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SHORTEST PATHS AMONG OBSTACLES, ZERO-COST REGIONS, AND ROADS

By

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Abstract

We introduce a new class of shortest path problems by considering the terrain navigation problem in which a point is required to navigate on a surface in the presense of obstacles, "zero-cost regions" regions, and "roads". Obstacles are forbidden regions for travel, zero-cost regions allow travel at no cost, and roads are piecewise-linear one-dimensional networks which permit travel at various speeds. This problem is a generalization of the usual shortest path problem among obstacles and is a special case of the weighted region problem studied by Mitchell and Papadimitriou [MP]. Here we are able to give an exact polynomial-time algorithm to find the shortest path from one point to another. Our approach is to use the local optimality criteria to build a special type of "visibility graph" which can be searched for optimal paths. The complexity of our algorithm in the case of no roads is quadratic, with various improvements given for special cases of convex obstacles or convex zero-cost regions. When roads are present, the worst-case complexity of our algorithm becomes $O(n^2 \log n)$.

Key Words: computational geometry, shortest paths, terrain navigation, visibility graphs, Snell's Law

1. Introduction

One of the basic tasks required of an autonomous vehicle is to move from one location to another. Assume that a map is given to us as a set of *regions* in the plane, each of which has an associated *weight*, or cost, α. The weight of a region specifies the "cost per unit distance" of a vehicle (considered to be a point) travelling in that region. Our objective is then to find a path in the plane which minimizes total cost according to this weighted Euclidean metric.

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The regions can be modelled as polygonal patches. There may be roads given on the map as well. Roads can be modelled as very skinny regions or as linear features (with an assigned weight which is presumably much less than the surrounding regions). In military applications, there may be regions which correspond to high threat risk, perhaps because the enemy has good visibility of you when you are in these regions. Costs can be assigned to travelling in these risky regions as well. More precisely, we assume that we are given a straight-line planar subdivision (without loss of generality, we can assume it to be a triangulation) with n vertices. Each face f has a weight $\alpha_f \in [0, +\infty]$ assigned for the cost of travel interior to the face, and each edge e has a cost $\alpha_e \in [0, +\infty]$ assigned for the cost of travel along the edge. Additionally, we will be able to handle the case in which there is a cost $\xi_e \in [0, +\infty]$ for *crossing* edge e. (If $\xi_e = \infty$, then e is an obstacle.)

The problem we have just described is termed the *Weighted Region Problem* [MP]. A special case of this problem has been studied extensively: find the shortest path for a point moving in the plane which must avoid a given set of *obstacles* (see [AAGHI], [Le], [Mi1], [RS], [SS], [We]). This is easily seen to be the special case of the weighted region problem in which the weights are either 1 or $+\infty$ depending on whether a region is "free space" or obstacle, respectively.

In this paper we consider a more general special case of the weighted region problem: Assume now that all weights on regions are taken from the set $\{0, 1, +\infty\}$and that weights on edges (or "linear features") may still be arbitrary nonnegative numbers. Thus, we assume that $\alpha_f \in \{0, 1, +\infty\}$ and $\alpha_e \in [0, +\infty]$ (also, that $\xi_e \in [0, +\infty]$). Then the resulting problem is the case of finding shortest paths among obstacles ($\alpha = +\infty$), zero-cost regions ($\alpha = 0$), and roads (line segment boundaries between regions for which α may be arbitrary). (Note: we will refer to edges as "roads", even though they may represent other types of linear features.)

The weighted region problem was first posed and solved in [MP]. Their algorithm found a decomposition of the edges of a triangulation into "intervals of optimality" which gave the necessary information to find an ϵ-optimal shortest path from a source point to any point on an edge. The algorithm used the continuous Dijkstra technique (see also [MMP]) of sweeping the plane. Because of the complexity of the local optimality criterion (Snell's Law of Refraction), it does not appear possible to solve exactly the weighted region problem; rather, the algorithm of [MP] finds a path whose length is at most $(1 + \epsilon)$ times the length of an optimal path. The algorithm of [MP] requires time $O(n^7 L)$, where L is the precision of the problem instance (the number of bits to describe the largest coordinate, the largest finite ratio of weights, and the ratio $1/\epsilon$).

Note that the algorithm that solves the weighted region problem is general enough to handle the case of obstacles, zero-cost regions, and roads. However, in this special case, we develop here algorithms that exploit the special discrete structure of optimal paths to give a more efficient, exact algorithm. Our approach is to build a special type of "extended" visibility graph (see [Le] or [Mi]) which exploits the local optimality properties of shortest paths. Our algorithm solves the problem exactly (under a model of computation which allows square roots) in polynomial time (at most *quadratic* in the size of the scene). (In the case that the visibility graph is sparse, we use results of [GM] to reduce the running time to an output-sensitive complexity.) We also consider the more general case in which there are roads in addition to obstacles, zero-cost regions, and free space. The roads may have arbitrary weights assigned to them and are assumed to be represented by a straight-line graph embedded in the plane. Again, a visibility graph approach yields an algorithm whose time bound ($O(n^2 \log n)$) considerably improves the bounds of [MP] given for the general case.

Our motivation for studying this class of problems is many-fold. Below, we list some of the applications that arise for our algorithms.

(i). We are interested in generalizing the standard shortest path problem with obstacles in the plane.

(ii). We are interested in further investigations into the complexity of the weighted region problem: In what cases can we get very efficient exact algorithms?

(iii). We were originally motivated by the *maximum concealment problem* of finding paths that minimize exposure to one or more threats. See Figure 1. There are regions of the plane which are "visible" to one or more enemy observers, and these are generally to be avoided. (The figure shows a single observer at the point O.) There are also obstacles, which must be avoided. Then there are regions (e.g., between mountain ranges, or behind rocks) which are hidden from the enemy's view, and are hence extremely cheap in comparison with the visible regions. The problem is to find a path between two points which maximizes the concealment (i.e., minimizes the length of time in the exposed regions). This problem falls into the category of a shortest path problem among obstacles and zero-cost regions.

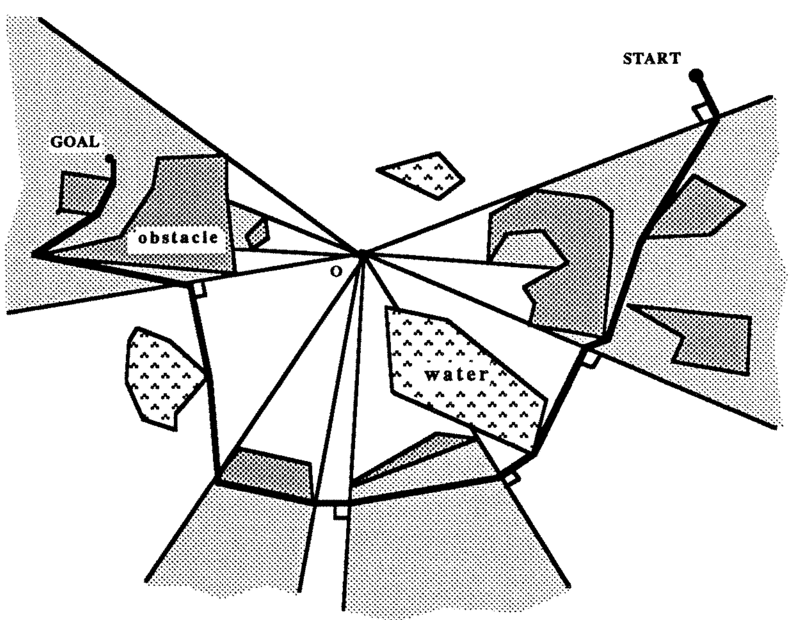

Figure 1. The maximum concealment problem.

For a more extensive discussion of problems in maximum concealment (including problems in which the *cost* of being seen varies with distance from the observer), see the forthcoming paper [Mi3]. Also, [GMN] examine the maximum concealment problem for a set of threats in a simple polygonal region with holes. They also consider the case of optimal ("least risk") watchman routes in the presense of threats.

(iv). The problems solved here can serve as valuable approximations to some weighted region problems. Considering the set of weights to be $\{0, 1, +\infty\}$ may be a reasonable approximation to the case of the weighted region problem in which there are obstacles and at most two other types of weighted regions, one being extremely cheap with respect to the other. Such would be the case for a runner (who is not a very good swimmer) trying to find a route from one point (say, on a flat island) to another point (say, in the water). There are many islands around. Some are flat and easy to run across. Others have huge cliffs surrounding them which make them impassable. Swimming through the water is possible, but the runner would much prefer to be running across a flat island. So how can he get from one point to another in the shortest time? His path will "hop" from one flat island to another, swimming in between, and avoiding the mountainous islands.

In particular, we can use our results to find lexicographically shortest paths through weighted regions, and these paths may provide good starting points for algorithms that do local search for shortest paths in weighted regions. See Section 5 for a further discussion.

(v). Shortest paths among obstacles and zero-cost regions yield a solution to the problem of finding the shortest path from a *region* (rather than a point) to another *region* (rather than a point). This generalizes the shortest path problem from that of finding the shortest path from point A to point B to finding the shortest path from region A to region B. The idea is simply to place a start point in region A, and a goal point in region B, and then to consider regions A and B as zero-cost regions. Since the original appearance of our research, another paper ([AAI]) has independently addressed this problem directly.

(vi). The results given here for shortest paths among obstacles and zero-cost regions are closely related to the dual of the shortest path problem: that of finding the maximum *flow* through polyhedral domains. This problem is extensively discussed in the paper [Mi4], where the duality relationship is clearly illustrated. Results from this paper are used in computing minimum *cuts* for the maximum flow problem.

This paper is organized as follows. In Section 2, we concentrate on the case of just obstacles and zero-cost regions. Section 3 includes a discussion of improvements in the case of convex obstacles and/or zero-cost regions. Section 4 introduces the effect of roads and other linear features to the model. Section 5 describes how our methods can be used to produce an algorithm to find the lexicographically optimal path through weighted regions. Finally, Section 6 mentions some further improvements and generalizations that may be made to the problem.

2. The $\{0, 1, +\infty\}$ Weighted Region Problem

Consider the weighted region problem in which all weights are restricted to be in the set $\{0, 1, +\infty\}$. Specifically, we are given a straight-line planar subdivision (perhaps bounded inside a simple polygon), and we are given a mapping of weights α_f on the interior of each face and α_e on each edge. The subdivision has

n vertices, $V = \{v_1, \ldots, v_n\}$, and hence also has $n_e = O(n)$ edges, $E = \{e_1, \ldots, e_{n_e}\}$. The special case we consider is that in which $\alpha_e, \alpha_f \in \{0, 1, +\infty\}$. (In Section 4 we will relax the assumption on α_e, letting it be an arbitrary nonnegative integer.) The interpretation of a weight α on a region is that the cost per unit distance in that region is α. Thus, if time is our measure of cost, then α is the reciprocal of the maximum velocity in the region.

We are also given a starting point s (also known as the "source") and a destination point t. Our objective is to find a path from s to t which minimizes the weighted length of path. (The weighted length of a path is simply the path integral of the piecewise-constant weight function assigned to regions of the plane.)

Regions with weight $+\infty$ are called *obstacles*, as travel through them is prohibited. Regions of zero weight are called *zero-cost regions* (or *0-regions*, for short), as it is "free" to move through them. We can think of 0-regions as places where we are able to travel at infinity speed. Finally, the regions of weight 1 are called *1-regions*, and they represent the "background" through which we are able to travel, but we pay some constant positive amount per unit distance. (Note that the choice of 1 as the background weight is arbitrary; the situation is obviously the same if the background had some other weight as long as it has the same weight everywhere.) If there are no 0-regions, then we get the usual obstacle avoidance problem in the plane. The 0-regions behave like desirable "islands" between which one may wish to "hop" to get from the source to the destination.

As already mentioned, the algorithm described in [MP] for the weighted region problem does in fact solve the special case with $\alpha \in \{0, 1, +\infty\}$. But the special case has structure which allows an alternative approach which is faster and easier to implement. Our approach here is to build a special kind of "visibility graph" which takes advantage of the local optimality criteria. The critical fact that we use is that locally optimal paths must enter a 0-region either at a vertex or perpendicularly at an edge.

A typical picture is shown in Figure 2, where we show obstacles as dark gray regions, 0-regions as white, and 1-regions as a light gray. Here, the entire maps lives inside a simple polygon.

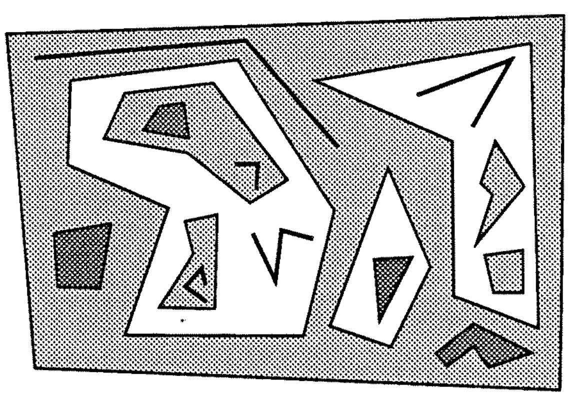

Figure 2. A map for the $\{0, 1, +\infty\}$ weighted region problem.

Note that, in general, the connected 0-regions will be *generalized* polygons with generalized polygonal holes, where the holes are occupied either by 1-regions or by obstacles. For the definition of generalized polygons, see [RS]. An example of such a polygon is shown in Figure 3. Similarly, 1-regions and obstacles will be generalized polygons with generalized polygonal holes. For brevity, we will refer to a generalized polygon with generalized polygonal holes as a *region*. (So a region is just a connected component of a planar straight-line subdivision.)

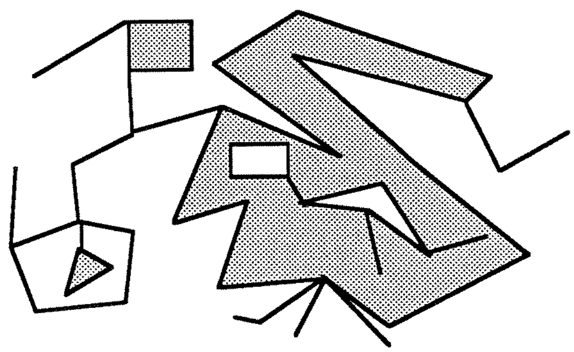

Figure 3. A generalized polygon.

Actually, our algorithm will be able to solve the lexicographic version of the problem: *Subject to the length of the path in the background being minimum*, and subject to obstacle avoidance, minimize the length of the path in the zero-cost regions. See Section 5.

We begin by recalling the results that are known for shortest paths among obstacles without the presense of 0-regions. By the recent results of [GM], we can build the *visibility graph* induced by a set of obstacles in time $O(n \log n + K)$, where K is the number of edges in the visibility graph. (Recall that the visibility graph is the graph whose nodes are vertices of obstacles and whose edges join vertices for which the line segment between them does not intersect the interior of any obstacle.) Once the visibility graph is constructed, we can search it using Dijkstra's algorithm to find a shortest path in time $O(n \log n + K)$ (see [AAGHI, Le, Mi1, SS, We].

Next, we state some rather obvious facts about the local optimality properties of shortest paths between regions.

Lemma 2.1 *A shortest path from a vertex v to a region R not containing v is a line segment $\overline{vw}$ from v to a point w on the boundary of R such that the relative interior of $\overline{vw}$ does not intersect R, and w is either a vertex of R, or w lies interior to an edge of R such that $\overline{vw}$ is perpendicular to that edge. In case w is a vertex of R, all edges of R incident to w must lie in the closed halfplane H, where H is defined by the line through w perpendicular to $\overline{vw}$ and H does not contain v. (The line defining H is "locally supporting" at point w.)*

Lemma 2.2 *A shortest path from a region R to a region R' is given by a line segment $\overline{uu'}$ such that the relative interior of $\overline{uu'}$ does not intersect $R \cup R'$, and one of the following cases must hold: (i). u and u' are vertices of R and R', respectively, in which case $\overline{uu'}$ makes an angle $\geq \pi/2$ with all edges incident to u or u'; (ii). u is a vertex of R and u' lies interior to an edge of R' such that $\overline{uu'}$ is perpendicular to that edge; (iii). u' is a vertex of R' and u lies interior to an edge of R such that $\overline{uu'}$ is perpendicular to that edge; or (iv). u and u' both lie interior to edges, and $\overline{uu'}$ is perpendicular to both of these edges. Furthermore, there always exists a shortest path which satisfies one of the first three cases.*

We define a line segment to be *locally optimal* if its interior is disjoint from the obstacles and 0-regions and it is either (i). a shortest path between two 0-regions, or (ii). a shortest path from an obstacle vertex (or s or t) to a 0-region such that it is locally tangent to the obstacle at that vertex, or (iii). a segment joining two obstacle vertices (or s or t) such that it is locally tangent at each of its endpoints. For a definition and discussion of local tangency, see [Mil]. Let $\mathcal{S}$ be the set of locally optimal line segments. Various locally optimal segments are illustrated in Figure 4.

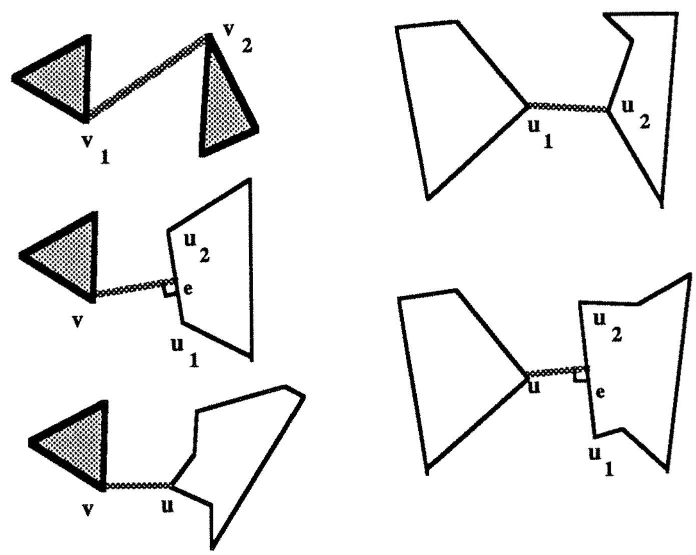

Figure 4. The various types of locally optimal segments $\mathcal{S}$.

We define the set W of *0-entrance points* to be the set of endpoints w of locally optimal segments $\overline{vw}$ for which w is an interior point of a boundary edge of a 0-region (and, hence, $\overline{vw}$ is perpendicular to the edge at point w).

It is easy to see that there are at most a quadratic number of locally optimal segments (and hence also at most a quadratic number of 0-entrance points).

Lemma 2.3 *There are at most $O(n^2)$ locally optimal line segments.*

Note that our definition of locally optimal segments rules out segments which cross 0-regions. In effect, we are saying that we can treat 0-regions as if they were "obstacles" for purposes of local optimality. The reason for this is that no shortest path can just "cut across" a 0-region: it can enter such a region at a vertex of the region or at a normal to an edge of the region; however, in such a case we would be able to use other locally optimal segments in the path. Thus, we have no need for segments which pass through 0-regions, and we can treat the 0-regions as if they were obstacles. (Once in a 0-region, a path needs to get from one point to another, but an arbitrary path, for instance a path along the boundary of the 0-region, will suffice. Later, when we consider lexicographic minimization, we will want to get through 0-regions in the shortest possible way.)

Lemma 2.4 *We do not need to consider segments whose interior intersects 0-regions.*

Proof: Consider a segment $e = \overline{pq}$ whose interior intersects 0-region R (but does not intersect any obstacle). Let $p' \in R$ be the point of $int(e) \cap R$ that that is closest to p. Similarly, let $q' \in R$ be the point of $int(e) \cap R$ closest to q. If $p' = q'$, then p' is a vertex of R, so the segments from p to p' and from p' to q would be considered separately as locally optimal segments. So, assume that $p' \neq q'$. Then, letting $d(p,q)$ indicate the weighted Euclidean length of segment $\overline{pq}$, we have $d(p,q) \leq d(p,p') + d(p',q') + d(q',q) = d(p,p') + d(q',q)$, since $p', q' \in R$. Thus, $d(p,q) \leq |\overline{pp'}| + |\overline{q'q}| < |\overline{pq}|$, showing that we need not consider segment e, whose Euclidean length is $|\overline{pq}|$. ∎

Lemma 2.5 *Shortest paths will consist of a union of locally optimal line segments through the 1-regions, together with arbitrary paths through 0-regions.*

Define the *extended visibility graph*, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, as follows:

$$\mathcal{V} = V \cup \{s,t\} \cup W$$

is the set of nodes, and

$$\mathcal{E} = E \cup \{(v,v') : v, v' \in V \cup \{s,t\}, \overline{vv'} \in \mathcal{S}\}$$
$$\cup \{(v,p) : v, p \in V, \overline{vw} \in \mathcal{S}, w \in int(\overline{pq})\}$$

is the set of edges. For edges of the third type ((v,p)), we keep a pointer to the 0-entrance point w, so that the path from v to w to p can be reconstructed should the edge (v,p) appear on the shortest path through $\mathcal{G}$. For the set of obstacles and 0-regions of Figure 5, we show the extended visibility graph in Figure 6.

Lengths are assigned to edges of $\mathcal{G}$ as follows. For $e \in E$, we assign the length 0 if e borders a 0-region, and otherwise we simply assign the Euclidean length of e. Edges of the form (v,v') get assigned their Euclidean length. Edges of the form (v,p) get assigned the Euclidean length of $\overline{vw}$.

Lemma 2.6 *If there exists a feasible path from s to t, then there is a shortest path from s to t, and one such shortest path must lie among paths in the extended visibility graph $\mathcal{G}$, where edges of the form (v,p) on the shortest path through $\mathcal{G}$ are replaced by the two segments $\overline{vw}$ and $\overline{wp}$.*

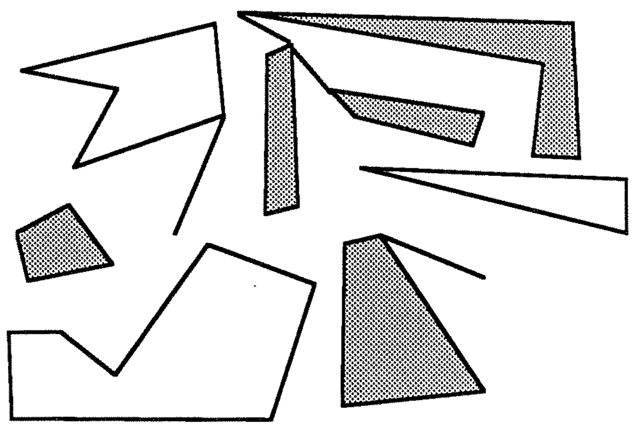

Figure 5. A collection of obstacles and zero-cost regions.

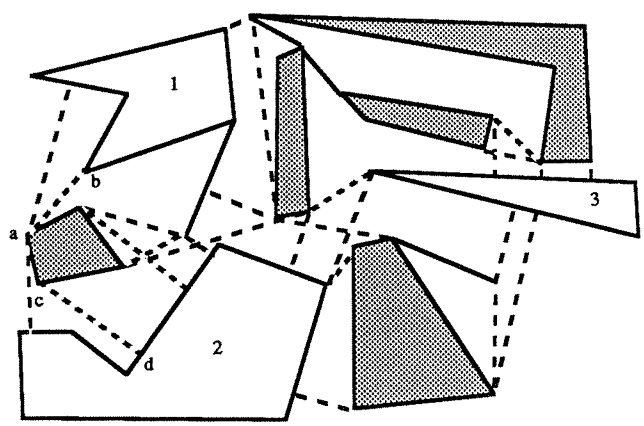

Figure 6. The extended visibility graph.

The graph $\mathcal{G}$ can be constructed in much the same way as standard visibility graphs. We appeal to the results of [AAGHI, We] to get a quadratic time bound and to the recent results of [GM] to get an output-sensative bound.

Lemma 2.7 *The extended visibility graph can be constructed in time $O(n^2)$. In fact, if $K = |\mathcal{E}|$, then we can build $\mathcal{G}$ in time $O(n \log n + K)$.*

Proof: We apply the visibility graph techniques of [AAGHI, We] or of [GM]. We first compute the visibility graph for the set of all edges E, where we treat obstacles and 0-regions as blockages to visibility. This is done in time $O(n^2)$ by the methods of [AAGHI, We], or by the new results of [GM] in time $O(n \log n + K)$, where K is the size of the output. Now, around each vertex v, we can output the visibility polygon (within the stated complexities). For edges of the visibility polygon which bound 0-regions, if the foot of the perpendicular, w, from v to the edge lies on the visible portion of the edge, then we establish an edge of type (v, p) from

v to an endpoint p of the edge and we record point w and distance $|\overline{vw}|$ along with this edge. Clearly, this additional work can be done in time $O(K)$. The concepts of the construction are shown in Figures 7(a–c), where obstacles are shown shaded gray, and 0-regions are shown in white. ∎

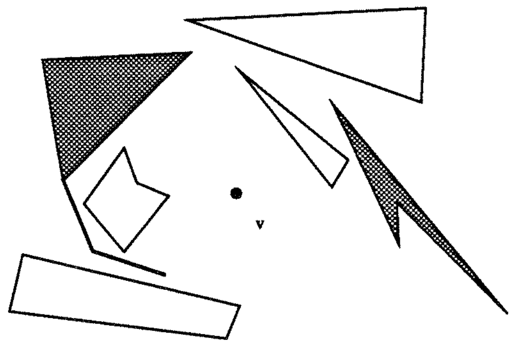

Figure 7(a). Obstacles and 0-regions about vertex v.

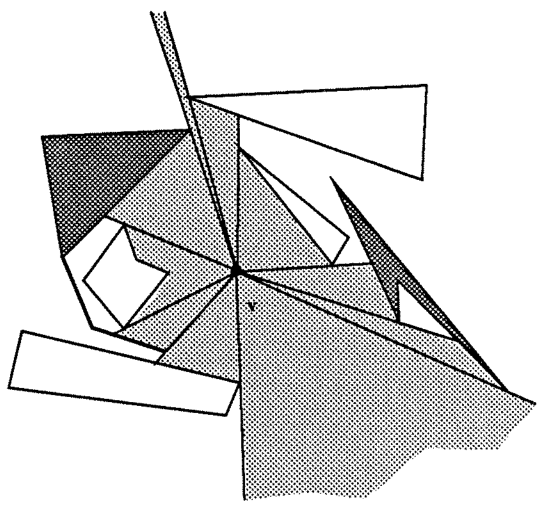

Figure 7(b). The visibility polygon about v.

Once we have constructed the extended visibility graph $\mathcal{G}$, we can search it using Dijkstra's algorithm. We have, then, by the results of [FT], the following theorem.

Theorem 2.8 *The shortest path problem in the presence of* $\{0, 1, +\infty\}$ *weighted regions can be solved in time* $O(n \log n + K)$*, where* $K = |\mathcal{E}|$ *is the number of edges in the visibility graph of the* n_e *edges defining the subdivision.*

Note that any two points of a 0-region can be joined by a path of length 0. The way we have defined $\mathcal{G}$, connections are made through 0-regions by following the boundaries. (Edges in E which bound 0-

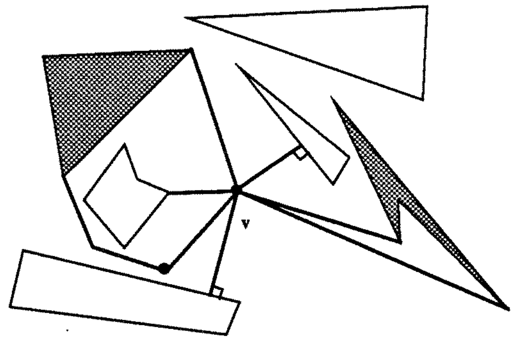

Figure 7(c). Edges of $\mathcal{G}$ out of v.

regions have length zero, and vertices v which connect to 0-entrance points are connected to endpoints of the bounding edge.) We could think of it as though we are shrinking 0-regions to single nodes, making zero-length connections to vertices and 0-entrance points of the region.

3. Special Cases

Convex Zero-Cost Regions

In some applications, we know that the 0-regions are convex. For example, in the case of the maximum concealment problem with multiple observers, the shadow regions will either be convex or they will be "bays" of the obstacles that cast shadows.

For the case of m_0 convex 0-regions with a total of n_0 vertices, and a collection of obstacles with a total of n vertices, we have the following result, which is an improvement in the case that $m_0 << n_0$.

Theorem 3.1 *The relevant part of the extended visibility graph can be computed in time* $g(m_0, n_0, n) = O(K + n \log n + (n + m_0)m_0 \log(n_0 + \frac{n_0}{m_0} n))$, *where K is the number of edges in the visibility graph among the obstacles alone. Furthermore, shortest paths can then be computed within this same time bound.*

In the case that all the zero-cost regions are *convex*, we get a simplification. Namely, now we can define $VG^* = (V^*, E^*)$, where is the set of nodes, and

$$E^* = E \cup \{(v, R) : v \in V \cup \{s, t\}, R \in \mathcal{R}_0\}$$
$$\cup \{(R, R') : R, R' \in \mathcal{R}_0 \ R \neq R'\}$$

is the set of edges. Basically, we have shrunk the zero-cost regions to single nodes, and connected them to obstacle vertices and other zero-cost regions. The basic reason for the simplification possible here is that the shortest obstacle-avoiding path between two convex regions will always consist of a piecewise linear path along vertices of the obstacles such that the first and last segments of the path are shortest paths from the respective points to the respective regions.

Lengths are assigned to edges as follows: edges in E get their usual Euclidean length; edges of the form (v, R) are assigned the distance from point v to polygon R (that is, the length of the shortest line segment from v to a point of R), if the shortest segment from v to R is obstacle-free and does not cross a zero-cost region, otherwise the edge is assigned a length of $+\infty$; and edges of the form (R, R') are assigned the distance between R and R' (that is, the length of the shortest line segment from a point of R to a point of R'), if the shortest segment is obstacle-free and does not cross a zero-cost region, otherwise the edge is assigned a length of $+\infty$.

Lemma 3.1 *If an optimal path exists from s to t, then there is an optimal path which lies on the extended visibility graph VG^*.*

Proof: Omitted here. ▮

Lemma 3.25 *The construction of VG^* in the case of convex zero-cost regions can be done in time* $g(m_0, n_0, n) = O(n^2 + (n + m_0)[\min\{m_0 \log \frac{n_0}{m_0}, n_0\} + m_0 \log(n + m_0)]) = O(n^2 + (n + m_0)m_0 \log(n_0 + \frac{n_0 n}{m_0})$.

Proof: The VG among the obstacles can be found in time $O(n^2)$ ([AAGHI], [LC], [We]). Between each pair of convex zero-cost regions we must find the minimal-distance segment. This can be done in logorithmic cost per pair of regions, at a total of $O(m_0^2 \log \frac{n_0}{m_0})$, or can be done in $O(n_0)$ per region, at a total cost of $O(m_0 n_0)$. Perform a similar task to compute the segments of tangency between every pair of zero-cost regions. (These will be needed to determine visibility among the zero-cost regions.) The shortest distance from each obstacle vertex to each zero-cost region can be found in total time $O(nm_0 \log \frac{n_0}{m_0})$ or $O(nn_0)$, whichever is smaller. Now, to determine which of the segments is feasible (ie, does not intersect the interior of any zero-cost region or any obstacle), we sort the segments (including the segments from obstacle vertices to zero-cost regions, segments between zero-cost regions, and segments of tangency between zero-cost regions) by slope around each convex zero-cost region, and perform a standard visibility algorithm on the result. (This is similar to the technique used in [Mil] to compute the necessary part of the visibility graph among a collection of convex obstacles.) The sorting step will require $O(m_0(n + m_0)\log(n + m_0))$. ▮

Once we have constructed VG^*, we can find shortest paths from s by running Dijkstra's algorithm in time $O((m_0+n)^2 + (m_0+n)\log(m_0+n)) = O((m_0+n)^2)$ (since $|V^*| = O(m_0+n)$ and $|E^*| = O(n^2 + nm_0 + m_0^2)$.

Theorem 3.3 *The shortest path problem in the presence of obstacles and convex zero-cost regions can be solved in time $O((m_0 + n)^2 + g(m_0, n_0, n))$.*

Note that this is an improvement over the general case handled by Theorem 2.3 when $m_0 << n_0$.

Convex Zero-Cost Regions and Convex Obstacles

Sometimes we also know that the obstacles are convex. We can get a further improvement here, if the number of obstacles is very small in comparison with the number of vertices defining them. The details appear in the full paper.

A further improvement is possible if both the obstacles and the zero-cost regions are convex. We will give the details of the improvement in the full paper. We comment here that the result depends on computing the "projection tangents" between obstacles (convex polygons) and zero-cost regions (also convex polygons). The notion is illustrated in Figure 8, where $\overline{rR}$ is the "right" projection tangent and $\overline{lL}$ is the "left" projection tangent. The only obstacle vertices for which we will need to compute shortest paths to the convex zero-cost region (which contains points r and l) are the points R and L.

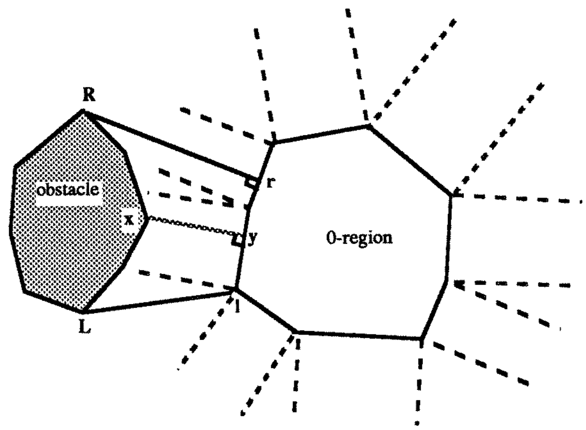

Figure 8. The projection tangents between two convex polygons.

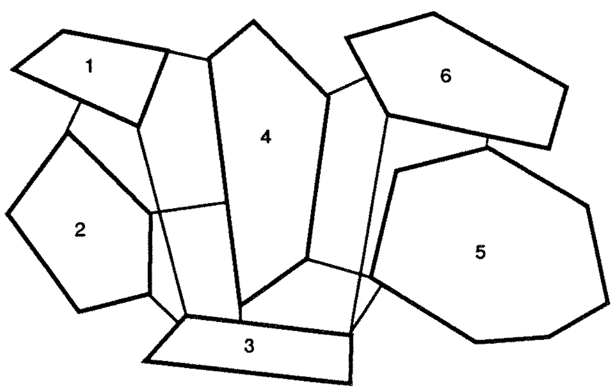

Figure 9. Critical graph for a set of convex 0-regions.

4. Roads

We describe now the important case of path planning among a set of linear features. We think of these one-dimensional features as "roads", although many other interpretations are possible. Linear features may be roads, rivers, ditches, power lines, very skinny weighted regions, etc. Applications include optimal path

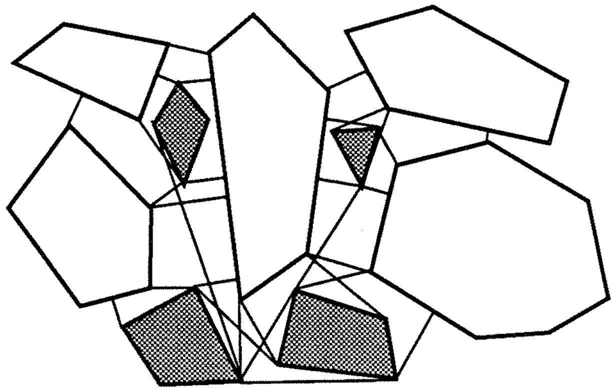

Figure 10. Critical graph for a set of convex 0-regions and convex obstacles.

planning for a mobile robot or optimal layout of pipelines, power lines or phone lines. The model is that the weight attached to the feature indicates a cost of motion or of construction (per unit length) along the feature.

We will begin our discussion by considering only the effect of linear features (thus, there are no obstacles or zero-cost regions, but only linear features and the background). So we assume for now that the background represents terrain all of which is equally costly to traverse. In the context of the weighted region problem, we are assuming that $\alpha_f = 1$ for all faces f, and that α_e is arbitrary on the edges. (We have slightly generalized the weighted region problem to allow for a fixed cost ξ_e of crossing edge e.) At the end of this section, we will generalize to the case in which there may be obstacles and/or zero-cost regions present as well as roads (i.e., $\alpha_f \in \{0, 1, +\infty\}$ and α_e arbitrary).

We are given a starting point s and a destination point t. We are also given a planar straight-line graph consisting of n straight edges, as shown in Figure 11. Each edge e has a weight $\alpha_e \in [0, +\infty]$ attached to it. (In the figure, roads with greater weight are drawn with a bolder line.) The weight α_e indicates the cost per unit distance of motion along the edge. We shall furthermore assume that there is a cost $\xi_e \in [0, +\infty]$ for *crossing* edge e. (Such a cost models, for instance, the cost of crossing a ditch or ravine.) The cost ξ_e is added to the length of any path which crosses the interior of e. Note that if $\xi_e = \infty$, then edge e acts as an obstacle.

Note that any edge e for which $\xi_e = 0$ and $\alpha_e \geq 1$ can be eliminated from consideration, since we will never want to travel along such an edge.

Our techniques for finding shortest paths are again to reduce the continuous problem to a discrete problem of searching an appropriate "visibility graph". We are able to prove that there exists a discrete, polynomially-sized graph which contains all shortest paths. The proof relies on exploiting the local optimality criteria of how optimal paths must enter and leave road edges. This will allow us to obtain an $O(n^2 \log n)$ algorithm for our problem.

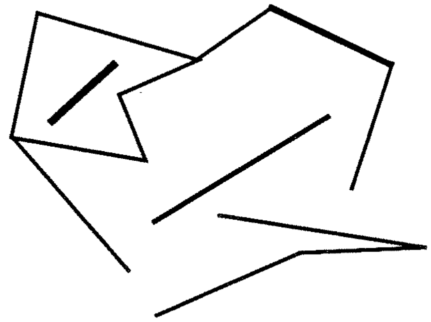

Figure 11. A network of linear features.

The key idea is that shortest paths will get onto the road network in a known way: either by entering at a vertex of a road segment or by hitting the interior of a road segment at the "critical angle", thus satisfying the local optimality criterion (assuming the road is "cheap").

For a road edge e, we define the *critical angle* as $\theta_c(e) = \sin^{-1}(\alpha_e)$. The angle that a path makes with an edge e at a bend point $w \in e$ of the path is defined to be the angle between the path and the normal line to the edge at the point w. We call w a critical point of entrance of a path on edge e if the path enters e at point w at the critical angle $\theta_c(e)$. We similarly define a critical point of exit of a path on an edge.

For an endpoint p of an edge $e = \overline{pq}$, we define the *critical cone* at p to be the cone pointed at p of all points r for which $\angle rpq \geq \pi/2 + \theta_c(e)$. Refer to Figure 12.

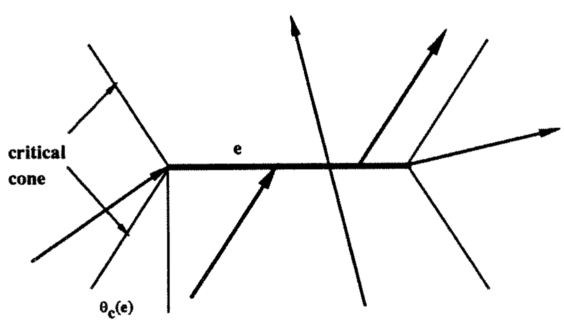

Figure 12. Local optimality conditions at a road edge.

The formal local optimality criteria are contained in the following lemmas. Figure 12 illustrates examples of each.

Lemma 4.1 *An optimal path that enters or leaves a road edge at a point interior to that edge must do so at the critical angle, $\theta_c(e)$, defined by the edge.*

Lemma 4.2 *An optimal path that enters or leaves a road edge at an endpoint of the edge must lie locally within the critical cone at that endpoint.*

Lemma 4.3 *An optimal path which crosses a road edge must do so at an angle of incidence less than the critical angle.*

We must also appeal to the following result of [MP] which states that between a critical point of exit and a critical point of entry there must be a vertex. This assures that we will not go from road segment to road segment (entering and leaving at critical angles), and therefore that we will not have to deal with too many critical points of entrance and exit. (Note that in order to go from one edge to another at critical angles would require that the edges be in a degenerate position, as it specifies an exact angle relationship between the two edges.)

Lemma 4.4 *There always exists a shortest path which does not contain segments that join a critical point of exit of edge e to a critical point of entrance of another edge e'. In other words, in our search for a shortest path, we can assume that between a critical point of exit and a critical point of entrance, there must exist a vertex (an endpoint of an edge).*

Line segments which join vertices to vertices or vertices to critical points that additionally satisfy the local optimality conditions of Lemmas 4.1–4.3 are called *locally optimal segments.* Our algorithm contructs the *critical graph* $\mathcal{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$, defined as follows. $\mathcal{V}^*$ is the set of all endpoints of locally optimal segments (these include vertices and critical points), and $\mathcal{E}^*$ is the set of locally optimal segments, together with segments joining all pairs of critical points that are on the same edge and are adjacent. Segments that lie on road edges are assigned a length equal to the product of the edge weight and the segment's Euclidean length; segments that pass through the background are assigned their Euclidean length plus the sum of all the fixed costs of all of the edges that the segment crosses.

Theorem 4.5 *Optimal paths can be found among paths in the critical graph defined above.*

Algorithmically, we can compute the critical graph as follows. For each vertex $v \in V \cup \{s, t\}$ and for each edge e, we determine the two critical points on the line through e with respect to v. Portions of the edge e which lie beyond the critical points (for which the angle of incidence from v would be greater than $\theta_c(e)$) can now be treated as obstacles from v. Compute the visibility polygon from v among these $O(n)$ line segments. This requires $O(n \log n)$ per vertex v. Now, instantiate a node of $\mathcal{G}^*$ at the critical points which lie inside the visibility polygon about v, and connect v to these nodes via an edge. Also, connect v to any other vertex v' visible from v for which the segment $\overline{vv'}$ lies within the critical cone at v'. Assign lengths to these connections according to the distance from v to the point, plus the sum of the fixed costs of any edges that are crossed by the segment. All of this can be done within the $O(n \log n)$ time bound for each vertex v. An example is shown in Figure 13. The result of making connections from each vertex is the critical graph, an example of which is shown in Figure 14.

Theorem 4.6 *The complexity of finding shortest paths among linear features is $O(n^2 \log n)$.*

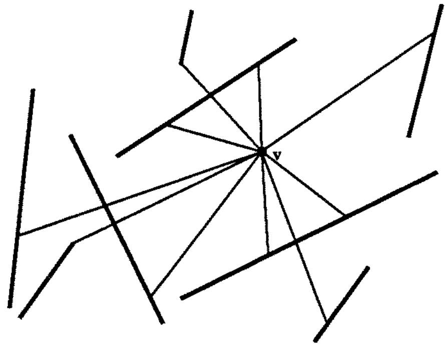

Figure 13. Making connections between a vertex and road edges.

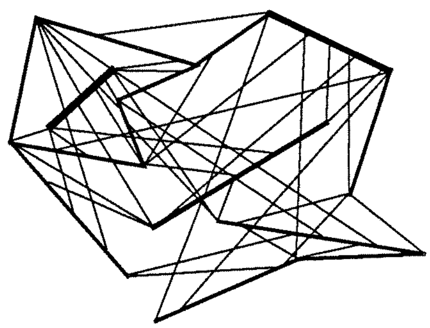

Figure 14. The critical graph.

We now consider the case in which there are obstacles and 0-regions in addition to roads. Basically all we have to do is to combine the results of this section and the last. We build a composite critical graph which takes into account the local optimality criteria of how optimal paths behave at obstacle vertices, how they behave at 0-region vertices and edges, and how they behave at road edges and vertices. The overall result is the following theorem.

Theorem 4.7 *The complexity of solving the weighted region problem in the case that $\alpha_f \in \{0, 1, +\infty\}$ for each face f and $\alpha_e \in [0, +\infty]$ for each edge e is $O(n^2 \log n)$, where n is the total number of vertices describing the planar subdivision. Within this same complexity, we can allow there to be a fixed cost $\xi_e \in [0, +\infty]$ of crossing an edge e.*

It is straightforward to generalize these results to the case in which edges may be directed, so that there is a different cost of motion in each direction along an edge. We could also consider edges to have two "sides", with a possibly different speed on each side, and the sides are not accessible to each other. For example, we may be modelling a divided highway whose median is not traversable. This can be modelled in our framework by sandwiching an obstacle edge (with $\xi = +\infty$) between two edges of possibly different speeds.

5. Solving the Lexicographic Weighted Region Problem

An important application of our techniques is to solve the "lexicographic" weighted region problem. We want to minimize the length of path in the most expensive region, then subject to this being minimized, we want to minimize the length of path in the second most costly region, then subject to this being minimized, etc. More formally, let the region weights be $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_W < +\infty$. Then, for a path $\mathcal{P}$ let $(d_W(\mathcal{P}), d_{W-1}(\mathcal{P}), \ldots, d_1(\mathcal{P}))$ be the vector of subpath lengths, where d_i is the length of the path $\mathcal{P}$ which passes through regions of weight α_i. Our goal is to minimize, lexicographically, the vector $(d_W(\mathcal{P}), d_{W-1}(\mathcal{P}), \ldots, d_1(\mathcal{P}))$ over all paths $\mathcal{P}$ from s to t.

Given a planar straight-line subdivision with weights assigned to regions, rank the weights from smallest to largest: $0, \alpha_1, \alpha_2, \ldots, \alpha_W, +\infty$. Zero-weight regions will always have weight zero, and infinite-cost regions will always have an infinite cost (and thus be obstacles). Rescale so that $\alpha_W = 1$. Treat all regions with weights less than α_W as if their weights were zero, and run our algorithm for the $\{0, 1, +\infty\}$ problem to find the shortest path in time $O(n^2)$. Now take the resulting solution path and look at its subpaths through regions with weight less than α. Solve each subpath problem again by selecting the highest weight (which is now the second-highest weight) and continue recursively until we reach the lowest non-zero weight. The result is a path which is lexicographically optimal. The overall complexity will be $O(n^2 W)$.

Theorem 5.1 *The complexity of finding the lexicographically shortest path through weighted regions is $O(n^2 W)$, where n is the number of vertices in the planar subdivision and W is the number of differently weighted regions.*

As a corollary, we get the fact that we can solve the lexicographic shortest path problem in the $\{0, 1, +\infty\}$ weighted region problem in time $O(n^2)$.

This approach is potentially very valuable to region planners, as it quickly (in polynomial time) finds a "good" path which can be further improved then by applying Snell's Law of refraction at each region boundary. This gives a fast heuristic for solving the weighted region problem. Note, however, that the weighted length of a lexicographically minimal path may be arbitrarily bad in comparison with the shortest weighted length path. The problem is that in order to save a few inches in a weight α region, we are willing to go arbitrarily many miles in a weight $\alpha' < \alpha$ region, and α' may be very close to α. An idea which improves the performance of the heuristic is to subdivide the weight α' regions into smaller pieces (with, perhaps a bounded ratio of length to width) and to put fixed costs on the edges used for the subdivision. Then, the very long paths through the weight α' regions are not "free" anymore. This approach is similar to the technique of using a "region graph", as explained in [MP, Mi2].

6. Generalizations and Extensions

Several generalizations to our problem are possible.

1). We could allow roads to be curved. Then, the local optimality condition at an interior road point y must be applied as if the boundary at y is the straight line tangent to the boundary at y.

2). We could allow the obstacles, 0-regions, and roads to reside on a nonplanar surface (e.g., it could be the surface of a polyhedron). We would then have to modify the algorithm of [MMP] to handle the case of roads and 0-regions. This generalization should yield an $O(n^2 \log n)$ algorithm for the problem on a surface.

3). An important open problem is to solve the special case of the weighted region problem with weights in the set $\{1, k\}$. That is, can the problem be solved exactly for two non-zero finite weights? Note that it would then be possible to solve the case with weights in the set $\{0, 1, k, +\infty\}$, even with arbitrary weight roads.

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