FOURTH MOMENTS IN THE GENERAL LINEAR MODEL; AND THE VARIANCE OF TRANSLATION INVARIANT QUADRATIC FORMS

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Abstract

Using vec and Kronecker product operators, a detailed derivation is given of fourth moments in the general linear model and of the variance of translation invariant quadratic forms.

Introduction

We consider the general linear model

$$\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{\beta} + \underset{\sim}{Z}_1\underset{\sim}{u}_1 + \underset{\sim}{Z}_2\underset{\sim}{u}_2 + \cdots + \underset{\sim}{Z}_k\underset{\sim}{u}_k \tag{1}$$

where $\underset{\sim}{X}$ of order $n \times p$ and $\underset{\sim}{Z}_m$ of order $n \times c_m$, $m = 1, \cdots, k$ are known incidence matrices, $\underset{\sim}{\beta}$ is an unknown vector of p fixed effects, and the $\underset{\sim}{u}_m$, of order $c_m \times 1$ for $m = 1, \cdots, k$, are unknown vectors of random effects such that

(i) the elements of $\underset{\sim}{u}_m$ are independent having common variance σ_m^2 and kurtosis γ_m, and

(ii) $\underset{\sim}{u}_m$ and $\underset{\sim}{u}_{m'}$ are independent for $m \neq m'$.

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Accordingly, the variance-covariance matrix of the vector $\underset{\sim}{y}$ is

$$\mathrm{Var}(\underset{\sim}{y}) = \sum_{m=1}^{k} \sigma_m^2 \underset{\sim}{Z}_m \underset{\sim}{Z}_m' = \underset{\sim}{V} .$$

As far as the ensuing algebra is concerned, a more convenient representation of (1) is

$$\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{\beta} + \underset{\sim}{Z}\underset{\sim}{u} \tag{2}$$

where $\underset{\sim}{Z}$ is the $n \times c_.$ $\left(c_. = \sum_{m=1}^{k} c_m\right)$ partitioned matrix $\underset{\sim}{Z} = [\underset{\sim}{Z}_1 \cdots \underset{\sim}{Z}_k]$, and $\underset{\sim}{u}$ is the $c_. \times 1$ vector $\underset{\sim}{u}' = [\underset{\sim}{u}_1' \cdots \underset{\sim}{u}_k']$. Corresponding to model (2), the variance-covariance matrix of $\underset{\sim}{y}$ is then

$$\underset{\sim}{V} = \underset{\sim}{Z}\underset{\sim}{D}\underset{\sim}{Z}'$$

where

$$\underset{\sim}{D} = E(\underset{\sim}{u}\underset{\sim}{u}') = \sum_{m=1}^{k}{}^{+} \sigma_m^2 \underset{\sim}{I}_{c_m} \tag{3}$$

and $\Sigma^+ \underset{\sim}{A}_i$ denotes the direct sum of matrices $\underset{\sim}{A}_i$.

Fourth Moments

The matrix of central fourth moments of the vector $\underset{\sim}{y}$ is, by definition,

$$\underset{\sim}{F} = \mathrm{Var}[(\underset{\sim}{y} - \underset{\sim}{X}\underset{\sim}{\beta}) * (\underset{\sim}{y} - \underset{\sim}{X}\underset{\sim}{\beta})] \tag{4}$$

where $\underset{\sim}{A} * \underset{\sim}{B}$ is the direct (Kronecker) product of $\underset{\sim}{A}$ and $\underset{\sim}{B}$. Substituting in terms of (2),

$$\underset{\sim}{F} = \text{Var}[(\underset{\sim}{Zu}) * (\underset{\sim}{Zu})]$$

$$= \text{Var}[(\underset{\sim}{Z} * \underset{\sim}{Z})(\underset{\sim}{u} * \underset{\sim}{u})]$$

$$= (\underset{\sim}{Z} * \underset{\sim}{Z})[\text{Var}(\underset{\sim}{u} * \underset{\sim}{u})](\underset{\sim}{Z} * \underset{\sim}{Z})' \ . \tag{5}$$

Defining the vec of a matrix to be the $\underset{\sim}{M}^c$ matrix first introduced by Roth [1934], namely, the vector obtained from stacking the columns of the matrix one beneath the other in a single vector, and noting that $\text{vec}(\underset{\sim}{uu}') = \underset{\sim}{u} * \underset{\sim}{u}$ it follows that

$$\text{Var}(\underset{\sim}{u} * \underset{\sim}{u}) = E[(\underset{\sim}{u} * \underset{\sim}{u})(\underset{\sim}{u} * \underset{\sim}{u})'] - [E(\underset{\sim}{u} * \underset{\sim}{u})][E(\underset{\sim}{u} * \underset{\sim}{u})]'$$

$$= E[(\underset{\sim}{uu}') * (\underset{\sim}{uu}')] - E[\text{vec}(\underset{\sim}{uu}')]E[\text{vec}(\underset{\sim}{uu}')]'$$

$$= E[(\underset{\sim}{uu}') * (\underset{\sim}{uu}')] - \text{vec}E(\underset{\sim}{uu}')[\text{vec}E(\underset{\sim}{uu}')]'$$

$$= E[(\underset{\sim}{uu}') * (\underset{\sim}{uu}')] - \text{vec}\underset{\sim}{D}(\text{vec}\underset{\sim}{D})' \ , \tag{6}$$

on using $\underset{\sim}{D}$ of (3).

To simplify (6) we define

$$c \equiv c_. = c_1 + c_2 + \cdots + c_k \tag{7}$$

and

$$\underset{\sim}{w} \equiv \underset{\sim}{D}^{-\frac{1}{2}}\underset{\sim}{u} = \{w_i\} \quad i = 1, \cdots, c \ . \tag{8}$$

Then $\underset{\sim}{w}$ has the properties

$$E(\underset{\sim}{w}) = 0, \quad E(\underset{\sim}{ww}') = \text{var}(\underset{\sim}{w}) = \underset{\sim}{I}_c \tag{9}$$

and

$$E(w_i^4) = 3 + \dot{\gamma}_i \quad \text{for } i = 1, \cdots, c \tag{10}$$

where

$$\dot{\gamma}_i = i^{th} \text{ diagonal element of } \sum_{m=1}^{k}{}^{+} \gamma_m \underset{\sim}{I}_{c_m} . \tag{11}$$

Then for (6)

$$E[(\underset{\sim}{u}\underset{\sim}{u}') * (\underset{\sim}{u}\underset{\sim}{u}')] = (\underset{\sim}{D}^{\frac{1}{2}} * \underset{\sim}{D}^{\frac{1}{2}})E[(\underset{\sim}{w}\underset{\sim}{w}') * (\underset{\sim}{w}\underset{\sim}{w}')](\underset{\sim}{D}^{\frac{1}{2}} * \underset{\sim}{D}^{\frac{1}{2}})' ,$$

$$= (\underset{\sim}{D}^{\frac{1}{2}} * \underset{\sim}{D}^{\frac{1}{2}})\underset{\sim}{\Sigma}(\underset{\sim}{D}^{\frac{1}{2}} * \underset{\sim}{D}^{\frac{1}{2}}) \tag{12}$$

on defining

$$\underset{\sim}{\Sigma}_{c^2 \times c^2} \equiv \{\underset{\sim}{\Sigma}_{ij}\} \quad \text{for } i,j = 1, 2, \cdots, c$$

$$= \{E(w_i w_j \underset{\sim}{w}\underset{\sim}{w}')\}$$

$$= \{E(w_i w_j w_k w_\ell)\} \quad \text{for } i,j,k,\ell = 1, 2, \cdots, c . \tag{13}$$

Now for $i = j$

$$E(w_i w_i w_k w_\ell) = \begin{cases} 3 + \dot{\gamma}_i & \text{when } i = k = \ell \\ 1 & \text{when } i \neq k = \ell \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

and for $i \neq j$

$$E(w_i w_j w_k w_\ell) = \begin{cases} 1 & \text{when } i = k,\ j = \ell \\ 1 & \text{when } i = \ell,\ j = k \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

so that, on defining

$$\underset{\sim}{e}_i = i^{th} \text{ column of } \underset{\sim}{I}_c , \tag{16}$$

(14) and (15) give the sub-matrices of (13) as

$$\underset{\sim}{\Sigma}_{ii} = \underset{\sim}{I} + (2 + \dot{\gamma}_i)\underset{\sim}{e}_i\underset{\sim}{e}_i' \tag{17}$$

and

$$\underset{\sim}{\Sigma}_{ij} = \underset{\sim}{e}_i\underset{\sim}{e}_j' + \underset{\sim}{e}_j\underset{\sim}{e}_i' , \quad \text{for } i \neq j . \tag{18}$$

Therefore in (13), noting that $2\underset{\sim}{e}_i\underset{\sim}{e}_i' = \underset{\sim}{e}_i\underset{\sim}{e}_j' + \underset{\sim}{e}_j\underset{\sim}{e}_i'$ for $i = j$,

$$\underset{\sim}{\Sigma} = \{\underset{\sim}{\Sigma}_{ij}\} = \underset{\sim}{I} + \{\underset{\sim}{e}_i\underset{\sim}{e}_j' + \underset{\sim}{e}_j\underset{\sim}{e}_i'\} \text{ for } i,j = 1, \cdots, c, + \sum_{i=1}^{c}{}^{+} \dot{\gamma}_i\underset{\sim}{e}_i\underset{\sim}{e}_i' . \tag{19}$$

In (19) it is important to note that for the matrix $\{\underset{\sim}{e}_i\underset{\sim}{e}_j' + \underset{\sim}{e}_j\underset{\sim}{e}_i'\}$ the sequence of subscripts is $j = 1, \cdots, c$ within $i = 1, \cdots, c$. This being so, it can be noted that

$$\{\underset{\sim}{e}_j\underset{\sim}{e}_i'\} \text{ for } i,j = 1, \cdots, c, = \underset{\sim}{I}_{(c,c)} , \tag{20}$$

the permuted identity matrix of order c^2, as used by Tracy and Dwyer [1969] and MacRae [1974]; and

$$\{\underset{\sim}{e}_i\underset{\sim}{e}_j'\} \text{ for } i,j = 1, \cdots, c, = \text{vec}\underset{\sim}{I}(\text{vec}\underset{\sim}{I})' . \tag{21}$$

Furthermore, using (11) in the last term of (19) gives, for $t_{m-1} = c_1 + c_2 + \cdots + c_{m-1}$

$$\sum_{i=1}^{c}{}^{+} \dot{\gamma}_i\underset{\sim}{e}_i\underset{\sim}{e}_i' = \sum_{m=1}^{k}{}^{+} \gamma_m\left(\sum_{i=t_{m-1}+1}^{t_m}{}^{+} \underset{\sim}{e}_i\underset{\sim}{e}_i'\right) . \tag{22}$$

Now for $i = 1, \cdots, c$

$\underset{\sim}{e}_i\underset{\sim}{e}_i'$ = a diagonal matrix of order c with its only non-zero element being 1 in the (i,i) position

and

$$\begin{bmatrix} \underset{\sim}{e}_i\underset{\sim}{e}_i' & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{e}_{i+1}\underset{\sim}{e}_{i+1}' \end{bmatrix} = \text{a diagonal matrix of order } 2c \text{ with its only non-zero elements being 1 in the } (i,i) \text{ and } (c+i+1,\ c+i+1) \text{ positions.}$$

Consider just the diagonal elements of this last matrix. Between the two 1's there are c - i + i = c zeros; and this is true for all i. Denote a row vector of c zeros by $\underset{\sim}{0}'_c$. Then the diagonal elements of (22) are

$$[\gamma_1\ \underset{\sim}{0}'_c\ \gamma_1\ \underset{\sim}{0}'_c\ \cdots\ \underset{\sim}{0}'_c\ \gamma_1\ \underset{\sim}{0}'_c\ \gamma_2\ \underset{\sim}{0}'_c\ \cdots\ \gamma_2\ \underset{\sim}{0}'_c\ \cdots\ \cdots\ \underset{\sim}{0}'_c\ \gamma_k\ \underset{\sim}{0}'_c\ \cdots\ \underset{\sim}{0}'_c\ \gamma_k] \tag{23}$$

where γ_m occurs c_m times for m = 1, 2, $\cdots$, k. This is a vector of c^2 elements and by its nature is $\text{vec}\left(\sum_{m=1}^{k}{}^{+}\ \gamma_m \underset{\sim}{I}_{c_m}\right)$. Hence, on using the definition

$$\text{diag}\ \underset{\sim}{x} \equiv \begin{array}{l}\text{diagonal matrix with diagonal elements}\\ \text{being the elements of the vector } \underset{\sim}{x}\ ,\end{array}$$

we have (22) as

$$\sum_{i=1}^{c}{}^{+}\ \dot{\gamma}_i \underset{\sim}{e}_i \underset{\sim}{e}'_i = \text{diag}\{\text{vec}(\Sigma^+\ \gamma_m \underset{\sim}{I}_{c_m})\}\ . \tag{24}$$

Substituting (20), (21) and (24) into (19) gives

$$\underset{\sim}{\Sigma} = \underset{\sim}{I}_{c^2} + \underset{\sim}{I}_{(c,c)} + (\text{vec}\underset{\sim}{I}_c)(\text{vec}\underset{\sim}{I}_c)' + \text{diag}\{\text{vec}(\Sigma^+\ \gamma_m \underset{\sim}{I}_{c_m})\}\ . \tag{25}$$

Using $\underset{\sim}{\Sigma}$ in (12) now gives

$$\begin{aligned} &E[(\underset{\sim}{u}\underset{\sim}{u}' * \underset{\sim}{u}\underset{\sim}{u}')] \\ &\quad = (\underset{\sim}{D}^{\frac{1}{2}} * \underset{\sim}{D}^{\frac{1}{2}})[\underset{\sim}{I} + \underset{\sim}{I}_{(c,c)} + (\text{vec}\underset{\sim}{I}_c)(\text{vec}\underset{\sim}{I}_c)' + \text{diag}\{\text{vec}(\Sigma^+\ \gamma_m \underset{\sim}{I}_{c_m})\}](\underset{\sim}{D}^{\frac{1}{2}} * \underset{\sim}{D}^{\frac{1}{2}}) \\ &\quad = \underset{\sim}{D} * \underset{\sim}{D} + (\underset{\sim}{D}^{\frac{1}{2}} * \underset{\sim}{D}^{\frac{1}{2}})\underset{\sim}{I}_{(c,c)}(\underset{\sim}{D}^{\frac{1}{2}} * \underset{\sim}{D}^{\frac{1}{2}}) + \underset{\sim}{z}\underset{\sim}{z}' + \underset{\sim}{\Gamma} \end{aligned} \tag{26}$$

where we define

$$\underset{\sim}{\Gamma} \equiv (\underset{\sim}{D}^{\frac{1}{2}} * \underset{\sim}{D}^{\frac{1}{2}})[\text{diag}\{\text{vec}\left(\sum_{m=1}^{k}{}^{+}\ \gamma_m \underset{\sim}{I}_{c_m}\right)\}](\underset{\sim}{D}^{\frac{1}{2}} * \underset{\sim}{D}^{\frac{1}{2}})\ . \tag{27}$$

Also, use is made of the results in MacRae [1974] that

$$\underset{\sim}{I}_{(p,p)}(\underset{\sim}{A}_{p\times q} * \underset{\sim}{B}_{p\times q})\underset{\sim}{I}_{(q,q)} = \underset{\sim}{B}_{p\times q} * \underset{\sim}{A}_{p\times q} \tag{28}$$

and

$$[\underset{\sim}{I}_{(p,p)}]^2 = \underset{\sim}{I}_{p^2} \tag{29}$$

so that

$$\underset{\sim}{I}_{(p,p)}(\underset{\sim}{A}_{p\times q} * \underset{\sim}{A}_{p\times q}) = (\underset{\sim}{B}_{p\times q} * \underset{\sim}{A}_{p\times q})\underset{\sim}{I}_{(q,q)} \ . \tag{30}$$

Hence for (26)

$$(\underset{\sim}{D}^{\frac{1}{2}} * \underset{\sim}{D}^{\frac{1}{2}})\underset{\sim}{I}_{(c,c)}(\underset{\sim}{D}^{\frac{1}{2}} * \underset{\sim}{D}^{\frac{1}{2}}) = (\underset{\sim}{D} * \underset{\sim}{D})\underset{\sim}{I}_{(c,c)} \ . \tag{31}$$

Furthermore, in (26)

$$\underset{\sim}{z} \equiv (\underset{\sim}{D}^{\frac{1}{2}} * \underset{\sim}{D}^{\frac{1}{2}})\mathrm{vec}\underset{\sim}{I} = \mathrm{vec}\underset{\sim}{D} \tag{32}$$

because, in general,

$$\mathrm{vec}(\underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{C}) = (\underset{\sim}{C}' * \underset{\sim}{A})\mathrm{vec}\underset{\sim}{B} \tag{33}$$

as in Neudecker [1969].

Using (31) and (32) in (26) therefore gives

$$E[(\underset{\sim}{u}\underset{\sim}{u}') * (\underset{\sim}{u}\underset{\sim}{u}')] = (\underset{\sim}{D} * \underset{\sim}{D})(\underset{\sim}{I} + \underset{\sim}{I}_{(c,c)}) + (\mathrm{vec}\underset{\sim}{D})(\mathrm{vec}\underset{\sim}{D})' + \underset{\sim}{\Gamma} \tag{34}$$

where $\underset{\sim}{\Gamma}$ is as defined in (27), and so substitution into (6) gives

$$\mathrm{var}(\underset{\sim}{u} * \underset{\sim}{u}) = (\underset{\sim}{D} * \underset{\sim}{D})(\underset{\sim}{I} + \underset{\sim}{I}_{(c,c)}) + \underset{\sim}{\Gamma} \ . \tag{35}$$

Putting (35) into (5) gives the matrix of fourth moments as

$$\underset{\sim}{F} = (\underset{\sim}{Z} * \underset{\sim}{Z})[(\underset{\sim}{D} * \underset{\sim}{D})(\underset{\sim}{I} + \underset{\sim}{I}_{(c,c)}) + \underset{\sim}{\Gamma}](\underset{\sim}{Z}' * \underset{\sim}{Z}')$$

and because $\underset{\sim}{ZDZ}' = \underset{\sim}{V}$ this is:

$$\underset{\sim}{F} = \underset{\sim}{V} * \underset{\sim}{V} + (\underset{\sim}{ZD} * \underset{\sim}{ZD})\underset{\sim}{I}_{(c,c)}(\underset{\sim}{Z}' * \underset{\sim}{Z}') + (\underset{\sim}{Z} * \underset{\sim}{Z})\underset{\sim}{\Gamma}(\underset{\sim}{Z}' * \underset{\sim}{Z}') .$$

Using (30) again leads to

$$\underset{\sim}{F} = (\underset{\sim}{V} * \underset{\sim}{V})(\underset{\sim}{I} + \underset{\sim}{I}_{(n,n)}) + (\underset{\sim}{Z} * \underset{\sim}{Z})\underset{\sim}{\Gamma}(\underset{\sim}{Z}' * \underset{\sim}{Z}') \qquad (36)$$

and on using (27) this has the equivalent form

$$\underset{\sim}{F} = (\underset{\sim}{V} * \underset{\sim}{V})(\underset{\sim}{I} + \underset{\sim}{I}_{(n,n)}) + (\underset{\sim}{ZD}^{\frac{1}{2}} * \underset{\sim}{ZD}^{\frac{1}{2}})[\text{diag}\{\text{vec}\left(\sum_{m=1}^{k}{}^{+} \gamma_m \underset{\sim}{I}_{c_m}\right)\}](\underset{\sim}{D}^{\frac{1}{2}}\underset{\sim}{Z}' * \underset{\sim}{D}^{\frac{1}{2}}\underset{\sim}{Z}'). \qquad (37)$$

This, then, is the general expression for the matrix of fourth central moments of the vector of observations in linear model theory.

In the special case of normality assumptions, i.e., $\underset{\sim}{u}_m \sim N(\underset{\sim}{0}, \sigma^2_m \underset{\sim}{I}_{c_m})$, we have $\gamma_m = 0$ and (37) reduces to

$$\underset{\sim}{F} = (\underset{\sim}{V} * \underset{\sim}{V})(\underset{\sim}{I} + \underset{\sim}{I}_{(n,n)}) \qquad (38)$$

Variance of Translation Invariant Quadratic Forms

The quadratic form $\underset{\sim}{y}'\underset{\sim}{A}\underset{\sim}{y}$ is called translation invariant when $\underset{\sim}{A}$, as well as being symmetric, satisfies $\underset{\sim}{AX} = 0$. Then the variance of the translation invariant quadratic form is

$$\begin{aligned} v(\underset{\sim}{y}'\underset{\sim}{A}\underset{\sim}{y}) &= v[(\underset{\sim}{y} - \underset{\sim}{X\beta})'\underset{\sim}{A}(\underset{\sim}{y} - \underset{\sim}{X\beta})] \\ &= v(\underset{\sim}{u}'\underset{\sim}{Z}'\underset{\sim}{AZu}) \\ &= v\{tr[\underset{\sim}{A}(\underset{\sim}{Zuu}'\underset{\sim}{Z}')]\} . \end{aligned}$$

Now use the general result for any product $\underset{\sim}{P}\underset{\sim}{Q}$, that

$$\operatorname{tr}(\underset{\sim}{P}\underset{\sim}{Q}) = (\operatorname{vec}\underset{\sim}{P}')'\operatorname{vec}\underset{\sim}{Q} \tag{39}$$

and so

$$\begin{aligned} v(\underset{\sim}{y}'\underset{\sim}{A}\underset{\sim}{y}) &= v\{(\operatorname{vec}\underset{\sim}{A})'\operatorname{vec}(\underset{\sim}{Z}\underset{\sim}{u}\underset{\sim}{u}'\underset{\sim}{Z}')\} \\ &= (\operatorname{vec}\underset{\sim}{A})'\operatorname{var}[\operatorname{vec}(\underset{\sim}{Z}\underset{\sim}{u}\underset{\sim}{u}'\underset{\sim}{Z}')]\operatorname{vec}\underset{\sim}{A} \\ &= (\operatorname{vec}\underset{\sim}{A})'\operatorname{var}[(\underset{\sim}{Z}\underset{\sim}{u}) * (\underset{\sim}{Z}\underset{\sim}{u})']\operatorname{vec}\underset{\sim}{A} \\ &= (\operatorname{vec}\underset{\sim}{A})'\underset{\sim}{F}(\operatorname{vec}\underset{\sim}{A}), \quad \text{using (4)} \end{aligned} \tag{40}$$

and on using (36) this gives

$$v(\underset{\sim}{y}'\underset{\sim}{A}\underset{\sim}{y}) = \theta_1 + \theta_2 \tag{41}$$

for

$$\theta_1 = (\operatorname{vec}\underset{\sim}{A})'(\underset{\sim}{V} * \underset{\sim}{V})(\underset{\sim}{I} + \underset{\sim}{I}_{(n,n)})\operatorname{vec}\underset{\sim}{A} \tag{42}$$

and

$$\theta_2 = (\operatorname{vec}\underset{\sim}{A})'(\underset{\sim}{Z}\underset{\sim}{D}^{\frac{1}{2}} * \underset{\sim}{Z}\underset{\sim}{D}^{\frac{1}{2}})[\operatorname{diag}\{\operatorname{vec}(\sum_{m=1}^{k}{}^{+} \gamma_m \underset{\sim}{I}_{c_m})\}](\underset{\sim}{D}^{\frac{1}{2}}\underset{\sim}{Z}' * \underset{\sim}{D}^{\frac{1}{2}}\underset{\sim}{Z}')\operatorname{vec}\underset{\sim}{A} . \tag{43}$$

In θ_1 of (42), the elements of the $[(i-1)n+j]^{th}$ row of $\underset{\sim}{I}_{(n,n)}$ are all zero except for a 1 in the $[(j-1)n+i]^{th}$ column (and vice versa); and also, because $\underset{\sim}{A}$ is symmetric, the $[(i-1)n+j]^{th}$ and $[(j-1)n+i]^{th}$ elements of $\operatorname{vec}\underset{\sim}{A}$ are the same. Hence

$$\underset{\sim}{I}_{(n,n)}(\operatorname{vec}\underset{\sim}{A}) = \operatorname{vec}\underset{\sim}{A} . \tag{44}$$

Also using (33) and (39)

$$(\operatorname{vec}\underset{\sim}{A})'(\underset{\sim}{V} * \underset{\sim}{V})\operatorname{vec}\underset{\sim}{A} = (\operatorname{vec}\underset{\sim}{A})'\operatorname{vec}(\underset{\sim}{V}\underset{\sim}{A}\underset{\sim}{V}) = \operatorname{tr}(\underset{\sim}{A}\underset{\sim}{V})^2 , \tag{45}$$

so that

$$\theta_1 = 2\mathrm{tr}(\underset{\sim}{A}\underset{\sim}{V})^2 . \tag{46}$$

Simplification of θ_2 in (43) starts with using (33) to get

$$\theta_2 = [\mathrm{vec}(\underset{\sim}{D}^{\frac{1}{2}}\underset{\sim}{Z}'\underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{D}^{\frac{1}{2}})]'[\mathrm{diag}\{\mathrm{vec}\Big(\sum_{m=1}^{k}{}^{+}\ \gamma_m \underset{\sim}{I}_{c_m}\Big)\}]\mathrm{vec}(\underset{\sim}{D}^{\frac{1}{2}}\underset{\sim}{Z}'\underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{D}^{\frac{1}{2}})$$

which is of the form

$$\theta_2 = (\mathrm{vec}\underset{\sim}{H})'[\mathrm{diag}\{\mathrm{vec}\underset{\sim}{L}\}]\mathrm{vec}\underset{\sim}{H} \tag{47}$$

for

$$\underset{\sim}{H} = \underset{\sim}{D}^{\frac{1}{2}}\underset{\sim}{Z}'\underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{D}^{\frac{1}{2}} \quad \text{and} \quad \underset{\sim}{L} = \sum_{m=1}^{k}{}^{+}\ \gamma_m \underset{\sim}{I}_{c_m} . \tag{48}$$

The nature of the vec and diag operators means that (47) is

$$\theta_2 = \sum_{i}\sum_{j} h_{ij}^2 \ell_{ij} \tag{49}$$

for $\underset{\sim}{H} = \{h_{ij}\}$ and $\underset{\sim}{L} = \{\ell_{ij}\}$ of (48). But with this $\underset{\sim}{L}$, the only non-zero ℓ_{ij}'s are the diagonal ones, $\ell_{tt} = \gamma_m$ for $t = 1, \cdots, c_m$ and $m = 1, \cdots, k$. Furthermore, as in (23), these diagonal elements have c zeros between them in $\mathrm{vec}\underset{\sim}{L}$ so that the use of (48) in (49) gives

$$\theta_2 = \sum_{m=1}^{k} \gamma_m \sum_{t=1}^{c_m} h_{tt}^2$$

for

$$h_{tt}^2 = t^{th} \text{ diagonal element of the } m^{th} \text{ diagonal sub-matrix of } \underset{\sim}{D}^{\frac{1}{2}}\underset{\sim}{Z}'\underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{D}^{\frac{1}{2}}$$

$$= \sigma_m^2(t^{th} \text{ diagonal element of the } m^{th} \text{ diagonal sub-matrix of } \underset{\sim}{Z}'\underset{\sim}{A}\underset{\sim}{Z}).$$

Therefore

$$\theta_2 = \sum_{m=1}^{k} \gamma_m \sigma_m^4 (\text{sum of squares of diagonal elements of } \underset{\sim}{Z}_m' \underset{\sim}{A} \underset{\sim}{Z}_m) \; . \tag{50}$$

Substituting (46) and (50) into (41) gives

$$v(\underset{\sim}{y}'\underset{\sim}{A}\underset{\sim}{y}) = 2\mathrm{tr}(\underset{\sim}{A}\underset{\sim}{V})^2 + \sum_{m=1}^{k} \gamma_m \sigma_m^4 (\text{sum of squares of diagonal elements of } \underset{\sim}{Z}_m' \underset{\sim}{A} \underset{\sim}{Z}_m). \tag{51}$$

This is the variance, under non-normality, of a translation invariant ($\underset{\sim}{A}\underset{\sim}{X} = \underset{\sim}{0}$) quadratic form $\underset{\sim}{y}'\underset{\sim}{A}\underset{\sim}{y}$. Under normality, $\gamma_m = 0$ for all m and (51) reduces to the familiar form

$$v(\underset{\sim}{y}'\underset{\sim}{A}\underset{\sim}{y}) = 2\mathrm{tr}(\underset{\sim}{A}\underset{\sim}{V})^2 \; . \tag{52}$$

Equation (51) is, of course, equivalent to the result given by Rao [1971] where he writes $\underset{\sim}{\Delta}_1$ for $\underset{\sim}{D}$ and $\underset{\sim}{\Delta}_2 = \sum_{m=1}^{k}{}^{+} \gamma_m \sigma_m^4 \underset{\sim}{I}_{c_m}$ and $\tilde{\underset{\sim}{B}} = \mathrm{diag}\{\text{diagonal elements of } \underset{\sim}{B}\}$ and so gets

$$v(\underset{\sim}{y}'\underset{\sim}{A}\underset{\sim}{y}) = 2\mathrm{tr}(\underset{\sim}{B}\underset{\sim}{\Delta}_1)^2 + \mathrm{tr}(\tilde{\underset{\sim}{B}}\underset{\sim}{\Delta}_2\tilde{\underset{\sim}{B}}) \; ,$$

for $\underset{\sim}{B} = \underset{\sim}{Z}'\underset{\sim}{A}\underset{\sim}{Z}$. With $\underset{\sim}{V}$ being $\underset{\sim}{Z}\underset{\sim}{D}\underset{\sim}{Z}'$ this is readily seen to be the same as (51).

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