THE NUMBER OF POISSON ARRIVALS LOST TO BALKING IN A SINGLE SERVER QUEUE WITH FIXED SERVICE TIME

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Abstract

Customers are assumed to arrive singly and randomly during the work shift of a single server and are either provided a service of fixed duration (d) if the server is free upon arrival or, otherwise, immediately depart without service. Conditional upon the number (n) of arrivals during a work shift of unit length the probability distribution of the number (X_n) served (assuming overtime is allowed to complete the last service) is

$$P\{X_n \geq x | N = n\} = P\{U_{(x)}^{(n)} \leq 1 - (x-1)d\}$$

where $U_{(x)}^{(n)}$ denotes the x'th smallest order statistic in a sample of n random numbers from the unit interval. The conditional mean value of X_n is approximately

$$E(X_n | N = n) \doteq \frac{n}{1 + (n-1)d}$$

the approximation being exact when $(n-1)d = 0, 1$, or $(n-1)$.

The problem

As an idealized model of an interviewer stationed in a shopping mall stopping passersby to complete a short questionnaire, we consider the case of a single server working a shift of fixed length W during which randomly arriving customers either receive the service, of fixed duration w, or pass on by if the server is already occupied with a customer. Similar models arise in animal behavior studies where W is the observer's work shift duration and w is the fixed duration of an observation; an example might be monitoring the behavior of foraging honeybees after returning to the (experimental) hive.

This represents a simplified special case of "systems with balking" (see Taylor and Karlin, 1984) in which long queues discourage customers. To further simplify the system we permit overtime; if a service is in progress when the server's work shift ends then the server works overtime to complete that service. Assuming that a (possibly hidden) counter records the number (N) of arrivals during a work shift [0,W], our objective is to calculate the conditional (given N) and unconditional probability distribution of the number (X_N) of services or, equivalently, of the number $Y_N = N - X_N$ of missed customers during the work shift.

The solution

The conditional probability distribution of the number (X_N) of services per work shift is expressible in terms of the distribution of order statistics $U_{(1)}^{(n)} < \cdots < U_{(n)}^{(N)}$ of a sample of N independent and identically distributed uniform random numbers in the interval $0 \le U_i \le 1$:

Theorem: $$P\left\{X_N \geq x | N = n\right\} = P\left\{U_{(x)}^{(n)} \leq 1 - (x-1)\frac{w}{W}\right\} .$$

An immediate corollary is that if the rate parameter of the Poisson arrival process is λ then

Corollary: $$P\{X_N \geq x\} = P\{\chi_{2x}^2 < 2\lambda[W - (x-1)w]\} .$$

The theorem is easily proved by considering an equivalent queueing process in which service is instantaneous but earns the server a duty break of fixed duration w. Since arrivals are random then no generality is lost if the server in such a system is allowed the option of accumulating off-time during the current work shift and curtailing the work shift, accordingly. Under this latter strategy it becomes clear that if at least x services are performed then the x'th arrival must have occurred prior to time t = W - (x-1)w and, conversely, if the x'th arrival did occur prior to this time then at least x services must have been accomplished in that shift. The conditional probability of at least x services among the n random arrivals in [0,W] is thus calculable as the probability that the time ($T_{(x)}^{(n)}$, say) of the x'th arrival does not exceed W - (x-1)w:

$$P\{X_N \geq x \mid N = n\} = P\left\{ \frac{T_{(x)}^{(n)}}{W} \leq 1 - (x-1)\frac{w}{W} \right\} = \frac{n!}{(x-1)!1!(n-x)!} \int_0^{1-(x-1)\frac{w}{W}} u^{x-1}(1-u)^{n-x}du$$

$$= \begin{cases} \sum_{r=0}^{n-x} \binom{n}{r} \left[(x-1)\frac{w}{W}\right]^r \left[1 - (x-1)\frac{w}{W}\right]^{n-r} & \text{if } 0 \leq (x-1)\frac{w}{W} < 1 \\ 0 \text{ otherwise} & \end{cases}$$

Multiplying this incomplete Beta probability by the Poisson probability

$$P\{N = n\} = e^{-\lambda W}(\lambda W)^n/n!$$

and summing over n, $n \geq x$, then yields the cumulative chi square probability specified in the corollary; $P(X_N \geq 0) = 1$ and for $x \geq 1$

$$P\{X_N \geq x\} = \int_0^{W-(x-1)w} \lambda^x t^{x-1} e^{-\lambda t} dt/(x-1)! ,$$

which is understood to vanish when $(x-1)w \geq W$.

Statistical considerations

From a statistical viewpoint the above theorem has somewhat greater utility than its corollary, since the latter specifies a distribution depending on an unknown arrival rate parameter λ which may vary from one work shift to another. If data (n_i, x_{n_i}) are available from k independent work shifts, $i = 1,\cdots,k$, then various statistical analyses become available, including goodness-of-fit tests. If service time w is specified and shift duration W is, likewise, a known constant then the probability distribution (incomplete Beta probability) of the theorem is completely specified and a test statistic such as R. A. Fisher's

$$-2 \sum_{i=1}^{k} \ln P\{X_N \geq x_i | N = n_i\} \mathrel{\dot\sim} \chi^2_{2k}$$

is available for testing the model. Likelihood ratio tests are also available, for example, for testing whether a specified w is compatible with the ML estimate, $\hat{w}_{ML}$, or for testing homogeneity of w over the k shifts involving possibly different servers.

The maximum likelihood procedure is cumbersome in this context and so might be replaced by the less efficient method of moments. To this end we note the approximation

$$E(X_N | N = n) = \sum_{x=1}^{n} P\{X_N \geq x | N = n\} \doteq \frac{n}{1 + (n-1)\frac{w}{W}}$$

which is exact at $w = 0$, $w = W/(n-1)$ and $w = W$. (At $w = W/(n-1)$ the p.m.f. of X_n is seen to be symmetric around $n/2$, with zero mass at $X_n = n$.) This approximation is a finite - W analogue of the long run ($W \to \infty$) fraction of customers $(1 - \pi_1)$ serviced; for example,

$$1 - \pi_1 = \frac{1}{1 + \frac{\lambda}{\mu}}$$

when service time is exponentially distributed with mean $1/\mu$. Perhaps, more generally, if interarrival times are exchangeable and service time is uncorrelated with arrival time then

$$E(X_N | N = n) \doteq \frac{n}{1 + (n-1)\frac{\bar{w}}{W}}$$

If balking is unobservable so that n is unknown then n may become the target of statistical inference. Point and interval estimators of n may be constructed using the conditional $P(X_n \geq x | N = n)$ as the basis for inference when w and W are known constants. When w = .2W, for example, the integer-valued ML estimator and the upper (one-tailed) 95 percent confidence limits are given by

X	$\hat{n}_{ML}$	upper confidence limit on n	$\frac{X(1 - .2)}{1 - .2X}$
1	1	4	1
2	3	8	2 2/3
3	6	17	6
4	14	44	16
5	∞	∞	∞

At X = 2, for example,

$$P(X_2 = 2 | N = 2) = .64$$

$$P(X_3 = 2 | N = 3) = .848$$

$$P(X_4 = 2 | N = 4) = .4976$$

showing that the likelihood of X = 2 is largest when N = 3. When N = 8 the lower tail probability at X = 2 is

$$P(X_8 \leq 2 | N = 8) = 1 - P(X_8 \geq 3 | N = 8) = 1 - \sum_{r=0}^{5} \binom{8}{r} (.4)^r (.6)^{8-r}$$

$$= 1 - .95019264$$

showing that 8 is the 95 percent upper confidence limit when X = 2.

Note that this stands in agreement with the result obtained from Snedecor's F-table. The incomplete beta integral on page 3 gives the upper (one-tailed) 95 percent confidence limit on n as that value n* satisfying the equation

$$\frac{n^* - x}{x + 1}\left(\frac{W}{xw} - 1\right) = F^*_{2(x+1),2(n^*-x)}$$

where $F^*_{f_1,f_2}$ denotes the tabulated 5 percent critical value of F on f_1 and f_2 degrees of freedom. At x = 2 and n* = 8 we find $F^*_{6,12} = 3.00$, and

$$\frac{8 - 2}{2 + 1}\left(\frac{1}{2(.2)} - 1\right) = 3$$

while at the neighboring integers

$$\frac{7 - 2}{2 + 1}\left(\frac{1}{2(.2)} - 1\right) = 2.5 \qquad \text{and} \qquad \frac{9 - 2}{2 + 1}\left(\frac{1}{2(.2)} - 1\right) = 3.5$$

$$F^*_{6,10} = 3.22 \qquad\qquad F^*_{6,14} = 2.85$$

so that n* = 8 is the closest integer solution to the equation. At x = 3 the l.h.s. becomes (n*-3)/6, giving

$n^* =$	16	17	18
$(n^*-3)/6 =$	2 1/6	2 1/3	2 1/2
$F^*_{8,2n^*-6} =$	2.32	2.29	2.27

showing that n* = 17 is the closest integer solution.

The last column in the above table is a point estimator of n based on the earlier approximation to $E(X_N | N = n)$,

$$\tilde{n} = \frac{x\left(1 - \frac{w}{W}\right)}{1 - \frac{xw}{W}} \quad .$$

These inference procedures neglect any information that might be available in the recorded amount of overtime.