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LIFE-TESTING AND ESTIMATION WITH
ARBITRARY DISTRIBUTION FUNCTION

by

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Abstract. We consider the problem of estimating the life-distribution F from censored lifetimes. Two observation schemes are considered, namely renewal testing over a long time horizon and survival testing with repetitions. In each case we exhibit a product-limit estimator of F which is shown to be consistent and to converge weakly to a Gaussian process. To do this we first extend these properties of the Nelson-Aalen martingale estimator to the family of Poisson-type counting processes. Our proof of weak convergence is based on the general functional central limit theorems for semimartingales of Jacod et al. (1982). The result for renewal testing is new while for survival testing we give an alternative proof of weak convergence, to that of Gill (1980b), which does not rely on special constructions.

Key Words: Life-testing, renewal testing, product-limit estimator, censoring, martingale, Poisson type counting process.

1.   **Introduction.** Life-testing situations arise in various fields such as medical clinical trials, industrial reliability and biological experimentation. In general a lifetime is a positive random variable X with distribution function (df) F. This paper considers the problem of estimating F under two observation schemes, namely testing with replacement and testing without replacement. In the former, called renewal testing, one observes a renewal process $\{S_n, n \geq 0\}$ induced by the distribution F over a period of time $[0,\tau]$, say, $\tau > 0$. Lifetimes correspond to the interrenewal times $\{S_n - S_{n-1}, n \geq 1\}$. In testing without replacement one observes n independent lifetimes $X_1,...,X_n$, called survival times, each with df F.

We treat the problem of estimating the distribution function F when the observations are censored and when F is arbitrary. Our approach is, based on the general theory of counting processes and relies on their martingale dynamics. For each observation scheme we exhibit a product-limit estimator of F which is consistent and whose normalized difference converges weakly to a Gaussian process. These properties are established as n and τ tend to infinity in testing without replacement and renewal testing, respectively. To handle the generality of F we appeal to results on weak convergence of semimartingales in Jacod et al. (1982).

The problem of estimating the life-distribution in renewal testing is treated by Gill (1981) for the two cases where F is purely discrete and F is continuous. His observation scheme is to observe n independent copies of an uncensored renewal process over a fixed horizon $[0,\tau]$ and to let n increase to infinity. The case of random right censoring under the same observation scheme is covered by more general results in Gill (1980a). Our observation scheme has n = 1, random right censoring and τ increasing to infinity. These results are new and generalize the results of Moore and Pyke (1968) who treat the case $\tau \to \infty$ without censoring. Moreover, our approach shows that contrary to the discussion in section 8 of Andersen and Borgan (1985), the counting process methods they review do extend to renewal-type processes.

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The problem of estimating a general life-distribution is treated by Gill (1980b, 1983). We given an alternative proof of weak convergence to that of Gill (1980b) based on our Theorem 3.2. This proof has the appeal of being a direct generalization of methods used in connection with the well known multiplicative intensity model (see Aalen (1977, 1978)) and does not rely on any special construction such as the one employed by Gill (1980b). The method of proof is based on generalizations of results of Lipster and Shiryaev (1980) found in Jacod, et al. (1982), a fact which was anticipated by Gill (1980b) (see comments, p. 78), but not actually used by him.

The organization is as follows. Section 2 introduces Poisson type counting processes and their compensators whose pathwise Radon-Nikodym derivative relative to a fixed Borel measure is an observed predictable process. Section 3 introduces the martingale estimator of this Borel measure and develops its main asymptotic properties of consistency and weak convergence. Our results extend these properties of the Nelson-Aalen martingale estimator to the family of Poisson type counting processes. The weak convergence result is a novel application of the theory of Jacod et al. (1982) and is of independent interest. Section 4 applies these results to estimating general life-distributions.

2. Poisson Type Counting Processes and Life-Testing

We define the family of Poisson type counting processes giving examples from life-testing. Let $(\Omega, \mathcal{F}, P)$ denote a probability space and $H = \{\mathcal{H}_t, t \geq 0\}$ a given family of sub-σ-algebras of $\mathcal{F}$ such that H is nondecreasing, right continuous and complete relative to P. For definitions of some standard terminology, such as adapted, predictable or compensator, we refer the reader to, for example, Metevier (1982) and Lipster and Shiryaev (1978).

Let $N = \{N_t, t \geq 0\}$ denote a counting process defined on $(\Omega, \mathcal{F}, P)$ and adapted to $\{\mathcal{H}_t, t \geq 0\}$ so that the sample paths of N are right continuous step functions with jumps of size $+1$, $N_0 = 0$ and for each $t \geq 0$ N_t is an $\mathcal{H}_t$-measurable random variable. Let B denote a

fixed Borel measure over the Borel sets in $\mathbb{R}_+ = [0, \infty)$ and let $Y = \{Y_t, t \geq 0\}$ denote a nonnegative $\mathcal{H}_t$-predictable process. Suppose N has compensator $A = \{A_t, t \geq 0\}$ relative to H given by

$$A_t = \int_{(0,t]} Y_s B\{ds\} , \qquad (2.1)$$

then in the terminology of Lipster and Shiryaev (1978) N is called a *Poisson type counting process*. Note that if Y_t is a constant $\lambda > 0$ for all $t \geq 0$ and B denotes the Lebesgue measure, then N is a simple Poisson point process. By definition the process $M = \{M_t, t \geq 0\}$ given by

$$M_t = N_t - A_t \qquad (2.2)$$

is a local square integrable martingale relative to H and, in the terminology of Jacod (1975), the kernel A{dt] is the dual predictable projection to N. When A is pathwise absolutely continuous (i.e. $A\{dt\} = Y_t b(t)dt$ where dt is Lebesgue measure and b is a nonnegative rate function) then the model (2.1) gives rise to the multiplicative intensity family of counting processes considered by Aalen (1978).

Model 1. Renewal Testing. Let $(\Omega, \mathcal{F}, P)$ denote a probability space on which a renewal process $\{S_n, n \geq 0\}$ induced by the distribution function F is defined. For each $n \geq 1$ define $X_n = S_n - S_{n-1}$ so that $\{X_n, n \geq 1\}$ denote the interrenewal times or successive lifetimes in testing with replacement. Define $H = \{\mathcal{H}_t, t \geq 0\}$ by

$$\mathcal{H}_t = \mathcal{A} \vee \mathcal{N} \vee \sigma(X_n \leq s, s \leq t, n \geq 1) \qquad (2.3)$$

where $\mathcal{N}$ contains the P-null sets of $\mathcal{F}$ and $\mathcal{F} \supseteq \mathcal{A}$ is independent of $\sigma(X_n, n \geq 1)$. Suppose there is a sequence of nonnegative random variables $\{U_n, n \geq 1\}$ defined on $(\Omega, \mathcal{F}, P)$ such that for each n U_n is a stopping time relative to H.

For each $n \geq 1$ define $N(n) = \{N_t(n), t \geq 0\}$ and $A(n) = \{A_t(n), t \geq 0\}$ by

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$$N_t(n) = 1(X_n \le t) \qquad (2.4)$$

$$A_t(n) = \int_{(0,t]} 1(X_n \ge s)\, B\{ds\} \qquad (2.5)$$

where $B(t) = \int_0^t (1 - F(s-))^{-1} F\{ds\}$, $t \ge 0$. It is an easy exercise to show that $N(n)$ has compensator $A(n)$ relative to H. Define $K(n) = \{K_t(n), t \ge 0\}$ by

$$K_t(n) = 1(U_n \ge t) . \qquad (2.6)$$

The process $K(n)$ is bounded and predictable, and plays the role of right censoring for the nth interrenewal time X_n. Thus if we define the censored processes

$$\tilde{N}_t(n) = \int_0^t K_s(n) dN_s(n) = 1(X_n \wedge U_n \le t,\, X_n \le U_n), \quad t \ge 0 \qquad (2.7)$$

$$\tilde{A}_t(n) = \int_0^t K_s(n) dA_s(n) = \int_0^t 1(X_n \wedge U_n \ge s)\, B\{ds\}, \quad t \ge 0 \qquad (2.8)$$

where $a \wedge b = \min(a,b)$, then Theorem 18.7 of Lipster and Shiryaev (1978) implies that $\{\tilde{N}_t(n), t \ge 0\}$ has compensator $\{\tilde{A}_t(n), t \ge 0\}$ relative to H, and therefore is of the Poisson type. ¤

Model 2. Testing without replacement and random censorship. For each $n \ge 1$ X_i and $U_i{}^n$, $i = 1,...,n$ are $2n$ independent random variables. X_i has distribution function F and $U_i{}^n$ has (sub)-distribution function $L_i{}^n$. The observable random variables $\tilde{X}_i{}^n$ and $\delta_i{}^n$ are given by $\tilde{X}_i{}^n = X_i \wedge U_i{}^n$ and $\delta_i{}^n = 1(X_i \le U_i{}^n)$. Thus $\tilde{X}_i{}^n$ are right censored lifetimes and $1 - \delta_i{}^n$ is an indicator of censoring. We assume without loss of generality below (cf. Rebolledo (1980)) that for each $n \ge 1$, $X_1,...,X_n, U_1{}^n,...,U_n{}^n$ are defined on the same probability space $(\Omega, \mathcal{F}, P)$.

For each $n \ge 1$ define $H^n = \{\mathcal{H}_t{}^n, t \ge 0\}$ by

$$\mathcal{H}_t{}^n = \mathcal{N} \vee \sigma(\tilde{X}_i{}^n \le s,\, \delta_i{}^n 1(\tilde{X}_i{}^n \le t),\quad s \le t,\, i = 1,...,n) \qquad (2.9)$$

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where $\mathbf{N}$ contains the P-null sets of $\mathcal{F}$ and their subsets. For $i = 1,...,n$ define the H^n-adapted processes $N^n(i) = \{N_t^n(i), t \geq 0\}$, $Y^n(i) = \{Y_t^n(i), t \geq 0\}$ and $M^n(i) = \{M_t^n(i), t \geq 0\}$ by

$$N_t^n(i) = 1(\tilde{X}_i^n \leq t, \delta_i^n = 1) \tag{2.10}$$

$$Y_t^n(i) = 1(\tilde{X}_i^n \geq t) \tag{2.11}$$

$$M_t^n(i) = N_t^n(i) - \int_{(0,t]} Y_s^n(i) \, B\{ds\} \tag{2.12}$$

where B is defined as in model 1. It is a consequence of Theorem 3.1.1 of Gill (1980b) that $M^n(1),...,M^n(n)$ are orthogonal square integrable martingales relative to H^n so that for each i, $N^n(i)$ has compensator $A^n(i) = \{A_t^n(i), t \geq 0\}$ given by $N_t^n(i) - M_t^n(i)$. Thus $N^n(i)$ is of the Poisson type. It is worth pointing out that if we aggregate (2.12) over i the resulting sum is still a martingale. However, the term given by aggregation over i of (2.10) is not in general a simple counting process as multiple points occur with positive probability whenever F has points of positive mass.   ¤

Models 1 and 2 are examples of random right censorship of lifetimes. More general censoring patterns may be considered as in Chapter 6 of Gill (1980b). We note that unfortunately our approach to censoring in model 1 is not general enough to include the sequential decision procedure of Bather (1977), but will include sequential procedures which give rise to H-stopping times.

3.   Estimation from Poisson Type Counting Processes.

This section is devoted to estimating the Borel measure B from observations of Poisson type counting processes. The results of this section are of independent interest and are used in section 4 to solve the main problem of estimating a life-distribution F.

For $n \geq 1$, $H^n = \{\mathcal{H}_t^n, t \geq 0\}$ denotes a given filtration and let $N^n = \{N_t^n, t \geq 0\}$ denote a Poisson type counting process with compensator $A^n = \{A_t^n, t \geq 0\}$ defined on a

probability space $(\Omega, \mathcal{F}, P)$ (cf. Rebolledo (1980), pp. 271). Let B denote a fixed Borel measure on $(\mathbb{R}_+, \sigma(\mathbb{R}_+))$ and let $Y^n = \{Y_t^n, t \geq 0\}$ denote the pathwise Radon-Nikodym derivative of A^n relative to B(i.e. for $t \geq 0$ $A_t^n = {}_0\!\int^t Y_s^n \, B\{ds\}$). In view of our applications we allow for N^n to have multiple points but these may only occur with positive probability on $\{t: \Delta B(t) > 0\}$ (cf. Brown (1978)). Here as below $\Delta B(t) = B(t) - B(t-)$ for all $t \geq 0$.

For each n assume that the bivariate process (N^n, Y^n) is observable over a period of time and consider the problem of estimating B. Define the empirical process $\hat{B}^n = \{\hat{B}_t^n, t \geq 0\}$ by the Stieltjes integral

$$\hat{B}_t^n = \int_0^t X_s^n \, dN_s^n \tag{3.1}$$

where $X_s^n = (Y_s^n)^{-1} \, 1(Y_s^n > 0)$ $(0/0 = 0$ by convention). Consider the process $M^n = \{M_t^n, t \geq 0\}$ given by the Stieltjes integral

$$M_t^n = \int_0^t X_s^n (dN_s^n - dA_s^n). \tag{3.2}$$

If for $t \geq 0$, $B(t) < \infty$ or more generally ${}_0\!\int^t X_s^n \, dA_s^n < \infty$, then by Theorem 18.7 of Lipster and Shiryaev (1978), M^n is a local martingale relative to H^n. Thus $\hat{B}^n$ is called the martingale estimate of B and is an extension of the so-called Nelson-Aalen estimator to the family of Poisson type counting processes (cf. Andersen and Borgan (1985)). We assume below that M^n is a locally square integrable local martingale on $[0, \infty)$ with quadratic variation $<M^n> = \{<M^n>_t, t \geq 0\}$. In our applications ${}_0\!\int^t X_s^n \, B\{ds\} < \infty$ $(t \geq 0)$ is sufficient for M^n to be locally square integrable (Theorem 18.8, Lipster and Shiryaev (1978)) and $<M^n>$ will be explicitly calculated. In Theorem 3.1 below we show that $\hat{B}^n$ is a consistent estimator of B.

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Theorem 3.1.   Consistency. For $n \geq 1$, suppose $M^n = \{M_t^n, t \geq 0\}$ is a locally square integrable local martingale. For $t \geq 0$ suppose that as $n \to \infty$

a) $\displaystyle \int_0^t 1(Y_s^n = 0)B\{ds\} \underset{p}{\to} 0$,

and

b) $\displaystyle <M^n>_t \underset{p}{\to} 0$.

Then $\displaystyle \sup_{s \leq t} |\hat{B}_s^n - B(s)| \underset{p}{\to} 0$ as $n \to \infty$.

Proof. Let $t \geq 0$ be such that (a) and (b) hold and observe that for each $s \geq 0$

$$\hat{B}_s^n - B(s) = M_s^n + [\int_0^s 1(Y_u^n > 0) \, B\{du\} - B(s)] .$$

By (a) the second term in brackets converges to zero in probability uniformly in $[0,t]$ as n tends to infinity. Next for $n \geq 1$, let $\{\tau_k^n, k \geq 1\}$ denote a localizing sequence of H^n-stopping times for M^n. Then for $k \geq 1$ and $n \geq 1$ the Lenglart-inequality (Lenglart (1977)) implies that for $a > 0$, $b > 0$

$$P(\sup_{s \leq t \wedge \tau_k^n} |M_s^n| > a^{1/2}) \leq \frac{b}{a} + P(<M^n>_{t \wedge \tau_k^n} \geq b) .$$

Since M^n is a local martingale on $[0,\infty)$, $\tau_k^n \to \infty$ almost surely as $k \to \infty$. Therefore the theorem is proved upon letting k increase to infinity and invoking condition (b).   ¤

We now turn in Theorem 3.2 below to the problem of the asymptotic distribution theory of the martingale estimator (3.1), where it is shown that the normalized difference converges weakly to a Gaussian process of independent increments.   Let $\{t_k, k \geq 1\} = \{t: \Delta B(t) > 0\}$ and let $X = \{X_t, t \geq 0\}$ denote a Gaussian process having covariance function

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$$<X>(t) = G^c(t) + \sum_{k:\, t_k \le t} \sigma_k^2, \quad t \ge 0 \qquad (3.3)$$

where G^c denotes a continuous function and $\{\sigma_k^2, k \ge 1\}$ are positive constants. Let $\{a_n, n \ge 1\}$ denote a sequence of nonnegative constants increasing to infinity as n increases. For each $n \ge 1$ define the process $Z^n = \{Z_t^n, t \ge 0\}$ by

$$Z_t^n = a_n (Y_t^n)^{-1} \, 1(Y_t^n > 0) . \qquad (3.4)$$

Theorem 3.2. Weak convergence. For $n \ge 1$ suppose $M^n = \{M_t^n, t \ge 0\}$ is a locally square integrable local martingale and that the following conditions hold. As $n \to \infty$,

(a) $\quad a_n \displaystyle\int_0^t 1(Y_s^n = 0)B\{ds\} \xrightarrow[p]{} 0$,

(b) $\quad$ For $t \ge 0$ and all $\varepsilon \in (0,1]$,

$$\int_0^t (Z_s^n)^2 \, 1(Z_s^n > \varepsilon) \, dA_s^{nc} \xrightarrow[p]{} 0 ,$$

where $A^{nc} = A^n - \Sigma \Delta A^n$ denotes the continuous part of A^n;

(c) $\quad$ For $t \ge 0$, $\displaystyle\int_0^t (Z_s^n)^2 \, dA_s^{nc} \xrightarrow[p]{} G^c(t)$,

(d) $\quad$ For $k \ge 1$ and $u \in \mathbb{R}$, $E(e^{iuW_k^n} | \mathcal{H}_{t_{k^-}}^n) \xrightarrow[p]{} e^{-u^2 \sigma_k^2 / 2}$,

where $W_k^n = Z_{t_k}^n (\Delta N_{t_k}^n - \Delta A_{t_k}^n)$. Here $i = \sqrt{-1}$.

(e) $\quad$ For $t \ge 0$, $\ell = 1,2$ $\displaystyle\sum_{k:\, t_k \le t} (a_n^2 \Delta <M^n>_{t_k})^\ell \xrightarrow[p]{} \sum_{k:\, t_k \le t} (\sigma_k^2)^\ell$.

Let $\mathbb{D}$ denote the space of functions from $\mathbb{R}_+$ into $\mathbb{R}$; right-continuous with limits from the left. Then as $n \to \infty$, $\{a_n(\hat{B}_t^n - B(t)), t \ge 0\}$ converges weakly to X in $\mathbb{D}$ endowed with the

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Skorohod topology (see Billingsley (1968)), where X is a Gaussian process having variance function (3.3).

Proof. See appendix. ¤

Theorems 3.1 and 3.2 extend the properties of consistency and asymptotic normality of the martingale estimator to the family of Poisson type counting processes. In our case the limiting Gaussian process has fixed points of discontinuity on $\{t_k, k \geq 1\}$. For the purpose of data analysis or inference from Poisson type counting processes plots of $\hat{B}^n$ and confidence bands for B are particularly easy to construct (see Andersen and Borgan (1985) for a discussion along these lines). To construct confidence bands we need a consistent estimator of the asymptotic variance function <X>. This is illustrated in our application below.

4. Estimating the Life-distribution.

The general results of section 3 are now applied to our main problem of estimating the life-distribution from censored lifetimes. We begin with some preliminaries.

Let F and B be defined as in model 1, and let y denote a nonnegative monotone decreasing function on $\mathbb{R}_+$ with $I = \{t: y(t) > 0\}$. Then define a function G by

$$G(t) = \int_0^t y(s)^{-1}(1 - \Delta B(s))B\{ds\} , \quad t \in I \qquad (4.1)$$

and a function L by

$$L(t) = \int_0^t 1(\Delta B(s) < 1)(1 - \Delta B(s))^{-2} G\{ds\} , \quad t \in I . \qquad (4.2)$$

Let $X = \{X_t, t \in I\}$ and $Z = \{Z_t, t \in I\}$ denote zero-mean Gaussian processes with variance functions G and L, respectively. Note that X and Z will have fixed points of discontinuity whenever B has jumps.

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4.1. Renewal Testing. Consider model 1, section 2, which describes testing with replacement and random censorship. Let $\{S_n, n \geq 0\}$ denote the underlying renewal process and define the renewal counting process $\pi = \{\pi_t, t \geq 0\}$ by

$$\pi_t = \sum_{n=1}^{\infty} 1(S_n \leq t) . \qquad (4.1.1)$$

Also define the process $K = \{K_t, t \geq 0\}$ by

$$K_t = K_{t-S_{n-1}} (n) \quad \text{if} \quad S_{n-1} \leq t < S_n \qquad (4.1.2)$$

where $K(n)$ is defined by (2.6). For each $\tau > 0$ define the *observable* processes $Y(\tau) = \{Y_t(\tau), t \geq 0\}$, $N(\tau) = \{N_t(\tau), t \geq 0\}$ and $\tilde{\pi}(\tau)$ by

$$\tilde{\pi}(\tau) = \int_0^{\tau} K_s d\pi_s \qquad (4.1.3)$$

$$Y_t(\tau) = \sum_{n=1}^{\pi_\tau} 1(X_n \wedge U_n \geq t) \qquad (4.1.4)$$

$$N_t(\tau) = \sum_{n=1}^{\pi_\tau} \tilde{N}_t (n) \qquad (4.1.5)$$

where $\tilde{N}_t(n)$ and U_n are defined in model 1 (see (2.6) and (2.7)).

Consider the empirical processes $\hat{B}(\tau) = \{\hat{B}_t(\tau), t \geq 0\}$ and $\hat{F}(\tau) = \{\hat{F}_t(\tau), t \geq 0\}$ defined by

$$\hat{B}_t(\tau) = \int_0^t X_s(\tau) dN_s(\tau) \qquad (4.1.6)$$

where $X_s(\tau) = (Y_s(\tau))^{-1} 1(Y_s(\tau) > 0)$, and

$$\hat{F}_t(\tau) = 1 - \prod_{s \leq t} (1 - \Delta \hat{B}_s(\tau)) . \qquad (4.1.7)$$

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We propose $\hat{B}(\tau)$ and $\hat{F}(\tau)$ as an estimator of B and F, respectively. These estimators extend the Nelson-estimator and the product-limit estimator to this observation scheme. Note that by Theorem 2.1 of Prabhu (1965) π_τ is finite almost surely so we ignore the null event $\{\tilde{\pi}(\tau) = \pi_\tau = \infty\}$.

Theorem 4.1.1. Consistency. Suppose that $\mu = {}_0\!\int^\infty (1 - F(y))dy < \infty$ and let $t > 0$ such that $Y_t(\tau)$ converges to infinity in probability as $\tau \to \infty$, then

(a) $\quad \sup_{s \le t} |\hat{B}_s(\tau) - B(s)| \underset{p}{\to} 0 , \quad \tau \to \infty ,$

(b) $\quad \sup_{s \le t} |\hat{F}_s(\tau) - F(s)| \underset{p}{\to} 0 , \quad \tau \to \infty .$

Proof. For $\tau > 0$ let $\Gamma = \Gamma(\tau) = [\tau/\mu]$ denote the largest integer in τ/μ and define $Y(\Gamma) = \{Y_t(\Gamma), t \ge 0\}$ and $N(\Gamma) = \{N_t(\Gamma), t \ge 0\}$ by substituting Γ for π_τ in (4.1.4) and (4.1.5), respectively. Let $H = \{\mathcal{H}_s, s \ge 0\}$ denote the history defined by (2.3) of model 1. It is obvious that relative to H, $N(\Gamma)$ is a Poisson type counting process with fixed Borel measure B and predictable auxiliary process $Y(\Gamma)$. Thus let $\hat{B}(\Gamma) = \{\hat{B}_s(\Gamma), s \ge 0\}$ denote the martingale estimator of B derived according to equation (3.1) from $N(\Gamma)$ and $Y(\Gamma)$. Next observe that for $0 \le s \le t$, $Y_s(\tau)$ is monotone decreasing so that

$$|\hat{B}_s(\Gamma) - \hat{B}_s(\tau)| \le |\pi_\tau - \Gamma| X_t(\tau)[1 + \hat{B}_t(\Gamma)]$$

where $X_t(\tau) = (Y_t(\tau))^{-1} 1(Y_t(\tau) > 0)$, and recall that π_τ/τ converges to μ^{-1} almost surely as $\tau \to \infty$. Moreover, our assumption implies that $F(t-) < 1$ and $B(t) < \infty$, and that $X_t(\tau)$ converges to zero in probability as $\tau \to \infty$. Therefore to prove (a) it suffices to prove the same result for $\hat{B}(\Gamma)$.

Let $M(\Gamma) = \{M_s(\Gamma), s \ge 0\}$ denote the square integrable martingale defined by substituting $N(\Gamma)$ and $Y(\Gamma)$ for N^n and Y^n in equation (3.2). According to Theorem 3.1, to prove the result (a) for $\hat{B}(\Gamma)$ it suffices to show that as $\tau \to \infty$

$$\int_0^t 1(Y_s(\Gamma) = 0)B\{ds\} \le 1(Y_t(\Gamma) = 0)B(t) \underset{p}{\to} 0 \,,$$

and

$$<M(\Gamma)>_t = \int_0^t X_s(\Gamma)(1 - \Delta B(s))B\{ds\} \underset{p}{\to} 0 \,,$$

where $X_s(\Gamma) = (Y_s(\tau))^{-1} 1(Y_s(\tau) > 0)$ $(0 \le s \le t)$. But $B(t) < \infty$ and our assumption together with the monotonicity of $Y(\tau)$ implies that as $\tau \to \infty$ $1(Y_s(\Gamma) = 0)$ and $X_s(\Gamma)$ converge to zero in probability uniformly over $[0,t]$. This implies the desired result.

To prove (b) we note that since $F(t-) < 1$, Lemma 3.2.1 of Gill (1980b) implies that

$$\sup_{s<t} |\hat{F}_s(\tau) - F(s)| \le (1 - F(t-))^{-1} \int_0^t |d\hat{B}_u(\tau) - dB(u)| \,,$$

and moreover that

$$\Delta \hat{F}_t(\tau) - \Delta F(t) = (1 - \hat{F}_{t-}(\tau)) \Delta \hat{B}_t(\tau) - (1 - F(t-)) \Delta B(t) \,.$$

Thus the result (b) easily follows from (a) and Lemma 2 of Gill (1980a). ¤

Theorem 4.1.2. Weak Convergence. Suppose that $\mu = \int_0^\infty (1-F(y))dy < \infty$ and that as $\tau \to \infty$,

   (a) $\tilde{\pi}(\tau)/\pi_\tau \underset{p}{\to} \theta > 0 \,,$

   (b) $\sup_s |Y_s(\tau)/\tilde{\pi}(\tau) - y(s)| \underset{p}{\to} 0 \,,$

   (c) $\sup_s |\tilde{\pi}(\tau)^{1/2} (Y_s(\tau))^{-1} 1(Y_s(\tau) > 0)| \underset{p}{\to} 0 \,.$

Let $\mathbb{D}(E)$ denote the space of right-continuous functions from E into $\mathbb{R}$ with limits from the left, $I = \{t: y(t) > 0\}$, and $\sigma = \sup\{t: y(t) > 0\}$. Consider the processes $W(\tau) = \{\tilde{\pi}(\tau)^{1/2}(\hat{B}_t(\tau) - B(t)), 0 \le t < \sigma\}$ and $Z(\tau) = \{\tilde{\pi}(\tau)^{1/2}(\hat{F}_t(\tau) - F(t))1(F(t) < 1) / (1 - F(t)), t \in I\}$. Then conditions (a), (b), and (c) imply that as $\tau \to \infty$, $W(\tau)$ and $Z(\tau)$

converge weakly to X and Z1(F < 1) in $\mathbb{D}([0,\sigma))$ and $\mathbb{D}(I)$, respectively, endowed with the Skorohod topology. If $\sigma \in I$ and $F(\sigma) < 1$ then the result for $W(\tau)$ extends to all of I.

Proof. We have that for $t < \sigma$, $F(t) < 1$ so that for all $t \in I$

$$\frac{\hat{F}_t(\tau) - F(t)}{1 - F(t)} 1(F(t) < 1) = \left[\int_0^t \frac{1 - \hat{F}_{s-}(\tau)}{1 - F(s-)} \frac{1(\Delta B(s) < 1)}{1 - \Delta B(s)} (d\hat{B}_s(\tau) - dB(s)) \right] 1(F(t) < 1)$$

where if $\sigma \in I$ and $F(\sigma) = 1$ the expressions above are set equal to zero. By virtue of Theorem 4.1.1, conditions (a) and (b) imply that $(1-\hat{F}_{s-}(\tau))/(1-F(s-))$ converges to one in probability uniformly over I as $\tau \to \infty$. Thus to prove the desired result for $Z(\tau)$ it suffices to prove the desired result for $W(\tau)$ (see Corollary A.1 of the appendix).

Since $\mu < \infty$, condition (a) implies that there is a constant, say, $\rho > 0$ such that as $\tau \to \infty$, $\tilde{\pi}(\tau)/\tau$ converges to ρ in probability. For $\tau > 0$ define $\Lambda = \Lambda(\tau) = [\rho\tau]$ and let $\hat{B}(\Gamma) = \{\hat{B}_t(\Gamma), t \geq 0\}$ be the martingale estimator of B defined in the proof of Theorem 4.1.1. Consider the normalized difference $W(\Gamma) = \{\Lambda^{1/2}(\hat{B}_t(\Gamma) - B(t)), 0 \leq t < \sigma\}$ and observe that for $0 \leq t < \sigma$

$$|W_t(\Gamma) - W_t(\tau)| \leq |\tilde{\pi}(\tau)^{1/2} - \Lambda^{1/2}| \; |B(t) - \hat{B}_t(\tau)| + \Lambda^{1/2} X_t(\tau)[1 + \hat{B}_t(\Gamma)] |\pi_\tau - \Gamma|$$

where $X_t(\tau) = (Y_t(\tau))^{-1} 1(Y_t(\tau) > 0)$. Recall that $y(t) > 0$ for $0 \leq t < \sigma$, so that conditions (a) and (b) imply that $Y_t(\tau)$ converges to infinity uniformly on $[0,\sigma)$ as $\tau \to \infty$. Thus, by Theorem 4.1.1, $\hat{B}_t(\Gamma)$ and $\hat{B}_t(\tau)$ converge to B(t) in probability uniformly on $[0,\sigma)$ as $\tau \to \infty$. Moreover, conditions (a) and (c) imply that $\Lambda^{1/2} X_t(\tau)$ converges to zero uniformly as $\tau \to \infty$. Finally, as $B(t) < \infty$ for $0 \leq t < \sigma$, it follows that, in particular, $|W_t(\Gamma) - W_t(\tau)|$ converges to zero uniformly on $[0,\sigma)$ as $\tau \to \infty$. Therefore, to prove the desired result for $W(\tau)$ it suffices to prove the result for $W(\Gamma)$. Note that if $\sigma \in I$ and $F(\sigma) < 1$ the argument above extends to all of I.

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For any function f let f^c denote the continuous part of f. To prove weak convergence of $W(\Gamma)$ we identify Z^n of equation (3.4) with $Z^\Gamma = \{Z_t^\Gamma, t \geq 0\}$ given by

$$Z_t^\Gamma = \Lambda^{1/2}(Y_t(\Gamma))^{-1} 1(Y_t(\Gamma) > 0) = \Lambda^{1/2} X_t(\Gamma)$$

and verify the conditions of Theorem 3.2. Direct calculation reveals that conditions (a), (b), (c), and (e) of Theorem 3.2 are equivalent, respectively, to the following four conditions: for $0 \leq t < \sigma$ and as $\tau \to \infty$

(1) $\displaystyle \Lambda^{1/2} \int_0^t 1(Y_s(\Gamma) = 0) \, B\{ds\} \xrightarrow{p} 0$,

(2) $\displaystyle \Lambda \int_0^t X_s(\Gamma) \, 1(Z_s(\Gamma) > \varepsilon) \, dB^c(s) \xrightarrow{p} 0$,

(3) $\displaystyle \Lambda \int_0^t X_s(\Gamma) dB^c(s) \xrightarrow{p} G^c(t)$,

where G is defined by (4.1), and for $\mathcal{L} = 1,2$

(4) $\displaystyle \sum_{k: \, t_k \leq t} [\Lambda X_{t_k}(\Gamma)(1 - \Delta B(t_k))\Delta B(t_k)]^{\mathcal{L}} \xrightarrow{p} \sum_{k: \, t_k \leq t} [y(t_k)^{-1}(1 - \Delta B(t_k))\Delta B(t_k)]^{\mathcal{L}}$

where $t_k \in \{t: \Delta B(t) > 0\}$. It is an easy exercise to show that conditions (b) and (c) of Theorem 4.1.2 hold if we replace $Y_s(\tau)$ and $\tilde{\pi}(\tau)$ with $Y_s(\Gamma)$ and Λ, respectively. Thus, condition (a) together with the uniformity of conditions (b) and (c) of Theorem 4.1.2 directly imply (1) - (4) above. Therefore, it remains to verify condition (d) of Theorem 3.2.

Consider the history $H = \{\mathcal{H}_t, t \geq 0\}$ defined by (2.3), and let $t_k \in \{t: \Delta B(t) > 0\}$ with $t_k < \sigma$. By virtue of the independence of the interrenewal times it follows that conditional on $\mathcal{H}_{t_k-}$, $\Delta N_{t_k}(\Gamma)$ is a binomial random variable with parameter $Y_{t_k}(\Gamma)$ and probability $\Delta B(t_k)$ (here the fact that $Y_{t_k}(\Gamma) \in \mathcal{H}_{t_k-}$ is used). Moreover, when $Y_{t_k}(\Gamma) > 0$ the random variable W_k^Γ in (d) of Theorem 3.2 is given by

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$$W_k^{\Gamma} = \Lambda^{1/2}(\Delta \hat{B}_{t_k}(\Gamma) - \Delta B(t_k)) = \Lambda^{1/2} \left(\frac{\Delta N_{t_k}(\Gamma)}{Y_{t_k}(\Gamma)} - \Delta B(t_k) \right).$$

Therefore, as a consequence of condition (b) of Theorem 4.1.2 and the Gaussian approximation to binomial sampling, it follows that the conditional law of $\Lambda^{1/2}(\Delta \hat{B}_{t_k}(\Gamma) - \Delta B(t_k))$ given $\mathcal{H}_{t_k-}$ converges to a Gaussian law with mean zero and variance $(1 - \Delta B(t_k)) \Delta B(t_k)/y(t_k)$ in probability as $\tau \to \infty$. As this verifies condition (d) of Theorem 3.2, the desired result for $W(\Gamma)$ and hence $W(\tau)$ now follows. Again if $\sigma \in I$ and $F(\sigma) < 1$ all the arguments above extend to all of I. This completes our proof. ¤

In practice the asymptotic variance functions G and L defined by (4.1) and (4.2), respectively, will be unknown. It turns out that under the assumptions of the theorem these have consistent estimators given by

$$\hat{G}_t = \tilde{\pi}_{\tau} \int_0^t Y_s^{-1}(\tau)(1 - \Delta \hat{B}_s(\tau))d\hat{B}_s(\tau) , \quad t \geq 0 , \tag{4.1.8}$$

and

$$\hat{L}_t = \tilde{\pi}_{\tau} \int_0^t (Y_s(\tau) - \Delta N_s(\tau))^{-1} 1(\Delta N_s(\tau) < Y_s(\tau))d\hat{B}_s(\tau) , \quad t \geq 0 . \tag{4.1.9}$$

4.2. Testing Without Replacement.

Consider the setup of model 2, section 2 which describes testing without replacement and random censorship. For $n \geq 1$ let $\{\mathcal{H}_t^n, t \geq 0\}$, $N(i)$, $Y(i)$ and $M(i)$ be defined by (2.9)-(2.12), respectively, $i = 1,...,n$. Next define the aggregate process $\bar{N} = \{\bar{N}_t, t \geq 0\}$, $\bar{Y} = \{\bar{Y}_t, t \geq 0\}$ and $\bar{M} = \{\bar{M}_t, t \geq 0\}$ by summing (2.10)-(2.12), respectively, over i. Then $\bar{N}$ is a Poisson type counting process in the extended sense of section 3 so we define the martingale estimator $\hat{B}^n = \{\hat{B}_t^n, t \geq 0\}$ of B by

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$$\hat{B}^n_t = \int_0^t X^n_s \, d\bar{N}_s \qquad (4.2.1)$$

where $X_s{}^n = (\bar{Y}_s)^{-1} 1(\bar{Y}_s > 0)$; the dependence of $\bar{Y}$ and $\bar{N}$ on n is implicit. The estimator $\hat{B}^n$ is the Nelson-estimator of the cumulative hazard and is used to define the product-limit estimator $\hat{F}^n = \{\hat{F}_t{}^n, t \geq 0\}$ by

$$\hat{F}_t{}^n = 1 - \prod_{s \leq t} (1 - \Delta\hat{B}_s{}^n) . \qquad (4.2.2)$$

Recall that if $t > 0$ such that $\bar{Y}_t$ converges to infinity in probability as $n \to \infty$, then Theorem 4.1.1 of Gill (1980b) implies that $\hat{B}^n$ and $\hat{F}^n$ converge uniformly over $[0,t]$ to B and F, respectively, as $n \to \infty$. We give an alternative proof of weak convergence to that of Theorem 4.2.2 of Gill (1980b) which is based on our Theorem 3.2.

Theorem 4.2.1. Weak convergence of the Nelson-estimator and the product-limit estimator. Let the space $\mathbb{D}$ be defined as in Theorem 4.1.2 and suppose that $\bar{Y}/n$ converges uniformly on $[0,\infty)$ to a function y in probability as $n \to \infty$. Let $\sigma = \sup\{t: \quad y(t) > 0\}$ and $I = \{t: \quad y(t) > 0\}$, and consider the processes $W(n) = \{n^{1/2}(\hat{B}_t{}^n - B(t)), 0 \leq t < \sigma\}$ and $Z(n) = \{n^{1/2}(\hat{F}_t{}^n - F(t))1(F(t) < 1)/(1 - F(t)),$ $t \in I\}$. Then as $n \to \infty$, $W(n)$ and $Z(n)$ converge weakly to X and $Z1(F < 1)$ in $\mathbb{D}([0,\sigma))$ and $\mathbb{D}(I)$, respectively, endowed with the Skorohod topology.

Proof. Since $\hat{B}^n$ and $\hat{F}^n$ are the martingale estimators of B and F, respectively, it is a corollary of the proof of Theorem 4.1.2 that it suffices to verify the equivalent of conditions (b) and (c) of that Theorem, and finally to verify condition (d) of Theorem 3.2. Here, of course, condition (a) of Theorem 4.1.2 is irrelevant. But the assumption above is equivalent to conditions (b) and (c) of Theorem 4.1.2 with $\bar{Y}_t$ and n in place of $Y_t(\tau)$ and $\tilde{\pi}_\tau$, respectively. Hence, it remains to verify condition (d) of Theorem 3.2.

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Suppose $t_k \in \{t: \Delta B(t) > 0\}$ and $t_k < \sigma$. Then according to Theorem 3.1.1 of Gill (1980b), the conditional law of $\Delta \overline{N}_{t_k}$ given $\mathcal{H}_{t_k^-}^n$ is Binomial with parameter $\overline{Y}_{t_k}$ and probability $\Delta B(t_k)$. Moreover, when $\overline{Y}_{t_k} > 0$ the random variable W_k^n of condition (d) in Theorem 3.2 is given by

$$W_k^n = n^{1/2} \left(\frac{\Delta \overline{N}_{t_k}}{\overline{Y}_{t_k}} - \Delta B(t_k) \right).$$

Therefore, by the assumption and the Gaussian approximation to Binomial sampling it follows that the conditional law of W_k^n given $\mathcal{H}_{t_k^-}^n$ converges to a Gaussian law with mean zero and variance $(1 - \Delta B(t_k))\Delta B(t_k)/y(t_k)$ in probability as $n \to \infty$. As this verifies condition (d) of Theorem 3.2, this completes our proof. ¤

The method of proof in Theorem 4.2.1 is based on functional central limit theorems for semimartingales, overcomes the need for elaborate constructions, and integrates the proof for continuous and discrete distributions.

Appendix: Proof of Theorem 3.2 on Weak Convergence

Before proceeding to a formal proof of Theorem 3.2 we develop some preliminary results regarding the martingale estimator (3.1). We first observe that

$$\hat{B}_t^n - B(t) = M_t^n + \int_0^t 1(Y_s^n = 0)B\{ds\}$$

$$= M_t^n + \tilde{B}_t^n, \quad t \geq 0, \tag{A.1}$$

where $M^n = \{M_t^n, t \geq 0\}$ is the square integrable martingale defined by (3.2), and the term $\tilde{B}^n = \{\tilde{B}_t^n, t \geq 0\}$ is a process of local bounded variation. Hence $\hat{B}^n - B$ is a semi-martingale, so to prove weak convergence for the normalized process we apply the general functional central limit theorems for semimartingales of Jacod, Klopotowski and Memin (1982) (JKM).

Let χ_I denote the characteristic function of the set $I = \{t_k : k \geq 1, \Delta B(t_k) > 0\}$ and consider the decomposition

$$a_n M_t^n = a_n \int_0^t (1 - \chi_I(s))dM_s^n + a_n \sum_{k: t_k \leq t} \Delta M_{t_k}^n$$

$$= U_t^n + V_t^n, \quad t \geq 0, \tag{A.2}$$

with obvious correspondence between terms. Here $U^n = \{U_t^n, t \geq 0\}$ and $V^n = \{V_t^n, t \geq 0\}$ decompose $a_n M^n$ into orthogonal local martingales defined over the set I^c (where B is continuous) and I (where B has jumps), respectively. The process U^n is a purely discrete local martingale (cf. Shiryaev (1981)) so that there exists a random measure $\mu^n - \nu^n$ on $\mathbb{R}_+ \times \mathbb{R}$ such that

$$U_t^n = \int_{\mathbb{R}} x(\mu^n((0,t] \times dx) - \nu^n((0,t] \times dx)), \quad t \geq 0. \tag{A.3}$$

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In the terminology of Jacod (1975), μ^n is a random counting measure with predictable projection v^n. For $n \geq 1$ and $u \in \mathbb{R}$ define the complex, predictable process $A(U^n,u) = \{A_t(U^n,u), t \geq 0\}$ given by

$$A_t(U^n,u) = \int_{\mathbb{R}} (e^{iux} - 1 - iux)\, v^n((0,t] \times dx) . \qquad (A.4)$$

By virtue of the definition of U^n in (A.2), it is easily verified that for each $t \geq 0$ $v^n(\{t\} \times \bullet) = 0$ so that, in particular, $\Delta A_t(U^n,u) = 0$. Therefore equation (2.8) of JKM becomes

$$\mathcal{E}(A(U^n,u))_t = \exp\{A_t(U^n,u)\} . \qquad (A.5)$$

Let G^c denote the continuous function defined in Theorem 3.2. We have the following.

Lemma A.1. Conditions (a), (b), and (c) of Theorem 3.2 imply that for $t \geq 0$

$$\mathcal{E}(A(U^n,u))_t \xrightarrow{p} \exp\{G^c(t)\} \text{ as } n \to \infty . \qquad (A.6)$$

Proof. We begin by showing that conditions (a), (b), and (c) of Theorem 3.2 imply conditions $[\beta]$ and $[\delta]$ of Theorem 2.21 of JKM, and condition $[\gamma\,']$ of Proposition 2.22 of JKM with $Z^n = 0$, $\tilde{B}^{Y}n = a_n\tilde{B}^n$ (see (A.1)), $\tilde{B}^X = 0$, $v^X = 0$, $C^Y n = 0$, $C^X = G^c$, and $Y^n = U^n$. Then lemma 2.24 of JKM will imply the desired result.

Obviously condition (a) implies condition $[\beta]$, so it remains to show that conditions (b) and (c) imply conditions $[\delta]$ and $[\gamma']$. Since $v^X = 0$, condition $[\delta]$ is equivalent to the following condition. For all $\varepsilon > 0$, $t \geq 0$

$$v^n((0,t] \times \{x: |x| > \varepsilon\}) \xrightarrow{p} 0 \quad \text{as } n \to \infty . \qquad (A.7)$$

Moreover, it follows that condition (A.7) implies that condition $[\gamma']$ is equivalent to the condition

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$$\int_{\mathbb{R}} v^n((0,t] \times dx)x^2 \, 1(|x| \le \varepsilon) \underset{p}{\to} G^c(t) \quad \text{as} \quad n \to \infty, \qquad (A.8)$$

for $t \ge 0$ and $\varepsilon > 0$. Hence, it suffices to show that (b) implies (A.7), and that (b) and (c) together imply (A.8). To prove these statements we first observe that by definition the quadratic characteristic of U^n has the representation

$$<U^n>_t = a_n^2 \int_0^t (1 - \chi_I(s))d<M^n>_s$$

$$= \int_0^t (Z_s^n)^2 \, dA_s^{nc}$$

$$= \int_{\mathbb{R}} v^n((0,t] \times dx)x^2, \quad t \ge 0, \qquad (A.9)$$

where A^{nc} denotes the continuous part of A^n (see section 3). Secondly, it is easily verified that

$$\int_0^t (Z_s^n)^2 \, 1(Z_s^n > \varepsilon)dA_s^{nc} \ge \varepsilon^2 \int_0^t 1(Z_s^n > \varepsilon)dA_s^{nc}$$

$$= \varepsilon^2 \, v^n((0,t] \times \{x: \, |x| > \varepsilon\}), \quad t \ge 0. \qquad (A.10)$$

The inequality (A.10) shows that condition (b) implies (A.7). Also, by virtue of equation (A.9), condition (b) together with condition (c) implies (A.8). Now Lemma 2.24 of JKM implies the desired result. ¤

We are now in a position to give a proof of Theorem 3.2.

Proof of Theorem 3.2. By virtue of equation (A.1) and condition (a), it suffices to show that $W^n = \{a_n M_t^n, t \ge 0\}$ converges weakly to X. Our plan is to demonstrate that the finite dimensional distributions converge and that the sequence $\{W^n\}$ is tight.

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To prove finite dimensional convergence we apply Theorem 3.4 of JKM, where in Theorem 3.4 we take $X = X$ and $X^n = W^n$. By virtue of the decomposition (A.2), the sequence $\{R^n(u,m), m \geq 1\}$ of predictable stopping times in Theorem 3.4 is given by the deterministic sequence $\{t_m : t_m \in I\}$ for all u and n. In addition, using the structure of the Gaussian process X in our Theorem 3.2, the sequence $\{R(u,m), m \geq 1\}$ in Theorem 3.4 is also given by $\{t_m : t_m \in I\}$. Hence, conditions (i) and (ii) of Theorem 3.4 of JKM are trivially satisfied. Now, by invoking the definitions of all the quantities involved, it is immediately seen that condition (d) of Theorem 3.2 is equivalent of condition (iii) of Theorem 3.4 of JKM. Finally, since in Theorem 3.4 of JKM we have $X'^n = U^n$, and $G'(u)_t = \exp\{G^c(t)\}$, Lemma A.1 shows that conditions (a), (b), and (c) imply condition (iv) of Theorem 3.4 of JKM. Therefore, according to Theorem 3.4 of JKM conditions (a), (b), (c), and (d) of Theorem 3.2 imply that the finite dimensional distributions of W^n converge to those of X.

We proceed to showing that $\{W^n\}$ is tight with an application of Theorem 2.8 of Jacod and Memin (1980). To apply Theorem 2.8 we identify X^n with W^n, and the predictable process $G^n = \{G_t^n, t \geq 0\}$ in Theorem 2.8 is defined by

$$G_t^n = <a_n M^n>_t = \int_0^t (Z_s^n)^2 \, dA_s^{nc} + \sum_{k:\, t_k \leq t} a_n^2 \Delta <M^n>_{t_k}. \qquad (A.11)$$

First observe that in Theorem 2.8 of Jacod and Memin (1980) $X_0^n = W_0^n = 0$, so that condition (i) of Theorem 2.8 is trivially satisfied. Secondly, in condition (ii) of Theorem 2.8 of Jacod and Memin (1980) we have $F^n = G^n$, so we proceed to verify condition (2.5) of Jacod and Memin (1980). In their condition (2.5) we take G^∞ to be the deterministic function $<X>$ defined by our equation (3.3). Now it is easily seen that conditions (c) and (e) together imply (2.5) of Jacod and Memin (1980). Therefore, by Theorem 2.8 of Jacod and Memin (1980) the sequence $\{W^n\}$ is relatively compact. Since $\mathbb{D}$ is separable and complete, it follows from Theorem 6.2 of Billingsley (1968) that $\{W^n\}$ is tight. Finally,

by combining this result with the previous one on finite dimensional convergence, we obtain the desired result.   ¤

In our application of Theorem 3.2 we use the following corollary.

Corollary A.1.   Suppose $H^n = \{H^n_t, t \geq 0\}$ is a bounded predictable process such that

H^n converges to a function h uniformly on $[0, \infty)$. Then under the conditions of Theorem 3.2, the sequence of martingales $a_n \int H^n(d\hat{B}^n - dB)$ converges weakly to a Gaussian process $\int h dX$.   ¤

Corollary A.1 is easily proved by noting that the conditions of Theorem 3.2 remain valid if we replace Z^n by $Z^n H^n$. The proof is omitted.

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