Two New Proofs of Afriat's Theorem *

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Abstract

We provide two new, simple proofs of Afriat's celebrated theorem stating that a finite set of price-quantity observations is consistent with utility maximization if, and only if, the observations satisfy a variation of the Strong Axiom of Revealed Preference known as the Generalized Axiom of Revealed Preference.

1 Introduction

The neoclassical theory of demand supposes that a consumer, facing a price vector $p \in \Re^l_{++}$ and with income $I > 0$, chooses his demand bundle $x \in \Re^l_+$ to maximize some utility function $u : \Re^l_+ \to \Re$ over his budget set $B(p, I) := \{x \in \Re^l_+ : p \cdot x \leq I\}$. We assume we have been presented with a finite data set $D := \{(p_i, x_i) : i \in N\}$, where $N := \{1, 2, \ldots, n\}$, of price vectors $p_i \in \Re^l_{++}$ and corresponding demand vectors $x_i \in \Re^l_+$. The basic question raised by Afriat is whether this data set is consistent with the maximization of a locally non-satiated utility function u in the sense that for each $i \in N$, x_i maximizes u over $B(p_i, p_i \cdot x_i)$. A locally non-satiated utility function is one for which every neighborhood of a commodity bundle contains another bundle with a higher utility. With such a utility function the consumer will have spent all his income, so that we can use $p_i \cdot x_i$ as the income for situation i.

If the set of price and quantity observations is derived from utility maximization it will surely satisfy the variation of the Strong Axiom of Revealed Preference, known as the Generalized

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Axiom of Revealed Preference, which states that, for any list $(q_1, y_1), \ldots, (q_m, y_m)$ of observed prices and associated demand vectors with the property that

$$q_j \cdot y_{j+1} \leq q_j \cdot y_j, \text{ for all } j \leq m-1,$$

we must have $q_m \cdot y_1 \geq q_m \cdot y_m$.[1]

The argument for the Generalized Axiom is straightforward. If $q_j \cdot y_{j+1} \leq q_j \cdot y_j$ then y_{j+1} could have been purchased at prices q_j. Since y_{j+1} was not purchased it cannot be strictly preferred to y_j so that $y_j \succsim y_{j+1}$. Thus y_j is "revealed preferred" to y_{j+1}. The entire sequence of inequalities therefore implies that $y_1 \succsim y_m$. If, on the other hand, $q_m \cdot y_1 < q_m \cdot y_m$ and the utility function is locally non-satiated, we could find a commodity bundle ξ close to y_1 with $q_m \cdot \xi < q_m \cdot y_m$ and $\xi \succ y_m$, violating the assumption that y_m maximizes utility at prices q_m and income $q_m \cdot y_m$.

The Generalized Axiom may be stated in a slightly different fashion which is more appropriate for our needs. If the inequalities

$$\begin{aligned} q_j \cdot y_{j+1} &\leq q_j \cdot y_j \text{ hold for all } j \leq m-1, \text{ and if} \\ q_m \cdot y_1 &\leq q_m \cdot y_m \text{ as well,} \end{aligned}$$

then we must have $q_m \cdot y_1 = q_m \cdot y_m$. But in this form there is no distinction between the last observation and any of the other observations, so that

$$q_j \cdot y_{j+1} = q_j \cdot y_j$$

holds for all j. This is the variation of the Strong Axiom which we shall adopt, not only for the full set of n observations but for any ordered subset as well.

Definition 1 *We say that the observations satisfy* the Generalized Axiom of Revealed Preference (GARP) *if for every ordered subset* $\{i, j, k, \ldots, r\} \subset N$ *with*

$$\begin{aligned} p_i \cdot x_j &\leq p_i \cdot x_i, \\ p_j \cdot x_k &\leq p_j \cdot x_j, \\ &\vdots \\ p_r \cdot x_i &\leq p_r \cdot x_r, \end{aligned}$$

it must be true that each inequality is, in fact, an equality.

[1] There is a great variety of terminology associated with the concept of revealed preference. The original definition offered by Samuelson [4], now known as the Weak Axiom of Revealed Preference (WARP), was thought by the author to be sufficient to recover a utility function generating the data. Houthakker's definition of the Strong Axiom (SARP) [3] provided the additional conditions necessary for recovery. But Houthakker's statement of the Strong Axiom is motivated by a single valued demand function rather than a finite list of observations and is, as a consequence, somewhat awkward. Afriat [1] used the terminology Cyclical Consistency (CC) for the simpler concept of the current paper. Cyclical Consistency is identical with the Generalized Axiom of Revealed Preference (GARP) introduced by Varian [5]. This does not exhaust the list of variations in terminology.

We have chosen to use the term GARP rather than Cyclical Consistency. Our purpose is to use a definition in which the phrase "Revealed Preference" actually appears rather than the earlier, equivalent terminology used by Afriat.

From the data set we can compute the square matrix $A = A(D)$ of order n defined by

$$a_{ij} := p_i \cdot (x_j - x_i) \text{ for all } i, j \in N.$$

Hence, a_{ij} negative means that x_i is revealed preferred to x_i. In this more condensed notation, the observations satisfy the Generalized Axiom if for every chain $\{i, j, k, \ldots, r\} \subset N$, $a_{ij} \leq 0, a_{jk} \leq 0, \ldots, a_{ri} \leq 0$ implies that all the terms are zero. It is clear that this condition is necessary for observations arising from utility maximization. What is less clear, and indeed surprising, is that it is also sufficient.

We state the result as the equivalence between several conditions on the data. We say that a utility function u *rationalizes* the data set D if, for each $i \in N$, x_i maximizes u over $B(p_i, p_i \cdot x_i)$.

Theorem 1 (Afriat) *The following are equivalent:*

(a) The data set D can be rationalized by a locally non-satiated utility function u.

(b) The matrix $A := A(D)$ has zero diagonal entries and satisfies the Generalized Axiom above.

(c) The matrix $A := A(D)$ has zero diagonal entries and there are ϕ_i's and positive λ_i's for $i \in N$ such that the Afriat inequalities hold:

$$\text{(AI)} \qquad \phi_j \leq \phi_i + \lambda_i a_{ij}, \qquad \text{for all } i, j \in N.$$

(d) The data set D can be rationalized by a continuous, concave, piecewise-linear, strictly monotonic utility function u.

This is a remarkable result because it gives succinct, testable conditions that a finite data set must satisfy in order to be consistent with utility maximization. Moreover, from the result, it follows that the assumptions of continuity, monotonicity and concavity are not refutable by a finite data set.

We have argued above that (a) implies (b). Clearly, (d) implies (a). Afriat gives a short simple argument that (c) implies (d), by explicitly constructing the desired utility function — we will provide the reasoning momentarily. The crux of the theorem is thus the implication from (b) to (c): for this, we establish a simple case in Section 2 and give two new proofs in Sections 3 and 4, based on induction and linear programming respectively. Section 5 contains some concluding remarks on a graph-theoretic interpretation of the hard implication and on the complexity of finding the ϕ_i's and positive λ_i's (and hence a suitable utility function) given the data D.

Suppose the Afriat Inequalities (AI) hold. Then consider the utility function

$$u(x) = \min\{\phi_1 + \lambda_1 p_1 \cdot (x - x_1), \ldots, \phi_n + \lambda_n p_n \cdot (x - x_n)\}.$$

We notice that each term in this expression is linear (and hence continuous and concave) and strictly monotone. Therefore, u, as their pointwise minimum, is continuous, concave, and strictly monotone as well. Finally, as is shown in the next two steps, u indeed generates the observations in the data set D.

1. $u(x_j) = \phi_j$, for all $j \in N$.

   By definition $u(x_j) = \min_i\{\phi_i + \lambda_i p_i \cdot (x_j - x_i)\} = \phi_j + \lambda_j p_j \cdot (x_j - x_j) = \phi_j$, where the minimum is taken by the index j from the Afriat inequalities.

2. $p_j \cdot x \leq p_j \cdot x_j \Rightarrow u(x) \leq u(x_j)$.

   $u(x) \leq \phi_j + \lambda_j p_j \cdot (x - x_j) \leq \phi_j = u(x_j)$, where the first inequality follows from the definition of u, the second from the fact that x is feasible at prices p_j and the last equality from Step 1. $\square$

The reader should be cautioned against imputing too much meaning to the utility function u constructed as above. Indeed, as an example, if all a_{ij}'s, i different from j, are positive, then for *any* permutation $(i_1, i_2, ..., i_n)$ of $(1, 2, ..., n)$, it is possible to choose ϕ_i's and positive λ_i's satisfying the Afriat inequalities above and with $\phi_{i_1} > \phi_{i_2} > \ldots > \phi_{i_n}$, and hence, by Step 1 above, with a similar ordering of the utilities. We show this at the end of Section 2.

2 A Simple Case.

We have shown that the Afriat inequalities imply the existence of a nice utility function that generates the data. What is less straightforward is to show that if the observations satisfy the Generalized Axiom then the Afriat inequalities have a solution, i.e., that (b) implies (c) in Theorem 1. Afriat's original proof is an inductive one, which is correct in the case in which $a_{ij} \neq 0$, $i \neq j$. Indeed in this case the proof is quite simple.[2]

Claim 1. There is an index $i \in N$ with $a_{ij} \geq 0$ for all $j \in N$.

> **Proof of Claim 1:** If this were not so, then every row would have a strictly negative entry. Start with row i, say, and suppose that $a_{ij} < 0$. Now consider row j, and identify a negative entry, say $a_{jk} < 0$. Continue to generate the sequence i, j, k, ..., until an index is repeated. Then a subsequence of this sequence yields a contradiction to the Generalized Axiom. $\square$

The existence of λ_j and ϕ_j is trivially true for $n = 1$; we can choose $\lambda_1 = 1$ and ϕ_1 arbitrarily. For the induction let us begin by renumbering the observations (and hence the rows and columns of A) so that $a_{nj} > 0$ for $j = 1, ...n-1$ (using Claim 1). Now suppose, by induction, that there exist $\phi_1, ..., \phi_{n-1}$; $\lambda_1, ..., \lambda_{n-1} > 0$ such that

$$\phi_j \leq \phi_i + \lambda_i a_{ij},\ i \neq j,\ i, j = 1, ..., n-1.$$

[2] A similar version was presented in an informal communication by M. Weitzman.

Let us select ϕ_n such that

$$\phi_n \leq \min_{i=1,\ldots,n-1} \{\phi_i + \lambda_i a_{in}\},$$

and then choose $\lambda_n > 0$ so that

$$\phi_j \leq \phi_n + \lambda_n a_{nj}, \text{ for } j = 1, ..., n-1.$$

Since all the off-diagonal elements of the nth row are strictly positive, λ_n can be chosen large enough so that these $n-1$ inequalities hold. Note the difficulty that arises if any a_{nj} is zero: increasing λ_n will not help to fix the inequality for this n and j. This completes the proof that the Afriat inequalities have a solution in this simple case.

Note that we can choose ϕ_n smaller than all the other ϕ_i's. But if also $a_{n-1,j} > 0$ for $j = 1, 2, \ldots, n-2$, then we could choose ϕ_{n-1} smaller than all previous ϕ_i's. Indeed, if all a_{ij}'s for i different from j are positive, then we can choose by induction the ϕ_i's in strictly decreasing order. But since *any* permutation of the indices maintains the positivity of the off-diagonal a_{ij}'s, it follows that with a suitable choice of the order of indices, we can obtain ϕ_i's and positive λ_i's satisfying the Afriat inequalities and with *any* prescribed ordering of the ϕ_i's, as claimed at the end of the previous section.

The general case, in which off-diagonal elements are allowed to be zero, is related to the issue of indifference classes in the revealed preference ordering. Two authors, Varian [5] and Diewert [2], have given correct proofs in this general case. They prove the result using an inductive argument which manages to handle the subtle issue of indifference classes. Unfortunately, the induction in each of these presentations is complex and may involve the introduction of more than one price-quantity observation at each step.

3 A General Inductive Proof.

We now provide a straightforward proof for the statement that (b) implies (c) in Afriat's theorem in the general case where $a_{nj} \geq 0$ for $j = 1, ...n-1$, but with some of these entries possibly zero. The argument is inductive, but as in the simple case the inductive step introduces a single observation at a time.

The key is to apply the inductive hypothesis to a different $(n-1) \times (n-1)$ matrix A'. Specifically, for $j = 1, ...n-1$, we define

$$a'_{ij} := \begin{cases} a_{ij} & \text{if } a_{nj} > 0, \\ \min\{a_{ij}, a_{in}\} & \text{if } a_{nj} = 0. \end{cases} \tag{1}$$

Claim 2. A' satisfies the Generalized Axiom.

Proof of Claim 2: First note that, if $a_{nj} = 0$, then $a_{jn} \geq 0$ by the Generalized Axiom, so that $a'_{jj} = a_{jj} = 0$ for $j = 1, \ldots, n-1$. Now suppose that A' has a cycle $(i, j, k, \ldots, r, i)$ with

$$a'_{ij} \;\leq\; 0,$$

$$
\begin{aligned}
a'_{jk} &\leq 0, \\
&\vdots \\
a'_{ri} &\leq 0,
\end{aligned}
$$

and at least one term strictly negative. Since A does satisfy the Generalized Axiom by hypothesis, there must be a term, say that corresponding to the pair (p, q), with

$$a'_{pq} \neq a_{pq}.$$

But if $a'_{pq} = a_{pn}$ and $a_{nq} = 0$, then we can replace the cycle $(\ldots, p, q, \ldots)$ by $(\ldots, p, n, q, \ldots)$ with two new terms

$$
\begin{aligned}
a_{pn} &\leq 0, \\
a_{nq} &= 0,
\end{aligned}
$$

and, as before, at least one of the terms in the new sequence is strictly negative. Continuing in this way we can construct a cycle in A that violates the Generalized Axiom, contrary to our assumption. Hence A' must satisfy the Generalized Axiom. $\square$

We can therefore apply our inductive assumption to A' to guarantee the existence of ϕ_i and positive λ_i for $i \in N_- := \{1, 2, ..., n-1\}$ so that

$$\phi_j \leq \phi_i + \lambda_i a'_{ij} \tag{2}$$

for $i, j \in N_-$. Since $a'_{ij} \leq a_{ij}$ from (1), this ensures that the Afriat inequalities hold also for A for $i, j \in N_-$. Next, set

$$\phi_n = \min_{i \in N_-} \{\phi_i + \lambda_i a_{in}\}$$

(note that we choose equality, not less than or equal to), to achieve the inequalities for $i < n$, $j = n$. Finally, set

$$\lambda_n := \max\{1, \max_{j \in N_-, a_{nj} > 0} [(\phi_j - \phi_n)/a_{nj}]\}.$$

As in the simple case, this choice makes sure that the inequalities hold for $i = n$ and $j < n$ in the case that $a_{nj} > 0$. To complete the proof, suppose that $a_{nj} = 0$. Then we have

$$
\begin{aligned}
\phi_j &\leq \min\nolimits_{i \in N_-} \{\phi_i + \lambda_i a'_{ij}\} && \text{(by (2))} \\
&\leq \min\nolimits_{i \in N_-} \{\phi_i + \lambda_i a_{in}\} && \text{(by (1))} \\
&= \phi_n && \text{by definition of } \phi_n \\
&= \phi_n + \lambda_n a_{nj} && \text{since } a_{nj} = 0.
\end{aligned}
$$

Clearly the inequality holds for $i = j = n$, and so the inductive step is complete. This finishes the proof. $\square$

4 A Proof using Linear Programming.

Diewert's proof [2] relates the Afriat inequalities to a particular linear programming problem. However the programming problem is not directly used in his proof. The argument presented here makes use of a linear program which is essentially identical with Diewert's, but uses the Duality Theorem of Linear Programming to show that the Generalized Axiom implies that the Afriat inequalities have a solution.[3]

Consider the following linear programming problem:

$$\begin{array}{lrcll} \min_{\lambda,\phi} & 0\cdot\lambda + 0\cdot\phi & & & \\ & \lambda_i & \geq & 1, & \text{for all } i \in N, \\ & a_{ij}\lambda_i + \phi_i - \phi_j & \geq & 0, & \text{for all } i,j \in N \text{ with } i \neq j \end{array}$$

in which the objective function is zero and the constraints are the Afriat inequalities. We shall show that the dual linear program is feasible and has a maximum of zero. The Duality Theorem then implies that the original problem is also feasible, and therefore the Afriat inequalities have a solution. Although the argument may seem a bit eccentric, the procedure is a standard trick to verify that a system of linear inequalities is consistent.

The matrix associated with the linear program is

$$\begin{array}{r} \textit{objective} \\ \\ \\ \vdots \\ \\ \\ \vdots \\ \\ \vdots \\ \\ \vdots \\ \\ \textit{variables} \end{array}
\left[\begin{array}{cccccccccc}
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & RHS \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\
a_{12} & 0 & \cdots & 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{1n} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{n1} & -1 & 0 & \cdots & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{n,n-1} & 0 & 0 & \cdots & -1 & 1 & 0 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_n & \phi_1 & \phi_2 & \cdots & \phi_{n-1} & \phi_n &
\end{array}\right]
\begin{array}{l} \\ y_1 \\ y_2 \\ \vdots \\ y_n \\ x_{12} \\ \vdots \\ x_{1n} \\ \vdots \\ x_{n1} \\ \vdots \\ x_{n,n-1} \\ \\ \end{array}$$

In this matrix the top row describes the coefficients of the objective function, the bottom row the variables associated with the columns and the last column the right hand side of the inequalities. The slack variables have been omitted.

If the dual variable associated with the inequality $\lambda_i \geq 1$ is $y_i (\geq 0)$ and the dual variable associated with the inequality $a_{ij}\lambda_i + \phi_i - \phi_j \geq 0$, for $i \neq j$, is $x_{ij} (\geq 0)$, the dual problem can

[3]Our colleague, John Geanakoplos, has shown us an elegant proof that the Afriat inequalities have a solution using the Min-Max theorem for two-person zero-sum games.

be stated as

$$\begin{array}{llllll}
\max_{y,x} & \sum_{i\in N} y_i & & & & \\
 & & \sum_{h\in N} x_{hi} & -\sum_{j\in N} x_{ij} & = & 0, \quad \text{for all } i \in N, \\
 & y_i & & +\sum_{j\in N} a_{ij}x_{ij} & = & 0, \quad \text{for all } i \in N, \\
\text{with } y_i, x_{ij} \geq 0 \text{ for all } i, j. & & & & &
\end{array}$$

The dual variables x_{ij} can be viewed as the entries in an $n \times n$ matrix X, whose diagonal entries are zero and whose off-diagonal elements are nonnegative. The first set of constraints in the dual problem state that for each i the sum of the entries in row i of X equals the sum of the entries in column i.

In order to use the Duality Theorem to prove that the Afriat inequalities have a solution, we need to show that $x = 0, y = 0$ is the optimal solution to the dual problem. Clearly $x = 0, y = 0$ is feasible for the dual and 0 is an lower bound for the optimal value of the dual objective function.

Claim 3. Let (x, y) be a feasible solution to the dual linear program. Then there is a feasible solution, possibly different, with the same objective function value and with no cycle $(i, j), (j, k) \ldots, (r, i)$ on which all x_{pq}'s are positive and all a_{pq}'s zero.

Proof of Claim 3: If there is such a cycle in a feasible solution, we can decrease each x_{pq} on the cycle by the minimum value of these x_{pq}'s, so that at least one such value becomes zero. In this procedure, the perturbed matrix X will still satisfy the constraints of the dual problem, and the variables y_p and hence the objective function value are unchanged since we are only modifying those x_{pq}'s whose corresponding a_{pq} coefficient is zero. $\square$

Now let us show that an optimal solution to the dual problem is $x = 0, y = 0$. Suppose, to the contrary, that $y_i > 0$ in some feasible solution (x, y), which without loss of generality we can assume satisfies the property of Claim 3. Then the sum

$$\sum_{q\in N} a_{iq}x_{iq} < 0$$

and at least one term is negative, say $a_{ij}x_{ij}$. Therefore a_{ij} is negative and x_{ij} positive. By the first set of constraints,

$$\sum_{q\in N} x_{jq} > 0,$$

while

$$\sum_{q\in N: x_{jq}>0} a_{jq}x_{jq} \leq 0$$

by the second set of constraints. We can therefore choose $k \neq j$ with x_{jk} positive and a_{jk} nonpositive. Continuing in this way, we must eventually repeat an index, and therefore we construct a cycle $(l, m, \ldots, r, l)$ on which all x_{pq}'s are positive and all a_{pq}'s nonpositive.

If the index we repeat is the first one with which we started, we immediately get a contradiction since the Generalized Axiom implies that all the terms a_{pq} in the cycle must be zero, but the first one is strictly negative by construction.

In the case that the cycle we construct does not include the first term, again, the Generalized Axiom implies that all terms must be zero, but this was already ruled out by our assumption that (x, y) satisfies the property of Claim 3.

We have demonstrated that the dual linear program is feasible and its maximum value is 0. By the Duality Theorem of Linear Programming the original problem is feasible, which means that the Afriat inequalities have a solution.

5 Graph-Theoretic Interpretation and Complexity

Here we describe the conclusion of Afriat's theorem in graph-theoretic terms and discuss the complexity of determining whether the data D is consistent with utility maximization and, if so, computing a possible utility function u. From a mathematical viewpoint, the hard part of the theorem claims that if a matrix A with zero diagonal entries satisfies the GARP condition, then there are ϕ's and positive λ's so that the Afriat inequalities hold. It is hard to comprehend these conditions intuitively: the graph-theoretic interpretation may make them more understandable.

Let us consider a graph with nodes N, with a directed edge from each $i \in N$ to each $j \in N$ with cost c_{ij}. We call a sequence of nodes $(i, j, k, \ldots, r, i)$ (with possible repeats) a circuit; if all nodes (except the first and last) are distinct, we call it a cycle. The circuit or cycle is *negative* if $c_{ij} + c_{jk} + \cdots + c_{ri}$ is negative, and *obviously negative* if all these summands are nonpositive, with at least one negative. Thus the Generalized Axiom asserts that, with costs given by $c_{ij} = a_{ij}$ for all $i, j \in N$, there is no obviously negative cycle; it is not hard to see that this is equivalent to there being no obviously negative circuit. However, the Generalized Axiom does not preclude the existence of a negative cycle in A.

The Afriat inequalities also have a graph-theoretic interpretation. It is not hard to see that, for given positive λ's, these inequalities hold for some ϕ's if and only if there is no negative circuit when the costs are given by $c_{ij} = \lambda_i a_{ij}$. Indeed, adding the inequalities $\phi_j - \phi_i \leq \lambda_i a_{ij}$ around any circuit shows that there are no negative circuits with these costs. On the other hand, the ϕ's correspond in some sense to distances: if we maximize $\phi_j - \phi_i$ over all ϕ's satisfying the inequalities, we get the least cost of a path from i to j using these edge costs, which exists if there is no negative circuit. We can obtain a solution to the inequalities by setting ϕ_j, for each j, to be the least cost of a path from some fixed node to j.

Thus Afriat's theorem states that, if the original costs $\{a_{ij}\}$ yield no obviously negative circuit, then the costs can be scaled using a positive weight for each tail node to give costs $\{\lambda_i a_{ij}\}$ for which there is no negative circuit at all. The converse is easy to see: if there is no negative circuit when the costs are $\{\lambda_i a_{ij}\}$, then there are no obviously negative circuits with these costs; but then there are no obviously negative circuits with the costs $\{a_{ij}\}$, since the sign of a_{ij} is the same as that of $\lambda_i a_{ij}$.

Finally, we can give a graph-theoretical interpretation to the other "cost" matrices that arise, such as A' defined in (1). After k steps of the procedure we obtain an $(n-k) \times (n-k)$

matrix $A^{(k)}$ ($A^{(0)} = A$ and $A^{(1)} = A'$). It is easy to see, by induction on k, that $a_{ij}^{(k)}$ is the least cost of a path $(i, p, q, \ldots, s, j)$ from i to j where $p, \ldots, s$ lie in $\{n-k+1, \ldots, n\}$ and all costs $a_{pq}, \ldots, a_{sj}$ are zero.

We remarked in the introduction that the Generalized Axiom gives testable conditions for the data D to be consistent with utility maximization. But how hard is it to check whether the axiom holds, and if so, to find a possible utility function? At first sight, we need to check every possible cycle, and while this is a finite procedure, there are exponentially many possible cycles (but see the discussion of Varian's approach below). If we knew the $2n$ numbers $\phi_1, \ldots, \phi_n$ and $\lambda_1, \ldots, \lambda_n > 0$, potentially satisfying the Afriat inequalities, then we would merely have to check these n^2 relations, and from these a suitable utility function is at hand. Diewert [2] proposed to find these numbers by solving a linear programming problem, but this is computationally burdensome. Varian's proof [5] gives an $O(n^3)$ algorithm to find the ϕ's and λ's. Indeed, Varian first defines x_i to be directly revealed preferred to x_j if $p_i \cdot x_j \leq p_i \cdot x_i$. He then computes the transitive closure R of this relation by a graph-theoretic algorithm in $O(n^3)$ time. Then the Generalized Axiom can be checked simply: for each i and j, see if $x_i R x_j$ and $p_j \cdot x_i < p_j \cdot x_j$; if so the Generalized Axiom is violated. If this does not occur for any such pair, the Generalized Axiom is satisfied. Armed with the transitive closure, Varian finds the ϕ's and λ's by an algorithm that must consider together every subset of observations with each pair related by R. Our inductive proof in Section 3 provides a simple alternative $O(n^3)$ method that determines these parameters one by one. (Of course, we also need $O(n^2 l)$ work to compute the entries of A from D.)

At each step of the inductive process, we search the current matrix A to find a nonnegative row, say the ith, which takes $O(n^2)$ time. (If there is no such row, then we can find an obviously negative cycle by the argument in the proof of Claim 1, also in $O(n^2)$ time.) We then interchange the ith and nth rows and columns of A, in $O(n)$ time, and calculate the reduced matrix A', in $O(n^2)$ time. When we receive information back from the smaller problem, we can find ϕ_n and λ_n each in $O(n)$ time. (If the smaller problem returns an obviously negative cycle in A', we can expand this to an obviously negative cycle in A using the argument in the proof of Claim 2, also in $O(n)$ time.) This gives a total amount of work at each stage of $O(n^2)$, for a total complexity of $O(n^3)$.

However, if at each stage we can find a positive row (except for its diagonal entry), then we can avoid the per stage $O(n^2)$ work and complete all the computation in a total of $O(n^2)$ time. Clearly we do not require the $O(n^2)$ work to calculate A' so we only need to show how the search for a positive row can be performed in only $O(n)$ time at each stage. Initially, let us compute the number of negative and zero entries in each row, at a one-time cost of $O(n^2)$. Then at each stage we can scan these counts to find a positive row, and then after permuting that row and the associated column to the end, we can update the counts for the submatrix containing all but the last row and column in just $O(n)$ work. Hence there is only $O(n)$ work per stage for a total of $O(n^2)$. (This complexity also holds if there are only a fixed number of times that a positive row cannot be found.)

When can we use this simplified algorithm? Clearly, if A contains no zero elements outside its diagonal, then the Generalized Axiom implies the existence of a positive row. More generally, note that, if the Generalized Axiom holds vacuously, i.e., there are no obviously negative *nor* all-zero-cost cycles, then the argument of the proof of Claim 1 shows that a positive row exists. This condition (assuming that all demand vectors x_i are distinct) is usually called the Strong Axiom of Revealed Preference (see, e.g., Varian [5]). Thus either the simple case considered in Section 3 or the Strong Axiom leads to the reduced complexity of $O(n^2)$ time to compute the ϕ's and λ's satisfying the Afriat inequalities and hence a possible utility function.

References

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