(A)

Solving for sensitivity of Directions to η_{j-1}, η_j

Sensitivity of $\underline{v}_{out\, j-1}$ and $\underline{v}_{in\, j}$

Let $\eta_{j-1} = \eta_0$ $\qquad$ $\underline{v}_{out\, j-1} = \underline{v}_0$

$\eta_j = \eta_F$ $\qquad$ $\underline{v}_{in\, j} = \underline{v}_F$

Nonlinear TPBVP Solved using Newton's method

$\underline{p}$: vector of Iono params

Diffeq Nonlinear ODE

$$\frac{d\underline{X}}{dP'} = F[\underline{X}, \underline{p}]$$

$$\underline{X} = \begin{bmatrix} \vec{r} \\ \vec{k} \end{bmatrix}$$

P' range-equivalent group delay

Initial B.C.

$$\left.\begin{aligned} \eta_0 &= [I \quad 0]\, \underline{X}(P'_0) \\ 0 &= H[\underline{X}(P'_0), \underline{p}] \end{aligned}\right\} \text{4 equations}$$

Final B.C.

$\cancel{\eta_F = \#\#}$

$$\eta_F = [I \quad 0]\, \underline{X}(P'_F) \Big\} \text{ 3 equations}$$

$$\overbrace{\text{Find} \quad \underline{X}(p'_0) \text{ and } p'_F}^{\text{7 unknowns}} \quad \text{that,}$$

(B)

after integrating from p'_0 to p'_F satisfies the 7 Boundary conditions, 4 I.C. BCs and 3 Final B.C.s.

Sensitivity with respect to jth element of $\underline{p}$, call it p_j

Linear time-varying ODE

$$\frac{d}{dp'}\left[\frac{\partial \underline{X}}{\partial p_j}\right] = F(p')\frac{\partial \underline{X}}{\partial p_j} + g_j(p')$$

where

$$F(p') = \underbrace{\frac{\partial \underline{f}}{\partial \underline{X}}}_{\text{6x6 matrix that depends on } p'}\Bigg|_{[\underline{X}(p'),\,\underline{p}]}$$

and

$$g_j(p') = \underbrace{\frac{\partial \underline{f}}{\partial p_j}}_{\text{6x1 vector that depends on } p'}\Bigg|_{\{\underline{X}(p'),\,\underline{p}\}}$$

(c)

Linear I.C. B.Cs:

$$\text{4 eqs}\left\{\begin{aligned} 0 &= [I,\ 0]\,\frac{\partial \underline{X}}{\partial p_j}\bigg|_{p_0'} \\ 0 &= \frac{\partial H}{\partial \vec{r}}\bigg|_{[\underline{X}(p_0'),\,\underline{p}]}\,[0\ \ I]\,\frac{\partial \underline{X}}{\partial p_j}\bigg|_{p_0'} + \frac{\partial H}{\partial p_j}\bigg|_{[\underline{X}(p_0'),\,\underline{p}]} \end{aligned}\right.$$

Final B.C.s

$$0 = [I\ \ 0]\left[\frac{\partial \underline{X}}{\partial p_j}\bigg|_{p_f'} + \frac{d\underline{X}}{dp'}\bigg|_{p_f'}\frac{\partial p_f'}{\partial p_j}\right]$$

$$\text{3 eqs}\left\{\text{or}\quad 0 = [I\ \ 0]\left[\frac{\partial \underline{X}}{\partial p_j}\bigg|_{p_f'} + \underline{F}\{\underline{X}(p_f'),\underline{p}\}\frac{\partial p_f'}{\partial p_j}\right]\right.$$

$$\left.\text{Find}\quad \frac{\partial \underline{X}}{\partial p_j}\bigg|_{p_0'} \quad\text{and}\quad \frac{\partial p_f'}{\partial p_j}\right\}\ \text{7 unknowns}$$

such that we satisfy the 7 B.C. equations after integrating the differential equation from $p' = p_0'$ to $p' = p_f'$

D

Initial and final directions

from nonlinear ~~TB~~ TPBVP

$$\underline{v}_0 = [I \quad 0]\, \underline{f}\{\underline{X}(p'_0), \underline{p}\}$$

$$\underline{v}_f = [I \quad 0]\, \underline{f}\{\underline{X}(p'_f), \underline{p}\}$$

Sensitivities

$$\frac{\partial \underline{v}_0}{\partial p_i} = [I, 0]\left[\frac{\partial f}{\partial X}\Big|_{\{X(p'_0), p\}} \frac{\partial X}{\partial p_i}\Big|_{p'_0} + \frac{\partial f}{\partial p_i}\Big|_{[X(p'_0), p]}\right]$$

$$\cancel{= [I, 0][F}$$

$$= [I, 0]\left[F(p'_0)\frac{\partial X}{\partial p_i}\Big|_{p'_0} + g_i(p'_0)\right]$$

$$\cancel{\frac{\partial \underline{v}_f}{\partial p_i} = [I \quad 0]\left[F(p'_f)\frac{\partial X}{\partial p_i}\Big|_{p'_f} + g_i(p'_f) + f\{X(p'_f), p\}\frac{\partial p'_f}{\partial p_i}\right]}$$

$$\frac{\partial v_f}{\partial p_i} = \begin{bmatrix} I & 0 \end{bmatrix} \left[F(p_f') \left\{ \frac{\partial X}{\partial p_j}\bigg|_{p_f'} + f[X(p_f'), p] \frac{\partial p_f'}{\partial p_i} \right\} + g_j(p_f') \right]$$

Sensitivities with respect to η_0

Linearized time-varying ODEs

$$\frac{d}{dp'}\left[\frac{\partial X}{\partial \eta_0}\right] = F(p') \frac{\partial X}{\partial \eta_0}$$

State-transition matrix ODE:

(STM)

~~Initial value p_0'~~

I.C.s:

$$\cancel{0 = [I, 0] \frac{\partial X}{\partial}}$$

Yeys

$$I = \begin{bmatrix} I & 0 \end{bmatrix} \frac{\partial X}{\partial \eta_0}\bigg|_{p_0'}$$

$$\cancel{0 = \frac{\partial H}{\partial r}, \frac{\partial H}{}}$$

$$0 = \left[\frac{\partial H}{\partial r}\bigg|_{[X(p_0'), p]}, \frac{\partial H}{\partial \dot{r}}\bigg|_{[X(p_0'), p]} \right] \frac{\partial X}{\partial \eta_0}\bigg|_{p_0'}$$

(F)

Final B.Cs

$\}$ eqs $\quad 0 = \begin{bmatrix} I & 0 \end{bmatrix} \left[\frac{\partial X}{\partial \eta_0}\Big|_{p_F} + f\{X(p_F'), p\} \frac{\partial p_F'}{\partial \eta_0} \right]$

find $\quad \frac{\partial X}{\partial \eta_0}\Big|_{p_0'}$ and $\frac{\partial p_F'}{\partial \eta_0}$

to solve the initial and terminal boundary conditions after integrating the linear STM ODE

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finally

$$\frac{\partial u_0}{\partial \eta_0} = \begin{bmatrix} I & 0 \end{bmatrix} \left[F(p_0') \frac{\partial X}{\partial \eta_0}\Big|_{p_0'} \right]$$

$$\frac{\partial u_F}{\partial \eta_0} = \begin{bmatrix} I & 0 \end{bmatrix} \left[F(p_F') \left\{ \frac{\partial X}{\partial \eta_0}\Big|_{p_0'} + f[X(p_F'), p] \frac{\partial p_F'}{\partial \eta_0} \right\} \right]$$

(6)

Sensitivities with respect to ~~t~~ η_F

(Linearized time-varying STM)

$$\frac{\partial}{\partial p'}\left[\frac{\partial \underline{X}}{\partial \eta_F}\right] = F(p')\frac{\partial \underline{X}}{\partial \eta_F}$$

I.C.s:

4 eqs

$$\begin{cases} 0 = [I \;\; 0]\left.\frac{\partial \underline{X}}{\partial \eta_F}\right|_{p'_0} \\ 0 = \left[\left.\frac{\partial H}{\partial \underline{r}}\right|_{[\underline{X}(p'_0), p]} , \left.\frac{\partial H}{\partial E}\right|_{(\underline{X}(p'_0), p)}\right]\left.\frac{\partial \underline{X}}{\partial \eta_F}\right|_{p'_0} \end{cases}$$

Final B.C.,

$$I = [I \; 0]\left[\frac{\partial X}{\partial \eta_F}\Big|_{p'_F} + f\{X(p'_F), p\}\frac{\partial p'_F}{\partial \eta_F}\right]$$

Find $\frac{\partial X}{\partial \eta_F}\Big|_{p'_0}$ and $\frac{\partial p'_F}{\partial \eta_F}$

to solve the initial and terminal boundary conditions after integrating the linear STM ODE

Finally

$$\frac{\partial \nu_0}{\partial \eta_F} = [I, 0]\left[F(p'_0)\,\frac{\partial X}{\partial \eta_F}\Big|_{p'_0}\right]$$

$$\frac{\partial \nu_F}{\partial \eta_F} = [I, 0]\left[F(p'_F)\left\{\frac{\partial X}{\partial \eta_F}\Big|_{p'_F} + f\{X(p'_F), p\}\frac{\partial p'_F}{\partial \eta_F}\right\}\right]$$