**Latin Squares and F Squares from Euler until Now**

(Seminar, SUNY Binghamton, November 16, 1984)

by

Walter T. Federer

Biometrics Unit, Cornell University, Ithaca, NY 14853

BU-863-M November 1984

Overhead No. 1 — Combinatorial Methods of Mathematics

Overhead No. 2 — Latin Square Developments Since Euler

Overhead No. 3 — F Square Developments

Overhead No. 4 — F Rectangle Developments

Overhead No. 5 — POFRDs

Overhead No. 6 — That Number 6!

Overhead No. 7 — Other Special Numbers 12 and 15

Overhead No. 8 — Other Developments

Some Selected Literature (mostly Cornell University)

Anderson, D. A., W. T. Federer, and E. Seiden (1974). On the construction of orthogonal F(2k,2)-squares. No. BU-500-M in the Mimeo Series of the Biometrics Unit, Cornell University, February.

Cheng, C.-S. (1977). Optimal designs for the elimination of heterogeneity. Ph.D. Thesis, Cornell University.

Cheng, C.-S. (1980). Orthogonal arrays with variable numbers of symbols. *Annals of Stat.* 8:447-453.

Denes, J. and A. D. Keedwell (1974). *Latin Squares and Their Applications.* Akadémiai Kiadó, Budapest; English Universities Press, London; Academic Press, New York.

Euler, L. (1782). Recherches sur une nouvelle espèce des quarrés magiques. Verh. Zeeuwsch Genoat. Wetenschappen, Vlissigen, 9, 85-239. (Translation BU-405-M in the Mimeo Series of the Biometrics Unit, Cornell University and Leonardi Euleri Opera Omnia, Series 1, 7(1923), 291-392.

Federer, W. T. (1955). *Experimental Design — Theory and Application.* Macmillan, New York. (Republished by Oxford and IBH Publishing Co., New Delhi, 1967, 1974.)

Federer, W. T. (1972). Construction of classes of experimental designs using transversals in Latin squares and Hedayat's sum composition method. In *Statistical Papers in Honor of George W. Snedecor* (T.A. Bancroft, ed.), Iowa State University Press, Ames, pp. 91-114.

Federer, W. T. (1975). On the construction of F-squares and single degree-of-freedom contrasts (preliminary). No. BU-566-M in the Mimeo Series of the Biometrics Unit, Cornell University, August.

Federer, W. T. (1977). On the existence and construction of a complete set of orthogonal F(4t;2t,2t)-squares design. *Annals of Stat.* 5:561-564.

Federer, W. T. (1984). Statistical analyses for multistage experiments. *Biometrical J.* 26(5), 535-553.

Federer, W. T., A. S. Hedayat, and J. P. Mandeli (1984). Pairwise orthogonal F-rectangle designs. *J. Statistical Planning and Inference* 10 (to appear).

Federer, W. T., A. Hedayat, E. T. Parker, B. L. Raktoe, Esther Seiden, and R. J. Turyn (1970). Some techniques for constructing mutually orthogonal latin squares. *Proc. 15th Conference Design Experiments Army Research Development Testing.* ARO-D Report 70-2, pp. 673-796. (Also Math. Res. Center Tech. Summary Report #1030, U.S. Army and University of Wisconsin.)

Federer, W. T. and H. C. Kirton (1984). Incomplete block and lattice rectangle designs for v=36 using F-square theory. BU-850-M in the Technical Report Series of the Biometrics Unit, Cornell University, September.

Federer, W. T., F.-C. H. Lee, and J. P. Mandeli (1976). F-squares geometries for n=3,4,5, and 6. BU-591-M in the Mimeo Series of the Biometrics Unit, Cornell University, August.

Federer, W. T. and J. P. Mandeli (1983). Orthogonal F-rectangles, orthogonal arrays, and codes. BU-822-M in the Technical Report Series of the Biometrics Unit, Cornell University, June.

Finney, D. J. (1945). Some orthogonal properties of the 4 × 4 and 6 × 6 Latin squares. *Annals Eugenics* 12:213-219.

Finney, D. J. (1946a). Orthogonal partitions of the 5 × 5 Latin squares. *Annals Eugenics* 13:1-3.

Finney, D. J. (1946b). Orthogonal partitions of the 6 × 6 Latin squares. *Annals Eugenics* 13:184-196.

Finney, D. J. (1982). Some enumerations for the 6 × 6 Latin squares. *Utilitas Mathematica* 21(A):137-153.

Freeman, G. H. (1966). Some non-orthogonal partition of 4 × 4, 5 × 5, and 6 × 6 latin squares. *Annals Math. Stat.* 37:661-681.

Golomb, S. W. and E. C. Posner (1964). Rook domains, Latin squares, affine planes, and error-distributing codes. *IEEE Transactions on Information Theory* IT-10:196-208.

Hedayat, A. (1969). On the theory of the existence, non-existence, and the construction of mutually orthogonal F-squares and Latin squares. Ph.D. Thesis, Cornell University.

Hedayat, A. (1971). A set of three mutually orthogonal latin squares of order 15. *Technometrics* 13:696-698.

Hedayat, A. (1973a). Self-orthogonal latin square designs and their importance. *Biometrics* 29:393-396.

Hedayat, A. (1973b). An application of sum composition: A self-orthogonal latin square of order ten. *J. Comb. Theory* 14:256-260.

Hedayat, A. and W. T. Federer (1969). An application of group theory to the existence and non-existence of orthogonal Latin squares. *Biometrika* 56:547-561.

Hedayat, A. and W. T. Federer (1970a). An easy method of constructing partially replicated latin square designs or order n for all n>2. *Biometrics* 26:327-330.

Hedayat, A. and W. T. Federer (1970b). On the equivalence of Mann's group automorphism method of constructing an OL(n,n-1) set and Raktoe's collineation method of constructing a balanced set of $\ell$-restrictional prime-powered lattice design. *Annals Math. Statistics* 41:1530-1540.

Hedayat, A. and W. T. Federer (1971). On embedding and enumeration of orthogonal latin squares. *Annals Math. Statistics* 42:509-516.

Hedayat, A. and W. T. Federer (1975). On the nonexistence of Knut Vik designs for all even orders. *Annals Statistics* 2:445-447.

Hedayat, A. and W. T. Federer (1984). Orthogonal F-rectangles for all even v. *Calcutta Statistical Assoc. Bulletin* (to appear).

Hedayat, A., E. T. Parker, and W. T. Federer (1970). The existence and construction of two families of designs for successive experiments. *Biometrika* 57:351-355.

Hedayat, A., D. Raghavarao, and E. Seiden (1975). Further contributions to the theory of F-squares design. *Annals Stat.* 3:712-716.

Hedayat, A. and E. Seiden (1970). F-square and orthogonal F-square designs: A generalization of Latin square and orthogonal Latin square design. *Proc. 2nd Chapel Hill Conference Combinatorial Mathematics Applications*, Chapel Hill, N.C., pp. 261-275 and *Annals Math. Stat.* 41:2035-2044.

Hedayat, A. and E. Seiden (1971). On a method of sum composition of orthogonal latin squares. I. *Proc. Conference Combinatorial Geometry Applications*, Perugia, Italy, pp. 239-256.

Hedayat, A., E. Seiden, and W. T. Federer (1972). Some families of designs for multistage experiments: mutually balanced Youden designs when the number of treatments is prime power or twin primes. I. *Annals Math. Statist.* 43:1517-1527.

Hedayat, A. and S. S. Shrikhande (1971). Experimental designs and combinatorial systems associated with latin squares and sets of mutually orthogonal latin squares. *Sankhyā*, Series A, 33(4):423-432.

Mandeli, J. P. (1975). Complete sets of orthogonal F-squares. M.S. Thesis, Cornell University.

Mandeli, J. P. (1978). Contributions to the theory of F-square and F-cube designs. Ph.D. Thesis, Cornell University.

Mandeli, J. P. and W. T. Federer (1983). An extension of MacNeish's theorem to the construction of sets of pairwise orthogonal F-squares of composite order. *Utilitas Mathematica* 24:87-96.

Mandeli, J. P. and W. T. Federer (1984a). Complete sets of orthogonal F-squares of prime power order with differing numbers of symbols. In *Experimental Design, Statistical Models, and Genetic Statistics* (K. Hinkelmann, ed.), Marcel Dekker, Inc., New York and Basel, Chapter 5, pp. 45-59.

Mandeli, J. P. and W. T. Federer (1984b). On the construction of mutually orthogonal F-hyperrectangles. *Utilitas Mathematica* 25:315-324.

Mandeli, J. P., F.-C. H. Lee, and W. T. Federer (1981). On the construction of orthogonal F-squares of order n from an orthogonal array (n,k,s,2) and an OL(s,t) set. *J. Statist. Planning and Inference* 5:267-272.

Raktoe, B. L. (1967). Application of cyclic collineations to the construction of balanced $\ell$-restrictional prime power lattice designs. *Annals Math. Statistics* 38:1127-1141.

Rao, C. R. (1946). Hypercubes of strength d leading to confounded designs in factorial experiments. *Bull. Calcutta Math. Soc.* 38:67-78.

Rao, C. R. (1973). Some combinatorial problems of arrays and applications to design of experiments. In *A Survey of Combinatorial Theory* (J.N. Srivastava, ed.), North-Holland, Amsterdam, pp. 349-359.

Schellenberg, P. J., G. H. J. vanRees, and S. A. Vanstone (1978). Four pairwise orthogonal Latin squares of order 15. *Ars Combinatoria* 6:141-150.

Schwager, S. J., W. T. Federer, and J. P. Mandeli (1984). Embedding cyclic Latin squares of order $2^n$ in a complete set of orthogonal F-squares. *J. Statist. Planning and Inference* 10:207-218.

Schwager, S. J., W. T. Federer, and B. L. Raktoe (1984). Nonisomorphic complete sets of orthogonal F-squares and Hadamard matrices. *Communications in Statistics* 13(11):1391-1406.

Street, D. J. (1979). Generalized Hadamard matrices, orthogonal arrays, and F-squares. *Ars Combinatoria* 8:131-141.

Def'n: Let $A = [a_{ij}]$ be an $n \times n$ matrix and let $\Sigma = \{c_1, c_2, \ldots, c_m\}$ be an ordered set of distinct elements of $A$. Further, suppose for $k = 1, 2, \ldots, m$, $c_k$ appears $\lambda_k$ ($\lambda_k \geq 1$) times in each row and each column of $A$. Denote $A$ as a frequency sq. F-square of order $n$ and frequency vector $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m)$.

When $m = n$ $\lambda_k = 1$, a Latin square results.

$$LS(5) = FS(5; 1^5) = FS(5; 1,1,1,1,1) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix} \qquad FS(5; 2, 3) = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 2 & 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 & 2 \\ 2 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 \end{bmatrix}$$

Def'n: Given an FS $F_1(n; \lambda_1, \lambda_2, \ldots, \lambda_r)$ on an $r$-set $\Sigma_1 = \{a_1, a_2, \ldots, a_r\}$ and an FS $F_2\{n; u_1, u_2, \ldots, u_t\}$ on a $t$-set $\Sigma_2 = \{b_1, b_2, \ldots, b_t\}$.

We say $F_2$ is an $\perp$-mate for $F_1$ if upon superposition of $F_2$ on $F_1$, $a_i$ with frequency $\lambda_i$ in $F_1$ appears $\lambda_i u_j$ times with $b_j$ with frequency $u_j$ in $F_2$.

$$LS(6) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \\ 3 & 4 & 1 & 2 & 6 & 5 \\ 4 & 6 & 5 & 1 & 3 & 2 \\ 5 & 3 & 2 & 6 & 1 & 4 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix} \perp FS(6; 1, 2, 1, 1, 1) = \begin{bmatrix} 1 & 2 & 3 & 4 & 2 & 5 \\ 4 & 5 & 2 & 3 & 1 & 2 \\ 2 & 3 & 2 & 5 & 4 & 1 \\ 2 & 1 & 4 & 2 & 5 & 3 \\ 5 & 4 & 1 & 2 & 3 & 2 \\ 3 & 2 & 5 & 1 & 2 & 4 \end{bmatrix}$$

$FS(6: 1,1,1,1,1,1)$ $\qquad$ $FS(6; 1, 2, 1^3)$

# SOME LANDMARKS IN THE DEVELOPMENT OF LATIN SQUARE THEORY

1782 - EULER (1782)

1807

1832

- CLAUSEN + SCHUMAKER (1944)

1857

1882

- CAYLEY (1890)
- TARRY (1900 etc.)

1907 - VEBLEN + BUSSEY (1906)

- MACNEISH (1922)

1932 - FISHER + YATES (1934)
- NORTON (1939); BOSE (1938); STEVENS (1938); Fisher (1942)
- SADE (1948, 1951); SAXENA (1950) MANN (1942, 1943, 1944)
FINNEY (1945-6)

1957 - BOSE, PARKER, + SHRIKHANDE (1960)

- HEDAYAT (1969); HEDAYAT et al. (Federer, Parker, Seiden, Raktoe, Raghavarao, Shrikhande)
- DENES + KEEDWELL (1974)

1982

# SOME LANDMARKS IN THE DEVELOPMENT OF F-SQUARE THEORY

1940

1945 Finney (1945, 1946)

1950

1955

LAKSHMINARAYAN (1958)

1960

1965

1970 Hedayat (1969); Hedayat + Seiden (1970)

Mandeli (1975)

1975 Hedayat, Raghavarao, + Seiden (1975)

Federer (1977); Mandeli (1978); Cheng (1977)

Street (1979)

1980 Cheng (1980); Mandeli, Lee, & Federer (1981)

Mandeli + Federer (1983, 1984); F & M (1983)

1985 Schager *et al*. (1984, 1984)

1935 – Yates (1935–1940); Yates & Hale (1939)
Youden (1937, 1940, 1951)
– Brandt (1938)
1940 – Cochran, Autrey & Cannon (1941) LATIN RECTANGLES & F-RECTANGLES

1945 –

1950 – Williams (1949), " (1950, '52) Patterson (1950, '51)
Lucas (1951, 1957), Quenouille (1953)

YAMAMOTO (1956)
1955 – Finney (1956); Finney & Outhwaite (1956)
– Samford (1957) Bradley (1958)
– Patterson & Lucas (1959, 1962)
1960 – Sheehe & Bross (1961)
– Linnerud ~~et al~~ (1962)
1965 – Berenblat (1964); Federer & Atkinson (1964)
Federer and Raktoe (1965); Atkinson (1966)
– Nair (1967)

1970 – Patterson (1970, 1973)
– Hedayat, Seiden, & Federer (1972)
– Hall and Williams (1973)
1975 – Hedayat & Afsarinejad (1975)
Raghavarao, Lakatos, Mercado (1976, 1978)

1980 – Kershner (1980)
Kershner & Federer (1982)
– Federer, Hedayat, & Mandeli (1984); Hedayat & Federer (1984)
1985 –

POFRDS

1945

1950

1955

1960
- Federer (consulting design problem)

1965

1970

1975

- Cheng (1977); Mandeli (1978)

1980
- ~~Cheng (1984)~~

; Federer & Mandeli (1983)

~~Mandel in~~ Federer (1984)

- Federer, Hedayat, & Mandeli (1984); Hedayat & Federer (1984)

1985

# The # 6

1782 - Euler

1900 - Tarry (1900)

1935 - Fisher and Yates (1934)

1940

1945 - Finney (1945, 1946); YAMAMOTO (1954)

970 - Hedayat (1969)

- Anderson et al. (1974)

975 - Federer (1975)

1980

- Finney (1982)

- Kirton (1984); Federer & Kirton (1984)

1985

## LS(12)

- POLS(12, 5) - BOSE, CHAKRAVARTI, & KNUTH (1960)

  - JOHNSON, DULMAGE, & MENDELSOHN (1959)

POLS(12, 5) + POFS(12; 6, 6; 6)
- MANDELI (1978)

## LS(15)

POLS(15, 3) - Hedayat (1971)

- POLS(15, 4) - Schellenberg, van Rees, and Vanstone (1978)

## LS(n) - Wilson (1972, 1974)

POLS($n \geq 6$, 2)
POLS($n \geq 42$, 3)
POLS($n \geq 52$, 4)
POLS($n \geq 62$, 5)
POLS($n \geq 90$, 6)

- - Wojtas (1978)
  POLS($n \geq 4922$, 7)

NONISOMORPHIC SETS

$n = 9$: 4 sets Fisher (1942, 1944)
Parker (1959)

Sub latin squares wn a LS (intercalates)

Partial latin squares

Repeated Measures Designs

Partial Geometries

F-square Geometries

Projective Planes

Codes

Block and Row-Column Designs

LATIN and F HYPERCUBES and HYPERRECTANGLES