},\boldsymbol{\alpha}>} - 1 - i < \boldsymbol{\beta}, \tau(\boldsymbol{\alpha}) >]\nu_3(d\boldsymbol{\alpha})\},$$

where $\mathbf{b_1}$ is chosen such that

$$\mathbf{X} \stackrel{d}{=} \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3,$$

cf. (1.1). We shall first show that

$$P\{\phi(\mathbf{X}_1) > x\} \sim H(x) \text{ as } x \to \infty. \tag{2.13}$$

Set $\lambda = \nu_1(\mathbf{R}^T) < \infty$ and let $\{\mathbf{Y}_j\}$ be an i.i.d. sequence in $\mathbf{R}^T$ with the common distribution $\lambda^{-1}\nu_1$. By (2.12) $\mathbf{X}_1$ can be represented as

$$\mathbf{X}_1 \stackrel{d}{=} \Sigma_{j=1}^N \mathbf{Y}_j, \tag{2.14}$$

where N is independent of $\{\mathbf{Y}_j\}$ Poisson random variables with parameter λ. We have, by the subadditivity of ϕ and Proposition 2.1(v),

$$\begin{aligned} P\{\phi(\mathbf{X}_1) > x\} &= P\{\phi(\Sigma_{j=1}^N \mathbf{Y}_j) > x] \\ &\leq P\{\Sigma_{j=1}^N \phi^+(\mathbf{Y}_j) > x\} \sim \lambda P\{\phi^+(\mathbf{Y}_1) > x\}, \end{aligned}$$

as $x \to \infty$. Indeed,

$$P\{\phi^+(\mathbf{Y}_1) > x\} = \lambda^{-1}\nu_3(\{\boldsymbol{\alpha} : \phi^+(\boldsymbol{\alpha}) > x\}) = \lambda^{-1}H(x), \tag{2.15}$$

for $x \geq r_0$, thus the distribution of $\phi^+(\mathbf{Y}_1)$ belongs to $\mathcal{S}$ by Proposition 2.1 (iii) and the assumption of the theorem. We have shown that

$$\overline{\lim}_{x\to\infty} \frac{P\{\phi(\mathbf{X}_1) > x\}}{H(x)} \leq 1. \tag{2.16}$$

To obtain a lower bound we proceed as follows. Fix $n \geq 1$, $M \in (0, \infty)$ and consider

$$A_j(x) = \{\phi^+(\mathbf{Y}_j) > x + (n-1)M,\ q(\mathbf{Y}_i) \leq M \text{ for all } i \neq j, i = 1, \ldots, n\},$$

where $x > M$. Set $A_1(x), \ldots, A_n(x)$ are pairwise disjoint and, on each $A_j(x)$, we have

$$\begin{aligned} \phi^+(\Sigma_{i=1}^n \mathbf{Y}_i) &\geq \phi^+(\mathbf{Y}_j) - \Sigma_{i\neq j}\phi^+(-\mathbf{Y}_i) \\ &> x + (n-1)M - \Sigma_{i\neq j} q(\mathbf{Y}_i) \geq x. \end{aligned}$$

Thus we get

$$\begin{aligned}\underline{\lim}_{x\to\infty}\frac{P\{\phi^+(\Sigma_{i=1}^n \mathbf{Y}_i) > x\}}{H(x)} &\geq n\underline{\lim}_{x\to\infty}\frac{P(A_1(x))}{H(x)} \\ &= n\underline{\lim}_{x\to\infty}\frac{P\{\phi^+(\mathbf{Y}_1) > x+(n-1)M\}}{H(x)}[P(q(\mathbf{Y}_1)\leq M)]^{n-1} \\ &= n\lambda^{-1}[P\{q(\mathbf{Y}_1)\leq M\}]^{n-1},\end{aligned}$$

by (2.15) and Proposition 2.1 (i). Letting $M\to\infty$ we obtain

$$\underline{\lim}_{x\to\infty}\frac{P\{\phi^+(\Sigma_{i=1}^n \mathbf{Y}_i) > x\}}{H(x)} \geq n\lambda^{-1}.$$

Hence, using Fatou's lemma, we have

$$\underline{\lim}_{x\to\infty}\frac{P\{\phi^+(\Sigma_{i=1}^N \mathbf{Y}_i) > x\}}{H(x)} \geq \Sigma_{n=1}^\infty e^{-\lambda}\frac{\lambda^n}{n!}\underline{\lim}_{x\to\infty}\frac{P\{\phi^+(\Sigma_{i=1}^n \mathbf{Y}_i) > x\}}{H(x)} \geq 1.$$

This completes the proof of (2.13).

Now we shall prove that

$$P\{\phi^+(\mathbf{X}_2) > x\} = o(H(x)), \text{ as } x\to\infty. \tag{2.17}$$

In view of (2.12), $\mathbf{X}_2$ can be represented in the form (2.14), where $\mathbf{Y}_j$'s have now the common distribution $\lambda^{-1}\nu_2$ and $\lambda = \nu_2(\mathbf{R}^T) < \infty$. By the definition of ν_2, $\phi(\mathbf{Y}_j)\leq r_0$ a.s. Hence

$$\begin{aligned}P\{\phi(\mathbf{X}_2) > x\} &= P\{\phi(\Sigma_{j=1}^N \mathbf{Y}_j) > x\} \\ &\leq P\{r_0 N > x\} \leq \exp(-r_0^{-1}x)\psi(1),\end{aligned}$$

where ψ is the moment generating function of N. (2.17) follows by Proposition 2.1 (ii).

Now we shall show that

$$P\{\phi^+(\mathbf{X}_3) > x\} = o(H(x)), \text{ as } x\to\infty. \tag{2.18}$$

First we notice that $P\{q(\mathbf{X}_3) < \infty\} = 1$. Indeed, assume to the contrary that $P\{q(\mathbf{X}_3) = \infty\} > 0$. Since $P\{\mathbf{X}_1 = \mathbf{X}_2 = 0\} > 0$, by the independence we get

$$0 < P\{\mathbf{X}_1 = 0, \mathbf{X}_2 = 0, q(\mathbf{X}_3) = \infty\} \leq P\{q(\mathbf{X}) = \infty\},$$

which contradicts the assumption of our theorem. Therefore $\mathbf{X}_3$ satisfies all the assumptions of Lemma 2.2 (with $\nu = \nu_3$) and by that lemma we have that $E\exp(\epsilon q(\mathbf{X}_3)) < \infty$ for some $\epsilon > 0$. Since

$$
\begin{aligned}
P\{\phi^+(\mathbf{X}_3) > x\} &\le P\{q(\mathbf{X}_3) > x\} \\
&\le e^{-\epsilon x} E \exp(\epsilon q(\mathbf{X}_3)),
\end{aligned}
$$

Proposition 2.1 (ii) yields (2.18).

Now we can complete the proof of the theorem. In view of (2.13) and Proposition 2.1 (iii), the distribution of $\phi^+(\mathbf{X}_1)$ is subexponential. Therefore by Proposition 2.1 (iv) we get

$$
\begin{aligned}
P\{\phi(\mathbf{X}) > x\} &\le P\{\phi^+(\mathbf{X}_1) + \phi^+(\mathbf{X}_2) + \phi^+(\mathbf{X}_3) > x\} \\
&\sim P\{\phi^+(\mathbf{X}_1) > x\},
\end{aligned}
$$

as $x \to \infty$. This establishes

$$
\overline{\lim}_{x\to\infty} \frac{P\{\phi(\mathbf{X}) > x\}}{H(x)} \le 1.
$$

Now, using again subadditivity of ϕ, Fatou's lemma and Proposition 2.1 (i) we get

$$
\begin{aligned}
\underline{\lim}_{x\to\infty} \frac{P\{\phi(\mathbf{X}) > x\}}{H(x)} &\ge \underline{\lim}_{x\to\infty} \frac{P\{\phi(\mathbf{X}_1) - \phi(-\mathbf{X}_2 - \mathbf{X}_3) > x\}}{H(x)} \\
&\ge \int_{\mathbf{R}^T} \underline{\lim}_{x\to\infty} \frac{P\{\phi(\mathbf{X}_1) > x + \phi(\boldsymbol{\alpha})\}}{H(x)} \mu(d\boldsymbol{\alpha}) = 1,
\end{aligned}
$$

where μ is the distribution of $-\mathbf{X}_2 - \mathbf{X}_3$. This completes the proof of Theorem 2.1. $\square$

3 Infinitely divisible processes given by a stochastic integral.

In this section we apply the results of Section 2 to i.d. stochastic processes; it is the possibility of this application that motivated the present research. Specifically, we consider i.d. processes given in the form

$$
X(t) = \int_S f(t,s) M(ds), \; t \in T, \tag{3.1}
$$

where $(\mathcal{S}, \mathcal{A})$ is a measurable space and M is an i.d. random measure on $(\mathcal{S}, \mathcal{A})$ with Lévy measure F and shift measure ν_0. That is, F is a σ-finite measure on $(S \times \mathbf{R}, \mathcal{A} \times \mathcal{B})$ and ν_0 is a σ-finite signed measure on $(\mathcal{S}, \mathcal{A})$. The random measure M is a stochastic process of the type $\{M(A), A \in \mathcal{A}_0\}$, where

$$\mathcal{A}_0 = \{A \in \mathcal{A} : \lambda(A) := |\nu_0|(A) + \int_A \int_{\mathbf{R}} \min(1, x^2) F(ds, dx) < \infty\},$$

such that M is independently scattered (i.e. for any disjoint $\mathcal{A}_0$ sets $A_1, \ldots, A_n$, $M(A_1), \ldots, M(A_n)$ are independent), σ-additive (i.e. for any disjoint $\mathcal{A}_0$ sets $A_1, A_2, \ldots$ such that $\cup_{i=1}^{\infty} A_i \in \mathcal{A}_0$ we have $M(\cup_{i=1}^{\infty} A_i) = \Sigma_{i=1}^{\infty} M(A_i)$ a.s.) and for every $A \in \mathcal{A}_0$, $M(A)$ is a real i.d. random variable with

$$E \exp(i\theta M(A)) = \exp\{i\theta\nu_0(A) + \int_A \int_{\mathbf{R}} (e^{i\theta x} - 1 - i\theta\tau(x)) F(ds, dx)\}, \tag{3.2}$$

where $\tau(x) = x/(1 + x^2)$.

We refer the reader to Rajput and Rosinski (1987) for more details on i.d. random measures and on conditions on the kernel $f(t,s)$ in (3.1) ensuring that the stochastic integral is well defined. We record at this time that the Lévy measure ν of the i.d. process $\{X(t), t \in T\}$ is given by $\nu = F \circ V^{-1}$, where $V : S \times \mathbf{R} \longrightarrow \mathbf{R}^T$ is defined by $T(s, x) = \{xf(t, s), t \in T\}$.

The following theorem follows immediately from Theorem 2.1.

THEOREM 3.1 *Let $\{X(t), t \in T\}$ be an i.d. process given by (3.1), where T is a countable set. Let ϕ and q satisfy (2.1), (2.2) and let $P\{q(X(\cdot)) < \infty\} = 1$. Define $H(y) = F(\{(s, x) \in S \times \mathbf{R} : \phi(xf(\cdot, s)) > y\}), y > 0$. If the distribution function $1 - min\{H(y), 1\}$, $y > 0$, belongs to the subexponential class $\mathcal{S}$, then the distribution of $\phi^+(X(\cdot))$ is in $\mathcal{S}$ and $P(\phi(X(\cdot)) > y) \sim H(y)$, as $y \to \infty$.*

Example. *Lévy motion.* Let $\mathbf{X} = \{X(t) : 0 \le t \le 1\}$ be a stationary independent increment process without Gaussian component and let ρ be the Lévy measure of $X(1)$. Clearly

$$X(t) = \int_0^1 1_{(0,t]}(s) M(ds), \; 0 \le t \le 1,$$

where M is a random measure induced by X and $F = Leb \otimes \rho$. Berman (1986) has proved that if $\mathbf{X}$ is also symmetric and the right tail of the Lévy measure ρ, $\rho((y, \infty))$, is regular varying of index -α, $0 < \alpha < 2$, then, as $y \to \infty$,

$$P(\sup_{0 \le t \le 1} X(t) > y) \sim P(X(1) > y) \sim \rho((y, \infty)).$$

Using Theorem 3.1 with

$$\phi(\boldsymbol{\alpha}) = \sup_{\substack{t \in [0,1] \\ t \text{ rational}}} \alpha(t), \quad q(\boldsymbol{\alpha}) = \sup_{\substack{t \in [0,1] \\ t \text{ rational}}} |\alpha(t)|$$

and computing easily $H(y) = \rho((y, \infty))$, we extend immediately the above asymptotic equivalences to all (not necessarily symmetric) Lévy processes for which the right tail of the Lévy measures $\rho, \rho((y,\infty))$, are subexponential. (It has been shown in Willekens (1987) that the first part of the equivalence above extends, actually to all Lévy processes with Lévy measure with a "long" right tail (i.e. $\lim_{y\to\infty} \rho((y+L,\infty))/\rho((y,\infty)) = 1$ for every $L > 0$). $\square$

Since the shift measure ν_0 does not enter either the condition or the conclusion of Theorem 3.1 (other than its role in the assumption that the integrals in (3.1) are well defined), we will assume in the remainder of the paper that, unless specified otherwise, $\nu_0 = 0$.

The structure of the i.d. random measure M and of the i.d. stochastic process $\{X(t),\ t \in T\}$ becomes, usually, more transparent when we represent the measure F in the form

$$F(A \times B) = \int_A \rho(s,B)\lambda(ds),\ A \in \mathcal{A}, B \in \mathcal{B}, \tag{3.3}$$

where λ is a probability measure on $(S,\mathcal{A})$ and $\rho(s,\cdot)$, $s \in S$ is a family of Lévy measures on R. We regard λ and $\rho(s,\cdot)$, $s \in S$ as a parametric representation of the random measure M. This representation is, obviously, not unique, and we refer (adding, therefore, another usage to the name) to the probability measure λ in (3.3) as a control measure of the random measure M.

An important class of functionals ϕ consists of *homogeneous* functionals, satisfying, in addition to (2.1),

$$\phi(c\boldsymbol{\alpha}) = c\phi(\boldsymbol{\alpha}),\ c > 0,\ \boldsymbol{\alpha} \in \mathbf{R}^T.$$

Examples of such functionals ϕ include seminorms on $\mathbf{R}^T$ (e.g. $\|\cdot\|_p$, $1 \le p \le \infty$), $\sup_{t\in T}$, $\overline{\lim}_{n\to\infty}\alpha(t_n)$, etc.

For homogeneous functionals ϕ we now proceed to develop conditions sufficient for the conclusion of Theorem 3.1 in terms of a particular parametrization of the random measure M. It will be seen subsequently that these conditions are more explicit in the sense that they exhibit explicitly the role of the kernel $f(t,s)$ in (3.1) and the parameters λ and $\rho(s,\cdot), s \in S$. We will also be able to express the property of subexponentiality in a natural language of random variables.

For $s \in S$ and $u \neq 0$ define

$$R(u,s) = \begin{cases} \inf\{x > 0 : \rho(s,(x,\infty)) \le u\} & \text{if } u > 0, \\ \inf\{x > 0 : \rho(s,(-\infty,-x)) \le u\} & \text{if } u < 0. \end{cases}$$

Let now ε, U and V be three independent random variables: let ε be a Rademacher random variable, U be uniformly distributed on $(0,1)$ and λ be S-valued with distribution λ. For an $a > 0$ let

$$Z_a = \phi^+(\varepsilon f(\cdot,V))R(a\varepsilon U,V). \tag{3.4}$$

A straightforward calculation shows that for $y > 0$

$$
\begin{aligned}
& P(Z_a > y) \\
= \; & \frac{1}{2a} \int_S [\rho(s, (\frac{y}{\phi^+(f(\cdot, s))}, \infty)) \wedge a + \rho(s, (-\infty, \frac{-y}{\phi^+(-f(\cdot, s))})) \wedge a] \lambda(ds).
\end{aligned} \tag{3.5}
$$

Comparing (3.5) with an alternative (when ϕ is homogeneous) expression for $H(y)$

$$
H(y) = \int_S [\rho(s, (\frac{y}{\phi^+(f(\cdot, s))}, \infty)) + \rho(s, (-\infty, \frac{-y}{\phi^+(-f(\cdot, s))}))\lambda(ds), \tag{3.6}
$$

we notice that it is natural to state sufficient conditions for the conclusion of Theorem 3.1 in terms of the random variable Z_a.

The following statement, of course, does not require a proof.

PROPOSITION 3.1 *Suppose that ϕ is homogeneous. If Z_a belongs to the subexponential class $\mathcal{S}$ and $P(Z_a > y) \sim \frac{1}{2a} H(y)$ as $y \to \infty$, then $P(\phi(X(\cdot)) > y) \sim H(y)$ as $y \to \infty$.*

In many cases dealing with subexponentiality of Z_a in (3.4) rather than with subexponentiality of $H(y)$ directly can greatly facilitate verifying the latter. We will return to this point in the sequel.

Example: *Stable Processes:*

A typical representation of a stable process is that of (3.1) with random measure M being an α-stable random measure with control measure m and skewness intensity β. That is, for any $A \in \mathcal{A}_0 = \{B \in \mathcal{A} : \; m(B) < \infty\}$, $M(A)$ is an α-stable random variable with scaling parameter $m(A)^{\frac{1}{\alpha}}$ and skewness parameter $\frac{1}{m(A)} \int_A \beta(s) m(ds)$.

It is straightforward to verify that, in the language of canonical representation (3.3), it amounts to taking $\lambda \sim m$, and $\rho(s, dx) = (d\lambda/dm)(s) \alpha c_\alpha \frac{1+\beta(s)}{2} x^{-(\alpha+1)} dx$ for $x > 0$ and $(d\lambda/dm)(s) \alpha c_\alpha \frac{1-\beta(s)}{2} (-x)^{-(\alpha+1)} dx$ for $x < 0$, where $c_\alpha = (\int_0^\infty x^{-\alpha} \sin x dx)^{-1}$.

Now, for any homogeneous subadditive functional ϕ, we obtain immediately from (3.6) and Theorem 2.1 that

$$
P(\phi(X(\cdot)) > y) \sim K y^{-\alpha}, \tag{3.7}
$$

provided $P(q(X(\cdot)) < \infty) = 1$; where $K = c_\alpha \int_S [\frac{1+\beta(s)}{2} \phi^+(f(\cdot, s))^\alpha + \frac{1-\beta(s)}{2} \phi^+(-f(\cdot, s))^\alpha] m(ds)$.

We now consider an application in which (3.6) is only partly of help. Let $\{X(t), t \in T\}$ be a measurable α-stable process on a separable metric space T; we assume, as before, that the process is given by (3.1) (with T no longer countable). Moreover, we assume that the kernel $f(t, s)$: is jointly measurable in t and s (this introduces no loss of generality, see Proposition 6.1 of Rosinski and Woyczynski (1986) and Proposition 3.1 of Samorodnitsky (1990)). Let ν be a σ-finite Borel measure on T. Let $p > 0$. Assume that

$$I_p(\mathbf{X}) = (\int_T |X(t)|^p \nu(t))^{1\wedge 1/p} < \infty \text{ a.s.} \tag{3.8}$$

We will show using Theorem 3.1 that

$$P(I_p(\mathbf{X}) > y) \sim K y^{-\alpha(1\wedge 1/p)}, \quad \text{as } y \to \infty, \tag{3.9}$$

where $K = c_\alpha \int_S (\int_T |f(t,s)|^p \nu(dt))^{\alpha/p} m(ds)$.

In the case $p \geq 1$ this can be obtained also from Corollary 6.20 of Araujo and Giné (1980). In the case $0 < p < 1$ (3.9) improves upon Theorem 5.1 of Samorodnitsky (1980).

We start with recalling that (3.8) implies that

$$\int_S (\int_T |f(t,s)|^p \nu(dt))^{\alpha/p} m(ds) < \infty. \tag{3.10}$$

In particular, for m-almost every s, $\int_T |f(t,s)|^p \nu(dt) < \infty$. Let μ be a Borel probability measure on T, equivalent to ν, and let $h(t) = \frac{\nu(dt)}{\mu(dt)}$. Let $\{r_n : n \geq 1\}$ be a sequence of i.i.d. T-valued random variables with common distribution μ, living on another probability space $(\Omega_1, \mathcal{F}_1, P_1)$. It follows by the Strong Law of Large Numbers that, for almost every $\omega \in \Omega$,

$$I_p(\mathbf{X}) = \lim_{n\to\infty} (\frac{1}{n} \Sigma_{i=1}^n |X(r_i)|^p h(r_i))^{1\wedge 1/p}. \tag{3.11}$$

By Fubini's theorem, for almost every $\omega_1 \in \Omega_1$, (3.11) holds P-a.s. Another application of Fubini's theorem shows that there is an event $\Omega_1^0 \subseteq \Omega_1$ with $P_1(\Omega_1^0) = 0$ such that for every $\omega_1 \notin \Omega_1^0$ both (3.11) holds P-a.s. and for m-almost every $s \in S$

$$\lim_{n\to\infty} (\frac{1}{n} \Sigma_{i=1}^n |f(r_i, s)|^p h(r_i))^{1\wedge 1/p} = (\int_T |f(t,s)|^p \nu(dt))^{1\wedge 1/p} \tag{3.12}$$

To simplify notation, we consider the case $0 < p < 1$ in the sequel. The case $p \geq 1$ can be treated in an identical way.

Fix once and for all an $\omega_1 \notin \Omega_1^0$, and consider an α-stable process $\{X(r_i), i \geq 1\}$. Define $q : \mathbf{R}^{\{r_i, i\geq 1\}} \to [0, \infty]$ and $\phi : \mathbf{R}^{\{r_i, i \geq 1\}} \to [0, \infty]$ by

$$\begin{aligned} q(\boldsymbol{\alpha}) &= \sup_n \frac{1}{n} \Sigma_{i=1}^n |\alpha(r_i)|^p h(r_i), \\ \phi(\boldsymbol{\alpha}) &= \overline{\lim}_{n\to\infty} \frac{1}{n} \Sigma_{i=1}^n |\alpha(r_i)|^p h(r_i). \end{aligned}$$

Obviously, q and ϕ satisfy (2.1) and (2.2). The function $H(y)$ is easily computable, it is equal to

$$H(y) = y^{-\alpha/p} c_\alpha \int_S (\int_T |f(t,s)|^p \nu(dt))^{\alpha/p} m(ds).$$

Applying Theorem 3.1 in this setting we get (3.9) for the case $0 < p < 1$. The case $p \geq 1$ can be treated similarly; ϕ is homogeneous in this case, so that (3.6) can be used to find H.

Example ξ – *radial processes*

The notion was introduced by Marcus (1987) and it refers to the i.d. processes (3.1) such that

1) $\sup_{t\in T}|f(t,s)| = 1$ for every $s \in S$,

2) $\rho(s,\cdot) = \rho(\cdot)$ (independent of s) in (3.3).

This class of i.d. processes includes, in particular, symmetric α-stable processes with bounded sample paths, $0 < \alpha < 2$.

Let $\phi(\boldsymbol{\alpha}) = \sup_{t\in T}|\alpha(t)|$. It follows from (3.6) that

$$H(y) = \rho((y,\infty)) + \rho((-\infty,-y)),\ y > 0.$$

We obtain immediately the following conclusion. Let $\{X(t), t \in T\}$ be a ξ-radial i.d. process with bounded sample paths. If the tail of $\rho((y,\infty)) + \rho((-\infty,-y))$ belongs to the subexponential class $\mathcal{S}$, then

$$\lim_{y\to\infty} \frac{P(\sup_{t\in T}|X(t)| > y)}{\rho((y,\infty)) + \rho((-\infty,-y))} = 1.$$

Marcus (1987) has obtained this conclusion in the symmetric case under more restrictive conditions (including regular varying tail of the Lévy measure).

Example *Oscillation of i.d. processes*

Let $\{X(t), t \in T\}$ be a stochastic process on a separable metric space (T,d), and let T_0 be a countable dense subset of T. Given a nonempty set $C \subseteq T$ the *oscillation* of the process $\{X(t), t \in T\}$ on C is defined by

$$W_{\mathbf{X}}(C) = \overline{\lim}|X(t_1) - X(t_2)|,$$

where the limit is taken over $t_1, t_2 \in T_0$, $d(t_1, C) \to 0$ and $d(t_1,t_2) \to 0$. We may regard $W_{\mathbf{X}}(C)$ as the highest jump of $\mathbf{X}$ on the set C.

Obviously if $\{X(t), t \in T\}$ is an i.d. process with bounded sample paths, our results apply with $q(\boldsymbol{\alpha}) = 2\sup_{t\in T_0}|\alpha(t)|$ and $\phi(\boldsymbol{\alpha}) = W_{\boldsymbol{\alpha}}(C)$, $\boldsymbol{\alpha} \in \mathbf{R}^{T_0}$.

As an example, let us consider oscillation of i.d. *moving averages* that is of i.d. processes given by (3.1) with $T \subseteq \mathbf{R}$, $S = (-\infty,+\infty), f(t,s) = f(s-t)$ for some measurable $f : \mathbf{R} \to \mathbf{R}$, and the measure F of the form $F(A \times B) = \int_A \rho(s,B)ds, A$ and B Borel sets. Let $\{X(t), 0 < t < 1\}$ be an i.d. moving average with bounded sample paths. If the function f is continuous, then $W_{\mathbf{X}}(C)$ is a degenerate (constant) random variable (Cambanis, Nolan and Rosinski (1990)). Let us examine the case when f has one discontinuity, and it is of the first kind. That is, for some $u_0 \in R$,

$$\Delta = |\lim_{u\to u_0-} f(u) - \lim_{u\to u_0+} f(u)| > 0.$$

Define for $y > 0$ and $C \subseteq (0,1)$,

$$\overline{G}_C(y) := \int_{\overline{C}} [\rho(s+u_0,(y,\infty)) + \rho(s+u_0,(-\infty,-y)]ds,$$

where $\overline{C}$ is the closure of C. We will show that, if the tail of $\overline{G}_C(y)$ belongs to the subexponential class $\mathcal{S}$, then

$$P(W_{\mathbf{X}}(C) > y) \sim \overline{G}_C(y/\Delta), \text{ as } y \to \infty. \tag{3.13}$$

In particular, if $\rho(s,B) = \rho(B)$ for some fixed Lévy measure ρ (which implies, in particular, that $\mathbf{X}$ is stationary), then (3.13) reduces to

$$P(W_{\mathbf{X}}(C) > y) \sim Leb(\overline{C})[\rho((\frac{y}{\Delta},\infty)) + \rho((-\infty,\frac{-y}{\Delta}))], y \to \infty,$$

and, similarly, for $\mathbf{X}$ being Lévy motion, where we have $f(u) = 1_{(-\infty,0]}(u)$ and $\rho(s,B) = \rho(B)$ if $s \geq 0$ and $= 0$ if $s < 0$, we obtain

$$P(W_{\mathbf{X}}(C) > y) \sim Leb(\overline{C})[\rho((y,\infty)) + \rho((-\infty,-y))], \ y \to \infty,$$

provided, of course, the assumption of subexponentiality holds.

To prove (3.13), note that in our case,

$$H(y) = \int_{-\infty}^{\infty} [\rho(s,(\frac{y}{W_{f(s-\cdot)}(C)},\infty) + \rho(s,(-\infty,\frac{-y}{W_{-f(s-\cdot)}(C)}))]ds$$

(use (3.6) with $\lambda(ds) = \frac{1}{2}e^{-|s|}ds$ (say) and ρ redefined accordingly). Obviously, $W_{f(s-\cdot)}(C) = \Delta$ if $s \in \overline{C} + u_0$ and $= 0$ if $s \notin \overline{C} + u_0$. Thus,

$$H(y) = \int_{\overline{C}+u_0} [\rho(s,(\frac{y}{\Delta},\infty)) + \rho(s,(-\infty,-\frac{y}{\Delta}))]ds = \overline{G}_C(\frac{y}{\Delta}).$$

Therefore, if the tail of $\overline{G}_C(y)$ belongs to the subexponentialial class $\mathcal{S}$, the tail of H also does so, and thus Theorem 3.1 applies. □

4 Tail of $\phi(\mathbf{X})$ and subexponentiality of the product of independent random variables.

In the previous section we have exhibited numerous examples of applications of our general result to particular i.d. processes. Of course, there are situations in which it is not easy to verify whether or not the tail of H belongs to the subexponential class $\mathcal{S}$. Even in the case of a homogeneous ϕ with $H(y)$ given by (3.6) this verification might be technically involved. In this section we describe an important situation in which we are able to "separate terms", and to obtain sufficient conditions for subexponentiality of H which are easier to verify.

Assume, therefore, that ϕ is a homogeneous functional. Assume, moreover, that the $F(A \times B) = \lambda(A)\rho(B)$ in (3.3) for some fixed Lévy measure ρ (thus, we are including all ξ-radial processes in this discussion). We will also assume, for simplicity, that the Lévy measure ρ is symmetric.

In the notation of (3.4) we have then $R(a\varepsilon U, V) = R(aU)$ (independent of ε and V), and so we obtain

$$Z_a = \phi^+(\varepsilon f(\cdot, V))R(aU), \tag{4.1}$$

a product of two independent random variables.

THEOREM 4.1 *Under conditions of Theorem 3.1 assume additionally that ϕ is homogeneous and that $F(A \times B) = \lambda(A)\rho(B)$ in (3.3), where λ is a probability measure. Let $\eta = \phi^+(\varepsilon f(\cdot, V))$. If Z_a belongs to the subexponential class $\mathcal{S}$ and the following condition holds*

$$\int_{R(a)}^{\infty} P(\eta > u^{-1}y)\rho(du) \sim \int_0^{\infty} P(\eta > u^{-1}y)\rho(du), \text{ as } y \to \infty, \tag{4.2}$$

then $\phi(X(\cdot))$ belongs to the subexponential class $\mathcal{S}$, and

$$P(\phi(X(\cdot)) > y) \sim 2aP(Z_a > y) \sim H(y) \text{ as } y \to \infty.$$

Proof. We have only to check that conditions of Proposition 3.1 hold. Since by (3.5)

$$\begin{aligned} P(Z_a > y) &= \frac{1}{a}[\int_{R(a)}^{\infty} P(\eta > u^{-1}y)\rho(du) + P(\eta > yR(a)^{-1})(a - \rho(R(a), \infty))] \\ &\geq \frac{1}{a}\int_{R(a)}^{\infty} P(\eta > u^{-1}y)\rho(du), \end{aligned}$$

and by (3.6)

$$H(y) = 2\int_0^{\infty} P(\eta > u^{-1}y)\rho(du),$$

it follows that $P(Z_a > y) \sim \frac{1}{2a}H(y)$, as $y \to \infty$, and we may now appeal to Proposition 3.1. □

Example. Suppose that $\rho((y, \infty)) \in RV_{-p}, p \geq 0$, as $y \to \infty$. If for some $\delta > 0$, $E((\eta^+)^{p+\delta}) < \infty$ and $\int_0^1 x^{p+\delta}\rho(dx) < \infty$ then for any $a > 0$ $P(Z_a > y) \sim \frac{1}{a}\rho((y, \infty))E((\eta^+)^p)$ as $y \to \infty$. It is trivial to check that (4.2) holds. Therefore,

$$\begin{aligned} P(\phi(X(\cdot)) > y) &\sim 2aP(Z_a > y) \\ &\sim \rho(y, \infty)\int_S[\phi^+(f(\cdot, s))^p + \phi^+(-f(\cdot, s))^p]\lambda(ds). \square \end{aligned}$$

In the previous example it was easy to check that $Z_a = \eta R(aU)$ belongs to the class RV_{-p}, thus also to the subexponential class $\mathcal{S}$. Applicability of Theorem 4.1 is greatly enhanced by the fact that, in general, there are many situations in which one can relatively easily verify that the product of two independent random variables belongs to the subexponential class $\mathcal{S}$. The following result is quoted from Cline and Samorodnitsky (1991).

THEOREM 4.2 *Let X and Y be independent random variables such that X belongs to the subexponential class $\mathcal{S}$.*

If there is a function $a(t) : (0,\infty) \to (0,\infty)$ such that, as $t \to \infty$,

(i) $a(t) \uparrow \infty$

(ii) $t/a(t) \uparrow \infty$,

(iii) $\frac{P(X>t-a(t))}{P(X>t)} = 1$,

(iv) $P(Y > a(t)) = o(P(XY > t))$

then the product XY belongs to the subexponential class $\mathcal{S}$.

Example. Suppose that $\rho([y,\infty)) \in \mathcal{S}$ and that η is a <u>bounded</u> random variable. Then Theorem 4.2 implies that $Z_a \in \mathcal{S}$ for any $a > 0$. Also (4.2) holds trivially for any $a > 0$. Thus, Theorem 4.1 applies. □

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