TESTING NON-TESTABLE HYPOTHESES IN LINEAR MODELS: A CORRECTION

S. R. Searle

Biometrics Unit, Cornell University
Ithaca, New York

BU-501-M

March, 1974

Abstract

Inconsistent statements in Searle [1971] about testing non-testable hypotheses are corrected.

In the linear model $\underset{\sim}{y} = \underset{\sim}{Xb} + \underset{\sim}{e}$ the F-statistic for testing H: $\underset{\sim}{K}'\underset{\sim}{b} = \underset{\sim}{m}$ is

$$F = Q/s\hat{\sigma}^2 \tag{1}$$

where

$\underset{\sim}{K}'$ is of full row rank s, and $\underset{\sim}{K}'\underset{\sim}{b}$ is estimable;

$\hat{\sigma}^2 = \underset{\sim}{y}'(\underset{\sim}{I} - \underset{\sim}{XGX}')\underset{\sim}{y}/[N - r(\underset{\sim}{X})]$ for $\underset{\sim}{X}'\underset{\sim}{XGX}'\underset{\sim}{X} = \underset{\sim}{X}'\underset{\sim}{X}$;

and

$$Q = (\underset{\sim}{K}'\underset{\sim}{b}^\circ - \underset{\sim}{m})'(\underset{\sim}{K}'\underset{\sim}{GK}')^{-1}(\underset{\sim}{K}'\underset{\sim}{b}^\circ - \underset{\sim}{m}) . \tag{2}$$

These results are derived in Searle [1971], hereafter referred to simply as LM, at Sec. 5.5b.

Suppose we define a hypothesis

$$H:\ \underset{\sim}{K}'\underset{\sim}{b} = \underset{\sim}{m} \quad \text{as} \quad H:\ \begin{bmatrix} \underset{\sim}{K}_1'\underset{\sim}{b} \\ \underset{\sim}{k}'\underset{\sim}{b} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{m}_1 \\ m_2 \end{bmatrix} \tag{3}$$

or more generally as

$$H:\ \begin{bmatrix} \underset{\sim}{K}_1'\underset{\sim}{b} \\ \underset{\sim}{K}_2'\underset{\sim}{b} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{m}_1 \\ \underset{\sim}{m}_2 \end{bmatrix} \tag{4}$$

where $\underset{\sim}{K}_1'\underset{\sim}{b}$ is estimable and $\underset{\sim}{K}_2'\underset{\sim}{b}$ is not [in (3) $\underset{\sim}{k}'\underset{\sim}{b}$ is not].

Exercise 7 and the end of Section 5d of Chapter 5 of LM suggest that the F-statistic for testing (3) is indistinguishable from that for testing just its testable part H_1: $\underset{\sim}{K}_1'\underset{\sim}{b} = \underset{\sim}{m}_1$. This suggestion is false, as pointed out by Hinkelmann [1972]. It is also contrary to the general result of Sec. 5.5e that when H: $\underset{\sim}{K}'\underset{\sim}{b} = \underset{\sim}{m}$ is a non-testable hypothesis the F-statistic calculated by (2) is identical to that for testing H: $\underset{\sim}{K}'\underset{\sim}{H}\underset{\sim}{b} = \underset{\sim}{m}$ where $H = \underset{\sim}{G}\underset{\sim}{X}'\underset{\sim}{X}$. This general result is proved in Sec. 5.5e and is correct.

Without reference to the correct general result we show the falsity of the false suggestion; i.e., we show that the F-statistic for the non-testable hypothesis (4) is not the F-statistic for its testable part H_1: $\underset{\sim}{K}_1'\underset{\sim}{b} = \underset{\sim}{m}_1$.

Recall that $\underset{\sim}{K}'\underset{\sim}{b}$ being estimable is only a sufficient condition for $(\underset{\sim}{K}'\underset{\sim}{G}\underset{\sim}{K})^{-1}$ and hence Q of (2) to exist. Estimability is not a necessary condition. Hence whenever $\underset{\sim}{K} = (\underset{\sim}{K}_1\ \underset{\sim}{K}_2)$ of (4) is such that $(\underset{\sim}{K}'\underset{\sim}{G}\underset{\sim}{K})^{-1}$ exists Q can be calculated even if $\underset{\sim}{K}'\underset{\sim}{b}$ is not estimable. Then from (2) it is clear that

$$Q = \begin{bmatrix} \underset{\sim}{K}_1'\underset{\sim}{b}^o - \underset{\sim}{m}_1 \\ \underset{\sim}{K}_2'\underset{\sim}{b}^o - \underset{\sim}{m}_2 \end{bmatrix}' \begin{bmatrix} \underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}_1 & \underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}_2 \\ \underset{\sim}{K}_2'\underset{\sim}{G}\underset{\sim}{K}_1 & \underset{\sim}{K}_2'\underset{\sim}{G}\underset{\sim}{K}_2 \end{bmatrix}^{-1} \begin{bmatrix} \underset{\sim}{K}_1'\underset{\sim}{b}^o - \underset{\sim}{m}_1 \\ \underset{\sim}{K}_2'\underset{\sim}{b}^o - \underset{\sim}{m}_2 \end{bmatrix} \tag{5}$$

We assume that $\underset{\sim}{G}$ is symmetric. If it is not, use $\underset{\sim}{G}\underset{\sim}{X}'\underset{\sim}{X}\underset{\sim}{G}'$ in its place. Then the inverse matrix in Q is symmetric and since $\underset{\sim}{K}_1'\underset{\sim}{b}$ is estimable, $\underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}$ is non-singular and so

$$\begin{bmatrix} \underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}_1 & \underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}_2 \\ \underset{\sim}{K}_2'\underset{\sim}{G}\underset{\sim}{K}_1 & \underset{\sim}{K}_2'\underset{\sim}{G}\underset{\sim}{K}_2 \end{bmatrix}^{-1} = \begin{bmatrix} (\underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}_1)^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} + \begin{bmatrix} -(\underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}_1)^{-1}\underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}_2 \\ \underset{\sim}{I} \end{bmatrix} \underset{\sim}{W}^{-1} \begin{bmatrix} -\underset{\sim}{K}_2'\underset{\sim}{G}\underset{\sim}{K}_1(\underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}_1)^{-1} & \underset{\sim}{I} \end{bmatrix}$$

where

$$\underset{\sim}{W} = \underset{\sim}{K}_2'\underset{\sim}{G}\underset{\sim}{K}_2 - \underset{\sim}{K}_2'\underset{\sim}{G}\underset{\sim}{K}_1(\underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}_1)^{-1}\underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}_2 \; . \tag{6}$$

Substituting this in (5) gives

$$Q = (\underset{\sim}{K}_1'\underset{\sim}{b}^\circ - \underset{\sim}{m}_1)'(\underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}_1)^{-1}(\underset{\sim}{K}_1'\underset{\sim}{b}^\circ - \underset{\sim}{m}_1) + \underset{\sim}{t}'\underset{\sim}{W}^{-1}\underset{\sim}{t} \tag{7}$$

for $\underset{\sim}{t} = \underset{\sim}{K}_2'\underset{\sim}{b}^\circ - \underset{\sim}{m}_2 - \underset{\sim}{K}_2'\underset{\sim}{G}\underset{\sim}{K}_1(\underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}_1')^{-1}(\underset{\sim}{K}_1'\underset{\sim}{b}^\circ - \underset{\sim}{m}_1)$

$= \underset{\sim}{K}_2'\underset{\sim}{b}^\circ - \underset{\sim}{m}_2 - \underset{\sim}{K}_2'(\underset{\sim}{b}^\circ - \underset{\sim}{b}^\circ_{H_1})$

$= \underset{\sim}{K}_2'\underset{\sim}{b}^\circ_{H_1} - \underset{\sim}{m}_2$

where $\underset{\sim}{b}^\circ_{H_1}$ is the solution vector under the hypothesis H_1: $\underset{\sim}{K}_1'\underset{\sim}{b} = \underset{\sim}{m}_1$ as in (72), LM p. 191. Hence in (7)

$$Q = Q_1 + (\underset{\sim}{K}_2'\underset{\sim}{b}^\circ_{H_1} - \underset{\sim}{m}_2)'\underset{\sim}{W}^{-1}(\underset{\sim}{K}_2'\underset{\sim}{b}^\circ_{H_1} - \underset{\sim}{m}_2) \tag{8}$$

for Q_1 being the numerator sum of squares for testing H_1: $\underset{\sim}{K}_1'\underset{\sim}{b} = \underset{\sim}{m}_1$. Thus Q, and hence the F-statistic for testing H of (4), is <u>not</u> the same as that for testing H_1.

Numerical illustration

The illustration in Chapter 5 has the following characteristics

$$\underset{\sim}{G} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \underset{\sim}{H} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \underset{\sim}{b}^\circ = \begin{bmatrix} 0 \\ 100 \\ 86 \\ 32 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

We consider the hypotheses of Exercise 7. First, the testable hypothesis

$$H_1: \alpha_1 = \alpha_2$$

for which

$$\underset{\sim}{K}_1' = [0 \quad 1 \quad -1 \quad 0] \qquad \underset{\sim}{m}_1 = 0 \tag{10}$$

$$\underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}_1 = \tfrac{1}{3} + \tfrac{1}{2} = 5/6 \qquad \underset{\sim}{K}_1'\underset{\sim}{b}^\circ - \underset{\sim}{m}_1 = 100 - 86 - 0 = 14 \tag{11}$$

and so

$$Q_1 = 14(5/6)^{-1}14 = 6(39.2) = 235.2 \; . \tag{12}$$

To test the non-testable hypothesis

$$\begin{aligned} H: \quad \alpha_1 &= \alpha_2 \\ \alpha_1 &= 110, \end{aligned}$$

the first part of which is the testable H_1, we have

$$\underset{\sim}{K}' = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \underset{\sim}{m} = \begin{bmatrix} 0 \\ 110 \end{bmatrix} \tag{13}$$

$$\underset{\sim}{K}'\underset{\sim}{G}\underset{\sim}{K} = \begin{bmatrix} \frac{1}{3} + \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\underset{\sim}{K}'\underset{\sim}{b}^{\circ} - \underset{\sim}{m} = \begin{bmatrix} 100 - 86 \\ 100 \end{bmatrix} - \begin{bmatrix} 0 \\ 110 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \end{bmatrix} .$$

Hence

$$\begin{aligned} Q &= [14 \quad -10](6/6)\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 14 \\ -10 \end{bmatrix} = 2(14^2) + 5(-10)^2 + 2(-2)14(-10) \\ &= 1452, \end{aligned} \tag{14}$$

as derived by Dempfle [1974]. We confirm that the same value is given by (8):

$$\underset{\sim}{b}^{\circ}_{H_1} = \underset{\sim}{b}^{\circ} - \underset{\sim}{G}\underset{\sim}{K}_1(\underset{\sim}{K}_1'\underset{\sim}{G}\underset{\sim}{K}_1)^{-1}(\underset{\sim}{K}_1'\underset{\sim}{b}^{\circ} - \underset{\sim}{m}_1)$$

$$= \begin{bmatrix} 0 \\ 100 \\ 86 \\ 32 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{3} \\ -\frac{1}{2} \\ 0 \end{bmatrix}(5/6)^{-1}(14), \quad \text{using (9), (10) and (11)}$$

$$= \begin{bmatrix} 0 \\ 100 - 28/5 \\ 86 + 42/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 94.4 \\ 94.4 \\ 0 \end{bmatrix}$$

From (13), $\mathbf{K}_2' = [0\ 1\ 0\ 0]$ and so

$$\mathbf{K}_2'\mathbf{b}^\circ_{H_1} - \mathbf{m}_2 = 94.4 - 110 = -15.6 \ .$$

And in (6),

$$\mathbf{W} = \tfrac{1}{3} - \tfrac{1}{3}(5/6)^{-1}\tfrac{1}{3} = \tfrac{1}{3} - \frac{2}{15} = \frac{1}{5} \ ,$$

so that in (8), using (12)

$$Q = 235.2 + (-15.6)(1/5)^{-1}(-15.6) = 235.2 + 1216.8 = 1254$$

as in (14).

The general result can also be confirmed, that Q is the numerator sum of squares for the testable hypothesis H: $\mathbf{K}'\mathbf{Hb} = \mathbf{m}$, which with $\mathbf{K}$ of (13) and hence

$$\mathbf{K}'\mathbf{H} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

is the hypothesis H: $\begin{bmatrix} \alpha_1 - \alpha_2 \\ \mu + \alpha_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 110 \end{bmatrix}$. Then

$$\mathbf{K}'\mathbf{Hb}^\circ - \mathbf{m} = \begin{bmatrix} 14 \\ 110 \end{bmatrix} - \begin{bmatrix} 0 \\ 110 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \end{bmatrix}$$

$$(\mathbf{K}'\mathbf{HGH}'\mathbf{K})^{-1} = \begin{bmatrix} 5/6 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}^{-1} = (6/6)\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

Hence, from (2) the Q for H: $\mathbf{K}'\mathbf{Hb} = \mathbf{m}$ is

$$Q = [14 \quad -10]\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 14 \\ -10 \end{bmatrix} = 1452$$

as in (14).

References

Dempfle, L. [1974]. Personal communication.

Hinkelmann, K. [1972]. Book review, of Searle [1971]. _Jour. Statist. Computation and Simulation_ 1, 195-6.

Searle, S. R. [1971]. _Linear Models_, Wiley, N. Y.