ON THE REACHABILITY PROBLEM
FOR 5-DIMENSIONAL
VECTOR ADDITION SYSTEMS[+]

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Abstract:

The reachability set for vector addition systems of dimension less than or equal to five are shown to be effectively computable semilinear sets. Thus reachability, equivalence and containment are decidable up to dimension 5. An example of a non-semilinear reachability set is given for dimension 6.

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Introduction

Vector addition systems or equivalent formalisms like Petri Nets have been studied extensively as a model for parallelism and resource allocation in operating systems [6,7]. Hack [4] and Rabin (see [1]) have shown that the equivalence and containment problems for arbitrary vector addition systems are undecidable. A number of other properties such as finiteness are known to be decidable [5,7]. However, the reachability problem, i.e., given an initial configuration, can one reach a specified configuration, has been left open. Some partial results have been obtained, notably by Van Leeuwen [8] who proved that the reachability problem is decidable up to dimension 3 and several authors (see Cardoza [2]) who showed that for reversible or self-dual vector addition systems, the reachability problem (as well as the equivalence problem) is decidable. (This particular case corresponds to the word problem for commutative semigroups.) Both of these partial results depend, at least implicitly on the fact that the reachability set is semilinear which is not in general true.

In this paper we show that the reachability set is an effectively computable semilinear set for dimensions less than or equal to 5. This proves that reachability, equivalence and containment are decidable up to dimension 5. An example of a non-semilinear reachability set is given for dimension 6. Thus results for higher dimension will need basically new approaches.

An important concept which we use extensively is that of a semilinear set. For C and $P \subseteq \mathbb{N}^n$ let

$$\mathscr{L}(C,P) = \{x \mid \exists\ c \text{ in } C,\ \alpha_1,\ldots,\alpha_k \in \mathbb{N} \text{ and } p_1,\ldots,p_k \in P,\ x = c + \sum_{i=1}^{k} \alpha_i p_i\}.$$

For convenience we write $\mathscr{L}(c,P)$ for $\mathscr{L}(\{c\},P)$. If P is finite, then $\mathscr{L}(c,P)$ is said to be a <u>linear</u> set. P is the set of <u>periods</u>. A set is <u>semilinear</u> if it is a finite union of linear sets. The class of semilinear sets is closed under union, intersection and complement [3]. For $L \subseteq \mathbb{N}^n$ and v in $\mathbb{Z}^n$ the <u>shift</u> of L with respect to v, denoted L+v, is the set $\{x+v \mid x \text{ in } L\} \cap \mathbb{N}^n$. The class of semilinear sets is closed under shift as seen in the following technical lemma.

<u>Lemma 1.1</u>: If L is semilinear, then L+v is semilinear.

<u>Proof</u>: Since the class of semilinear sets is closed under union we need only show that L+v is semilinear for L a linear set. Note that for $v \geq 0$, $\mathscr{L}(c,P)+v$ is just $\mathscr{L}(c+v,P)$. Let $P = \{p_1,\ldots,p_k\}$. Associate with each element x of $\mathscr{L}(c,P)+v$ any k-tuple $(\alpha_1,\ldots,\alpha_k)$ such that $x = c + \sum_{i=1}^{k} \alpha_i p_i + v$. Let B be the set of elements of $\mathscr{L}(c,P)+v$ corresponding to minimal k-tuples. Then $\mathscr{L}(c,P)+v = \mathscr{L}(B,P)$. Since B is finite, $\mathscr{L}(c,P)+v$ is semilinear. ∎

The <u>cone</u> generated by a set of vectors $P = \{p_1,\ldots,p_k\}$ and

We introduce a variation of the vector addition system by adding a finite state control. The addition of states often reduces the number of dimensions needed to model a given system. For a vector addition system with states the reachability set is semilinear up to dimension 2 but not, in general, semilinear for dimension 3 or higher. Since this model reduces the dimension at which non-semilinear sets arise, it is our hope that it will make it easier to prove further results for non-semilinear cases.

I. Preliminaries

We first give basic definitions and notation used throughout the paper. Let $\mathbb{N}$ denote the set of nonnegative integers $\{0, 1, \ldots\}$, $\mathbb{Z}$ denote the set of all integers $\{\ldots, -1, 0, 1, \ldots\}$ and $\mathbb{R}$ denote the rationals. Let $\mathbb{N}^n(\mathbb{Z}^n)$ denote the set of n-tuples of elements of $\mathbb{N}(\mathbb{Z})$. If t is an n-tuple, $\Pi_i(t)$ is the ith component of t. Unless otherwise specified, operations on tuples are componentwise extensions of the usual operations (e.g. for v and w in $\mathbb{N}^n$, v+w is defined by $\Pi_i(v+w) = \Pi_i(v)+\Pi_i(w)$ for i=1 to n). An important exception is the relation < between elements of $\mathbb{N}^n$. $v \leq w$ means $\Pi_i(v) \leq \Pi_i(w)$ $i=1,\ldots,n$ but $v < w$ means $\Pi_i(v) \leq \Pi_i(w)$ $i=1,\ldots,n$ and $\Pi_j(v) < \Pi_j(w)$ for some j, $1 \leq j \leq n$. Also an obvious but important fact is that for $n > 1$, $\leq$ is not a total order on $\mathbb{N}^n$. We use $v \between w$ to express that v and w are incomparable. Any set of pairwise incomparable elements of $\mathbb{N}^n$ is finite, hence the minimum of a subset of $\mathbb{N}^n$ is finite.

a point b is the set $\mathscr{C}(b,P) = \{x | x \in \mathbb{N}^n, x = b + \sum_{i=1}^{k} \alpha_i p_i, \alpha_i \text{ in } \mathbb{R}\}$.
We will make use of the fact that $\mathscr{L}(B,P)$ is semilinear even if B and P are infinite provided there exists a finite subset $P_f = \{p_1,\ldots,p_k\} \subseteq P$ such that $B \in \mathscr{C}(x_0,P_f)$ for some x_0, and $P \in \mathscr{C}(0,P_f)$. We first show that if P is finite, then $\mathscr{L}(B,P)$ is semilinear.

<u>Lemma 1.2</u>: Let $B \subseteq \mathbb{N}^n$ be a possibly infinite set and $P = \{p_1,\ldots,p_k\} \subseteq \mathbb{N}^n$ a finite set such that B is contained in the cone C generated by P and some x_0. Then $\mathscr{L} = \mathscr{L}(B,P)$ is semilinear.

<u>Proof</u>: Let $B = \{b_1,b_2,\ldots\}$. Delete from B each b_i such that there exists $j < i$, $b_i \in \mathscr{L}(b_j,P)$. This does not change the set $\mathscr{L}$. If B becomes finite clearly $\mathscr{L}$ is semilinear by definition of a semilinear set.

Assume B is still infinite. Each b_i can be expressed

$$b_i = x_0 + \sum_{i=1}^{k} \alpha_i p_i + \sum_{i-1}^{k} n_i p_i \quad 0 \leq \alpha_i < 1, n_i \in N$$

There is a finite number of constants c, $c \in \mathbb{N}^n$ such that $\sum_{i=1}^{k} \alpha_i p_i = c$, $0 \leq \alpha_i < 1$. So an infinite subset of the b_i's must be of the form $b_i = x_0 + c + \sum_{i=1}^{k} m_i p_i$. Represent the <u>jth</u> element of this subset by the vector $\langle n_1^j,\ldots,n_k^j \rangle$. Since there is an infinite number of these vectors, there must be an infinite increasing sequence, hence there exists b_i, b_j, $i < j$ such that the representation of b_i is less than that of b_j. Hence $b_j \in \mathscr{L}(b_i,P)$, a contradiction. Hence we conclude that the new B is finite and $\mathcal{L}$ is semilinear.

Lemma 1.3: Let B and P be possibly infinite subsets of N^n such that for some finite set $P_f \subseteq P$, $B \in \mathcal{C}(x_0,P_f)$ for some x_0, and $P \in \mathcal{C}(0,P_f)$. Then $\mathcal{L}(B,P)$ is semilinear.

Proof: $\mathcal{L}(B,P) = \mathcal{L}(\mathcal{L}(B,P),P_f)$ and hence is semilinear by the previous lemma since $\mathcal{L}(B,P) \subseteq \mathcal{C}(x_0,P_f)$.

Lemma 1.4: Consider an infinite sequence of linear sets $\mathcal{L}(x_i,P_i)$ such that P_1 nonempty and for each i, $P_i \subseteq P_{i+1}$, contained in a one dimensional space ℓ. Then $\bigcup_{i \in \mathbb{N}} \mathcal{L}(x_i,P_i) = \mathcal{L}(x_i,P_i)$ for some finite set F.

Proof: Without loss of generality we can assume that P_1 contains some vector a. Thus ℓ is partitioned into a finite number of equivalence classes modulo a. Since a must be in each P_i, if x is in $\mathcal{L}(x_i,P_i)$ then all $y \geq x$ in the same equivalence class must also be in $\mathcal{L}(x_i,P_i)$. But there are only finitely many equivalence classes and for a given x there are only finitely many $y < x$ in the same class. Thus there are only a finite number of i such that $\mathcal{L}(x_i,P_i)$ contains an x not in any $\mathcal{L}(x_j,P_j)$, $j < i$. ∎

We use the term boundary to designate an hyperplane of the form $\{x \mid \Pi_i(x) = 0,\ x \in \mathbb{N}^n\}$ for some i. Boundaries separate $\mathbb{N}^n$ from the rest of $\mathbb{Z}^n$.

An n-dimensional vector addition scheme W is a finite subset of $\mathbb{Z}^n$. An n-dimensional vector addition system (VAS for short) is a pair (x,W) where x in $\mathbb{N}^n$ is called the <u>start</u> point and $W \subseteq \mathbb{Z}^n$. The <u>reachability set</u> of the VAS (x,W), denoted $R(x,W)$ is the set of all z, $z = x+v_1+ \ldots +v_j$, where each v_i is in W and for $1 \leq i \leq j$, $x+v_1+ \ldots +v_i \geq 0$. The sequence $v_1,\ldots,v_j$ is called a <u>W-path</u> or path when W is understood, <u>valid</u> at x, $v_1+\ldots v_j$ is the <u>displacement</u> of the path. A W-path is sometimes noted $p \in W^*$, using the notation of regular expressions. If $p \in W^*$, $W = \{w_1,\ldots,w_k\}$ we define $\chi(p) \in \mathbb{N}^k$, the <u>folding</u> of p, by $\pi_i(\chi(p))$ equals the number of occurrences of w_i in p. Of course, a folding corresponds to many paths, and for a given start point some (or all) may be nonvalid.

The reachability problem is to determine for a VAS (x,W) and a point y whether y is in $R(x,W)$. It is an open problem whether there is a decision procedure for solving all instances of the problem. The problem is solvable up to dimension 3 [8] and in various special cases, for example when W is self-dual ($v \in W \iff -v \in W$) (see for example Cardoza [2]).

II. <u>Vector Addition Systems with States (VASS).</u>

In this section we present a new model for vector addition systems that includes a finite state control. We first show that an n-dim VASS can be simulated by an (n+3)-dim VAS, hence the two formulations have the same power. Next we prove that a 2-dim VASS has an effective semilinear reachability set. Finally we give an example of a 3-dim VASS that generates exponentiation, hence its reachability set is not semilinear.

A <u>vector addition scheme with states</u> is a vector addition scheme W, together with a finite state control S. Transitions are in $S \to S \times W$. The transition $p \to (q,v)$ can be applied at the point x in state p and yields the point $x+v$ in state q, provided that $x+v \geq 0$.

A <u>vector addition system with states</u> (<u>VASS</u> for short) is a vector addition scheme with states $\langle W,S \rangle$ together with a starting point x_0 and a starting state $p_0 \in S$.

<u>Lemma 2.1</u>: An n-dim VASS can be simulated by an (n+3)-dim VAS.

<u>Proof</u>: We give the construction of the VAS. The last three coordinates encode the state while the first n coordinates are as in the VASS. Assume that the VASS has k states $q_1,\ldots,q_k$. Let $a_i=i$ for $i=1$ to k, $b_k=k+1$ and $b_i=b_{i+1}+k+1$ for $i=1$ to $k-1$. If the VASS is at v in state q_i then the VAS will be at $(v,a_i,b_i,0)$. For each i the VAS has two dummy transitions t_i and t'_i defined so that t_i goes from $(v,a_i,b_i,0)$ to $(v,0,a_{k-i+1},b_{k-i+1})$ and t'_i goes from $(v,0,a_{k-i+1},b_{k-i+1})$ to $(v,b_i,0,a_i)$. Note that t_i and t'_i modify only the last three components. In addition there is a transition t''_i for each transition $i \to (j,w)$ of the VASS, defined so that t''_i goes from $(v,b_i,0,a_i)$ to $(v+w,a_j,b_j,0)$ provided $v+w \geq 0$.

Clearly any path of the VASS can be mimicked by the VAS. It remains to be shown that the VAS cannot do something unintended. We will only show that t''_i can only be applied if the last three components are b_i, 0 and a_i respectively. The other cases are similar. Observe that for each i and j, $a_i < a_{i+1}$, $b_i > b_{i+1}$, $a_i < b_j$ and $b_i - b_{i+1} = k+1 > a_j$. Let v''_i be the vector $(w, a_j - b_i, b_j, -a_i)$ which accomplishes the transition t''_i. Note that the $n-1^{st}$ and last components are negative. Hence t''_i cannot be applied when the last three coordinates are $(a_i, b_i, 0)$ or $(0, a_{k-i+1}, b_{k-i+1})$ since either the first or third components are null. Let the last three coordinates be $(b_m, 0, a_m)$. Then if $m < i$, t''_i cannot be applied since $a_m - a_i < 0$. If $m > i$, then t''_i cannot be applied since $b_m + a_j - b_i \leq a_j - (k+1) < 0$ ∎

Since an n-dim VASS can trivially simulate an n-dim VAS, these two models have essentially the same power.

We are now going to show that the reachability set for each 2-dim VASS is semilinear. The idea is the following. Start enumerating paths. On encountering a path containing a subpath starting and ending in the same state from some x to some y, $y \geq x$, we observe that the subpath can be repeated as often as we like, giving an infinite set of paths. Thus if z is any point reachable from y, we can reach the set of points $\{z + i(y-x) \mid i = 1, 2, \ldots\}$. We enumerate the reachability set by enumerating such linear sets continuing this process until a collection of linear sets is constructed which is closed under transitions of the VASS. Even though the above process does not in general terminate, in dimension 2 it does terminate implying that the reachability set of a 2-dim VASS is semilinear.

The intuitive reason why the enumeration terminates in the 2-dimensional case is as follows. If the process does not terminate, then there is an infinite path such that points along this path are not in previously computed linear sets. The set of periods for the linear sets corresponding to points on this path must eventually have arbitrarily large cardinality. By Lemma 1.3 this implies that the cones generated by the periods must be "widening" infinitely often. In dimension 2 this implies that eventually periods parallel to the axis vectors can be added and hence the cones cannot widen further.

In the following we make these ideas precise. A <u>short path</u> is a path with no repeating state except that the first and last state are the same. A <u>short positive path</u> is a short path with a positive displacement. Note that there is only finitely many short paths. An <u>axis</u> is a vector with one positive component and all other components zero.

We give an algorithm that constructs a tree labelled by 3-tuples $[x,p,A_x]$ where x is in $\mathbb{N}^2$, p is a state and $A_x \subseteq \mathbb{N}^2$. The label $[x,p,A_x]$ denotes the fact that every point in the linear set $\mathscr{L}(x,A_x)$ can be reached in state p. When a new vertex is added with label $[x,p,A_x]$ the displacement of any short positive path which is valid at x is added to the set of periods

A_x. Also if there is a path valid at x whose displacement is an axis, the axis is added to A_x if a parallel axis vector is not already present. Each vertex inherits the periods of its father. If $\mathscr{L}(x,A_x)$ is contained in $\mathscr{L}(z,A_z)$ where the vertex labelled $[z,p,A_z]$ is an ancestor of $[x,p,A_x]$, the path is terminated at $[x,p,A_x]$ since any descendant of $[x,p,A_x]$ is equivalent to a descendant of $[z,p,A_x]$ which is closer to the root. In this case $[x,p,A_x]$ is marked.

Algorithm

Input: The set of transitions and the start point x_0 and start state p_0 forming a VASS.

$A_{x_0} \leftarrow \emptyset$;

Create root labelled $[x_0, p_0, A_{x_0}]$;

while there are unmarked leaves do

begin

Pick an unmarked leaf $[x, p, A_x]$;

Add to A_x all displacements of short positive paths from p to p valid at x;

if there exists $c \in \mathbb{N}^2$, $c = \binom{0}{\gamma}$ or $\binom{\gamma}{0}$ such that

a) c is not colinear to any vector of A_x, and

b) either i) there exists an ancestor $[z,p,A_z]$ of $[x,p,A_x]$ such that $x-z = c$, or

ii) for some non positive short path from p to p valid at x, with displacement a, and some $b \in A_x$, there exists α, $\beta \in \mathbb{N}$ such that $\alpha a + \beta b = c$

<u>then</u> add c to A_x;

<u>If</u> there exists an ancestor $[z,p,A_z]$ of $[x,p,A_x]$ such that $\mathcal{L}(z,A_z)$ contains x and $A_z = A_x$

<u>then</u> mark $[x,p,A_x]$

<u>else</u> <u>for</u> each transition $p \to (q,v)$ <u>do</u>

<u>begin</u>

Let $a = \alpha_1 v_1 + \ldots + \alpha_k v_k$ for a in $\mathcal{L}(0,A_x)$ where $A_x = \{v_1,\ldots,v_k\}$

<u>for</u> each a in $\mathcal{L}(0,A_x)$ corresponding to a minimum tuple $(\alpha_1,\ldots,\alpha_k)$ such that $x+a+v \geq 0$ <u>do</u>

Construct a son $[y,q,A_y]$ where $y = x+a+v$ and $A_x = A_y$;

<u>end</u>;

<u>if</u> $[x,p,A_x]$ has no son <u>then</u> mark $[x,p,A_x]$;

<u>end</u>

<u>Lemma 2.2</u>: There exists a constant b such that for each label $[x,p,A_x]$ of the tree, $|A_x| \leq b$. Moreover if $[x,p,A_x]$ is an ancestor of $[y,q,A_y]$ then $A_x \subseteq A_y$.

<u>Proof</u>: A_x contains only displacements of short positive paths and at most 2 axis vectors, hence there is a bound on $|A_x|$. If $[x,p,A_x]$ is an ancestor of $[y,q,A_y]$ then $A_x \subseteq A_y$ since sons inherit the periods of the fathers.

<u>Lemma 2.3</u>: The preceding algorithm always terminates, and the corresponding tree is finite and effectively computable.

<u>Proof</u>: Assume the algorithm never terminates. All instructions inside the while loop are finite, so the only possibility is that the while loop itself never terminates. But each time the loop is executed a new vertex is visited, hence an infinite tree is constructed. Since the fan-out of the tree is finite, because A_x is bounded, there must be an infinite path, by application of König's Lemma. We are now going to show that all paths must be finite, hence that the algorithm terminates.

Assume that there is an infinite path with vertices $[x_i,p_i,A_{x_i}]$, $i = 0,1,\ldots$. By Lemma 2.2 the A_{x_i}'s remain unchanged beyond some finite i_0. Thus there is an infinite path $[x_i,p_i,A]$, $i = i_0, i_0+1,\ldots$ for some A. We will show that there exists a cone $\mathscr{C}(y_0,A)$ such that all but a finite number of x_i's lie in the cone. But by Lemma 1.2 only a finite number of x_i may lie in the cone, a contradiction. From this we conclude the path is finite. It remains to show the existence of the cone $\mathscr{C}(y_0,A)$.

We first show that only a finite number of the x_i from the path may lie on the same horizontal or vertical line. Suppose $x_{i_1},x_{i_2},\ldots$ lie on the same horizontal or vertical line. The sequence $x_{i_1},x_{i_2},\ldots$ has a minimum say x_{i_m}. Also an axis vector colinear to the horizontal or vertical line must be in A since there must exist a pair of indices $i_j < i_{j'}$ for which $x_{i_j} < x_{i_{j'}}$. Hence for all k, $\mathcal{L}(x_{i_k},A)$ lie in the cone $\mathscr{C}(x_{i_m},A)$ and by Lemma 1.2 $\bigcup_{i_k} \mathcal{L}(x_{i_k},A) = \bigcup_{i_k \in F} \mathcal{L}(x_{i_k},A)$ for some finite F. Thus there exists $i_j < i_{j'}$ such that $x_{i_{j'}}$ is in $\mathcal{L}(x_{i_j},A)$. Hence the last vertex should be marked and the sequence terminated, a contradiction. Thus only a finite number of x_i may lie on the same horizontal or vertical line.

Consider any fixed c_0 in $\mathbb{N}^2$ and let $D = \{x | x \geq c_0\}$. The region $\mathbb{N}^2-D$ is composed of a finite number of vertical and horizontal lines and thus by the previous argument contains only a finite number of x_i from any infinite path in the tree. Choose c_0 sufficiently large so that all transitions and short paths are valid at c_0, hence at any point of D. Since the path in the tree has only a finite number of points outside D, there exists an index j_0 such that $x_{j_0},x_{j_0+1},\ldots$ are in D. In general $x_i = x_{i-1}+a_i+v_i$ where v_i is a transition and a_i is a minimal displacement such that $x_i \geq 0$. However at any point in the region D, v_i is valid, hence $a_i=0$ and the path in the tree is also a path of the VASS.

We will show that for each state p, there can be only a finite number of vertices $[x_i,p,A]$ with x_i outside the cone $\mathscr{C}(x_{i_0},A)$ where x_{i_0} is the first point on the path in state p, $i_0 \geq j_0$. Let $[x_i,p,A]$, $i > i_0$ be another point on the path in state p. If i does not exist, then our claim is trivially true. Clearly $x_i - x_{i_0} = \sum_{j=1}^{k} w_j$ where the w_j's are displacements of short paths since there is a W-path from x_{i_0} to x_i and from state p to state p. We consider several cases.

<u>Case 1</u>: A does not contain any axis vector. Then either

i) all w_j's are positive hence $w_j \in A$ and $x_i \in \mathscr{C}(x_{i_0},A)$ (hence no x_i outside the cone) or

ii) all w_j's and all vectors of A are colinear, so $x_i \in \mathscr{C}(x_{i_0},A)$ or $x_i < x_{i_0}$ (hence a finite number of x_i's outside the cone).

<u>Case 2</u>: A contains both axis. Then $\mathbb{N}^2 - \mathscr{C}(x_{i_0},A)$ is composed of finitely many vertical and horizontal lines, and by an earlier argument these lines can contain only finitely many points of the path.

Case 3: A contains one axis, say $c = (0,\alpha)$. Let v_1 be the vector of A with smallest slope (possibly infinite if $v_1 = c$, but not zero). Let $v_1 = (a,b)$ and $v_1^{\perp} = (-b,a)$ ($v_1^{\perp}$ is orthogonal to v_1, and points "above" v_1. Then all w_j's make a positive dot product with $v_1^{\perp}$, because otherwise if say $w_j^T \cdot v_1^{\perp} < 0$, then either $w_j \geq 0$, w_j smaller slope than v_1, and w_j must be in A, or $w_j \leq 0$ and we can introduce a second axis. In both cases there is a contradiction. So $(x_i - x_{i_0})^T \cdot v_1^{\perp} \geq 0$ and x_i is above the line $x_{i_0} + \beta v_1$. Hence either $x_i \in \mathscr{C}(x_{i_0}, A)$ if $x_i \geq x_{i_0}$, or x_i lies on one of a finite number of vertical lines.

In all three cases, there can be only a finite number of the x_i's reached in state p outside $\mathscr{C}(x_{i_0}, A)$. Let X be the set of x_i's such that x_i is reached in state p, $x_i \in \mathscr{C}(x_{i_0}, A)$. If X is finite then there is only a finite number of x_i's reached in state p. Suppose X is infinite by Lemma 1.2 $\mathscr{L}(X,A) = \mathscr{L}(B,A)$ where B is a finite subset of X. Hence there exists $x_i \in B$, $x_j \in X$, $i < j$ such that $x_j \in \mathscr{L}(x_i, A)$. But then the path should terminate at x_j. A contradiction. So there can be only a finite number of the x_i's reached in state p. Since this is true for all states, the path is finite. Hence the tree is finite and the algorithm terminates. ∎

Let $T_p = \cup \mathcal{L}(x,A_x)$ where the union is over all vertices $[x,p,A_x]$ of the tree.

Lemma 2.4: T_p is an effectively computable semilinear set.

Proof: Clear since the tree is finite by previous lemma.

The next lemma shows that indeed we compute the reachability set.

Lemma 2.5: Let R_p be the set of points reachable in state p. Then $R_p = T_p$ for all states p.

Proof:

Part 1: $T_p \subseteq R_p$. We show by induction on the depth of x in the tree that for any node $[x,p,A_x]$, $\mathcal{L}(x,A_x) \subseteq R_p$.

Basis: Let $[x_0,p_0,A_{x_0}]$ be the label of the root. Let $w \in A_{x_0}$. Either w is the displacement of a short positive path from p_0 to p_0, valid at x_0, or w is an axis vector, $w = \alpha a+\beta b$ where a is the displacement of a nonpositive short path from p_0 to p_0, valid at x_0 and b is the displacement of a short positive path from p_0 to p_0, valid at x_0. In this case, we can apply first the β copies of b, followed by the α copies of a, hence w is also valid at x_0, and $w \geq 0$. In both cases, x_0+w is reachable in state p_0, and $x_0+w \geq x_0$, hence x_0+a, $a \in \mathcal{L}(0,A_{x_0})$ is reachable in state p_0, from x_0 and $\mathcal{L}(x_0,A_{x_0}) \subseteq R_{p_0}$.

Induction hypothesis: Assume that for each vertex $[x,p,A_x]$ of depth at most n-1, $\mathcal{L}(x,A_x) \subseteq R_p$. Let $[y,q,A_x]$ be a vertex of depth n, and let $[x,p,A_x]$ be its father. So $y = x+a_1+v$, where $a_1 \in \mathcal{L}(0,A_x)$, and $p \rightarrow (q,v)$ is a transition. Let $z \in \mathcal{L}(y,A_y)$. We will show that $z \in R_q$. $z = y+a$, $a \in \mathcal{L}(0,A_y)$. Since $A_x \subseteq A_y$, $a = a_2+a_3$, $a_2 \in \mathcal{L}(0,A_x)$, $a_3 \in \mathcal{L}(0,A_y-A_x)$. Now $z = x+a_1+a_2+v+a_3$, and $x+a_1+a_2 \in \mathcal{L}(x,A_x)$ is reachable in state p, by induction hypothesis. $p \rightarrow (q,v)$ is valid at $x+a_1$, so it is also valid at $x+a_1+a_2$, hence $x+a_1+a_2+v = y+a_2 \in R_q$. Now, by an argument similar to the one used in the basis part, all vectors of A_y-A_x are positive displacements of paths valid at y, hence at $y+a_2$. So $z = y+a_2+a_3 \in R_q$. ∎

<u>Part 2</u>: $R_p \subseteq T_p$ for all p. We show by induction on the length of the W-path from x_0 to z (in state p) that $z \in T_p$.

<u>Basis</u>: if $z = x_0$ then $z \in \mathcal{L}(x_0,A_{x_0}) \subseteq T_{p_0}$.

<u>Induction hypothesis</u>: Assume that for each state p and any point z reachable in state p by a path of length n-1, $z \in \mathcal{L}(x,A_x)$ for some vertex $[x,p,A_x]$.

Let z be reachable in state p by a path of length n. Let $q \rightarrow (p,v)$ be the last transition of the path and $z = y+v$. By induction hypothesis, $y \in \mathcal{L}(x,A_x)$ for some vertex $[x,q,A_x]$. So $y = x+c$, $c \in \mathcal{L}(0,A_x)$. We may assume that x is not a leaf. Otherwise either

i) no transition is applicable at x, a contradiction, or

ii) $\mathcal{L}(x,A_x) \subseteq \mathcal{L}(x',A_x)$ for some interior vertex $[x',q,A_x]$, and we may replace x by x'.

Let $A_x = \{v_1,\ldots,v_k\}$. Then x has sons x+a+v for each $a = \sum_{i=1}^{k} \alpha_i v_i$ corresponding to a minimum tuple of α's such that $x+a+v \geq 0$. Since $x+c+v \geq 0$ we can write $c = a_1+a_2 \in \mathscr{L}(0,A_x)$ such that $t = x+a_1+v$ is a son of x labelled $[t,p,A_t]$ and $z = t+a_2$. But a_2 is in $\mathscr{L}(0,A_x) \subseteq \mathscr{L}(0,A_t)$. Hence z is in $\mathscr{L}(t,A_t)$.

By parts 1 and 2, $R_p = T_p$ for all p. ∎

Theorem 2.6: In a 2-dim VASS, the set of points reachable in any given state is semilinear and effectively computable.

Proof: Clear from Lemmas 2.4 and 2.5.

Corollary 2.7: Equivalence and reachability are decidable for 2-dim VASS.

We now give an example of a 3-dim VASS that generates exponentiation, hence in general, 3-dim VASS have non-semilinear reachability sets.

Lemma 2.8: There exists a 3-dim VASS with a non-semilinear reachability set.

Proof: Consider the following 3-dim VASS with two states, p and q. The start point and state are $x_0 = (0,0,1)$ and, $p_o = p$. The transitions are:

t_1: p → (p,(0,1,-1)) t_2: p → (q,(0,0,0))

t_3: q → (q,(0,-1,2)) t_4: q → (p,(1,0,0)).

Let condition (1) be $0 < x_2+x_3 \leq 2^{x_1}$ and condition (2) be

$0 < 2x_2+x_3 \le 2^{x_1+1}$.

<u>Claim</u>: $x = (x_1,x_2,x_3)$ is reachable in state p if and only if (1) holds, and $x = (x_1,x_2,x_3)$ is reachable in state q if and only if (2) holds.

We first show =>. Note that initially (1) holds and

i) if (1) holds and we apply t_1, (1) still holds,

ii) if (1) holds and we apply t_2, (2) holds,

iii) if (2) holds and we apply t_3, (2) holds,

iv) if (2) holds and we apply t_4, (1) holds.

Hence any reachable point satisfies (1) or (2) depending upon the state. We now show <=, i.e. if (1) or (2) holds, then we can reach x in the appropriate state. The proof is by induction on the first coordinate x_1.

<u>Basis</u>: If $x_1 = 0$ and (1) holds then either

i) $x = (0,0,1)$ and x is reachable in state p by the null path or

ii) $x = (0,1,0)$ and x is reachable in state p by transition t_1.

If $x_1 = 0$ and (2) holds then either

i) $x = (0,1,0)$ and x is reachable in state q by t_1t_2

ii) $x = (0,0,1)$ and x is reachable in state q by t_2

iii) $x = (0,0,2)$ and x is reachable in state q by $t_1t_2t_3$.

So in all cases x is reachable in the right state.

Induction hypothesis: Assume that all points satisfying (1) or (2) and $x_1 \le a_1-1$ can be reached in the appropriate state. Let $a = (a_1,a_2,a_3)$ and assume that (1) holds, i.e. $0 < a_2+a_3 \le 2^{a_1}$. We will show that a is reachable in state p.

Assume that $0 < a_2+a_3 \le 2^{a_1-1}$. Then by the induction hypothesis, $a' = (a_1-1,a_2,a_3)$ is reachable in state p, and by applying t_2t_4, we reach a in state p. So now we assume that $2^{a_1-1} < a_2+a_3 \le 2^{a_1}$. Let $a_2+a_3 = 2^{a_1-1}+b$ where $0 < b \le 2^{a_1-1}$. By the induction hypothesis, there is a path to $a' = (a_1-1,b,2^{a_1-1}-b)$ since a' satisfies (1). But now, at a' we can apply $t_2(t_3)^b t_4(t_1)^{a_2}$ and we get $(a_1,a_2,2^{a_1-1}+b-a_2)$ in state p. Since $2^{a_1-1}+b-a_2 = a_3$, we have reached a in state p.

A similar argument shows that if (2) holds we can reach a in state q. Hence, by induction, our claim is true. Clearly the reachability set is not semilinear, thus our lemma. ∎

Note that although the reachability set is not semilinear, we can specify it completely by recursive relations, hence reachability is decidable for this particular example. Indeed, a possible way to solve the reachability problem would be to prove that the reachability set can always be effectively represented by some recursive relations.

Remarks: The example we have presented is in some sense the simplest non-semilinear VASS. If we reduce by one the dimension, or the number of states, or even the number of transitions, we get a semilinear reachability set. Also, there is a very similar 3-dim VASS (2 states, 4 transitions) that generates squaring.

Conclusion: In this section we have introduced VASS and showed that they have non-semilinear reachability sets for dimension as low as 3 (In the next section we will see that for VAS this happens only at dimension 6). So it might be easier to use VASS to prove results on non-semilinear systems. Also with VASS, it is possible to reduce the dimension of a system when one coordinate remains bounded, replacing each value of that coordinate by a state. In fact we use this property in the proof that 5-dim VAS have semilinear reachability sets.

Short of solving the general reachability problem, it would be interesting to solve the reachability problem for 3-dim VASS, since they have non-semilinear reachability sets.

III. 5-dim VAS have a semilinear reachability set.

In this section we show that the set of all z reachable from a given x is semilinear provided we restrict x and z to be sufficiently large. The first part, based on Van Leeuwen's results, is the case where x and z have n-1 large coordinates (larger than some computable constant b). In the second part we extend this to points having n-2 large coordinates (larger than some computable constant c). Finally, in part three we use these results to show that 5-dim VAS have semilinear reachability sets.

In this part we investigate some properties of paths and reachability sets when either endpoint of a path, or even an intermediate point, is sufficiently far from n-1 boundaries, in a sense to be defined latter. This part is inspired from Van Leeuwen [8]. We first give some of his notations and results and then give some generalizations and improvements of these results. Many of these results use the fact that a nonvalid path can always be reordered so that at least one coordinate remains positive.

A principal arcone is a subset of $\mathbb{N}^n$ of the form $\{x \mid x \geq v\}$ for some $v \in \mathbb{N}^n$. It can be viewed as $\mathbb{N}^n$ shifted by a positive vector v.

Let $x \in \mathbb{N}^n$, $A \subseteq \mathbb{N}^n$. A web of x with respect to A is a set L of W-paths such that:

i) each path in L is a valid path from x to some y in A.

ii) if p_1 is a valid path from x to some y in A, then there exists p_2 in L, $\chi(p_2) \leq \chi(p_1)$.

iii) if p_1, p_2 are in L, then $\chi(p_1)$ and $\chi(p_2)$ are incomparable.

Informally speaking, a web is a minimal set of shortest paths from x to A. Webs are always finite no matter what A is. However it is not always known how to compute them.

Lemma 3.1: [van Leeuwen] For each $x \in \mathbb{N}^n$ and principal arcone A, the web of x with respect to A can be effectively determined.

A W-transformation area S is a subset of $\mathbb{N}^n$ such that any (nonvalid) W-path between two points x and y of S can be rearranged into a valid path from x to y.

We now give a slightly generalized version of one of Van Leeuwen's theorems.

Lemma 3.2: For each j, $1 \leq j \leq n$, there is an effectively computable v_j with $\Pi_j(v_j) = 0$ such that $A = \{x \mid x \geq v_j\}$ is a W-transformation area. Moreover, v_j can be chosen independent of the positive vectors of W.

Proof: The first part of the lemma is Van Leeuwen's theorem. It remains to be shown that v_j can be chosen independent of the positive vectors of W. Note that this property is useful when dealing with linear starting sets $\mathscr{L}(x,P)$: We can then just add P to W, and consider x as the starting point. The transformation areas are unchanged.

Assume that we partition W into W_1 and W_2, W_1 containing all positive vectors of W and W_2 the rest. Using Van Leeuwen's construction, find a W_2-transformation area $A = \{x \mid x \geq v_j\}$. Then A is also a W-transformation area. Consider a (nonvalid) W-path from x to y, both x and y in A. Rearrange the path so that all positive vectors are at the beginning. We get a valid path from x to z, z in A, followed by a nonvalid W_2-path from z to y. Since both z and y are in a W_2-transformation area, this path can be rearranged into a valid path.

We can find transformation areas with an even more general form as the next lemma shows:

Lemma 3.3: There is an effectively computable constant $B = (b,\ldots,b)$ such that the set of points having n-1 coordinates larger than b is a W-transformation area. Again, B is independent of the positive vectors of W.

Proof: Note that this transformation area is of the form $S = \bigcup_{j=1}^{n} \{x \mid x \geq v'_j\}$ where $\Pi_i(v'_j) = b$, $\Pi_j(v'_j) = 0$, $i \neq j$. Let v_j be as in the previous lemma. Let $c = \max_{i,j} \Pi_i(v_j)$ and $d = \max \{-\Pi_i(w) \mid \Pi_i(w) \leq 0,\ w \in W,\ 1 \leq i \leq n\}$, and $b = c(d+1)$. We are going to show that this choice of b satisfies our lemma.

Assume there is a nonvalid path p from x to y, both in S. If x and y are larger than the same v_j, then by Lemma 3.2 we can rearrange p into a valid path. So without loss of generality, we can assume that $x \geq v'_1$, $\Pi_1(x) < \Pi_1(v_2) \leq c$, and $y \geq v'_2$, $\Pi_2(y) < \Pi_2(v_1) \leq c$. (This is because $v'_i > v_i$.)

Now since $\Pi_1(x) < \Pi_1(v_2) \leq c$ and $\Pi_1(y) \geq \Pi_1(v_2') = b \geq c$, p must contain some vectors with a positive first coordinate. In fact, a sequence of at most c vectors must increase the first coordinate from $\Pi_1(x)$ to at least c. Let p_1 be this sequence. If we put the sequence p_1 at the beginning, we get a valid path from x to some z followed by some path from z to y. But then $\Pi_1(z) \geq c \geq \Pi_1(v_2)$ and $\Pi_i(z) \geq \Pi_i(x) - c.d \geq b - c.d \geq c \geq \Pi_i(v_2)$. Hence both z and y are larger than v_2 so the path from z to y can be rearranged into a valid path. Hence our lemma.

We now use these results to show that in some cases, the reachability set is semilinear. First when the starting point has n-1 sufficiently large coordinates, and then when any intermediate point has n-1 large coordinates.

Given a vector addition scheme W, the set of points reachable from 0, by not necessarily valid paths is an effective linear set $L_0 = \mathcal{L}(0,P)$. Note that each period p of P can be generated by a nonvalid path from 0 to p. For any $x \in \mathbb{N}^n$, the set of points $y \geq x$ reachable from x by a (nonvalid) path is $L_0+x = \mathscr{L}(x,P)$. If x is sufficiently large, the paths generating each period become valid when applied at x. Hence there exists a constant c_w^j such that for all $x_0 \geq c_w^j$ $R(x_0,W) \cap \{x|x \geq x_0\}$ is equal to $\mathscr{L}(x_0,P)$. Moreover c_w^j can be chosen so that $\pi_j(c_w^j) = 0$, by reordering paths generating the periods of P, so that they are always valid in the j^{th} dimension. We show that c_w^j has a stronger property.

<u>Lemma 3.4</u>: There exists an effectively computable constant $c_w^j \in \mathbb{N}^n$, $\pi_j(c_w^j) = 0$ such that for each $x_0 \geq c_w^j$, $R(x_0,W) = \mathscr{L}(B,P)$ for some finite, effectively computable B and P.

<u>Proof</u>: Take c_w^j and P as defined before, and let $x_0 \geq c_w^j$. We already have $\mathscr{L}(x_0,P) = R(x_0,W) \cap \{x|x \geq x_0\}$. Also if $x \in R(x_0,W)$, then $\mathscr{L}(x,P) \subseteq R(x_0,W)$. To see this, note that if $p \in \mathscr{L}(0,P)$, $x+p = x_0+p+(x-x_0)$. But $x_0+p \in R(x_0,W)$ and $x-x_0$ is a valid path at x_0+p since it is valid at x_0, so $x+p \in R(x_0,W)$. To find B, we are going to close $\mathscr{L}(x_0,P)$ under shifts by vectors of W. To do that we construct a tree labelled by points x, where a son is a shift of its father. More precisely:

i) The root is labelled x_0.

ii) If x is an unmarked leaf, for all $w \in W$, shift $\mathscr{L}(x,P)$ by w. The shifted set is of the form $\mathscr{L}(D,P)$ for some finite D. If D is empty, mark x.

iii) Create a son y for each $y \in D$. If $y \geq z$ for some ancestor z of y, mark y.

Clearly this tree is finite since the fan-out is finite, and the paths of the tree are made of decreasing or incomparable points, hence are finite. We want to show that $\mathcal{L}(B,P) = R(x_0,w)$ where B is the set of all labels of the tree.

First $\mathcal{L}(B,P) \subseteq R(x_0,W)$ since all labels of the tree are clearly reachable from x_0. Also $\mathcal{L}(B,P)$ contains x_0, so it suffices to show that $\mathcal{L}(B,P)$ is closed under shift by any vector of W. Let x be a vertex of the tree. Then either

i) x is unmarked, then the shift of $\mathcal{L}(x,P)$ is included in $\mathcal{L}(B,P)$ since we create a son y of x for each constant of $\mathcal{L}(D,P)$, the shift of $\mathcal{L}(x,P)$.

ii) x is marked and the shift of $\mathcal{L}(x,P)$ is empty, or

iii) x is marked because it has some ancestor y, $y \leq x$. But then, since x is reachable from y, $x \geq y$, $x \in \mathcal{L}(y,P)$ and $\mathcal{L}(x,P) \subseteq \mathcal{L}(y,P)$ so the shift of $\mathcal{L}(x,P)$ is contained in the shift of $\mathcal{L}(y,P)$.

In each case, the shift of $\mathcal{L}(x,P)$ is included in $\mathcal{L}(B,P)$. ■

In the next lemma we give a stronger result, namely the set of points reachable through a point having n-1 large coordinates is semilinear, no matter where x_0 is.

Lemma 3.5: There exists an effectively computable constant K_W^j, $\Pi_j(K_W^j) = 0$ such that for all x_0 in $\mathbb{N}^n$, the set of points reachable from x_0 through a point y, $y \geq K_W^j$, is an effective semilinear set.

Proof: Let v_j, $\Pi_j(v_j) = 0$ define a W-transformation area as in Lemma 3.2. Let c_w^j with $\Pi_j(c_w^j) = 0$, be defined as in Lemma 3.4. Define K_w^j by $\Pi_j(K_w^j) = 0$ and $K_w^j = \max(v_j, c_w^j)$. Let $A = \{x | x \geq K_w^j\}$. We want to show that the set of points reachable through paths having at least one point in A is semilinear. Pick some starting point x_0, $x_0 \in \mathbb{N}^n$. Note that if $x_0 \in A$, we are done by lemma 3.4. We can determine the web of x_0 with respect to A, by Lemma 3.1. Let $S = \{p_1,\ldots,p_m\}$ be this web. Let z_i, $i = 1,\ldots,m$ be the points of A reached from x_0 by the paths p_i, $i = 1,\ldots,m$. By definition of a web the p_i are valid paths at x_0. By Lemma 3.4, $R(z_i,W)$ is an effective semilinear set. We will show that any point reachable through A is also reachable through some z_i.

Let z be reachable from x_0, through some $y \in A$. Let p be a valid path from x_0 to y. By definition of a web, there exists $p_i \in S$ such that $\chi(p) \geq \chi(p_i)$, hence y is reachable from z_i by a (nonvalid) path. However A is also a W-transformation area, so this path can be rearranged into a valid path, hence y and z are in $R(z_i,w)$. So the set we are looking for is $\bigcup_{i=1}^{m} R(z_i,W)$, which is semilinear by Lemma 3.4. ◻

We can generalize this result even more.

Theorem 3.6: There exists a constant K_w, such that for all $x_0 \in \mathbb{N}^n$, the set of points reachable from x_0, through a point having any n-1 coordinates larger than K_w's is an effective semilinear set.

<u>Proof</u>: Let $k = \max\limits_{\substack{1\le j\le n\\ 1\le i\le n}} \Pi_i(K_w^j)$. Then $K_w = (k,k,\ldots,k)$ clearly satisfies the theorem since if x has n-1 coordinates larger than K_w's (all but the j<u>th</u>) one) then $x \geq K_w^j$ and Lemma 3.5 applies.

The interesting point of this theorem is that when we compute a reachability set we can restrict paths to have at least one small coordinate, that is paths running in some finite number of n-1 dimensional spaces. We will see next that for paths with at least two small coordinates, we can only get a weaker result. Furthermore, these results do not hold at all for paths with 3 small coordinates, since we can simulate states.

We are now concerned with points far from n-2 boundaries. We are going to show (Theorem 3.11) that the set of points z reachable from some point x_0 is semilinear when z's and x_0 are sufficiently far from the same n-2 boundaries. Without loss of generality, we assume in the following that the n-2 large coordinates are the last n-2. We first prove (Lemma 3.9) the existence and computability of a constant $C \in \mathbb{N}^n$, $\Pi_1(C) = \Pi_2(C) = 0$ such that for all x and y greater than C there is a valid path from x to y if and only if there is a path valid in the first two dimensions. We then prove (Lemma 3.10) that the set of points reachable from some x_0 with a path valid in two dimensions is an effective semilinear set.

We first define C and then prove it has the required properties. Let $B = (b,b,\ldots,b) \in \mathbb{N}^n$ be a constant such that the set of points having n-1 coordinates larger than b is a W-transformation area, as in Lemma 3.3. Let $V = \{v_1, v_2, \ldots, v_e\}$ be the set of displacements v_i of paths of length at most b^2, such that $\Pi_1(v_i) = \Pi_2(v_i) = 0$ (i.e. the projection of the path generating v_i along first 2 coordinates is a loop). Find b' such that the set $\{x \mid \Pi_i(x) \geq b', i = 3,\ldots,n\}$ is a V-transformation area as in Lemma 3.3. Note that V is equivalent to an (n-2)-dim system. Finally we define c by: $c \pm \Pi_i(p) \pm \Pi_i(p') \geq \mathrm{Max}(b,b')$ for all $i = 3,\ldots,n$ and for all W-paths p, p' of length at most b^2. Let $C = (0,0,c,\ldots,c)$ and consider a path from x to z, valid in first two dimensions, x and y larger than C.

Lemma 3.7: If the path from x to z contains a point y such that $\Pi_1(y) \geq b$ or $\Pi_2(y) \geq b$ then the path can be rearranged onto a valid path.

Proof: Let y_1 (possibly equal to x) be the first point along the path such that $\Pi_1(y_1) \geq b$ or $\Pi_2(y_1) \geq b$. Let y_2 (possibly equal to y_1 or z) be the last such point. Now consider the projection of the path along the first two dimensions. $\bar{x}$ means the projection of x.

From the path from $\bar{x}$ to $\bar{y}_1$ extract a loop-free path p_1 from $\bar{x}$ to $\bar{y}_1$. In $\mathbb{N}^n$ this path goes from x to some point y_1^1 where $\bar{y}_1^1 = \bar{y}_1$. Similarly extract from the path from $\bar{y}_2$ to $\bar{z}$ a loop-free path from $\bar{y}_2$ to $\bar{z}$. In $\mathbb{N}^n$ this path goes from some y_2^1 to z where $\bar{y}_2^1 = \bar{y}_2$. Now we have a path from x to y_1^1 to y_2^1 to z. But the paths from x to y_1^1 and from y_2^1 to z are of length less than b^2, hence they are valid. They are valid in first two dimensions because they are obtained from valid paths by removing loops. They are valid in other dimensions because $\Pi_i(x) \geq c$, $\Pi_i(z) \geq c$ and $c \pm \Pi_i(p) \geq 0$ for all p of length less than b^2, $i = 3,\ldots,n$.

Moreover $\Pi_i(y_1^1) \geq b$, $\Pi_i(y_2^1) \geq b$ for $i = 3,\ldots,n$ and by assumption $\Pi_1(y_1^1) \geq b$ or $\Pi_2(y_1^1) \geq b$ and $\Pi_1(y_2^1) \geq b$ or $\Pi_2(y_1^1) \geq b$. So y_1^1 and y_2^1 are in the same W-transformation area and the path from y_1^1 to y_2^1 can be rearranged into a valid path.

<u>Lemma 3.8</u>: Assume that the path from x to z is such that its projection along the first two dimensions lies in the square $[0,b] \times [0,b]$ and consists of a loop-free path from $\bar{x}$ to $\bar{y}$ followed by a number of simple loops from $\bar{y}$ to $\bar{y}$, followed by a loop-free path from $\bar{y}$ to $\bar{z}$. Then the path can be rearranged into a valid path.

<u>Proof</u>: Paths from $\bar{x}$ to $\bar{y}$ and from $\bar{y}$ to $\bar{z}$ are of length less than b^2 since they are loop-free. Also simple loops from $\bar{y}$ to $\bar{y}$ are vectors of V. Let y_i be the <u>ith</u> point along the path such that $\bar{y}_i = \bar{y}$, $i = 1,\ldots,m$ for some arbitrary m. In the n-dimensional space, we have a path from x to y_1 to y_2 ... to y_m to z. Also $\Pi_i(y_1) + \Pi_i(p) \geq \mathrm{Max}(b,b')$, $\Pi_j(y_m) + \Pi_j(p) \geq \mathrm{Max}(b,b')$ for $j = 3,\ldots,n$ and any p of length at most b^2. •

Note that there is a V-path from y_1 to y_m and both points are in a V-transformation area, hence the V-path can be rearranged into a valid V-path. In fact we have a stronger property. Since

$\Pi_i(y_1) + \Pi_i(p) \geq b'$, $\Pi_i(y_m) + \Pi_i(p) \geq b'$, all points y' of the valid V-path are such that $\Pi_i(y') + \Pi_i(p) \geq 0$, for $i = 3,\ldots,n$, and any p of length at most b^2. Consider the W-path induced by the valid V-path, i.e. consider each vector of V as a path in W. Any point on the W-path is of the form y'+p where y' is on the V path and p is a portion of a simple loop, hence of length at most b^2. So the W-path is also valid in the ith dimension, $i = 3,\ldots,n$. Moreover the W-path is still valid in the first two dimensions since we have just reordered loops around $\bar{y}$. Hence we have a valid path from x to z.

Lemma 3.9: There exists an effective constant $C \in \mathbb{N}^n$, $\Pi_1(C) = \Pi_2(C) = 0$ such that for all x, z greater than C if there is a path from x to z valid in the first two dimensions, it can be rearranged into a valid path from x to z.

Proof: We take C as before and consider a path from x to z valid in the first two dimensions, $x, z \geq C$. We are going to show that we can rearrange the path so that it satisfies either Lemma 3.7 or Lemma 3.8. In both cases, we can then rearrange the path into a valid one.

Assume that the projection of the path along the first two dimensions lies entirely in the square $[0,b] \times [0,b]$. If not, conditions of Lemma 3.7 are satisfied.

Let y be a point along the path with largest first coordinate. Extract a loop free path from $\bar{x}$ to $\bar{y}$ and from

$\bar{y}$ to $\bar{z}$. The rest of the original path can be decomposed into simple loops. All we have to prove is that these loops can be applied at $\bar{y}$ so that the new path is valid in the first two dimensions. This is clearly true if we apply these loops at $\bar{y}$ starting with the point on the loop with the smallest second coordinate.

Once this is done, either one loop goes outside the square $[0,b] \times [0,b]$ and we can apply Lemma 3.7, or all loops remain in the square and we can apply Lemma 3.8. In both cases the path from x to z can be rearranged into a valid path.

Remark: The region $\{x | x \geq C\}$ has a property similar to, but weaker than a W-transformation area. A path from x to z greater than C can be reordered into a valid path, but only if it is already valid in two dimensions.

Lemma 3.10: The set of points reachable from some point x_0 with a path valid in the first two dimensions is an effective semi-linear set.

Proof: Let $W = \{v_1, \ldots, v_k\}$ be the vector addition scheme and let A be the $n \times k$ matrix whose columns are the vectors of W. If p is a W-path define $y \in \mathbb{N}^k$ by $\Pi_i(y) = \Pi_i(\chi(p))$. Then a path from x_0 to x must satisfy $x_0 + Ay = x$.

We now characterize the projection of such a path along the first two dimensions. Let $S = [0,b] \times [0,b]$ where b defines a $\bar{W}$-transformation area as in Lemma 3.3. Consider a valid path (in these 2 dimensions) from $\bar{x}_0$ to $\bar{x}$ and let $\bar{z}_1$ be the first point out of S ($\bar{z}_1$ possibly equal to $\bar{x}_0$) let $\bar{z}_2$ be the last point out of S ($\bar{z}_2$ possibly equal to $\bar{z}_1$ or $\bar{x}$). If there is no point out of S, let $\bar{z}_1 = \bar{z}_2 = \bar{x}$. The path from $\bar{x}_0$ to $\bar{z}_1$ is either null or inside S (except for last point). Hence the set of foldings of such paths can be expressed as a semilinear set $L_1(\bar{x}_0)$. L_1 is the union of $b^2 + 1$ sets, one for each point of S, and the empty path for x_0 outside S. There is a similar set $L_2(\bar{x})$ for paths from $\bar{z}_2$ to $\bar{x}$. But then the conditions:

$$\bar{x}_0 + \bar{A}.y = \bar{x}, \quad y = y_1 + y_2 + y_3, \; y_1 \in L_1(\bar{x}_0), \; y_2 \in L_2(\bar{x})$$

are verified if and only if there exists a valid path $\bar{p}$ from x_0 to $\bar{x}$ with $\lambda(\bar{p}) = y$. Together with the first condition we get:

$$x_0 + A.y = x, \qquad y = y_1 + y_2 + y_3$$

$$y_1 \in L_1(\bar{x}_0) \;, \qquad y_2 \in L_2(\bar{x})$$

which is true if and only if there exists a path p from x_0 to x, such that $\lambda(p) = y$, and the path is valid in the first two dimensions.

The conditions above have a semilinear set of solutions, hence our lemma.

Using Lemmas 3.9 and 3.10 we are able to prove the next theorem.

Theorem 3.11: There exists a constant C, effectively computable, with $H_1(C) = H_2(C) = 0$ such that the set of points z reachable from a given x_0, x_0 and z greater than C, is an effective semilinear set.

Proof: Let x_0 be greater than C, where C is defined as before. By Lemma 3.9 the set of z's, z greater than C, reachable by a valid path, is the same as the set of z's reachable by a path valid in the first two dimensions. By Lemma 3.10, this set is the intersection of a semilinear set and $\{x | x \geq C\}$, hence it is semilinear. ☒

Corollary 3.12: For the same constant C, and any set of periods P, the set of z's reachable from $\mathcal{L}(x_0,P)$, x_0 and z greater than C, is an effective semilinear set.

Proof: $R(\mathcal{L}(x_0,P),W) = R(x_0,W \cup P)$. So the only question is whether the constant C is the same for the schemes W and $W \cup P$. However if C satisfies Lemma 3.9 for W, it clearly satisfies Lemma 3.9 for $W \cup P$, hence our Corollary. ☒

We are now ready to prove that a 5-dim VAS has an effective semilinear reachability set. Informally, we can view $\mathbb{N}^5$ as a (nondisjoint) union of 5-dim subspaces, each one far from n-2 (i.e. 3) boundaries, together with a finite union of subspaces where three dimensions are bounded. The subspaces where three dimensions are bounded can be simulated by a 2-dim VASS. Since we can compute the reachability set in both types of subspaces (in a somewhat restricted sense), the main problem is with paths crossing from one subspace to another.

Let c be a constant such that for all pairs i,j, $i \neq j$ the set of points x reachable from x_0 where $\Pi_k(x_0) \geq c$, $\Pi_k(x) \geq c$ for all k distinct from i and j, is an effective semi-linear set as guaranteed by theorem 3.11. Define C_{ij}, $i \neq j$ by $\Pi_k(C_{ij}) = c$, $k \neq i$, $k \neq j$ and $\Pi_i(C_{ij}) = \Pi_j(C_{ij}) = 0$. Let $R_{ij} = \{x \mid x \geq C_{ij}\}$ and $R = \bigcup_{i,j} R_{ij}$. Let $\bar{R}_{ijk} = \{x \mid \Pi_i(x) \leq c,$ $\Pi_j(x) \leq c,\ \Pi_k(x) \leq c\}$ for $i \neq j \neq k$, and let $\bar{R} = \bigcup_{i,j,k} \bar{R}_{ijk}$. Note that $R \cup \bar{R} = N^5$, and each $\bar{R}_{ijk}$ is the union of c^3 "parallel" planes, hence $\bar{R}_{ijk}$'s can be simulated by 2-dim VASS, where the states record the ith, jth and kth coordinate.

The extended intersection E of the subspaces $\bar{R}_{ijk}$ is the set of points x such that $x \in \bar{R}_{ijk}$ and $x + v \in \bar{R}_{pqr}$ for some transition v, and index p, q, r $p \neq i$ or $q \neq j$ or $k \neq r$. Note that E contains the usual pairwise intersections of the $\bar{R}_{ijk}$'s, plus some other points. A W-path going directly from a region $\bar{R}_{ijk}$ to a region $\bar{R}_{pqr}$ must contain a point of E. Paths can also change region by going through R. E is a finite union of lines parallel to one of the axis.

A path crosses through a line $\ell \in E$ if the path has a point in $\ell \in \bar{R}_{ijk}$ and the next point along the path is in another region $\bar{R}_{pqr}$.

Lemma 3.13: Let L_0 be a semilinear starting set. The set of points reachable from L_0 through paths that never cross through a line of E is an effective semilinear set.

Proof: We use a tree-based argument: We construct a tree labelled by semilinear sets. Edges correspond to shifts and closures. More precisely:

(1) The root is labelled by L_0. We assume without loss of generality that L_0 lies entirely in a region R_{ij} or $\bar{R}_{ijk}$.

(2) Let L be the label of an unmarked leaf. Then either

i) $L \subseteq \bar{R}_{ijk}$ for some i,j,k. Add to L the set of points, L', of $\bar{R}_{ijk}$ reachable from L by paths remaining in $\bar{R}_{ijk}$, using a 2-dim VASS. Then shift L by all vectors of W, getting L". Create a new son of L, $L_{pq} = L'' \cap R_{pq}$ for each pair p,q. Or

ii) $L \subseteq R_{ij}$ for some i,j. If L has an ancestor included in the same R_{ij}, mark L else add to L the set of points L' of R_{ij} reachable from L by any path, as guaranteed by Lemma 3.11. Then shift

L by all vectors of W, getting L". Create a new son of L, $L_{pq} = L'' \cap R_{pq}$ or $L_{pqr} = L'' \cap \bar{R}_{pqr}$ for each p,q,r.

Note that the tree is finite since along any path, closure under a region R_{pq}, for fixed p and q, can occur only once, and closures under region $\bar{R}_{ijk}$'s must lie between closures under regions of the form R_{pq}. Clearly the union of the labels of the tree is a semilinear set, and it is the set of all points reachable from L_0 by paths that never crosses through E. ∎

Let m be the number of lines of E. We are going to show, by induction on i, $0 \leq i \leq m$ that the set of points reachable from L_0 by paths crossing through i lines of E is an effective semilinear set. The previous lemma is the basis, and the result for i = m gives our main Theorem:

<u>Theorem 3.14</u>: Given a semilinear starting set L_0, the reachability set of a 3-dim VAS is an effective semilinear set.

<u>Proof</u>: By induction on the number of lines of E crossed.

<u>Basis</u>: This is the previous lemma, 3.13.

<u>Inductive Step</u>: Assume that for any set of k lines of E, $\{\ell_1,\ldots,\ell_k\}$, $0 \leq k \leq m-1$, the set of points reachable from any semilinear set L_0, though paths crossing only through lines $\ell_1,\ldots,\ell_k$, is an effective semilinear set.

Consider a set of k+1 lines of E, $\ell_0, \ell_1, \ldots, \ell_k$ and a starting set L_0. We are going to alternate closure under paths crossing through $\ell_1, \ldots \ell_k$, and shifts $w \in W$ that create crossings through ℓ_0. Hence we keep getting new crossing points in ℓ_0 until the whole process is closed under both operations. The problem is to do that in a finite number of steps.

For that we observe that if a path crosses infinitely often through ℓ_0, then there must be a pair x,y, $x \in \ell_0$, $y \in \ell_0$, $x < y$ and x occurs before y along the path. If so the linear set $\mathcal{L}(x,\{y-x\})$ is reachable. Moreover any successor of x inherits the period y-x.

Again we are going to use a tree-based argument, since we have to keep track of the ancestors of a given set. Let L_0 be the starting set. By induction hypothesis, we can compute the set L_1 of points reachable from L_0 through paths crossing only through $\ell_1,\ldots,\ell_k$. We intersect ℓ_0 and L_1 and get a semilinear set. We create a separate tree for each linear component $\mathcal{L}(x_0,P_0)$ of this set.

Consider one tree. The root is labelled $\mathcal{L}(x_0,P_0)$ (P_0 possibly empty). Let $\mathcal{L}(x_j,P_j)$ be an unmarked leaf. We shift $\mathcal{L}(x_j,P_j)$ by any vector of W that causes only crossing through ℓ_0. Let L be the shifted set. Compute L', the set of points reachable from L by crossing through $\ell_1,\ldots,\ell_k$, and let L" be $L' \cap \ell_0$. For each linear component $\mathcal{L}(x_{j+1},P_{j+1})$ of L", create a new son of $\mathcal{L}(x_j,P_j)$. If $\mathcal{L}(x_{j+1},P_{j+1})$ is contained in the union of its ancestors, mark it. If not, add P_j to P_{j+1}. If P_{j+1} is still empty but $x_{j+1} \geq x_i$ for some ancestor x_i, add $x_{j+1}-x_i$ to P_{j+1}. Continue this process until all leaves are marked.

We are going to show that this tree is finite. Assume there is an infinite path labelled by $\mathcal{L}(x_j,P_j)$ $j = 0,\ldots,\infty$. If the P_j's remain empty then the x_j's must be strictly decreasing which is impossible. So P_j contains at least one vector v for sufficiently large j's. But by Lemma 1.4, there can be only a finite number of distinct $\mathcal{L}(x_j,P_j)$ in the cone generated by v, hence one of the x_j's should be marked, a contradiction.

We conclude that the tree is finite, hence the process is effective. It should be clear that if L is the union of the labels of the tree, then L is the set of all points of ℓ_0 reachable from L_0, by paths crossing through $\ell_0,\ell_1,\ldots,\ell_k$. By closing L under paths crossing through $\ell_1,\ldots,\ell_k$, we get all the points reachable from L_0, by paths crossing through $\ell_0,\ell_1,\ldots,\ell_k$. Hence the induction hypothesis is true for k+1, and by induction, for k=n. So our theorem holds.

Since the exponential example of Section II can be expressed with a 6-dim VAS (using Lemma 2.1), we have:

<u>Theorem 3.15</u>: The reachability set of an n-dim VAS is an effective semilinear set for $n \leq 5$, and is not in general semilinear for $n \geq 6$.

<u>Corollary 3.16</u>: Reachability, Equivalence, Containment are all decidable for n-dim VAS, $n \leq 5$.

From these results, we can draw some conclusions. First any new results on VAS must come from at least 6-dimensional systems. However, it is very hard to have any intuition on what can happen in $\mathbb{N}^6$ (but not in $\mathbb{N}^5$!). So the VASS model might be more interesting, since open problems arise as low as dimension 3.

Also, most results so far are based, at least implicitly, on the fact that the reachability set is semilinear. Clearly such approaches cannot be used directly for further results.

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