European Option Pricing with Fixed Transaction Costs

Ajay Subramanian Aiyer (Center for Applied Mathematics)

August 21, 1996

Abstract

In this paper, we study the problem of European option pricing in the presence of *fixed* transaction costs. The problems of optimal portfolio selection and option pricing in the presence of *proportional* transaction costs has been extensively studied in the mathematical finance literature. However, much less is known when we have fixed transaction costs. In this paper, we show that calculating the price of an European option involves calculating the value functions of two stochastic impulse control problems and we obtain the explicit expressions for the resultant quasi-variational inequalities satisfied by the value functions and then carry out a numerical calculation of the option price.

1 Introduction

Option pricing has been extensively studied in the mathematical finance literature since the publication of the Black-Scholes formula in 1973. The analysis of Black and Scholes however assumes a perfect market with frictionless trading. More recently, option pricing has been investigated in the presence of imperfect markets with transaction costs. Several authors have studied the problems of option pricing and optimal portfolio selection in this setting.

In 1993, Davis, Panas and Zariphopolou, studied the problem of European option pricing in the presence of proportional transaction costs and showed how to obtain the price of the option in terms of the value functions of two different singular stochastic control problems. In this paper, we study the problem of option pricing when the transaction costs are a fixed fraction of the portfolio value along with a fixed exogenously specified fee. Adapting the ideas of Davis et al to this setting, we show that the option price can be expressed in terms of the value functions of two different stochastic *impulse* control problems and derive the quasi-variational inequalities of the value function.

The general problem of impulse control and quasi-variational inequalities has been thoroughly investigated in the mathematical literature.(see Bensoussan and Lions(1984) and the references cited therein). Eastham and Hastings (1988) provided one of the first rigorous adaptations of the theory to the problem of optimal impulse control of portfolios.

In section 2, we provide a rigorous introduction to the problem of option pricing and present a definition for the price of a European option in terms of the value functions of two stochastic impulse control problems. As in Davis et al(1992), we define the *replicating portfolio* for a European option and prove that when the space of admissible trading strategies is linear, the price of the option is the initial endowment associated with the replicating portfolio. In section 3, we carry out a detailed investigation of the general situation with fixed transaction costs and define the admissible trading strategies for the investor. We also prove theorems analogous to those in EH giving conditions under which optimal admissible trading policies exist. In section 4, we actually carry out a calculation of the price of the European option when the drifts and volatilities of the bond and stock are constants which is the Black-Scholes market. We analyze the quasi-variational inequalities satisfied by the value function of the associated impulse control problem and prove that, under the conditions of the model under investigation, the optimal trading policy for the investor entails *at most* one intermediate transaction in the interval $[0.T)$ in addition to liquidation at the terminal date T.

In section 5, we conclude the paper.

2 Option Pricing

In this section, we adapt the ideas of Davis et al(1992) to our setting and give a definition of the option price in terms of utility maximization. In order to facilitate comparison with their definitions and results, we shall attempt to retain their notation.

$\mathbf{P} = (P_0, P_1, ..., P_n)$ is a vector-valued stochastic process describing the prices of a risk-free asset or bond P_0 and n risky assets or stocks. These are assumed to be defined on a probability space $(\Omega, \mathcal{F}, P)$ and a time interval $[0, T]$. The right continuous complete filtration $\mathcal{F}_t$ of the probability space is generated by the price process $\mathbf{P}(t)$. We shall define a price at time zero for a European option with exercise time T on one of the stocks, say $P_1(t)$.

Let $\mathcal{T}(S_0)$ denote the set of *admissible trading strategies* for an investor who starts with S_0 shares of the risk-free asset and no shares in stocks. At any time t, the value of the investor's portfolio is given by

$$V_t = \mathbf{S}(t).\mathbf{P}(t) \tag{2.1}$$

where $\mathbf{S}(t)$ is the vector-valued "share" process. If the investor carries out a transaction at time t, then

$$V_{t+} = \mathbf{S}(t).\mathbf{P}(t) - (1 - \alpha)|\mathbf{S}(t).\mathbf{P}(t)| - \beta \tag{2.2}$$

where $0 < \alpha \leq 1$ and $\beta \geq 0$.

Thus, we have defined the effect of transactions for positive and negative values of the portfolio.

In this paper, we shall make the following important assumption about the nature of the market.

Assumption : Transaction costs incurred as soon as the investor decides to make a transaction, i.e. the investor pays the transaction cost *before* carrying out the transaction.

An option on the stock $P_1(t)$ is the right to buy one share at time T at a price E, which may be, in general, an $\mathcal{F}_T-$ measurable random variable. The option writer forms a portfolio to hedge the option and liquidates it at time T. If $P_1 \leq E$ the option is not exercised and the cash value of the portfolio is $\mathbf{P}.\mathbf{S}(T) - (1-\alpha)|\mathbf{S}(t).\mathbf{P}(t)| - \beta$. If $P_1 > E$, the buyer pays the writer E in cash, and the writer given one share to the buyer. The cash value of the portfolio is therefore $\mathbf{S}(t).\mathbf{P}(t) - (1-\alpha)|\mathbf{S}(t).\mathbf{P}(t)| - \beta + E - P_1(T)$ where $\hat{\mathbf{e}}_1$ is the vector $(0, 1, 0, ..., 0)$.

For notational convenience, let us henceforth assume that the value of the writer's portfolio at time T is positive. Let $\mathcal{U} : R \to R$ be the writer's utility function which is concave and increasing. Just as in Davis et al, we can define the following value functions :

$$\begin{aligned} V_w(S_0) = sup_{\pi \in \mathcal{T}(S_0)} E[\mathcal{U}(I_{(P_1(T) \leq E)}(\mathbf{P}.\mathbf{S}(T) - (1-\alpha)|\mathbf{S}(t).\mathbf{P}(t)| - \beta) \\ + I_{(P_1(T) > E)}(E - P_1(T) + \mathbf{S}(t).\mathbf{P}(t) - (1-\alpha)|\mathbf{S}(t).\mathbf{P}(t)| - \beta)] \end{aligned} \tag{2.3}$$

and

$$V_1(S_0) = sup_{\pi \in \mathcal{T}(S_0)} E[\mathcal{U}(\mathbf{P}.\mathbf{S}(T) - (1-\alpha)|\mathbf{S}(t).\mathbf{P}(t)| - \beta)] \tag{2.4}$$

$V_w(S_0)$ is the maximum utility available to the writer if he hedges the option and $V_1(S_0)$ is the maximum utility if he enters the market with an initial endowment of S_0 shares of the risk-free asset. Let

$$S_w = inf\{S_0 : V_w(S_0) \geq 0\} \tag{2.5}$$

and

$$S_1 = inf\{S_0 : V_1(S_0) \geq 0\} \tag{2.6}$$

If we assume that V_w and V_1 are continuous and monotone increasing functions of their argument, then just as in Davis et al, we can define the price of the option as

$$p_w = S_w - S_1 \tag{2.7}$$

A *replicating portfolio* for the option contract is an element $\hat{\pi} \in \mathcal{T}(\hat{S}_0)$ with initial endowment $\hat{S}_0$, such that $\mathbf{S}(T) = I_{(P_1(T)>E)}(-E, 1, 0, ..., 0)$.
Note: The form of the replicating portfolio differs from the one in Davis et al because of the different transaction cost model under consideration.

For notational convenience, we shall assume that $\mathbf{P}.\mathbf{S}(\mathbf{T}) \geq 0$ in the following :
We shall now prove a proposition exactly analogous to theorem 1 in Davis et al.

Proposition 1 *Suppose $\mathcal{T}$ is a linear space and $V_w(S_0)$ and $V_1(S_0)$ are both continuous and strictly increasing functions of S_0. Then $p_w = \hat{S}_0$ if a replicating portfolio $\hat{\pi} \in \mathcal{T}(\hat{S}_0)$ exists.*

Proof.
The proof follows exactly along the lines of the one given in Davis et al. By linearity, an arbitrary trading strategy π can be written in the form $\pi = \hat{\pi} + \tilde{\pi}$ with $\hat{\pi} \in \mathcal{T}(\hat{S}_0)$ and $\tilde{\pi} \in \mathcal{T}(\tilde{S}_0)$. Then,

$$0 = V_w(S_w)$$

$$V_w(S_w) = \sup_{\pi \in \mathcal{T}(S_w)} E[\mathcal{U}(I_{P_1(T) \leq E}(\alpha \mathbf{P}.\mathbf{S} - \beta) + I_{P_1(T)>E}(\alpha(\mathbf{P}.\mathbf{S}) - \beta - P_1(T) + E))] \tag{2.8}$$

Therefore,

$$V_w(S_w) = \sup_{\tilde{\pi} \in \mathcal{T}(S_w - \hat{S}_0)} E[\mathcal{U}(\alpha \mathbf{P}.\mathbf{S} - \beta + I_{P_1(T)>E}(E - P_1(T)))] \tag{2.9}$$

By the definition of the replicating portfolio, we easily see that

$$0 = V_w(S_w) = V_1(S_w - \hat{S}_0) \tag{2.10}$$

Therefore,

$$\hat{S}_0 = S_w - S_1 = p_w \tag{2.11}$$

This completes the proof.

Just as in the paper by Davis et al, we see that the Black-Scholes model satisfies the conditions of proposition 1. Therefore, in the absence of transaction costs, the price of a European option reduces to the Black-Scholes price.

3 Option Pricing with Fixed Transaction Costs

In this section, we shall introduce a model to price a European option in the presence of fixed transaction costs. We adopt the notation introduced by Eastham and Hastings(1988).

3.1 Price Processes for the Bond and Stocks

If $\mathbf{P} = (P_0, P_1, ..., P_n)$ is the vector-valued price process for the bond and n stocks, then we assume that

$$dP_0(r) = \mu_0(P_0(r), r)dr \tag{3.1}$$

with $P_0(t) = p_0 > 0$ for $t \in [0, T]$ and $\mu_0 > 0$

$$dP_i(r) = \mu_i(\mathbf{P}(r), r)dr + \sigma_{ij}(\mathbf{P}(r), r)dW_j(r) \tag{3.2}$$

for $i = 1...n$ and $j = 1...d$. The initial conditions are given by $P_i(t) = p_i > 0$ for $t \in [0, T]$. Thus, we have modeled the stock price processes as diffusion processes.

3.2 The Admissible Trading Strategies

The admissible trading strategies for the option writer can be broadly described as follows. At any time $t \in [0, T]$, the investor holds a portfolio of shares of each asset represented by the $(n+1)-$ dimensional vector $\mathbf{s} \in R^{n+1}$. He intervenes at various instants to rebalance his portfolio and at each time of intervention, he pays a fixed transaction cost so that the value of this portfolio satisfies :

$$\mathbf{p}(t+).\mathbf{s}'(t+) = \mathbf{p}(t).\mathbf{s}(t) - (1-\alpha)|\mathbf{p}.\mathbf{s}(t)| - \beta \tag{3.3}$$

where $t \in [0, T)$ is a time at which he intervenes or trades and $\mathbf{s}$ and $\mathbf{s}'$ are the initial and final share vectors respectively. At the final time T he dissolves his portfolio and *consumes* it. In order to ensure the existence of optimal strategies, we have to however, impose additional conditions on the share process and the trading strategies available to the investor.

We assume that there exist upper and lower bounds on the number of shares of each risky asset available to the investor. Therefore, for each $i = 1...n$ there exist $B_i > 0$ and $B_i < \infty$ such that

$$-B_i \leq s_i \leq B_i$$

We also assume that there exists $\Gamma < 0$ and $\Gamma > -\infty$ such that the investor is permitted to carry out a transaction if and only if the value of his portfolio *after* he pays the transaction cost does not fall below Γ. Following the notation of Eastham and Hastings(1988), let

$$\mathbf{B} = (-\infty, \infty) \times \Pi_{i=1}^{n}[-B_i, B_i]$$

The *feasible set* $K(\mathbf{p}, \mathbf{s})$ corresponding to the price vector $\mathbf{p}$ and share vector $\mathbf{s}$ is given by

$$K(\mathbf{p}, \mathbf{s}) = \{\mathbf{s}' \in \mathbf{B} : \mathbf{p}.\mathbf{s}' = \mathbf{p}.\mathbf{s} - (1-\alpha)|\mathbf{p}.\mathbf{s}| - \beta\}$$

provided $\mathbf{p}.\mathbf{s} \geq -\Gamma$ and $\mathbf{p}.\mathbf{s}' \geq -\Gamma$ and $K(\mathbf{p}, \mathbf{s}) = \Phi$ otherwise.

Let $\mathbf{K} = \{(\mathbf{p}, \mathbf{s}) : K(\mathbf{p}, \mathbf{s}) \neq \Phi\}$. From the above definitions, it is clear that $\mathbf{K}$ is a *closed* set.

3.3 The Impulse Control Problem

From the model and the definitions introduced in the previous section, it is clear that the investor's optimization problem is an *impulse control* problem. An *impulse control* $v = \{(\theta_k, \mathbf{S}_k)\}$ is a sequence such that (θ_k) is a nondecreasing sequence of stopping times and $(\mathbf{S}_k$ is a sequence of R^{n+1} valued random variables such that $\mathbf{S}_k \in \mathcal{F}_{\theta_k}$ where $\mathcal{F}_t$ is the complete, right continuous filtration on the underlying probability space. If $\mathbf{X}$ is the joint price-share process $(\mathbf{P}, \mathbf{S})$, the impulse control v is called *admissible* if

$$\theta_k \leq \theta_{k+1} \text{ for any } k$$

$$\theta_k \text{ is a stopping time of } \mathbf{X}$$

$$P[lim_{k\to\infty}\theta_k \leq T] = 0$$

$$\mathbf{S}_k \in K(\mathbf{P}(\theta_k), \mathbf{S}_{k-1})$$

$$\mathbf{S}_k \in \mathcal{F}_{\theta_k}$$

If $\mathcal{U}$ is the investor's utility function, then we can define the *value* of the admissible impulse control v by

$$J(\mathbf{p}, \mathbf{s}, t, v) = E_{\mathbf{p},\mathbf{s},t}[\mathcal{U}(C_T)] \tag{3.4}$$

where $C_T \in R$ is the *consumption* at the terminal time T.

Note In the problem we are investigating, there is consumption only at the terminal time T in contrast with the portfolio selection problem studied by Eastham and Hastings(1988).

The impulse control problem reduces to finding an admissible v^* such that $J(\mathbf{p}, \mathbf{s}, t, v^*) \geq J(\mathbf{p}, \mathbf{s}, t, v)$ for each $\mathbf{p} \in R_+^{n+1}$ and $\mathbf{s} \in \mathbf{B}$ and each admissible v.

From the definitions above, we see that the admissible impulse controls are exactly the admissible trading strategies available to the investor and the impulse control problem is exactly the optimization problem we need to solve in order to determine the price of an option. In the next subsection, we shall introduce the quasi-variational inequalities that describe the solution of the impulse control problem and prove that a sufficiently regular solutionn of the quasi-variational inequalities (if it exists) gives rise to an admissible impulse control.

3.4 The Optimal Impulse Control Policy

We shall now introduce the quasi-variational inequalities satisfied by a sufficiently regular solution of the impulse control problem. Let $u = u(\mathbf{p}, \mathbf{s}, r)$ be a continuous real-valued function on $\mathbf{R}_+^{n+1} \times \mathbf{B} \times \mathbf{R}_+$ with continuous first partial derivatives in r and continuous second partial derivatives in $\mathbf{p}$. We define the operator

$$\mathbf{L}u(\mathbf{p}, \mathbf{s}, r) = u_r(\mathbf{p}, \mathbf{s}, r) + \nabla u(\mathbf{p}, \mathbf{s}, r).\boldsymbol{\mu}(\mathbf{p}, r) + (1/2)\sum_{i=1}^{n}\sum_{j=1}^{n} u_{ij}(\mathbf{p}, \mathbf{s}, r)A_{ij}(\mathbf{p}, r) \tag{3.5}$$

where $A = \sigma\sigma'$.

We also define the operator

$$\mathbf{M}u(\mathbf{p}, \mathbf{s}, r) = sup_{s' \in K(\mathbf{p},\mathbf{s})} u(\mathbf{p}, \mathbf{s}', r) \tag{3.6}$$

If $\mathbf{p}, \mathbf{s} \notin \mathbf{K}$ or $K(\mathbf{p}, \mathbf{s}) = \Phi$, we define $\mathbf{M}u(\mathbf{p}, \mathbf{s}, r) = -\infty$.

We note that the operator $\mathbf{M}$ is well-defined since for each $(\mathbf{p}, \mathbf{s}) \in \mathbf{K}$, $K(\mathbf{p}, \mathbf{s})$ is compact and non-empty. Further, we will show later that K is upper semicontinuous as a multifunction. Since u is continuous and $\mathbf{B}$ is locally compact, there exists a Borel measurable function $\phi : \mathbf{K} \times [0, T) \to \mathbf{B}$ such that

$$\phi(\mathbf{p}, \mathbf{s}, r) \in K(\mathbf{p}, \mathbf{s}) \tag{3.7}$$

and

$$\mathbf{M}u(\mathbf{p}, \mathbf{s}, r) = u(\mathbf{p}, \phi(\mathbf{p}, \mathbf{s}, r), r) \tag{3.8}$$

for all $(\mathbf{p}, \mathbf{s}, r) \in [0, T)$.

Let $\mathcal{C}$ be the space of continuous real-valued functions on $\mathbf{R}_+^{n+1} \times \mathbf{B} \times \mathbf{R}_+$ with continuous first partial derivatives in r , continuous second partial derivatives in $\mathbf{p}$ and satisfying **Dynkin's formula**, i.e.

$$E_{\mathbf{p},\mathbf{s},t} u(\mathbf{p}(\tau), \mathbf{s}(\tau), \tau) = u(\mathbf{p}, \mathbf{s}, t) + E_{\mathbf{p},\mathbf{s},t} \int_0^\tau \mathbf{L}u(\mathbf{p}, \mathbf{s}, r)dr \tag{3.9}$$

Suppose there exists $u \in \mathcal{C}$ such that

$$u \geq \mathbf{M}u, \mathbf{L}u \leq 0$$

$$(u - \mathbf{M}u).(\mathbf{L}u) = 0$$

$$u(\mathbf{p}, \mathbf{s}, T) = \mathcal{U}(C_T(\mathbf{p}, \mathbf{s}))$$

We can define the *continuation set* $\mathbf{C} = \{u > \mathbf{M}u\}$ and the *action set* $\mathbf{A} = \mathbf{C}^c$

Lemma 1 *The multifunction $K : \mathbf{K} \to \mathbf{B}$ is upper semicontinuous.*

Proof.

The proof follows along the lines of the proof of the analogous proposition in Eastham and Hastings(1988). We need to show that

$$\limsup_{k \to \infty} K(\mathbf{p_k}, \mathbf{s_k}) \subseteq K(\mathbf{p}, \mathbf{s})$$

when $(\mathbf{p_k}, \mathbf{s_k}) \to (\mathbf{p}, \mathbf{s})$ in $\mathbf{K}$.

Given any subsequence (k_i) and $\mathbf{s'_i} \in K(\mathbf{p_{k_i}}, \mathbf{s_{k_i}})$ converging to some $\mathbf{s}'$, it is enough to show that $\mathbf{s}' \in K(\mathbf{p}, \mathbf{s})$.

But $K(\mathbf{p_k}, \mathbf{s_k}) = \{\mathbf{s'_k} \in \mathbf{B} : \mathbf{p_k}.\mathbf{s'_k} = \alpha \mathbf{p_k}.\mathbf{s_k} - \beta\}$

Since $\mathbf{p_{k_i}} \to \mathbf{p}$ and $\mathbf{s_{k_i}} \to \mathbf{s}'$, by continuity $\mathbf{p}.\mathbf{s}' = \alpha \mathbf{p}.\mathbf{s} - \beta$. This completes the proof.

Lemma 2 $\mathbf{M}u$ *is upper semicontinuous on* $\mathbf{K} \times [0, T)$.

The proof follows exactly along the lines of lemma 2.8 in Eastham and Hastings (1988) using the result of the previous lemma. In particular, this lemma implies that the *continuation set* $\mathbf{C} = \{u > \mathbf{M}u$ is *open*

Proposition 2 *u is the minimum element of the set of functions $v \in \mathcal{C}$ satisfying*

$$v \geq \mathbf{M}u \textit{ and } \mathbf{L}v \leq 0$$

$$v(\mathbf{p}, \mathbf{s}, T) = \mathcal{U}(C_T(\mathbf{p}, \mathbf{s}, T))$$

Proof.

Since u is continuous and $\mathbf{M}u$ is upper semicontinuous, the *continuation set* $\mathbf{C} = \{u > \mathbf{M}u\}$ is an *open* set. Let $v \in \mathcal{C}$ be as in the statement of the proposition. Clearly, $v \geq \mathbf{M}u$ implies that $v \geq u$ on $\mathbf{C}$. It remains to show that $v \geq u$ on the set $u > \mathbf{M}u$. Suppose we assume the contrary, i.e there exists $(\mathbf{p}, \mathbf{s}, r) \in \mathbf{B} \times [0, T) : u(\mathbf{p}, \mathbf{s}, r) > v(\mathbf{p}, \mathbf{s}, r)$. Since $\mathbf{L}u = 0$ on $u > \mathbf{M}u$ and $\mathbf{L}v \leq 0$, it follows that $\mathbf{L}(u - v) \geq 0$ on $u > \mathbf{M}u$.

Define $\tau = \inf \{t > r : u(\mathbf{p}(t), \mathbf{s}(t), t) = v(\mathbf{p}(t), \mathbf{s}(t), t)\}$ with $\mathbf{p}(r) = \mathbf{p}$ and $\mathbf{s}(r) = \mathbf{s}$. Since $u(\mathbf{p}, \mathbf{s}, T) = v(\mathbf{p}, \mathbf{s}, T)$, clearly $\tau \leq T$. Further τ is a stopping time since u and v are continuous functions and therefore $\{u > v\}$ is an open set. Applying Dynkin's formula to $u - v$, we see that

$$E_r(u - v)(\tau) = (u - v)(r) + E_r \int_r^\tau \mathbf{L}(u - v)(s)ds$$

where the other arguments of $u - v$ have been suppressed. By the definition of τ, and the fact that $(u - v)(r) > 0$, it follows that we must have $\mathbf{L}(u - v)(s) < 0$ for some $s \in [r, \tau)$. Therefore, $\mathbf{L}v(s) > 0$ which contradicts the assumptions on v. Hence, $v \geq u$ on $u > \mathbf{M}u$. This completes the proof. **Note**: We have used the fact that $u \leq v$ on $u > \mathbf{M}u$ so that for $t \in [r, \tau)$,$u(\mathbf{p}(t), \mathbf{s}(t), t) > \mathbf{M}u(\mathbf{p}(t), \mathbf{s}(t), t)$.

Thus, if u is a sufficiently smooth solution of the quasi-variational inequalities,we can construct an impulse control policy as follows:

Using the notation of Eastham and Hastings(1988), if the initial data is $(\mathbf{p}, \mathbf{s}, t)$, we set $\theta_0^* = t$ and $\mathbf{S_0^*} = \mathbf{s}$. Define

$$\theta_k^* = \inf \{r \geq \theta_{k-1}^* : (\mathbf{P}(r), \mathbf{S_{k-1}^*}, r) \in \mathbf{A}\} \tag{3.10}$$

where $\theta_k^* = \infty$ if process does not enter $\mathbf{A}$ before T.

$$\mathbf{S_k^*} = \phi(\mathbf{P}(\theta_k^*), \mathbf{S_{k-1}^*}, \theta_k^*) \tag{3.11}$$

By the upper semicontinuity of $\mathbf{M}u$ and the continuity of u, it is easy to see that $\mathbf{A}$ is a closed set. Therefore, θ_k^* is a stopping time for each k. By the continuity of the price process, $(\mathbf{P}(\theta_k^*), \mathbf{S}_{k-1}^*, \theta_k^*) \in \mathbf{A}$ and $\mathbf{S_k^*}$ is well defined. We shall now check that the control policy $v^* = \{(\theta_k^*, \mathbf{S_k^*})\}$ is *admissible.*

Proposition 3 *If the price process* $\mathbf{P}$ *is almost surely finite and quasi-left continuous,*

$$P[lim_{k \to \infty} \theta_k^* \leq \infty] = 0$$

Note*: In the problem we are considering ,* $\mathbf{P}$ *is, in fact, a continuous process. But the assertion holds even if* $\mathbf{P}$ *has jumps where the jump times are totally inaccessible. (see Elliott(1982)).*

Proof.

We shall again adopt the notation of Eastham and Hastings(1988).

Let

$$D = \{lim_{k\to\infty}\theta_k^* \le T\}$$

$$M = \max\{B_i; i = 1, ..., n\}$$

$$W_k(t) = \sum_{i=0}^{n} P_i(t).S_{k,i}^*, t \ge \theta_k^*$$

and, as in Eastham and Hastings, let us drop the subscript '$*$'.

We shall first prove that $S_{k,0}$, the number of shares of the riskless asset is almost surely finite. By the definition of the admissible trading strategies, we know that

$$W_{k-1}(\theta_k) - W_k(\theta_k) \ge \beta.$$

Therefore,

$$S_{k,0} \le S_{k-1,0} + \sum_{i=1}^{n}(P_i(\theta_k)/P_0(\theta_k))(S_{k-1,i} - S_{k,i}) - \beta/P_0(\theta_k)$$

Iterating the above, we see that

$$S_{k,0} \le s_0 + \sum_{j=1}^{k}\sum_{i=1}^{n}(P_i(\theta_j)/P_0(\theta_j))(S_{j-1,i} - S_{j,i}) - \sum_{j=1}^{k} c/P_0(\theta_j)$$

Interchanging the order of summation and using the monotonicity of P_0, we finally obtain

$$S_{k,0} \le s_0 + \sum_{i=1}^{n}(P_i(\theta_1)/p_0)s_i + (M/p_0)\sum_{i=1}^{n}[\sum_{j=1}^{k-1}|P_i(\theta_{j+1}) - P_i(\theta_j)| - kc/(nP_0(T))]$$

This proves that $S_0 < \infty$ almost surely. We shall now show that $S_0 > -\infty$ almost surely. By the definition of admissible trading strategies,

$$W_k(\theta_k) \ge -\Gamma$$

Therefore,

$$P_0(\theta_k)S_0(\theta_k) + \sum_{i=1}^{n} P_i(\theta_k)S_i(\theta_k) \ge -\Gamma$$

Therefore,

$$S_0(\theta_k) \ge -\Gamma - M\sum_{i=1}^{n} P_i(\theta_k)$$

Since $P_0 > 0$, $S_0 > -\infty$ almost surely.

By the finiteness of $\mathbf{P}$ and S_0, we see that $-\infty < W_k < \infty$ almost surely. We shall now show that

$$D \subseteq \limsup_{k\to\infty}\{W_k(\theta_{k+1} - W_k(\theta_k) \ge \beta/2\} \text{ a.s} \tag{3.12}$$

Let us assume the contrary. Therefore, suppose $\omega \in D$ and there exists $I = I(\omega)$ such that for all $k \geq I$,

$$W_k(\theta_{k+1}) - W_k(\theta_k) \leq -\beta$$

Therefore, for $j > I$,

$$W_{j-1}(\theta_j) \leq -(j - I - 1)\beta/2 + W_I(\theta_I)$$

Since $W_I(\theta_I) < \infty$ a.s., by choosing j sufficiently large , we see that $W_{j-1}(\theta_j) < -\Gamma$ almost surely which contradicts our definition of the choice of trading strategies. Therefore, we have a contradiction and this proves the statement.

The rest of the proof now proceeds along the lines of the proof of lemma (2.12) in Eastham and Hastings(1988). We know that

$$D \subseteq \limsup_k \{W_k(\theta_{k+1} - W_k(\theta_k) \geq \beta/2\} \cap D$$

Therefore,

$$\begin{aligned} D \subseteq \limsup_k \{&[P_0(\theta_{k+1}) - P_0(\theta_k)][s_0 + \sum_{i=1}^{n}(P_i(\theta_1)/p_0)s_i \\ &+ [P_0(\theta_{k+1}) - P_0(\theta_k)].[(M/p_0)\sum_{i=1}^{n}(\sum_{j=1}^{k-1}|P_i(\theta_{j+1}) - P_i(\theta_j)| - k\beta/(nP_0(T)))] \\ &+ \sum_{i=1}^{n}[P_i(\theta_{k+1}) - P_i(\theta_k)].S_{k,i} \geq \beta/2\} \cap D \end{aligned} \tag{3.13}$$

By the pigeonhole principle,

$$\begin{aligned} D \subseteq \limsup_k \{&[P_0(\theta_{k+1}) - P_0(\theta_k)][s_0 + \sum_{i=1}^{n}(P_i(\theta_1)/p_0)s_i] \\ &\geq \beta/(2(2n+1))\} \cap D \cup \bigcup_{i=1}^{n} \limsup_k \{[P_0(\theta_{k+1}) - P_0(\theta_k)] \\ &.[(M/p_0)\sum_{j=1}^{k-1}|P_i(\theta_{j+1}) - P_i(\theta_j)| - kc/(nP_0(T))]// \geq \beta/(2(2n+1))\} \cap D \\ &\cup \bigcup_{i=1}^{n} \limsup_k \{[P_i(\theta_{k+1}) - P_i(\theta_k)]S_{k,i} \geq \beta/(2(2n+1))\} \cap D \end{aligned} \tag{3.14}$$

The first lim sup is contained in

$$\limsup_k \{P_0(\theta_{k+1}) - P_0(\theta_k) \geq cp_0/(2(2n+1)(s_0p_0 + \sum P_i(\theta_1)s_i)); // s_0p_0 + \sum P_i(\theta_1)s_i > 0\} \cap D \tag{3.15}$$

Since $D = \{lim_{k\to\infty}\theta_k^* \leq T\}$, we easily see that the above is contained in the set where P_0 becomes infinite at some $t \leq T$ which has probability 0 by hypothesis. Since $S_{k,i}$ is bounded, for each $i = 1, ..., n$

$$\limsup_k \{[P_i(\theta_{k+1}) - P_i(\theta_k)]S_{k,i} \geq c/(2(2n+1))\} \cap D$$
$$\subseteq \limsup_k \{P_i(\theta_{k+1}) - P_i(\theta_k) \geq c/(2M(2n+1))\} \cap D \quad (3.16)$$

The above is contained in the set where finiteness or quasi-left continuity of $\mathbf{P}$ is violated and therefore has probability zero.

By the monotonicity of P_0 , each set in the middle of the union in **??** is contained in

$$\limsup_k \{(1/k)\sum_{j=1}^{k-1} |P_i(\theta_{j+1}) - P_i(\theta_j)|$$
$$\geq p_0c/(MnP_0(T) + p_0c/(2kM(2n+1)(P_0(T) - p_0))\} \cap D$$
$$\subseteq \{|P_i(\theta_{j+1}) - P_i(\theta_j)| \not\to 0 \text{ as } j \to \infty\} \cap D \quad (3.17)$$

which has probability zero by hypothesis. This completes the proof of the assertion !

Proposition 4 *If u is a sufficiently smooth solution of the quasi-variational inequalities giving rise to an admissible control $v^* = \{(\theta_k^*, \mathbf{S}_k^*)\}$. Also suppose that u satisfies Dynkin's formula, i.e.*

$$E[u(\mathbf{P}(\theta'_k), \mathbf{S}_{k-1}, \theta'_k) - u(\mathbf{P}'_{k-1}, \mathbf{S}_{k-1}, \theta'_{k-1})] = E\int_{\theta'_{k-1}}^{\theta'_k} \mathbf{L}u(\mathbf{P}(r), \mathbf{S}_{k-1}, r)dr$$

where $v = \{(\theta_k, \mathbf{S}_k)\}$ is any admissible impulse control and $\theta'_k = \theta_k \wedge T$. Then v^* is an optimal impulse control.

Proof.

The proof of this proposition follows exactly along the lines of the proof of Theorem 3.2 in EH using the results of the previous propositions and we shall not repeat it here.

4 Pricing of the European Option

In this section, we shall use the results of the previous section to explicitly calculate the price of a European option in the situation where the parameters of the bond and stock price processes are constants. We shall consider the situation where we have one bond and one stock which is basically the Black-Scholes situation. We shall then compare the price of the option with the Black-Scholes price and thus demonstrate the effect of fixed transaction costs.

4.1 Price processes for the bond and stock

The price processes for the bond and stock are given by

$$dP_0(t) = \mu_0 P_0(t)dt \tag{4.1}$$

$$dP_1(t) = \mu_1 P_1(t)dt + \sigma P_1(t)dW(t) \tag{4.2}$$

where $W(t)$ is one-dimensional Brownian motion.

The partial differential operator associated with the optimal impulse control policy is given by

$$\mathbf{L}u = u_t + \mu_0 p_0 u_{p_0} + \mu_1 p_1 u_{p_1} + (1/2)\sigma^2 p_1^2 u_{p_1 p_1} \tag{4.3}$$

In this paper, we shall assume that the investor has the risk-neutral utility function

$$\mathcal{U}(x) = x$$

In the notation of **section 3.2**, we have

$$\mathbf{B} = (-\infty, \infty) \times [-B_1, B_1]$$

and

$$\begin{aligned} K(\mathbf{p}, \mathbf{s}) = \{\mathbf{s}' \in \mathbf{B} : \mathbf{p}.\mathbf{s}' = \alpha \mathbf{p}.\mathbf{s} - \beta \geq -\Gamma \text{ for } \mathbf{p}.\mathbf{s} > 0, \\ \mathbf{p}.\mathbf{s}' = (2-\alpha)\mathbf{p}.\mathbf{s} - \beta \geq -\Gamma \text{ for } \mathbf{p}.\mathbf{s} < 0\} \end{aligned} \tag{4.4}$$

$$K = \{(\mathbf{p}, \mathbf{s}) : K(\mathbf{p}, \mathbf{s}) \neq \Phi\} \tag{4.5}$$

By the above definition, we see that

$$K = \{(\mathbf{p}, \mathbf{s}) : \mathbf{p}.\mathbf{s}' \geq -\Gamma\}$$

4.2 The Impulse Control Problems

By the results of **sections 2 and 3.4**, the value functions $V_w(S_0)$ and $V_1(S_0)$ are the solutions of the following problems :

$$V_1(S_0) = \sup_{\Pi \in \mathcal{T}(S_0)} E[\mathbf{P}.\mathbf{S}(T) - (1-\alpha)|\mathbf{P}.\mathbf{S}(T)| - \beta] \tag{4.6}$$

$$V_w(S_0) = \sup_{\Pi \in \mathcal{T}(S_0)} E[\mathbf{P}.\mathbf{S}(T) - (1-\alpha)|\mathbf{P}.\mathbf{S}(T)| - \beta + I_{(P_1(T)>E)}(E - P_1(T))] \tag{4.7}$$

If we set

$$\eta = E[I_{(P_1(T)>E)}(P_1(T) - E)]$$

we easily see that

$$V_w(S_0) = V_1(S_0) - \eta \tag{4.8}$$

Therefore,

$$S_1 = \inf \{S_0 : V_1(S_0) \geq 0\} \tag{4.9}$$

$$S_w = \inf \{S_0 : V_1(S_0) \geq -\eta\} \tag{4.10}$$

Thus, we only need to calculate $V_1(S_0)$.

4.3 Calculation of $V_1(S_0)$

By the results of section 3.4,

$$V_1(S_0) = u(P_0(0), P_1(0), S_0, 0, 0) \tag{4.11}$$

where $u \in \mathcal{C}$ (defined in section 3.4) and

$$u \geq \mathbf{M}u, \mathbf{L}u \leq 0$$

$$(u - \mathbf{M}u).(\mathbf{L}u) = 0$$

$$u(\mathbf{p}, \mathbf{s}, T) = \mathbf{p}.\mathbf{s} \tag{4.12}$$

We shall solve the above *implicit obstacle* problem by the usual method of "iterating upon the obstacle". We choose u_0 as the solution of the p.d.e

$$\mathbf{L}u_0 = 0$$

satisfying

$$u_0(\mathbf{p}, \mathbf{s}, T) = \mathbf{p}.\mathbf{s} - (1-\alpha)|\mathbf{p}.\mathbf{s}| - \beta$$

and define $u_i, i = 1, 2, 3, ...$ by

$$\mathbf{L}u_i \leq 0, u_i \geq u_{i-1}$$

and

$$(\mathbf{L}u_i)(u_i - \mathbf{M}u_{i-1}) = 0$$

Therefore, $u_0 \leq u_1 \leq u_2 \leq ... \leq u$.

For notational convenience, we shall assume in the following that $\mathbf{p}.\mathbf{s} \geq 0$. The expressions for $\mathbf{p}.\mathbf{s} < 0$ can be obtained from those for $\mathbf{p}.\mathbf{s} \geq 0$ by replacing the parameter α by $2 - \alpha$.

It is easy to see that

$$u_0 = \alpha p_0 s_0 \exp(\mu_0(T-t)) + \alpha p_1 s_1 \exp(\mu_1(T-t)) - \beta \tag{4.13}$$

If $\mu_1 = \mu_0$, $u_0 > \mathbf{M}u_0$ everywhere. Therefore,

$$u = u_0$$

In this case,

$$V_1(s_0) = \alpha p_0 s_0 \exp(\mu_0 T) - \beta \text{ for } s_0 \geq 0 \tag{4.14}$$

and

$$V_1(s_0) = (2-\alpha) p_0 s_0 \exp(\mu_0 T) - \beta \text{ for } s_0 < 0 \tag{4.15}$$

In the notation of section 1, $S_1 = (\beta/\alpha)\exp(-\mu_0 T)$ and $S_w = (\beta - \eta)/alpha \exp(-\mu_0 T)$. Therefore, the price of the European option is given by

$$p_w = S_w - S_1 = -(\eta/\alpha)\exp(-\mu_0 T) = (E[I_{(P_1(T)>E)}(P_1(T) - E)]/\alpha)\exp(-\mu_0 T) \tag{4.16}$$

We shall now consider the case when $\mu_1 > \mu_0$.

Proposition 5 *If $\mu_1 \neq \mu_0$, the optimal trading policy involves at most one transaction in $[0, T)$, i.e. $u = u_1$.*

Proof.

We shall describe the proof for the case where $\mu_1 > \mu_0$. The proof for the case where $\mu_1 < \mu_0$ is analogous.

In this case ,we have

$$u_0 = (\alpha p_0 s_0 + \alpha p_1 s_1)\exp(\mu_0(T-t)) + \alpha p_1 s_1(\exp(\mu_1(T-t)) - \exp(\mu_0(T-t))) - \beta \tag{4.17}$$

It is easy to see that

$$\mathbf{M}u_0 = (\alpha^2(p_0 s_0 + p_1 s_1 0) - \alpha\beta)\exp(\mu_0(T-t)) + \alpha p_1 B(\exp(\mu_1(T-t)) - \exp(\mu_0(T-t))) - \beta \tag{4.18}$$

Define

$$\tilde{u} = (\alpha(p_0 s_0 + p_1 s_1))\exp(\mu_0(T-t)) + \alpha p_1 B(\exp(\mu_1(T-t)) - \exp(\mu_0(T-t))) - \beta \tag{4.19}$$

It is easy to see that

$$\mathbf{L}\tilde{u} = \alpha(\mu_1 - \mu_0)\exp(\mu_0(T-t))p_1(s_1 - B) \leq 0$$

since $s_1 \leq B$. It is also easy to see that

$$\tilde{u} \geq \mathbf{M}\tilde{u}$$

and

$$\tilde{u}(T) = \alpha(p_0 s_0 + p_1 s_1) - \beta$$

By the result of proposition 2,

$$\tilde{u} \geq u$$

In particular,

$$\tilde{u} \geq u_1$$

Since the operator $\mathbf{M}$ is monotone

$$\mathbf{M}\tilde{u} \geq \mathbf{M}u_1 \geq \mathbf{M}u_0$$

$$\mathbf{M}\tilde{u} = \sup_{(s_0', s_1') \in K(\mathbf{p}, \mathbf{s})} [\alpha \mathbf{p}.\mathbf{s}' \exp(\mu_0(T-t)) + p_1 B(\exp(\mu_1(T-t) - \exp(\mu_0(T-t)))) - \beta]$$

Therefore,

$$\mathbf{M}\tilde{u} = (\alpha^2 \mathbf{p}.\mathbf{s} - \alpha\beta)\exp(\mu_0(T-t)) + p_1 B(\exp(\mu_1(T-t)) - \exp(\mu_0(T-t))) - \beta$$

It is also easy to show that

$$\mathbf{M}u_0 = \mathbf{M}\tilde{u}$$

Therefore,

$$\mathbf{M}u_1 = \mathbf{M}u_0$$

Substituting the above in the quasi-variational inequalities defining $u_0, u_1, ...$, we see that u_1 satisfies

$$\mathbf{L}u_1 \leq 0$$

$$(\mathbf{L}u_1)(u_1 - \mathbf{M}u_1) = 0$$

Therefore, $u = u_1$! Thus, we have shown that the optimal policy involves at most one transaction in $[0, T)$

This completes the proof. !

Therefore, by the result of the above proposition,

$$V_1(S_0) = u_1(P_0(0), P_1(0), S_0, 0, 0) \tag{4.20}$$

Using the above, we can calculate the price of the European option.