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OPTIMAL POLICIES FOR TWO PRODUCT INVENTORY SYSTEMS, WITH AND WITHOUT SETUP COSTS

A Thesis

Presented to the Faculty of the Graduate School

of Cornell University for the Degree of

Doctor of Philosophy

by

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September 1966

BIOGRAPHICAL SKETCH

Edward Joel Ignall was born in New York City in April 1938 and attended school in and around New York, being graduated from Teaneck (N.J.) High School in 1955. He entered Cornell University in September 1955 and received a B.M.E. (with distinction) in June 1960.

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He married the former Judith Horton in April 1960 and they have a son, David.

ACKNOWLEDGMENTS

I am grateful and indebted to Professor Donald Iglehart for introducing me to inventory theory and guiding this thesis; to Professor Richard Conway for guiding my graduate studies; to Professor Robert Bechhofer for advice and support in all matters for as long as I can remember; to Professor Ta-Chung Liu for making a minor in econometrics so satisfying; and to Professor Arthur Veinott, Jr. for providing information, comments, and criticism.

I thank Mrs. Barbara Walker, Mrs. Edith Otway, and Mrs. Shirley Crants for typing the manuscript.

My thanks also go to the Ford Foundation for support of my entire graduate program and to the Office of Naval Research for partial support of this thesis.

And I thank my wife Judy for helping me in more ways than I can count.

TABLE OF CONTENTS

Chapter 1

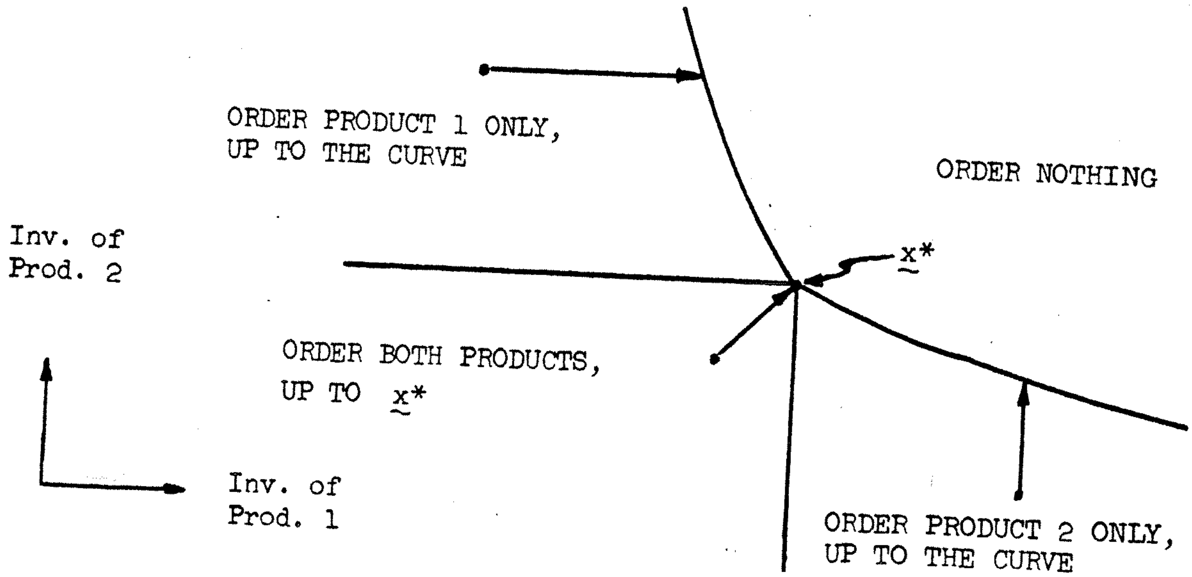
Introduction and Summary

1.1 Summary The object of this study is the discovery of optimal ordering policies for multiproduct inventory systems. The following features characterize the systems that we consider. It is assumed that, at the start of each of N periods of equal length, an order for one or more products can be placed. Demands in the periods are independent vector random variables with known probability density functions. Demands for individual products within a period are assumed to be non-negative, but they need not be independent. Whenever demand exceeds inventory, their difference is backlogged rather than lost. There are costs for holding inventory and for being out of stock. The purchasing cost is linear in the amount ordered, and there may or may not be a setup cost for ordering. Delivery may be instantaneous or after a lag of a fixed number of periods. There is a discount factor, which may be unity. An optimal policy is defined to be one that minimizes the expected discounted costs over the N periods.

No Setup Cost: Chapters 2 and 3 For the case where there is no setup cost, this thesis extends some work of A. F. Veinott, Jr.

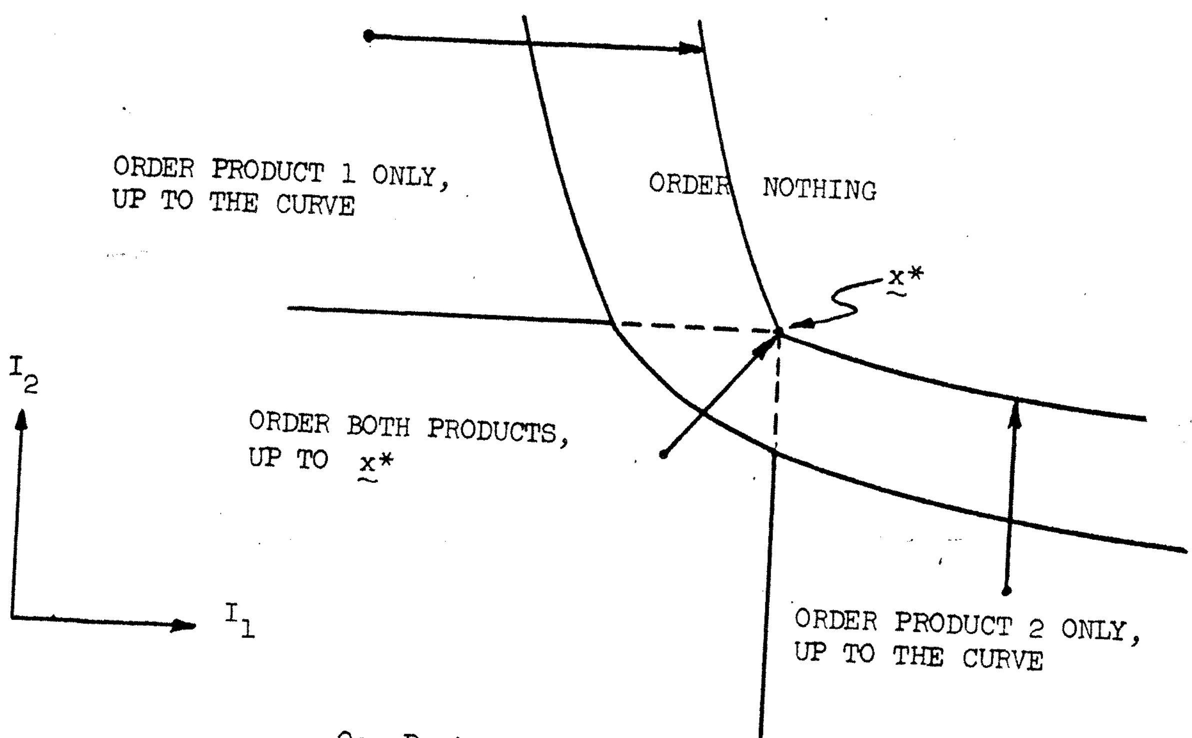
For an N period, m product inventory system with stationary cost functions and demand distribution, Veinott has shown that if inventory is below an m dimensional $\underset{\sim}{x}^*$ at the start, it is optimal to order up to that same point in the first period and each of the subsequent periods. If the starting inventory is not below $\underset{\sim}{x}^*$, Veinott finds the optimal policy when the products must be stocked in fixed proportions [23]. By making some assumptions on the cost functions, we have found the optimal policy for $m=2$ products when the proportional stocking restriction is removed. That policy turns out to be: Order none of the "overstocked" product(s), and order less of the other product(s) than would be ordered if there were no overstocked products. See Figure 1.1. The policy is stationary, as is $\underset{\sim}{x}^*$, in the sense that the optimal action at any point in time does not depend either on the number of periods remaining or on N. In addition, in Veinott's case and ours, the solution of a single one period problem is all that is required to obtain the "parameters" of the policy.

Motivated in part by our results, Veinott subsequently found the optimal policy for m products without the proportional stocking restriction, under conditions slightly stronger than ours [20]. His analysis in turn led to our generalizing of his conditions for 3 products.



N Period Policy, No Setup Cost

Figure 1.1



One Period Policy, Setup Cost

Figure 1.2

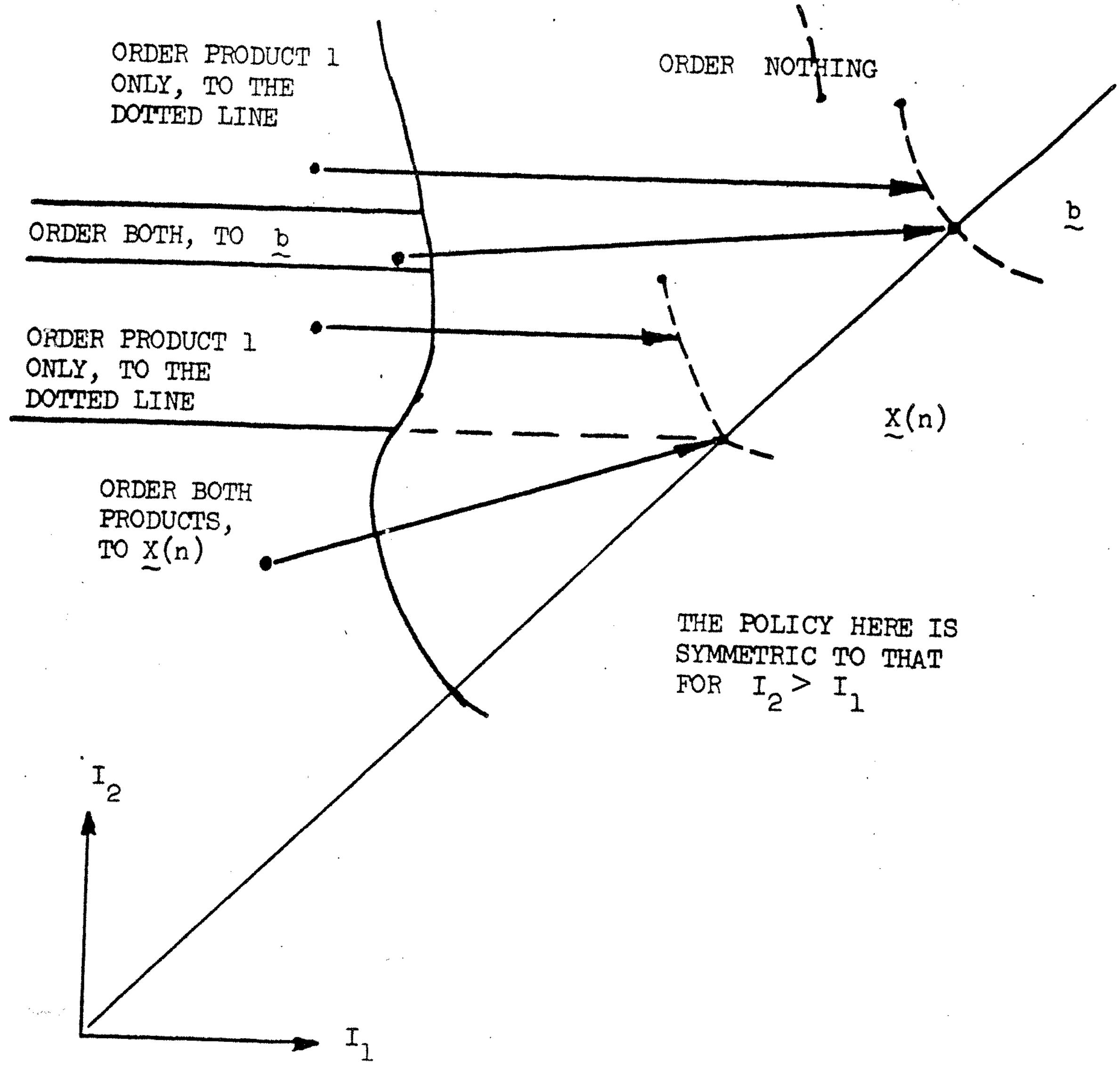
In [23], Veinott permits the cost functions and demand distributions to be non-stationary if they are such that the points that minimize expected cost for the associated one period problems "nest". By nesting, we mean that for $t=1,2,\ldots,N-1$, the optimal point for period t considered by itself is below the optimal point for period $t+1$ considered by itself. If these points do nest, and if starting inventory is below the first of them, Veinott proves that the optimal policy is to order up to each of them in turn. We are able to obtain a similar result for the case where the costs and demand are such that there is nesting of the single period "order-to" curves.

Setup Cost: Chapter 4 For the case where there is a setup cost, we confine ourselves to 2 product systems. We assume that there is some setup cost advantage to ordering both products together. In other words, the setup cost for ordering both products together is not greater than the sum of the two setup costs for separate ordering. We obtain optimal one period policies which are roughly of the (s,S) type. See Figure 1.2 for an example. For $N > 1$ we are able to obtain the optimal policy for the case where (1) the cost functions and demand distribution are symmetric in the 2 products and (2) the setup cost is the same for ordering one, the other, or both products. The symmetry restriction is quite strong; however the result is, to our knowledge, the first multiperiod, multiproduct, setup cost result. See Figure 1.3 for an example.

1.2 Discussion of Some of our Assumptions

A Crucial Assumption In this thesis we follow Veinott in making the assumption that at the end of the N periods (1) any excess inventory can be returned with full refund and (2) any excess demand must be filled, at the usual purchase price. This assumption, which Veinott and Wagner [26] credit to Beckmann [6], is crucial to the niceness of some of the results obtained here and by Veinott. For example, without it, even for one product, no setup cost, the optimal "order to" point is non-increasing with time (see page 3 of Iglehart [10]); that is, not necessarily stationary. A possible intuitive explanation of the non-stationarity that results without the assumption is the following: Compare the decision with two periods to go with that in the last period. The possible holding and shortage cost consequences next period are the same; but in the last period, demand less than inventory implies the loss of the purchase cost investment on their difference. With two periods to go, this period's demand being less than inventory is not so tragic, since there is always the last period, during which their difference might be sold.

The assumption makes it possible to include the linear part of purchasing cost in holding cost, thus converting the no setup cost problem into one with no purchase cost, only holding and shortage cost.



N Period Policy, Setup Cost

Figure 1.3

For the setup cost problem, setup, holding, and shortage costs remain.

Of course, the assumption is not plausible, but if N is large, as it would be in most applications, the cost that is neglected will be small relative to other costs. We feel that the advantages of relative ease in obtaining the optimal policy and in finding its "parameters" justify the assumption.

Why Multiproduct? The bulk of the literature in mathematical inventory theory deals with single product problems while most firms stock many products. Is explicit treatment of multiproduct problems necessary, or are solutions to single product problems sufficient? Our feeling is that single product solutions are often insufficient, because they cannot adequately take into account constraints or costs that depend on total inventory. In many cases they effectively assume away a potentially important part of the problem, that of optimum allocation of a given total inventory among the products. We deal with this issue in more detail in Sections 11 and 12 of Chapter 2.

Choice of Review Period A drawback to the utility of much of the development in inventory theory is the assumption that the length of the period between reviews is given. In practical situations, the choice of period length can be as important as the choice of policy given the period length. Shorter review periods normally bring reduced inventory. The resulting holding cost reduction must be balanced against the cost of reviewing more often. Although aware of this problem, we will not explicitly deal with the choice of review period length.

In many real situations, for order quantity in a given range, the amount paid to the supplier is a linear function of the amount ordered. If this order quantity is sizable, the setup cost associated with placing the order (if there is one) will be small compared to the amount paid to the supplier, and can be neglected. This leads to the common assumption of a linear purchasing cost. However, for linear purchasing cost, the optimal policy is to place an order in every period. Consequently, if the periods are short enough, the resulting order quantities will be small and setup costs can no longer be neglected.

This same line of reasoning calls for the explicit consideration of quantity discounts when the review period is short. However, with a general concave purchase cost function, even for one product, the optimal policy is not simple: Although a single point divides the ordering region from the region where no order is placed, in the former region the point that is ordered up to is no longer a constant, but is a function of inventory before ordering. See Karlin, pp. 120-4 and pp. 149-5 in [3]. This complexity in the one product case has deterred us from investigating quantity discounts in the multi-product case.

1.3 <u>The Literature</u> We will not systematically review the literature of inventory theory, but instead refer here to publications that are closely related to our work. Recent and excellent reviews of the field have been written by Scarf [15] and Veinott [25].

When there is no setup cost, we have already referred to the multiproduct work of Veinott [23]. Bellman, Glicksberg, and Gross [7] have treated the multiproduct problem when the products have independent costs but (possibly) correlated demands. They show that the optimal policy is the "sum" of the one product solutions obtained using the marginal demand distributions. (See Section 12 of Chapter 2.) Evans [9] has investigated multiproduct systems where there is a maximum order quantity, as could be the case when the products are made rather than bought.

When there is a setup cost, we known of no published periodic review results. Zangwill's treatment of a multiproduct situation [29] permits setup costs, but assumes that demands are known constants. The one product work on which our results are based is that of Scarf [14], Veinott and Wagner [26], and Veinott [22]. Balintfy [5] has proposed a policy for continuous review that is similar to our optimal one period policy. (See Section 5 of Chapter 4.)

Chapter 2

Two Product Systems, No Setup Cost

2.1 Summary This chapter is devoted to finding optimal ordering policies for two product systems when there is no setup cost associated with placing an order. We first explicitly introduce notation for the costs and demand distribution and general assumptions about the management of the system. Then we convert the linear purchasing cost into holding cost. The introduction of Property A2 follows, and it is assumed that expected one period holding and shortage cost has that property. A derivation of the one period optimal policy and the N period optimal policy for the stationary instant delivery case comes next. Then the case where each order is delivered a fixed number of periods after it is placed is considered, followed by the case of non-stationary cost functions and demand distributions. A comparison of Property A2 with convexity and other properties, a look at whether reasonable holding and shortage cost functions imply any of these properties, and a discussion of the problems encountered in specifying these cost functions follow. The chapter is concluded by an assessment of the need for explicit treatment of the two product aspect.

2.2 Introduction It will be convenient to number the periods so that from a calendar viewpoint period 1 is the last and period N is the first. Vectors, which will be underlined, will be assumed to be column vectors. The holding and shortage costs in a period are determined by functions $h(\cdot)$ and $p(\cdot)$ respectively. To allow holding cost to be based on inventory on hand at the start of the period, or that on hand at the end, or both, the holding cost function $h(\cdot,\cdot)$ (which will be written as $h(\cdot)$ for brevity) will be defined on both starting inventory and demand in the period. For example, if holding cost is linear in the average of starting and ending inventories, the $h(\underset{\sim}{x},\underset{\sim}{D}) = h_1\left(\frac{x_1 + (x_1-D_1)}{2}\right) + h_2\left(\frac{x_2+(x_2-D_2)}{2}\right)$. The shortage cost function $p(\cdot)$ will be defined on the end of period inventory vector. For example, if the shortage cost is linear in the amount short, $p(\underset{\sim}{x}) = p_1(\max(0,-x_1)) + p_2(\max(0,-x_2))$.

Demand in a period is a vector random variable with probability density function $\varphi(\cdot)$. Demands for individual products within a period must be non-negative but need not be independent. The demand vectors, $\underset{\sim}{D}_N, \underset{\sim}{D}_{N-1}, \ldots, \underset{\sim}{D}_1$, are assumed to be independent of one another. Excess demand is backlogged - that is, shipped as soon as stock becomes available-rather than lost.

There is a discount factor, α, which gives the present worth of a dollar that becomes available one period from now. It will be assumed that $0 \leq \alpha \leq 1$.

The purchasing cost is linear in the amount ordered, so that if $\underset{\sim}{x}$ is the order quantity, the purchase cost is $\underset{\sim}{c}'\underset{\sim}{x}$.

If $h(\cdot)$, $p(\cdot)$, $\varphi(\cdot)$, and $\underset{\sim}{c}$ change with time, they will be subscripted with the time period in which they apply.

2.3 Converting Purchasing Cost into Holding Cost

Let $\underset{\sim}{q}_n \equiv$ inventory before ordering in period n

$\underset{\sim}{x}_n \equiv$ inventory after ordering in period n

$\underset{\sim}{D}_n \equiv$ demand in period n

We assume the following sequence of events in period n.

(a) Review of inventory position

(b) Order placement and delivery

(c) Demand.

Note that we are assuming instant delivery.

The quantity ordered in period n is $\underset{\sim}{x}_n - \underset{\sim}{q}_n$. We assume that if demand exceeds supply in any period, their difference is backlogged, so that $\underset{\sim}{q}_{n-1} = \underset{\sim}{x}_n - \underset{\sim}{D}_n$. We assume that any excess inventory after demand in period 1 is returnable and any backlog must be purchased, and we specify a price vector $\underset{\sim}{c}_0$ for these transactions. Then we can write the total discounted purchasing cost over the N periods as TDPC = $\sum_{n=1}^{N} \alpha^{N-n} \underset{\sim}{c}_n'(\underset{\sim}{x}_n - \underset{\sim}{q}_n) - \alpha^N \underset{\sim}{c}_0'$. Substituting for $\underset{\sim}{q}_n$ for $n=0,1,\ldots,N-1$,

$$\text{TDPC} = \sum_{n=1}^{N-1} \alpha^{N-n} \underset{\sim}{c}_n' (\underset{\sim}{x}_n - (\underset{\sim}{x}_{n+1} - \underset{\sim}{D}_{n+1}))$$

$$+ \alpha^0 c_N' (\underset{\sim}{x}_N - \underset{\sim}{q}_N) - \alpha^N \underset{\sim}{c}_0' (\underset{\sim}{x}_1 - \underset{\sim}{D}_1)$$

$$= \sum_{n=1}^{N} \alpha^{N-n+1} \underset{\sim}{c}_{n-1}' \underset{\sim}{D}_n - \underset{\sim}{c}_N \underset{\sim}{q}_N$$

$$+ \sum_{n=1}^{N} \alpha^{N-n} (\underset{\sim}{c}_n - \alpha\, \underset{\sim}{c}_{n-1})' \underset{\sim}{x}_n .$$

Since $\underset{\sim}{q}_N$, the initial inventory, is assumed to be given, and since ordering policy does not influence demand, the first and second terms of TDPC are unaffected by the choice of policy and can be (and are) discarded. The third term depends only on $\underset{\sim}{x}_N, \underset{\sim}{x}_{N-1}, \ldots, \underset{\sim}{x}_1$ and we will henceforth consider it as a holding cost.

2.4 Expected Holding and Shortage Cost; Property A2

Now the expected holding and shortage cost, this period, if $\underset{\sim}{x}$ is the inventory on hand after ordering can be specified. We will call it $L(\cdot)$, and if the costs and demand distribution are stationary, we have

$$L(\underset{\sim}{x}) = \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} [(1-\alpha)\, \underset{\sim}{c}'\underset{\sim}{x} + h(\underset{\sim}{x},\underset{\sim}{t}) + p(\underset{\sim}{x}-\underset{\sim}{t})]\varphi(t)\, dt$$

Note that in addition to assuming $\underset{\sim}{c}_1 = \underset{\sim}{c}_2 = \cdots = \underset{\sim}{c}_n = \underset{\sim}{c}$, we have assumed that $\underset{\sim}{c}_0 = \underset{\sim}{c}$. We also assume that the integral exists; that is, that $\varphi(\underset{\sim}{t})$ gets small faster than $p(\underset{\sim}{x} - \underset{\sim}{t})$ gets large as $\underset{\sim}{t}$ increases, for any $\underset{\sim}{x}$. (It is not sufficient for all moments of $\varphi(\cdot)$ to exist: In the one product case, $\varphi(t) \equiv e^{-t}$ for $t \geq 0$ implies that all moments exist, yet $p(y) \equiv e^{-y}-1$ for $y \leq 0$ and $p(y) \equiv 0$ for $y > 0$ implies that, for $x < 0$, $\int_0^\infty p(x-t)\varphi(t)dt = \int_0^\infty (e^{+t-x}-1)(e^{-t})dt = e^{-x} \int_0^\infty d_t - 1$, and $\int_0^\infty dt$ does not exist. Therefore $L(\cdot)$ would not exist for $x < 0$.)

We now define Property A2, and hereafter it will be assumed that $L(\cdot)$ has this property. The property, as we define it, is stronger than is necessary to prove the results we obtain. We have chosen not to generalize it, however, to avoid lengthening the proofs. A discussion of what is and is not essential in A2, and the relationship of A2 to other properties (convexity, etc.) is deferred to Sections 9 and 10.

Partial derivatives will be written in the form

$D_1L(\underset{\sim}{x})$, $D_2L(\underset{\sim}{x})$, $D_{12}\, L(\underset{\sim}{x})$ where $D_1L(\underset{\sim}{x}) \equiv \dfrac{\partial L(x_1,x_2)}{\partial x_1}$, etc.

This notation makes it easy to distinguish a partial derivative from a total derivative when both are evaluated at a point where one component is a function of the other.

<u>Property A2</u>: Consider a two product system. Let $L(\underset{\sim}{x})$ be the expected holding and shortage cost, this period, if $\underset{\sim}{x}$ is the amount on hand after delivery but before demand. We say that $L(\cdot)$ has Property A2 if

(a) $L(\cdot)$ is non-negative everywhere

(b) $L(\cdot)$ has continuous second partial derivatives

(c) The set of points $\{(z_1(x_2),x_2) \mid D_1L(z_1(x_2),x_2) = 0\}$ constitute a single-valued, continuously differentiable, non-increasing function of x_2. For (x_1,x_2) such that $x_1 < z_1(x_2)$, $D_1L(x_1,x_2) < 0$ and for (x_1,x_2) such that $x_1 > z_1(x_2)$, $D_1L(x_1,x_2) > 0$.

(d) The set of points $\{(x_1,z_2(x_1)) \mid D_2L(x_1,z_2(x_1)) = 0\}$ constitute a single-valued, continuously differentiable, non-increasing function of x_1. For (x_1,x_2) such that $x_2 < z_2(x_1)$, $D_2L(x_1,x_2) < 0$, and for (x_1,x_2) such that $x_2 > z_2(x_1)$, $D_2L(x_1,x_2) > 0$.

(e) The two functions described in (c) and (d) have exactly one point of intersection, call it $\underset{\sim}{x}^* \equiv (x_1^*, x_2^*)$. For $x_1 > x_1^*$, $D_1L(x_1,z_2(x_1)) > 0$ and for $x_1 < x_1^*, D_1L(x_1,z_2(x_1)) < 0$. In other words, the curve $D_1L = 0$ lies below the curve $D_2L = 0$ for $x_1 > x_1^*$ and above $D_2L = 0$ for $x_1 < x_1^*$.

See Figure 2.1 for an illustration of property A2.

We define $C_n(\underset{\sim}{q})$ to be the minimum expected total discounted cost over n periods if $\underset{\sim}{q}$ is the inventory on hand, before ordering, in period n. We set $C_0(\underset{\sim}{q}) = 0$ for every $\underset{\sim}{q}$. We define

$$G_n(\underset{\sim}{q}) \equiv L(\underset{\sim}{q}) + \alpha \int_{\underset{\sim}{t} \geq 0} C_{n-1}(\underset{\sim}{q}-t)\ \varphi\ (t)dt.$$

Since $C_n(\cdot)$ is the expected cost when an optimal policy is followed, it must satisfy

$$C_n(\underset{\sim}{q}) = \min_{\underset{\sim}{x} \geq \underset{\sim}{q}} \{L(\underset{\sim}{x}) + \alpha \int_{t \geq \underset{\sim}{0}} C_{n-1}(\underset{\sim}{x}-\underset{\sim}{t})\varphi(t)dt\}$$

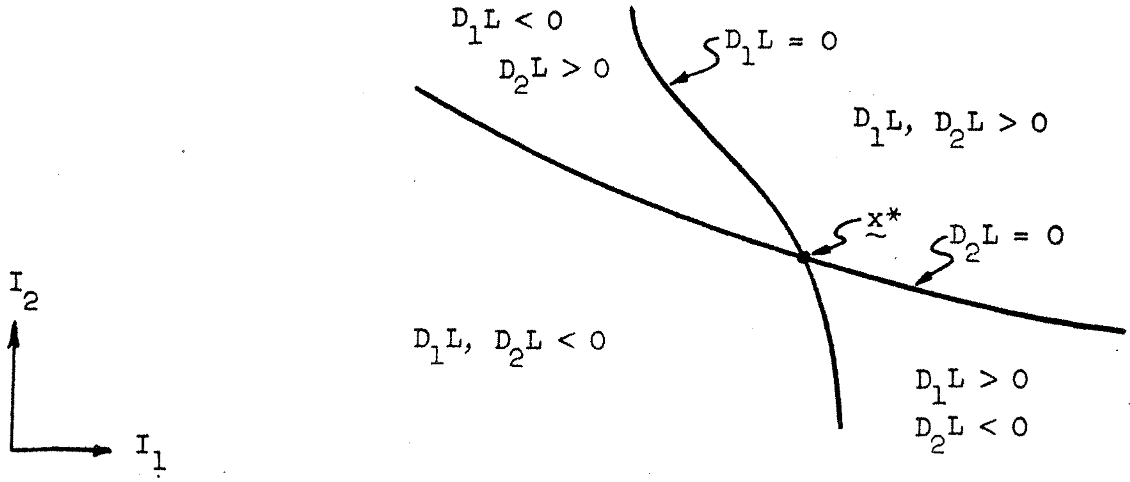
$$= \min_{\underset{\sim}{x} \geq \underset{\sim}{q}} G_n(x)\ .$$

2.5 The One Period Problem, Stationary Costs and Demand

We now obtain the one period optimal policy and $C_1(\cdot)$.

Lemma 2.1. If $L(\cdot)$ has Property A2 then $L(\cdot)$ is non-decreasing as we move away from $\underset{\sim}{x}^*$ on either $\{z_1(x_2),x_2\}$ or $\{x_1,z_2(x_1)\}$.

Proof: We consider the case of $z_1(x_2)$ for $x_2 > x^*_2$. The other three cases are analogous and will be omitted. Consider any u and v such that $u > v \geq x^*_2$. Speaking loosely, we could construct a series of line segments of non-positive slope, lying between $D_1L=0$ and $D_2L=0$, which connect $(z_1(u),u)$ and $(z_1(v),v)$. See Figure 2.2. Since $D_1L \leq 0$ and $D_2L \geq 0$ everywhere in the area through which these segments pass, successive application of the mean value theorem implies that $L(z_1(u),u) \geq L(z_1(v),v)$. Speaking rigorously, we can use a line integral theorem, see Apostol [1], page 280. Since $z_1(\cdot)$ is



Property A2 Illustrated

Figure 2.1

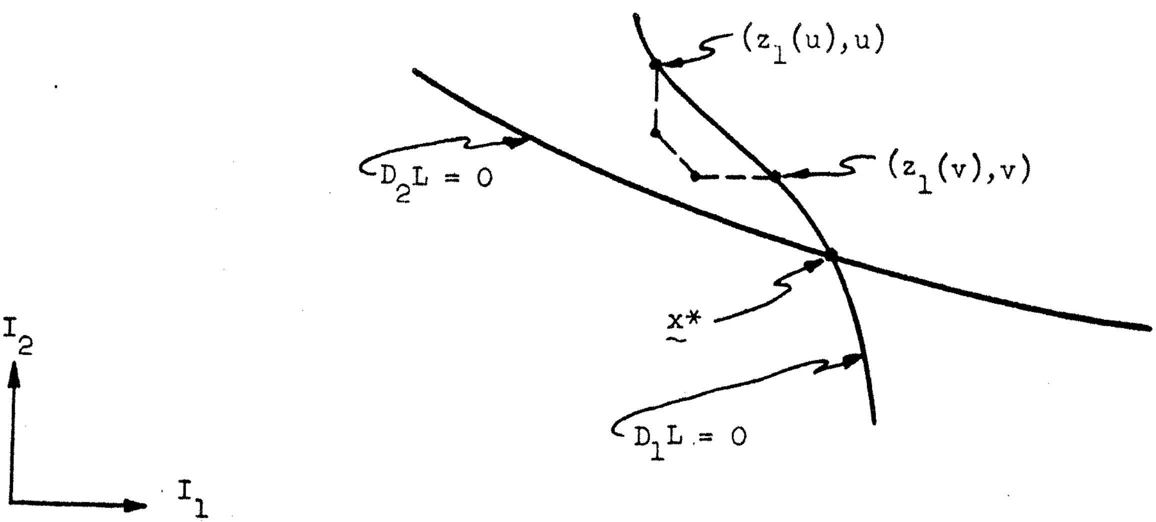


Figure 2.2

differentiable and its derivative is continuous, it is piecewise smooth and we can write

$$L(z_1(u),u) - L(z_1(v),v) = \int_{x_2=v}^{u} \nabla L(\underset{\sim}{x})\, dz_1(x_2).$$

Now $\nabla L(\underset{\sim}{x}) = (D_1 L(\underset{\sim}{x}),\ D_2 L(\underset{\sim}{x}))$ and $D_1 L(\underset{\sim}{x}) = 0$ for $\underset{\sim}{x} \in \{(z_1(Y_2),Y_2)\}$ and $D_2 L(\underset{\sim}{x}) \geq 0$ for $\underset{\sim}{x} \in \{(z_1(Y_2),Y_2)\}$ and $x_2 > x_2^*$, so the integrand is non-negative. Therefore, since $u > v$ the line integral is non-negative and the lemma is proved.

This lemma implies that $\underset{\sim}{x}^*$ is the point at which $L(\cdot)$ attains its minimum. That is, we have

Lemma 2.2. If $L(\cdot)$ has Property A2, then $L(\underset{\sim}{x}) \geq L(\underset{\sim}{x}^*)$ for every $\underset{\sim}{x}$.

Proof:

(1) Suppose $\underset{\sim}{x}$ lies on one of the two curves, $D_1L=0$ and $D_2L=0$. Then by Lemma 2.1, $L(\underset{\sim}{x}) \geq L(\underset{\sim}{x}^*)$.

(2) Suppose $\underset{\sim}{x}$ does not lie on either of the curves. Then it is clear from the mean value theorem that $L(\underset{\sim}{x}) \geq L(z_1(x_2),x_2)$. Therefore, by Lemma 2.1, $L(\underset{\sim}{x}) \geq L(x^*)$, and the proof is complete.

In the one period problem, $G_1(\underset{\sim}{q}) = L(\underset{\sim}{q})$ and the optimal policy, when inventory before ordering is $\underset{\sim}{q}$, is to order to the point that minimizes $L(\cdot)$ in the region $R_q \equiv \{\underset{\sim}{x} | \underset{\sim}{x} \geq \underset{\sim}{q}\}$. Therefore, if $\underset{\sim}{q} \leq \underset{\sim}{x}^*$, it is optimal to order up to $\underset{\sim}{x}^*$. If $q_2 > x_2^*$ and $q_1 \leq z_1(q_2)$, it is desirable to order up to $(z_1(q_2), q_2)$, since $D_1L(x_1,q_2) < 0$ for all $x_1 < z_1(q_2)$ and $D_1L(x_1,q_2) > 0$ for all $x_1 > z_1(q_2)$. It is also optimal, since if any amount of product 2, say Δ, is ordered, it would be desirable to go to $(z_1(q_2+\Delta), q_2+\Delta)$, and $L(z_1(q_2+\Delta), q_2+\Delta) \geq L(z_1(q_2), q_2)$ by Lemma 2.1. By symmetry, if $q_1 > x_1^*$ and $q_2 \leq z_2(q_1)$, only product 2 should be ordered, up to $z_2(q_1)$.

The only remaining region is that where D_1L and $D_2L > 0$, and clearly the optimal policy here is to do nothing.

If we define

$$R_{12} \equiv \{\underset{\sim}{q} \mid \underset{\sim}{q} \leq \underset{\sim}{x}^*\}$$
$$R_1 \equiv \{\underset{\sim}{q} \mid q_2 > x_2^*,\ q_1 \leq z_1(q_2)\}$$
$$R_2 \equiv \{\underset{\sim}{q} \mid q_1 > x_1^*,\ q_2 \leq z_2(q_1)\}$$
$$R_0 \equiv \{\underset{\sim}{q} \mid q_1 > x_1^*,\ q_2 > z_2(q_1) \text{ or } q_2 > x_2^*,\ q_1 > z_1(q_2)\}$$

then we have just proved

Lemma 2.3. If L has Property A2, then the optimal one period policy, when inventory before ordering is $\underset{\sim}{q}$ is to:

(i) order both products, up to $\underset{\sim}{x}^*$, if $\underset{\sim}{q} \in R_{12}$

(ii) order product 1 only, up to $z_1(q_2)$, if $\underset{\sim}{q} \in R_1$

(iii) order product 2 only, up to $z_2(q_1)$, if $\underset{\sim}{q} \in R_2$

(iv) order nothing, if $\underset{\sim}{q} \in R_0$.

The regions and the optimal policy are pictured in Figure 2.3.

Now we write down $C_1(\cdot)$ and obtain its first partial derivatives.

(a) For $\underset{\sim}{q} \leqq \underset{\sim}{x}^*$, we have $C_1(\underset{\sim}{q}) = L(\underset{\sim}{x}^*)$, and $D_1C_1(q) = D_2C_2(\underset{\sim}{q}) = 0$.

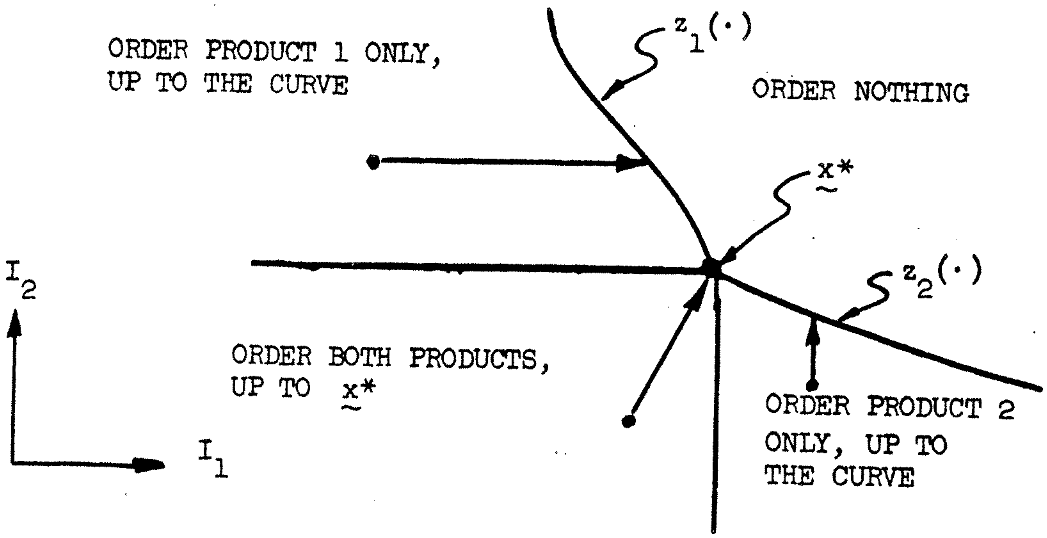
(b) For $q_2 > x_2^*$ and $q_1 \leqq z_1(q_2)$, we have $C_1(\underset{\sim}{q}) = L(z_1(q_2),q_2)$.

Therefore, $D_1C_1(\underset{\sim}{q}) = 0$ and

$$D_2C_1(\underset{\sim}{q}) = \lim_{\Delta \to 0} \left[\frac{L(z_1(q_2+\Delta),q_2+\Delta) - L(z_1(q_2),q_2)}{\Delta}\right]$$

$$= \lim_{\Delta \to 0} \frac{[L(z_1(q_2+\Delta),q_2+\Delta)-L(z_1(q_2+\Delta),q_2)]+[L(z_1(q_2+\Delta),q_2)-L(z_1(q_2),q_2)]}{\Delta}$$

$$= D_2L(z_1(q_2), q_2) + \left(\frac{dz_1(q_2)}{dq_2}\right) D_1L(z_1(q_2),q_2).$$



The Optimal One Period Policy

Figure 2.3

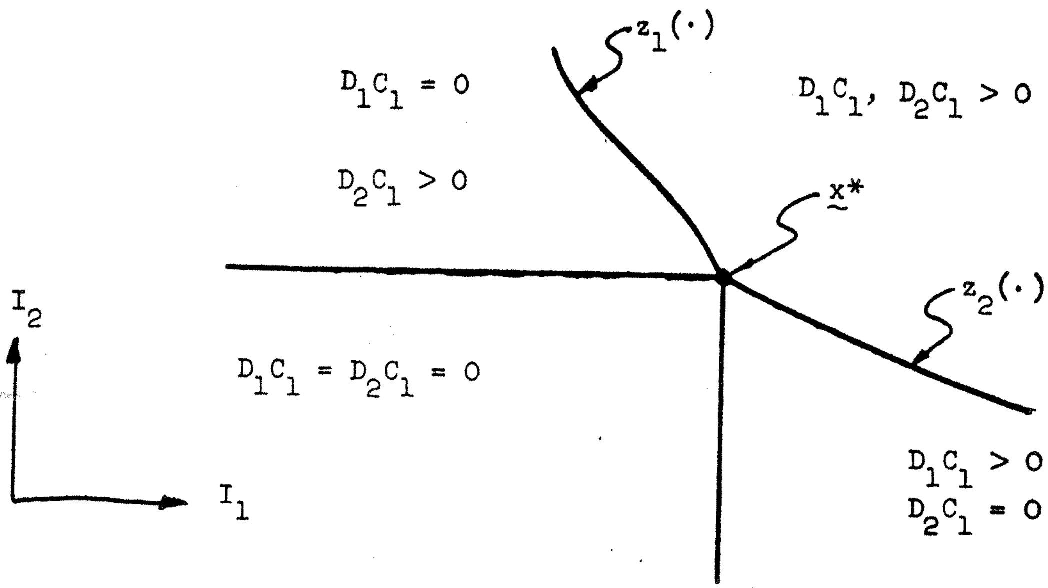


Figure 2.4

Since $D_1L(z_1(q_2),q_2) = 0$ and $D_2L(z_1(q_2),q_2) > 0$,
$D_2C_1(\underset{\sim}{q}) > 0$.

(c) For $q_1 > x_1^*$ and $q_2 \leq z_2(q_1)$, we have, by symmetry, $D_2C_1(\underset{\sim}{q}) = 0$ and $D_1C_1(\underset{\sim}{q}) > 0$.

(d) For either $q_1 > x_1^*$ and $q_2 > z_2(q_1)$ or $q_2 > x_2^*$ and $q_1 > z_1(q_2)$, we have $C_1(\underset{\sim}{q}) = L(\underset{\sim}{q})$, and $D_1C_1(\underset{\sim}{q}) > 0$ and $D_2C_1(\underset{\sim}{q}) > 0$.

This proves the following two lemmas

<u>Lemma 2.4.</u> If $L(\cdot)$ has Property A2, then if inventory before ordering is $\underset{\sim}{q}$, the behavior of the first partials of the one period expected cost function under an optimal policy is given by

(i) if $\underset{\sim}{q} \in R_{12}$, $D_1C_1(\underset{\sim}{q}) = D_2C_1(\underset{\sim}{q}) = 0$

(ii) if $\underset{\sim}{q} \in R_1$, $D_1C_1(\underset{\sim}{q}) = 0$, $D_2C_1(\underset{\sim}{q}) > 0$

(iii) if $\underset{\sim}{q} \in R_2$, $D_1C_1(\underset{\sim}{q}) > 0$, $D_2C_1(\underset{\sim}{q}) = 0$

(iv) if $\underset{\sim}{q} \in R_0$, $D_1C_1(\underset{\sim}{q}) > 0$, $D_2C_1(\underset{\sim}{q}) > 0$.

and

<u>Lemma 2.5.</u> If $L(\cdot)$ has Property A2, then if $\underset{\sim}{q}$, the inventory before ordering, is less than or equal to $\underset{\sim}{x}^*$, $C_1(\underset{\sim}{q}) = L(\underset{\sim}{x}^*)$.

The behavior of the partials of C_1 is pictured in figure 2.4.

The following result allows us to interchange differentiation and integration operations in Theorem 2.7. Since the proof gives no insights into inventory problems, it is deferred to Appendix A.

Lemma 2.6. If L has Property A2, then the first partials of $C_1(\cdot)$ are continuous.

2.6. The N Period Problem, Stationary Costs and Demand

Now we prove a theorem which shows that the optimal policy for the N period problem is given by $\underline{x}^*$ (which was shown by Veinott [23]) and by the functions $z_1(\cdot)$ and $z_2(\cdot)$. That is, the policy is of the same type as the one period policy, and further, the "parameters" of the policy are exactly those of the one period policy.

Theorem 2.7. If $L(\cdot)$ has Property A2 and if the first partials of $C_n(\cdot)$ are continuous and satisfy conditions (i) (ii) (iii) (iv) of Lemma 2.4, then

(a) the optimal policy with $n+1$ periods remaining is given by (i) (ii) (iii) (iv) of Lemma 2.3,

(b) the first partials of $C_{n+1}(\cdot)$ satisfy (i) (ii) (iii) (iv) of Lemma 2.4, and

(c) the first partials of $C_{n+1}(\cdot)$ are continuous.

Note: The proof of (c) is like that of Lemma 2.6 in giving no insight into inventory problems. Although straightforward, it is lengthy, and we omit it.

Proof:

$$G_{n+1}(\underset{\sim}{q}) \equiv L(\underset{\sim}{q}) + \alpha \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n(\underset{\sim}{q}-\underset{\sim}{t}) \varphi(\underset{\sim}{t})\, d\underset{\sim}{t}$$

and

$$C_{n+1}(\underset{\sim}{q}) = \min_{\underset{\sim}{x} \geq \underset{\sim}{q}} [G_{n+1}(\underset{\sim}{x})] .$$

First we show that $\underset{\sim}{x}^*$ is a global minimizer of $G_{n+1}(\cdot)$. We minimize $G_{n+1}(\cdot)$, here and later on, by minimizing the terms separately. This procedure works only if both minimizations yield the same point, which, though unusual in general, does occur here. Consider any point $\underset{\sim}{x}$. $L(\underset{\sim}{x}) \geq L(\underset{\sim}{x}^*)$ by Lemma 2.1. Consider any $\underset{\sim}{t} \geq \underset{\sim}{0}$. Since $C_n(\cdot)$ is constant below $\underset{\sim}{x}^*$ and is non-decreasing in x_1 and in x_2 everywhere, we have $C_n(\underset{\sim}{x}-\underset{\sim}{t}) \geq C_n(\underset{\sim}{x}^*-\underset{\sim}{t})$.
Therefore $\alpha \int_{\underset{\sim}{t}\geq\underset{\sim}{0}} C_n(\underset{\sim}{x}-\underset{\sim}{t}) \varphi(\underset{\sim}{t})\, d\underset{\sim}{t} \geq \alpha \int_{\underset{\sim}{t}\geq\underset{\sim}{0}} C_n(\underset{\sim}{x}^*-\underset{\sim}{t}) \varphi(\underset{\sim}{t})\, d\underset{\sim}{t}$ so

that $G_{n+1}(\underset{\sim}{x}) \geq G_{n+1}(\underset{\sim}{x}^*)$.[1] At this point, we can conclude that if $\underset{\sim}{q} \leq \underset{\sim}{x}^*$, it is optimal to order up to $\underset{\sim}{x}^*$.

Now suppose $q_2 > x_2^*$ and $q_1 \leq z_1(q_2)$. We show that

$\min_{\underset{\sim}{x} \geq \underset{\sim}{q}} G_{n+1}(\underset{\sim}{x}) = G_{n+1}(z_1(q_2),q_2)$, so that only product 1 is ordered, up to $z_1(q_2)$. First, by Lemma 2.1, $L(z_1(q_2),q_2) = \min_{\underset{\sim}{x} \geq \underset{\sim}{q}} L(x)$.

---

1 - This presumes that the integrals exist. That they do can be seen from the following. Let $T_1, T_2, \ldots$ be square regions defined by $T_k \equiv \{\underset{\sim}{x} | x_1, x_2 \in [0,k]\}$, $k=1,2,\ldots$. Since $L(\underset{\sim}{x}^*) \geq 0$, $C_n(y) > 0$ for every $\underset{\sim}{y}$. Therefore, since $\varphi(\cdot)$ is non-negative everywhere,

$$\int_{T_k} C_n(\underset{\sim}{x}-\underset{\sim}{t})\ \varphi(\underset{\sim}{t})d\underset{\sim}{t} \leq \int_{T_{k+1}} C_n(\underset{\sim}{x}-\underset{\sim}{t})\ \varphi(\underset{\sim}{t})d\underset{\sim}{t}.$$

If we define $\Omega_k(\underset{\sim}{x}) \equiv \int_{T_k} C_n(\underset{\sim}{x}-\underset{\sim}{t})\ \varphi\ (\underset{\sim}{t})d\underset{\sim}{t}$, then we have just proved that $\Omega_1(\underset{\sim}{x}), \Omega_2(\underset{\sim}{x}), \ldots$ are a monotone non-decreasing sequence. Since for any $\underset{\sim}{x}$, $C_n(\underset{\sim}{x}-\underset{\sim}{t}) \leq C_n(\underset{\sim}{x})$ for any $\underset{\sim}{t} \geq \underset{\sim}{0}$, $\Omega_k(\underset{\sim}{x}) \leq \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n(\underset{\sim}{x})\ \varphi\ (\underset{\sim}{t})\ d\underset{\sim}{t} = C_n(\underset{\sim}{x})$ for every k. Consequently the sequence is bounded, and so it has a limit. But

$$\int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n(\underset{\sim}{x}-\underset{\sim}{t})\varphi(t)dt = \lim_{k \uparrow \infty} \Omega_k(x),$$ so we have shown that the integral exists.

Consider $\underset{\sim}{x} \geq \underset{\sim}{q}$ and any $\underset{\sim}{t} > \underset{\sim}{0}$. Then, because $z_1(\cdot)$ has non-positive slope, $C_n(\underset{\sim}{x}-\underset{\sim}{t}) \geq C_n((z_1(q_2),q_2)-\underset{\sim}{t})$. (Figure 2.5 is an aid to seeing why this is true.) Therefore

$$\int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n((z_1(q_2),q_2)-\underset{\sim}{t})\ \varphi\ (\underset{\sim}{t})d\underset{\sim}{t} \leq \min_{\underset{\sim}{x} \geq \underset{\sim}{q}} [\int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n(\underset{\sim}{x}-\underset{\sim}{t})\varphi(\underset{\sim}{t})d\underset{\sim}{t}].$$

Again the two terms are minimized at the same point, namely $(z_1(q_2),q_2)$, so that only product 1 should be ordered, up to $z_1(q_2)$.

By symmetry, if $q_1 > x_1^*$ and $q_2 \leq z_2(q_1)$, only product 2 should be ordered, up to $z_2(q_1)$.

Now suppose $q_1 > x_1^*$ and $q_2 > z_2(q_1)$. By Lemma 2.1, $L(\underset{\sim}{q}) = \min\limits_{\underset{\sim}{x} \geq \underset{\sim}{q}} L(\underset{\sim}{x})$. Consider any $\underset{\sim}{x} \geq \underset{\sim}{q}$, and any $\underset{\sim}{t} \geq \underset{\sim}{0}$. $C_n(\underset{\sim}{x}-\underset{\sim}{t}) \geq C_n(\underset{\sim}{q}-\underset{\sim}{t})$, so that $\int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n(\underset{\sim}{x}-\underset{\sim}{t})\varphi(\underset{\sim}{t})d\underset{\sim}{t} > \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n(\underset{\sim}{q}-\underset{\sim}{t})\varphi(\underset{\sim}{t})d\underset{\sim}{t}$, and therefore $G_{n+1}(\underset{\sim}{q}) = \min\limits_{\underset{\sim}{x} \geq \underset{\sim}{q}} G_{n+1}(\underset{\sim}{x})$. Consequently the optimal policy here is to order nothing. If $q_2 > x_2^*$ and $q_1 > z_1(q_2)$, then by symmetry, the optimal policy is to order nothing.

This establishes (a). Now we show that the first partials of $C_{n+1}(\cdot)$ behave as stated in (b).

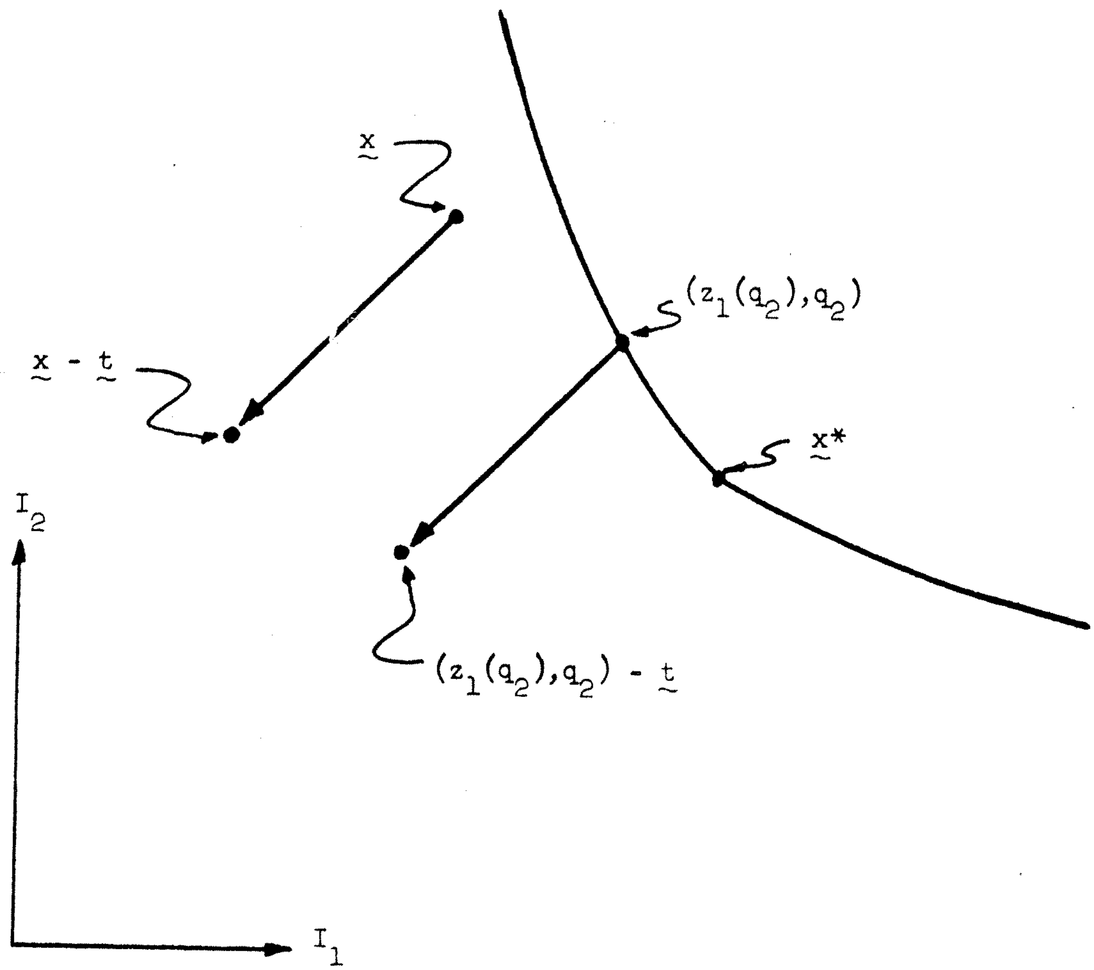


Figure 2.5

(i) Suppose $\underset{\sim}{q} \leq \underset{\sim}{x}^*$. Then $C_{n+1}(\underset{\sim}{q}) = L(\underset{\sim}{x}^*) + \alpha \int C_n(\underset{\sim}{x}^* - \underset{\sim}{t})\varphi(\underset{\sim}{t})d\underset{\sim}{t}$

$$= L(\underset{\sim}{x}^*) + \alpha C_n(\underset{\sim}{x}^*).$$

Therefore $D_1C_{n+1}(\underset{\sim}{q}) = D_2C_{n+1}(\underset{\sim}{q}) = 0$.

(ii) Suppose $q_2 > x_2^*$ and $q_1 \leq z_1(q_2)$. Then

$$C_{n+1}(\underset{\sim}{q}) = L(z_1(q_2),q_2) + \alpha \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n(z_1(q_2),q_2)-\underset{\sim}{t})\varphi(\underset{\sim}{t})d\underset{\sim}{t}.$$

Clearly, $D_1C_{n+1}(\underset{\sim}{q}) = 0$. To get $D_2C_{n+1}(\underset{\sim}{q})$, consider the two terms separately. For the first term we get $\frac{d}{dq_2} L(z_1(q_2),q_2)$ which was shown to be > 0 in Lemma 2.4. For the second term we get $\frac{d}{dq_2} \alpha \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n((z_1(q_2),q_2)-\underset{\sim}{t})\varphi(\underset{\sim}{t})d\underset{\sim}{t}$. The continuity of the partials of $C_n(\cdot)$ ensures that it equals

$$\alpha \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} \frac{d}{dq_2} [C_n((z_1(q_2),q_2)-t)]\, \varphi(\underset{\sim}{t})d\underset{\sim}{t}$$

and that this integral exists.

$$\frac{d}{dq_2} C_n((z_1(q_2),q_2)-\underset{\sim}{t}) = D_2C_n((z_1(q_2)-t)$$
$$+ \frac{d(z_1(q_2)-t_1)}{dq_2} \cdot D_1C_n((z_1(q_2),q_2)-\underset{\sim}{t}).$$

For $\underset{\sim}{t} \geq \underset{\sim}{0}$, $D_1C_n((z_1(q_2),q_2)-\underset{\sim}{t}) = 0$ and $D_2C_n((z_1(q_2),q_2)-\underset{\sim}{t}) \geq 0$.

Therefore

$$\frac{d}{dq_2} \alpha \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n((z_1(q_2),q_2)-\underset{\sim}{t})\varphi(\underset{\sim}{t})d\underset{\sim}{t} \geq 0$$ and so $D_2C_{n+1}(\underset{\sim}{q}) > 0$.

(iii) Suppose $q_1 > x_1^*$ and $q_2 \leq z_2(q_1)$. Then by symmetry,

$$C_{n+1}(\underset{\sim}{q}) = L(q_1, z_2(q_1)) + \alpha \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n((q_1, z_2(q_1)) - \underset{\sim}{t})\varphi(\underset{\sim}{t})d\underset{\sim}{t},$$

and $D_1 C_{n+1}(\underset{\sim}{q}) > 0$ and $D_2 C_{n+1}(\underset{\sim}{q}) = 0$.

(iv) Suppose $q_1 > x_1^*$ and $q_2 > z_2(q_1)$. The case $q_2 > x_2^*$ and $q_1 > z_1(q_2)$ is analogous and will be omitted.

$$C_{n+1}(\underset{\sim}{q}) = L(\underset{\sim}{q}) + \alpha \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n(\underset{\sim}{q} - \underset{\sim}{t})\varphi(\underset{\sim}{t})d\underset{\sim}{t}.$$

$D_1 L(\underset{\sim}{q}) > 0$ since L has Property A2. Again the continuity of the partials of $C_n(\cdot)$ implies that

$$D_1[\alpha \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n(\underset{\sim}{q} - \underset{\sim}{t})\ \varphi\ (\underset{\sim}{t})d\underset{\sim}{t}] = \alpha \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} D_1 C_n(\underset{\sim}{q} - \underset{\sim}{t})\varphi(\underset{\sim}{t})d\underset{\sim}{t},$$

with the second integral existing. This integral is non-negative because $D_1 C_n(\cdot)$ is non-negative everywhere, and therefore $D_1 C_{n+1}(\underset{\sim}{q}) > 0$. By symmetry, $D_2 C_{n+1}(\underset{\sim}{q}) > 0$, and the proof of (b) is complete.

An interesting consequence of the theorem is <u>Corollary 2.8</u>. If $L(\cdot)$ has Property A2, then if $\underset{\sim}{q} \leq \underset{\sim}{x}^*$, $C_n(\underset{\sim}{q}) = (\frac{1-\alpha^n}{1-\alpha}) \cdot L(\underset{\sim}{x}^*)$ for $n=1,2,\ldots$

Proof: Theorem 2.7 implies that

$$C_n(\underset{\sim}{q}) = L(\underset{\sim}{x}^*) + \alpha C_n(\underset{\sim}{x}^*) \quad \text{for} \quad n=1,2,\ldots$$

Now $C_1(\underset{\sim}{x}^*) = L(\underset{\sim}{x}^*)$, so that by successive substitution,

$$C_n(\underset{\sim}{q}) = L(\underset{\sim}{x}^*) \, [1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1}]$$

$$= L(\underset{\sim}{x}^*) \, . \, (\frac{1-\alpha^p}{1-\alpha}).$$

2.7 A Fixed Delivery Lag

In this section, we show that the results for the instant delivery case apply to the fixed delivery lag problem if Property A2 obtains for a redefined expected holding and shortage cost function.

Suppose there is a delivery lag of λ periods, where λ is a positive integer. In this case, an order is delivered at the start of the period λ periods after it is placed. The lag is assumed to be fixed; it does not change with time, and there is no possibility of an order being delivered early or late. The lag is assumed to be the same for each of the products.

We assume the following sequence of events in period n.

(1) Review of inventory position

(2) Order placement, for delivery λ periods hence

(3) Arrival of order placed λ periods ago

(4) Demand.

Then, continuing our period numbering convention, we <u>define</u>

$\underset{\sim}{q}_n$ ≡ inventory on hand at review time in period n

$\underset{\sim}{t}_n$ ≡ amount ordered in period n, for delivery in period $n-\lambda$

$\underset{\sim}{x}_n$ ≡ inventory on hand in period n just after the arrival of the order placed in period $n + \lambda$.

$\underset{\sim}{D}_n$ ≡ demand in period n.

Then $\underset{\sim}{x}_n$ is the stock available to meet demand in period n. Consequently holding and shortage cost incurred in period n both depend only on $\underset{\sim}{x}_n$ and $\underset{\sim}{D}_n$.

After the conversion of purchasing cost into holding cost, which follows below, the redefinition of expected holding and shortage cost is possible, and the applicability of the instant delivery results is immediate.

We assume that the firm intends to sell the two products for the next N periods. Consequently, the firm has ordering decisions to make in each of the next $N-\lambda$ periods, there being no point in placing any orders after that since delivery would be too late to be of use. We assume, for convenience only, that the items are paid for in the period in which they are received.[1] Then the

---

1 - All that need be assumed is that the items are paid for r periods after they are ordered, where r is non-negative and remains fixed through time.

total discounted purchasing cost for the N periods can be written as TDPC =

$$\sum_{n=\lambda+1}^{N} \underset{\sim}{c}_n' \, \underset{\sim}{t}_n \, \alpha^{N-n+\lambda} - \underset{\sim}{c}_\lambda' \underset{\sim}{q}_0 \, \alpha^N,$$ where $\underset{\sim}{c}_n$ is the purchase

price for items ordered in period n and where excess inventory at the end is returnable and backlog must be purchased. We observe that $\underset{\sim}{q}_n = \underset{\sim}{x}_n - \underset{\sim}{t}_{n+\lambda}$ and, because of the backlogging assumption, $\underset{\sim}{q}_n = \underset{\sim}{x}_{n+1} - \underset{\sim}{D}_{n+1}$. Therefore we can substitute $\underset{\sim}{x}_{n-\lambda} - \underset{\sim}{x}_{n+1-\lambda} + \underset{\sim}{D}_{n+1-\lambda}$ for $\underset{\sim}{t}_n$ and $\underset{\sim}{x}_1 - \underset{\sim}{D}_1$ for $\underset{\sim}{q}_0$. Making these substitutions and collecting terms, we get TDPC = $\sum_{n=\lambda}^{N} \alpha^{N-n+\lambda} \underset{\sim}{c}_n' \underset{\sim}{D}_{n-\lambda+1} - \alpha^{\lambda} \underset{\sim}{c}_N' \underset{\sim}{x}_{N-\lambda+1}$

$+ \sum_{n=1}^{N-\lambda} \alpha^{N-n} (\underset{\sim}{c}_{n+\lambda} - \alpha \underset{\sim}{c}_{n+\lambda-1})' \underset{\sim}{x}_n$. Since demand is not affected by

ordering policy, the first term can be neglected. Similarly, $\underset{\sim}{x}_{N-\lambda+1}$ depends only on demand and on orders placed prior to period N, so the second term can be neglected. This leaves the third term, which can and will be treated as if it were a holding cost. For the remainder of this section, we also set $\underset{\sim}{c}_n = \underset{\sim}{c}$ for $n = \lambda, \lambda+1, \ldots, N$.

To begin the redefinition of expected holding and shortage cost, we define

$$\underset{\sim}{y}_n = \underset{\sim}{q}_n + \underset{\sim}{t}_n + \underset{\sim}{t}_{n+1} + \ldots + \underset{\sim}{t}_{n+\lambda} = \underset{\sim}{x}_n + \underset{\sim}{t}_n + \ldots + \underset{\sim}{t}_{n+\lambda-1}.$$

We call $\underset{\sim}{y}_n$ the stock on hand and on order in period n. The backlogging of demand implies that $\underset{\sim}{x}_{n-\lambda} = \underset{\sim}{y}_n - (\underset{\sim}{D}_n + \underset{\sim}{D}_{n-1} + \ldots + \underset{\sim}{D}_{n-\lambda+1})$. We define $\Phi^*_\lambda(\tilde{y}) = \Pr(\tilde{D}_1+\ldots+\tilde{D}_\lambda \leq \tilde{y})$ and let $\varphi^*_\lambda(\cdot)$ be its probability density function. Then the pdf of $\tilde{D}_n+\ldots\tilde{D}_{n-\lambda+1}$ is $\varphi^*_\lambda(.)$. We define $L^{(\lambda)}(\tilde{y})$ to be the expected holding and shortage cost, incurred λ periods from now, if stock on hand and on order now is $\tilde{y}$.

$$L^{(\lambda)}(\tilde{y}) = \int_{\tilde{z}\geq \tilde{0}} \int_{\tilde{t}\geq \tilde{0}} \begin{bmatrix} h(\tilde{y}-\tilde{z},\tilde{t}) \\ +p(\tilde{y}-\tilde{z}-\tilde{t}) \\ +(1-\alpha)\tilde{c}'(\tilde{y}-\tilde{z}) \end{bmatrix} \varphi^*_\lambda(\tilde{z})\ \varphi(\tilde{t})d\tilde{z}d\tilde{t}$$

which can be rewritten as

$$\int_{\tilde{z}\geq\tilde{0}} L(\tilde{y}-\tilde{z})\ \varphi^*_\lambda(\tilde{z})d\tilde{z}.$$

We assume that $L^{(\lambda)}(\cdot)$ has Property A2. (This neither implies that $L(\cdot)$ has A2 nor it is implied by $L(\cdot)$ having A2. See Appendix B for examples.) Then treating stock on hand and on order as we did inventory in the instant delivery case, the entire development for that case applies here. That is, in any period, if the total of inventory on hand and all orders previously placed and yet undelivered is less than $\tilde{x}^*$, the global minimizer of $L^{(\lambda)}(\cdot)$, it is optimal to place an order for the difference between $\tilde{x}^*$ and what is on hand and on order. If their total is not below $\tilde{x}^*$ but is to the left of $D_1L^{(\lambda)}(\underset{\sim}{x}) = 0$, it is optimal to order product 1 only, up to the curve. Similarly, if it is not below $\underset{\sim}{x}^*$ but is below $D_2L^{(\lambda)}(\underset{\sim}{x}) = 0$, it is optimal to order product 2 only, up to the curve. Finally, if their total is both to the right of $D_1L^{(\lambda)}(\underset{\sim}{x}) = 0$ and above $D_2L^{(\lambda)}(\underset{\sim}{x}) = 0$, it is optimal to not place any order.

2.8 Non Stationary Costs and Demand

Under certain conditions, the cost functions and demand distribution can change with time without changing the form of the optimal policy from that obtained for stationary costs and demand. The conditions are

(1) the order to points, $\underset{\sim}{x}_1^*, \underset{\sim}{x}_2^*, \ldots, \underset{\sim}{x}_n^*$, obtained by considering each period by itself, form a non-increasing sequence, and

(2) the order to curves, $z_{1,1}(\cdot)$, $z_{2,1}(\cdot)$, $\ldots$, $z_{N,1}(\cdot)$ and $z_{1,2}(\cdot)$, $z_{2,2}(\cdot), \ldots,$ $z_{N,2}$, obtained by considering each period by itself, are each non-increasing sequences of functions.

Then these points and curves, each determined by reference to a one period problem, give the optimal policy for the N period problem. That is, if the conditions obtain, behaving in each period as if it were the last one is, in fact, optimal behavior for the entire sequence of periods. Consequently, this result may be interpreted as a planning horizon theorem, in the sense of Modigliani and Hohn [12] and Wagner and Whitin [27].

To gain insight into why these conditions permit this separation into one period problems, consider the following. With purchase cost incorporated into holding cost, from the standpoint of expected cost in period 1, there is no disadvantage in having as little as possible on hand at the start of period 1. The conditions ensure that, for any starting inventory in period 2, the optimal policy for period 2 considered by itself calls for ordering up to a point that would allow, whatever the demand in period 2, ordering up to the same point in period 1 as ordering nothing in period 2 would. So behaving in period 2 as if it were the only period cannot increase cost in period 1. This reasoning, applied to periods n and $n-1$ instead of 2 and 1 for $n = 3, 4, \ldots, N$ implies our result. Theorem 2.7 can be considered as a special case of this result.

In [23], Veinott discusses conditions on the $h_n(\cdot)$'s, $p_n(\cdot)$'s, $\varphi_n(\cdot)$'s and $\underset{\sim}{c}_n$'s that ensure nesting of the $\underset{\sim}{x}_n{}^*$'s. He leaves detailed results for "subsequent papers." There is some question in our mind about the ability (in a logical sense) to specify period specific holding cost functions. This issue is considered in Section 11.

To begin the formal analysis, we define

$$L_n(\underset{\sim}{x}) \equiv \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} \begin{bmatrix} (\underset{\sim}{c}_n - \alpha\, \underset{\sim}{c}_{n-1})'\underset{\sim}{x} \\ +h_n(\underset{\sim}{x},\underset{\sim}{t}) + p_n(\underset{\sim}{x}-\underset{\sim}{t}) \end{bmatrix} \varphi_n(t)dt$$

for $n = 1,2,\ldots,N.$

If $L_n(\cdot)$ has Property A2, then

$$R_{12}(n) \equiv \{\underset{\sim}{q} \mid \underset{\sim}{q} \leq \underset{\sim}{x}_n{}^*\}$$

$$R_1\,(n) \equiv \{\underset{\sim}{q} \mid q_2 > x^*_{n,2},\ q_1 \leq z_{n,1}\,(q_2)\}$$

$$R_2\,(n) \equiv \{\underset{\sim}{q} \mid q_1 > x^*_{n,1},\ q_2 \leq z_{n,2}\,(q_1)\}$$

$$R_0\,(n) \equiv \{\underset{\sim}{q} \mid q_1 > x^*_{n,1},\ q_2 > z_{n,2}\,(q_1) \text{ or } q_2 > x^*_{n,2}, q_1 > z_{n,1}(q_2)\}$$

We will need

Hypothesis B:

(1) $L_N(\cdot)$, $L_{N-1}(\cdot),\ldots,L_1(\cdot)$ have Property A2

(2) $\underset{\sim}{x}_n{}^* \leq \underset{\sim}{x}_{N-1}^* \leq \ldots \leq \underset{\sim}{x}_1{}^*$

(3) For every t, $z_{N,1}(t) \leq z_{N-1;1}(t) \leq \ldots \leq z_{1,1}(t)$

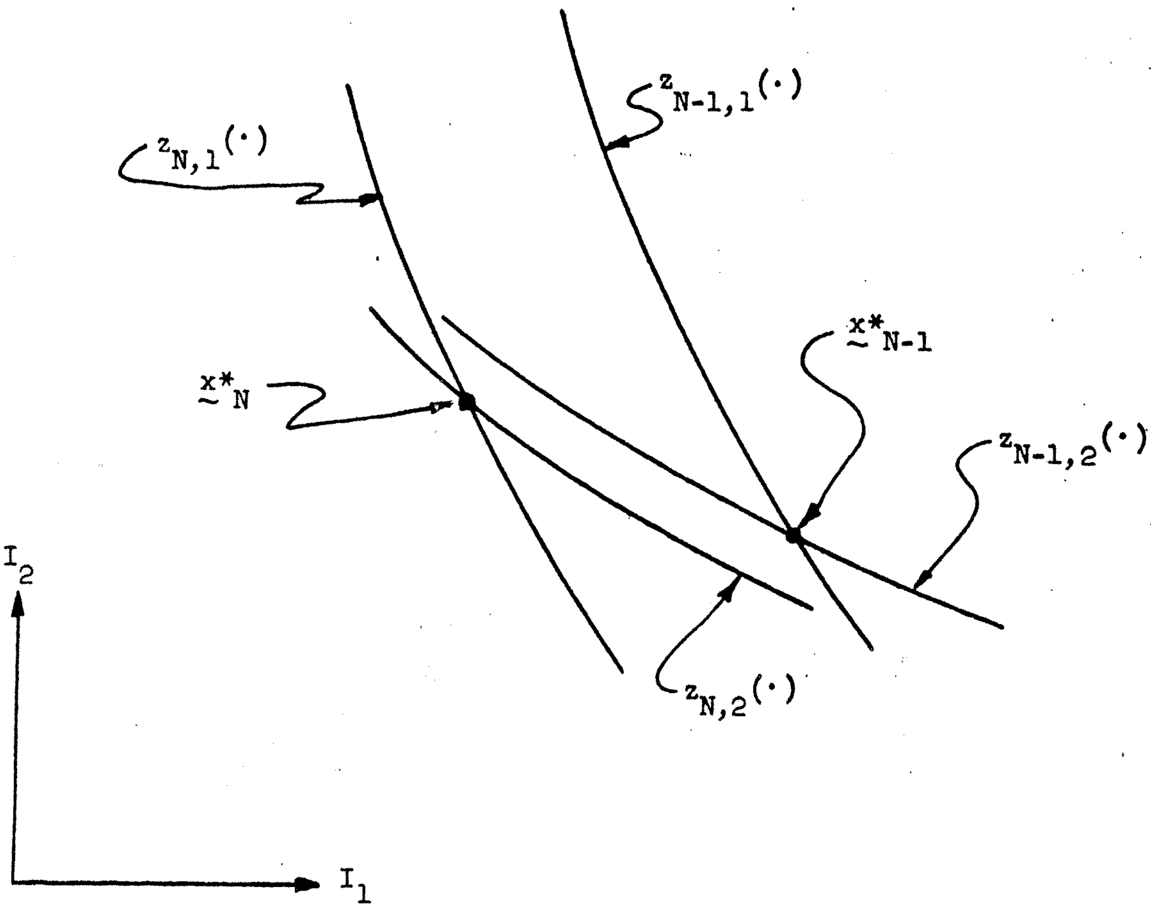
and $z_{N,2}(t) \leq z_{N-1,2}(t) \leq \ldots \leq z_{1,2}(t)$.

Note that (3) does not imply (2), as figure 2.6 illustrates.

Lemmas 2.1 through 2.6 apply to $L_1(\cdot)$, so we proceed directly to the proof of the analog of Theorem 2.7, namely

Theorem 2.9 If $L_{n+1}(\cdot), L_n(\cdot),\ldots L_1(\cdot)$ satisfy Hypothesis B and if the partials of $C_n(\cdot)$ are continuous and satisfy (i) through (iv) of Lemma 2.3 (for $R_{12}(n)$, $R_1(n)$, $R_2(n)$, $R_0(n)$), then

(a) the optimal policy with n+1 periods remaining is given by (i) through (iv) of Lemma 2.3 for $R_{12}(n+1)$, $R_1(n+1)$, $R_0(n+1)$, $R_2(n+1)$.



The Need for (2) of Hypothesis B

Figure 2.6

(b) the first partials of $C_{n+1}(\cdot)$ satisfy (i) through (iv) of Lemma 2.4 for $R_{12}(n+1)$, $R_1(n+1)$, $R_2(n+1)$, $R_0(n+1)$

and (c) the first partials of $C_{n+1}(\cdot)$ are continuous.

Note: Again the proof of (c) is unimformative, and again we omit it.

Proof: This proof is straightforward generalization of Theorem 2.7.

$$G_{n+1}(\underset{\sim}{q}) \equiv L_{n+1}(\underset{\sim}{q}) + \alpha \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n(\underset{\sim}{q}-\underset{\sim}{t})\ \varphi_n(\underset{\sim}{t})\ d\underset{\sim}{t}$$

$$C_{n+1}(\underset{\sim}{q}) = \min_{\underset{\sim}{x} \geq \underset{\sim}{q}} [G_{n+1}(\underset{\sim}{x})]$$

First we show that $\underset{\sim}{x}^*_{n+1}$ is a global minimizer of $G_{n+1}(\cdot)$. Consider any $\underset{\sim}{x}$. $L_{n+1}(\underset{\sim}{x}) \geq L_{n+1}(x^*_{n+1})$ by Lemma 2.1. Now consider any $\underset{\sim}{t} \geq \underset{\sim}{0}$. Since the partials of $C_n(\cdot)$ are non-negative and since they are zero for any $\underset{\sim}{y} \leq \underset{\sim}{x}_n^*$ and since $\underset{\sim}{x}^*_{n+1} \leq \underset{\sim}{x}_n^*$, we have

$C_n(\underset{\sim}{x}-\underset{\sim}{t}) \geq C_n(\underset{\sim}{x}^*_{n+1}-\underset{\sim}{t})$. Therefore

$$\alpha \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n(\underset{\sim}{x}-\underset{\sim}{t})\varphi_n(\underset{\sim}{t})\, d\underset{\sim}{t} \geq \alpha \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n(\underset{\sim}{x}^*_{n+1} - \underset{\sim}{t})\varphi_n(\underset{\sim}{t})\, d\underset{\sim}{t} \quad \text{so}$$

that $G_{n+1}(\underset{\sim}{x}) \geq G_{n+1}(\underset{\sim}{x}^*_{n+1})$. At this point, we can conclude that if $\underset{\sim}{q} \leq \underset{\sim}{x}^*_{n+1}$, it is optimal to order up to $\underset{\sim}{x}^*_{n+1}$.

Now suppose $q_2 > x^*_{n+1,2}$ and $q_1 \leq z_{n+1,1}(q_2)$. We show that $\min\limits_{\underset{\sim}{x} \geq \underset{\sim}{q}} G_{n+1}(\underset{\sim}{x}) = G_{n+1}(z_{n+1,1}(q_2), q_2)$, so that only product 1 is ordered, up to $Z_{n+1}(q_2)$. First, by Lemma 2.1, $L(z_{n+1}(q_2), q_2) = \min\limits_{\underset{\sim}{x} \geq \underset{\sim}{q}} L(\underset{\sim}{x})$.

Consider any $\underset{\sim}{x} \geq \underset{\sim}{q}$ and any $\underset{\sim}{t} \geq \underset{\sim}{0}$. Then, because $z_{n,1}(\cdot)$ has non-positive slope, $C_n(\underset{\sim}{x}-\underset{\sim}{t}) \geq C_n((z_{n,1}(q_2),q_2)-\underset{\sim}{t})$. Since $z_{n+1,1}(t) \leq z_{n,1}(t)$ for every t, the latter equals $C_n((z_{n+1,1}(q_2),q_2)-\underset{\sim}{t})$. Therefore

$$\int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n((z_{n+1,1}(q_2),q_2)-\underset{\sim}{t})\varphi_n(\underset{\sim}{t})d\underset{\sim}{t} = \min_{\underset{\sim}{x} \geq \underset{\sim}{q}} \left[\int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n(\underset{\sim}{x}-\underset{\sim}{t})\varphi_n(\underset{\sim}{t})d\underset{\sim}{t}\right].$$

We see that the two terms are minimized at the same point, namely $(z_{n+1,1}(q_2),q_2)$, so that only product 1 should be ordered, up to $z_{n+1,1}(q_2)$.

To complete the proof of (a), areas $R_2(n+1)$ and $R_0(n+1)$ must be considered. However the modifications of the proof of (a) in Theorem 2.7 that are necessary should be clear from the treatment of areas $R_{12}(n+1)$ and $R_1(n+1)$, so we omit the rest of the proof of (a).

Once (a) has been established, the proof of (b) in Theorem 2.7 carries over directly to establish (b) here, and the proof is complete.

2.9 Generalizing Property A2

In discussing how Property A2 can be generalized, it is helpful to make use of the properties to which A2 will later be compared. Consequently, they will be defined now.

Convexity: Consider a function $f(\cdot)$, defined on a convex set S. Then $f(\cdot)$ is *convex* if, for every $\underset{\sim}{x}, \underset{\sim}{y} \in S$,

$$\theta f(\underset{\sim}{x}) + (1-\theta) f(\underset{\sim}{y}) \geq f(\theta \underset{\sim}{x} + (1-\theta)\underset{\sim}{y}) \tag{2.9.1}$$

for every $0 \leq \theta \leq 1$. If $f(\cdot)$ satisfies the added condition that, for every $\underset{\sim}{x}, \underset{\sim}{y} \in S$ such that $\underset{\sim}{x} \neq \underset{\sim}{y}$, (2.9.2) $$\theta f(\underset{\sim}{x}) + (1-\theta)f(\underset{\sim}{y}) > f(\theta \underset{\sim}{x} + (1-\theta)\underset{\sim}{y})$$

for every $0 < \theta < 1$, then $f(\cdot)$ is said to be strictly convex.

Quasiconvexity: A condition weaker than convexity is quasiconvexity, which is defined below. Quasiconvexity is entirely analogous to quasiconcavity, which was defined and explored by Arrow and Enthoven [2].

For a function $f(\cdot)$, defined on a convex set S, two equivalent definitions of quasiconvexity, which are taken from Wolfe [28], page 7, are

(1) $f(\cdot)$ is quasiconvex, if, for all real c,

(2.9.3) $$\Gamma(c) \equiv \{\underset{\sim}{x} \mid \underset{\sim}{x} \epsilon S, f(\underset{\sim}{x}) \leq c\} \text{ is a convex set}$$

(2) $f(\cdot)$ is quasiconvex if, for every $\underset{\sim}{x}, \underset{\sim}{y} \in S$,

(2.9.4) $$\max\,(f(\underset{\sim}{x}), f(\underset{\sim}{y})) \geq f(\theta \underset{\sim}{x} + (1-\theta)\underset{\sim}{y}) \text{ for every } 0 \leq \theta \leq 1.$$

By analogy with strict convexity, $f(\cdot)$ is said to be strictly quasiconvex if, for every $\underset{\sim}{x}, \underset{\sim}{y} \in S$ such that $\underset{\sim}{x} \neq \underset{\sim}{y}$,

(2.9.5) $$\max\,(f(\underset{\sim}{x}), f(\underset{\sim}{y})) > f(\theta \underset{\sim}{x} + (1-\theta)\underset{\sim}{y}) \text{ for every } 0 < \theta < 1.$$

Some examples of convex and quasiconvex functions of one variable are given in Figure 2.7.

From these definitions, it should be clear that in part (c) of the definition of Property A2, the statement that $D_1L(x_1,x_2) < 0$ for $x_1 < z_1(x_2)$ and $D_2L(x_1,x_2) > 0$ for $x_1 > z_1(x_2)$ can be replaced by a statement that $L(x_1,x_2)$ is strictly quasiconvex in x_1. A similar replacement can be made in (d).

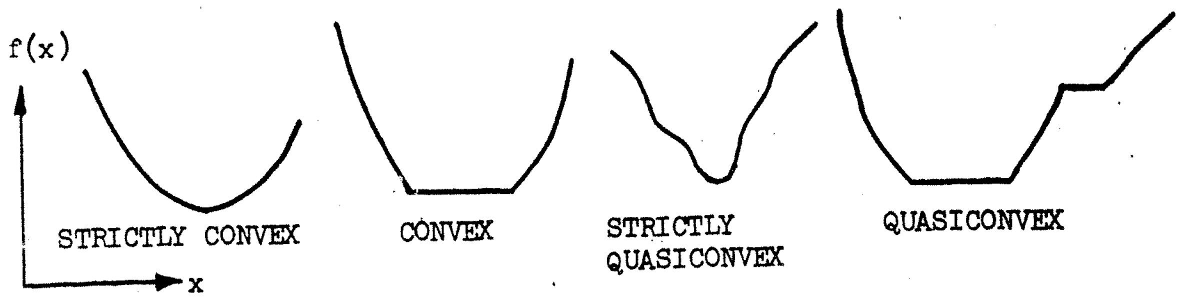
We now define a weaker property than Property A2, which will be called A2*, under which an optimal N period policy will be the same as a one period policy. For brevity, many of the details are omitted from the discussion which follows the definition.

Property A2*: $L(\cdot)$ has Property A2* if

(a) (b) $L(\cdot)$ is continuous

(c) For any fixed x_2, $L(\cdot)$ is a quasiconvex function of x_1. Further, there exists a nonincreasing function $z_1(\cdot)$, such that $z_1(x_2)$ is one of the points at which $L(\cdot)$ attains its minimum.

(d) For any fixed x_1, $L(\cdot)$ is a quasiconvex function of x_2. Further, there exists a nonincreasing function $z_2(\cdot)$, such that $z_2(x_1)$ is one of the points at which $L(\cdot)$ attains its minimum.



Examples of Convex and Quasiconvex Functions

Figure 2.7

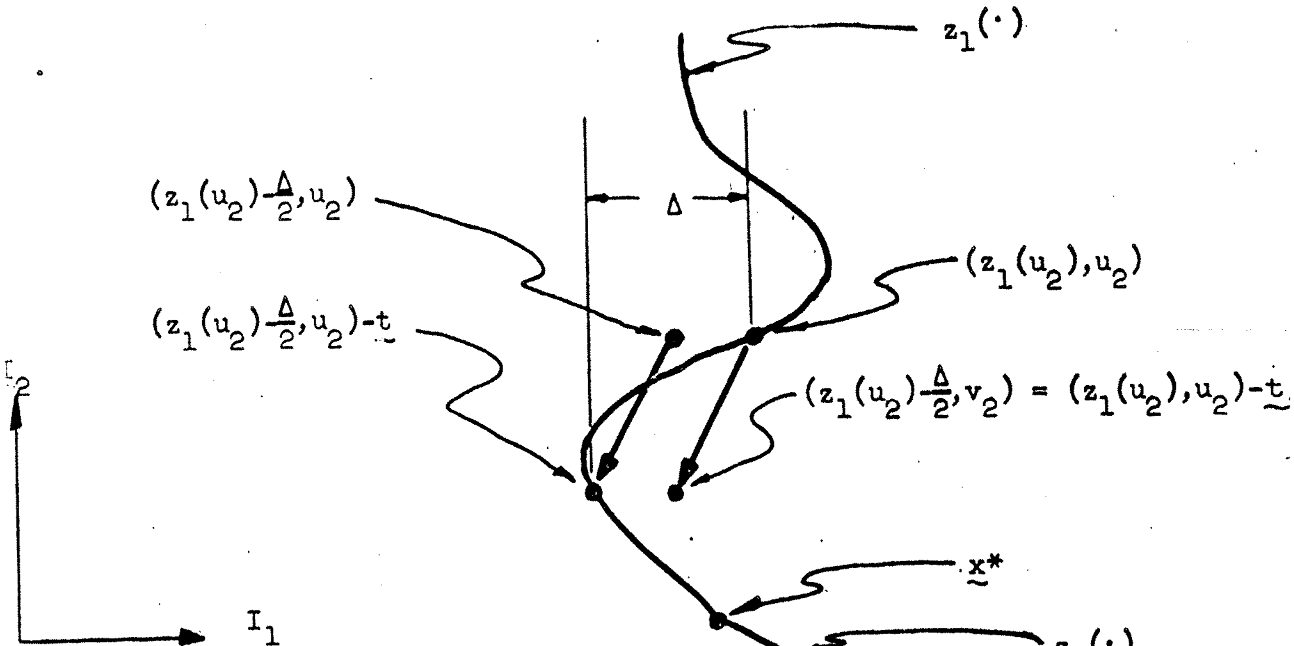


Figure 2.8

(e) (A) The set $H \equiv \{x_1 | z_1(z_2(x_1)) = x_1\}$ is a set of isolated points, and

(B) one point of H, call it x_1^*, is such that for $x_1 > x_1^*$, $z_1(z_2(x_1)) \leq x_1$, and for $x_1 < x_1^*$, $z_1(z_2(x_1)) \geq x_1$.

The existence of the partial derivatives of $L(\cdot)$ only makes the statements and proofs of the results of Section 5 and 6 more compact, and is otherwise not essential. (Even continuity of $L(\cdot)$ is not essential, but we find it hard to conceive of an $L(\cdot)$ that is not continuous.)

To see why $z_1(\cdot)$ and $z_2(\cdot)$ must be non-increasing, suppose that $L(\cdot)$ satisfied the original definition of Property A2, except that for some $u_2 > v_2 \geq x_2^*$, $z_1(u_2) > z_1(u_1)$, as illustrated in Figure 2.8. It would still be possible to prove Lemmas 2.1 through 2.6. However, the current proof of Theorem 2.7. would fail. Let $\Delta \equiv z_1(u_2)-z_1(v_2)$. For $\underset{\sim}{t} \equiv (\frac{\Delta}{2}, u_2-v_2)$, $C_1((z_1(u_2),u_2)-\underset{\sim}{t}) = C_1(z_1(v_2)+\frac{\Delta}{2},v_2)$ which is greater than $C_1(z_1(v_2),v_2) = C_1((z_1(u_2)-\frac{\Delta}{2},u_2)-\underset{\sim}{t})$. Therefore, for $\underset{\sim}{x} \equiv (z_1(u_2)-\frac{\Delta}{2},u_2)$, it cannot be said that $G_2(\underset{\sim}{x}) \geq G_2(z_1(u_2),u_2)$, so that it may not be optimal, in period 2, to order from $\underset{\sim}{x}$ up to $(z_1(u_2),u_2)$.

Although A2* permits multiple intersections of $D_1L = 0$ and $D_2L=0$, it does not permit any set of these points where $z_1(z_2(x_1)) = x_1$ to be connected. The reason is: If the first partials of $L(\cdot)$ did not exist and a set of such points were connected, then a situation of the type pictured in Figure 2.9 could occur; and if such a situation did occur, then the optimal one period policy would no longer be given by Lemma 2.3.

If the first partials of $L(\cdot)$ exist, then by the line integral theorem used in Lemma 2.1, $L(\cdot)$ would be constant along any connected set of points where $z_1(z_2(x_1)) = x_1$. If not, then we have been unable to prove from the remaining parts of A2* that, for example, $L(\cdot)$ is not concave in x_1 along such a set. Therefore, we cannot rule out $L(u,z_2(u)) > L(a,z_2(a)),L(b,z_2(b))$ where $u = \frac{a+b}{2}$, as pictured in Figure 2.9.

2.10 Comparing Property A2 to Convexity and Quasiconvexity

In comparing Property A2 to convexity and quasiconvexity, we continue to assume that purchasing cost has been converted to holding cost and is therefore included in $L(\cdot)$.

What determines the form of the optimal policy is the behavior of $L(\cdot)$, the expected one period holding and shortage

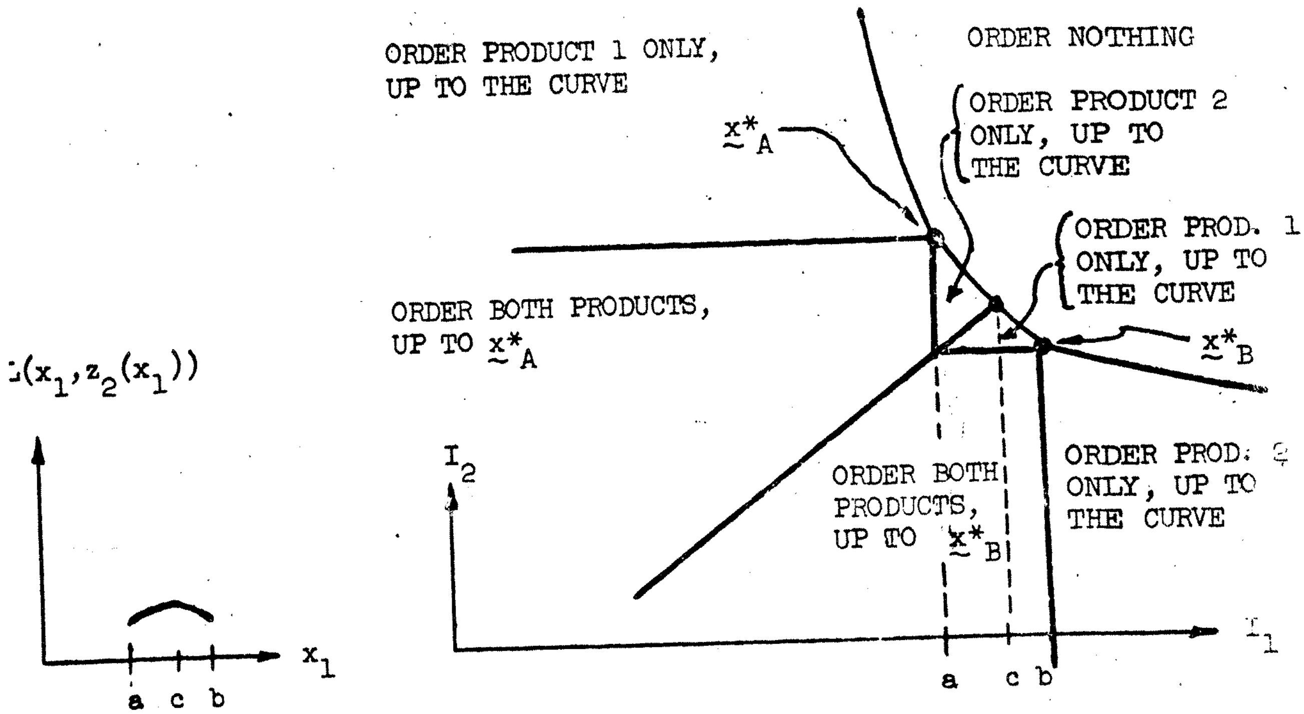
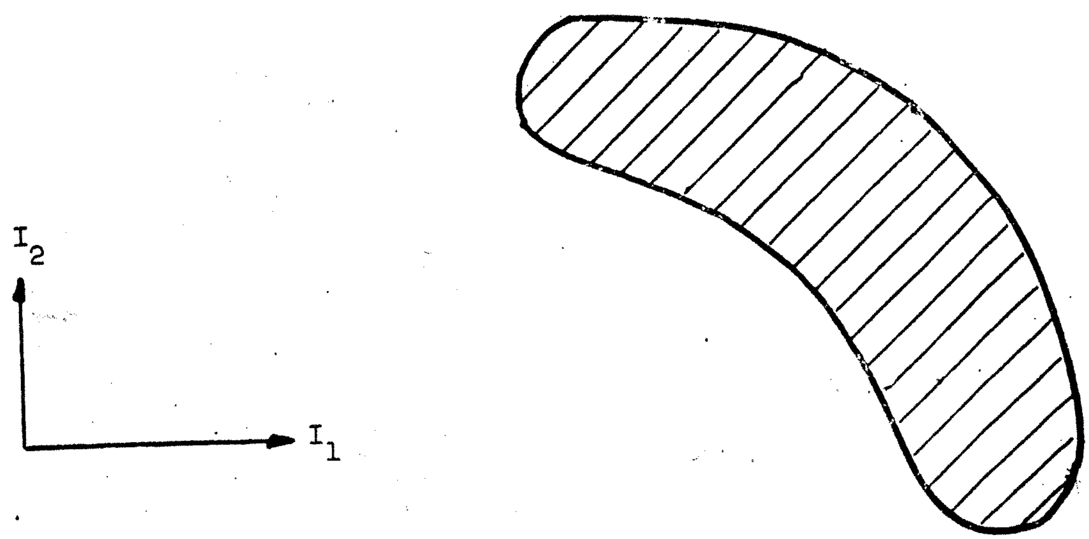


Figure 2.9



A Non-Convex Set that is Convex on Horizontal and Vertical Lines

Figure 2.10

cost function. The optimality of the policy described in Lemma 2.3 and Theorem 2.7, which Veinott [23] calls the base stock policy (since, eventually, one orders up to $\underset{\sim}{x}^*$ in every period), depends on $L(\cdot)$'s satisfying A2. In the one product analog of our work, Veinott [21] assumes that $L(\cdot)$ is quasiconvex. However, it would be desirable if conditions could be imposed on $h(\cdot)$, $p(\cdot)$, $\phi(\cdot)$, and $\underset{\sim}{c}$ directly, since they could be checked more easily than can conditions on $L(\cdot)$. This has been done by Karlin ([3], pp 113-7 and pp 137-42) for the one product case. He indicates that convexity of $h(\cdot)$ and $p(\cdot)$ implies convexity of $L(\cdot)$ and hence the optimality of a base stock policy. He also shows that if $\phi(\cdot)$ is a Polya frequency function, certain sign change patterns on c and the derivatives of $h(\cdot)$ and $p(\cdot)$ are sufficient to imply that a base stock policy is optimal.

However, to the extent that $L(\cdot)$ may satisfy the required conditions on it even where $h(\cdot)$, $p(\cdot)$, $\phi(\cdot)$ and $\underset{\sim}{c}$ do not, direct checking of $L(\cdot)$ is desirable. For example, in the one product case, $h(\cdot)$ and $p(\cdot)$ need not be convex for $L(\cdot)$ to be convex.

In generalizing to two products, convexity of $h(\cdot)$ and $p(\cdot)$ is still sufficient but not necessary for the convexity of $L(\cdot)$. We know of no analog of the Polya frequency function in two dimensions. And the extension of the one product quasiconvexity assumption can be done in at least two ways, namely two dimensional quasiconvexity and Property A2.

The comparisons that follow will be between quasiconvexity and A2. These will be sufficient to indicate the relationships between convexity and A2, since quasiconvexity is implied by convexity. Unfortunately, there is no easy relationship between quasiconvexity and A2.

<u>Quasiconvexity does not imply A2</u>. The major reason is the requirement in A2 that $z_1(\cdot)$ and $z_2(\cdot)$ be the non-increasing. This implies that the cross partials of $L(\cdot)$ are non-negative, at least near the two curves. Non-negative cross partials are not essential to quasiconvexity. (For example, $f(x,y) = (x-y)^2$ is quasiconvex and $D_{12}f(x,y) = -2$ for every (x,y).) In Section 9, the necessity for $z_1(\cdot)$ and $z_2(\cdot)$ being non-increasing was shown. Therefore, it should not be surprising that quasiconvexity of $L(\cdot)$ is <u>not</u> sufficient to imply the optimality of an N period policy of the form given in Lemma 2.3 and Theorem 2.7. However, if $L(\cdot)$ is assumed to have non-negative cross partials and be quasiconvex, a policy of that form will be optimal, as the following outline indicates.

Define $z_1(x_2)$ as the smallest x_1 that minimizes $L(\cdot)$ on the line where x_2 is constant, and define $z_2(\cdot)$ similarly. It is easy to show that these two curves intersect at least once and that at least one of these points of intersection is an absolute minimizer of $L(\cdot)$. Let $\underset{\sim}{x}^*$ be any of these last named points. Then with these definitions, it is possible to prove modified versions of Lemmas 2.1 through 2.5 and Theorem 2.7 in the same way as the originals are proved. The modifications would allow for the non-existence of derivatives and would drop the strictness from the inequalities.

Quasiconvexity and non-negative cross partials still do not imply Property A2 (or Property A2*). The difficulty is a minor one relating to part (e) of the definition of A2. For example, $L(x_1,x_2) = (x_1+x_2)^2$ is quasiconvex with non-negative cross partials, and $z_1(z_2(x_1)) = x_1$ on the entire line $x_1+x_2 = 0$. Therefore, $L(\cdot)$ does not satisfy A2 (or A2*).

Property A2 does not imply quasiconvexity. The difficulty here is major, and results from the fact that quasiconvexity in x_1 for any x_2 and in x_2 for any x_1 do not imply quasiconvexity in the two-dimensional vector $\underset{\sim}{x}$. See Figure 2.10, where a set

$\Gamma(c) \equiv \{\underset{\sim}{x} | L(\underset{\sim}{x}) \leq c\}$ that is convex in x_1 for any x_2 and in x_1 for any x_2 but not convex in two space is illustrated. Furthermore, the possibility that $L(\cdot)$ will have A2 and not be quasiconvex is not just theoretical -- we feel there are reasonable inventory situations where it will happen. (See Section 11.) This explains our choice of A2 over quasiconvexity and non-negative cross partials in the original development.

Now that the non-negativity of the cross partials of $L(\cdot)$, either explicity or implicit, has been brought into the open, there may be some question as to its reasonableness. This issue is treated in Section 11.

2.11 Relating Holding and Shortage Costs to Property A2 And Actual Systems to Our Model

If this work is to have any practical significance, the assumptions we have made about $L(\cdot)$ must be reasonable. And reasonableness should be judged in terms of the demand distributions and holding and shortage cost functions that imply or are implied by our assumptions about $L(\cdot)$. The model that we have assumed must also be a reasonable fit to real world inventory systems.

We feel that primarily because of the difficulties involved in specifying shortage costs, the question of whether or not $L(\cdot)$ will have A2 cannot in general be resolved. In many cases it will be clear from $h(\cdot)$ and $p(\cdot)$ that $L(\cdot)$ does have A2. In other cases, it will be clear from $h(\cdot)$, $p(\cdot)$, and $\phi(\cdot)$ that it does not. In still other cases, it will be necessary to construct $L(\cdot)$ in order to find out whether or not it has A2.

The framework of this section will be a discussion of shortage cost followed by a discussion of holding cost. Our primary aim will be separating the cases where it is clear from $h(\cdot)$ and $p(\cdot)$ that $L(\cdot)$ has A2 from those where $L(\cdot)$ must be constructed.

Comments on the fit of the model to actual systems will be woven into this framework. In the absence of a two-dimensional analog of the Polya frequency function, demand distributions will not be discussed.

Of particular relevance to our treatment of multiproduct systems are the discussions of the plausibility of $D_{12}h(\cdot)$ being non-negative and the plausibility $L(\cdot)$ having A2 but not being quasiconvex. The possible problems of specifying a non-stationary $h(\cdot)$ function bear on the potential usefulness of the results of Section 7 of this chapter.

Shortage Costs:

To begin, we seek assumptions on $p(\cdot)$ that, along with reasonable ones on $h(\cdot)$, imply that $L(\cdot)$ has A2. Suppose that $h(\underset{\sim}{x},\underset{\sim}{t})$ can be written as $\hat{h}(\underset{\sim}{x})$ or $\tilde{h}(\underset{\sim}{x}-\underset{\sim}{t})$. In the discussion of holding costs we give reasons for supposing that $D_{12}h(\cdot)$ is non-negative and that $h(\cdot)$ is convex and increasing in x_1 for any x_2 and in x_2 for any x_1. Even with these assumptions, unless $p(\cdot)$ is convex and decreasing in x_1 for any x_2 and in x_2 for any x_1, one cannot be sure that $L(\cdot)$ will be quasiconvex in x_1 for any x_2 and in x_2 for any x_1. And even if it is, unless $D_{12}p(\cdot)$ is non-negative, one cannot be sure that $z_1(\cdot)$ and $z_2(\cdot)$ will be non-increasing.

If, in the two product case, the shortage cost $p(\underset{\sim}{x})$ can be written as $p_1(x_1) + p_2(x_2)$ for every $\underset{\sim}{x}$, then $D_{12}p(\cdot)$ is always zero and hence non-negative. If in addition, $p_1(\cdot)$ and $p_2(\cdot)$ are both convex functions, then $p(\cdot)$ is convex in two space. This separable and convex shortage cost is at least superficially plausible and is probably correct in many cases. However, this is not always the case. Suppose, for example, that emergency replenishment is made if the firm runs out of either or both products, and that the added cost is independent of the quantity and is the same whether one product or both must be acquired in this way. In this case, the discontinuity in $p(\cdot)$ makes examination of $D_{12}p(\cdot)$ inappropriate, so we look at its analog,

$$[(p(\underset{\sim}{x}+\underset{\sim}{\Delta}_1+\underset{\sim}{\Delta}_2)-p(\underset{\sim}{x}+\underset{\sim}{\Delta}_1))-(p(\underset{\sim}{x}+\underset{\sim}{\Delta}_2)-p(\underset{\sim}{x}))] \text{ for } \underset{\sim}{\Delta}_1 = (d_1,0),\ \underset{\sim}{\Delta}_2 = (0,d_2)$$

and $d_1,d_2 > 0$. This quantity must be non-negative everywhere to ensure that $z_1(\cdot)$ and $z_2(\cdot)$ are non-increasing. But for $\underset{\sim}{x} = (-1,-1)$ and $d_1=d_2=2$, we have $p(-1,-1) = p(1,-1) = p(-1,1) =$ the cost of emergency shipment while $p(+1,+1)$ is zero, and our analog of $D_{12}p(\cdot)$ is negative.

As an introduction to the difficulties of specifying a shortage cost function, consider the following example. Suppose a firm reviews its inventory position once a week. It has 90 units in stock at the start of the week and will receive (say) 150 units next Monday. Suppose 5 separate orders, each requesting shipment of 20 units this week, arrive. Then one of the orders cannot be filled this week. On that order, the firm can ship 10 units now and 10 units next week, or ship the entire order next week, or perhaps get an emergency shipment from its supplier and ship all 20 this week.

The cost of being out of stock--of paying for two shipments instead of one or of losing goodwill by shipping next week or of paying a premium for emergency supply -- may be independent of the size of the stock out. That is, the costs mentioned above might very well be the same if the firm were 5 units short on the last order rather than 10 units short. If this were the case, $p(\cdot)$ would not be convex.

For the three specified ways of handling stock outs, there are costs that are hard to measure. For the first two, the customer's original request is not satisfied on schedule. Should this cause him to take his business elsewhere in the future, the profits that the firm will not make as a result are a cost chargeable to being out of stock.[1] Satisfactory procedures for estimating the probability of this event and the amount of profit loss if it occurs are not yet available. When emergency supply is used to avoid delaying customer shipment, the cost in terms of relationship with the firm's supplier may far exceed the immediate out-of-pocket cost.

In some situations, when a customer's <u>current</u> demand cannot be satisfied immediately, that very demand is lost. The

1 - Loss of future business also implies that future demand distributions depend on current policy and on quantity demanded, rendering our model inappropriate.

backlogging assumption has been made because under the lost demand assumption, the analysis unavoidably becomes more complicated. Consider, for instance, the fixed delivery lag case. When excess demand is backlogged, expected holding and shortage cost λ periods hence is a function of stock on hand and on order now. When excess demand is lost, this is not the case, because inventory just before demand, λ periods hence, is no longer the difference between stock on hand and on order now and demand in the intervening periods, but will be greater than the difference if there are any stockouts between now and then. Expected holding and shortage cost λ periods hence is then a function of stock on hand and the individual order quantities; that is, it is a function of $\lambda+1$ variables.

Suppose a policy obtained from a backlogging model is used, when in the actual system excess demand is lost. If the optimal policy under backlogging results in a very small percentage of demand being backlogged, it would seem that this policy would be quite close to optimal for the lost sales model.

Holding Costs:

First let us consider what costs are included under the name holding cost in our model. Since interest costs relate to cash flows, that part of holding cost that is called "cost of money tied up in inventory" (see page 10 of Starr and Miller [18]) is covered by the $(1-\alpha)\underset{\sim}{c}$ term. (This term is linear in $\underset{\sim}{x}$. If the cost of capital increases with the amount required, then α will not be a constant, but a function of $\underset{\sim}{x}$, and this term will no longer be linear in $\underset{\sim}{x}$.) Consequently, $h(\cdot)$ includes costs related to the physical handling and storage of inventory: Handling costs, warehouse rental, pilferage, taxes, insurance, etc.

In many warehouses, the space required and the handling costs depend only on the total amount stored and not on the relative proportions of the different products that make up the total. To the extent that the products can be mixed at will without impairing their availability for shipment, this assumption is true. To the extent that the products must be stored in separate areas, it may not be true. It may be necessary for different products to be physically separated. Or it may only be necessary to avoid stacking one product on top of (or in front of) another.

If holding cost is a function of total inventory, then $h(\underset{\sim}{x})$ can be written as $\tilde{h}(x_1+x_2)$ for every $\underset{\sim}{x}$. If $\tilde{h}(\cdot)$ is convex and increasing, then $D_{12}h(\cdot)$ is non-negative and $h(\cdot)$ is convex in x_1 for any x_2 and convex in x_2 for any x_1. So if $p(\cdot)$ is convex and decreasing and separable, $L(\cdot)$ will have property A2.

This assumption on $\tilde{h}(\cdot)$ is quite plausible. It is equivalent to saying that the marginal cost of storage is a non-decreasing function of inventory level. In other words, the incremental cost of storing one more unit if 100 units are on hand is at least as large as the incremental cost of storing another unit if 50 are on hand.

In the short run this may not be true. On a given day, with a given number of men on the job, additional units can be stored with no increase in out of pocket cost as long as the total workload does not exceed the "capacity" of the workforce. As long as the total inventory does not exceed the capacity of the warehouse, there will be no increase in out of pocket cost for space either. However, in the longer run, where our main interest lies, it is reasonable to suppose that marginal handling, space and other costs that make up holding cost will increase with inventory level.

Consider a firm which is deciding how much to stock in its two-story warehouse. Starting with an empty warehouse, each additional unit that is put into inventory will be stored farther from the shipping dock, at increased handling time and hence cost.

After inventory hits a certain level, units will have to be stored on the second floor at still higher handling cost. When inventory exceeds the capacity of the warehouse, then public warehouse space must be rented, at even higher unit cost.

In the day to day operation of the warehouse, even when it is relatively full, an additional unit received may in fact be stored right next to the shipping dock. This results because a warehouse operating policy should specify storing the units that turn over the fastest nearest the shipping dock. Units stored on the second floor should be shipped only when high demand has cleaned out the first floor or when they would be too old to sell if not shipped now. This just indicates that a warehouse operating policy is implicit in any holding cost function. Since our emphasis is on stationary cost functions and demand distributions, this period's holding cost for a given inventory level and demand quantity should be interpreted as the average cost per period of operating the warehouse at that level with that demand.

Of course there are situations where holding cost cannot be written as a function of total inventory. In these situations, if for any given inventory of product 1, the marginal cost of storing product 1 does not decrease as the inventory of product 2 increases, then $D_{12}h(\cdot)$ is non-negative. If this should be the case, then one would expect the marginal cost of storing product 1 to be non-decreasing with inventory of product 1, for any given inventory of product 2. This would imply that $D_{11}h(\cdot)$ is non-negative, so that $h(\cdot)$ would be convex in x_1 for any fixed x_2. By the same reasoning, with products 1 and 2 reversed, $h(\cdot)$ would be convex in x_2 for any fixed x_1.

This convexity and the non-negativity of $D_{12}h(\cdot)$, along with $p(\cdot)$ being convex and decreasing and separable, imply that $L(\cdot)$ has A2. However they do <u>not</u> imply that $h(\cdot)$ is convex in two-dimensional space. For example, $h(\underset{\sim}{x}) = (x_1+x_2)^2 + x_1x_2$ has $D_{11}h(\underset{\sim}{x}) = D_{22}h(\underset{\sim}{x}) = 2$ and $D_{12}h(\underset{\sim}{x}) = D_{21}h(\underset{\sim}{x}) = 3$ for every $\underset{\sim}{x}$. But observe that $h(2,2) = 20 > 19 = h(1,3) = h(3,1)$, so that $h(\cdot)$ is not convex. We feel that this is more than just a mathematical possibility, that actual holding cost functions may behave this way.

One way for $h(\cdot)$ to be non-convex when the four second partials are non-negative, is for the warehouse to be cheaper to operate, for a given total inventory, when the inventory is predominantly one product or the other, than when the inventory is divided evenly between them. If this is true for <u>any</u> total inventory, and if demand is bounded (from above), then $L(\cdot)$ will not be quasiconvex.[1] However, the non-negativity of the four second partials of $h(\cdot)$, with $p(\cdot)$ convex, decreasing and separable, suggest that $L(\cdot)$ is very likely to have Property A2*. (The only question is whether H, defined in Section 9 of this chapter, is a set of isolated points or not.) This is our reason for preferring A2 to quasiconvexity in stating our results.

See Appendix C for an example where $L(\cdot)$ has A2 but is not quasiconvex.

It is possible that there will be a logical problem in specifying $h(\cdot)$. Suppose a firm with an existing warehouse is evaluating its inventory policy. The cost, in some future period, of holding an amount higher than current capacity depends on whether amounts that large will be held always or only occasionally.

---

1 - Even when demand is not bounded, if the cost of "even distribution" is high enough relative to the shortage cost, $L(\cdot)$ will fail to be either convex or quasiconvex.

If it will be always, then the cost should probably be based on expanding the existing warehouse. If it will be ocassionally, then the cost should probably be based on storing the excess in a public warehouse, since this is likely to be cheaper than expansion. The same ideas apply to holding an amount less than current capacity: If inventory will always be less than capacity, credit for some alternative use of the idle space should reduce holding cost. If inventory will be lower only occasionally, then there is no saving. In other words, in asking for the cost of holding a given amount, one must specify whether he wants long run or short run cost.

For the stationary, instant delivery case, the optimal policy is such that the inventory just after delivery is the same in every period[1] so that long run costs are appropriate. But if costs or demand are non-stationary of if there is a delivery lag, the optimal policy is such that inventory on hand just after delivery

---

1 - Except for the time, if any is required, to let inventory fall below $\underset{\sim}{x}^*$.

will change from period to period, and there is a question of whether to make $h(\cdot)$ a long run or a short run cost. This question may be more apparent than real since short run and long run costs may not be significantly different. We would think that short run and long run marginal interest costs would be the same, while long run marginal costs of physical storage and handling would be considerably less than the corresponding short run marginal costs. If so, and if $h(\cdot)$ is expanded to include the $(1-\alpha)\underset{\sim}{c}'\underset{\sim}{x}$ term, the problem of defining $h(\cdot)$ becomes more real as the ratio of product volume and/or weight to product value increases. For example, interest costs should dominate physical costs for portable transistor radios which retail for about $2000 per cubic foot, while the physical storing and handling costs might be dominant for soap flakes at about $5 per cubic foot.[1]

1 - On page 39 of Section 11 of the New York Times, March 20, 1966, a SONY radio, 2 1/4" x 4" x 1" is advertised for $11.99. At that time, a two pound carton of Ivory Snow, measuring 8 3/16" x 11" x 3", retailed for 79 cents.

2.12 Is Explicit Treatment of Two Product Problems Necessary

The optimality of two single product solutions for a two product problem will be taken to mean that the optimal ordering decision for product 1 can be made without reference to either the inventory position or the ordering decision for product 2, and vice-versa. For a problem posed in the two product framework, this implies that $z_1(\cdot)$ and $z_2(\cdot)$ are constants. The optimal policy would be as illustrated in Figure 2.11.

$z_1(\cdot)$ and $z_2(\cdot)$ would certainly be constants if $L(\underset{\sim}{x})$ could be written as $L_1(x_1) + L_2(x_2)$ for every $\underset{\sim}{x}$.[1] They might even be constants if $L(\cdot)$ is not separable.[2] However, it is unlikely that, in any real inventory situation, $z_1(\cdot)$ and $z_2(\cdot)$ will be constant unless $L(\underset{\sim}{x})$ can be written as $L_1(x_1)+L_2(x_2)$.

Whether it is mathematically possible or not, it is unlikely that $L(\cdot)$ will be separable unless both $h(\cdot)$ and $p(\cdot)$ are. Situations where $h(\cdot)$ will not be separable have been discussed in Section 11. In these situations then, cxplicit treatment of the two product problem is likely to be required if an optimal policy is to be obtained.

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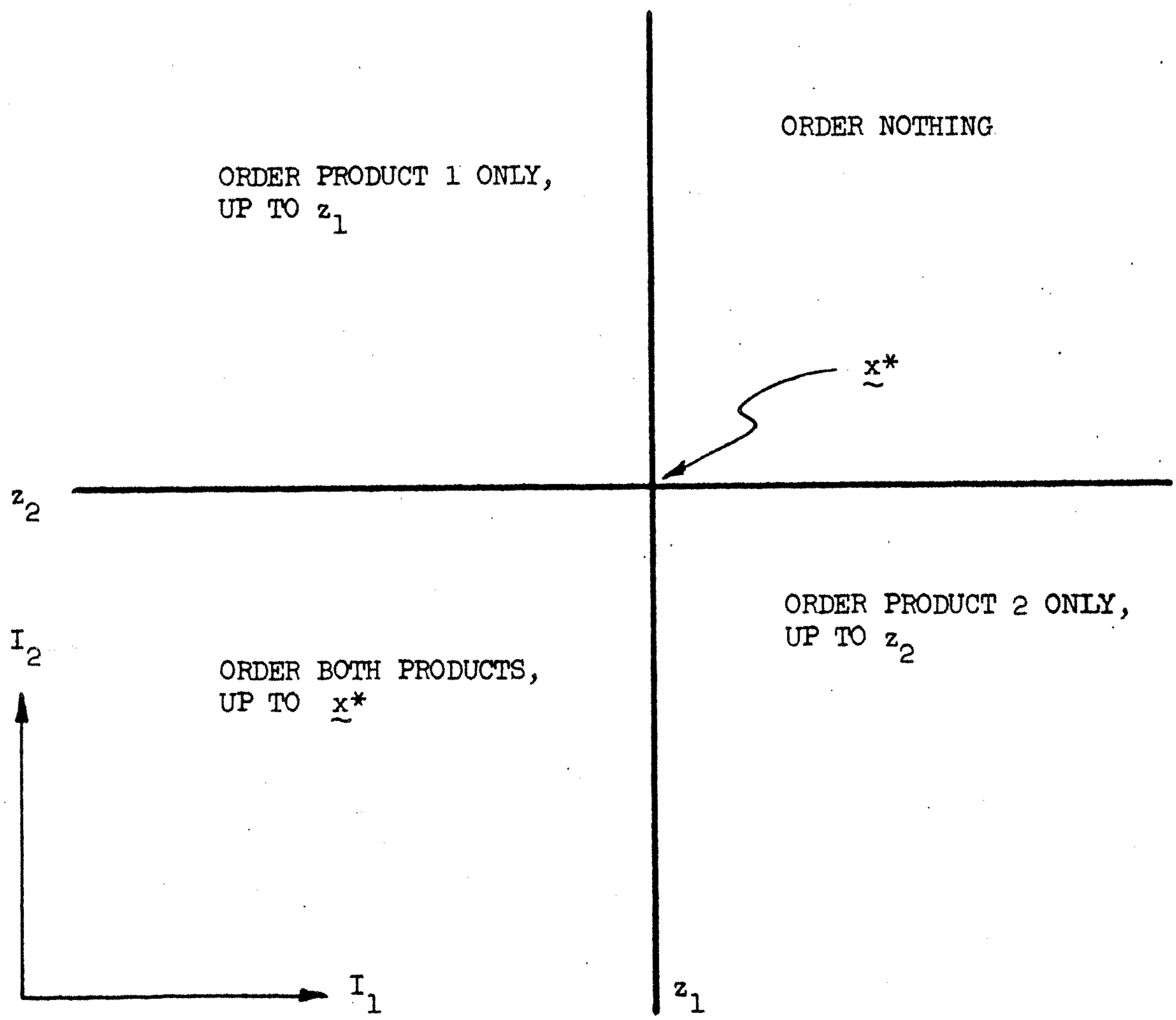
1 - This condition is part of the set given by Veinott [23], which imply that a two product problem can be "factored" into two one product problems.

2 - $L(x_1,x_2) = x_1^2x_2^2$ cannot be separated, yet $z_1(x_2)=0$ for every x_2 and $z_2(x_1)=0$ for every x_1.



The Optimal N Period Policy when $z_1(\cdot)$ and $z_2(\cdot)$ are Constant

Figure 2.11

From a practical point of view, single product solutions to two product problems may be fairly close to optimal. Since Theorem 2.7 implies that, eventually, the firm orders up to $\underset{\sim}{x}^*$ in every period, let us concentrate on $\underset{\sim}{x}^*$ and forget about the $z_1(\cdot)$ and $z_2(\cdot)$ curves. Suppose, for example, that $p(\cdot)$ is separable and that $h(\underset{\sim}{x},\underset{\sim}{t}) = h\cdot(x_1+x_2)^\beta$ for some $\beta \geqq 1$. And suppose that $h(\cdot)$ were approximated by $\tilde{h}(\underset{\sim}{x},t) = h \cdot [x_1^\beta+x_2^\beta]$, which is separable. Let $\underset{\sim}{x}^o = (x_1^o,x_2^o)$ be the point that minimizes $L(\cdot)$ when $\tilde{h}(\cdot)$ is substituted for $h(\cdot)$. Then x_1^o and x_2^o can be obtained by considering the problem as two separate one product problems.

For $\beta=1$, $x_1^o = x_1^*$. In an example where $p(\cdot)$ is also symmetric (see Appendix D), as β increases from 1, the ratio of x_1^* to x_1^o decreases. This suggests that the closer $h(\cdot)$ can be approximated by a separable function, the closer will be the "sum" of the single product policies to the true optimal policy.

Since $(1-\alpha)\underset{\sim}{c}'\underset{\sim}{x}$ is essentially a holding cost, the larger $(1-\alpha)\underset{\sim}{c}'\underset{\sim}{x}$ is compared to $h(\underset{\sim}{x})$, the closer will be the sum of the single product policies to the true optimal policy.

Chapter 3

Three or More Products, No Setup Cost

3.1 Introduction and Summary In this chapter we give some unpublished results of Veinott [20], obtained after our results of Chapter 2, for the multiproduct case where m, the number of products, can be any positive integer. The first of these is a theorem which states that if the optimal policy for the one period problem satisfies a certain hypothesis (described in the next paragraph) then that same policy, parameter values included, is optimal for the N period problem. If $L(\cdot)$ has Property A2, then his hypothesis is satisfied, so that our two product results of Section 4 through 6 of Chapter 2 are then a special case of this theorem.

Consider any two starting inventory vectors such that one is greater than or equal to the other. (Note that it may not be possible to "order" two arbitrary vectors in this way.) Veinott's hypothesis then is that there be an optimal one period policy which specifies an order quantity for the larger of the starting inventories that, for each product, is not larger than the order quantity specified for the smaller starting inventory.

Next we give a condition on $L(\cdot)$, obtained by Veinott, which implies that the one period policy satisfies his hypothesis[1]. Specifying this condition requires a preliminary definition and we defer both to Section 3.

Property A2 holding for every pair of products in a three product case is <u>not</u> sufficient to imply that the optimal one period policy satisfies Veinott's hypothesis. In Section 4, we introduce a three dimensional analog of Property A2 that is sufficient, and compare it to Veinott's condition of Section 3.

3.2 The Optimal N Period Policy for m Products

In this section we give a hypothesis that if satisfied by the one period optimal policy ensures that the policy is optimal for the N period problem. The proof that it does is given in Theorem 3.1. These results are Veinott's, and what follows is a close paraphrasing of his work in [20].

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1 - Veinott [20] gives another set of conditions: If $L(\underset{\sim}{x})$ can be written as $\sum_{i=1}^{m} b_i(x_i) + \sum_{i=1}^{m} h_i\left(\sum_{j=1}^{i} x_j\right)$ where the $b_i(\cdot)$'s $b_i(\cdot)$'s and $h_i(\cdot)$'s are convex, then the one period optimal policy satisfies his hypothesis. He proves this in [19], where $\underset{\sim}{x}$ is a vector of inventories of <u>a single product at different points in time</u>. In our problem, $\underset{\sim}{x}$ is a vector of inventories of the m products at a single point in time. Except when the $h_i(\cdot)$'s are all identically zero (in which case the problem can be solved by "adding" m single product solutions), there is no reason to suppose that $L(\cdot)$ can be written in this way.

Consider a one period, m product problem. We will say that y is optimal for x if $L(\underset{\sim}{y}) = \min_{\underset{\sim}{z} \geq \underset{\sim}{x}} L(\underset{\sim}{z})$. (Note that there may not be a unique $\underset{\sim}{y}$ for a given $\underset{\sim}{x}$.)

Hypothesis 3 (Veinott): If $\underset{\sim}{x}' \leq \underset{\sim}{x}$ and if $\underset{\sim}{y}$ is optimal for $\underset{\sim}{x}$, then there is a $\underset{\sim}{y}'$ that is optimal for $\underset{\sim}{x}'$ which satisfies $\underset{\sim}{y}' - \underset{\sim}{x}' \geq \underset{\sim}{y} - \underset{\sim}{x}$.

Consider any two starting inventories such that, product by product, one is larger than the other. Then Hypothesis 3 states that there is an optimal one period policy which specifies ordering less (or at least not more) of each product from the larger starting inventory.

In the following theorem we prove that if, in a given N period inventory problem, there is an optimal one period policy that satisfies Hypothesis 3, then that same policy is optimal in each of the N periods. For the purposes of the theorem, we reverse our convention and number the time periods in chronological order.

Theorem 3.1 (Veinott): Consider an N period inventory problem and suppose Hypothesis 3 is satisfied. Let $\underset{\sim}{x}_1$ denote inventory at the start of period 1. Let $\underset{\sim}{Y}^*$ be the ordering policy defined as follows: Let $\underset{\sim}{y}_1^*$ be optimal (in the sense of Hypothesis 3) for $\underset{\sim}{x}_1$. Suppose $\underset{\sim}{y}_t^*$ has been defined. Then let $\underset{\sim}{y}^*_{t+1}$ be optimal for

$\underset{\sim}{x}^*_{t+1} \equiv \underset{\sim}{y}^*_t - \underset{\sim}{D}_t$. Then Y^* is an optimal policy, in that it minimizes expected discounted costs over the N periods[1].

Proof: First we will show by induction that

$$\min_{\underset{\sim}{Z} \geq \underset{\sim}{x}_1 - \sum_{i=1}^{t-1} \underset{\sim}{D}_i} L(\underset{\sim}{Z}) = L(\underset{\sim}{y}^*_t) \quad \text{for } t=1,2,\ldots N \tag{3.1}$$

The result is trivially true for $t=1$. Suppose it holds for some integer t. Then y^*_t is optimal (in the sense of Hypothesis 3) for $\underset{\sim}{x}_1 - \sum_{i=1}^{t-1} \underset{\sim}{D}_i$.

Therefore, by Hypothesis 3, there is a point $\underset{\sim}{v}$ that is optimal (in the sense of Hypothesis 3) for $\underset{\sim}{x}_1 - \sum_{i=1}^{t} \underset{\sim}{D}_i$ which satisfies $\underset{\sim}{v} - (\underset{\sim}{x}_1 - \sum_{i=1}^{t} \underset{\sim}{D}_i) \geq \underset{\sim}{y}^*_t - (\underset{\sim}{x}_1 - \sum_{i=1}^{t-1} \underset{\sim}{D}_i)$ or $\underset{\sim}{v} \geq \underset{\sim}{y}^*_t - \underset{\sim}{D}_t$. Now by definition $L(\underset{\sim}{v}) = \min_{\underset{\sim}{Z} \geq \underset{\sim}{x}_1 - \sum_{i=1}^{t} \underset{\sim}{D}_i} (L(\underset{\sim}{Z}))$

which equals $\min_{\underset{\sim}{Z} \geq \underset{\sim}{y}^*_t - \underset{\sim}{D}_t} (L(\underset{\sim}{Z})) = L(\underset{\sim}{y}^*_{t+1})$

---

1 - Veinott allows N to be infinite in [20].

since $\underset{\sim}{v} \geq \underset{\sim}{y}^*_t - \underset{\sim}{D}_t$. This completes the induction.

Consider any ordering policy Y and let $\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots \underset{\sim}{y}_N$ be the associated inventory levels after ordering. Clearly $\underset{\sim}{y}_t \geq \underset{\sim}{x}_1 - \sum_{i=1}^{t-1} \underset{\sim}{D}_i$ for every t. Then by (3.1) we have, for any set of previous demands $\underset{\sim}{D}_1, \ldots, \underset{\sim}{D}_{t-1}$,

$$L(\underset{\sim}{y}_t) \geq L(\underset{\sim}{y}^*_t) \text{ for } t=1,2,\ldots,N.$$

Therefore, taking expectation with respect to the distribution of $\underset{\sim}{D}_1, \ldots, \underset{\sim}{D}_{t-1}$, $E(L(\underset{\sim}{y}_t)) \geq E(L(\underset{\sim}{y}^*{}_t))$ for $t=1,2,\ldots,N$.

If $f(\underset{\sim}{x}_1|Y)$ is defined to be expected discounted cost if initial inventory is $\underset{\sim}{x}_1$ and policy is Y, then for any policy Y

$$f(\underset{\sim}{x}_1|Y) = \sum_{t=1}^{n} \alpha^t \; E(L(\underset{\sim}{y}_t)) \geq$$

$$\sum \alpha^t E(L(\underset{\sim}{y}^*{}_t)) = f(\underset{\sim}{x}_1|Y^*),$$

which proves the theorem.

3.3 A Condition That Implies the Hypothesis of Theorem 3.1

In order to describe Veinott's condition on $L(\cdot)$ that implies his hypothesis, the concept of a nearly principal minor of a matrix must be introduced. We quote from Veinott, [20]:

"Let $H=(h_{ij})$ be an nxn matrix. Let

$$H\binom{i_1,\ldots,i_p}{j_1,\ldots,j_p} = |(hi_{kjt})| \quad \begin{matrix} 1 \leq i_1 \leq \ldots \leq i_p \leq n \\ 1 \leq j_1 \leq \ldots \leq j_p \leq n \end{matrix}$$

be the indicated minor of H. I will call the minor <u>nearly principal</u> if there is exactly one index i_r that is in $\{i_1,\ldots,i_p\}$ but not in $\{j_1,\ldots,j_p\}$, and exactly one index j_s that is in $\{j_1,\ldots,j_p\}$ but not in $\{i_1,\ldots,i_p\}$.

I will write $H\begin{pmatrix} i_1,\ldots i_p\ ;\ r \\ j_1,\ldots j_p\ ;\ s \end{pmatrix}$ to indicate the nearly principal minor in which i_r and j_s are as designated above."

Suppose that $L(\cdot)$ has continuous second partial derivatives, and let $H_{\underset{\sim}{y}}$ be the mxm matrix of these second partials, so that

$$h_{ij} = D_{ij}\, L(\underset{\sim}{y}).$$

Then Veinott's condition is:

(A) $H_{\underset{\sim}{y}}$ is positive definite (that is, $L(\cdot)$ is convex)

and

(B) every nearly principal minor of $H_{\underset{\sim}{y}}$ satisfies

$$(3.2)\qquad (-1)^{r+s}\, H_{\underset{\sim}{y}} \begin{pmatrix} i_1,\ldots,i_p\ ;\ r \\ j_1,\ldots,j_p\ ;\ s \end{pmatrix} \geq 0.$$

For two products, (B) reduces to $D_{12}L(\cdot)$ being non-negative, which as illustrated in Section 10 of Chapter 2 is necessary at least near the curves $D_1L=0$ and $D_2L=0$. For three products, (B) requires, in addition to non-negative cross partials,

$$D_{ij}(\underset{\sim}{x}) \cdot D_{ik}L(\underset{\sim}{x}) \leq D_{ii}L(\underset{\sim}{x}) \cdot D_{jk}L(\underset{\sim}{x}) \quad \text{for} \quad i \neq j \neq k \neq i.$$

The significance of this condition on the second partials will be discussed in the next section.

In [20], which is a personal letter in reply to my request for comments on a draft of Sections 1 through 6 of Chapter 2, Veinott does not include the proof that (A) and (B) imply his hypothesis. Therefore we are unable to include it here.

3.4 Three Products: Some Discussion

Suppose that Property A2 holds for every pair of products in a three product case. Consider the curves $D_1L=0$ and $D_2L=0$ as functions of x_3. For any given x_2, the value of x_1 at which $D_1L=0$ is a non-increasing function of x_3, since A2 holds for products 1 and 3. Similarly, for any given x_1, the x_2 at which $D_2L=0$ is a non-increasing function of x_3. Let $\underset{\sim}{z}_{12}(x_3)$ be the point at which $D_1L = D_2L = 0$ for any given x_3. Then, despite these two results, $x_3' \geq x_3$ does not imply that $\underset{\sim}{z}_{12}(x_3') \leq \underset{\sim}{z}_{12}(x_3)$. The situation has been illustrated in Figure 2.6.

If for some $x_3' > x_3$ it is true that $\underset{\sim}{z}_{12}(x_3') \leq \underset{\sim}{z}_{12}(x_3)$, then Hypothesis 3 will not be satisfied. If an attempt is made to prove directly that the one period optimal policy is optimal for the N period problem, then a situation exactly analogous to the illustration in Section 9 of Chapter 2 of what can result if $D_1L=0$ has positive slope in the two product case arises. Consequently, Property A2 holding for each pair of products is not sufficient to imply N period optimality for the optimal one period policy.

In the previus section it was noted that, for three products, Veinott's condition is

(i) $D_{ij}L(\underset{\sim}{x})\ D_{ik}L(\underset{\sim}{x}) \leq D_{ii}L(\underset{\sim}{x})\ D_{jk}L(\underset{\sim}{x})$
for $i \neq j \neq k \neq i$ and for every $\underset{\sim}{x}$

(ii) non-negative cross partials

(iii) convexity of $L(\cdot)$.

and that this condition implies Hypothesis 3. As we demonstrate below in Lemma 3.2, the addition of (i) and (ii) to A2 holding for every pair of products is sufficient to ensure that $\underset{\sim}{z}_{12}(x_3') \leq \underset{\sim}{z}_{12}(x_3)$ if $x_3' \geq x_3$ (and similar results for $\underset{\sim}{z}_{23}(\cdot)$ and $\underset{\sim}{z}_{13}(\cdot)$). (Actually, we change (ii) to require strictly positive cross partials, thus making the proof more compact.) Therefore, conditions (a) through (e) of Lemma 3.2 imply Hypothesis 3: This can be established in a straightforward way and we omit it.

Lemma 3.2 If

(a) $L(\cdot)$, considered as a function of x_1 and x_2, has Property A2 for any fixed x_3

(b) $L(\cdot)$, considered as a function of x_2 and x_3, has Property A2 for any fixed x_1

(c) $L(\cdot)$, considered as a function of x_1 and x_3, has Property A2 for any fixed x_2

(d) $D_{ij}L(\underset{\sim}{x}) D_{ik}L(\underset{\sim}{x}) \leq D_{ii}L(\underset{\sim}{x})\, D_{jk}L(\underset{\sim}{x})$ for every $\underset{\sim}{x}$ and $i \neq j \neq k \neq i$

and (e) $D_{ij}L(\underset{\sim}{x}) > 0$ for $i \neq j$ and for every $\underset{\sim}{x}$.

then
$$\underset{\sim}{z}_{12}(x_3') \leq \underset{\sim}{z}_{12}(x_3) \text{ if } x_3' \geq x_3 ,$$
$$\underset{\sim}{z}_{23}(x_1') \leq \underset{\sim}{z}_{23}(x_1) \text{ if } x_1' \geq x_1 ,$$
and
$$\underset{\sim}{z}_{13}(x_2') \leq \underset{\sim}{z}_{13}(x_2) \text{ if } x_2' \geq x_2 .$$

Proof: We will consider only the curve $D_1L = D_2L = 0$; proof for the other two cases is analogous and will be omitted.

Consider any two points on the curve $D_1L = D_2L = 0$. Label them $\underset{\sim}{p}$ and $\underset{\sim}{q}$ so that $p_3 > q_3$. We will show that $p_2 \leq q_2$. Symmetric arguments (which will be omitted) show that $p_1 \leq q_1$, so this will be sufficient to imply that $\underset{\sim}{z}_{12}(p_3) \leq \underset{\sim}{z}_{12}(q_3)$, the desired result.

Consider $\underset{\sim}{r}$ and $\underset{\sim}{s}$ such that $D_1L(\underset{\sim}{r}) = 0$, $r_2=q_2$, $r_3=p_3$ and $D_2L(\underset{\sim}{s}) = 0$, $s_2=q_2$, $s_3=p_3$. $D_{12}L(\cdot) > 0$ implies that $\underset{\sim}{s}$ is unique and (a) implies that $\underset{\sim}{r}$ is unique. See Figure 3.1. Now if $s_1 \leqq r_1$, then (a) of the hypothesis implies that $p_2 \leqq q_2$. Therefore we show that $s_1 \leqq r_1$.

Consider the x_1x_3 plane for $x_2 = q_2$. Define $\underset{\sim}{t}$ by $t_1=r_1$, $t_2=r_2$, $D_2L(\underset{\sim}{t})=0$; $D_{12}L(\cdot) > 0$ implies that t is unique. We show that $t_3 \leqq r_3$ which implies that $s_1 \leqq r_1$ since $D_{32}L(\cdot) > 0$. See Figure 3.2.

Let $z_1(x) \equiv z$ such that $D_1L(x,q_2,z) = 0$ and let $z_2(x) \equiv z$ such that $D_2L(x,q_2,z) = 0$. $D_{13}L(\cdot)$, $D_{23}L(\cdot) > 0$ imply that conditions for an implicit function theorem (p. 147 of Apostol [1]) are satisfied, so that $z_1(\cdot)$ and $z_2(\cdot)$ are functions and have first derivatives.

We show that $t_3 < r_3$ by contradiction. Suppose that $t_3 > r_3$; that is, that $z_2(r_1) > z_1(r_1)$. Then, since $z_1(q_1)=z_2(q_1)$, there must be at least one point with x_1 value $\epsilon(r_1,q_1]$ at which the $z_2(\cdot)$ curve crosses the $z_1(\cdot)$ curve. Let $\underset{\sim}{u}$ be any such point; then $\underset{\sim}{u}$ must satisfy $z_2'(u_1) < z_1'(u_1)$. But $z_2'(u_1) = -D_{12}L(\underset{\sim}{u})/D_{23}L(\underset{\sim}{u})$ which, by (d) and (e), is greater than or equal to $-D_{11}L(\underset{\sim}{u})/D_{13}L(\underset{\sim}{u}) = z_1'(u_1)$. Consequently there can be no such point $\underset{\sim}{u}$ and therefore $t_3 > r_3$ is contradicted.

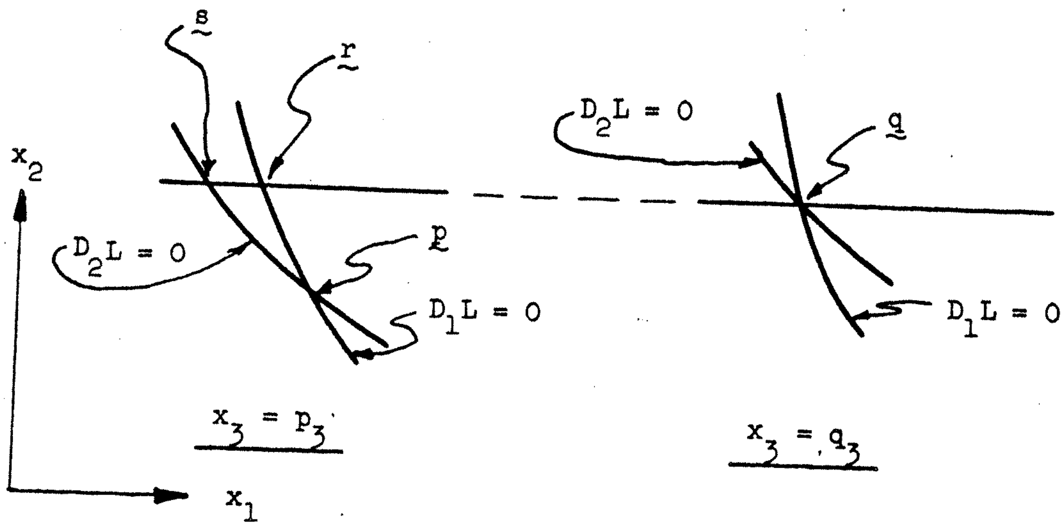


Figure 3.1

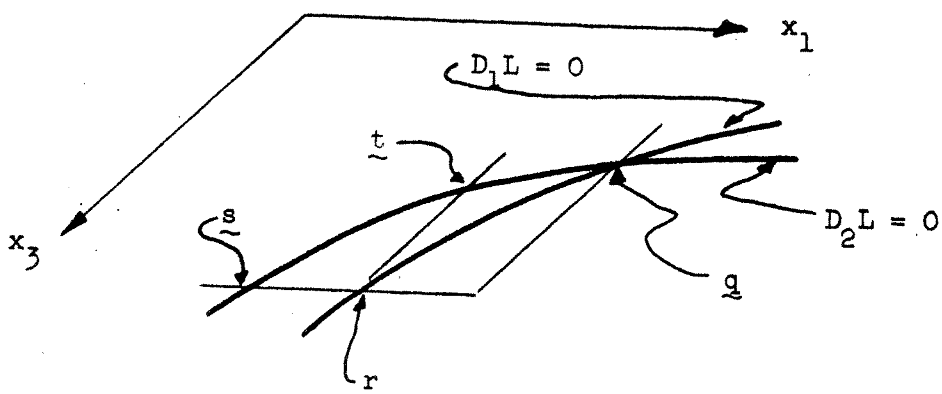


Figure 3.2

Chapter 4

Two Product Systems, Setup Cost

4.1 Summary This chapter is devoted to finding optimal ordering policies for two product systems when there is a positive setup cost associated with placing an order. In the introductory section we review some of the literature on both one product and multi-product problems with setup cost and give some assumptions and notation for the development which follows.

In Sections 3 and 4 we treat the case where there is one setup cost incurred if any order is placed. That is, the setup cost is independent of which product or products are ordered. We first obtain the optimal one period policy when $L(\cdot)$ has Property A2. Since we are unable to show that the N period policy is simple under A2, we introduce Hypothesis 4, which is stronger than A2. Under this hypothesis, the optimal N period policy is obtained.

Section 5 is devoted to the case where the setup cost incurred if both products are ordered exceeds the setup cost when only one product is ordered. We obtain the optimal one period policy: Its form depends on how much the setup cost is increased when both products rather than only one are ordered.

The chapter is concluded with a discussion which includes the difficulties of usefully extending K-convexity, so appropriate in the one product case, to two dimensions.

4.2 <u>Introduction</u> For one product, the classic paper is Scarf's [14], in which the concept of K-convexity is used to prove that the convexity of $L(\cdot)$ implies that a policy of the form "if $x \leq s_n$, order up to S_n; if $x > s_n$, order nothing" is optimal in period n. Such a policy is called an (s,S) policy. Iglehart [11] has shown that if $\alpha < 1$, each of the sequences $\{S_n\}$ and $\{s_n\}$ contains at least one convergent subsequence, and that using any pair of limit points is optimal in the infinite horizon problem. For $\alpha = 1$, Iglehart [10] has shown that there is a stationary (s,S) policy (that is, one with $s_1=s_2=\ldots$ and $S_1=S_2=\ldots$) such that as n gets large, the ratio of expected cost in n periods under an optimal policy to n times expected cost per period under the stationary policy approaches unity.

Scarf treated the case where inventory at the end of period 1 is <u>not</u> returnable. For the (simpler) case where it is returnable, Veinott and Wagner [26] have developed a computing procedure for finding the parameters of the optimal policy for the infinite horizon case. For $\alpha < 1$, they seek a pair of limit points of the sequences $\{S_n\}$ and $\{s_n\}$. For $\alpha = 1$, they seek a pair (s,S) that satisfy Iglehart's result. In the process, they obtain bounds on S_n and s_n. We obtain similar bounds on analogous parameters for a restricted version of the two product case.

Also assuming ending inventory is returnable, Veinott [22] shows that quasiconvexity of $L(\cdot)$ is sufficient for an (s,S) policy to be optimal. As Veinott points out, Scarf's development permits holding cost, shortage cost, and the linear part of purchase cost to vary with time as long as $L_n(\cdot)$ is convex for every n. Veinott's development permits non-stationarity of these costs as long as $L_n(\cdot)$ is quasiconvex for every n only if the $(\hat{s}_n, \hat{S}_n)$ that are optimal for period n considered by itself are such that $\hat{S}_n \leq \hat{S}_{n-1}$ for all n. (This nesting is similar to that required in Section 8 of Chapter 2.)

In the one product case, there is a setup cost K, incurred each time an order is placed. In the two product case, we <u>define</u> K_1 to be the setup cost incurred if only product 1 is ordered, K_2 to be the cost if only product 2 is ordered, and K_{12} to be the cost if both are ordered. We assume that $K_1, K_2, K_{12} > 0$. We also assume that $K_1 + K_2 \geq K_{12}$; that is, that there is no setup cost disadvantage to ordering both products simultaneously.

For two or more products, Naddor and Saltzman [13] and Balintfy [5] explore the possibilities for savings in ordering several products together, assuming that for each product, the demand in any time interval is a constant determined by the length of the interval and the fixed and known demand rate for the product. Naddor and Saltzman assume that when an order is placed which calls for the delivery of m different products, the setup cost for that order is given by $c_1 + c_2 m$ where $c_1, c_2 \geq 0$. Balintfy makes the same assumption in three of the four cases he treats. This assumption satisfies our requirement that $K_1+K_2 \geq K_{12}$, since it would specify $K_1=K_2= c_1+c_2$ and $K_{12}=c_1+2c_2$.

Balintfy also considers stochastic demand, under the assumption that inventory is reviewed continuously. He presents the concept of the "random joint order policy". Such a policy specifies, for each product, in addition to a reorder point and an "order-to" point (s and S), a "can-order" point which lies between the other two. Whenever the inventory of any product drops to its reorder point, an order is placed which brings the inventory of every product whose inventory was below its "can-order" point up to its "order-to" point. This type of policy, which seems very reasonable, should be compared with our one period periodic review results of Section 5.

Except for the addition of the setup cost, our assumptions about the operation of the system and the incurring of costs will be the same as those of Section 2 of Chapter 2. An optimal N period policy will continue to be one which minimizes expected discounted costs over the N period horizon.

We continue to assume that, at the end of the last period, any inventory on hand is returnable for full refund and any backlog must be purchased at the usual price. Again, this assumption is of little practical significance in an N period problem if N is large. For the one period problems that we solve, it is important. However, our results will hold when $L(\underset{\sim}{x}) \equiv (1-\alpha)\, \underset{\sim}{c}'\underset{\sim}{x}$

$$+ \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} [h(\underset{\sim}{x},\underset{\sim}{t}) + p(\underset{\sim}{x}-\underset{\sim}{t})]\, \varphi(\underset{\sim}{t})\, d\underset{\sim}{t} \text{ is replaced by}$$

$$G(\underset{\sim}{x}) = \underset{\sim}{c}'\underset{\sim}{x} + \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} [h(\underset{\sim}{x},\underset{\sim}{t}) + p(\underset{\sim}{x}-\underset{\sim}{t})]\, \varphi(\underset{\sim}{t})\, d\underset{\sim}{t},$$ if $G(\cdot)$ satisfies the conditions specified for $L(\cdot)$. Since $G(\cdot)$ plays the role of $L(\cdot)$ in determining the optimal one period policy when ending inventory is not returnable and ending backlog is not filled, our one period results apply in this situation.

It was hoped that K-convexity could profitably be extended to two dimensions. Unfortunately, we have not been able to do so. Our difficulties in attempting an extension are the subject of Section 6. Our approach then will be more like that of Veinott [22] and Veinott and Wagner [26] than that of Scarf [14].

In Section 12 of Chapter 2, the possibility of solving the two product problem as two one product problems when there is no setup cost was discussed. When there is a setup cost, $K_{12}=K_1+K_2$ is an additional requirement for the two single product solutions to be optimal for the original problem. Consequently, if $K_{12} \neq K_1+K_2$, the results that follow are of some interest even when $L(\cdot)$ is separable. In other words, for explicit treatment of the two product problem to be interesting when there is not setup cost, there must be some "jointness" in either holding cost or shortage cost. On the other hand, when there is a setup cost, a jointness in it alone requires explicit treatment of the two product problem. See the first example in Appendix E.

4.3 Equal Setup Cost; The One Period Problem

In this section (and the next), we assume that $K_1=K_2=K_{12} > 0$, and we define $K = K_1$. We obtain the optimal one period policy when $L(\cdot)$ has Property A2. The optimal policy is implicit in

$$C_1(\underset{\sim}{x}) \equiv \min[L(\underset{\sim}{x}), \min_{\underset{\sim}{y} > \underset{\sim}{x}} (L(\underset{\sim}{y}) + K)].$$

If $L(\underset{\sim}{x}) = C_1(\underset{\sim}{x})$ then it is optimal to place no order from x.
If $\underset{\sim}{y} > \underset{\sim}{x}$ and $L(\underset{\sim}{y}) + K = C_1(\underset{\sim}{x})$, then it is optimal to order up to $\underset{\sim}{y}$ from $\underset{\sim}{x}$.

The policy is pictured in Figure 4.1. In order to characterize it, we (implicitly) define the functions $q_{12}(\cdot)$, $q_1(\cdot)$, and $q_2(\cdot)$ as follows:

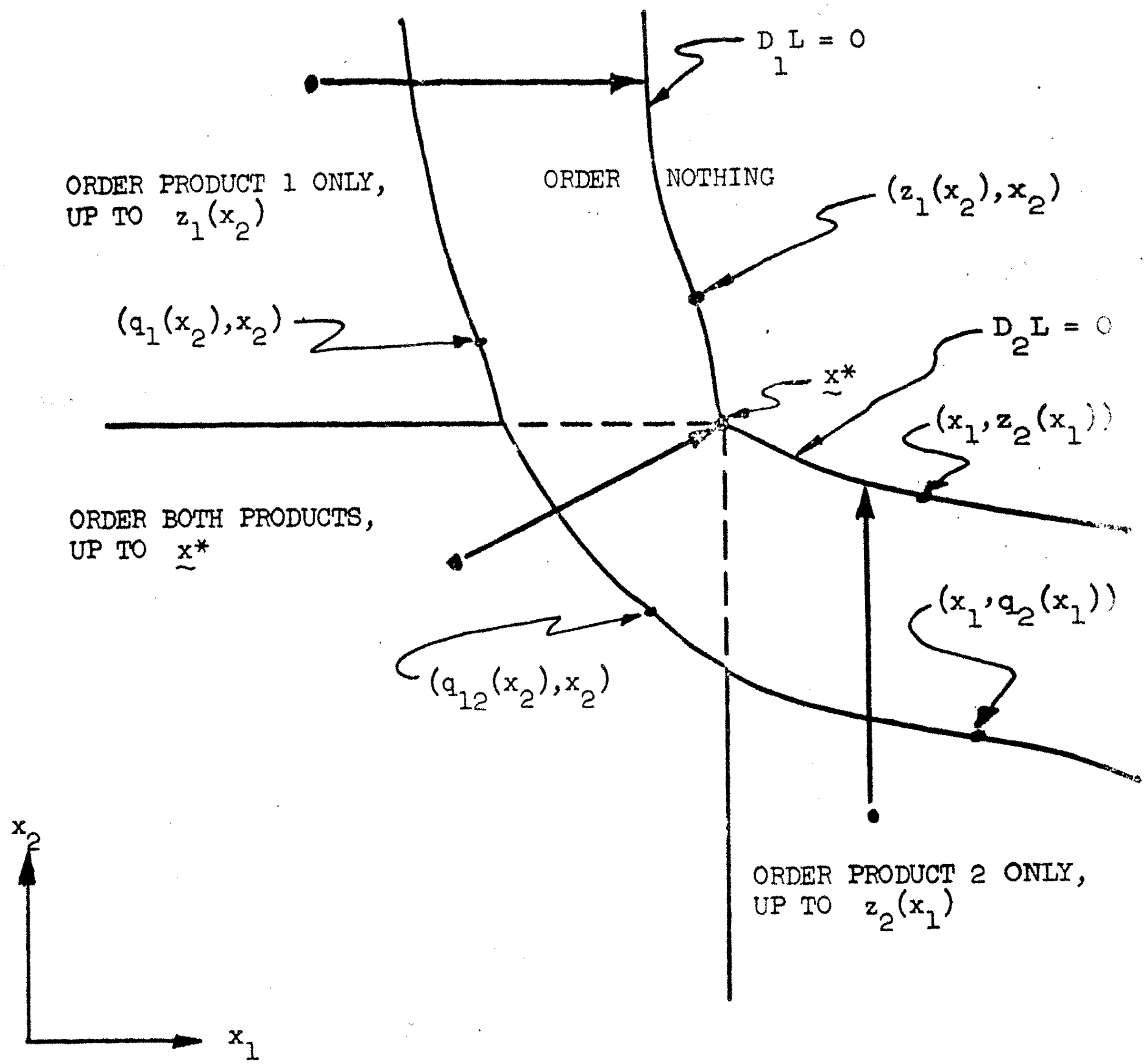
$$Q_{12} \equiv \{(q_{12}(v), v) \mid L(q_{12}(v),v) = L(\underset{\sim}{x}^*) + K_{12} \text{ and } (q_{12}(v), v) < \underset{\sim}{x}^*\}$$

$$Q_1 \equiv \{(q_1(v),v) \mid L(q_1(v),v) = L(z_1(v),v) + K_1 \text{ and } q_1(v) < z_1(v)\}$$

$$Q_2 \equiv \{(u, q_2(u)) \mid L(u, q_2(u)) = L(u, z_2(u)) + K_2 \text{ and } q_2(u) < z_2(u)\}$$

Since $D_1L(\underset{\sim}{x}) < 0$ for $\underset{\sim}{x} \leqq (z_1(x_2), x_2)$, specifying any x_2 yields a unique $q_1(x_2)$. Similarly, specifying any x_1 yields a unique $q_2(x_1)$. The function $q_{12}(\cdot)$ is defined on the interval $[q_2(x_1^*), x_2^*]$, and specifying any x_2 in this interval yields a unique $q_{12}(x_2)$ because $D_1L(\underset{\sim}{y}) < 0$ if $\underset{\sim}{y} \leqq \underset{\sim}{x}^*$.

Consider any $\underset{\sim}{x} \leqq \underset{\sim}{x}^*$. If $x_2 < q_2(x_1^*)$, then there is a point $(q_{12}(v),v)$ such that the line segment connecting $\underset{\sim}{x}$ to it is of non-negative slope. Then using the mean value theorem and the fact that $D_1L(\cdot)$ and $D_2L(\cdot)$ are negative below $\underset{\sim}{x}^*$, $L(\underset{\sim}{x}) > L(q_{12}(v),v)$. Since $L(q_{12}(v),v) = L(\underset{\sim}{x}^*) + K_{12}$, it is optimal to



The Optimal One period Policy, $K_1 = K_2 = K_{12}$

Figure 4.1

order up to $\underset{\sim}{x}^*$ from $\underset{\sim}{x}$. If $x_2 \geq q_2(x_1^*)$, then there are two possibilities: $x_1 \leq q_{12}(x_2)$ and $x_1 > q_{12}(x_2)$. If $x_1 \leq q_{12}(x_2)$, then $L(\underset{\sim}{x}) \geq L(q_{12}(x_2),x_2) = L(\underset{\sim}{x}^*) + K_{12}$, so it is optimal to order up to $\underset{\sim}{x}^*$ from $\underset{\sim}{x}$. If $x_1 > q_{12}(x_2)$, then the inequality is reversed, so that it is optimal to place no order from $\underset{\sim}{x}$.

Consider any $\underset{\sim}{x}$ such that $x_2 > x_2^*$, $x_1 \leq z_1(x_2)$. As shown in Lemma 2.3, $L(z_1(x_2),x_2) = \min\limits_{\underset{\sim}{y} \geq \underset{\sim}{x}} L(\underset{\sim}{y})$. Therefore, if $x_1 \leq q_1(x_2)$, $L(\underset{\sim}{x}) \geq L(q_1(x_2),x_2) = L(z_1(x_2),x_2) + K_1$, so that it is optimal to order up to $z_1(x_2)$ from $\underset{\sim}{x}$. If $x_1 > q_1(x_2)$, the inequality is reversed, so that it is optimal to place no order from $\underset{\sim}{x}$.

Consider any $\underset{\sim}{x}$ such that $x_1 > x_1^*$, $x_2 \leq z_2(x_1)$. By symmetry, if $x_2 \leq q_2(x_1)$, it is optimal to order up to $(x_1,z_2(x_1))$; if not, it is optimal to place no order.

If $\underset{\sim}{x}$ is such that either $x_2 > x_2^*$, $x_1 > z_1(x_2)$ or $x_1 > x_1^*$, $x_2 > z_2(x_1)$, then, as shown in Lemma 2.3, $L(\underset{\sim}{x}) = \min\limits_{\underset{\sim}{y} \geq \underset{\sim}{x}} L(\underset{\sim}{y})$, so that it is optimal to place no order from $\underset{\sim}{x}$.

This completes the proof of

Lemma 4.1: If $L(\cdot)$ has Property A2 and if $K_1=K_2=K_{12}=K > 0$, then the optimal one period policy, when inventory before ordering is $\underset{\sim}{x}$, is:

(a) if $x_2 < q_2(x_1*)$ and $x_1 \leqq x_1*$, order up to $\underset{\sim}{x}*$

(b) if $q_2(x_1*) \leqq x_2 \leqq x_2*$ and $x_1 \leqq q_{12}(x_2)$, order up to $\underset{\sim}{x}*$

(c) if $x_2 > x_2*$ and $x_1 \leqq q_1(x_2)$, order up to $(z_1(x_2),x_2)$

(d) if $x_1 > x_1*$ and $x_2 \leqq q_2(x_1)$, order up to $(x_1,z_2(x_1))$

(e) if $\underset{\sim}{x}$ satisfies none of the above, place no order from $\underset{\sim}{x}$.

That $q_{12}(\cdot)$ is a non-increasing function is clear because $D_1L(\cdot)$ and $D_2L(\cdot)$ are both negative below $\underset{\sim}{x}*$. It will be useful in the unequal setup cost cases for $q_1(\cdot)$ and $q_2(\cdot)$ to be non-increasing also. To ensure this, assuming that $D_{12}L(\cdot)$ is non-negative is sufficient.

<u>Lemma 4.2</u>: If $L(\cdot)$ has Property A2, $D_{12}(\cdot)$ is non-negative, and $K_1,K_2 > 0$, then $q_1(\cdot)$ and $q_2(\cdot)$ are non-increasing functions.

<u>Proof</u>: The proof will be carried out for $q_1(\cdot)$ only; the treatment of $q_2(\cdot)$ is analogous and will be omitted.

Consider any $y < w$. It will be shown that $q_1(y) \geqq q_1(w)$. (See Figure 4.2). Consider $\underset{\sim}{a} \equiv (q_1(w),y)$ and $\underset{\sim}{b} \equiv (z_1(w),y)$. By (c) of A2, $z_1(y) \geqq b_1$. It is true that

(A) $L(q_1(w),w)-L(\underset{\sim}{a}) \leqq L(z_1(w),w)-L(\underset{\sim}{b})$, because

(i) $L(q_1(w),w)-L(\underset{\sim}{a}) = \int_y^w D_2L(q_1(w),v)dv$

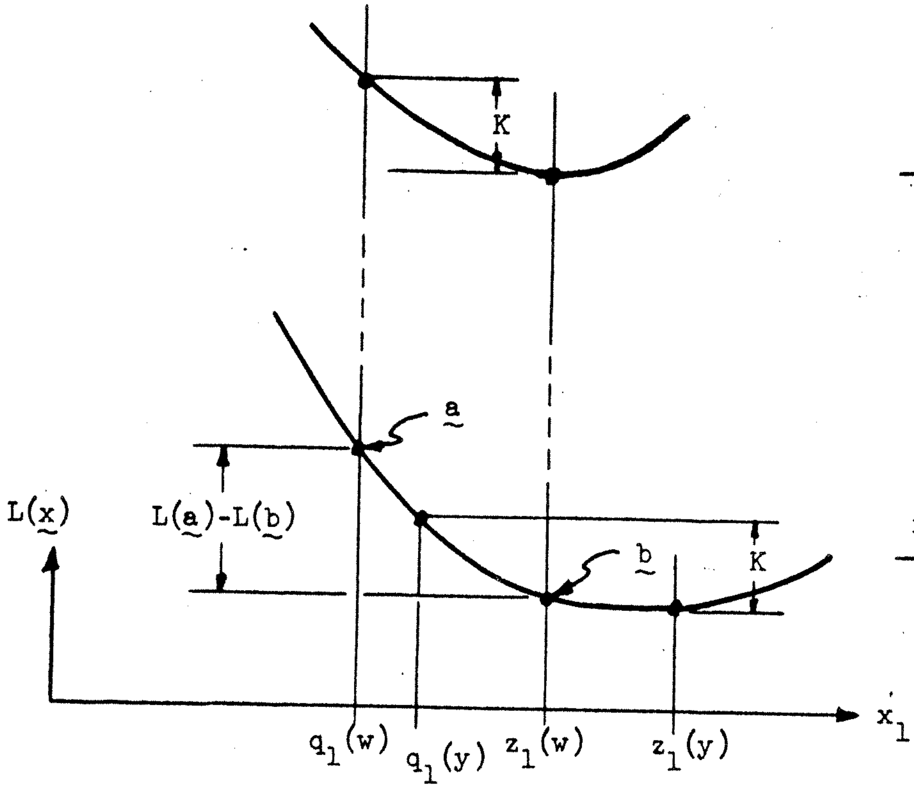
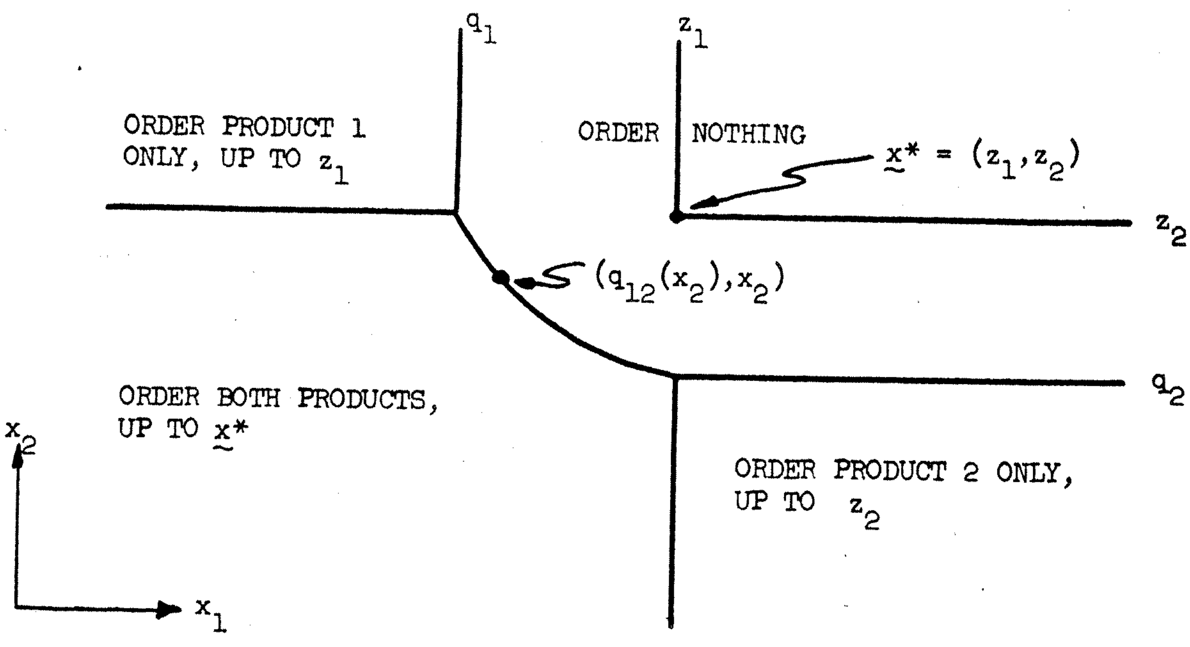


Figure 4.2



The Optimal One Period Policy when $L(\cdot)$ is Separable and $K_1 = K_2 = K_{12}$

Figure 4.3

(ii) $L(z_1(w),w)-L(\underset{\sim}{b}) = \int_y^w D_2L(z_1(w),v)dv$ and

(iii) $D_2L(z_1(w),v) \geq D_2L(q_1(w),v)$ for every v.

(Statement (iii) is true because $D_{12}(\cdot)$ is non-negative.) Rearranging (A), $L(\underset{\sim}{a})-L(\underset{\sim}{b}) \geq L(q_1(w),w)-L(z_1(w),w)=K$. Since $L(z_1(y),y) \leq L(\underset{\sim}{b})$, $L(\underset{\sim}{a}) - L(z_1(y),y) \geq K$. Consequently, $D_1L(u,y) < 0$ for $u < z_1(y)$ implies that $q_1(y) \geq a_1 = q_1(w)$, which completes the proof.

If $L(\cdot)$ is separable (that is, if $L(\underset{\sim}{x}) = L_1(x_1) + L_2(x_2)$ for every $\underset{\sim}{x}$), then $z_1(\cdot)$, $q_1(\cdot)$, $z_2(\cdot)$ and $q_2(\cdot)$ will be constants. But $q_{12}(\cdot)$ will not be a constant, and explicit recognition of the two product nature of the problem is necessary if the optimal one period policy is to be obtained. See Figure 4.3 and the first example in Appendix E.

4.4 Equal Setup Costs; Hypothesis 4, The N Period Problem

When $K_1=K_2=K_{12}$, we have not been able to show that Property A2 by itself is sufficient for a simple policy to be optimal in every period of an N period problem. We have needed a fairly restrictive set of assumptions, which we will call Hypothesis 4, in order to ensure the optimality of a simple policy. Before introducing this hypothesis, there are two statements which we can make about the policy in period n. The first states that an order need never be placed from the region where both $D_1L(\cdot)$ and $D_2L(\cdot)$ are positive. The second states that an order need never be placed to a region that is a subset of the one where $D_1L(\cdot)$ and $D_2L(\cdot)$ are positive.

<u>Lemma 4.3</u>: If $L(\cdot)$ has Property A2, then if $\underset{\sim}{x} \in R_0$, where

$$R_0 \equiv \{(x_1,x_2) \mid x_1 \geq x_1^*,\ x_2 \geq z_2(x_1) \ \text{ or } \ x_2 \geq x_2^*,\ x_1 \geq z_1(x_2)\},$$

there is <u>an</u> optimal policy such that, for $n=1,2,\ldots,N$, it is optimal to order nothing from $\underset{\sim}{x}$ when there are n periods remaining.

<u>Proof</u>: In this proof we use only two facts:

$D_1L(\cdot)$ and $D_2L(\cdot)$ are non-negative in R_0 and

$C_n(\cdot)$ is the expected n period cost under an <u>optimal policy</u>.

For $n=1$, it is clearly optimal to not order from any $\underset{\sim}{x} \in R_0$.

Consider any $n \geq 2$ and consider any $\underset{\sim}{x} \in R_0$. We will show, for any $\underset{\sim}{y}$ such that $\underset{\sim}{y} > \underset{\sim}{x}$, that there is no advantage to ordering up to $\underset{\sim}{y}$ from $\underset{\sim}{x}$. This requires the examination of three cases:

(1) $y_1 > x_1,\ y_2 = x_2$

(2) $y_1 = x_1,\ y_2 > x_2$

(3) $y_1 > x_1,\ y_2 > x_2$

We will omit cases (2) and (3) since they are analogous to case (1) which is treated below.

To show that there is no advantage to ordering from $\underset{\sim}{x}$ to $\underset{\sim}{y}$ in period n when $y_1 > x_1$, $y_2 = x_2$, we must show that $G_n(\underset{\sim}{y}) + K_1 \geq G_n(\underset{\sim}{x})$. First of all, $L(\underset{\sim}{y}) \geq L(\underset{\sim}{x})$. So if we can show that $\int C_{n-1}(\underset{\sim}{y}-\underset{\sim}{t})\,\varphi(\underset{\sim}{t})d\underset{\sim}{t} + K_1 \geq \int C_{n-1}(\underset{\sim}{x}-\underset{\sim}{t})\,\varphi(\underset{\sim}{t})d\underset{\sim}{t}$, then adding α times this inequality to the first one yields

$G_n(\underset{\sim}{y}) + \alpha K_1 \geq G_n(\underset{\sim}{x})$, so that $G_n(\underset{\sim}{y}) + K_1 \geq G_n(\underset{\sim}{x})$.

Consider any $\underset{\sim}{t}$. For any optimal policy, one of the following must be true in period n-1:

(a) order nothing from $\underset{\sim}{y} - \underset{\sim}{t}$

(b) order product 1 only from $\underset{\sim}{y} - \underset{\sim}{t}$

(c) order product 2 only from $\underset{\sim}{y} - \underset{\sim}{t}$

(d) order both products from $\underset{\sim}{y} - \underset{\sim}{t}$.

We establish that $C_{n-1}(\underset{\sim}{y} - \underset{\sim}{t}) + K_1 \geq C_{n-1}(\underset{\sim}{x} - \underset{\sim}{t})$ in all four situations. Therefore, weighting by $\varphi(\cdot)$ and integrating over $\underset{\sim}{t} \geq \underset{\sim}{0}$ yields

$\int C_{n-1}(\underset{\sim}{y}-\underset{\sim}{t})\,\varphi(\underset{\sim}{t})d\underset{\sim}{t} + K_1 \geq \int C_{n-1}(\underset{\sim}{x}-\underset{\sim}{t})\,\varphi(\underset{\sim}{t})d\underset{\sim}{t}$, which is the desired result.

(a) If nothing is ordered from $\underset{\sim}{y}-\underset{\sim}{t}$, then since $y_2-t_2 = x_2-t_2$ and $y_1-t_1 > x_1-t_1$, an order <u>can</u> be placed from $\underset{\sim}{x}-\underset{\sim}{t}$ to $\underset{\sim}{y}-\underset{\sim}{t}$ at a cost of K_1. Therefore

$C_{n-1}(\underset{\sim}{x}-\underset{\sim}{t}) \leq G_{n-1}(\underset{\sim}{y}-\underset{\sim}{t}) + K_1 = C_{n-1}(\underset{\sim}{y}-\underset{\sim}{t}) + K_1.$

(b) If only product 1 is ordered from $\underset{\sim}{y}-\underset{\sim}{t}$, up to $\underset{\sim}{q}$, then since $q_1 > x_1-t_1$ and $q_2=x_2-t_2$,

$$C_{n-1}(\underset{\sim}{x}-\underset{\sim}{t}) \leq G_{n-1}(\underset{\sim}{q}) + K_1 = C_{n-1}(\underset{\sim}{y}-\underset{\sim}{t}) \leq C_{n-1}(\underset{\sim}{y}-\underset{\sim}{t}) + K_1.$$

(c) If only product 2 is ordered from $\underset{\sim}{y} - \underset{\sim}{t}$, up to $\underset{\sim}{q}$, then we have

$$C_{n-1}(\underset{\sim}{x}-\underset{\sim}{t}) \leq G_{n-1}(\underset{\sim}{q}) + K_{12} =$$

$$(G_{n-1}(\underset{\sim}{q}) + K_2) + (K_{12}-K_2) =$$

$$C_{n-1}(\underset{\sim}{q}) + (K_{12}-K_2) \leq C_{n-1}(\underset{\sim}{q}) + K_1, \quad \text{since} \quad K_1+K_2 \geq K_{12}.$$

(d) If both products are ordered from $\underset{\sim}{y}-\underset{\sim}{t}$, up to $\underset{\sim}{q}$, then $C_{n-1}(\underset{\sim}{x}-\underset{\sim}{t}) \leq G_{n-1}(\underset{\sim}{q}) + K_{12} = C_{n-1}(\underset{\sim}{y}-\underset{\sim}{t}) \leq C_{n-1}(\underset{\sim}{y}-\underset{\sim}{t}) + K_1$.

This establishes that $C_{n-1}(\underset{\sim}{x}-\underset{\sim}{t}) \leq C_{n-1}(\underset{\sim}{y}-\underset{\sim}{t}) + K_1$ for every $\underset{\sim}{t}$ and completes the proof of the lemma.

To obtain the second result, that part of this region from which orders are not placed is such that no orders are placed from anywhere to it, we (implicitly) define $w_1(\cdot)$ and $w_2(\cdot)$ as follows:

$$W_1 \equiv \{(w_1(v),v) \mid L(w_1(v),v) = L(z_1(v),v) + K_1 \text{ and } w_1(v) > z_1(v)\}$$

$$W_2 \equiv \{(u,w_2(u)) \mid L(u,w_2(u)) = L(u,z_2(u)) + K_2 \text{ and } w_2(u) > z_2(u)\}.$$

Then, as proved in Lemma 4.4 below, there is an optimal policy such that one never orders up to any point that lies above both of these curves. (The fact that these curves might intersect more than once is not relevant.)

<u>Lemma 4.4</u>: If $L(\cdot)$ has Property A2 and if $K_1=K_2=K_{12}= K \geq 0$, then, for $n=1,2,\ldots,N$, for any point $\underset{\sim}{x}$ such that $x_1 \geq w_1(x_2)$ and $x_2 \geq w_2(x_1)$, there is an optimal policy such that there is no point $\underset{\sim}{y}$ from which it is optimal, in period n, to order up to $\underset{\sim}{x}$ from $\underset{\sim}{y}$.

<u>Proof</u>: Consider any $\underset{\sim}{x}$ such $x_1 \geq w_1(x_2)$ and $x_2 \geq w_2(x_1)$, and consider any $\underset{\sim}{y}$. If $\underset{\sim}{y} \nleq \underset{\sim}{x}$, then it is impossible to order from $\underset{\sim}{y}$ to $\underset{\sim}{x}$, so it cannot be optimal to do so. If $\underset{\sim}{y} \leq \underset{\sim}{x}$, then consider the line passing through both $\underset{\sim}{y}$ and $\underset{\sim}{x}$. Since $\underset{\sim}{y} \leq \underset{\sim}{x}$, this line has non-negative slope. Proceeding down and/or to the left from $\underset{\sim}{x}$ on the line, one of the following will be reached first: $\underset{\sim}{y}$, the curve $D_1L=0$, the curve $D_2L=0$. See Figure 4.4 for an example where $D_1L=0$ is reached first.

If $\underset{\sim}{y}$ is reached first, then $y_1 \geq z_1(y_2)$ and $y_2 \geq z_2(y_1)$, so that by Lemma 4.3, it is not optimal to order up to $\underset{\sim}{x}$ from $\underset{\sim}{y}$.

If $D_1L=0$ is reached first, at some point, call it $\underset{\sim}{v}$, then $v_2 \geq z_2(v_1)$, so that $v_2 \geq x_2^*$. We show that, for

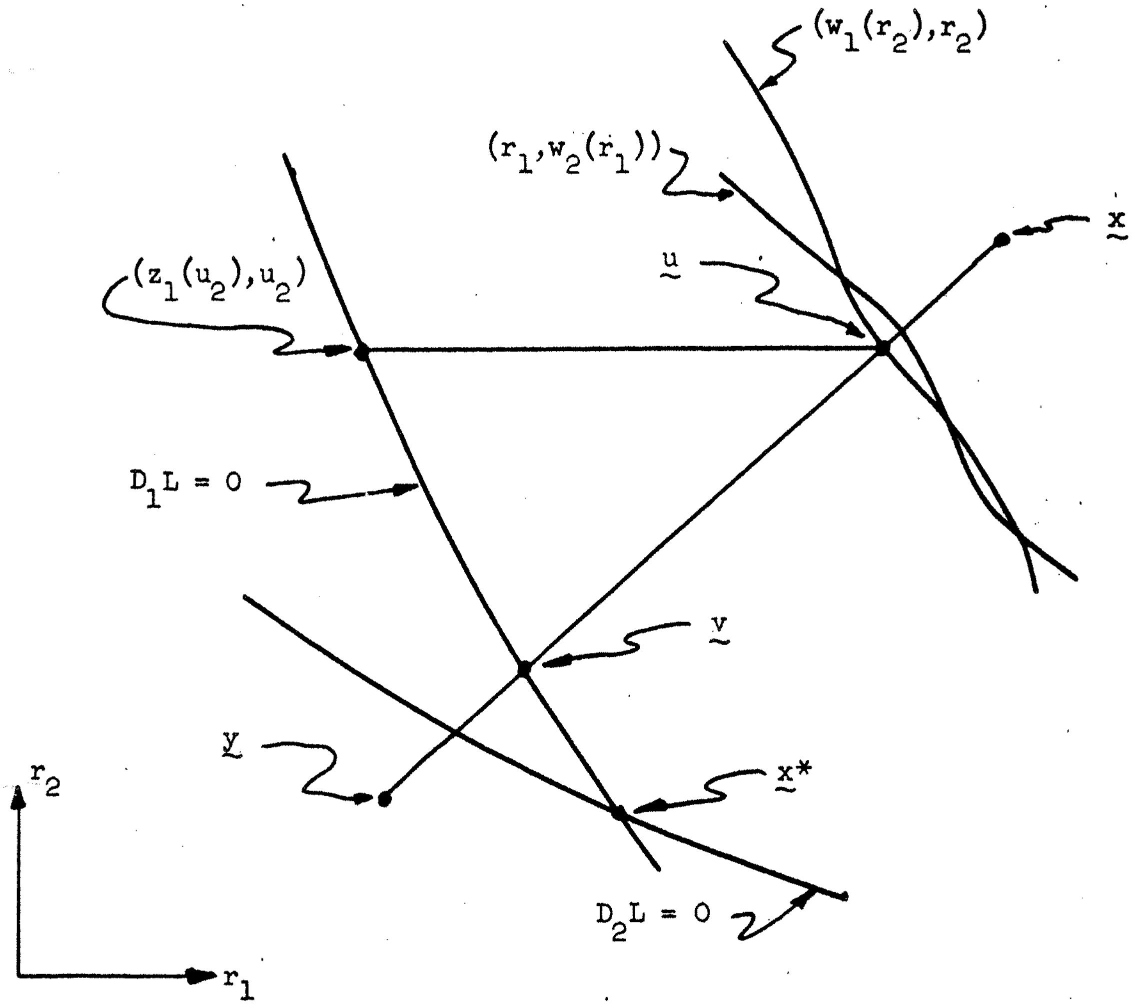

$(w_1(r_2),r_2)$
$(r_1,w_2(r_1))$
$(z_1(u_2),u_2)$
$\underset{\sim}{u}$
$\underset{\sim}{x}$
$D_1L = 0$
$\underset{\sim}{v}$
$\underset{\sim}{y}$
$\underset{\sim}{x}^*$
$D_2L = 0$
r_2
r_1

Figure 4.4

$n=1,2,\ldots,N$, $G_n(\underset{\sim}{v}) \leq G_n(\underset{\sim}{x})$, so that there is no advantage to ordering to $\underset{\sim}{x}$ rather than $\underset{\sim}{v}$. Since $\underset{\sim}{v} \geq \underset{\sim}{y}$, this is sufficient to show that there is an optimal policy which would not specify an order from $\underset{\sim}{y}$ to $\underset{\sim}{x}$ when $D_1L=0$ is reached first.

Let u be the point on the line at which $w_1(u_2) = u_1$. Then by Lemma 2.1, $u_2 \geq v_2 \geq x_2{*}$ implies that $L(\underset{\sim}{v}) \leq L(z_1(u_2),u_2)$. $L(z_1(u_2),u_2) + K \leq L(\underset{\sim}{u})$ by the definition of $w_1(\cdot)$. $L(\underset{\sim}{u}) \leq L(\underset{\sim}{x})$. Therefore $L(\underset{\sim}{v}) + K_1 \leq L(\underset{\sim}{x})$.

For $n=1$, $G_1(\underset{\sim}{r}) = L(\underset{\sim}{r})$ for every $\underset{\sim}{r}$, so $G_1(\underset{\sim}{v}) \leq G_1(\underset{\sim}{x})$. For $n > 1$, for every $\underset{\sim}{t}$, $C_{n-1}(\underset{\sim}{v}-\underset{\sim}{t}) \leq C_{n-1}(\underset{\sim}{x}-\underset{\sim}{t}) + K$ since $\underset{\sim}{v}-\underset{\sim}{t} \leq \underset{\sim}{x}-\underset{\sim}{t}$. Therefore

$$\alpha \int C_{n-1}(\underset{\sim}{v}-\underset{\sim}{t})\ \varphi(\underset{\sim}{t})d\underset{\sim}{t} \leq \alpha \int C_{n-1}(\underset{\sim}{x}-\underset{\sim}{t})\ \varphi(\underset{\sim}{t})d\underset{\sim}{t} + \alpha K \leq \alpha \int C_{n-1}(\underset{\sim}{x}-\underset{\sim}{t})\ \varphi(\underset{\sim}{t})d\underset{\sim}{t} + K.$$

Adding this inequality to $L(\underset{\sim}{v}) + K \leq L(\underset{\sim}{x})$ yields $G_n(\underset{\sim}{v}) \leq G_n(\underset{\sim}{x})$, the desired result.

If $D_2L=0$ is reached first, at some point $\underset{\sim}{v}$, then arguments analogous to those used for the case when $D_1L=0$ is reached first imply that $G_n(\underset{\sim}{v}) \leq G_n(\underset{\sim}{x})$ for $n=1,2,\ldots,N$. This then completes the proof of the lemma.

Before introducing Hypothesis 4, a definition is needed. A function $f(\cdot)$ of two variables that satisfies $f(x_1,x_2) = f(x_2,x_1)$ for every (x_1,x_2) will be said to be <u>symmetric</u>.

<u>Hypothesis 4</u>: Consider a two product system. We will say that Hypothesis 4 is satisfied if

(i) $L(\cdot)$ has Property A2

(ii) $L(\cdot)$ is quasiconvex

(iii) $L(\cdot)$ is symmetric

(iv) $K_1=K_2=K_{12} \geq 0$

(v) $\varphi(\cdot)$ satisfies $\varphi(u,r-u) \geq \varphi(v,r-v)$ if $\left|u-\frac{r}{2}\right| \leq \left|v-\frac{r}{2}\right|$, for every r.

The most restrictive requirement in Hypothesis 4 is that $L(\cdot)$ and $\varphi(\cdot)$ be symmetric. (That $\varphi(\cdot)$ must be symmetric can be seen by letting $v=r-u$ in (v).) $L(\cdot)$ must be symmetric if $h(\cdot)$, $p(\cdot)$ and $\varphi(\cdot)$ are, and it is unlikely that it will be if any of them are not. It is not unreasonable to suppose that inventory can be measured in units such that $h(\cdot)$ and $p(\cdot)$ will be symmetric if the products are physically similar, so the restrictive part is that $\varphi(\cdot)$ be symmetric. Symmetry of $\varphi(\cdot)$ implies that the (marginal) distribution of demand for product 1 is the same as that for product 2; same mean, same variance, same shape, etc. There can be no redefinition of units to achieve this since the symmetry of $h(\cdot)$ and $p(\cdot)$ would be destroyed.

Part (v) requires more than symmetry of $\varphi(\cdot)$. If $\varphi(\cdot)$ is symmetric and demand for product 1 is independent of demand for product 2 then $\varphi(\cdot)$ satisfies (v) if and only if the marginal density is a Polya frequency function of order 2.[1] We prove this below as Lemma 4.5. Since the exponential, gamma, uniform, truncated normal, and many other unimodal densities are Polya of order 2, the requirement (v) will often be met if $\varphi(\cdot)$ is symmetric and demands for the two products are independent.

If demands for the two products are not independent, we have no corresponding result as to when (v) will be met. However, when $\varphi(\cdot)$ is both symmetric and a truncated bivariate normal density[2], it is easy to verify that it is met.

<u>Lemma 4.5</u>: If, in a given period, demands for products 1 and 2 are independent and identically distributed non-negative random variables, then (I) their (common) density is a Polya frequency function of order 2 if and only if (II) their joint density $\varphi(\cdot)$ satisfies, for every r, $\varphi(u,r-u) \geq \varphi(v,r-v)$ if $\left|u-\frac{r}{2}\right| \leq \left|v-\frac{r}{2}\right|$.

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1 - For a detailed discussion of Polya frequency functions, see Schoenberg [17]. For their application in inventory theory, see Karlin, Chapters 8 and 9 of [3].

2 - That is, if

$$\varphi(x_1,x_2) = k\cdot\exp\left(-\frac{(x_1-\mu)^2 + (x_2-\mu)^2 - 2\rho(x_1-\mu)(x_2-\mu)}{2(1-\rho^2)\sigma^2}\right) \quad \text{for } x_1,x_2 > 0$$

$$= 0 \quad \text{elsewhere,}$$

and $k > 0$, $|\rho| \leq 1$, $\sigma > 0$, μ unrestricted.

Proof: Let $f(\cdot)$ be the marginal density of demand for product 1. By assumption, it is also the marginal density of demand for product 2.

First it will be shown that (I) implies (II). Consider any r and any u,v such that $\left|u-\frac{r}{2}\right| \leq \left|v-\frac{r}{2}\right|$ and consider $\varphi(u,r-u)$ and $\varphi(v,r-v)$. Since demand is non-negative, and since $\varphi(x,y) = \varphi(y,x)$, attention can be restricted to $v \geq u \geq \frac{r}{2} \geq 0$ with no loss of generality. Since demands for the two products are independent and identically distributed, $\varphi(u,r-u) = f(u)f(r-u)$ and $\varphi(v,r-v) = f(v)f(r-v)$.

Since $f(\cdot)$ is a Polya frequency function of order 2, for any $y_1 < y_2$, $z_1 < z_2$, (A) $f(y_1-z_1)\ f(y_2-z_2) \geq f(y_1-z_2)\ f(y_2-z_1)$. Let $z_1=0$, $y_1=u$, $y_2=v$, $z_2=u+v-r$. These satisfy $y_1 < y_2$ and $z_1 < z_2$. Substituting $y_1-z_1 = u$, $y_1-z_2 = r-v$, $y_2-z_1 = v$, $y_2-z_2 = r-u$ into (A) yields $f(u)f(r-u) \geq f(v)f(r-v)$, which is the desired result.

Now it will be shown that (II) implies (I). Consider any $y_1 < y_2$ and any $z_1 < z_2$. Let $u = y_1-z_1$, $v = y_2-z_1$, $r = y_1+y_2-z_1-z_2$. Then $r-u = y_2-z_2$ and $r-v = y_1-z_2$. Now $\left|u - \frac{r}{2}\right| = 1/2\ \left|2u-r\right| =$

$$1/2\ \left|y_1-y_2+z_2-z_1\right|$$

and $\left|v-\frac{r}{2}\right| = 1/2\ \left|y_2-y_1+z_2-z_1\right|$. Since $y_1 < y_2$ and $z_1 < z_2$, $\left|v-\frac{r}{2}\right| \geq \left|u-\frac{r}{2}\right|$. Therefore, by (II), $f(u)f(r-u) \geq f(v)f(r-v)$.

Substituting for u, v, and r, $f(y_1-z_1)f(y_2-z_2) \geq f(y_2-z_1)f(y_1-z_2)$, which shows that $f(\cdot)$ is a Polya frequency function of order 2, and the proof is complete.

The requirement (ii) that $L(\cdot)$ be quasiconvex is made in order that it be quasiconvex on any line of slope equal to -1. This property, along with symmetry and the requirement on $\varphi(\cdot)$, implies that $G_n(\cdot)$ and $C_n(\cdot)$ will also be quasiconvex on any line of slope -1. There is nothing to be gained by assuming convexity instead of quasiconvexity because convexity will not be reproduced on lines of slope -1.[1]

The following lemma will be needed in showing that quasiconvexity on any line of slope -1 is preserved.

<u>Lemma 4.6</u>: If $g(\cdot)$ is a quasiconvex function of one variable and symmetric about 0 and if $f(\cdot)$ is unimodal, non-negative, and symmetric about 0, with $\int_{-\infty}^{\infty} f(x)dx < \infty$ and with $f(x) = 0$ for all $x > M$ for some $M < \infty$, <u>then</u> $J(\cdot)$, defined by

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1 - For example, if $L(\underset{\sim}{x}) = x_1^2 + x_2^2$ and K=1, then $C_1(\underset{\sim}{x}) = L(\underset{\sim}{x})$ if $x_2 > 0, -1 \leq x_1 \leq 0$ and $C_1(\underset{\sim}{x}) = 1 + x_2^2$ if $x_2 > 0, x_1 < -1$. Therefore, $C_1(-1.1,1.1) = 2.21$, $C_1(-1,1) = 2$, $C_1(-.9,.9) = 1.62$ and $C_1(\cdot)$ is not convex on the line $x_1 = -x_2$. It is quasiconvex on that line.

$J(a) = \int_{-\infty}^{\infty} g(a-x)\, f(x)dx$, is a quasiconvex function, symmetric about 0.

Note: It is assumed that $f(x) = 0$ if $x > M$ in order to ensure the existence of the integral that defines $J(\cdot)$.

Proof: First it will be shown that $J(\cdot)$ is symmetric about 0. $J(-a) = \int_{-\infty}^{\infty} g(-a-x)f(x)dx$. Letting $y = -x$ and using the symmetry of $g(\cdot)$ and $f(\cdot)$ about 0, $J(-a) = \int_{+\infty}^{\infty} g(a-y)f(y)dy = J(a)$.

In proving that $J(\cdot)$ is quasiconvex, it will be assumed that both $g(\cdot)$ and $f(\cdot)$ have continuous first derivatives. (If this is not the case, then the proof is somewhat tedious, and it is deferred to Appendix F.) Since, $J(\cdot)$ has been shown to be symmetric about 0, it will suffice to show that $J'(a) \leqq 0$ if $a \leqq 0$.

Since $g(\cdot)$ and $f(\cdot)$ have continuous first derivatives, $J'(\cdot)$ exists and is given by $J'(a) = \int_{-\infty}^{\infty} g'(a-x)\, f(x)\, dx$. Substituting $y = a-x$, $J'(a) = \int_{-\infty}^{\infty} g'(y)\, f(a-y)dy$ which equals $\int_{0}^{\infty} g'(y)\, [f(a-y) - f(a+y)]dy$ since $g'(y) = -g'(-y)$. If $a \leqq 0$, then $y \geqq 0$ implies that $|a-y| \geqq |a+y|$, so that $f(a-y) \leqq f(a+y)$. Since $g'(y) \geqq 0$ for $y \geqq 0$, $J'(a) \leqq 0$ if $a \leqq 0$, and the proof is complete.

The following three lemmas pave the way for the specification of the optimal n period policy.

<u>Lemma 4.7</u>: If $L(\cdot)$ has Property A2, is quasiconvex, and is symmetric, and if $K_1=K_2=K_{12}=K \geq 0$ then, for any r, $\left|x-\frac{r}{2}\right| \geq \left|y-\frac{r}{2}\right|$ implies that $C_1(x,r-x) \geq C_1(y,r-y)$.

<u>Proof</u>: Since $L(\cdot)$ is quasiconvex and symmetric, and since $K_1=K_2$, $C_1(\cdot)$ is also symmetric. Therefore, it is sufficient to consider any $x \geq y \geq \frac{r}{2}$. One of the following three statements describes the optimal ordering policy.

(i) No order is placed from $(x,r-x)$

(ii) An order is placed from $(x,r-x)$ to a point $\underset{\sim}{t}$, $\underset{\sim}{t} \geq (y,r-y)$.

(iii) An order is placed from $(x,r-x)$ to a point $\underset{\sim}{t}$, $\underset{\sim}{t} \ngeq (y,r-y)$.

It will be shown that $C_1(x,r-x) \geq C_1(y,r-y)$ in all three cases.

(i) In this case, $C_1(x,r-x) = L(x,r-x)$ which is greater than or equal to $L(y,r-y) = C_1(y,r-y)$.

(ii) Since $\underset{\sim}{t} \geq (y,r-y)$, $C_1(y,r-y) \leq K+L(t)$.

$C_1(x,r-x) = K+L(\underset{\sim}{t})$, so $C_1(x,r-x) \geq C_1(y,r-y)$.

(iii) See Figure 4.5. Since $\underset{\sim}{t} > (x,r-x)$, $t_1+t_2 > r$.

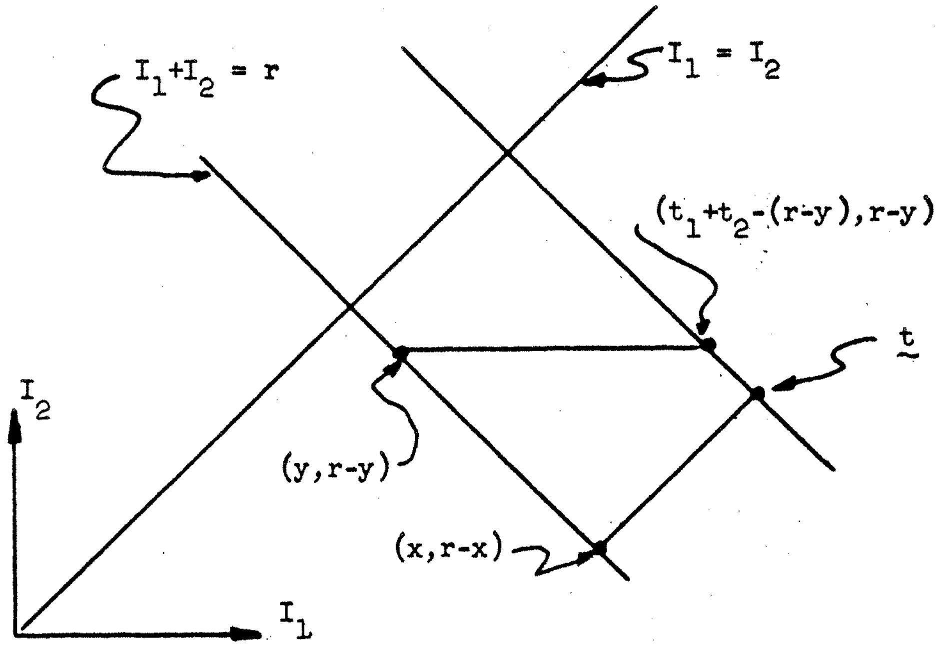


Figure 4.5

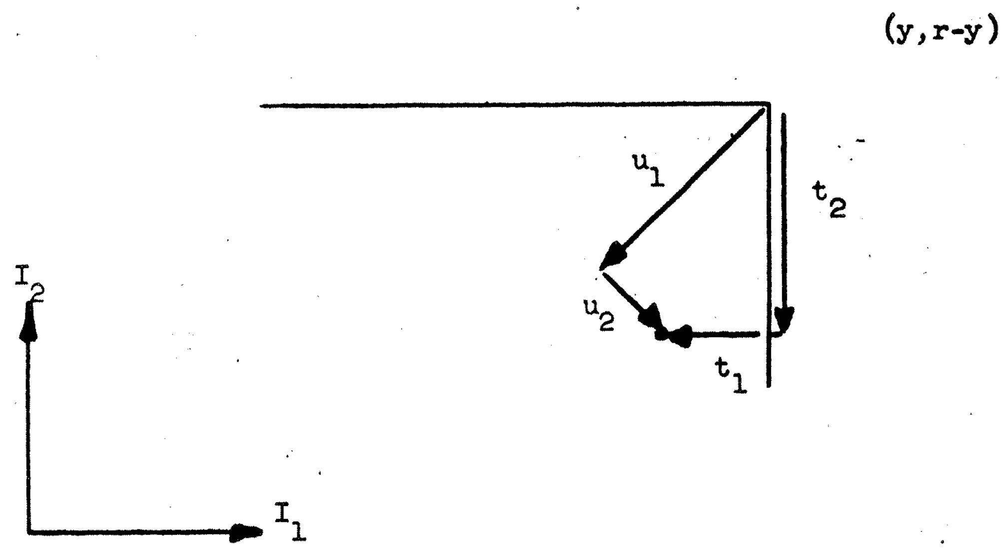


Figure 4.6

Since $\underset{\sim}{t} \nleq (y, r-y)$ and $t_1 \geq x > y$, $t_2 < r-y$. Therefore $t_1 > t_1+t_2 - (r-y)$. Since $t_1+t_2 - (r-y) > \frac{r}{2}$, $L(t_1+t_2 - (r-y), r-y) \leq L(\underset{\sim}{t})$. But $C_1(x, r-x) = L(\underset{\sim}{t})+K$ and since $t_1+t_2 - (r-y) > y$, $C_1(y, r-y) \leq L(t_1+t_2 - (r-y), r-y)+K$, so that $C_1(y, r-y) \leq C_1(x, r-x)$.

<u>Lemma 4.8</u>: If Hypothesis 4 is satisfied, then $G_n(x, r-x) \geq G_n(y, r-y)$ and $C_n(x, r-x) \geq C_n(y, r-y)$ for $n=1,2,\ldots,N$ and for every r, if $\left|x-\frac{r}{2}\right| \leq \left|y-\frac{r}{2}\right|$.

<u>Proof:</u> The proof will be by induction.

First consider $n=1$. $G_1(\underset{\sim}{x}) = L(\underset{\sim}{x})$ for every $\underset{\sim}{x}$, so that by Hypothesis 4, $G_1(\cdot)$ satisfies the condition. By Lemma 4.7, $C_1(\cdot)$ satisfies it.

Now suppose that $G_n(\cdot)$ and $C_n(\cdot)$ satisfy the condition. We will show that $G_{n+1}(\cdot)$ satisfies it. Then, by the arguments used to prove Lemma 4.7, (with $G_{n+1}(\cdot)$ playing the role of $L(\cdot)$ and $C_{n+1}(\cdot)$ the role of $C_1(\cdot)$) $C_{n+1}(\cdot)$ must satisfy it. These arguments will be omitted.

By definition,

$$G_{n+1}(\underset{\sim}{u}) = L(\underset{\sim}{u}) + \alpha \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n(\underset{\sim}{u}-\underset{\sim}{t})\, \varphi(\underset{\sim}{t})\, d\underset{\sim}{t}$$

and $C_{n+1}(\underset{\sim}{u}) = \min [\min_{\underset{\sim}{v} > \underset{\sim}{u}} G_{n+1}(\underset{\sim}{v}) + K, G_{n+1}(\underset{\sim}{u})]$. Since $L(\cdot)$ is symmetric by Hypothesis 4 and $C_n(\cdot)$ is symmetric by the induction hypothesis, $G_{n+1}(\cdot)$ is symmetric. Therefore, it will be sufficient to consider $y \geq x \geq \frac{r}{2}$. For this case then, we wish to show that $G_{n+1}(y,r-y) \geq G_{n+1}(x,r-x)$. $G_{n+1}(y,r-y) - G_{n+1}(x,r-x) =$
$L(y,r-y) - L(x,r-x) + \alpha[\int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n((y,r-y)-\underset{\sim}{t})\varphi(\underset{\sim}{t})d\underset{\sim}{t} - \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} C_n((x,r-x)-\underset{\sim}{t})\varphi(\underset{\sim}{t})d\underset{\sim}{t}]$.

Since $L(y,r-y) \geq L(x,r-x)$ and $\alpha \geq 0$, it will be sufficient to show that the difference between the two integrals, call it $D(y,x,r)$, is non-negative.

<u>Define</u> $u_1 = (t_1+t_2)/\sqrt{2}$ and $u_2 = (-t_1+t_2)/\sqrt{2}$, so that the Jacobian,

$$\begin{vmatrix} \frac{\partial u_1}{\partial t_1} & \frac{\partial u_2}{\partial t_1} \\ \frac{\partial u_1}{\partial t_2} & \frac{\partial u_2}{\partial t_2} \end{vmatrix},$$

is 1.

We will integrate with respect to u_1 and u_2:
See Figure 4.6. Then $D(u,x,r) =$

$$\int_{t_1=0}^{\infty} \int_{t_2=0}^{\infty} [C_n(y-t,r-y-t_2)-C_n(x-t_1,r-x-t_2)]\varphi(t_1,t_2)dt_2dt_1$$

$$= \int_{u_1=0}^{\infty} \left[\begin{array}{l} \int_{u_2=-u_1}^{u_1} C_n(y- \frac{u_1-u_2}{\sqrt{2}}, r-y- \frac{u_1+u_2}{\sqrt{2}})\varphi(\frac{u_1-u_2}{\sqrt{2}}, \frac{u_1+u_2}{\sqrt{2}})du_2 \\ - \int_{u_2=-u_1}^{u_1} C_n(x- \frac{u_1-u_2}{\sqrt{2}}, r-x- \frac{u_1+u_2}{\sqrt{2}})\varphi(\frac{u_1-u_2}{\sqrt{2}}, \frac{u_1+u_2}{\sqrt{2}})du_2 \end{array} \right] du_1.$$

We will show that the first inner integral is larger than the second for any u_1, thus implying that $D(y,x,r) \geqq 0$, the desired result. To do so, we apply Lemma 4.6, with $C_n(\cdot)$ playing the role of $g(\cdot)$ and $\varphi(\cdot)$ the role of $f(\cdot)$, and considering $y,x,r,$ and u_1 as fixed. The value of $\varphi(\frac{u_1-u_2}{\sqrt{2}}, \frac{u_1+u_2}{\sqrt{2}})$ depends on $\frac{u_1-u_2}{\sqrt{2}} - \frac{u_1+u_2}{\sqrt{2}} = -\sqrt{2}\,u_2$ and by Hypothesis 4, $\varphi(\cdot)$ is unimodal, non-negative, and symmetric about $u_2=0$. By the induction hypothesis, the value of $C_n(y - \frac{u_1-u_2}{2}, r-y - \frac{u_1+u_2}{2})$ depends on $(y - \frac{u_1-u_2}{\sqrt{2}}) - (r-y - \frac{u_1+u_2}{\sqrt{2}}) = 2y-r+\sqrt{2}\,u_2$, and $C_n(\cdot)$ is quasiconvex in this variable and symmetric about $2y-r+\sqrt{2}\,u_2=0$. Therefore, the parameter a in Lemma 4.6 is equal to $2y-r$ in this case. Similarly, the parameter a is $2x-r$ for the second inner integral. Therefore, since $2y-r \geqq 2x-r \geqq 0$, Lemma 4.6 implies that the first inner integral is at least as large as the second. The integrand of the outer integral is then non-negative, so $D(y,x,r) \geqq 0$ completing the proof of the lemma.

<u>Lemma 4.9</u>: Consider the sets T_1 and T_2

where $T_1 \equiv \{\underset{\sim}{x} | x_1 \leq \min (z_1(x_2), x_1^*)\}$.

and $T_2 \equiv \{\underset{\sim}{x} | x_2 \leq \min (z_2(x_1), x_2^*)\}$.

If Hypothesis 4 is satisfied then, for $n=1,2,\ldots,N$, $\underset{\sim}{x}, \underset{\sim}{y} \in T_1$ and $x_1 > y_1, x_2 = y_2$ imply that $G_n(\underset{\sim}{x}) \leq G_n(\underset{\sim}{y})$ and $C_n(\underset{\sim}{x}) \leq C_n(\underset{\sim}{y})$ and $\underset{\sim}{x}, \underset{\sim}{y} \in T_2$ and $x_2 > y_2$, $x_1 = y_1$ imply that $G_n(\underset{\sim}{x}) \leq G_n(\underset{\sim}{y})$ and $C_n(\underset{\sim}{x}) \leq C_n(\underset{\sim}{y})$. (The sets T_1 and T_2 are pictured in Figure 4.7.)

<u>Note</u>: If the first partial derivatives of $G_n(\cdot)$ and $C_n(\cdot)$ existed everywhere in T_1 and in T_2, then the lemma would state that $D_1G_n(\cdot)$ and $D_1C_n(\cdot)$ are non-negative on T_1 and $D_2G_n(\cdot)$ and $D_2C_n(\cdot)$ are non-negative on T_2. However the partials do not exist everywhere -- for example, if $x_2 > x_2^*$, $D_1C_1(q_1(x_2),x_2)$ does not exist, since the left hand partial is zero and the right hand one is $D_1L(q_1(x_2),x_2) > 0$.

<u>Proof</u>: The proof will be by induction. The details will be carried out for the assertion about $G_n(\cdot)$ and $C_n(\cdot)$ on T_1 only, although the induction does make use of the assertion about them on T_2. The treatment for set T_2 is analogous and will be omitted.

For $n=1$, $G_1(\underset{\sim}{x}) = L(\underset{\sim}{x})$, and by Property A2, $D_1G_1(\underset{\sim}{x}) \leq 0$ for any $\underset{\sim}{x} \in T_1$, so the assertion is true for $G_1(\cdot)$. Consider

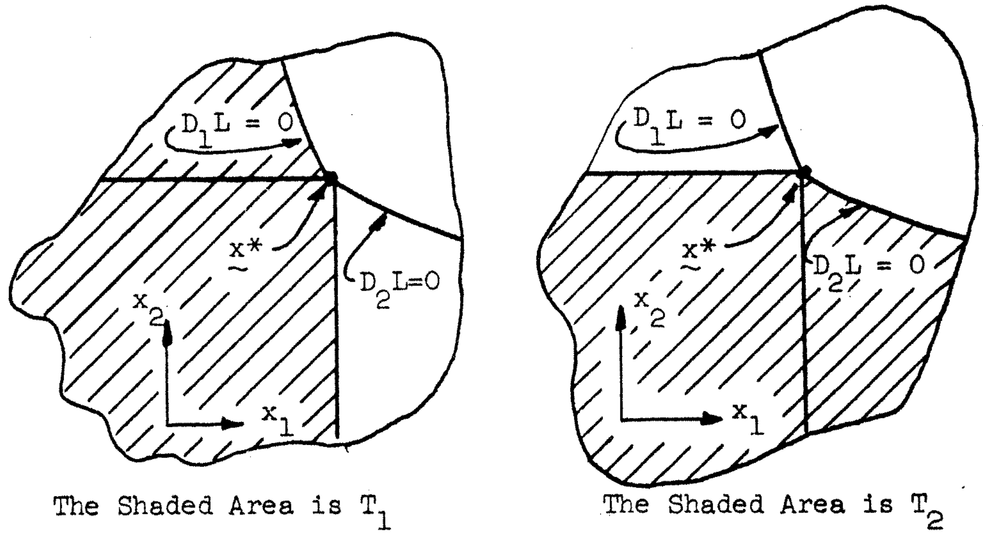


Figure 4.7

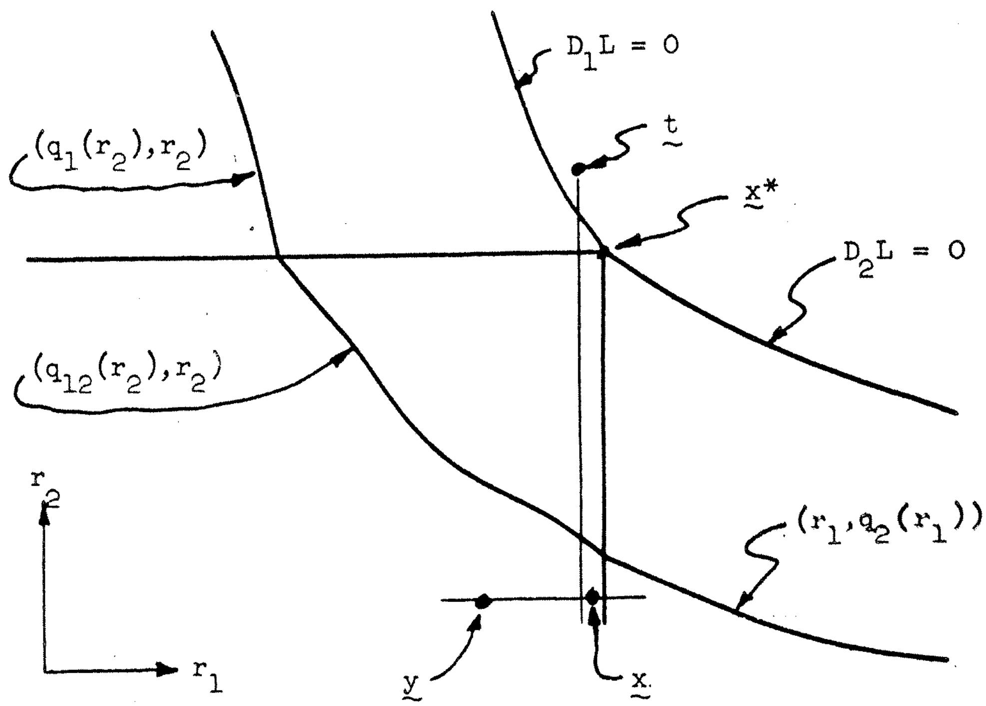


Figure 4.8

any $\underset{\sim}{x} \in T_1$. If $x_2 \leq x_2^*$, by Lemma 4.1, $C_1(\underset{\sim}{x}) = \min(L(\underset{\sim}{x}^*)+K, L(\underset{\sim}{x}))$, so that $x_1 > y_1$, $x_2=y_2$ implies that $C_1(\underset{\sim}{x}) \leq C_1(\underset{\sim}{y})$. Similarly, if $x_2 > x_2^*$, $x_1 > y_1$, $x_2=y_2$ implies that $C_1(\underset{\sim}{x}) \leq C_1(\underset{\sim}{y})$.

Now suppose that the assertion holds for $G_n(\cdot)$ and $C_n(\cdot)$. We will show that it holds for $G_{n+1}(\cdot)$ and $C_{n+1}(\cdot)$. First it will be shown that $G_{n+1}(\underset{\sim}{x}) - G_{n+1}(\underset{\sim}{y}) \leq 0$ if $x_1 > y_1$ and $x_2 = y_2$ and $\underset{\sim}{x}, \underset{\sim}{y} \in T_1$.

$$G_{n+1}(\underset{\sim}{x})-G_{n+1}(\underset{\sim}{y})=L(\underset{\sim}{x})-L(\underset{\sim}{y}) + \alpha \int_{\underset{\sim}{t} \geq \underset{\sim}{0}} [C_n(\underset{\sim}{x}-\underset{\sim}{t})-C_n(\underset{\sim}{y}-\underset{\sim}{t})]\varphi(\underset{\sim}{t})d\underset{\sim}{t}.$$

$L(\underset{\sim}{x}) \leq L(\underset{\sim}{y})$ by Property A2. Since $\underset{\sim}{t} \geq \underset{\sim}{0}$ implies that both $\underset{\sim}{x}-\underset{\sim}{t}$ and $\underset{\sim}{y}-\underset{\sim}{t}$ are elements of T_1 if $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are, $C_n(\underset{\sim}{x}-\underset{\sim}{t})-C_n(\underset{\sim}{y}-\underset{\sim}{t}) \leq 0$ for every $\underset{\sim}{t} \geq \underset{\sim}{0}$ by the induction hypothesis. Since $\varphi(\cdot)$ is non-negative, the integrand is always non-positive, so that $\alpha \int [C_n(\underset{\sim}{x}-\underset{\sim}{t})-C_n(\underset{\sim}{y}-\underset{\sim}{t})]\varphi(\underset{\sim}{t})d\underset{\sim}{t} \leq 0$ and $G_{n+1}(\underset{\sim}{x})-G_{n+1}(\underset{\sim}{y}) \leq 0$.

Now it will be shown that $C_{n+1}(\underset{\sim}{x}) \leq C_{n+1}(\underset{\sim}{y})$ if $x, y \in T_1$ and $x_1 > y_1$, $x_2=y_2$. If it is optimal to place no order from $\underset{\sim}{y}$, then $C_{n+1}(\underset{\sim}{y}) = G_{n+1}(\underset{\sim}{y}) \geq G_{n+1}(\underset{\sim}{x}) \geq C_{n+1}(\underset{\sim}{x})$. If it is optimal to order up to $\underset{\sim}{q}$ from $\underset{\sim}{y}$, two cases will be considered: $x_2 > x_2^*$ and $x_2 \leq x_2^*$. In either case, if $q_1 \geq x_1$, it is possible to place an order from $\underset{\sim}{x}$ to $\underset{\sim}{q}$, so that $C_{n+1}(\underset{\sim}{x}) \leq G_{n+1}(\underset{\sim}{q})+K=C_{n+1}(\underset{\sim}{y})$.

(1) $\underline{x_2 > x_2^*}$: If $q_2 = y_2$ then since $G_{n+1}(\cdot)$ is decreasing in product 1 up to $z_1(y_2)$, $q_1 \geq z_1(y_2) \geq x_1$, and $C_{n+1}(\underset{\sim}{x}) \leq C_{n+1}(\underset{\sim}{y})$. If $q_2 > y_2$, there are two possibilities: $q_1 \geq x_1$ and $q_1 \not\geq x_1$. For the former, $C_{n+1}(\underset{\sim}{x}) \leq C_{n+1}(\underset{\sim}{y})$. For the latter, we have $q_2 > y_2 = x_2$ and $x_2 \geq x_1$, implying that $q_1 < q_2$ and $q_1 < x_2$. Define $\underset{\sim}{r}$ by $r_2 = x_2$, $r_1 = q_1 + q_2 - x_2$. Now $q_1 < r_1$, $q_1 < r_2$, $q_2 > r_1$, $q_2 > r_2$ and $q_1 + q_2 = r_1 + r_2$, so that, by Lemma 4.8, $G_{n+1}(\underset{\sim}{r}) \leq G_{n+1}(\underset{\sim}{q})$. If $r_1 < x_1$, then $G_{n+1}(\underset{\sim}{r}) \geq G_{n+1}(\underset{\sim}{x})$, so that $C_{n+1}(\underset{\sim}{x}) \leq G_{n+1}(\underset{\sim}{r}) \leq G_{n+1}(\underset{\sim}{r}) + K \leq G_{n+1}(\underset{\sim}{q}) + K = C_{n+1}(\underset{\sim}{y})$. If $r_1 \geq x_1$, the $\underset{\sim}{r}$ is reachable from $\underset{\sim}{x}$ and $C_{n+1}(\underset{\sim}{x}) \leq G_{n+1}(\underset{\sim}{r}) + K \leq G_{n+1}(\underset{\sim}{q}) + K = C_{n+1}(\underset{\sim}{y})$.

(2) $\underline{x_2 \leq x_2^*}$: Since $G_{n+1}(\cdot)$ is decreasing in both variables below $\underset{\sim}{x}^*$, $G_{n+1}(\underset{\sim}{x}^*) = \min_{\underset{\sim}{u} \leq \underset{\sim}{x}^*} G_{n+1}(\underset{\sim}{u})$, so that we need only consider $\underset{\sim}{q} \not\leq \underset{\sim}{x}^*$. Applying Lemma 4.8 we have $G_{n+1}(\underset{\sim}{x}^*) =$

$$\min_{u_1 + u_2 \leq x_1^* + x_2^*} [G_{n+1}(\underset{\sim}{u})].$$

Since $q_1 \geq x_1$ implies $C_{n+1}(\underset{\sim}{x}) \leq C_{n+1}(\underset{\sim}{y})$, $q_1 < x_1$ is the only case left. Define $\underset{\sim}{t}$ by $t_1 = x_1$, $t_2 = q_1 + q_2 - x_1$. Since $q_2 > q_1$ and $q_2 > x_1$, $q_2 > t_2 > t_1 > q_1$, so that by Lemma 4.8, $G_{n+1}(\underset{\sim}{t}) \leq G_{n+1}(\underset{\sim}{q})$. Therefore $C_{n+1}(\underset{\sim}{x}) \leq G_{n+1}(\underset{\sim}{t}) + K \leq G_{n+1}(\underset{\sim}{q}) + K = C_{n+1}(\underset{\sim}{y})$. This completes the proof of the lemma.

It may seem that Lemma 4.9 could be proved if Property A2 is substituted for Hypothesis 4. However, this is not the case, because, without the quasiconvexity and symmetry, it is possible that $G_2(\cdot)$ will be minimized at a unique point t and that $\underset{\sim}{t} \ngeq \underset{\sim}{x}^*$, as illustrated if Figure 4.8. See Appendix E for a discrete demand example where this does occur. And if $\underset{\sim}{t} \ngeq \underset{\sim}{x}^*$, we can find $\underset{\sim}{x}$ and $\underset{\sim}{y}$ such that $x_2 = y_2 < x_2^*$, $x_1^* \geq x_1 > t_1 > y_1$ and $G_2(\underset{\sim}{y}) > G_2(\underset{\sim}{x}) > G_2(\underset{\sim}{t}) + K$.

By assumption, $\underset{\sim}{u} \neq \underset{\sim}{t}$ implies $G_2(\underset{\sim}{u}) > G_2(\underset{\sim}{t})$. Therefore $C_2(\underset{\sim}{x}) = \min [\min\limits_{\underset{\sim}{u} > \underset{\sim}{x}} G_2(\underset{\sim}{u}) + K, G_2(\underset{\sim}{x})] > G_2(\underset{\sim}{t}) + K = C_2(\underset{\sim}{y})$, and the lemma would not be true. This possibility that a unique $\underset{\sim}{t}$ that minimizes $G_2(\cdot)$ may not be greater than or equal to $\underset{\sim}{x}^*$ reveals a major aspect of the difficulty of characterizing the N period optimal policy under A2 alone.

In order to simplify notation, we will state and prove Theorem 4.10, which specifies the optimal N period policy under Hypothesis 4 for starting inventory $\underset{\sim}{x}$ such that $x_1 \leq x_2$. This will be sufficient since $G_n(\cdot)$ is symmetric. We will need the following definitions.

Let $\underset{\sim}{X}(n)$ be <u>defined</u> as the largest point satisfying

$$G_n(\underset{\sim}{X}(n)) = \min_{x_1 = x_2} G_n(\underset{\sim}{x}) .$$

Let $\underset{\sim}{m}(n)$ be <u>defined</u> as the smallest point satisfying $G_n(\underset{\sim}{m}(n)) = G_n(\underset{\sim}{X}(n)) + K$ and $m_1(n) = m_2(n)$. Since it is optimal to order from $\underset{\sim}{m}(n)$, by Lemma 4.3 $\underset{\sim}{m}(n) \ngeq \underset{\sim}{x}^*$ so that $\underset{\sim}{m}(n) \leq \underset{\sim}{x}^*$

For $x_2 > X_2(n)$, let $\underset{\sim}{S}^{(n)}(x_2)$ be <u>defined</u> by
$G_n(\underset{\sim}{S}^{(n)}(x_2)) = \min\limits_{y_2 \geq x_2} G_n(\underset{\sim}{y})$ and $S_1^{(n)}(x_2) \leq S_2^{(n)}(x_2)$ and if $G_n(\underset{\sim}{y}) = G_n(\underset{\sim}{S}^{(n)}(x_2))$ and $y_2 \geq x_2$ then $S_1^{(n)}(x_2) \geq y_1$.

For $m_2(n) < x_2 \leq X_2(n)$, let $s^{(n)}(x_2)$ be <u>defined</u> as the smallest value satisfying $G_n(s^{(n)}(x_2), x_2) = G_n(\underset{\sim}{X}(n)) + K$. For $x_2 > X_2(n)$, let $s^{(n)}(x_2)$ be <u>defined</u> as the smallest value satisfying $G_n(s^{(n)}(x_2), x_2) = G_n(\underset{\sim}{S}^{(n)}(x_2)) + K$.

Let $\underset{\sim}{M}$ be <u>defined</u> by $L(\underset{\sim}{M}) = L(\underset{\sim}{x}^*) + K$, $\underset{\sim}{M} > \underset{\sim}{x}^*$, $M_1 = M_2$.

Let $\underset{\sim}{P}$ be <u>defined</u> by $w_1(P_2) = P_1$, $w_2(P_1) = P_2$. That is, $\underset{\sim}{P}$ is the point at which $w_i(\cdot)$ curves intersect. It is easy to prove that $P_1 = P_2$ and $\underset{\sim}{P}$ is unique.

<u>Theorem 4.10</u>: If Hypothesis 4 is satisfied, then for $n = 1, 2, \ldots, N$, the optimal policy in period n is:

(1) <u>For $x_2 \leq m_2(n)$:</u> Order both products, up to $\underset{\sim}{X}(n)$.

(2) <u>For $m_2(n) < x_2 \leq X_2(n)$:</u> If $x_1 \leq s^{(n)}(x_2)$, order up to $\underset{\sim}{X}(n)$; if not, order nothing.

(3) <u>For $x_2 > X_2(n)$:</u> If $x_1 \leq s^{(n)}(x_2)$, order up to $\underset{\sim}{S}^{(n)}(x_2)$; if not, order nothing.

In addition, the "parameters" of the policy satisfy the following conditions:

(a) $\underset{\sim}{x}^* \leq \underset{\sim}{X}^{(n)} \leq \underset{\sim}{M}$. $\underset{\sim}{X}(n)$ minimizes $G_n(\cdot)$.

(b) $z_1(S_2^{(n)}(x_2)) \leq S_1^{(n)}(x_2) \leq w_1(S_2^{(n)}(x_2))$.

If $x_2 > P_2$, $S_2^{(n)}(x_2) = x_2$. If $x_2 \leq P_2$, then either $S_2^{(n)}(x_2) = x_2$ or $S_2^{(n)}(x_2) = S_1^{(n)}(x_2)$.

(c) For $x_2 > x_2^*$, $q_1(x_2) \leq s^{(n)}(x_2) \leq z_1(x_2)$.

For $m_2(n) \leq x_2 \leq x_2^*$, $q_{12}(x_2) \leq s^{(n)}(x_2)$.

Further $s^{(n)}(\cdot)$ is continuous, and is non-increasing for $m_2(n) \leq x_2 \leq x_2^*$.

See Figure 4.9.

Before proving the Theorem, let us try to see what it does and does not say. First we compare it to the one period result of Lemma 4.1.

In the last period, if $\underset{\sim}{x} \not\geq \underset{\sim}{x}^*$, the minimizer of $L(\cdot)$, then at most one of the products is ordered. With $n \geq 2$ periods remaining, no comparable statement can be made: We cannot rule out the ordering both products for some $\underset{\sim}{x}$ that is not less that $\underset{\sim}{X}(n)$, the minimizer of $G_n(\cdot)$. Only if $x_1 > P_1$ or $x_2 > P_2$ can we be sure that at most one product is ordered.

In the last period, the $q_{12}(\cdot)$ curve, which separates the order to $\underset{\sim}{x}*$ region from the order nothing region, is non-increasing. With $n \geqq 2$ periods remaining, $s^{(n)}(\cdot)$ is the analogous curve. We can say it is non-increasing only for $x_2 \leqq x_2^*$; it may be increasing for $x_2^* < x_2 < X_2(n)$.

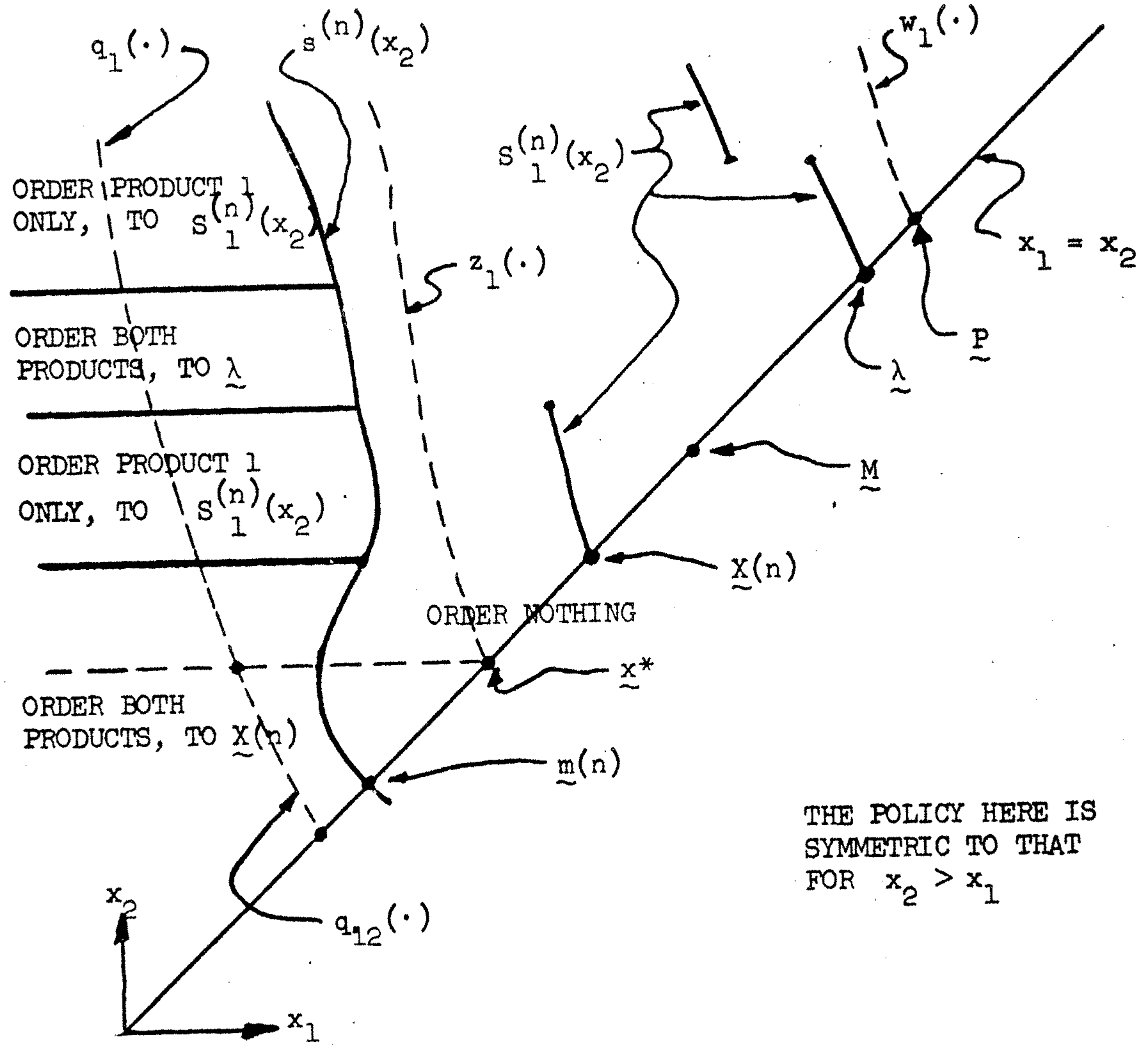
In the last period, the order to curve, $z_1(\cdot)$ is continuous. With $n \geqq 2$ periods remaining, and for $x_2 > P_2$ so that $S_2^{(n)}(x_2) = x_2$, its analog $S_1^{(n)}(\cdot)$ may not be continuous.

All these possibilities are portrayed in Figure 4.9, and the second of them is illustrated in the third example in Appendix E.

Next we compare it to the N period result when there is no setup cost. Consider $\underset{\sim}{x}_n$, the inventory before ordering with n periods remaining and suppose n is large. The region $\{\underset{\sim}{x} \mid \underset{\sim}{x} \nleqq \underset{\sim}{P}\}$ can be called <u>transient</u> because the optimal policy is such that:

(a) If $x_1 > P_1$ in period n, then none of product 1 is ordered in that period, so that eventually (since $E(D_1) > 0$) x_1 will be $< P_1$. The same is true for product 2.

(b) If $x_1 < P_1$ in period n, then b_1, the quantity of product 1 ordered in that period, is such that



The Optimal Policy in Period n

Figure 4.9

$x_1 + b_1 < P_1$, so that x_1 will be $< P_1$ in the next period and each succeeding period. The same is true for product 2.

The region $\{\underset{\sim}{x} \mid \underset{\sim}{x} \leq \underset{\sim}{P}\}$ can be called the <u>recurrent</u> region. Since the transient region is never entered if $\underset{\sim}{x}_N \leq \underset{\sim}{P}$ and is eventually left, never to be reentered if $\underset{\sim}{x}_N \nleq \underset{\sim}{P}$, there is justification for being really concerned only about the optimal policy in the recurrent region. Where there is no setup cost, in the recurrent region the policy is the ultimate in simplicity: Order to $\underset{\sim}{x}^*$ in every period. As we have seen, where there is a setup cost, even under Hypothesis 4, the optimal policy in the recurrent region is considerably more complicated when there is a setup cost.

<u>Proof of Theorem 4.10</u>: For writing ease, the period index n will be dropped from the parameters. It will be dropped from $G_n(\cdot)$ and $C_n(\cdot)$ also, except where needed for clarity.

We will assume that the bounds are correct and show that the policy given by (1),(2),(3) is optimal.

(1) <u>$x_2 \leq m_2(n)$</u>: If $x_2 \leq m_2$ then $x_1 \leq x_2$ implies $\underset{\sim}{x} < \underset{\sim}{m}$. Since by Lemma 4.9 $D_1G(\cdot)$ and $D_2G(\cdot)$

are non-positive[1] below $\underset{\sim}{x}^*$, $G_n(\underset{\sim}{x}) \geq G_n(\underset{\sim}{m}) = \min_{\text{all } \underset{\sim}{y}} G_n(\underset{\sim}{y}) + K$, so it is optimal to order to $\underset{\sim}{X}$.

(2) $\underline{m_2(n) < x_2 \leq X_2(n)}$: If $x_2 \leq s(x_2)$ then by Lemma 4.9, $G_n(\underset{\sim}{x}) \geq G_n(s(x_2), x_2) = G_n(\underset{\sim}{X}) + K$, so it is optimal to order to $\underset{\sim}{X}$. If $s(x_2) < x_1 \leq z_1(x_2)$, then the inequality is reversed and it is optimal to order nothing. If $x_1 > z_1(x_2)$, then by Lemma 4.3 it is optimal to order nothing.

(3) $\underline{x_2 > X_2(n)}$: The arguments given in (2) suffice here when $\underset{\sim}{X}$ is replaced by $\underset{\sim}{S}(x_2)$.

Now we turn to the bounds.

(a) We show $\underset{\sim}{X}$ minimizes $G(\cdot)$ by contradiction. Suppose $\underset{\sim}{x}$ such that $x_1 \neq x_2$ satisfies $G(\underset{\sim}{x}) < G(\underset{\sim}{X})$. By Lemma 4.8, $G(\underset{\sim}{x}) \geq G\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}\right)$. By definition $G\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}\right) \geq G(\underset{\sim}{X})$ which yields the contradiction. $\underset{\sim}{X} \geq \underset{\sim}{x}^*$ because $D_1G(\cdot)$ and $D_2G(\cdot)$

1 - We use this language for expository ease even though the partials do not exist.

are non-positive below $\underset{\sim}{x}^*$. To show that $\underset{\sim}{X} \leqq \underset{\sim}{M}$, we need only show that $G(\underset{\sim}{x}^*) < G_n(\underset{\sim}{x})$ if $\underset{\sim}{x} > \underset{\sim}{M}$. But $L(\underset{\sim}{x}^*) + K < L(\underset{\sim}{x})$ and $C_{n-1}(\underset{\sim}{x}^* - \underset{\sim}{t}) \leqq C_{n-1}(\underset{\sim}{x} - \underset{\sim}{t}) + K$ for every $\underset{\sim}{t}$, so it is true.

(b) $S_1(x_2) \geqq z_1(S_2(x_2))$ because $D_1G(\cdot)$ is non positive to the left of $z_1(\cdot)$. By the argument used in (a), $G(\underset{\sim}{y}) > G(z_1(y_2), y_2)$ if $y_1 > w_1(y_2)$, so $S_1(x_2) \leqq w_1(S_2(x_2))$.

Suppose $S_2(x_2) > x_2$ and $S_1(x_2) \neq S_2(x_2)$. By definition $S_1(x_2) < S_2(x_2)$. If $S_1(x_2) + S_2(x_2) \geqq 2x_2$, then $\underset{\sim}{R} \equiv \left(\frac{S_1(x_2) + S_2(x_2)}{2}, \frac{S_1(x_2) + S_2(x_2)}{2}\right)$ is such that $r_2 \geqq x_2$ and $r_1 > S_1(x_2)$. By Lemma 4.8, $G(\underset{\sim}{R}) \leqq G(\underset{\sim}{S})$; coupling this with $r_2 = r_1 > S_1$ implies $\underset{\sim}{S}$ does not satisfy its definition. If $S_1(x_2) + S_2(x_2) < 2x_2$, then using $\underset{\sim}{Q} \equiv (S_1(x_2) + S_2(x_2) - x_2, x_2)$ in place of $\underset{\sim}{R}$ yields a similar contradiction. Therefore, $S_2(x_2) > x_2$ implies $S_2(x_2) = S_1(x_2)$.

If $x_2 > P_2$ then $\underset{\sim}{u}$ such that $u_1 = u_2 > x_2$ satisfies $u_1 > w_1(u_2)$. Therefore, $G(\underset{\sim}{u}) > G(z_1(u_2), u_2) \geqq G(\underset{\sim}{S}(x_2))$, so that $\underset{\sim}{S}(x_2) \neq \underset{\sim}{u}$. Therefore $S_2(x_2) = x_2$.

(c) For $x_2 > x_2^*$, we have $L(q_1(x_2),x_2) = L(z_1(x_2),x_2) + K$ and by Lemma 4.9 $C_{n-1}((q_1(x_2)-\underset{\sim}{t}) \geq C_{n-1}((z_1(x_2),x_2)-\underset{\sim}{t})$ for every $\underset{\sim}{t} \geq \underset{\sim}{0}$. Therefore $G_n(q_1(x_2),x_2) \geq G_n(z_1(x_2),x_2) + K \geq G_n(\underset{\sim}{s}(x_2)) + K$. Since $D_1G(\underset{\sim}{x}) \leq 0$ for $x_1 \leq q_1(x_2)$, $s(x_2) \geq q_1(x_2)$. By Lemma 4.3, $G_n(z_1(x_2),x_2) \geq G_n(\underset{\sim}{S}(x_2)) + K$, so $s(x_2) \leq z_1(x_2)$.

For $x_2 < x_2^*$, a straight forward modification of the preceding shows that $s(x_2) \geq q_{12}(x_2)$. For $m_2 \leq u \leq v \leq x_2^*$, we have $G_n(s(u),u) = G_n(s(v),v)$. Since $D_1G(\cdot)$ and $D_2G(\cdot)$ are non-positive below $\underset{\sim}{x}^*$, $s(u) \geq s(v)$.

The continuity of $s(\cdot)$ is a straightforward consequence of the continuity of $G(\cdot)$ and the non-positivity of $D_1G(\cdot)$ on T_1.

4.5 Unequal Setup Costs; The One Period Problem

In this section we assume that $K_{12} > K_1 = K_2 > 0$. In other words, the setup cost is the same if only one product is ordered, regardless of whether it is product 1 or product 2, and there is an additional setup cost if both products are ordered simultaneously. The assumption that $K_{12} \leq K_1 + K_2$ will be retained. We obtain the optimal one period policy when $L(\cdot)$ has Property A2 and $D_{12}L(\cdot)$ is non-negative.

It is easy to obtain the optimal one period policy for $K_1 \neq K_2$, but we feel it is of marginal interest and do not include it.

The description of the one period optimal policy requires a certain amount of groundwork. The main results are pictured in Figures 4.10 and 4.11, which refer to Lemmas 4.12 and 4.13 respectively. Reference to these figures may halp reveal the rationale behind the definitions.

Consider the set of points R <u>defined</u> by

$$R \equiv \{(r_1,r_2) \mid L(r_1,z_2(r_1)) = L(z_1(r_2),r_2) \text{ and } \underset{\sim}{r} \leq \underset{\sim}{x}^*\}.$$

From Lemma 2.1, it is clear that if $\underset{\sim}{u},\underset{\sim}{v} \in R$ and $\underset{\sim}{u} \neq \underset{\sim}{v}$, then either $\underset{\sim}{u} > \underset{\sim}{v}$ or $\underset{\sim}{v} < \underset{\sim}{u}$. Therefore R defines a function $r(\cdot)$: for any $x_2 \leq x_2^*$, $r(x_2)$ is the unique point x_1 such that $(x_1,x_2) \in R$. It is clear that $r(\cdot)$ has an inverse, so that $r(\cdot)$ must be continuous.

It is also clear that to the left of the curve defined by R it is preferable to order product 1 rather than product 2 and to the right of it, it preferable to order 2 rather than 1.

Now attention will be turned to the behavior of $q_1(\cdot)$ and $q_2(\cdot)$. If they are non-increasing functions, the one period optimal policy is easier to describe than if they are not. To ensure that they are, it will be assumed for the remainder of this section that $D_{12}L(\cdot)$ is non-negative. One implication of $q_1(\cdot)$ and $q_2(\cdot)$ being non-increasing is:

Lemma 4.11: If $L(\cdot)$ has Property A2, $D_{12}L(\cdot)$ is non-negative and $K_1 = K_2 > 0$, then the two curves, $q_1(\cdot)$ and $q_2(\cdot)$ have exactly one point of intersection, call it $\underset{\sim}{\lambda}$, and $\underset{\sim}{\lambda} < \underset{\sim}{x}^*$.

Proof: Suppose $\underset{\sim}{u}, \underset{\sim}{v} \in Q_1 \cap Q_2$ and $\underset{\sim}{u} \neq \underset{\sim}{v}$. We obtain a contradiction. By Lemma 4.2 both $q_1(\cdot)$ and $q_2(\cdot)$ are non-increasing, so that both $\underset{\sim}{u} \nless \underset{\sim}{v}$ and $\underset{\sim}{v} \nless \underset{\sim}{u}$ are true. But clearly $\underset{\sim}{u}, \underset{\sim}{v} \in R$, so that either $\underset{\sim}{u} < \underset{\sim}{v}$ or $\underset{\sim}{v} < \underset{\sim}{u}$ must be true, which is a contradiction, and the proof is complete.

To see when it might be advantageous to order both products, define $\underset{\sim}{\beta}$ by:

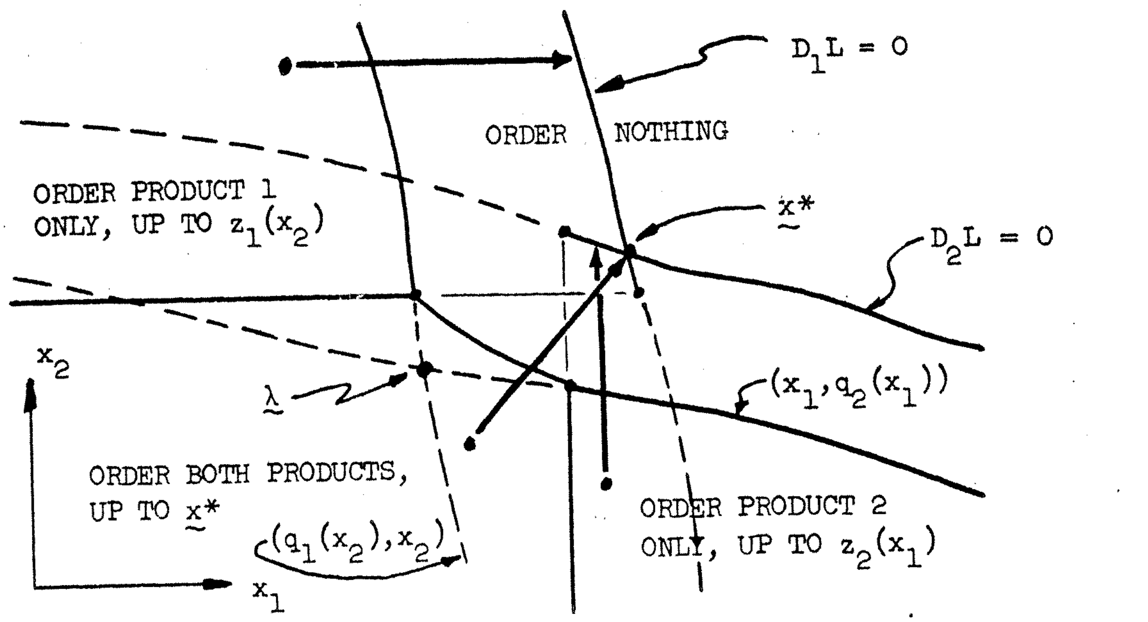
(1) $\underset{\sim}{\beta} \in R$

(2) $L(\underset{\sim}{\beta}) = L(\underset{\sim}{x}^*) + K_{12}$.

Since $D_1L(\cdot)$ and $D_2L(\cdot)$ are negative below $\underset{\sim}{x}^*$, $\underset{\sim}{\beta}$ is unique. To compare $\underset{\sim}{\beta}$ and $\underset{\sim}{\lambda}$, recall that $L(\underset{\sim}{\lambda}) = L(z_1(\lambda_2), \lambda_2) + K_1 = L(\lambda_1, z_2(\lambda_1)) + K_2$. Since $\underset{\sim}{\lambda}$ and $\underset{\sim}{\beta}$ are both in R, there are two possibilities: $\underset{\sim}{\lambda} < \underset{\sim}{\beta}$ and $\underset{\sim}{\lambda} \geq \underset{\sim}{\beta}$. If $K_1=K_2=K_{12}$ then $\underset{\sim}{\lambda} < \underset{\sim}{\beta}$ since $L(\underset{\sim}{\lambda}) > L(\beta)$. If $K_{12} = 2K_1 = 2K_2$ then $\underset{\sim}{\lambda} \geq \underset{\sim}{\beta}$: Since $\lambda_2 \leq x_2* \leq z_2(\lambda_1)$, $L(\underset{\sim}{\lambda}) \leq L(\lambda_1, x_2*) + K_2$, and since $q_1(x_2*) \leq \lambda_1 \leq x_1*$, $L(\lambda_1, x_2*) \leq L(\underset{\sim}{x}*) + K_1$. Consequently, $L(\underset{\sim}{\lambda}) \leq L(\underset{\sim}{x}*) + K_1+K_2 = L(\underset{\sim}{x}*) + K_{12} = L(\underset{\sim}{\beta})$ or $L(\underset{\sim}{\lambda}) \leq L(\underset{\sim}{\beta})$.

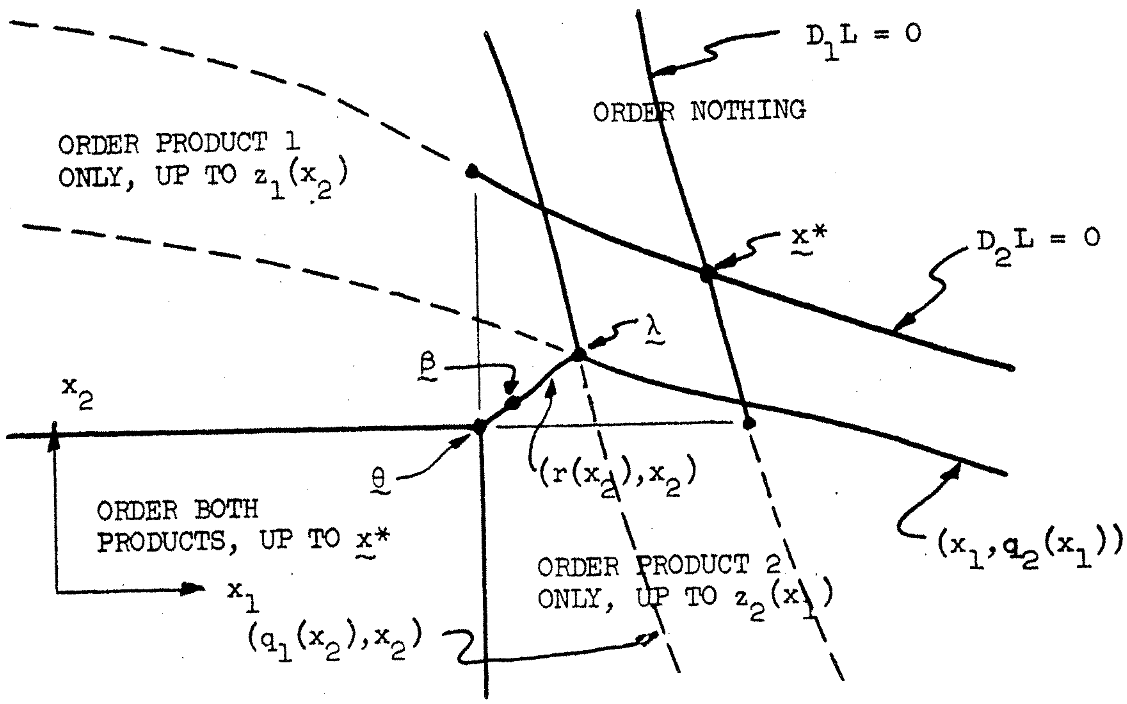
First consider the case $\underset{\sim}{\lambda} < \underset{\sim}{\beta}$. By Lemma 2.1, as x_2 decreases from x_2*, $L(q_1(x_2),x_2)$ increases, while $L(q_{12}(x_2),x_2)$ remains constant. Since $K_{12} \geq K_1$, $L(q_1(x_2*),x_2*) \leq L(q_{12}(x_2*),x_2*)$, and $q_1(x_2*) \geq q_{12}(x_2*)$. If $q_1(\cdot)$ is non-increasing, then $\underset{\sim}{\lambda} < \underset{\sim}{\beta}$ implies $q_1(\beta_2) < \beta_1$, so that since $\beta_1 = q_{12}(\beta_2)$, $L(q_1(\beta_2),\beta_2) > L(q_{12}(\beta_2),\beta_2)$ and $q_1(\beta_2) < q_{12}(\beta_2)$. Consequently there exists an x_2' such that $L(q_1(x_2'),x_2') = L(\underset{\sim}{x}*) + K_{12}$ which satisfies $\beta_2 < x_2' \leq x_2*$. Further, since $L(q_1(x_2),x_2)$ is strictly increasing as x_2 decreases from x_2*, x_2' is unique. For $x_2 = x_2'$, $q_{12}(x_2) = q_1(x_2)$; for $x_2 > x_2'$, $q_{12}(x_2) < q_1(x_2)$; for $x_2 < x_2'$, $q_{12}(x_2) > q_1(x_2)$. Similarly, there exists a unique x_1' such that $L(x_1', q_2(x_1')) = L(\underset{\sim}{x}*) + K_{12}$ which satisfies $\beta_1 < x_1' \leq x_1*$, and for $x_1 > x_1'$, $q_1(\cdot)$ lies above $q_{12}(\cdot)$, and for $x_1 < x_1'$, $q_1(\cdot)$ lies below $q_{12}(\cdot)$.

Now that x_1' and x_2' have been defined, the one period optimal policy when $\underset{\sim}{\lambda} < \underset{\sim}{\beta}$ can be specified. It is pictured in Figure 4.10.



The Optimal One Period Policy when K_{12} is relatively Small

Figure 4.10



The Optimal One Period Policy when K_{12} is relatively Large

Figure 4.11

implies $x_1 \leq r(x_2)$, so $L(z_1(x_2),x_2) \leq L(x_1,z_2(x_1))$. Consequently, $x_1 \leq q_1(x_2)$ implies it is optimal to order up to $(z_1(x_2),x_2)$ from $\underset{\sim}{x}$.

By symmetry, for $x_1' < x_1 \leq x_1^*$, it is optimal to order up to $(x_1,z_2(x_1))$ from $\underset{\sim}{x}$ if $x_2 \leq q_2(x_1)$ and to place no order from $\underset{\sim}{x}$ if $x_2 > q_2(x_1)$.

This leaves the set $\{\underset{\sim}{x} \mid \underset{\sim}{x} \leq (x_1',x_2')\}$ to be accounted for. Since the curve $q_{12}(\cdot)$ lies above both the $q_1(\cdot)$ and $q_2(\cdot)$ curves in this area, if $\underset{\sim}{x}$ is such that $x_2 \geq q_1(x_2^*)$ and $x_1 > q_{12}(x_2)$, it is optimal to place no order from $\underset{\sim}{x}$. If $\underset{\sim}{x}$ does not satisfy this condition, then $L(\underset{\sim}{x}) \geq L(\underset{\sim}{x}^*) + K_{12}$. Since $x_1 \leq x_1'$, $L(x_1,z_2(x_1)) \geq L(\underset{\sim}{x}^*)+K_{12}-K_2$. Therefore it is better to order to $\underset{\sim}{x}^*$ than to $(x_1,z_2(x_1))$. Similarly, $x_2 \leq x_2'$ implies it is better to order to $\underset{\sim}{x}^*$ than $(z_1(x_2),x_2)$. Consequently it is optimal to order to $\underset{\sim}{x}^*$ from $\underset{\sim}{x}$. This completes the proof of the lemma.

Now consider the case $\underset{\sim}{\lambda} \geq \underset{\sim}{\beta}$. $L(\underset{\sim}{\beta}) = K_{12} + L(\underset{\sim}{x}^*)$ by definition. $L(\beta_1,q_2(\beta_1)) = L(\beta_1,z_2(\beta_1))+K_2$ by definition. Since $\underset{\sim}{\lambda} \geq \underset{\sim}{\beta}$ and since $q_2(\cdot)$ is non-increasing, $q_2(\beta_1) \geq \beta_2$ so that $L(\underset{\sim}{\beta}) \geq L(\beta_1,q_2(\beta_1))$. Consequently, $L(\beta_1,z_2(\beta_1))+K_2 \leq L(\underset{\sim}{x}^*) +K_{12}$. Therefore, from $\underset{\sim}{\beta}$, there is no advantage in ordering to $\underset{\sim}{x}^*$ rather than $(\beta_1,z_2(\beta_1))$.

To obtain a point of indifference between ordering only one product and ordering both, define $\underset{\sim}{\theta}$ by

(1) $\underset{\sim}{\theta} \in R$

(2) $K_2 + L(\theta_1, z_2(\theta_1)) = L(\underset{\sim}{x}^*) + K_{12}$.

It has just been shown that $\underset{\sim}{\theta} \leq \underset{\sim}{\beta}$.

Now that $\underset{\sim}{\theta}$ has been defined, the one period optimal policy when $\underset{\sim}{\lambda} \geq \underset{\sim}{\beta}$ can be specified. It is pictured in Figure 4.11.

Lemma 4.13: If $L(\cdot)$ has Property A2, $D_{12}L(\cdot)$ is non-negative, $K_1=K_2 > 0$, and K_{12} is relatively large so that $\underset{\sim}{\lambda} \geq \underset{\sim}{\beta}$, then the one period optimal policy is:

(a) if $\underset{\sim}{x} \leq \underset{\sim}{\theta}$, order to $\underset{\sim}{x}^*$

(b) if $\underset{\sim}{x}$ is such that $x_1 \leq r(x_2)$ and $x_1 \leq q_1(x_2)$ and $x_2 > \theta_2$, order product 1 only, to $(z_1(x_2), x_2)$

(c) if $\underset{\sim}{x}$ is such that $x_1 > r(x_2)$ and $x_2 \leq q_2(x_1)$ and $x_1 > \theta_1$, order product 2 only, to $(x_1, z_2(x_1))$

(d) if $\underset{\sim}{x}$ satisfies none of the above, order nothing.

Proof: In common with Lemma 4.14, only $\underset{\sim}{x} \leq \underset{\sim}{x}^*$ need be considered.

Suppose $\underset{\sim}{x} \leq \underset{\sim}{\theta}$. Then, since $\underset{\sim}{\theta} \leq \underset{\sim}{\beta}$, $L(\underset{\sim}{x}) \geq L(\underset{\sim}{x}^*)+K_{12}$. Since $x_1 \leq \theta_1 \leq x_1^*$, $L(x_1, z_2(x_1)) \geq L(\theta_1, z_2(\theta_1))$, so that by the definition of $\underset{\sim}{\theta}$, $L(x_1, z_2(x_1))+K_2 \geq L(\underset{\sim}{x}^*)+K_{12}$. Similarly, $x_2 \leq \theta_2$ implies $L(z_1(x_2), x_2)+K_1 \geq L(\underset{\sim}{x}^*)+K_{12}$. Therefore, if $\underset{\sim}{x} \leq \underset{\sim}{\theta}$, it is optimal to order up to $\underset{\sim}{x}^*$.

Suppose $\underset{\sim}{x}$ is such that $x_1 \leq r(x_2)$ and $x_1 \leq q_1(x_2)$ and $x_2 > \theta_2$. Since $x_2 > \theta_2$, $L(z_1(x_2),x_2)+K_1 < L(\underset{\sim}{x}^*)+K_{12}$. Since $x_1 \leq q_1(x_2)$, $L(\underset{\sim}{x}) \geq L(z_1(x_2)x_2)+K_1$. And since $x_1 \leq r(x_2)$, $L(z_1(x_2),x_2)+K_1 \leq L(x_1,z_2(x_1))+K_2$. Therefore, it is optimal to order up to $(z_1(x_2),x_2)$ from $\underset{\sim}{x}$.

By symmetry, if $\underset{\sim}{x}$ is such that $x_1 > r(x_2)$ and $x_2 \leq q_2(x_1)$ and $x_1 > \theta_1$, it is optimal to order up to $(x_1,z_2(x_1))$.

The remaining region is $\underset{\sim}{x}$ such that $x_1 > q_1(x_2)$ and $x_2 > q_2(x_2)$. Consequently, $L(\underset{\sim}{x}) < L(Z_1(x_2),x_2)+K_1$ and $L(\underset{\sim}{x}) < L(x_1,z_2(x_1))+K_2$. Also $L(\underset{\sim}{x}) < L(\underset{\sim}{\lambda}) < L(\underset{\sim}{\beta}) = L(\underset{\sim}{x}^*)+K_{12}$. Therefore, from this region, it is optimal to order nothing, and the proof is complete.

In Lemma 4.3 we have an n period result when the setup costs are unequal. Lemma 4.4 can be modified to allow unequal setup costs. Both lemmas are concerned with the policy in the region above both the $z_1(\cdot)$ and $z_2(\cdot)$ curves. By themselves, these results are not very interesting: They would be interesting if they could be coupled with statements about the optimal policy in the region where either $D_1L(\cdot)$ or $D_2L(\cdot)$ is negative. Unfortunately, we have been unable to make much progress in this region. The difficulty of making strong statements about the optimal n period policy should be apparent from the relative complexity of the one period optimal policy.

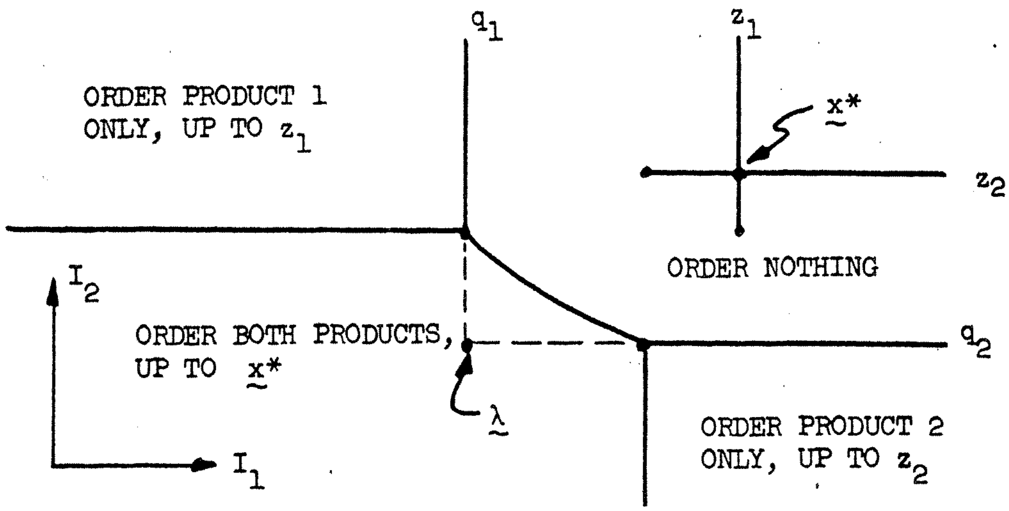
If $L(\underset{\sim}{x})$ can be written as $L_1(x_1)+L_2(x_2)$ for every $\underset{\sim}{x}$, it is interesting to compare our one period, **periodic** review result with Balintfy's [5] "random joint order policy" for continuous review. They are pictured in Figures 4.12 and 4.13 respectively. (It is apparent from Figure 4.12 that $\underset{\sim}{\lambda} \leq \underset{\sim}{\beta}$.)

4.6 <u>Discussion: K-Convexity</u> In this section we indicate how a fixed delivery lag can be handled, compare the bounds on the parameters of the N period policy with the one product bounds of Veinott and Wagner, and discuss K-convexity.

If there is a fixed delivery lag of λ periods, then operating as we did in Section 7 of Chapter 2, it can be seen that $L(\cdot)$ can be replaced by $L^{(\lambda)}(\cdot)$ in all of this chapter's results. If the setup cost is incurred at the time of order placement rather than delivery, then K should be replaced by $K\alpha^{-\lambda}$, since $L^{(\lambda)}(\cdot)$ is incurred at delivery.

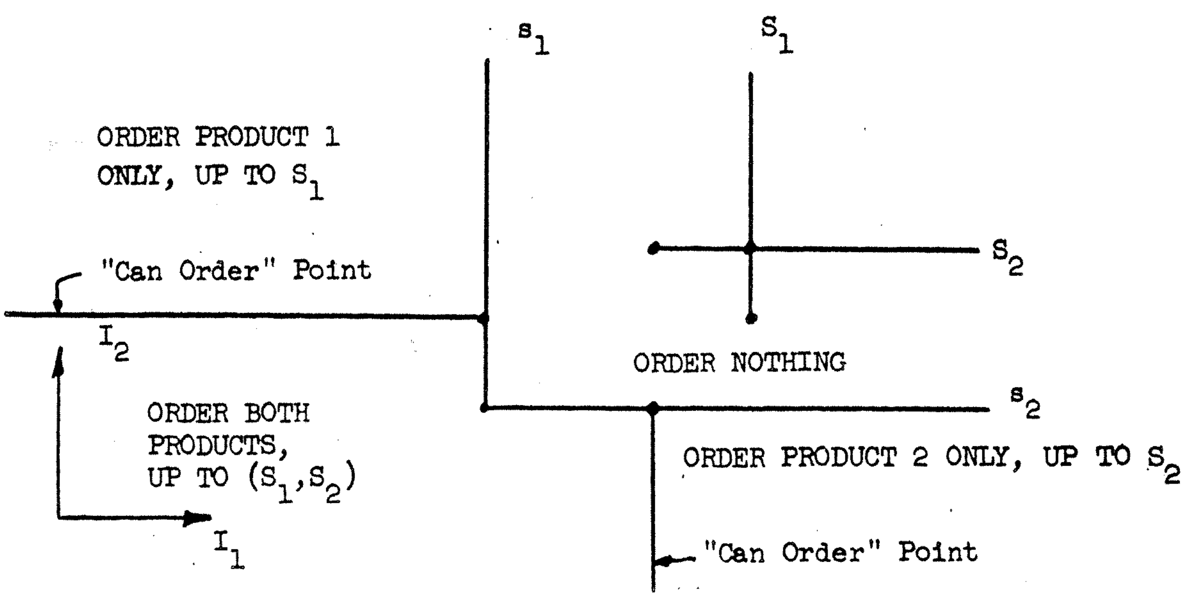
We now compare the bounds in Theorem 4.10 with those obtained by Veinott and Wagner [26] for the one product problem. They assume inventories, demands, and order quantities must be integers, so we adjust their definitions to a continuously divisible product. Let s_1 and S_1 be defined by

$$L(S_1) = \min_{\text{all } y} L(y) \quad \text{and} \quad L(s_1) = L(S_1) + K,\ s_1 < S_1.$$



The Optimal One Period Policy when $L(\cdot)$ is Separable

Figure 4.12



The Form of a "Random Joint Order Policy" for Two Products, Continuous Review

Figure 4.13

Let $\underline{s}$ and $\overline{s}$ be defined by $\underline{s} = s_1$ and $L(\overline{s}) = L(S_1)+(1-\alpha)K, \overline{s} < S_1$. Let $\underline{S}$ and $\overline{S}$ be defined by $\underline{S} = S_1$ and $L(\overline{S}) = L(S_1)+\alpha K$, $\overline{S} > S_1$. Then Veinott and Wagner show that $\underline{s} \leq s_n \leq \overline{s} \leq \underline{S} \leq S_n \leq \overline{S}$ for $n=1,2,\ldots$.

For $x_2 > P_2$, $s^{(n)}(\cdot)$ and $S_1^{(n)}(\cdot)$ are analogous to s_n and S_n, and they satisfy $q_1(x_2) \leq s^{(n)}(x_2) \leq z_1(x_2) \leq S_1^{(n)}(x_2) \leq w_1(x_2)$. Also analogous to s_n and S_n are $\underset{\sim}{m}(n)$ and $\underset{\sim}{X}(n)$, which satisfy $\underset{\sim}{a} \leq \underset{\sim}{m}(n) \leq \underset{\sim}{x}^* \leq \underset{\sim}{X}(n) \leq \underset{\sim}{M}$, where $a_1 = q_{12}(a_2)$, $a_1=a_2$. It should be apparent that these bounds could be improved by using αK rather than K to define $w_1(\cdot)$ and $w_2(\cdot)$ and by defining an analog of $\overline{s}$. These improvements, while straightforward, would be a complicating factor in an already complicated development, and we have omitted them.

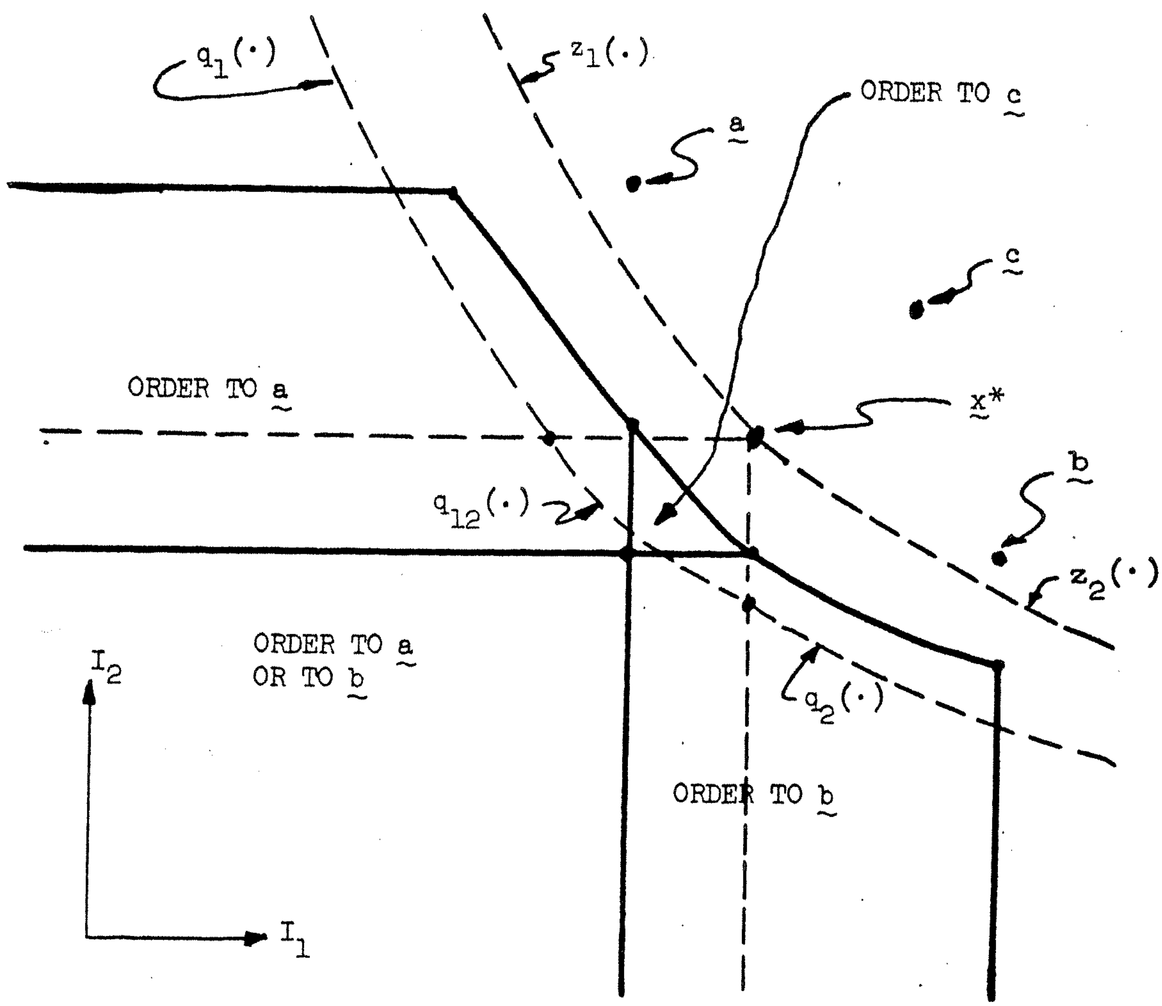
It should be noted that both our bounds and those of Veinott and Wagner say nothing about the relationship between values of the same parameter in two different periods.

For one product with setup cost, the concept of K-convexity, invented and exploited by Scarf [14], is extremely appropriate. If $L(\cdot)$ is convex, then it is easy to prove by induction that $G_n(\cdot)$ and $C_n(\cdot)$ are K-convex. The crucial steps are using the K-convexity of $G_n(\cdot)$ to show that a simple policy is optimal, and then using the policy and the K-convexity to show that $C_n(\cdot)$ is K-convex.

<u>K-convexity (Scarf)</u>: Consider any $K \geqq 0$. If, for every $x \geqq y$ and every $0 \leqq \lambda \leqq 1$, $\lambda f(y) + (1-\lambda)[f(x) + K] \geqq f(\lambda y + (1-\lambda)x)$, then $f(\cdot)$ is said to be K-convex. This definition is "directed": K is added to value of $f(\cdot)$ at the larger argument. One straightforward extension to functions of two variables would substitute $\underset{\sim}{x}$ and $\underset{\sim}{y}$ for x and y, requiring that $\underset{\sim}{x} \geqq \underset{\sim}{y}$.

Assume that $G_2(\cdot)$ is K-convex under the proposed extension. We have been unable to rule out the possibility that the set of points at which $G_2(\cdot)$ is minimized will consist of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, and $a_1 > b_1$, $a_2 < b_2$. Worse still, there may be points below the $q_{12}(\cdot)$ curve that are less than neither $\underset{\sim}{a}$ nor $\underset{\sim}{b}$. Since $G_2(q_{12}(x_2),x_2) \geqq G_2(\underset{\sim}{x}^*) + K$, an order is placed from these points, and we have the possibility of an optimal policy in period 2 being as illustrated in Figure 4.14.

Since our proposed extension of K-convexity does not imply a relatively simple policy, we feel that it is not the right concept for two product problems. We do not see at this point how this problem can be overcome. It does not seem that a definition that would compare $\underset{\sim}{x}$ and $\underset{\sim}{y}$ such that neither $\underset{\sim}{x} \not\geqq \underset{\sim}{y}$ nor $\underset{\sim}{y} \not\geqq \underset{\sim}{x}$ would solve the problem: There is nothing in the one-dimensional definition to prevent multiple absolute minima, and if this occurs on a line of negative slope, the possibility of Figure 4.14 is still present. Therefore we have pursued K-convexity no farther.



$q_1(\cdot)$
$z_1(\cdot)$
ORDER TO c
a
c
ORDER TO a
x*
b
$q_{12}(\cdot)$
$z_2(\cdot)$
I_2
ORDER TO a
OR TO b
$q_2(\cdot)$
ORDER TO b
I_1

Figure 4.16

Appendix A

Proof of Lemma 2.6

It must be shown that if $L(\cdot)$ has Property A2, then the first partial derivatives of $C_1(\cdot)$ are continuous.

The plane will be partitioned in the 4 sets, R_{12}, R_1, R_2, and R_0 defined in Lemma 2.3. The closure of each set will be considered: It will be shown that the first partials of $C_1(\cdot)$ are continuous on each of these closures. Operating on the closures implies continuity at the boundary points, so they need not and will not be treated separately.

Let $V_1 \equiv \{(x_1,x_2) \mid x_1 < x_1^*,\ x_2 = x_2^*\}$

$$V_1 \equiv \{(x_1,x_2) \mid x_1 = x_1^*,\ x_2 < x_2^*\}$$

$$W_1 \equiv \{(x_1,x_2) \mid D_1L(x_1,x_2) = 0, x_2 > x_2^*\}$$

$$W_2 \equiv \{(x_1,x_2) \mid D_2L(x_1,x_2) = 0, x_1 > x_1^*\}.$$

Then, if $\bar{R} \equiv$ the closure of R,

$$\bar{R}_{12} \equiv R_{12}$$

$$\bar{R}_1 = R_1 \cup V_1$$

$$\bar{R}_2 = R_2 \cup V_2$$

$$\bar{R}_0 = R_0 \cup W_1 \cup W_2$$

(i) $\underline{q \in \bar{R}_{12}}$: $C_1(\underset{\sim}{q}) = L(\underset{\sim}{x}^*)$ so that $D_1C_1(\underset{\sim}{q}) = D_2C_1(\underset{\sim}{q}) = 0$.

Since both first partials are constant, they are continuous on $\bar{R}_{12}$.

(ii) $\underline{\underset{\sim}{q} \in \bar{R}_1}$: By Lemma 2.4, $D_1C_1(\underset{\sim}{q}) = 0$, which is continuous on $\bar{R}_1$.

$D_2C_1(\cdot)$ is continuous at $\underset{\sim}{q} \in \bar{R}_1$ if

$$\lim_{\substack{\underset{\sim}{q}+\underset{\sim}{\beta}\in\bar{R}_1 \\ |\underset{\sim}{\beta}| \to 0}} | D_2C_1(\underset{\sim}{q}) - D_2C_1(\underset{\sim}{q}+\underset{\sim}{\beta}) | = 0, \text{ where } |\underset{\sim}{\beta}| \equiv \sqrt{\beta_1^2 + \beta_2^2}\ .$$ For

any $\underset{\sim}{q}\in V_1$, $D_2C_1(\underset{\sim}{q}) = D_2L(z_1(q_2),q_2) = D_2L(z_1(x_2^*)) = D_2L(\underset{\sim}{x}^*) = 0$.

Using the equality of the first and second terms, and Lemma 2.4, $\underset{\sim}{q}$ and $\underset{\sim}{q}+\underset{\sim}{\beta}\in\bar{R}_1$ implies

(a) $D_2C_1(\underset{\sim}{q}) - D_2C_1(\underset{\sim}{q}+\underset{\sim}{\beta}) = D_2L(z_1(q_2),q_2) - D_2L(z_1(q_2+\beta_2),q_2+\beta_2)$.

Now by a mean value theorem (Buck [8], page 199), which requires that the second partials of $L(\cdot)$ be continuous,

(b) $D_2L(z_1(q_2),q_2) - D_2L(z_1(q_2+\beta_2),q_2 + \beta_2) =$

$D_{12}L(\underset{\sim}{p}) \cdot (z_1(q_2)-z_1(q_2+\beta_2)) + D_{22}L(\underset{\sim}{p}) \cdot (-\beta_2)$ where

$\underset{\sim}{p} = \lambda(z_1(q_2),q_2) + (1-\lambda)(z_1(q_2+\beta_2),q_2+\beta_2)$ for some $\lambda\epsilon[0,1]$.

Define $\theta(\beta_2,q_2) \equiv \max\left(\max_{\lambda\epsilon[0,1]} |D_{12}L(\underset{\sim}{p})|, \max_{\lambda\epsilon[0,1]} |D_{22}L(\underset{\sim}{p})|\right)$.

Since the second partials are continuous, $\theta(\beta_2,q_2) < \infty$. Clearly,

(c) $|D_{12}L(\underset{\sim}{p}) \cdot (z_1(q_2)-z_1(q_2+\beta_2))+D_{22}L(\underset{\sim}{p}) \cdot (-\beta_2)| \leqq$

$\theta(\beta_2,q_2) \cdot (|z_1(q_2)-z_1(q_2+\beta_2)|+|\beta_2|)$.

Define $H(\beta_2,q_2) \equiv \theta(\beta_2,q_2)(|z_1(q_2)-z_1(q_2+\beta_2)|+|\beta_2|)$. Using (a), (b), and (c),

(d) $H(\beta_2,q_2) \geqq |D_2C(\underset{\sim}{q}) - D_2C(\underset{\sim}{q}+\underset{\sim}{\beta})|$.

For any given q_2, $\beta_2 \geqq B_2' \geqq 0$ or $\beta_2 \leqq \beta_2' \leqq 0$ implies that $H(\beta_2,q_2) \geqq H(\beta_2',q_2)$, because $\theta(\beta_2 q_2) \geqq \theta(\beta_2',q_2) \geqq 0$ by reference to the definition and because Property A2 implies that $|z_1(q_2)-z_1(q_2+\beta_2)| \geqq |z_1(q_2)-z_1(q_2+\beta_2')|$. Therefore, if

(e) $\max(H(+\beta_2^*,q_2),H(-\beta_2^*,q_2)) < \epsilon$, then

(f) $|D_2C(\underset{\sim}{q})-D_2C(\underset{\sim}{q}+\underset{\sim}{\beta})| < \epsilon$ if $|\beta_2|<|\beta_2^*|$.

But $\theta(\beta_2,q_2)$ bounded and non-increasing as $\beta_2\to 0$ and $|\beta_2|+|z_1(q_2)-z_1(q_2+\beta_2)|\to 0$ as $\beta_2\to 0$ imply that $H(\beta_2,q_2)\to 0$ as $\beta_2\to 0$, so for any $\epsilon > 0$, a β_2^* satisfying (e) can be found.

Therefore $D_2C_1(\cdot)$ is continuous at $\underset{\sim}{q}$.

(iii) $\underline{q \epsilon \bar{R}_2}$: This case is analogous to (ii) and the proof will be omitted.

(iv) $\underline{q \epsilon \bar{R}_1}$: By Lemma 2.4, for $\underset{\sim}{q} \epsilon R_0$, $C_1(\underset{\sim}{q}) = L(\underset{\sim}{q})$.

But for $\underset{\sim}{q} \epsilon W_1 \cup W_2$, $C_1(\underset{\sim}{q}) = L(\underset{\sim}{q})$ also. Therefore, continuity of the first partials of $L(\cdot)$ implies the continuity of the first partials of $C_1(\cdot)$.

Appendix B

Delivery Lag Examples

The following examples indicate that

(1) $L^{(\lambda)}(\cdot)$ having A2 does not imply that $L(\cdot)$ has A2

and

(2) $L(\cdot)$ having A2 does not imply that $L^{(\lambda)}(\cdot)$ has A2.

We do not prove statements (1) and (2). Instead we consider two single product, discrete demand examples which give the essence of the examples that would be required to prove (1) and (2). In both examples, it is assumed that $c = 0$. In both examples, $L(\cdot)$ and $L^{(\lambda)}(\cdot)$ are not differentiable everywhere.

In the first example, an $h(\cdot)$, $p(\cdot)$, and $\varphi(\cdot)$ are specified such that, for a delivery lag λ equal to 1 period, $L^{(\lambda)}(\cdot)$ is quasiconvex[1] while $L(\cdot)$ is not quasiconvex. In the second, an $h(\cdot)$, $p(\cdot)$ and $\varphi(\cdot)$ are specified such that $L(\cdot)$ is quasiconvex while, for $\lambda=1$, $L^{(\lambda)}(\cdot)$ is not. In both examples, $h(\cdot)$ is not a non-decreasing function of inventory, so there is no intention of suggesting that (1) and (2) are likely to be true in real inventory situations.

---

1 - See Section 9 of Chapter 2.

<u>First example</u>:

$$\varphi(0) = \varphi(4) = 1/2$$

$$\begin{array}{rlll} p(x) & = -4x + 2 & \quad & x < 0 \\ & = 0 & & x \geqq 0 \\ h(x) & = 0 & & x < 0 \\ & = -x + 2 & & 0 \leqq x < 2 \\ & = x - 2 & & 2 \leqq x < 4 \\ & = 2x - 6 & & 4 \leqq x < 6 \\ & = -2x + 18 & & 6 \leqq x < 8 \\ & = 4x - 30 & & 8 \leqq x \end{array}$$

See Figure B.1.

With instant delivery, we have:

$$L(8) = 1/2 \cdot h(8) + 1/2\, h(4) = 2$$

$$L(6) = 1/2 \cdot h(6) + 1/2\, h(2) = 3$$

$$L(4) = 1/2 \cdot h(4) + 1/2\, h(0) = 2.$$

Since $L(6) > \max\,(L(4), L(8))$, $L(\cdot)$ is not quasiconvex.

For a delivery lag λ equal to 1, we have $P(D_1+D_2=0) = 1/4$, $P(D_1+D_2=4) = 1/2$, and $P(D_1+D_2=8) = 1/4$, so for example, $L^{(1)}(0) = 1/4\, h(0) + 1/2\, p(-4) + 1/4\, p(-8) = 18$. In the same way, $L^{(1)}(2) = 11.5$, $L^{(1)}(4) = 6$, $L^{(1)}(6) - 4$, $L^{(1)}(8) = 2$, $L^{(1)}(10) = 6$, $L^{(1)}(12) = 6$, $L^{(1)}(14) = 13.5$, $L^{(1)}(16) = 18.5$. Also, $L(x) = 18 - 4x$ for $x < 0$ and $L(x) = -45.5 + 4x$ for $x > 16$.

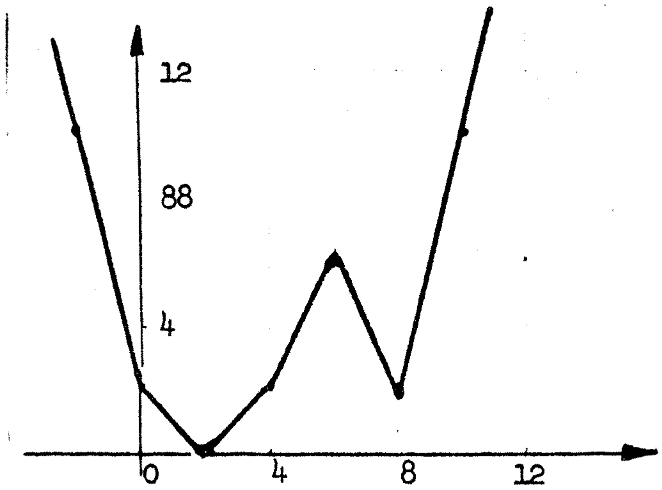


Figure B.1

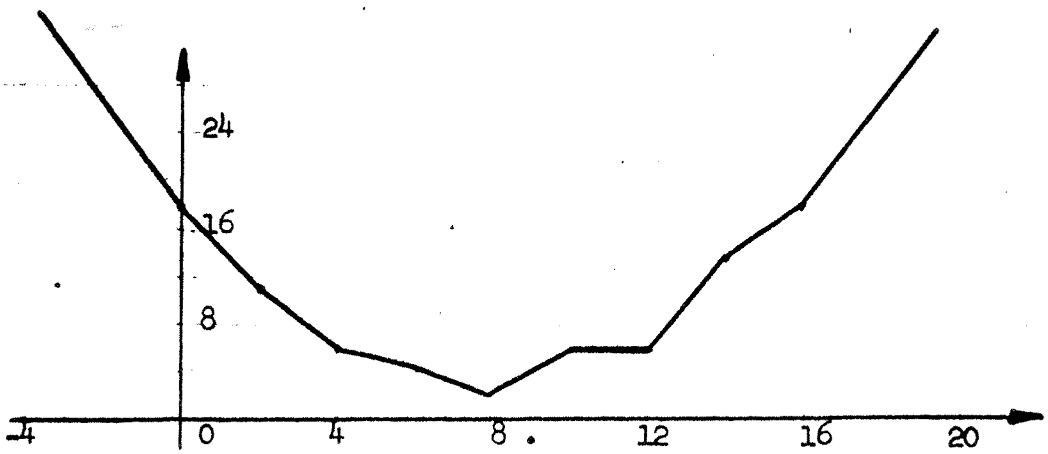


Figure B.2

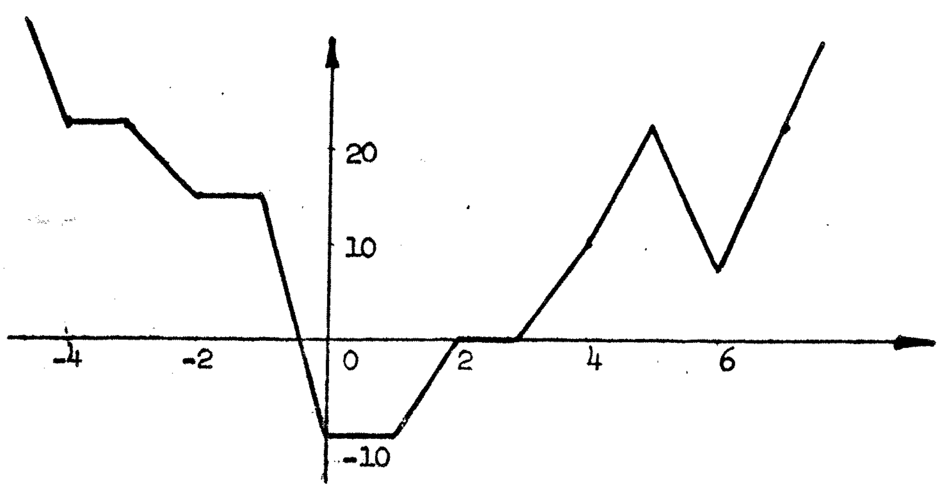


Figure B.3

Since $h(\cdot)$ and $p(\cdot)$ are piecewise linear, with changes in slope occurring only at the even integers, and since $\varphi(\cdot)$ is positive only on the even integers, $L^{(1)}(\cdot)$ is piecewise linear with changes in slope occurring only at the even integers. Therefore, the above calculations specify $L^{(1)}(\cdot)$ completely, and by inspection of Figure B.2, $L^{(1)}(\cdot)$ is quasiconvex.

Second Example:

$$\varphi(0) = \varphi(4) = 2/5, \qquad \varphi(2) = 1/5$$

$$\begin{aligned}
p(x) &= -30x - 97.5 && x < -4 \\
&= 22.5 && -4 \leq x < -3 \\
&= -7.5x && -3 \leq x < -2 \\
&= 15 && -2 \leq x < -1 \\
&= -25x - 10 && -1 \leq x < 0 \\
&= 0 && 0 \leq x \\
h(x) &= 0 && x < 0 \\
&= -10 && 0 \leq x < 1 \\
&= 10x - 10 && 1 \leq x < 2 \\
&= 0 && 2 \leq x < 3 \\
&= 10x - 30 && 3 \leq x < 4 \\
&= 12.5x - 40 && 4 \leq x < 5 \\
&= -15x + 97.5 && 5 \leq x < 6 \\
&= 30x - 172.5 && 6 \leq x
\end{aligned}$$

See Figure B.3.

From this we find that $L(\cdot)$ is quasiconvex by examining its slope:

$10 \leqq x$: $L(x) = 30x + \text{constant}$, so $\frac{dL(x)}{dx} = +30$.

$9 \leqq x < 10$: $L(x) = (30x)(2/5 + 1/5) + (-15x)(2/5) + \text{constant}$

$= 12x + \text{constant}$.

$8 \leqq x < 9$: $L(x) = 23x + \text{constant}$

$7 \leqq x < 8$: $L(x) = 13x + \text{constant}$

$6 \leqq x < 7$: $L(x) = 14.5x + \text{constant}$

$5 \leqq x < 6$: $L(x) = \text{constant}$

$4 \leqq x < 5$: $L(x) = 5x + \text{constant}$

$3 \leqq x < 4$: $L(x) = -4x + \text{constant}$

$2 \leqq x < 3$: $L(x) = \text{constant}$

$1 \leqq x < 2$: $L(x) = -4x + \text{constant}$

$x < 1$: $L(x)$ is a decreasing function of x.

For a delivery lag λ equal to 1, we have

$$P(D_1+D_2 = 0) = P(D_1+D_2 = 2) = P(D_1+D_2 = 6) = P(D_1+D_2 = 8) = \frac{4}{25}$$

and $P(D_1+D_2 = 4) = \frac{9}{25}$. Then

$$L^{(\lambda)}(6) = \frac{4}{25}(7.5 + 10 - 10 + 15) + \frac{9}{25}(0) = 3.6,$$

and

$$L^{(\lambda)}(5) = \frac{4}{25}(22.5 + 0 + 15 + 22.5) + \frac{9}{25}(-10) = 6,$$

and

$$L^{(\lambda)}(4) = \frac{4}{25}(10 + 0 + 15 + 22.5) + \frac{9}{25}(-10) = 4.$$

Consequently, $L^{(\lambda)}(\cdot)$ is not quasiconvex.

Appendix C

An Example where $L(\cdot)$ has Property A2 but is not Quasiconvex

In this example, $h(\cdot)$ and $p(\cdot)$ are convex in x_1 for any x_2 and in x_2 for any x_1. Demand is assumed to be uniformly distributed on the unit square. To be specific, we assume that

(i) $\varphi(x_1,x_2) = 1$ if $0 < x_1 < 1$ and $0 < x_2 < 1$

$= 0$ elsewhere

(ii) $p(x_1,x_2) = p \cdot (\max(0,-x_1) + \max(0,-x_2))$ where $p > 1$.

(iii) $$h(x_1,x_2) = \frac{[\max(x_1,0)+\max(x_2,0)]^2}{1+\max(x_1,0)+\max(x_2,0)} + 2\max(x_1,0)\cdot\max(x_2,0)$$

and $h(\cdot)$ is defined on the vector of ending inventory.

(iv) $\underset{\sim}{c} = \underset{\sim}{0}$

Then $L(x_1,x_2) = \int_0^\infty \int_0^\infty [p(x_1-u,x_2-v)+h(x_1-u,x_2-v)] \cdot \varphi(u,v)dudv.$

First we show that $L(\cdot)$ is neither convex nor quasiconvex. Clearly $L(3,1) = L(1,3)$. We show that $L(2,2) > L(3,1)$, which implies that $L(\cdot)$ is neither convex nor quasiconvex.

$$L(2,2) - L(3,1) = \int_0^1 \int_0^1 \frac{(4-u-v)^2}{5-u-v} + 2(2-u)(2-v) \quad dudv$$

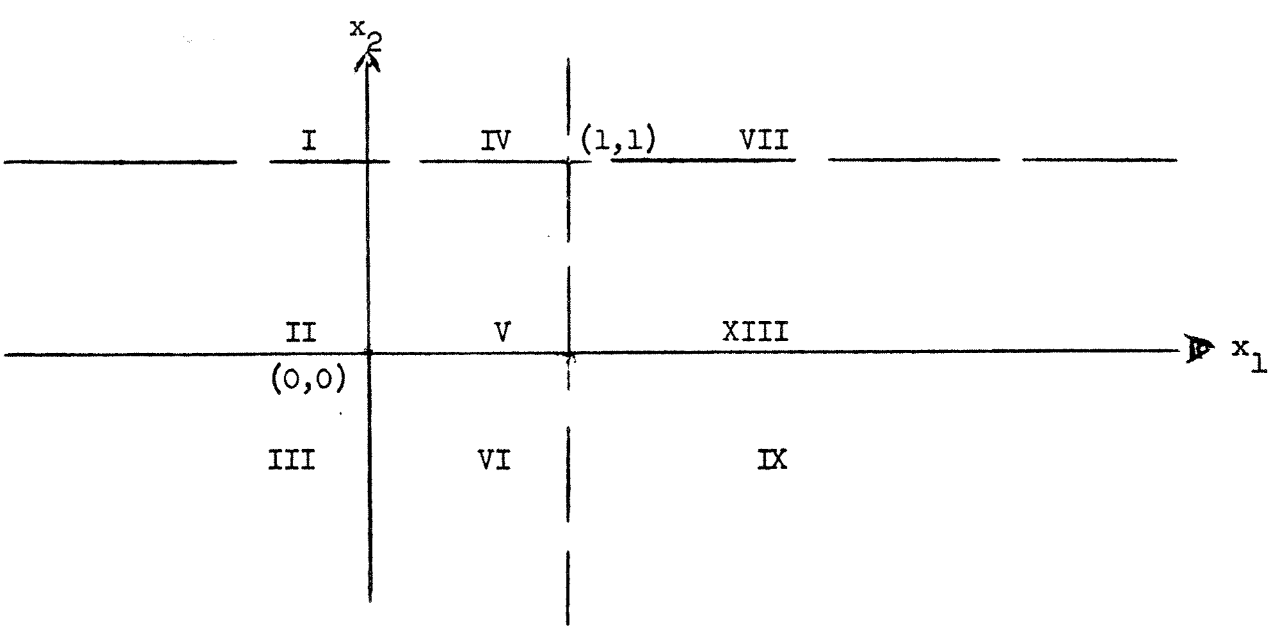
$$- \int_0^1 \int_0^1 \frac{(4-u-v)^2}{5-u-v} + 2(3-u)(1-v) \quad dudv$$

$$= \int_0^1 \int_0^1 (1-u+v)dudv = 2.$$

To begin the proof that $L(\cdot)$ has Property A2, observe that since $h(\cdot) + p(\cdot)$ is positive everywhere, $L(\cdot)$ is also. Further, since all partial derivatives of $h(\cdot)$ and $p(\cdot)$ exist, the second partial derivatives of $L(\cdot)$ exist and are continuous.

To continue the proof, the plane will be divided in nine areas, labeled as in Figure C.1.

<u>Figure C.1</u>



First $D_1L(\cdot)$ will be obtained in each area.

I: $L(x_1,x_2) = \int_0^1 \int_0^1 h(0,x_2-v)\ dudv + \int_0^1 p \cdot (u-x_1)du$

$$D_1L(x_1,x_2) = \int_0^1 (-p)\ du = -p$$

Similarly, in II and III, $D_1L(\underset{\sim}{x}) = -p$.

$$\text{IV:}\quad L(x_1,x_2) = \int_0^1 \int_0^{x_1} h(x_1-u,x_2-v)\ dudv$$

$$+ \int_0^1 \int_{x_1}^1 h(0,x_2-v)\ dudv$$

$$+ \int_{x_1}^1 p \cdot (u-x_1)du$$

Therefore
$$D_1L(x_1,x_2) = \int_0^1 \int_0^{x_1} \frac{\partial}{\partial x_1} h(x_1-u,x_2-v)\ dudv$$

$$+ \int_0^1 h(0,x_2-v)dv$$

$$- \int_0^1 h(0,x_2-v)dv$$

$$+ \int_{x_1}^1 -p\ du - p(1-x_1)$$

$$= \int_0^1 \int_0^{x_1} \frac{\partial}{\partial x_1} h(x_1-u,x_2-v)\ dudv - p(1-x_1).$$

Since $\frac{\partial}{\partial x_1} h(x_1-u,x_2-v) = -\frac{\partial}{\partial u} h(x_1-u,x_2-v)$,

integrating with respect to u yields

$$D_1L(x_1,x_2) = \int_0^1 [h(x_1,x_2-v)-h(0,x_2-v)]\ dv - p(1-x_1).$$

Using arguments similar to those in IV, we obtain:

V: $$D_1L(x_1,x_2) = \int_0^{x_2} [h(x_1,x_2-v)-h(0,x_2-v)]dv$$

$$+ (1-x_2)\ [h(x_1,0)] -p\ (1-x_1).$$

VI: $$D_1L(x_1,x_2) = h(x_1,0) -p(1-x_1).$$

VII: $$D_1L(x_1,x_2) = \int_0^1[h(x_1,x_2-v) -h(x_1-1,x_2-v)]dv.$$

VIII: $$D_1L(x_1,x_2) = \int_0^{x_2} [h(x_1,x_2-v) -h(x_1-1,x_2-v)]\ dv$$

$$+ [h(x_1,0) -h(x_1-1,0)](1-x_2).$$

IX: $$D_1L(x_1,x_2) = h(x_1,0) -h(x_1-1,0).$$

Now let us examine $D_1L(\cdot)$. In regions I,II, and III, $D_1L(\cdot)$ is negative. In regions VII,VIII, and IX, $D_1L(\cdot)$ is positive. In regions IV,V, and VI, $D_1L(x_1,x_2)$ is an increasing function of x_1, since $h(x_1,x_2)$ is an increasing function of x_1. In these three regions, $D_1L(x_1,x_2)$ is negative at $x_1=0$ and positive at $x_1=1$. Therefore $L(\cdot)$ is strictly quasiconvex in x_1 for any x_2. By symmetry, $L(\cdot)$ is strictly quasiconvex in x_2 for any x_1.

Let $z_1(x_2)$ be such that $D_1L(z_1(x_2),x_2) = 0$; and $z_2(x_1)$ be such that $D_2L(x_1,z_2(x_1)) = 0$. It has just been shown that $z_1(\cdot)$

and $z_2(\cdot)$ are single-valued functions, whose ranges are contained in [0,1].

Next we apply an implicit function theorem (Theorem 26, page 222 of Buck [8]) to show that $z_1(\cdot)$ and $z_2(\cdot)$ are continuously differentiable. <u>Define</u> $F(x_1,x_2,z) \equiv D_1L(x_1,x_2)+z$. And suppose that $F(\underset{\sim}{t}) = 0$. Then a continuously differentiable function $\varphi(x_2,z)$, satisfying $F(\varphi(y_2,y_3),y_2,y_3) = 0$, exists for every $\underset{\sim}{y}$ near $\underset{\sim}{t}$, if $F(\cdot)$ is continuously differentiable and if $D_1F(\underset{\sim}{t}) \neq 0$. The former is clearly true, and $D_1F(x_1,x_2,z) = D_{11}L(x_1,x_2)$ which equals $\int_0^1 D_1h(x_1,x_2-v)dv + p > 0$ anywhere in region IV and therefore at $(z_1(x_2),x_2,0)$. $D_1F(z_1(x_2),x_2,0)$ is similarly positive in regions V and VI, so $z_1(\cdot)$ is continuously differentiable. By symmetry, so is $z_2(\cdot)$.

Next we prove that (A) $z_1(\cdot)$ and $z_2(\cdot)$ are non-increasing, (B) there is only one point, $\underset{\sim}{x}^*$, for which it is true that $z_2(x_1^*) = x_2^*$ and (C) $z_2(x_1) > x_1$ for $x_1 < x_1^*$ and $z_2(x_1) < x_1$ for $x_1 > x_1^*$. Since we have already shown that $L(\cdot)$ is positive with continuous second partials, and that it is strictly quasiconvex when considered as a function of either variable, and that $z_1(\cdot)$ and $z_2(\cdot)$ are continuously differentiable, this will complete the proof that $L(\cdot)$ has Property A2.

To obtain $\frac{dz_1(x_2)}{dx_2}$, consider the implicit function $D_1L(z_1(x_2),x_2) = 0$. Taking the derivative with respect to x_2 yields $\frac{dz_1(x_2)}{dx_2} \cdot D_{11}L(z_1(x_2),x_2) + D_{21}L(z_1(x_2),x_2) = 0$.

In region IV, substituting into this formula yields

$$\frac{dz_1(x_2)}{dx_2} = (-1) \cdot \frac{\int_0^1 [D_2h(z_1(x_2),x_2-v)-D_2h(0,x_2-v)]dv}{p + \int_0^1 D_1h(z_1(x_2),x_2-v)dv}$$

Since $D_{12}h(y_1,y_2) = \frac{2}{(1+y_1+y_2)^3} + 2 > 0$ for $y_1,y_2 > 0$,

the integrand in the numerator is positive everywhere, so the numerator is positive. The denominator is clearly positive, so that $\frac{dz_1(x_2)}{dx_2} < 0$ in this region.

In region V, substitution yields

$$\frac{dz_1(x_2)}{dx_2} = (-1) \cdot \frac{\int_0^{x_2} [D_2h(z_1(x_2),x_2-v)-D_2h(0,x_2-v)]dv}{p+(1-x_2)D_1h(z_1(x_2),0)+ \int_0^{x_2} D_1h(z_1(x_2),x_2-v)dv} .$$

By the same arguments used for region IV= $\frac{dz_1(x_2)}{dx_2} < 0$. It will also be shown that $\frac{dz_1(x_2)}{dx_2} > -1$, which, by symmetry, will show that $z_1(\cdot)$ and $z_2(\cdot)$ intersect exactly once, with $z_1(\cdot)$ above $z_2(\cdot)$ to the left of the intersection and $z_1(\cdot)$ below $z_2(\cdot)$ to the right of it. First, since $D_2h(q,x_2-v) = -\frac{\partial}{\partial v}h(q_1,x_2-v)$, the numerator of the expression for $\frac{dz_1(x_2)}{dx_2}$ is equal to

$[h(z_1(x_2),x_2)-h(z_1(x_2),0)] - [h(0,x_2)-h(0,0)]$. Second, since

$D_1h(x_1,x_2-v) = D_2h(x_1,x_2-v)-2x_1+2(x_2-v)$,

$$\int_0^{x_2} D_1h(z_1(x_2),x_2-v)dv = -2z_1(x_2)x_2 + x_2^2$$
$$+ [h(z_1(x_2),x_2)-h(z_1)x_2),0)].$$

Therefore,

$$\frac{dz_1(x_2)}{dx_2} = (-1)$$
$$\cdot \frac{[h(z_1(x_2),x_2)-h(z_1(x_2),0)]-[h(0,x_2)-h(0,0)]}{h(z_1(x_2),x_2)-h(z_1(x_2),0)+[p+x_2^2-2z_1(x_2)x_2]+(1-x_2)D_1h(z_1(x_2),0)} .$$

Since $h(0,x_2)-h(0,0) > 0$ and $(1-x_2)D_1h(z_1(x_2),0) > 0$, if $p + x_2^2-2z_1(x_2)x_2 > 0$, then $\left|\frac{dz_1(x_2)}{dx_2}\right|$ will be <1. But

$p+x_2^2 - 2z_1(x_2)x_2 = p-(z_1(x_2))^2 + (x_2-z_1(x_2))^2$ which is positive since $p > 1 \geqq z_1(x_2)$.

In region VI, $D_1L(x_1,x_2) = h(x_1,0) - p(1-x_1)$, so that $z_1(x_2)$ is constant and $\dfrac{dz_1(x_2)}{dx_2} = 0$.

Appendix D

One Product Solutions to Two Product Problems

In this example, it is assumed that

(i) $\varphi(x_1,x_2) = 1$ if $0 < x_1 < 1$ and $0 < x_2 < 1$

$= 0$ elsewhere

(ii) $p(x_1,x_2) = p \cdot (\max(0,-x_1)+\max(0,-x_2))$ where $\underset{\sim}{x}$ is inventory just after demand,

(iii) $h(x_1,x_2) = h \cdot (x_1+x_2)^\beta$ where $\underset{\sim}{x}$ is inventory just before demand and $\beta > 1$,

(iv) $\underset{\sim}{c} = \underset{\sim}{0}$.

Then $L(x_1,x_2) =$

$$\int_0^\infty \int_0^\infty [h(x_1,x_2)+p(x_1-u,x_2-v)]\varphi(u,v)dudv.$$

The point that minimizes $L(\cdot)$ will be called $\underset{\sim}{x}^*$. From the symmetry of $h(\cdot)$, $p(\cdot)$, and $\varphi(\cdot)$, it is clear that $x_1^* = x_2^*$.

It will be assumed that, to solve this problem as two one product problems, $h(\cdot)$ will be replaced by $\tilde{h}(x_1,x_2) = h \cdot (x_1^\beta + x_2^\beta)$. We wish to emphasize that $\tilde{h}(\cdot)$ is one particular way of approximating $h(\cdot)$ with a separable function, and not necessarily the best one.

<u>Define</u>

$$\tilde{L}(x_1,x_2) = \int_0^\infty \int_0^\infty [\tilde{h}(x_1,x_2)+p(x_1-u,x_2-v)]\varphi(u,v)dudv.$$

The point that minimizes $\tilde{L}(\cdot)$ will be called $\tilde{x}^0$, and it should be clear that $x_1^0 = x_2^0$.

From the analysis in Appendix C, for $x_1 < 0$, $D_1L(\underset{\sim}{x})$ and $D_1\tilde{L}(\underset{\sim}{x})$ are negative, and for $x_1 > 1$, they are positive, while in between, for any x_2, they are increasing functions of x_1. For $0 < x_1 < 1$, $D_1L(x) = -p(1-x_1)$

$$+ h \cdot \beta(x_1+x_2)^{\beta-1}$$

and $D_1\tilde{L}(\underset{\sim}{x}) = -p(1-x_1) + h \cdot \beta(x_1)^{\beta-1}$. Remembering that $x_1^* = x_2^*$ and $x_1^0 = x_2^0$, $\underset{\sim}{x}^*$ and $\underset{\sim}{x}^0$ are defined by the following (implicit) functions:

(D.1) $h \cdot \beta \cdot (2x_1^*)^{\beta-1} + px_1^* - p = 0$

(D.2) $h \cdot \beta \cdot (x_1^0)^{\beta-1} + px_1^0 - p = 0$

When $\beta = 1$, $h(\cdot)$ and $\tilde{h}(\cdot)$ are identical, so $\underset{\sim}{x}^* = \underset{\sim}{x}^0$. As β increases from 1, $\tilde{h}(\cdot)$ becomes a poorer approximation to $h(\cdot)$. It will be shown than, if $\underset{\sim}{x}^*$ and $\underset{\sim}{x}^0$ are considered as functions of β, the ratio $\dfrac{x_1^*}{x_1^0}$ decreases from 1 as β increases from 1.

First, we illustrate this result numerically. Assume that $p = 5h$. For $\beta = 1$, (D.1) yields $h + px_1^* - p = 0$ or

$x_1^* = \frac{p-h}{p} = .8$. Also $x_1^0 = .8$, and $\frac{x_1}{x_1^0} = 1$. For $\beta = 2$, (D.1) yields $(4h+p)\,x_1^* - p = 0$ or $x_1^* = \frac{p}{4h+p} = \frac{5}{9}$. (D.2) yields $(2h + p)x_1^0 - p = 0$, or $x^0 = \frac{5}{7}$, so that $\frac{x_1^*}{x_1^0} = \frac{7}{9}$. For $\beta = 3$, (D.1) yields $12h(x_1^*)^2 + px_1^* - p = 0$, or $1](x_1^*)^2 + 5x_1^* -5 = 0$, so that $x_1^* = .469$. (D.2) yields $3h(x_1^*)^2 + px_1^* - p = 0$, or

$$x_1^0 = \frac{-5 \pm \sqrt{25+60}}{6} = .703, \text{ so that } \frac{x_1^*}{x_0^*} = .667.$$

To prove that $\frac{x_1^*}{x_1^0}$ does not increase as β increases,

we show that $\left|\frac{\frac{dx_1^*(\beta)}{d\beta}}{x_1^*(\beta)}\right| \geq \left|\frac{\frac{dx_1^0(\beta)}{d\beta}}{x_1^0(\beta)}\right|$ for all $\beta \geq 1$.

First we verify that conditions for an implicit function theorem are satisfied, so that $\frac{dx_1^*(\beta)}{d\beta}$ and $\frac{dx_1^0(\beta)}{d\beta}$ exist.

Differentiating (D.1) with respect to β yields

$$0 = h \cdot [2x_1^*(\beta)]^{\beta-1} + h\beta[2x_1^*(\beta)]^{\beta-1}\ln[2x_1^*(\beta)]$$
$$+ \frac{dx_1^*(\beta)}{d\beta}\,[h(\beta)(\beta-1)2^{\beta-1}[x_1^*(\beta)]^{\beta-2} + p], \text{ or}$$

$$(D.3)\quad \frac{dx_1^*(\beta)}{d\beta} = (-1)\,\frac{2^{\beta-1}h\_x_1^*(\beta)]^{\beta-1}(1+\beta\ln 2x_1^*(\beta))}{2^{\beta-1}h\beta(\beta-1)[x_1^*(\beta)]^{\beta-1}+p}\ .$$

Similarly, we get

$$(D.4)\quad \frac{dx_1^0(\beta)}{d\beta} = (-1)\,\frac{h[x_1^0(\beta)]^{\beta-1}(1+\beta\ln x_1^0(\beta))}{h\beta(\beta-1)[x_1^0(\beta)]^{\beta-1}+p}$$

Now it will be shown that $x_1^0(\beta) \geq x_1^*(\beta) \geq \frac{x_1^0(\beta)}{2}$ for any $\beta > 1$. The first of these inequalities is clear from comparing (D.1) and (D.2), since their only difference is a more positive multiplier for $[x_1^*]^{\beta-1}$ in (D.1). To obtain the second, consider $D_1L(\frac{x_1^0}{2},\frac{x_1^0}{2}) = -p+p\frac{x_1^0}{2} + h\beta(\frac{x_1^0}{2}+\frac{x_1^0}{2})\beta-1.$ Note that (D.2) implies that $-p + px_1^0 + h\beta(x_1^0)^{\beta-1} = 0$, so therefore $D_1L(\frac{x_1^0}{2},\frac{x_1^0}{2}) = \frac{px_1^0}{2} > 0$. Consequently $x_1^* \geq \frac{x_1^0}{2}$.

The first of these inequalities implies that

$$\left|\frac{dx_1^*(\beta)}{d\beta}\right| \geq \left|\frac{dx_1^0(\beta)}{d\beta}\right| \quad \text{is sufficient for} \quad \left|\frac{\frac{dx_1^*(\beta)}{d\beta}}{x_1^*(\beta)}\right| \geq \left|\frac{\frac{dx_1^0(\beta)}{d\beta}}{x_1^0(\beta)}\right|\ .$$

Now consider the right hand side of (D.3) when $x_1^*(\beta)$ is replaced by a free variable, x, and call it $G(x,\beta)$. Rewriting $G(x,\beta)$ as $(-1)2^{\beta-1}h\left(\frac{1 + \beta \text{ in } 2x}{\beta(\beta-1)+ \frac{p}{2^{\beta-1}hx^{\beta-1}}}\right)$

makes it clear that $|G(x,\beta)|$ increases as x increases from zero. Substituting $\frac{x_1^0(\beta)}{2}$ for x yields

$$G\left(\frac{x_1^0(\beta)}{2},\beta\right) = (-1)\,\frac{2^{\beta-1}h\left(\frac{x_1^0(\beta)}{2}\right)^{\beta-1}\left(1+\beta\ln 2\cdot\frac{x_1^0(\beta)}{2}\right)}{2^{\beta-1}h(\beta)(\beta-1)\left(\frac{x_1^0(\beta)}{2}\right)^{\beta-1} + p}$$

$$= \frac{dx_1^0(\beta)}{d\beta}\,.$$

Since $x_1^*(\beta) \geq \frac{x_1^0(\beta)}{2} \geq 0$, $\left|G\left(\frac{x_1^0(\beta)}{2},\beta\right)\right| \leq \left|\frac{dx_1^*(\beta)}{d\beta}\right|$, and we have completed the proof that $\frac{x_1^*(\beta)}{x_1^0(\beta)}$ is a non-increasing function of β.

Appendix E

Setup Cost Examples

E.1 An Example of the Two Period Optimal Policy

The optimal policy for a two period problem with an $h(\cdot)$, $p(\cdot)$ and $\varphi(\cdot)$ that essentially satisfy Hypothesis 4 is obtained. ($L(\cdot)$ has Property A2* but not Property A2.) The policy is illustrated in Figures E.2 and E.4. The costs and demand distribution have been chosen to make the computations relatively easy and not for realism.

The problem

(a) $\underset{\sim}{c} = \underset{\sim}{0}$

(b) $h(\underset{\sim}{x},\underset{\sim}{t}) = \max(x_1,0) + \max(x_2,0)$

(c) $p(\underset{\sim}{x}-\underset{\sim}{t}) = 0$ if $x_1-t_1 \geqq -1$ and $x_2-t_2 \geqq -1$

$= 2$ elsewhere

(d) $\varphi(\underset{\sim}{t}) = 1$ if $0 \leqq t_1 \leqq 1$ and $0 \leqq t_2 \leqq 1$

$= 0$ elsewhere

(e) $\alpha = 1$

(f) $K_1 = K_2 = K_{12} = 1$

Then $L(\cdot)$, pictured in Figure E.1, can be seen to satisfy Property A2*. It is also apparent that all but (a) of Hypothesis 4

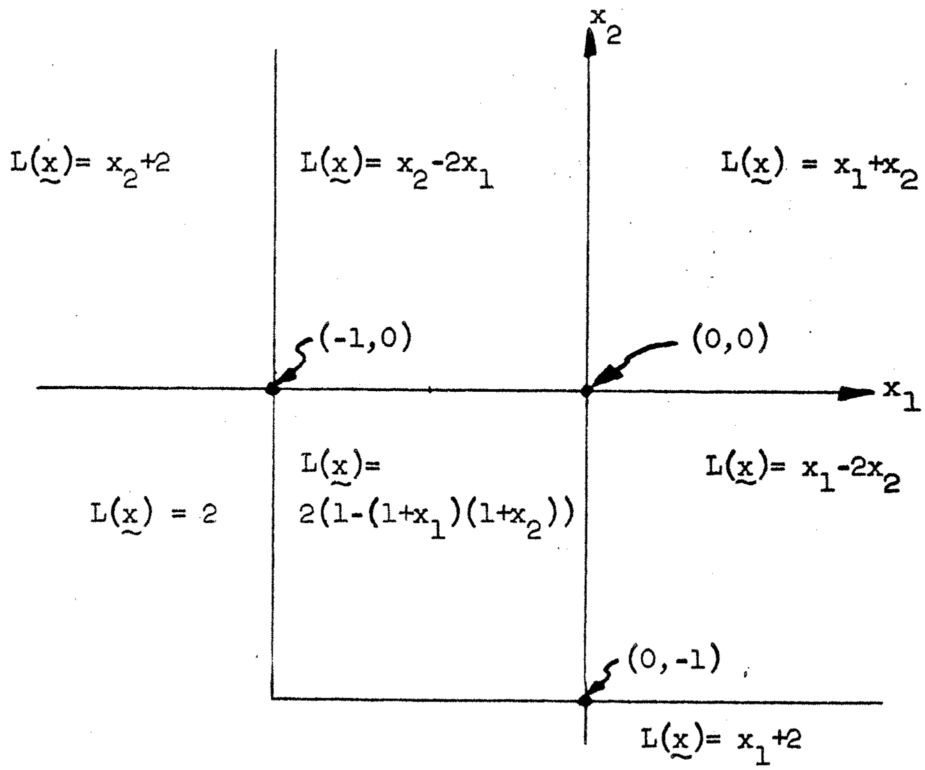


Figure E.1

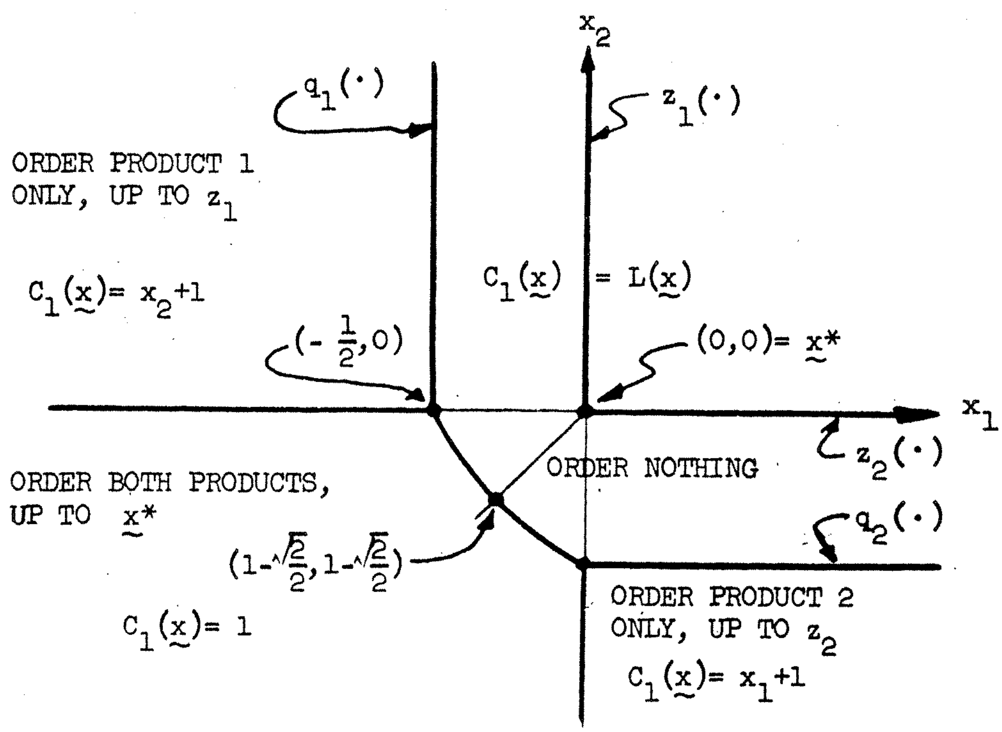


Figure E.2

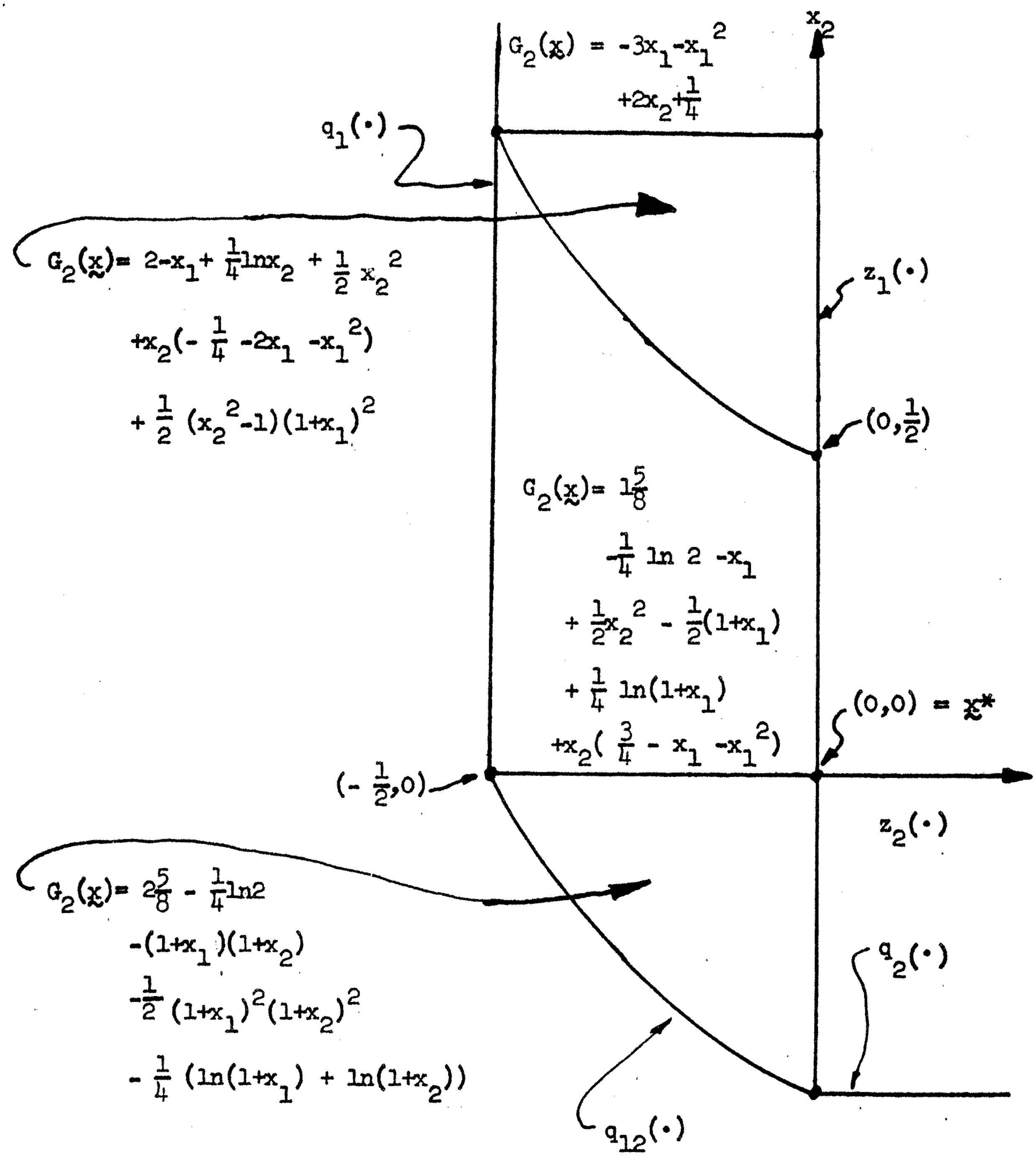
are satisfied. It is clear from Figure E.1 that $\underset{\sim}{x}^* = (0,0)$ and $z_1(x_2) = z_2(x_1) = 0$ for all x_1 and x_2. Therefore $q_1(x_2) = q_2(x_1) = -1/2$ for all $x_1 \geqq 0$ and all $x_2 \geqq 0$. For $-1/2 \leqq x_2 \leqq 0$, $q_{12}(x_2)$ is given by $2[1 - (1+q_{12}(x_2))] = 1$ or $(1+q_{12}(x_2)) = 1/2$, an hyperbola. Also, $w_1(x_2) = w_2(x_1) = 1$ for all $x_1 \geqq 1$ and all $x_2 \geqq 1$. The optimal policy in the last of the two periods then is pictured in Figure E.2, where $C_1(\cdot)$ is also specified.

In Figure E.3 we specify $G_2(\cdot)$, using the bounds of Theorem 4.10 to define the areas of interest. Since $G_2(\cdot)$ is symmetric, the area where $x_1 > x_2$ is omitted. It is not necessary to obtain $G_2(\cdot)$ explicitly between the $z_1(\cdot)$ and $w_1(\cdot)$ curves, since

(1) it is easy to show that $D_1G_2(\cdot)$ is non-negative in this area

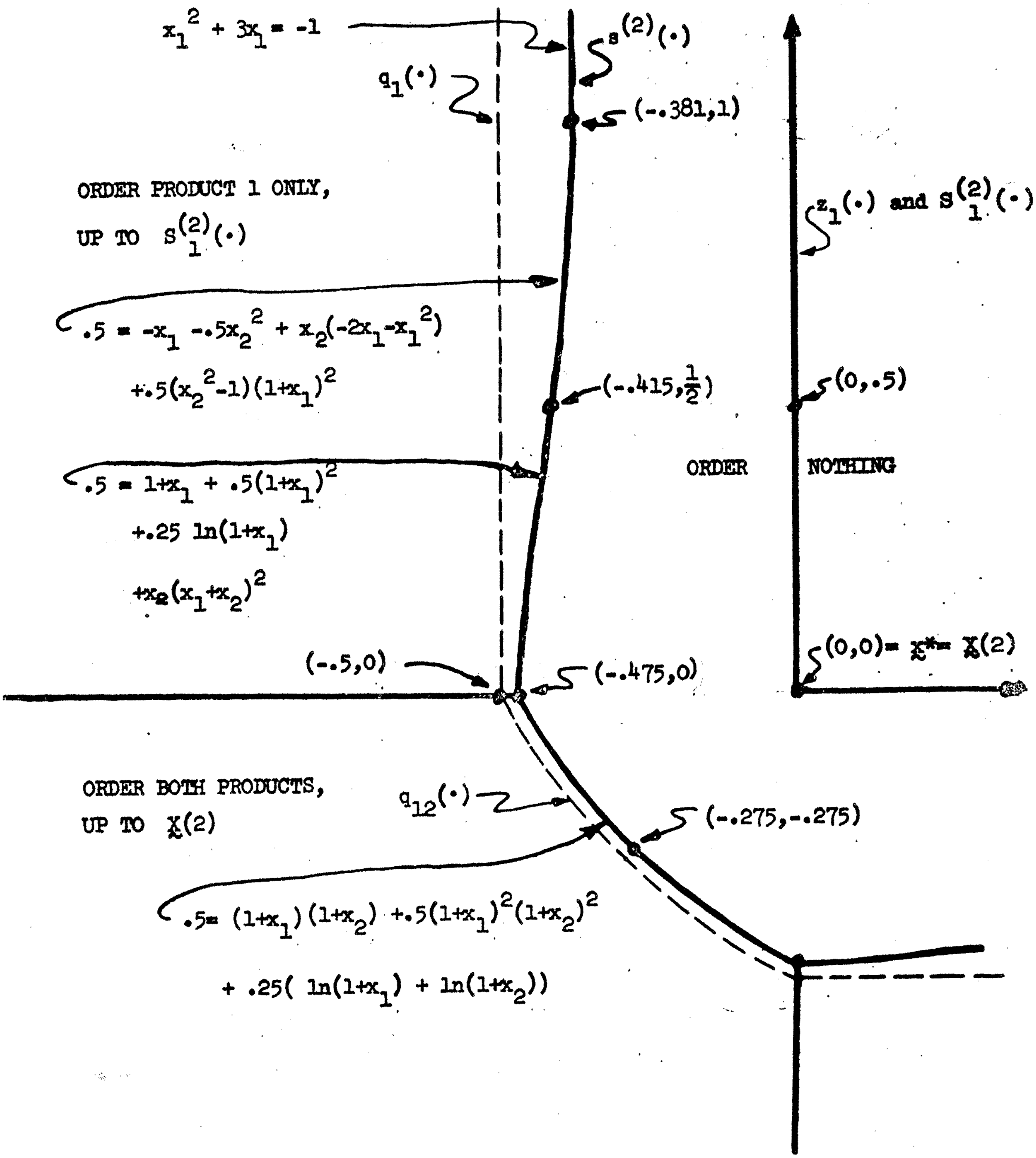
and (2) from the formulas for $G_2(\cdot)$ between $q_1(\cdot)$ and $z_1(\cdot)$, it can be seen than, above $x_1=0$, $D_2G_2(\cdot)$ is non-negative on the line $x_2=0$.

This implies that $\underset{\sim}{X}(2) = \underset{\sim}{x}^* = (0,0)$ and $\underset{\sim}{S}^{(2)}(x_2) = (0,x_2)$ for every $x_2 > 0$. Solving then for $S^{(2)}(\cdot)$, we obtain the result pictured in Figure E.4 as the optimal policy with two periods remaining.



x_2
$G_2(\underset{\sim}{x}) = -3x_1 - x_1^2 + 2x_2 + \frac{1}{4}$
$q_1(\cdot)$
$G_2(\underset{\sim}{x}) = 2 - x_1 + \frac{1}{4}\ln x_2 + \frac{1}{2}x_2^2 + x_2(-\frac{1}{4} - 2x_1 - x_1^2) + \frac{1}{2}(x_2^2 - 1)(1+x_1)^2$
$z_1(\cdot)$
$(0,\frac{1}{2})$
$G_2(\underset{\sim}{x}) = 1\frac{5}{8} - \frac{1}{4}\ln 2 - x_1 + \frac{1}{2}x_2^2 - \frac{1}{2}(1+x_1) + \frac{1}{4}\ln(1+x_1) + x_2(\frac{3}{4} - x_1 - x_1^2)$
$(0,0) = \underset{\sim}{x}^*$
$(-\frac{1}{2},0)$
$z_2(\cdot)$
$G_2(\underset{\sim}{x}) = 2\frac{5}{8} - \frac{1}{4}\ln 2 - (1+x_1)(1+x_2) - \frac{1}{2}(1+x_1)^2(1+x_2)^2 - \frac{1}{4}(\ln(1+x_1) + \ln(1+x_2))$
$q_2(\cdot)$
$q_{12}(\cdot)$

Figure E.3



The Optimal Policy with Two Periods Remaining

Figure E.4

Note: If $p(\underset{\sim}{x}-\underset{\sim}{t})$ is set equal to 4 for $\underset{\sim}{x}-\underset{\sim}{t} < (1,1)$ then

(a) $L(\cdot)$ becomes separable and

(b) $q_{12}(x_2) = -1/2 - x_2$, a straight line, for $-1/2 < x_2 < 0$.

E.2 An Example where the Minimizer of $G_2(\cdot)$ is not greater than or equal to $\underset{\sim}{x}^*$

Consider the following discrete demand example. Again, $h(\cdot)$, $p(\cdot)$, and $\varphi(\cdot)$ are chosen for convenience only. Further, we specify $h(\cdot)$ and $p(\cdot)$ only in the area of interest; their sum, $h(\underset{\sim}{x},\underset{\sim}{t}) + p(\underset{\sim}{x}-\underset{\sim}{t})$ is given as a function of $\underset{\sim}{x}-\underset{\sim}{t}$ in Figure E.5. We also assume that $\underset{\sim}{c}=\underset{\sim}{0}$, $\alpha=1$, $K_1 = K_2 = K_{12} = 2$, and $\varphi(1,1) = \varphi(0,1) = \varphi(1,0) = 1/3$.

$L(\cdot)$ is given in Figure E.6 and $C_1(\cdot)$ in Figure E.7. Then $G_2(\cdot)$ is given in Figure E.8. We note that $G_2(\cdot)$ is minimized at $(1,-1)$ and $(-1,1)$, with $G_2(\cdot)$ being larger at any point other than these, and that neither point is $\geq \underset{\sim}{x}^* = (0,0)$.

Note that $L(\cdot)$ is quasiconvex in x_1 for any x_2 and vice-versa, and satisfies the obvious discrete version of Property A2. The difficulty arises because $L(\cdot)$ is not quasiconvex along the lines $x_1 + x_2 = -4$ and $x_1 + x_2 = 0$.

x_2-t_2 \ x_1-t_1	-3	-2	-1	0	1	2
1	.9	.8	.1	14.	100.	100.
0	.2	.4	0	2.	14.	100.
-1	.2	.2	0	0	.1	100.
-2	100.	5.6	.2	.4	.8	100.
-3	100.	100.	.2	.2	.9	200.

$h(\underset{\sim}{x},\underset{\sim}{t}) + p(\underset{\sim}{x}-\underset{\sim}{t})$ vs. $\underset{\sim}{x}-\underset{\sim}{t}$

Figure E.5

x_2 \ x_1	-2	-1	0	1	2
1	.5	.4	.7	10.	71.3
0	.2	.2	0	.7	38.
-1	35.3	2.	.2	.4	33.6
-2	100.	35.3	.2	.5	67.

$L(\underset{\sim}{x})$ vs. $\underset{\sim}{x}$

Figure E.6

x_2 \ x_1	-2	-1	0	1	2
1	.5	.4	.7	10.	71.3
0	.2	.2	0	.7	38.0
-1	2.	2.	.2	.4	33.6
-2	2.	2.	.2	.5	35.6

$C_1(\underset{\sim}{x})$ vs. $\underset{\sim}{x}$

Figure E.7

x_2 \ x_1	-2	-1	0	1	2
1	2.	.7	.9	10.5	87.9
0	2.2	1.6	.8	.9	49.6
-1	37.3	4.	1.6	.7	45.8
-2	102.	37.3	2.2	2.	79.8

$G_2(\underset{\sim}{x})$ vs. $\underset{\sim}{x}$

Figure E.8

E.3 An Example where $s^{(2)}(\cdot)$ has Positive Slope below $\underset{\sim}{X}(2)$

Let $L(\cdot)$ be as pictured in Figure E.9 and let be uniform on the unit square. Assume that $p > 0$ and that $\alpha = 1$. Let $K = tp$ where $t < 1$. Hypothesis 4 is then satisfied, and $C_1(\cdot)$ is as pictured in Figure E.10. It can be shown that

$$D_2G_2(\underset{\sim}{x}) = \frac{1}{2+p}\left(-2-p-t^2p^2 - x_1^2(2+p)^2 - x_1(2+p)(2tp) + 4x_2^2 - x_2(2+p)\right)$$

in the shaded region on that figure.

If $p > 2$, it can be shown that $D_2G_2(\underset{\sim}{x}) < 0$ everywhere in the shaded region. Assume that $p=8$ and $t = 1/2$. Then $G_2(0,0) > 3.68 > 3 = G_2(1,1)$, so that $\underset{\sim}{X}(2) > (0,0)$. For $x_1 = x_2 > 0$, it can be shown that $G_2(\underset{\sim}{x}) \geq 3.68 - 6x + 4x^2$, so that $G_2(\underset{\sim}{x}) \geq 1.43$. Now $G_2(-.1,0) < 5 < 1.43 + K = 5.43 < G_2(\underset{\sim}{X}(2))$, so that $s^{(2)}(0) < -.1 <$ Since $s^{(2)}(\cdot)$ is continuous, it is in the shaded region for some $x_2 > 0$. Since $\underset{\sim}{X}(2) > (0,0)$, $D_2G_2(\cdot) < 0$ in this region implies that $s^{(2)}(\cdot)$ has positive slope at some point below $\underset{\sim}{X}(2)$.

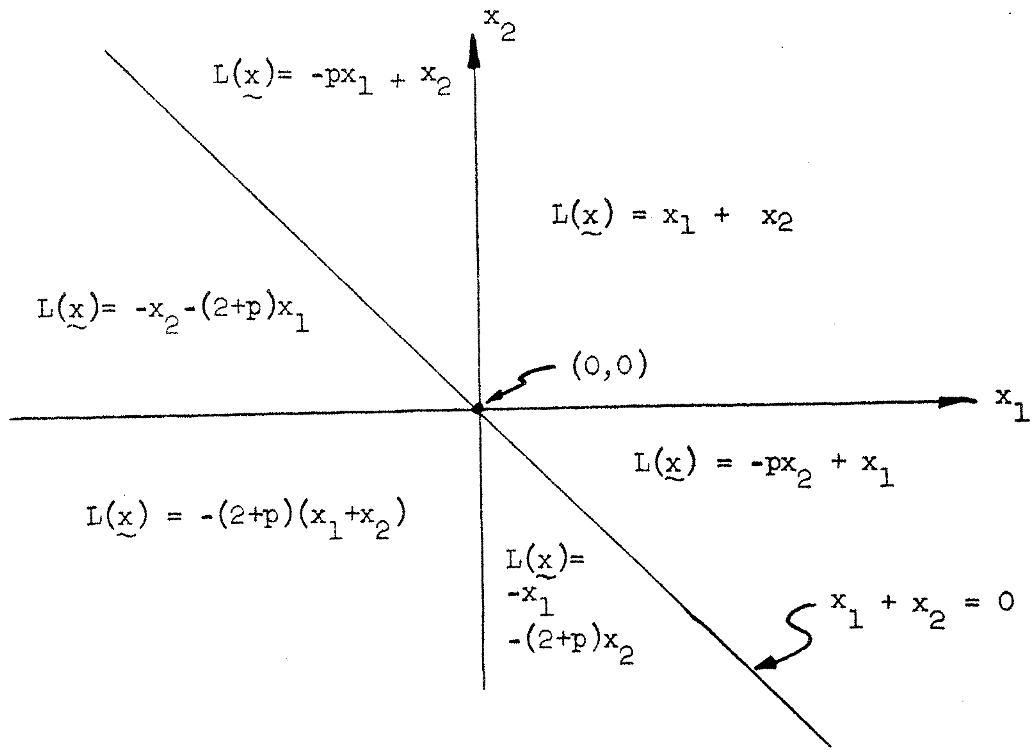


Figure E.9

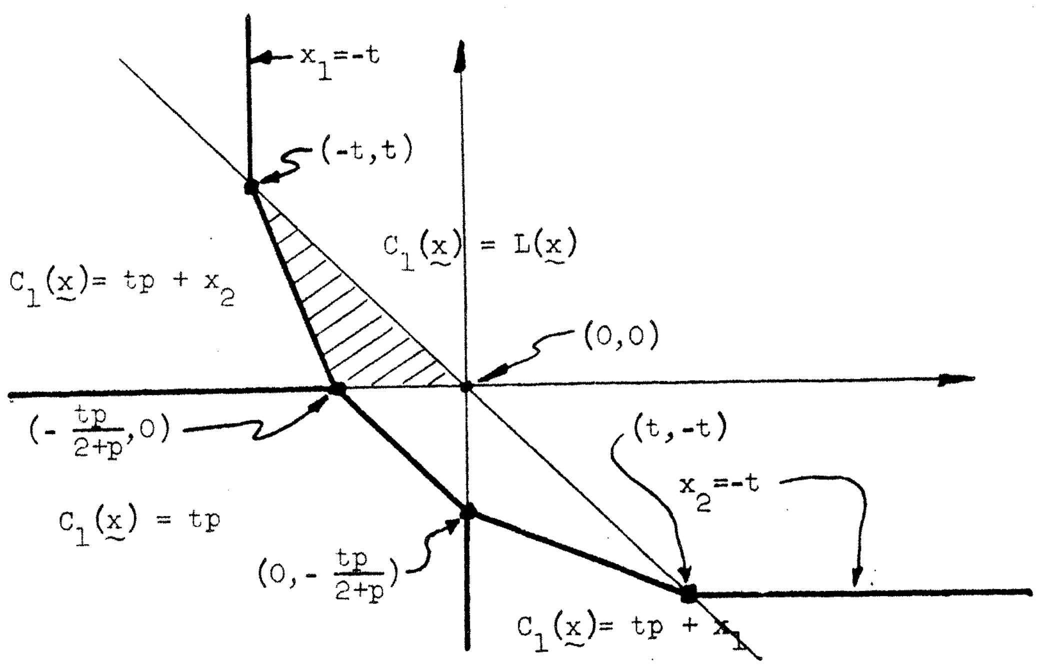


Figure E.10

Appendix F

Alternate Proof of Lemma 4.6

Since $J(\cdot)$ has been shown to be symmetric, it will suffice to show that $J(a) \geqq J(b)$ if $a \geqq b \geqq 0$.

$$J(a) - J(b) = \int_{\infty}^{\infty} g(a-x)\ f(x)dx - \int_{\infty}^{\infty} g(b-x)\ f(x)dx$$

$$= \int_{\infty}^{\infty} [g(a-x) - g(b-x)]f(x)dx.$$

If the real line is partitioned into four intervals,

$(-\infty,0)$ $[0, \frac{a+b}{2})$, $[\frac{a+b}{2}, a+b)$, and $[a+b, \infty)$, and if Q_1, Q_2, Q_3, and Q_4 are defined to be the values obtained by integrating over each interval, then $J(a) - J(b)$ can be written as $Q_1 + Q_2 + Q_3 + Q_4$. It will be shown that $Q_2 + Q_3 \geqq 0$ and $Q_1 + Q_4 \geqq 0$, thus implying $J(a) - J(b) \geqq 0$.

First consider $Q_2 + Q_3 =$

$$\int_0^{\frac{a+b}{2}} [g(a-x)-g(b-x)]f(x)dx + \int_{\frac{a+b}{2}}^{a+b} [g(a-x)-g(b-x)]f(x)dx.$$

Substituting $y = a+b-x$,

$$Q_3 = \int_0^{\frac{a+b}{2}} [g(y-b) - g(y-a)]\ f(a+b-y)\ dy$$

is obtained. Using the symmetry of $g(\cdot)$,

$$Q_3 = \int_0^{\frac{a+b}{2}} (-1)\ [g(a-y)-g(b-y)]\ f(a+b-y)\ dy.$$

Therefore

$$Q_2 + Q_3 = \int_0^{\frac{a+b}{2}} [g(a-y)-g(b-y)][f(y)-f(a+b-y)]\ dy.$$

Now $a \geq b \geq y$ implies that $g(a-y) \geq g(b-y)$. Further, $y \leq \frac{a+b}{2}$ implies that $y \leq a+b-y$ so that $f(y) \geq f(a+b-y)$. Consequently, the integrand is always non-negative, so $Q_2 + Q_3 \geq 0$.

Next consider $Q_1 + Q_4 =$

$$\int_{-\infty}^{0} [g(a-x)-g(b-x)]f(x)dx + \int_{a+b}^{\infty} [g(a-x)-g(b-x)]\ f(x)dx.$$

Substituting $y=a+b-x$,

$$Q_1 = \int_{a+b}^{\infty} [g(y-b)-g(y-a)]\ f(a+b-y)\ dy,$$

and, by the symmetry of $g(\cdot)$,

$$Q_1 = \int_{a+b}^{\infty} (-1)\ [g(a-y)-g(b-y)]\ f(a+b-y)\ dy.$$

Therefore $Q_1 + Q_4 = \int_{a+b}^{\infty} [g(a-y)-g(b-y)][f(y)-f(a+b-y)]\ dy.$

Now $y \geq a \geq b$ implies that $g(a-y) \leq g(b-y)$. Further, $y \geq a+b$

implies that $y \geqq |a+b-y|$, so that $f(y) \leqq f(a+b-y)$. Therefore the integrand, being the product of non-positive terms, is always non-negative, so $Q_1 + Q_4 \geqq 0$.

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Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Cornell University Department of Industrial Engineering and Operations Research - Ithaca, New York	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

OPTIMAL POLICIES FOR TWO PRODUCT INVENTORY SYSTEMS, WITH AND WITHOUT SETUP COSTS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report, August 1966

5. AUTHOR(S) (Last name, first name, initial)

Ignall, Edward J.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
August, 1966	168	29

8a. CONTRACT OR GRANT NO. Nonr-401(55) b. PROJECT NO. NR 047-055 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 13 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. AVAILABILITY/LIMITATION NOTICES

Releasable without limitations on dissemination

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Logistics and Mathematical Statistics Branch Office of Naval Research, Wash., D.C.

13. ABSTRACT

For the case of no setup cost, the structure of the transient portion of the optimal N period policy for two product inventory systems is obtained, the recurrent portion having been obtained by Veinott. The function L, the expected holding and shortage cost this period, is assumed to satisfy conditions roughly equivalent to (a) having a unique minimum and (b) being quasiconvex in each dimension with the relative minimum decreasing as the other value increases. The policy is: do not order the overstocked product and order less of the other product than would be ordered if there were no overstocked product. The policy is stationary, depending neither on N nor on the number of periods remaining.

For the setup cost case, the optimal N period policy is obtained under strong assumptions on L and the optimal one period policy under (a) and (b). Bounds analogous to those of Veinott and Wagner are obtained. The apparent inappropriateness of K-convexity is discussed.

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14. KEY WORDS	LINK A		LINK B		LINK C	
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