CONTRIBUTIONS TO THE STIELTJES MOMENT PROBLEM AND TO THE INTERTWINING OF MARKOV SEMIGROUPS

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CONTRIBUTIONS TO THE STIELTJES MOMENT PROBLEM AND TO THE INTERTWINING OF MARKOV SEMIGROUPS

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Each chapter of this thesis is represented as a metaphorical meeting between mathematicians and their ideas: Stieltjes meet Gauss; Berg meet Urbanik; Dynkin meet Jacobi; Dynkin meet Villani. In the "Stieltjes meet Gauss" part we provide some new criteria for the determinacy problem of the Stieltjes moment problem, starting with a Tauberian type criterion for indeterminacy that is expressed purely in terms of the asymptotic behavior of the moment sequence (and its extension to imaginary lines). Under an additional assumption this provides a converse to the classical Carleman's criterion, thus yielding an equivalent condition for determinacy. We also provide a criterion for determinacy that only involves the large asymptotic behavior of the distribution (or of the density if it exists), which can be thought of as an Abelian counterpart to the previous Tauberian type result. This latter criterion generalizes Hardy's condition for determinacy, and under some further assumptions yields a converse to Pedersen's refinement of Krein's celebrated theorem. The proofs utilize non-classical Tauberian results for moment sequences that are analogues of the ones developed by Feigin and Yaschin and Balkema et al. for the bi-lateral Laplace transform in the context of asymptotically parabolic functions, which generalize the classical Gaussian setting. We illustrate these results by studying the time-dependent moment problem for the law of a process whose logarithm is a Lévy process, which is a generalization of the log-normal distribution. Along the way, we derive the large asymptotic behavior of the density of spectrally-negative Lévy processes having a Gaussian component, which may be of independent interest. We continue the

study of this time-dependent moment problem in the "Berg meet Urbanik" part where we focus on Berg-Urbanik semigroups, a class of multiplicative convolution semigroups on $\mathbb{R}_+$ that is in bijection with the set of Bernstein functions. Berg and Durán proved that the law of such semigroups is determinate (at least) up to time $t = 2$, and for the Bernstein function $\phi(u) = u$ Berg made the striking observation that for time $t > 2$ the law of this semigroup is indeterminate. We extend these works by estimating the threshold time $\mathcal{T}_\phi \in [2, \infty]$ that it takes for the law of such Berg-Urbanik semigroups to transition from determinacy to indeterminacy in terms of simple properties of the underlying Bernstein function ϕ, such as its Blumenthal-Getoor index. In particular, we show that $\mathcal{T}_\phi = 2$ for any Bernstein function ϕ with a drift component, thereby generalizing Berg's result to this entire class. One of the several strategies we implement to deal with the different cases relies on the non-classical Abelian type criterion mentioned above. To implement this approach we provide detailed information regarding distributional properties of the semigroup such as existence and smoothness of a density, and, the large asymptotic behavior for all $t > 0$ of this density along with its successive derivatives, which are original results in the Lévy process literature. In the "Dynkin meet Jacobi" part we introduce and study non-local Jacobi operators, which generalize the classical (local) Jacobi operator. We show that these operators extend to the generator of an ergodic Markov semigroup with an invariant probability measure and study its spectral and convergence properties. In particular, we give a series expansion of the semigroup in terms of explicitly defined polynomials, which are counterparts of the classical Jacobi orthogonal polynomials. In addition, we give a complete characterization of the spectrum of the non-self-adjoint generator and semigroup. We show that the variance decay of the semigroup is hypocoercive in the sense of Villani, with explicit constants, which provides a natural generalization of the spectral gap estimate. After a random warm-up time the semigroup also decays exponentially in entropy, and is both hypercontractive and ultracontractive. Our proofs hinge on the development of intertwining relations—a notion for Markov semigroups introduced by Dynkin—between local and non-local Jacobi operators/semigroups, with the local Jacobi operator/semigroup serving as a reference object for transferring properties to the non-local ones. Finally, in the "Dynkin meet Villani" part, we offer an original and comprehensive spectral theoretical approach to the study of convergence to equilibrium, and in particular of the hypocoercivity phenomenon, for contraction semigroups in Hilbert spaces. Here we utilize intertwining to transfer spectral information from a known, reference semigroup $\widetilde{P} = (e^{-t\widetilde{\mathcal{A}}})_{t\geqslant 0}$ to a target semigroup P that is the object of study. This allows us to obtain conditions under which P satisfies a hypocoercive estimate with exponential decay rate given by the spectral gap of $\widetilde{\mathcal{A}}$. Along the way we also develop a functional calculus involving the non-self-adjoint resolution of identity induced by the intertwining relations. We apply these results to degenerate, hypoelliptic Ornstein-Uhlenbeck semigroups on $\mathbb{R}^d$, and non-local Jacobi semigroups on $[0,1]^d$; in both cases we obtain hypocoercive estimates and are able to explicitly identify the hypocoercive constants.

All work in this thesis was done in collaboration with P. Patie, and the work in Chapter 4 was done with the additional collaboration of P. Cheridito and A. Srapionyan. The contents of Chapters 2–5 have been submitted to peer-reviewed journals as follows:

- Non-classical Tauberian and Abelian type criteria for the moment problem, arXiv:1804.10721 [math.PR], 19pp., 2018;
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- On non-local ergodic Jacobi semigroups: spectral theory, convergence-to-equilibrium and contractivity, arXiv:1905.07832 [math.PR], 40pp., 2019;
- A spectral theoretical approach for hypocoercivity applied to some degenerate hypoelliptic, and non-local operators, arXiv:1905.07042 [math.PR], 22pp., 2019.

BIOGRAPHICAL SKETCH

Aditya Vaidyanathan was born in the ancient city of Tiruchirapalli, India and grew up variously in Singapore, Australia, and Switzerland. After completing his high school education at the International School of Basel he graduated from the University of Southern California in May 2013 with a B.S. in Aerospace Engineering and a B.A. in Mathematics. He began his doctoral studies at Cornell's Center for Applied Mathematics in the fall of 2013, starting working under the advisement of Prof. Pierre Patie in early 2015, and earned a M.S. in Applied Mathematics in January 2017. From May 2018 to August 2018 Aditya was part of the inaugural class of Summer Associates in the Quantitative Research – Machine Learning Program at J.P. Morgan where he worked in the QR – Rates team. In July 2019 he will return to J.P. Morgan in New York as a Full-Time Associate in the Quantitative Research – Machine Learning Program with a placement in QR – Rates/GEM.

To my Amma and Appa, my Paatti and Thatha; and, to Mischa.

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CHAPTER 1

INTRODUCTION

This thesis is split into two parts, consisting of two chapters each, and each of these chapters represents a metaphorical meeting between mathematicians and their ideas: Stieltjes meet Gauss; Berg meet Urbanik; Dynkin meet Jacobi; Dynkin meet Villani.

In the first half we investigate the determinacy problem of the Stieltjes moment problem, which asks under what conditions a probability measure ν supported on $[0,\infty)$ is moment determinate, i.e. uniquely determined within the space of probability measures supported on $[0,\infty)$ by its sequence of moments given, for any integer $n \geqslant 0$, by

$$\int_0^\infty x^n \nu(dx) < \infty.$$

The two most widely used critera for establishing moment determinacy and moment indeterminacy are, respectively, Carleman's summability criterion on the moment sequence [32], and Pedersen's refinement of Krein's integrability criterion [103] for the density, when it exists, of the probability measure itself. However, neither of these criteria are, in general, equivalent to moment determinacy and indeterminacy, respectively, and are complementary in the sense that the former involves conditions on the moment sequence, while the latter involves conditions on the measure itself. Thus, investigating new criteria for moment determinacy and indeterminacy has remained an active area of research for several decades, with deep connections to, and implications for, other branches of mathematics, see for instance the recent survey by Schmüdgen, [110]

In Chapter 2 we develop some new criteria for both moment determinacy and moment indeterminacy. Our first result is a criterion for moment indeterminacy that is expressed in terms of the large asymptotic behavior of the moment sequence (and its extension to imaginary lines), and under an additional assumption this criterion provides a converse to the classical Carleman's criterion, thus upgrading it to a necessary and sufficient condition in this context. We also provide a criterion for moment determinacy that only involves the large asymptotic behavior of the distribution (or of the density if it exists), which can be thought of as an Abelian counterpart to the previous Tauberian type result. This latter criterion generalizes Hardy's condition for determinacy, and under some further assumptions yields a converse to the Pedersen's refinement of the celebrated Krein's theorem. The proofs utilize non-classical Tauberian results for moment sequences that are analogues of the ones developed by Feigin and Yaschin [51] and Balkema et al. [11] for the bi-lateral Laplace transforms in the context of asymptotically parabolic functions; as the Tauberian results in [51, 11] generalize the classical Gaussian setting we term this part "Stieltjes meet Gauss". We illustrate these results by studying the time-dependent Stieltjes moment determinacy for multiplicative convolution semigroups $(\nu_t)_{t\geqslant 0}$, that is semigroups satisfying, for $n, t \geqslant 0$,

$$\int_0^\infty x^n \nu_t(dx) = \int_{-\infty}^\infty e^{ny}\,\mathbb{P}(Y_t \in dy) = e^{t\Psi(n)},$$

where $(Y_t)_{t\geqslant 0}$ is a one-dimensional Lévy process such that $\mathbb{E}[e^{nY_t}] < \infty$, for all $n, t \geqslant 0$. If $(Y_t)_{t\geqslant 0}$ has a Gaussian component then ν_t is indeterminate for all $t > 0$, which generalizes the famous result that the log-normal distribution is indeterminate. Along the way, we also derive the large asymptotic behavior of the density of spectrally-negative Lévy processes having a Gaussian component, which may be of independent interest in the Lévy process literature. We also provide an example of a Lévy process for which ν_t is moment determinate if and only if $t \leqslant 2$.

In Chapter 3, the "Berg meet Urbanik" part, we continue our study of this time-dependent moment problem via Berg-Urbanik semigroups, which are multiplicative convolution semigroups $(\nu_t)_{t\geqslant 0}$ such that

$$\int_0^\infty x^n \nu_t(dx) = \left(\prod_{k=1}^n \phi(k)\right)^t,$$

where ϕ is a Bernstein function. In [19], Berg and Durán proved that the law of such semigroups is moment determinate (at least) up to time $t = 2$, and for the Bernstein function $\phi(u) = u$ Berg [16] made the striking observation that for time $t > 2$ the law of this semigroup is moment indeterminate. We extend these works by estimating the threshold time $\mathcal{T}_\phi \in [2, \infty]$ that it takes for the law of such Berg-Urbanik semigroups to transition from moment determinacy to moment indeterminacy in terms of simple properties of the underlying Bernstein function ϕ, such as its Blumenthal-Getoor index. In particular, we show that $\mathcal{T}_\phi = 2$ for any Bernstein function ϕ with a drift component, thereby generalizing Berg's result to this entire class. One of the several strategies we implement to deal with the different cases relies on the non-classical Abelian type criterion mentioned above. To implement this approach we provide detailed information regarding distributional properties of the semigroup such as existence and smoothness of a density, and, the large asymptotic behavior for all $t > 0$ of this density along with all of its successive derivatives. In particular, these results are original in the Lévy processes literature and also may be of independent interest.

The theme for the second half of this thesis is intertwining of Markov semigroups, a notion that was introduced by Dynkin [48] while investigating the question of when functions of a Markov process remain Markovian. Informally, an intertwining is a commutation relationship of the form

$$P_t \Lambda = \Lambda \widetilde{P}_t$$

where $P = (P_t)_{t \geqslant 0}$ and $\widetilde{P} = (\widetilde{P}_t)_{t \geqslant 0}$ are suitable (e.g. Markov) semigroups of linear operators and Λ is a linear operator. As the investigations below reveal, it is desirable to establish an intertwining on the space $\mathrm{L}^2(\widetilde{\nu})$, where $\widetilde{\nu}$ is the invariant probability measure for $\widetilde{P}$, however, in practice it is often easier to establish such an identity on a more convenient function space, such as the space of polynomials $\mathcal{P}$. The extension of the intertwining to $\mathrm{L}^2(\widetilde{\nu})$ then requires that $\mathcal{P}$ be a dense subspace of $\mathrm{L}^2(\widetilde{\nu})$, a fact that is guaranteed when the probability measure $\widetilde{\nu}$ is moment determinate. This connection between the density of $\mathcal{P}$ in $\mathrm{L}^2(\widetilde{\nu})$ and the moment determinacy of $\widetilde{\nu}$ is what bridges the two halves of this thesis.

In Chapter 4 we introduce and study, via intertwining relations, non-local Jacobi operators, which generalize the classical (local) Jacobi operator on $[0, 1]$; hence "Dynkin meet Jacobi". We show that these operators extend to the generator of an ergodic Markov semigroup with an invariant probability measure, very much in the spirit of Dynkin's original work, and study the spectral and convergence properties of this semigroup. In particular, we give a series expansion of the semigroup in terms of explicitly defined polynomials, which are counterparts of the classical Jacobi orthogonal polynomials. In addition, we give a complete characterization of the spectrum of the non-self-adjoint generator and semigroup. We show that the variance decay of the semigroup is hypocoercive in the sense of Villani, with explicit constants, which provides a natural generalization of the spectral gap estimate. After a random warm-up time the semigroup also decays exponentially in entropy and is both hypercontractive and ultracontractive. Our proofs hinge on the development of intertwining relations between local and non-local Jacobi operators/semigroups, with the local Jacobi operator/semigroup serving as a reference object for transferring properties to the non-local ones.

The aim of Chapter 5 is to offer an original and comprehensive spectral theoretical approach to the study of convergence to equilibrium, and in particular of the hypocoercivity phenomenon, for contraction semigroups in Hilbert spaces. Again, our approach rests on exploiting intertwining relationships and thus "Dynkin meet Villani" for this part. We utilize intertwining to transfer spectral information from a known, reference semigroup $\widetilde{P} = (e^{-t\widetilde{\mathcal{A}}})_{t\geqslant 0}$ to a target semigroup P, which is the object of study. This allows us to obtain conditions under which P satisfies a hypocoercive estimate with exponential decay rate given by the spectral gap of $\widetilde{\mathcal{A}}$. Along the way we also develop a functional calculus involving the non-self-adjoint resolution of identity induced by the intertwining relations. We apply these results in a general Hilbert space setting to two cases: degenerate, hypoelliptic Ornstein-Uhlenbeck semigroups on $\mathbb{R}^d$, and non-local Jacobi semigroups on $[0, 1]^d$, which were introduced and studied for $d = 1$ in Chapter 4. In both cases we obtain hypocoercive estimates and are able to explicitly identify the hypocoercive constants.

CHAPTER 2

NON-CLASSICAL TAUBERIAN AND ABELIAN TYPE CRITERIA FOR THE MOMENT PROBLEM

In this chapter we provide some new criteria for the determinacy problem of the Stieltjes moment problem. We first give a Tauberian type criterion for moment indeterminacy that is expressed purely in terms of the asymptotic behavior of the moment sequence (and its extension to imaginary lines). Under an additional assumption this provides a converse to the classical Carleman's criterion, thus yielding an equivalent condition for moment determinacy. We also provide a criterion for moment determinacy that only involves the large asymptotic behavior of the distribution (or of the density if it exists), which can be thought of as an Abelian counterpart to the previous Tauberian type result. This latter criterion generalizes Hardy's condition for determinacy, and under some further assumptions yields a converse to the Pedersen's refinement of the celebrated Krein's theorem. The proofs utilize non-classical Tauberian results for moment sequences that are analogues to the ones developed in [51] and [11] for the bi-lateral Laplace transforms in the context of asymptotically parabolic functions. We illustrate our results by studying the time-dependent moment problem for the law of log-Lévy processes viewed as a generalization of the log-normal distribution. Along the way, we derive the large asymptotic behavior of the density of spectrally-negative Lévy processes having a Gaussian component, which may be of independent interest.

2.1 Introduction and main results

The determinacy problem for the Stieltjes moment problem asks under what conditions a measure ν supported on $[0, \infty)$ can be uniquely determined by its sequence of moments

$\mathcal{M}_\nu = (\mathcal{M}_\nu(n))_{n\geqslant 0}$ where, for any $n \geqslant 0$,

$$\mathcal{M}_\nu(n) = \int_0^\infty x^n \nu(dx) < \infty.$$

When a measure is uniquely determined by its moments we say it is moment determinate, otherwise it is moment indeterminate. Note that we consider only measures with unbounded support since otherwise the problem is trivial. For references on the moment problem see the classic monographs [2] and [111], the comprehensive exposition [112], and the more recent monograph [110], where the interested reader will find a nice description of its connections and interplay with many branches of mathematics, as well as its broad range of applications.

2.1.1 A Tauberian type moment condition for indeterminacy, and a converse for Carleman's criterion

One of the most widely used criteria for determinacy is Carleman's criterion, which states that if

$$\sum^{\infty} \mathcal{M}_\nu^{-\frac{1}{2n}}(n) = \infty,$$

then ν is moment determinate, where for a sequence $(a_n)_{n\geqslant 0}$ of real numbers $\sum^\infty a_n = \infty$ denotes $\sum_{n_0}^\infty a_n = \infty$ for some index $n_0 \geqslant 1$ whose choice does not impact the divergence property (the same notation holds for integrals of functions). However, it is well-known that the divergence of this series is not a necessary condition for moment determinacy, see e.g. Heyde [65] for an example. The main result in this section is a condition for indeterminacy that is entirely expressed in terms of the moment transform (and its extension to imaginary lines) of the measure, which under an additional assumption yields a converse to Carleman's criterion. In order to state this criterion we need to introduce some notation.

Let $C^2_+(I)$ denote the set of twice differentiable functions on an interval $I \subseteq \mathbb{R}$ whose second derivative is strictly positive on I. We define the set $\mathcal{A}$ of *asymptotically parabolic functions*, a notion which traces its origins to [10, 11], as

$$\left\{G \in C^2_+((a,\infty)),\ a \geqslant -\infty;\ G''\left(u + w(G''(u))^{-\frac{1}{2}}\right) \overset{\infty}{\sim} G''(u),\ \text{loc. unif. in } w \in \mathbb{R}\right\}, \quad (2.1)$$

where $f(u) \overset{\infty}{\sim} g(u)$ means that $\lim_{u\to\infty} \frac{f(u)}{g(u)} = 1$. We are now ready to state our Tauberian type criterion for the Stieltjes moment problem.

Theorem 2.1.1. *Let $\mathcal{M}_\nu$ be the Stieltjes moment sequence of a measure ν and assume that the following two conditions hold.*

(a) There exists $G \in \mathcal{A}$ such that

$$\mathcal{M}_\nu(n) \overset{\infty}{\sim} e^{G(n)}.$$

(b) There exists $n_0 \in [0,\infty)$ such that for $n \geqslant n_0$, writing $\eta^2(n) = (\log \mathcal{M}_\nu(n))''$, the functions

$$y \mapsto \left|\frac{\mathcal{M}_\nu\left(n + i\frac{y}{\eta(n)}\right)}{\mathcal{M}_\nu(n)}\right| \quad (2.2)$$

are uniformly (in n) dominated by a function in $\mathrm{L}^1(\mathbb{R})$.

(1) Then, the condition

$$\int^\infty (uG'(u) - G(u))G''(u)e^{-\frac{G'(u)}{2}}du < \infty \implies \nu \textit{ is moment indeterminate.} \quad (2.3)$$

(2) If in addition $\lim_{u\to\infty} ue^{-\frac{G'(u)}{2}} < \infty$ then

$$\begin{aligned} \nu \textit{ is moment determinate} &\iff \int^\infty (uG'(u) - G(u))G''(u)e^{-\frac{G'(u)}{2}}du = \infty \\ &\iff \int^\infty e^{-\frac{G'(u)}{2}}du = \infty \\ &\iff \sum^\infty \mathcal{M}_\nu^{-\frac{1}{2n}}(n) = \infty. \end{aligned} \quad (2.4)$$

This Theorem is proved in Section 2.3.1. We call it a Tauberian type result since assumptions on the moment transform alone give sufficient information regarding the measure for concluding indeterminacy. In Section 2.2.1 below we shall provide an application of this criterion to the time-dependent moment problem for the law of log-Lévy processes. Invoking now a useful result from Berg and Durán [19, Lemma 2.2 and Remark 2.3] regarding factorization of moment sequences in relation to the moment problem, we deduce the following corollary of Theorem 2.1.1(1).

Corollary 2.1.1. *Let $\mathcal{M}_{\mathcal{V}}$ be the Stieltjes moment sequence of a measure $\mathcal{V}$ and suppose that, for $n \geqslant 0$,*

$$\mathcal{M}_{\mathcal{V}}(n) = \mathcal{M}_{\nu}(n)\mathfrak{m}(n),$$

where $\mathcal{M}_{\nu}$ is a Stieltjes moment sequence that satisfies the assumptions of Theorem 2.1.1, and $(\mathfrak{m}(n))_{n\geqslant 0}$ is a non-vanishing Stieltjes moment sequence. Then,

$$\int^{\infty} (uG'(u) - G(u))G''(u)e^{-\frac{G'(u)}{2}}du < \infty \implies \mathcal{V} \textit{ is moment indeterminate.}$$

We proceed by offering a few remarks regarding our criterion in relation to the recent literature on the moment problem. In Theorem 2.1.1(1) we provide a checkable criterion for indeterminacy based solely on properties of the moment transform, which seems to be new in the context of the moment problem. For example, the assumption that $\mathcal{M}_{\nu}(n) \geqslant cn^{(2+\varepsilon)n}$ for some constants $c, \varepsilon > 0$ and n large enough, together with

$$\lim_{x\to\infty} \frac{x\nu'(x)}{\nu(x)} = -\infty, \tag{2.5}$$

where $\nu(dx) = \nu(x)dx$, allows one to conclude indeterminacy, see Theorem 5 in the nice survey [80]. The condition expressed by (2.5), which goes back to [78], is called Lin's condition in the literature, and involves the a priori assumption of the existence and differentiability of the density on a neighborhood of infinity.

In the same spirit, the integrability condition in Theorem 2.1.1(b) can be replaced by (but is not equivalent to) the assumption that $\nu(dx) = \nu(x)dx$ is such that

$$x \mapsto -\log \nu(e^x) \text{ is convex, for } x \text{ large enough.} \tag{2.6}$$

Under the assumption in (2.6), Pakes proved in [94] that Carleman's criterion becomes an equivalent condition for moment determinacy. However, as with Lin's condition, this involves assumptions on both the moment sequence and the density, and is a rather strong geometric requirement on the density itself. We point out that, as by-product of Theorem 2.1.1, we have $\nu(dx) = \nu(x)dx$, $x > 0$, and that $\nu(e^x)$ satisfies a less stringent asymptotic condition, which is in fact implied by (2.6), see [11, Theorem 2.2 and Equation (4.5)].

In Theorem 2.1.1(2) we are able to show, under a further mild assumption on G, that Carleman's criterion becomes necessary and sufficient for determinacy. The additional assumption on G is what allows us to connect the condition in (2.3) to the finiteness of the sum in (2.4), which is the harder of the two implications to prove. While both Lin's condition in (2.5) and Pakes' condition (2.6) yield converses to Carleman's criterion, we avoid having to make distinct assumptions on the moment transform and the density.

2.1.2 An Abelian type tail condition for determinacy, and a converse for Krein's criterion

The celebrated Krein's criterion, refined by Pedersen [103], states that if $\nu(dx) = \nu(x)dx$ and, for some $x_0 \geqslant 0$,

$$\int_{x_0}^{\infty} \frac{-\log \nu(x^2)}{1+x^2} dx < \infty,$$

then ν is moment indeterminate (the case $x_0 = 0$ yields the original version of Krein's theorem). It is also well-known that this condition is not necessary for moment indeterminacy, see the counterexample in [103]. In this section we provide conditions for moment determinacy that are stated in terms of the measure directly, which under some additional assumptions yields a converse to the refined Krein's criterion.

To state our result we define the set of *admissible* asymptotically parabolic functions as

$$\mathcal{A}_{\mathfrak{D}} = \left\{ G_* \in \mathcal{A};\ \lim_{x\to\infty} \frac{G_*(x)}{x} = \infty \right\},$$

and note that not all asymptotically parabolic functions are admissible, see e.g. the last row of Table 2.1 in Section 2.2.2. The admissibility condition is equivalent to the condition that, for large enough x, the function $x \mapsto e^{G_*(\log x)}$ grows faster than any polynomial. Hence our reason for assuming admissibility is to avoid trivial situations in terms of the moment problem. We suggestively write G_* as it will turn out that G_* will be the Legendre transform of a function $G \in \mathcal{A}$, see the beginning of Section 2.3 and in particular Lemma 2.3.2 for further details.

Next, we write, for suitable functions f and g, $f(x) \overset{\infty}{=} \mathrm{O}(g(x))$ if $\overline{\lim}_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| < \infty$ and $f(x) \overset{\infty}{\asymp} g(x)$ if $f \overset{\infty}{=} \mathrm{O}(g(x))$ and $g(x) \overset{\infty}{=} \mathrm{O}(f(x))$. We also write $\overline{\nu}(x) = \int_x^\infty \nu(dx)$ for the tail of a probability measure ν. The following result may be thought of as the Abelian counterpart to the Tauberian type result in Theorem 2.1.1.

Theorem 2.1.2. *Let ν be a probability measure with all positive moments finite.*

(1) Suppose that there exists $G_* \in \mathcal{A}_{\mathfrak{D}}$ such that either

$$\overline{\nu}(x) \overset{\infty}{=} \mathrm{O}(e^{-G_*(\log x)}), \tag{2.7}$$

or, if $\nu(dx) = \nu(x)dx$, that

$$\nu(x) \overset{\infty}{=} \mathrm{O}(e^{-G_*(\log x)}). \tag{2.8}$$

Then, writing γ for the inverse of the continuous, increasing function G_*',

$$\sum^{\infty} e^{-\frac{\gamma(n)}{2}} = \infty \implies \nu \textit{ is moment determinate}. \tag{2.9}$$

(2) If in addition

$$\nu(x) \overset{\infty}{\asymp} e^{-G_*(\log x)},$$

and $\lim_{x\to\infty} G_*'(x)e^{-\frac{x}{2}} < \infty$, *then*

$$\begin{aligned} \nu \textit{ is moment indeterminate} &\iff \sum^{\infty} e^{-\frac{\gamma(n)}{2}} < \infty \\ &\iff \int^{\infty} G_*(x)e^{-\frac{x}{2}}dx < \infty \\ &\iff \int^{\infty} \frac{-\log \nu(x^2)}{1+x^2}dx < \infty. \end{aligned}$$

This Theorem is proved in Section 2.3.2. It leads to a generalization of Hardy's condition for moment determinacy, which was proved by Hardy in a series of papers [63, 64] and seemed to have gone unnoticed in the probabilistic/moment problem literature until the recent exposition by Stoyanov and Lin [114], see also [80]. The criterion states that if

$$\int_0^{\infty} e^{c\sqrt{x}}\nu(dx) < \infty, \text{ for some } c > 0, \tag{2.10}$$

then ν is determinate.

Corollary 2.1.2. *Let ν be a probability measure with all positive moments finite.*

(1) Hardy's condition is satisfied, i.e. (2.10) *holds, if and only if*

$$\overline{\nu}(x) \overset{\infty}{=} \mathrm{O}(e^{-c\sqrt{x}-\frac{1}{2}\log x}).$$

Consequently Hardy's condition implies that (2.7) *and* (2.9) *of Theorem 2.1.2(1) are satisfied, with $x \mapsto G_*(x) = ce^{\frac{x}{2}} + \frac{1}{2}x \in \mathcal{A}_{\mathcal{D}}$.*

(2) If

$$\nu(x) \overset{\infty}{\asymp} e^{-\frac{\alpha\sqrt{x}}{\log x}} \tag{2.11}$$

then ν does not satisfy Hardy's criterion, i.e.

$$\int_0^\infty e^{c\sqrt{x}}\nu(x)dx = \infty, \quad \forall c > 0,$$

yet ν is moment determinate for all $\alpha > 0$.

This Corollary is proved in Section 2.3.3, and we proceed with some remarks concerning it as well as Theorem 2.1.2. The fact that Theorem 2.1.2(1) leads to a generalization of Hardy's condition shows that the assumptions we make are rather weak yet still yield the moment determinacy of ν. Note that the requirement in (2.7) or in (2.8) does not trivially imply moment determinacy since a function $G_* \in \mathcal{A}_{\mathfrak{D}}$ may be sublinear at the log-scale, e.g. $G_*(\log x) = x^\alpha$ for $\alpha > 0$.

It was shown in Stoyanov and Lin [114] that Hardy's condition implies Carleman's criterion, so that the same argument that disproves the necessity of Carleman's criterion also shows that Hardy's condition is not necessary for moment determinacy. This argument, which goes back to Heyde [65], involves the subtle manipulation of point mass at the origin. In Corollary 2.1.2(2) we are able to give explicit examples of densities, characterized only by their large asymptotic behavior, for which Hardy's condition fails yet, by Theorem 2.1.2(2), Carleman's criterion holds.

In Theorem 2.1.2(2) we give necessary and sufficient conditions on the density ν for moment indeterminacy, and also show that Krein's criterion becomes necessary and sufficient in our context. The existing criteria in the literature that give converses to Krein's theorem require either the differentiability of the density, such as Lin's condition in (2.5), or an exact representation for the density, e.g. [94, Theorem 4], neither of which we suppose.

Finally, we mention that we apply Theorem 2.1.2 to study the log-Lévy moment problem for so-called Berg-Urbanik semigroups in Chapter 3.

2.2 Applications

2.2.1 The log-Lévy moment problem

One of the most famous indeterminate measures is the log-normal distribution, and the indeterminacy of this measure has the consequence that the random variable e^{B_t} is moment indeterminate for all $t > 0$, where $B = (B_t)_{t \geqslant 0}$ is a standard Brownian motion. In this section we apply Theorem 2.1.1 to study this time-dependent moment problem when B is replaced by a Lévy process (admitting all exponential moments), which we call the log-Lévy moment problem.

We recall that a (one-dimensional) Lévy process $Y = (Y_t)_{t \geqslant 0}$ is a $\mathbb{R}$-valued stochastic process with stationary and independent increments, that is continuous in probability, and such that $Y_0 = 0$ a.s. Such processes are fully characterized by the law of Y_1, which is known to be infinitely divisible, and whose characteristic exponent is given by

$$\Psi(u) = bu + \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} \left(e^{ur} - 1 - ur\mathbf{1}_{\{|r| \leqslant 1\}}\right) \Pi(dr), \quad u \in i\mathbb{R}, \tag{2.12}$$

with $b \in \mathbb{R}$, $\sigma \geqslant 0$, and Π a σ-finite, positive measure satisfying $\Pi(\{0\}) = 0$ and the integrability condition

$$\int_{-\infty}^{\infty} \min(1, r^2)\, \Pi(dr) < \infty.$$

As we are interested in the log-Lévy moment problem we only consider Lévy processes admitting all positive exponential moments, i.e. $\mathbb{E}[e^{uY_t}] < \infty$ for all $u, t \geqslant 0$. This condition is equivalent to Ψ admitting an analytical extension to the right-half plane, still denoted by Ψ, which in terms of the Lévy measure can be expressed as

$$\int_1^\infty e^{ur}\Pi(dr) < \infty, \quad u \geqslant 0,$$

see [108, Theorem 25.3 and Lemma 25.7]. In this case we have that

$$\mathbb{E}[e^{uY_t}] = e^{t\Psi(u)}, \quad u \geqslant 0.$$

Theorem 2.2.1. *Let $Y = (Y_t)_{t\geqslant 0}$ be a Lévy process admitting all exponential moments.*

(1) If in (2.12) $\sigma^2 > 0$, then the random variable e^{Y_t} is moment indeterminate for any $t > 0$.

(2) If $\Psi(u) = u\log(u+1)$, $u \geqslant 0$, then the random variable e^{Y_t} is moment determinate if and only if $t \leqslant 2$.

This Theorem is proved in Section 2.3.4. In Theorem 2.2.1(2) we provide an example of a Lévy exponent such that the log-Lévy moment problem is determinate up to a threshold time, and then indeterminate afterwards. This phenomenon has been observed in the literature by Berg in [16] for the so-called Urbanik semigroup and in the next chapter we extend Berg's result to a large class of multiplicative convolution semigroups, which do not have a log-normal component. We also mention that we prove Theorem 2.2.1(2) also via an application of Theorem 2.1.1 and interestingly, the additional condition in Item (2) of Theorem 2.1.1 is only fulfilled for $t \geqslant 2$.

We point out that, as a by-product of the proof of Theorem 2.2.1(1), in the case of spectrally-negative Lévy processes, we get the following large asymptotic behavior of their densities, valid for all $t > 0$, which seems to be new in the Lévy literature. Note that, from [108, Ex. 29.14, p. 194], we have, for $\sigma^2 > 0$ and any fixed $t > 0$, $\mathbb{P}(Y_t \in dy) = f_t(y)dy$, $y \in \mathbb{R}$, where $y \mapsto f_t(y) \in C^\infty(\mathbb{R})$, all derivatives of which tend to

0 as $|y| \to \infty$, and where $C^\infty(\mathbb{R})$ stands for the space of infinitely differentiable functions on $\mathbb{R}$.

Corollary 2.2.1. *Assume that, in* (2.12)*,* $\sigma^2 > 0$ *and* $\overline{\Pi}(0) = \Pi(0,\infty) = 0$*. Then, for any fixed* $t > 0$*, we have the following large asymptotic behavior of the density*

$$f_t(t\Psi'(y)) \overset{\infty}{\sim} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2}t\sigma^2 y^2 + ty^2 \int_{-\infty}^0 e^{yr} r \Pi(-\infty,r)dr}. \tag{2.13}$$

The proof of this Corollary is given in Section 2.3.5. When $\Pi \equiv 0$ then one can easily invert Ψ' in (2.13) to reveal the classical asymptotic for the density of a Brownian motion with drift. We point out that if $\Pi(dr) = \alpha|r|^{-\alpha-1}dr, r < 0, 0 < \alpha < 2$, that is the Lévy measure of a spectrally-negative α-stable Lévy process, then $\int_{-\infty}^0 e^{yr} r\Pi(-\infty,r)dr = -\Gamma(2-\alpha)y^{\alpha-2}$. As an illustration, when $\alpha = \frac{3}{2}$ and we choose $\Psi(u) = \frac{1}{2}\sigma^2 u^2 + \frac{2}{3}u^{\frac{3}{2}}$, a straightforward computation allows one to get that, for $t > 0$,

$$f_t(y) \overset{\infty}{\sim} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{y^2}{2\sigma^2 t} - H(y,t)},$$

where H is given by

$$H(y,t) = \frac{\left(y + \frac{1}{2\sigma^2}\right)\sqrt{y + \frac{1}{4\sigma^2}}}{\sigma^3 t} + \frac{\left(y + \frac{1}{\sigma^2} - \sqrt{\frac{y}{\sigma^2} + \frac{1}{4\sigma^2}}\right)^{\frac{3}{2}}}{3\sigma^3\sqrt{t}}.$$

Note that, for fixed t,

$$H(y,t) \overset{\infty}{\sim} \frac{y^{\frac{3}{2}}}{\sigma^3 t}\left(1 + \frac{\sqrt{t}}{3}\right),$$

and that the two terms in the above asymptotic scale differently in t.

2.2.2 Some new and classical examples of asymptotic behavior for densities

In the following table we list some further examples of functions $G_* \in \mathcal{A}$, and state whether or not any probability density ν satisfying

$$\nu(x) \overset{\infty}{\asymp} e^{-G_*(\log x)}$$

admits all moments, and if so, whether it is moment determinate, possibly as a function of some parameter.

$\nu(x) \overset{\infty}{\asymp} e^{-G_*(\log x)}$	$\mathcal{M}_\nu(n) < +\infty$	parameter	moment (in)determinacy
$\exp\left(-\frac{\alpha\sqrt{x}}{\log x}\right)$	$n \in \mathbb{N}$	$\alpha > 0$	determinate $\forall \alpha > 0$
$\exp\left(-x^\beta\right)$	$n \in \mathbb{N}$	$\beta > 0$	determinate $\iff \beta \geqslant \frac{1}{2}$
$\exp\left(-(\log x)^\delta\right)$	$n \in \mathbb{N}$	$\delta > 1$	indeterminate $\forall \delta > 1$
$\exp\left(-\kappa(\log x)\log(\log x)\right)$	$n \in \mathbb{N}$	$\kappa > 0$	indeterminate $\forall \kappa > 0$
$\exp\left(-(\log x)^\lambda + \log x\right)$	$n \leqslant 1$	$\lambda \in (0, 1)$	

Table 2.1: Examples of asymptotically parabolic functions and moment (in)determinacy of ν.

The first row corresponds to (2.11) of Corollary 2.1.2. The example from the second row of Table 2.1 is well-known in the literature. The authors in [114] use it to illustrate that the exponent 1/2 (i.e. square root) in Hardy's condition cannot be improved, and in this sense Hardy's condition is the optimal version of Cramer's condition for moment determinacy. As can be readily checked, the function $x \mapsto e^{\beta x} \in \mathcal{A}_{\mathfrak{D}}$, for all $\beta > 0$, and the condition in (2.9) of Theorem 2.1.2(2) organically reveals the threshold value of $\beta = \frac{1}{2}$.

This example also illustrates how a natural transformation of functions $G_* \in \mathcal{A}$ influences the moment determinacy of $\nu(x) \overset{\infty}{\asymp} e^{-G_*(\log x)}$. For $c > 0$ let d_c denote the dilation operator, acting on functions $f : \mathbb{R}_+ \to \mathbb{R}$ via

$$\mathrm{d}_c\, f(x) = f(cx).$$

From the fact that $\mathcal{A}$ is a convex cone, we get that, for any $c > 0$ and $G_* \in \mathcal{A}$, $cG_* \in \mathcal{A}$. However, we also have, for any $c > 0$ and $G_* \in \mathcal{A}$, that $\mathrm{d}_c\, G_* \in \mathcal{A}$ since

$$(\mathrm{d}_c\, G_*)''\,(x) = c^2 G_*''(cx)$$

and hence the defining properties of G_* in (2.1) carry over to $\mathrm{d}_c\, G_*$. Now let $G_*(x) = e^{\beta x}$ and consider $\nu(x) \overset{\infty}{\asymp} \exp(-x^{\beta})$. Then taking cG_* leads to $\nu(x) \overset{\infty}{\asymp} \exp(-cx^{\beta})$, which is moment determinate if and only if $\beta \geqslant \frac{1}{2}$, independently of $c > 0$, while taking $\mathrm{d}_c\, G_*$ leads to $\nu(x) \overset{\infty}{\asymp} \exp(-x^{c\beta})$, which is moment determinate if and only if $c\beta \geqslant \frac{1}{2}$.

2.3 Proofs

Before we begin with the proofs of the main results we introduce some notation, then state and prove some preliminary lemmas that will be useful below. For $a \geqslant -\infty$, we say that a function $s : (a, \infty) \to (0, \infty)$ is *self-neglecting* if

$$\lim_{u\to\infty} \frac{s(u + ws(u))}{s(u)} = 1 \quad \text{locally uniformly in } w \in \mathbb{R}.$$

Hence a function $G \in \mathcal{A}$ if and only if $G \in C_+^2((a, \infty))$ and its *scale function* $s_G(u) = (G''(u))^{-\frac{1}{2}}$ is self-neglecting. Note that the self-neglecting property is closed under asymptotic equivalence, that is if $s(u) \overset{\infty}{\sim} p(u)$ and s is self-neglecting then p is self-neglecting. We refer to [22, Section 2.11] for further information regarding self-neglecting functions. Next, a function b is said to be *flat* with respect to $G \in \mathcal{A}$ if

$$\lim_{u\to\infty} \frac{b(u + ws_G(u))}{b(u)} = 1 \quad \text{locally uniformly in } w \in \mathbb{R},$$

where s_G is the scale function of G. It is immediate from the definition that both s_G and $1/s_G$ are flat with respect to G. In the following lemma we collect some results regarding self-neglecting and flat functions. They are essentially known in the literature, however we provide proofs for completeness sake. For two functions f and g we write $f(u) \stackrel{\infty}{=} o(g(u))$ if $\lim_{u\to\infty} \left|\frac{f(u)}{g(u)}\right| = 0$.

Lemma 2.3.1.

(1) Let $s : (a,\infty) \to (0,\infty)$, $a \geqslant -\infty$, be self-neglecting. Then $s(u) \stackrel{\infty}{=} o(u)$.

(2) Let b be flat with respect to $G \in \mathcal{A}$. Then,

$$\lim_{u\to\infty} \frac{\log b(u)}{G(u)} = 0.$$

Proof. The representation theorem for self-neglecting functions, see [22, Theorem 2.11.3], states that

$$s(u) = c(u)\int_0^u g(y)dy, \tag{2.14}$$

where c is measurable and $\lim_{u\to\infty} c(u) = \gamma \in (0,\infty)$, and $g \in C^\infty(\mathbb{R})$ is such that $\lim_{y\to\infty} g(y) = 0$. Now let $k > 0$ be fixed. From (2.14) we get

$$\frac{s(u)}{uc(u)} = \frac{1}{u}\int_0^u g(y)dy \leqslant \frac{k}{u}\sup_{y\in[0,u]}|g(y)| + \frac{u-k}{u}\sup_{y\in[k,u]}|g(y)|$$

and thus we deduce that

$$\lim_{u\to\infty}\frac{s(u)}{u} \leqslant \gamma \sup_{y\geqslant k}|g(y)|.$$

Since $k > 0$ was arbitrary and $\lim_{y\to\infty} g(y) = 0$ the conclusion of Lemma 2.3.1(1) then follows. For the next claim, we invoke [10, Proposition 3.2] to get that, on the one hand, there exists a $C^\infty(\mathbb{R})$-function β such that $b(u) \stackrel{\infty}{\sim} \beta(u)$ and

$$\lim_{u\to\infty}\frac{s_G(u)\beta'(u)}{\beta(u)} = 0.$$

On the other hand, Proposition 5.8 from the same paper yields

$$\lim_{u\to\infty}\frac{1}{s_G(u)G'(u)}=0.$$

Then, by L'Hospital's Rule,

$$\lim_{u\to\infty}\frac{\log\beta(u)}{G(u)}=\lim_{u\to\infty}\frac{\beta'(u)}{\beta(u)G'(u)}=\lim_{u\to\infty}\frac{s_G(u)\beta'(u)}{\beta(u)}\frac{1}{s_G(u)G'(u)}=0,$$

and the claim follows from $b(u)\overset{\infty}{\sim}\beta(u)$ together with $\lim_{u\to\infty}G(u)=\infty$, see [10, Theorem 5.4]. □

For the next lemma we recall that the Legendre transform of a convex function G is

$$G_*(x)=\sup_{u\in\mathbb{R}}\{xu-G(u)\}.$$

When G is differentiable the supremum is attained at the unique point $u=G'^{-1}(x)$, where G'^{-1} stands for the inverse of the continuous increasing function G', so that the Legendre transform is given by

$$G_*(x)=xG'^{-1}(x)-G(G'^{-1}(x)). \tag{2.15}$$

The function G_* is always convex, and the Legendre transform is an involution on the space of convex functions, i.e. for G convex one has $(G_*)_*=G$. In the next lemma we prove another closure property regarding the Legendre transform, pertaining to the set $\mathcal{A}_{\mathfrak{D}}$.

Lemma 2.3.2. *The set of admissible asymptotically parabolic functions is closed under Legendre transform, that is if $G_*\in\mathcal{A}_{\mathfrak{D}}$ then $(G_*)_*=G\in\mathcal{A}_{\mathfrak{D}}$. Consequently if $G_*\in\mathcal{A}_{\mathfrak{D}}$ then* $\lim_{x\to\infty}G_*'(x)=\lim_{u\to\infty}G'(u)=\infty$.

Proof. Let $G_*\in\mathcal{A}_{\mathfrak{D}}$. Since $G_*:(a,\infty)\to\mathbb{R}$, for some $a\in[-\infty,\infty)$, and G_* is convex we have the standard inequality

$$G_*(x)-G_*(y)\leqslant G_*'(x)(x-y),$$

for all $x, y \in (a, \infty)$. Fixing y and letting $x \to \infty$ we see that the admissibility property implies $\lim_{x\to\infty} G_*'(x) = \infty$. In [10] it was shown that the set of asymptotically parabolic functions is closed under Legendre transform, in the sense that the restriction of $(G_*)_* = G$ to the image of (a, ∞) under G_*' is asymptotically parabolic. Since, the image of (a, ∞) under G_*' is (b, ∞) for some $b \in [-\infty, \infty)$, it follows that G restricted to (b, ∞) is asymptotically parabolic. Hence it remains to show that G is admissible. To this end, we consider the function

$$f(x) = e^{-G_*(x)} \mathbf{1}_{\{x>a\}}.$$

The admissibility of G_* implies that, for any $n \geqslant 1$,

$$\lim_{x\to\infty} f(x)e^{nx} = \lim_{x\to\infty} e^{nx-G_*(x)} \mathbf{1}_{\{x>0\}} = 0$$

i.e. that f has a Gaussian tail in the sense of [11]. This in turn yields that, for any $n \geqslant 1$,

$$\lim_{u\to\infty} e^{nu}(1 - F(u)) = \lim_{u\to\infty} e^{nu} \int_u^\infty f(x)dx = 0$$

which is equivalent to

$$\lim_{u\to\infty} e^{-nu} \int_{-\infty}^\infty e^{ux} f(x)dx = \infty, \tag{2.16}$$

see e.g. the discussion after [11, Theorem C]. However, the Gaussian tail property of f allows us to invoke the same result to conclude that

$$\int_{-\infty}^\infty e^{ux} f(x)dx \overset{\infty}{\sim} \frac{\sqrt{2\pi}}{s_G(u)} e^{G(u)}.$$

This asymptotic, combined with $\log s_G(u) \overset{\infty}{=} \mathrm{o}(G(u))$ from Lemma 2.3.1(2) and the property in (2.16), allows us to conclude that G is admissible, from which $\lim_{u\to\infty} G'(u) = \infty$ follows as before. □

In the following we provide a Tauberian result on the moment transform which is an analogue to the one obtained for the bi-lateral Laplace transform, originally by Feigin and Yaschin, see [51, Theorem 3].

Proposition 2.3.1. *Let $\mathcal{M}_\nu$ be the Stieltjes moment sequence of a measure ν, and suppose that the conditions in Theorem 2.1.1 are satisfied. Then, ν is absolutely continuous with respect to Lebesgue measure and its density $\nu(dx) = \nu(x)dx$, $x > 0$, satisfies*

$$\nu(x) \overset{\infty}{\sim} \frac{1}{\sqrt{2\pi}} \frac{e^{-G_*(\log x)}}{x s_{G_*}(\log x)},$$

where G_* is the Legendre transform of G and s_{G_*} is its own scale function. Furthermore, $G_* \in \mathcal{A}_{\mathfrak{D}}$.

Proof. Let μ be the pushforward of the measure ν under the map $x \mapsto \log x$, meaning that $\mu(dy) = \nu(e^y)e^y dy$, $y \in \mathbb{R}$, when ν is absolutely continuous with a density ν. It is immediate that μ is a probability measure with $\text{supp}(\mu) = \mathbb{R}$, and for $u \geqslant 0$,

$$\mathcal{F}_\mu(-iu) = \int_{-\infty}^{\infty} e^{uy}\mu(dy) = \int_0^\infty x^u \nu(dx) = \mathcal{M}_\nu(u),$$

where the left-hand equality sets a notation. Since ν admits all positive moments we get that $\mathcal{F}_\mu(-iu) < \infty$ for all $u \geqslant 0$. Let $k \geqslant 1$ be fixed and choose M large enough such that $\log M > k$. Then,

$$\mathcal{M}_\nu(n) = \int_0^\infty x^n \nu(dx) \geqslant \int_M^\infty x^n \nu(dx) \geqslant \overline{\nu}(M)M^n,$$

and $\overline{\nu}(M) > 0$ by assumption on the support of ν. By the choice of M,

$$\lim_{n\to\infty} \mathcal{M}_\nu(n)e^{-kn} \geqslant \lim_{n\to\infty} \overline{\nu}(M)e^{n(\log M - k)} = \infty,$$

from which we conclude that $\lim_{u\to\infty} \mathcal{F}_\mu(u)e^{-ku} = \infty$. Let us write F for the cumulative distribution function of μ. Then the properties $\mathcal{F}_\mu(-iu) < \infty$, for all $u \geqslant 0$, and $\lim_{u\to\infty} \mathcal{F}_\mu(-iu)e^{-ku} = \infty$ for any $k \geqslant 1$, are equivalent to F having a very thin tail in the sense of [11]. Note that, since $\mathcal{M}_\nu(n) \overset{\infty}{\sim} e^{G(n)}$, the property of F having a very thin tail which by Lemma 2.3.2 and its proof is implies that both $G, G_* \in \mathcal{A}_{\mathfrak{D}}$. Next, for $n \geqslant 0$, we recall that the Esscher transform of μ is the probability measure whose cumulative distribution function is given by $\int_{-\infty}^{t} e^{nx}\mu(dx)/\int_{-\infty}^{\infty} e^{nx}\mu(dx)$, which is well-defined thanks to the fact that F, its distribution, has a very thin tail. Write $\mathcal{E}_\mu(n)$ for the normalized Esscher transform of μ, which means that its bi-lateral Laplace transform takes the form

$$\mathcal{F}_{\mathcal{E}_\mu(n)}(-iu) = \frac{\mathcal{M}_\nu\left(n + \frac{u}{\eta(n)}\right)}{\mathcal{M}_\nu(n)} \exp\left(-\frac{\mathcal{M}_\nu'(n)}{\mathcal{M}_\nu(n)}u\right). \tag{2.17}$$

By applying Taylor's theorem with the Lagrange form of the remainder to the right-hand side we get

$$\mathcal{F}_{\mathcal{E}_\mu(n)}(-iu) = \exp\left(\frac{\eta^2\left(n + \frac{\theta u}{\eta(n)}\right)}{\eta^2(n)} \frac{u^2}{2}\right), \tag{2.18}$$

where $\eta^2(n) = (\log \mathcal{M}_\nu(n))''$ and $\theta(u,n)$ is such that $|\theta| \leqslant 1$. Now, the fact that $\lim_{n\to\infty}(\log \mathcal{M}_\nu(n) - G(n)) = 0$ allows us to use [11, Theorem A] to conclude that $\frac{1}{\eta(n)} \overset{\infty}{\sim} s_G(n)$, where s_G is the scale-function of G. By assumption s_G is self-neglecting, and the self-neglecting property is closed under asymptotic equivalence, so we get that $1/\eta$ is self-neglecting. From (2.18) it then follows that $\lim_{n\to\infty} \mathcal{F}_{\mathcal{E}_\mu(n)}(-iu) = e^{u^2/2}$, where the convergence is uniform on bounded u-intervals. Since the convergence of the bi-lateral Laplace transform yields the convergence in distribution, we then also get that $\lim_{n\to\infty} \mathcal{F}_{\mathcal{E}_\mu(n)}(y) = e^{-y^2/2}$ uniformly on bounded y-intervals. However, note that substituting iy for u in (2.17) gives

$$\left|\mathcal{F}_{\mathcal{E}_\mu(n)}(y)\right| = \left|\frac{\mathcal{M}_\nu\left(n + i\frac{y}{\eta(n)}\right)}{\mathcal{M}_\nu(n)}\right|.$$

Therefore, the assumption in (2.2) of Theorem 2.1.1 is that, for all $n \geqslant n_0$, we have $|\mathcal{F}_{\mathcal{E}_\mu(n)}(y)| \leqslant h(y)$, for some $h \in \mathrm{L}^1(\mathbb{R})$ and uniformly in n. By the dominated convergence theorem, we get the stronger convergence property $\lim_{n\to\infty} ||\mathcal{F}_{\mathcal{E}_\mu(n)}(y) - e^{-y^2/2}||_{\mathrm{L}^1(\mathbb{R})} = 0$. This allows us to invoke [11, Theorem 5.1], from which we conclude that $\mu(dy) = \mu(y)dy$, $y \in \mathbb{R}$, and that the continuous density $\mu(y)$ has a Gaussian tail. Then, Theorem 4.4 in the aforementioned paper allows us to identify the asymptotic behavior of μ as

$$\mu(y) \stackrel{\infty}{\sim} \frac{1}{\sqrt{2\pi}} \frac{e^{-G_*(y)}}{s_{G_*}(y)},$$

where G_* is the Legendre transform of G, and s_{G_*} is its own scale function. By changing variables it follows that $\nu(dx) = \nu(x)dx$, $x > 0$, and

$$\nu(x) \stackrel{\infty}{\sim} \frac{1}{\sqrt{2\pi}} \frac{e^{-G_*(\log x)}}{x s_{G_*}(\log x)}.$$

□

2.3.1 Proof of Theorem 2.1.1

First we use Proposition 2.3.1 to get that $\nu(dx) = \nu(x)dx$, $x > 0$, with

$$\nu(x) \stackrel{\infty}{\sim} x^{-1} B(\log x) e^{-G_*(\log x)}, \tag{2.19}$$

where G_* is the Legendre transform of G, and the function B is flat with respect to G_*. To prove the indeterminacy of ν we apply a refinement of Krein's theorem due to Pedersen [103], which amounts to showing that there exists $x_0 \geqslant 0$ such that

$$\int_{x_0}^{\infty} \frac{-\log \nu(x^2)}{1+x^2} dx < \infty.$$

First take $\ell \geqslant 0$ large enough so that at least $G_*(x) > 0$ for $x \geqslant \ell$, which is possible since $G_* \in \mathcal{A}_{\mathfrak{D}}$ and thus $\lim_{x\to\infty} G_*(x) = \infty$. Given the large asymptotic behavior of ν in (2.19) it suffices to show that

$$\int_{x_0}^{\infty} \left(\frac{2\log x}{1+x^2} + \frac{\log(1 - \frac{B(2\log x)}{G_*(2\log x)})}{1+x^2} + \frac{G_*(2\log x)}{1+x^2} \right) dx < \infty,$$

for some suitably chosen $x_0 \geqslant \ell$. The integral of the first term in the sum is plainly finite for any $x_0 \geqslant 0$. Since B is flat with respect to G_*, Lemma 2.3.1(2) gives that $\lim_{x\to\infty} \frac{\log B(x)}{G_*(x)} = 0$, and therefore the integral of the second term is also finite for any

$x_0 \geqslant \ell$. Consequently it remains to bound the integral of the last term, for which, after performing a change of variables, we obtain

$$\frac{1}{2}\int_{y_0}^{\infty} G_*(y)e^{-\frac{y}{2}}\frac{e^y}{1+e^y}dy \leqslant \int_{y_0}^{\infty} G_*(y)e^{-\frac{y}{2}}dy,$$

where $y_0 = 2\log x_0$. Then, using (2.15) and making another change of variables yield

$$\int_{y_0}^{\infty} G_*(y)e^{-\frac{y}{2}}dy = \int_{u_0}^{\infty}(uG'(u) - G(u))G''(u)e^{-\frac{G'(u)}{2}}du, \tag{2.20}$$

where $u_0 = G'^{-1}(y_0)$. The assumption in Theorem 2.1.1(1) is that the integral on the right is finite for some x_0 (and thus u_0) large enough, and thus we conclude that ν is moment indeterminate, which completes the proof of Item (1). For the proof of the next claim say G is defined on (a,∞), for $a \geqslant -\infty$. Then, from the assumption in Item (a) we get that

$$\sum_{\max(1,\lceil a\rceil)}^{\infty} \mathcal{M}_\nu^{-\frac{1}{2n}}(n) \geqslant C_1 \sum_{\max(1,\lceil a\rceil)}^{\infty} e^{-\frac{G(n)}{2n}},$$

for some constant $C_1 > 0$. Since G is convex and differentiable it satisfies the inequality,

$$G(n) - G(s) \leqslant G'(n)(n-s),$$

for all $n, s \in (a,\infty)$. Choosing some fixed $s \in (a,\infty)$ and using this inequality we get

$$\begin{aligned}\sum_{\max(1,\lceil a\rceil)}^{\infty} e^{-\frac{G(n)}{2n}} \geqslant \sum_{\max(1,\lceil a\rceil)}^{\infty} e^{\frac{G'(n)s-G(s)}{2n}} e^{-\frac{G'(n)}{2}} &\geqslant C_2 \sum_{\max(1,\lceil a\rceil)}^{\infty} e^{-\frac{G(s)}{2n}} e^{-\frac{G'(n)}{2}} \\ &\geqslant C_3 \sum_{\max(1,\lceil a\rceil)}^{\infty} e^{-\frac{G'(n)}{2}},\end{aligned}$$

where $C_2, C_3 > 0$ are constants. Putting all of these facts together yields that

$$\sum_{\max(1,\lceil a\rceil)}^{\infty} \mathcal{M}_\nu^{-\frac{1}{2n}}(n) \geqslant C \sum_{\max(1,\lceil a\rceil)}^{\infty} e^{-\frac{G'(n)}{2}}, \tag{2.21}$$

for some positive constant $C > 0$. We wish to compare the sum in the right-hand side of (2.21) with the integral in (2.20). To this end we integrate the right-hand side of (2.20) by parts to obtain

$$\begin{aligned}\int_{u_0}^{\infty}(uG'(u)-G(u))G''(u)e^{-\frac{G'(u)}{2}}du &= -2(uG'(u)-G(u))e^{-\frac{G'(u)}{2}}\Big|_{u_0}^{\infty} \\ &\quad + 2\int_{u_0}^{\infty} uG''(u)e^{-\frac{G'(u)}{2}}du\end{aligned} \tag{2.22}$$

Using the assumption that $\lim_{u\to\infty} ue^{-\frac{G'(u)}{2}} < \infty$, an application of L'Hôpital's rule allows us to conclude that

$$\lim_{u\to\infty}\frac{uG'(u)-G(u)}{e^{\frac{G'(u)}{2}}} = 2\lim_{u\to\infty}\frac{u}{e^{\frac{G'(u)}{2}}} < \infty,$$

and hence the boundary term in (2.22) is a finite constant, say b_1. Integrating by parts again the right-hand integral in (2.22), we obtain

$$b_1 + 2\int_{u_0}^{\infty} uG''(u)e^{-\frac{G'(u)}{2}}du = b_1 - 4ue^{-\frac{G'(u)}{2}}\Big|_{u_0}^{\infty} + 4\int_{u_0}^{\infty} e^{-\frac{G'(u)}{2}}du. \tag{2.23}$$

The limit at infinity for the boundary term in (2.23) above can be controlled by the assumption that $\lim_{u\to\infty} ue^{-\frac{G'(u)}{2}} < \infty$, and hence this boundary term, say b_2, is also finite. Thus we get

$$\int_{u_0}^{\infty}(uG'(u)-G(u))G''(u)e^{-\frac{G'(u)}{2}}du = b_1 + b_2 + 4\int_{u_0}^{\infty} e^{-\frac{G'(u)}{2}}du.$$

Now, since G' is non-decreasing we have, taking $u_0 \geqslant \max(1, \lceil a \rceil)$,

$$\int_{u_0}^{\infty} e^{-\frac{G'(u)}{2}}du \leqslant \sum_{n=\lfloor u_0 \rfloor}^{\infty} e^{-\frac{G'(n)}{2}},$$

so that finally we establish the inequalities

$$\begin{aligned}\int^{\infty}(uG'(u)-G(u))G''(u)e^{-\frac{G'(u)}{2}}du &= b_1 + b_2 + 4\int^{\infty} e^{-\frac{G'(u)}{2}}du \\ &\leqslant b_1 + b_2 + \frac{4}{C}\sum^{\infty}\mathcal{M}_{\nu}^{-\frac{1}{2n}}(n),\end{aligned}$$

where $b_1, b_2 \in \mathbb{R}$ and $C > 0$ are finite constants. Hence, if the Carleman's sum on the right-hand side of (2.24) is finite we conclude that $\int^{\infty} e^{-\frac{G'(u)}{2}}du < \infty$, which implies that $\int^{\infty}(uG'(u)-G(u))G''(u)e^{-\frac{G'(u)}{2}}du < \infty$, which in turn yields the moment indeterminacy of ν. Conversely, if the integral on the left-hand side of (2.24) is infinite then $\int^{\infty} e^{-\frac{G'(u)}{2}} du = \infty$, which forces Carleman's sum to be infinite, thus yielding the moment determinacy of ν. This completes the proof of Item (2) and hence of the Theorem.

2.3.2 Proof of Theorem 2.1.2

We first note that it suffices to prove Theorem 2.1.2(1) under the assumption that $\nu(dx) = \nu(x)dx$ such that

$$\nu(x) \overset{\infty}{=} \mathrm{O}(e^{-G_*(\log x)}).$$

This is because establishing the result in this case allows us to apply it to the probability density

$$x \mapsto \frac{\overline{\nu}(x)}{\mathcal{M}_\nu(1)},$$

which is the density of the so-called stationary-excess distribution of ν (of order 1), and whose moment determinacy implies the moment determinacy of ν, see [15, Section 2]. Hence, we suppose that there exist constants $A', C' > 0$ such that

$$\nu(x) \leqslant C' e^{-G_*(\log x)}$$

for $x \geqslant A'$. Without loss of generality we can replace $G_*(x)$ by $G_*(x) - x$, since the addition of linear functions does not affect the asymptotically parabolic property or the other conditions of the Theorem. Hence we may assume that there exist constants $A, C > 0$ such that

$$\nu(x) \leqslant C e^{-G_*(\log x) - \log x} \tag{2.24}$$

for $x \geqslant A$. Set, for $n \geqslant 0$,

$$\mathfrak{s}(n) = \int_A^{\infty} x^n e^{-G_*(\log x) - \log x} dx.$$

By a change of variables

$$\mathfrak{s}(n) = \int_{\log A}^{\infty} e^{ny - G_*(y)} dy, \tag{2.25}$$

and since $G_* \in \mathcal{A}_{\mathfrak{D}}$ the right-hand side is finite for all $n \geqslant 0$. This combined with (2.24) allow us to obtain the bound

$$\mathcal{M}_\nu(n) = \int_0^\infty x^n \nu(x) dx \leqslant \int_0^A x^n \nu(x) dx + C \int_A^\infty x^n e^{-G_*(\log x) - x} dx \leqslant A^n + C\mathfrak{s}(n),$$

for $n \geqslant 0$. Since $\mathrm{supp}(\nu) = [0, \infty)$ it is straightforward that $A^n \stackrel{\infty}{=} \mathrm{o}(\mathcal{M}_\nu(n))$ and hence, for n large enough,

$$C\mathfrak{s}(n) \geqslant \mathcal{M}_\nu(n) - A^n = \mathcal{M}_\nu(n) \left(1 - \frac{A^n}{\mathcal{M}_\nu(n)}\right) \geqslant c\mathcal{M}_\nu(n),$$

where $c \in (0, 1)$ is a constant. Therefore to prove moment determinacy it suffices to show the divergence of the sum $\sum^\infty \mathfrak{s}^{-\frac{1}{2n}}(n)$ for a suitable lower index, since this would imply the divergence of Carleman's sum $\sum^\infty \mathcal{M}_\nu^{-\frac{1}{2n}}(n)$. To this end, we note that $G_* \in \mathcal{A}_{\mathfrak{D}}$ implies that the function

$$f(y) = e^{-G_*(y)} \mathbf{1}_{\{y > \log A\}}$$

satisfies the conditions of [11, Theorem C], see e.g. the proof of Proposition 2.3.1. The expression for $\mathfrak{s}$ given in (2.25) allows us to invoke this quoted result to conclude that

$$\mathfrak{s}(n) \stackrel{\infty}{\sim} \frac{\sqrt{2\pi}}{s_G(n)} e^{G(n)},$$

where $(G_*)_* = G$ is the Legendre transform of G_*, and s_G is its own scale function. Hence, choosing a such that G is well-defined on (a, ∞), we have

$$\sum_{\max(\lceil a \rceil, 1)}^{\infty} \mathfrak{s}^{-\frac{1}{2n}}(n) \geqslant C_1 \sum_{\max(\lceil a \rceil, 1)}^{\infty} s_G(n)^{\frac{1}{2n}} e^{-\frac{G(n)}{2n}},$$

for some constant $C_1 > 0$. By combining Lemma 2.3.1(1) and Lemma 2.3.2 we have that $s_G(n) \stackrel{\infty}{=} \mathrm{o}(n)$, and thus

$$\sum_{\max(\lceil a \rceil, 1)}^{\infty} s_G(n)^{\frac{1}{2n}} e^{-\frac{G(n)}{2n}} = \sum_{\max(\lceil a \rceil, 1)}^{\infty} \left(\frac{n}{s_G(n)}\right)^{-\frac{1}{2n}} n^{\frac{1}{2n}} e^{-\frac{G(n)}{2n}} \geqslant C_2 \sum_{\max(\lceil a \rceil, 1)}^{\infty} e^{-\frac{G(n)}{2n}}$$

for $C_2 > 0$ a constant. Next, let γ denote the inverse of G'_*, so that by (2.15) the function G can be written as

$$G(n) = n\gamma(n) - G_*(\gamma(n)).$$

Using this expression we get

$$\sum_{\max(\lceil a\rceil,1)}^{\infty} e^{-\frac{G(n)}{2n}} = \sum_{\max(\lceil a\rceil,1)}^{\infty} e^{-(\frac{\gamma(n)}{2}-\frac{G_*(\gamma(n))}{2n})} \geqslant C_3 \sum_{\max(\lceil a\rceil,1)}^{\infty} e^{-\frac{\gamma(n)}{2}},$$

for some constant $C_3 > 0$. Putting all of these observations together gives us the inequality

$$\sum_{\max(\lceil a\rceil,1)}^{\infty} \mathfrak{s}(n)^{-\frac{1}{2n}} \geqslant \widetilde{C} \sum_{\max(\lceil a\rceil,1)}^{\infty} e^{-\frac{\gamma(n)}{2}},$$

for some constant $\widetilde{C} > 0$, depending only on A. If the sum on the right-hand side diverges, which is the condition of Item (1), it follows that ν is moment determinate, which concludes the proof in this case. Next, for the proof of Item (2), we again assume, without loss of generality, that

$$\frac{1}{C}e^{-G_*(\log x)-\log x} \leqslant \nu(x) \leqslant Ce^{-G_*(\log x)-\log x},$$

for some constants $C, A > 0$ (which may be different from the ones above) and for $x \geqslant A$. Then, for $K > 0$ another constant,

$$\begin{aligned}\int_A^\infty \frac{-\log \nu(x^2)}{1+x^2}dx &\leqslant \int_A^\infty \frac{G_*(2\log x)+\log(Cx^2)}{1+x^2}dx \\ &\leqslant K + \int_A^\infty \frac{G_*(2\log x)}{1+x^2}dx = K + \int_{2\log A}^\infty G_*(y)\frac{e^{\frac{y}{2}}}{2(1+e^y)}dy \\ &\leqslant K + \frac{1}{2(1+A^{-2})}\int_{2\log A}^\infty G_*(y)e^{-\frac{y}{2}}dy.\end{aligned}$$

Thus it suffices to show that if the right-most integral is infinite then ν is moment determinate. To this end, we wish to perform a similar integration by parts calculation as in Theorem 2.1.1, which requires us to have

$$\lim_{u\to\infty} ue^{-\frac{G'(u)}{2}} < \infty.$$

By properties of the Legendre transform the functions G'_* and G' are inverses of each other, so that

$$\lim_{u\to\infty} ue^{-\frac{G'(u)}{2}} < \infty \iff \lim_{x\to\infty} G'_*(x)e^{-\frac{x}{2}} < \infty,$$

and the rest of the proof proceeds as in the proof of Theorem 2.1.1.

2.3.3 Proof of Corollary 2.1.2

Suppose that ν satisfies Hardy's condition, that is, for some $c > 0$,

$$\int_0^\infty e^{c\sqrt{x}}\nu(dx) < \infty.$$

Then an application of Fubini's theorem yields

$$\int_0^\infty e^{c\sqrt{x}}\nu(dx) = \int_0^\infty \left(\int_0^x \frac{c}{2\sqrt{r}}e^{c\sqrt{r}}dr\right)\nu(dx) = \int_0^\infty \frac{c}{2\sqrt{r}}e^{c\sqrt{r}}\overline{\nu}(r)dr < \infty,$$

so that in particular $\overline{\nu}(r) \stackrel{\infty}{=} \mathrm{O}(e^{-c\sqrt{r}-\frac{1}{2}\log r})$. Conversely suppose $\overline{\nu}(x) \leqslant Ke^{-c\sqrt{x}-\frac{1}{2}\log x}$ for some $K, c, A > 0$ and $x \geqslant A$. Then, for $0 < c' < c$,

$$\int_0^\infty e^{c'\sqrt{x}}\nu(dx) \leqslant e^{c'\sqrt{A}} + \int_A^\infty e^{c'\sqrt{x}}\nu(dx).$$

By applying Fubini's theorem to the integral on the right-hand side we get

$$\int_A^\infty e^{c'\sqrt{x}}\nu(dx) = \int_0^A \frac{c'}{2\sqrt{y}}e^{c'\sqrt{y}}\overline{\nu}(A)dy + K\int_A^\infty e^{(c'-c)\sqrt{x}-\frac{1}{2}\log x}dx < \infty,$$

and thus Hardy's condition is satisfied for $c' \in (0, c)$. The fact that $x \mapsto ce^{\frac{x}{2}} + \frac{1}{2}x \in \mathcal{A}_{\mathfrak{D}}$ is readily checked, which completes the proof of the first item. Next, from (2.11), there exist $M, \underline{x} > 0$ such that, for $x \geqslant \underline{x}$,

$$\nu(x) \geqslant Me^{-\frac{\alpha\sqrt{x}}{\log x}}.$$

Consequently, for $c > 0$,

$$\int_0^\infty e^{c\sqrt{x}}\nu(x)dx \geqslant M\int_{\underline{x}}^\infty e^{\sqrt{x}\left(c-\frac{\alpha}{\log x}\right)}dx$$

and clearly the right-hand side is infinite for any $c > 0$. Write $G_*(x) = \alpha x^{-1} e^{\frac{x}{2}}$ and note that, by (2.11), we have

$$\nu(x) \overset{\infty}{\asymp} e^{-G_*(\log x)}.$$

To show moment determinacy of ν, we will show that G_* satisfies the assumptions of Theorem 2.1.2(2). A straightforward computation gives,

$$s_{G_*}(x) = \frac{2e^{-\frac{x}{4}}}{\sqrt{\alpha f(x)}}, \text{ where } f(x) = \left(\frac{1}{x} - \frac{1}{x^2} + \frac{8}{x^3}\right),$$

which is plainly positive for $x > 0$. Since $\mathcal{A}$ is a convex cone, $f(x) \overset{\infty}{\sim} x^{-1}$, and self-neglecting functions are closed under asymptotic equivalence, it suffices to show that the function

$$s(x) = \frac{e^{-\frac{x}{4}}}{x}$$

is self-neglecting. However, this is immediate, as $\lim_{x\to\infty} s(x) = 0$ and

$$\frac{s(x + ws(x))}{s(x)} = e^{-ws(x)} \left(1 + \frac{ws(x)}{x}\right).$$

Next, since $G_*'(x) = \alpha x^{-1} e^{\frac{x}{2}} \left(\frac{1}{2} - \frac{1}{x}\right)$ it follows that $\lim_{x\to\infty} G_*'(x) e^{-\frac{x}{2}} = 0$. Finally, for any $x_0 \geqslant 0$,

$$\int_{x_0}^{\infty} G_*(x) e^{-\frac{x}{2}} dx = \alpha \int_{x_0}^{\infty} \frac{1}{x} dx = \infty,$$

so that by Theorem 2.1.2(2) ν is moment determinate for all $\alpha > 0$.

2.3.4 Proof of Theorem 2.2.1

Proof of Theorem 2.2.1(1)

First we shall prove the claim in the case $\overline{\Pi}(0) = 0$ and $\sigma^2 > 0$. Since $Y = (Y_t)_{t\geqslant 0}$ admits all exponential moments its characteristic exponent Ψ admits an analytical extension to the right-half plane, which we still denote by Ψ, and takes the form (2.12) for $u \geqslant 0$. Let $t > 0$ be fixed. Differentiating Ψ in (2.12), see e.g. [108, p. 347], one gets

$$\Psi'(u) = b + \sigma^2 u + \int_{-\infty}^{0} \left(e^{ur} - \mathbf{1}_{\{|r|\leqslant 1\}}\right) r\Pi(dr) \tag{2.26}$$

and

$$\Psi''(u) = \sigma^2 + \int_{-\infty}^{0} r^2 e^{ur}\Pi(dr) > 0, \tag{2.27}$$

where the integrability conditions on Π also ensure that Ψ'' is well-defined on $\mathbb{R}_+$. Next, invoking the dominated convergence theorem, we have $\lim_{u\to\infty}\int_{-\infty}^{0} r^2 e^{ur}\Pi(dr) = 0$ and hence $\frac{1}{\sqrt{\Psi''(u)}} \overset{\infty}{\sim} \frac{1}{\sigma}$. Since constants are trivially self-neglecting, and self-neglecting functions are closed under asymptotic equivalence, it follows that $\Psi \in \mathcal{A}$. Furthermore, since $\mathcal{A}$ is a convex cone we get that $t\Psi \in \mathcal{A}$, and thus the condition in Theorem 2.1.1(a) is fulfilled. Let us now write $\nu_t(dx) = \mathbb{P}(e^{Y_t} \in dx)$, $x > 0$, and, for all $n \geqslant 1$,

$$\eta^2(n) = (\log \mathcal{M}_{\nu_t}(n))'' = t\Psi''(n). \tag{2.28}$$

Then, for all $n \geqslant 1$ and $y \in \mathbb{R}$, we have

$$\begin{aligned}
\log\left|\frac{\mathcal{M}_{\nu_t}\left(n + i\frac{y}{\eta(n)}\right)}{\mathcal{M}_{\nu_t}(n)}\right| &= t\,\mathrm{Re}\left(\Psi\left(n + i\frac{y}{\eta(n)}\right) - \Psi(n)\right) \\
&= -\frac{t\sigma^2}{2\eta^2(n)}y^2 + t\int_{-\infty}^{0} e^{nr}\left(\cos\left(\frac{yr}{\eta(n)}\right) - 1\right)\Pi(dr) \\
&\leqslant -\frac{t\sigma^2}{2\eta^2(n)}y^2,
\end{aligned}$$

where we simply use the trivial bound for the integral term. By combining (2.28) with (2.27), one easily gets that, for any $n \geqslant 1$, $\eta^2(n) \geqslant t\sigma^2$ and thus

$$\left|\frac{\mathcal{M}_{\nu_t}\left(n + \frac{iy}{\eta(n)}\right)}{\mathcal{M}_{\nu_t}(n)}\right| \leqslant e^{-\frac{y^2}{2}},$$

which shows that the condition in Theorem 2.1.1(b) is satisfied. From (2.27), one observes, since $\sigma^2 > 0$, that

$$\Psi''(u) \overset{\infty}{\sim} \sigma^2, \tag{2.29}$$

and thus by integration, see [92, Section 1.4], $\Psi'(u) \overset{\infty}{\sim} \sigma^2 u$. Therefore

$$\lim_{u\to\infty} u e^{-\frac{t\Psi'(u)}{2}} < \infty.$$

Finally, noting that $\Psi(n) = \frac{\sigma^2}{2}n^2 + \Psi_0(n)$ where Ψ_0 as a Laplace exponent of another spectrally negative Lévy process (possibly the negative of a subordinator) is such that $\Psi_0(n) > 0$ for n large enough, we obtain the following upper bound

$$\sum^{\infty} \mathcal{M}_{\nu_t}^{-\frac{1}{2n}}(n) \leqslant \sum^{\infty} e^{-\frac{t\sigma^2}{4}n} < \infty.$$

By Theorem 2.1.1(2) it follows that ν_t, the law of e^{Y_t}, is moment indeterminate for all $t > 0$. In the general case when $\overline{\Pi}(0) \neq 0$ we may separate the terms and write

$$\begin{aligned}
\Psi(u) &= bu + \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{0} (e^{ur} - 1 - ur\mathbf{1}_{\{|r|\leqslant 1\}})\Pi(dr) + \int_0^{\infty} (e^{ur} - 1 - ur\mathbf{1}_{\{|r|\leqslant 1\}})\Pi(dr) \\
&= \Psi_-(u) + \int_0^{\infty} (e^{ur} - 1 - ur\mathbf{1}_{\{|r|\leqslant 1\}})\Pi(dr) \\
&= \Psi_-(u) + \Psi_+(u),
\end{aligned}$$

where Ψ_- is a characteristic exponent whose Lévy measure Π_- satisfies $\overline{\Pi}_-(0) = 0$. Thus, for any $n \geqslant 0$,

$$\mathcal{M}_{\nu_t}(n) = \int_0^{\infty} x^n \,\mathbb{P}(X_t \in dx) = \int_{-\infty}^{\infty} e^{ny} \,\mathbb{P}(Y_t \in dy) = e^{t(\Psi_-(n)+\Psi_+(n))},$$

and from the earlier observations $(e^{t\Psi_-(n)})_{n\geqslant 0}$ is an indeterminate moment sequence. Since $e^{t\Psi_+(n)} > 0$ for all $n, t \geqslant 0$, Corollary 2.1.1 gives that the random variable e^{Y_t} is moment indeterminate for all $t > 0$.

Proof of Theorem 2.2.1(2)

First, for any $n, t \geqslant 0$, writing again $\nu_t(dx) = \mathbb{P}(e^{Y_t} \in dx)$, $x > 0$, we have

$$\mathcal{M}_{\nu_t}(n) = e^{tn\log(n+1)},$$

and hence

$$\sum^{\infty} \mathcal{M}_{\nu_t}^{-\frac{1}{2n}}(n) = \sum^{\infty}(n+1)^{-\frac{t}{2}}.$$

The latter series diverges if and only if $t \leqslant 2$, which by Carleman's criterion yields the moment determinacy of ν_t for $t \leqslant 2$. For the proof of indeterminacy we resort to an application of Theorem 2.1.1, and to this end we first check that the function $\Psi(u) = u\log(u+1)$ is asymptotically parabolic on $\mathbb{R}^+$. Plainly, Ψ is twice differentiable and taking derivatives we have, for any $u \geqslant 0$,

$$\Psi''(u) = \frac{u+2}{(u+1)^2} > 0, \tag{2.30}$$

and thus $\Psi \in C_+^2(\mathbb{R}_+)$. Clearly $\frac{1}{\sqrt{\Psi''(u)}} \overset{\infty}{\sim} \sqrt{u}$ and it is readily checked that $u \mapsto \sqrt{u}$ is self-neglecting. Since self-neglecting functions are closed under asymptotic equivalence it follows that $\Psi \in \mathcal{A}$, and since $\mathcal{A}$ is a convex cone we get that $t\Psi \in \mathcal{A}$, for any $t > 0$. We proceed by verifying that the condition in Theorem 2.1.1(b) is fulfilled for all $t > 0$. Write $\mathrm{Log} : \mathbb{C} \to \mathbb{C}$ for the holomorphic branch of the complex logarithm such that $\mathrm{Log}(1) = 0$, and let η be defined by

$$\eta^2(n) = (\log \mathcal{M}_{\nu_t}(n))'' = t\frac{n+2}{(n+1)^2}.$$

Then, for all $n \in \mathbb{N}$ and $y \in \mathbb{R}$,

$$\left| \frac{\mathcal{M}_{\nu_t}\left(n + \frac{iy}{\eta(n)}\right)}{\mathcal{M}_{\nu_t}(n)} \right| = e^{t\,\mathrm{Re}\left(\left(n+\frac{iy}{\eta(n)}\right)\mathrm{Log}\left(n+1+\frac{iy}{\eta(n)}\right) - n\log(n+1)\right)}.$$

Focusing on the term inside the exponential, we have

$$\begin{aligned}
&\mathrm{Re}\left(\left(n + \tfrac{iy}{\eta(n)}\right)\mathrm{Log}\left(n+1+\tfrac{iy}{\eta(n)}\right) - n\log(n+1)\right) \\
&= n\log\left(\frac{\sqrt{(n+1)^2 + \frac{y^2}{\eta^2(n)}}}{(n+1)}\right) - \frac{y}{\eta(n)}\arctan\left(\frac{y}{\eta(n)(n+1)}\right).
\end{aligned}$$

Simplifying within the logarithm and substituting for the definition of η then yields

$$\begin{aligned}&\mathrm{Re}\left(\left(n+\tfrac{iy}{\eta(n)}\right)\mathrm{Log}\left(n+1+\tfrac{iy}{\eta(n)}\right)-n\log(n+1)\right)\\&=\frac{n}{2}\log\left(1+\frac{y^2}{t(n+2)}\right)-\frac{y(n+1)}{\sqrt{t(n+2)}}\arctan\left(\frac{y}{\sqrt{t(n+2)}}\right).\end{aligned}$$

Since $\log(1+x^{-1})\overset{\infty}{=}x^{-1}+\mathrm{o}(x^{-1})$ (resp. $\arctan(x^{-1})\overset{\infty}{=}x^{-1}+\mathrm{o}(x^{-1})$) , we have that

$$\lim_{n\to\infty}\frac{n}{2}\log\left(1+\frac{y^2}{t(n+2)}\right)=\frac{y^2}{2t}\quad\left(\text{resp. }\lim_{n\to\infty}\frac{y(n+1)}{\sqrt{t(n+2)}}\arctan\left(\frac{y}{\sqrt{t(n+2)}}\right)=\frac{y^2}{t}\right).$$

It follows that there exists $n_0>0$ such that for all $n\geqslant n_0$,

$$\left|\frac{\mathcal{M}_{\nu_t}\left(n+\frac{iy}{\eta(n)}\right)}{\mathcal{M}_{\nu_t}(n)}\right|\leqslant e^{-Cy^2},$$

where $0<C<\frac{1}{2}$ is a constant depending only on n_0. Hence the integrability condition in Theorem 2.1.1(b) is satisfied for any $t>0$. The proof will be completed if we can show that the additional condition in Theorem 2.1.1(2) holds, namely that

$$\lim_{u\to\infty}ue^{-\frac{t\Psi'(u)}{2}}<\infty\quad\text{for}\quad t\geqslant 2.$$

However, simple algebra yields that for $t\geqslant 2$

$$\lim_{u\to\infty}ue^{-\frac{t\Psi'(u)}{2}}=\lim_{u\to\infty}ue^{-\frac{t}{2}\left(\frac{u}{u+1}+\log(u+1)\right)}=\lim_{u\to\infty}u(u+1)^{-\frac{t}{2}}e^{-\frac{tu}{2(u+1)}}<\infty.$$

2.3.5 Proof of Corollary 2.2.1

In this case, we write $\nu_t(x)dx=\mathbb{P}(e^{Y_t}\in dx), x>0$, see the comments before the statement. In the proof of Theorem 2.2.1(1) it was shown that, for any $t>0$, $\mathcal{M}_{\nu_t}$ fulfills the assumptions of Proposition 2.3.1 when $\sigma^2>0$ and $\Pi(0,\infty)=0$. Invoking this result, noting that $(t\Psi)_*(y)=t\Psi_*(\frac{y}{t})$, and changing variables, we get for $f_t(y)dy=\mathbb{P}(Y_t\in dy), y\in\mathbb{R}$, and any $t>0$,

$$f_t(y)\overset{\infty}{\sim}\frac{1}{\sqrt{2\pi t}}\sqrt{\Psi_*''\left(\frac{y}{t}\right)}e^{-t\Psi_*(\frac{y}{t})}.$$

Next, we have, for $y > 0$,

$$\begin{aligned}
\Psi_*(\Psi'(y)) &= y\Psi'(y) - \Psi(y) \\
&= \frac{1}{2}\sigma^2 y^2 + y\int_{-\infty}^{0}\left(e^{yr} - \mathbf{1}_{\{|r|\leqslant 1\}}\right) r\Pi(dr) \\
&\quad - \int_{-\infty}^{0}\left(e^{yr} - 1 - yr\mathbf{1}_{\{|r|\leqslant 1\}}\right)\Pi(dr) \\
&= \frac{\sigma^2 y^2}{2} + H(y),
\end{aligned}$$

where the first equality follows from (2.15), the second follows from (2.26) and some straightforward algebra, and the third equality serves as a definition for the function H. Observe that an integration by parts yields

$$\begin{aligned}
H(y) &= \int_{-\infty}^{0}(1 - e^{yr}(1 - yr))\,\Pi(dr) \\
&= -y^2\int_{-\infty}^{0} e^{yr} r\Pi(-\infty, r)dr + (1 - e^{yr}(1 - yr))\,\Pi(-\infty, r)\big|_{-\infty}^{0} \\
&= -y^2\int_{-\infty}^{0} e^{yr} r\Pi(-\infty, r)dr,
\end{aligned}$$

where we used that $\lim_{r\to-\infty}\Pi(-\infty, r) = 0$ and $\lim_{r\to 0} r^2\Pi(-\infty, r) = 0$. Finally, since

$$\Psi_*''(\Psi'(y)) = \frac{1}{\Psi''(y)} \quad \text{and} \quad \Psi''(y) \overset{\infty}{\sim} \sigma^2,$$

we conclude that

$$f_t(t\Psi'(y)) \overset{\infty}{\sim} \frac{1}{\sqrt{2\pi t}}\sqrt{\Psi_*''(\Psi'(y))}e^{-t\Psi_*(\Psi'(y)))} \overset{\infty}{\sim} \frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{1}{2}t\sigma^2 y^2 + ty^2\int_{-\infty}^{0} e^{yr} r\Pi(-\infty,r)dr}.$$

CHAPTER 3

THE LOG-LÉVY MOMENT PROBLEM VIA BERG-URBANIK SEMIGROUPS

We consider the Stieltjes moment problem for the Berg-Urbanik semigroups which form a class of multiplicative convolution semigroups on $\mathbb{R}_+$ that is in bijection with the set of Bernstein functions. In [19], Berg and Durán proved that the law of such semigroups is moment determinate (at least) up to time $t = 2$, and, for the Bernstein function $\phi(u) = u$, Berg [16] made the striking observation that for time $t > 2$ the law of this semigroup is moment indeterminate. We extend these works by estimating the threshold time $\mathcal{T}_\phi \in [2, \infty]$ that it takes for the law of such Berg-Urbanik semigroups to transition from moment determinacy to moment indeterminacy in terms of simple properties of the underlying Bernstein function ϕ, such as its Blumenthal-Getoor index. One of the several strategies we implement to deal with the different cases relies on the non-classical Abelian type criterion for the moment problem proved in the previous chapter. To implement this approach we provide detailed information regarding distributional properties of the semigroup such as existence and smoothness of a density, and, the large asymptotic behavior for all $t > 0$ of this density along with its successive derivatives. In particular, these results, which are original in the Lévy processes literature, may be of independent interest.

3.1 Introduction

In this chapter we continue our study of the Stieltjes moment determinacy for multiplicative convolution semigroups $(\nu_t)_{t\geqslant 0}$, that is semigroups satisfying, for $n, t \geqslant 0$,

$$\int_0^\infty x^n \nu_t(dx) = \int_{-\infty}^\infty e^{ny}\,\mathbb{P}(Y_t \in dy) = e^{t\Psi(n)}$$

where $(Y_t)_{t\geqslant 0}$ is a one-dimensional Lévy process such that $\mathbb{E}[e^{nY_t}] < \infty$, for all $n, t \geqslant 0$. In other words, we study the moment determinacy of the law of a process whose logarithm is a Lévy process, and we call this problem the log-Lévy moment problem, for short.

We first point out that if $\Psi(n) = \frac{1}{2}n^2$ then $(\nu_t)_{t\geqslant 0}$ boils down to the semigroup of the geometric Brownian motion, whose law is indeterminate by its moments for all $t > 0$. This is because for any $t > 0$ the geometric Brownian motion is log-normally distributed, and it is well-known that a log-normal distribution is indeterminate by its moments. More generally, in Theorem 2.2.1(1) it is proved that the log-Lévy moment problem is indeterminate for all $t > 0$ whenever the associated Lévy process has a Gaussian component, a case that we exclude from our analysis.

Moreover, Urbanik, in [121], introduced the multiplicative convolution semigroup of probability densities $(e_t)_{t\geqslant 0}$ satisfying, for $n, t \geqslant 0$,

$$\begin{aligned}\int_0^\infty x^n e_t(x)dx = (n!)^t &= \exp\left(t\sum_{k=1}^n \log k\right) \\ &= \exp\left(t\int_0^\infty (e^{-ny} - 1 - n(e^{-y} - 1))\frac{dy}{y(e^y - 1)}\right),\end{aligned} \tag{3.1}$$

and Berg [16, Theorem 2.5] discovered that the measure $e_t(x)dx$ is moment determinate if and only if $t \leqslant 2$. This interesting fact reveals that the log-Lévy moment problem can be non-trivial, since there can exist a threshold time $\mathfrak{T} \in [0, \infty]$ such that ν_t is moment determinate for $0 \leqslant t \leqslant \mathfrak{T}$ and moment indeterminate for $t > \mathfrak{T}$.

In the same paper, Berg defined a family of multiplicative convolution semigroups $(\nu_t)_{t\geqslant 0}$ that are in bijection with the set of Bernstein functions $\mathcal{B}$, see (3.5) below for definition. In particular, for any $\phi \in \mathcal{B}$, the moments of ν_t are given, for $n, t \geqslant 0$, by

$$\mathcal{M}_{\nu_t}(n) = \int_0^\infty x^n \nu_t(dx) = \left(\prod_{k=1}^n \phi(k)\right)^t \tag{3.2}$$

where $\mathcal{M}_{\nu_t}$ is called the moment transform of ν_t and for $n = 0$ the product is assumed to be 1. We call these the Berg-Urbanik semigroups, since (3.1) corresponds to the specific case $\phi(u) = u$ of (3.2). Note that, for a probability measure λ, there is the notion of Urbanik decomposability semigroups $\mathbb{D}(\lambda)$, which have also been referred to as Urbanik semigroups in the literature, see e.g. [71, 70], and are distinct from the semigroups $(\nu_t)_{t\geqslant 0}$ defined via (3.2). Furthermore in [17, Theorem 2.2] it was also shown that, $\mathcal{M}_{\nu_t}$ admits an analytical extension to the right-half plane, and, for $\mathrm{Re}(z) \geqslant 0$ and $t \geqslant 0$,

$$\mathcal{M}_{\nu_t}(z) = e^{t\Psi(z)}$$

where

$$\Psi(z) = z \log \phi(1) + \int_0^\infty (e^{-zy} - 1 - z(e^{-y} - 1)) \frac{\kappa(dy)}{y(e^y - 1)} \tag{3.3}$$

and

$$\int_0^\infty e^{-uy} \kappa(dy) = \frac{\phi'(u)}{\phi(u)},$$

with $\kappa(dy) = \int_0^y U(dy - r)(r\mu(dr) + \delta_{\mathtt{d}}(dr))$, where U is the potential measure, μ the Lévy measure and $\mathtt{d}$ the drift of ϕ, see (3.6) and (3.4) below for definitions. This is the general form of the right-most equality in (3.1), and we note that Hirsch and Yor have also derived (3.3) using different means, see [66, Theorem 3.1]. We mention that Hirsch and Yor also offer a nice exposition on the wealth of results by Urbanik in [122], which continues the investigations started in [121].

The log-Lévy moment problem for general Berg-Urbanik semigroups is only partially understood. It is known that any Berg-Urbanik semigroup is moment determinate for $t \leqslant 2$, see [16], and that there are Berg-Urbanik semigroups that are moment determinate for all $t \geqslant 0$, see [17], however much less is known concerning moment indeterminacy. We were inspired by Berg's results, in particular his remarkable discovery of the threshold for the classical Urbanik semigroup $(e_t)_{t\geqslant 0}$, to further study the log-Lévy moment problem in this setting. In particular, our aim was to understand how to estimate the threshold time $\mathcal{T}$ from simple properties of the underlying Bernstein function, and our main contribution in this regard is Theorem 3.2.1 below, which provides several new and original results in this area.

One of our approaches stems on a recent Abelian type criterion for the moment problem that gives a necessary and sufficient condition for moment indeterminacy, see Theorem 2.1.2. To utilize this criterion we resort to proving the existence of densities for certain Berg-Urbanik semigroups and study their large asymptotic behavior. To obtain such asymptotics we apply, in a novel and non-standard way, a closure result for Gaussian tails obtained by Balkema et al. [10] combined with some recent Gaussian tail asymptotics estimates due to Patie and Savov [99].

The remaining part of the chapter is organized as follows. In Section 3.2 we state our main result for the log-Lévy moment problem, as well some auxiliary results on Berg-Urbanik semigroups and Lévy processes. In Section 3.3 we discuss some illustrative examples of Berg-Urbanik semigroups. Finally, Section 3.4 is devoted to the proofs of the results stated in Section 3.2.

3.2 Main results

We start with some preliminaries. Let $\phi : [0,\infty) \to [0,\infty)$ be the function defined by

$$\phi(u) = \mathsf{k} + \mathsf{d}u + \int_0^\infty (1 - e^{-uy})\mu(dy), \tag{3.4}$$

where $\mathsf{k}, \mathsf{d} \geqslant 0$ and μ is a Radon measure on $(0,\infty)$ that satisfies $\int_0^\infty (1 \wedge y)\mu(dy) < \infty$. We write $\mathcal{B}$ for the set of Bernstein functions, which is defined as

$$\mathcal{B} = \{\phi : [0,\infty) \to [0,\infty);\ \phi \text{ is of the form (3.4)}\}. \tag{3.5}$$

Note that $\mathcal{B}$ is a convex cone, i.e. for $\phi_1, \phi_2 \in \mathcal{B}$ and $c_1, c_2 > 0$ one has $c_1\phi_1 + c_2\phi_2 \in \mathcal{B}$, and also that the triplet $(\mathsf{k}, \mathsf{d}, \mu)$ in (3.4) uniquely determines any $\phi \in \mathcal{B}$. We recall that the mapping $u \mapsto \phi'(u)$ is completely monotone, i.e. $\phi' \in C^\infty(\mathbb{R}^+)$, the space of infinitely continuously differentiable functions on $\mathbb{R}^+$ and for all $n \in \mathbb{N}$ and $u \geqslant 0$, $(-1)^n \phi^{(n+1)}(u) \geqslant 0$. It is well-known that the mapping $u \mapsto \frac{1}{\phi(u)}$ is also completely monotone and the corresponding Radon measure U is the so-called potential measure of (the subordinator associated to) ϕ, i.e. for any $u \geqslant 0$,

$$\int_0^\infty e^{-uy} U(dy) = \frac{1}{\phi(u)}. \tag{3.6}$$

We refer to the excellent monograph [109] for further information on Bernstein functions, and also to [99, Section 4] and [98, Section 3], in which several properties of Bernstein functions that are used in the proofs are collected. In what follows we systematically exclude the trivial Bernstein function $\phi \equiv 0$ since this yields the degenerate convolution semigroup of a Dirac mass at 1 for all time.

A family of measures $(\nu_t)_{t \geqslant 0}$ is said to be a multiplicative convolution semigroup if, for $t, s \geqslant 0$ we have $\nu_t \diamond \nu_s = \nu_{t+s}$, where $\diamond$ denotes the product convolution on the multiplicative group $(\mathbb{R}_+, \times)$. Next, we define the moment transform of an integrable function $f : \mathbb{R}_+ \to \mathbb{R}$, and of a probability measure ρ supported on $[0, \infty)$, for (at least) $z \in i\mathbb{R}$ as

$$\mathcal{M}_f(z) = \int_0^\infty x^z f(x) dx, \quad \text{and} \quad \mathcal{M}_\rho(z) = \int_0^\infty x^z \rho(dx),$$

and observe that the moment transform is simply a shift of the classical Mellin transform. The moments of ρ, if they exist, are given, for $n \geqslant 0$, by

$$\mathcal{M}_\rho(n) = \int_0^\infty x^n \rho(dx).$$

We say that a measure ρ supported on $[0, \infty)$ is Stieltjes moment determinate, or simply moment determinate for short, if the sequence $(\mathcal{M}_\rho(n))_{n \geqslant 0}$ uniquely characterizes the measure ρ among all probability measures supported on $[0, \infty)$ and admitting all moments. Otherwise, we say ρ is moment indeterminate. The moment problem for probability measures supported on $[0, \infty)$ has been intensively studied for many years, going back to the original memoir by Stieltjes [113]. For excellent references on aspects of the Stieltjes (and other) moment problems see the classic texts [2] and [111], as well as the more recent monograph [110].

We now state the definition of Berg-Urbanik semigroups, whose validity is justified by [16, Theorem 1.8].

Definition 3.2.1. Let $\phi \in \mathcal{B}$. Then the *Berg-Urbanik semigroup associated to* ϕ is the unique multiplicative convolution semigroup $(\nu_t)_{t \geqslant 0}$ of probability measures characterized, for any $t \geqslant 0$ and $\mathrm{Re}(z) > 0$, by

$$\mathcal{M}_{\nu_t}(z) = e^{t\Psi(z)}$$

where Ψ was defined in (3.3). Recall that, for any $n \in \mathbb{N}$ and $t > 0$, $e^{t\Psi(n)} = \left(\prod_{k=1}^{n} \phi(k)\right)^t$.

Occasionally we write $(\nu_t^{\phi})_{t \geqslant 0}$ to emphasize the dependence of the Berg-Urbanik semigroup on the Bernstein function, but will mostly drop this superscript for convenience. In such cases the Bernstein function will be clear from the context.

3.2.1 The log-Lévy moment problem for Berg-Urbanik semigroups

To describe our first main result we introduce the *threshold index*. For each $\phi \in \mathcal{B}$ we let $\mathcal{T}_\phi \in [0, \infty]$ be defined by

$$\mathcal{T}_\phi = \inf\{t > 0;\ \nu_t^{\phi} \text{ is indeterminate}\} = \sup\{t > 0;\ \nu_t^{\phi} \text{ is determinate}\},$$

where we utilize the bijection between $\mathcal{B}$ and the set of Berg-Urbanik semigroups, as well as the convention that $\sup \emptyset = 0$. It is justified to call $\mathcal{T}_\phi$ a threshold index since $(\nu_t)_{t \geqslant 0}$ is a multiplicative convolution semigroup and according to [19, Lemma 2.2 and

Remark 2.3], a measure $\mu \diamond \sigma$ is moment indeterminate if μ is indeterminate and $\sigma \neq c\delta_0$, $c > 0$. Since, for any $\phi \in \mathcal{B}$, ν_t is moment determinate for $t \leqslant 2$, it follows that $\mathcal{T}_\phi \geqslant 2$. In the case when $\mathcal{T}_\phi = \infty$ we say the Berg-Urbanik semigroup is *completely determinate*, otherwise if $\mathcal{T}_\phi \in [2, \infty)$ we say the semigroup is *threshold determinate*. We proceed by defining some subsets of $\mathcal{B}$ that will be useful to state our main results. First, let

$$\mathcal{B}_d = \{\phi \in \mathcal{B};\ \mathsf{d} > 0\}$$

denote the set of Bernstein functions with a positive drift. Next, write

$$\mathcal{B}_{\mathcal{J}} = \{\phi \in \mathcal{B};\ \mu(dy) = v(y)dy \text{ with } v \text{ non-increasing}\}$$

and note that this is sometimes referred to as the Jurek class of Bernstein functions, due to [69], see also [109, Chapter 10]. For a Bernstein function ϕ we write $\phi(\infty) = \lim_{u\to\infty} \phi(u) \in (0, \infty]$, and define its Blumenthal-Getoor index as

$$\beta_\phi = \inf\left\{\beta \geqslant 0;\ \overline{\lim_{u\to\infty}}\, u^{-\beta}\phi(u) < \infty\right\} \in [0, 1], \tag{3.7}$$

noting that this definition coincides with the original one in [23] for driftless subordinators. We also define the lower index of ϕ

$$\delta_\phi = \sup\left\{\delta \geqslant 0;\ \underline{\lim_{u\to\infty}}\, u^{-\delta}\phi(u) > 0\right\},$$

which has appeared in the study of shift-Harnack inequalities for subordinate semigroups, see [42]. From these definitions it is clear that $0 \leqslant \delta_\phi \leqslant \beta_\phi \leqslant 1$, and moreover one can construct an example for which strict inequality is possible, see [23, Section 6]. In view of this, we set

$$\mathcal{B}_{\asymp} = \{\phi \in \mathcal{B};\ \delta_\phi = \beta_\phi\}.$$

We are now ready to state our main result regarding the log-Lévy moment problem for Berg-Urbanik semigroups.

Theorem 3.2.1. *Let $(\nu_t)_{t\geqslant 0}$ be the Berg-Urbanik semigroup associated to $\phi \in \mathcal{B}$.*

(1) *The inequality*

$$\mathfrak{T}_\phi \geqslant \frac{2}{\beta_\phi}$$

holds, and if $\beta_\phi > 0$ *and* $\overline{\lim}_{u\to\infty} u^{-\beta_\phi}\phi(u) < \infty$ *then* $\nu_{\mathfrak{T}_\phi}$ *is moment determinate. In particular, if* $\phi(\infty) < \infty$ *then* $\beta_\phi = 0$ *and* $(\nu_t)_{t\geqslant 0}$ *is completely determinate.*

Moreover, the following hold.

(2) *If* $\phi \in \mathcal{B}_d$ *then* $\mathfrak{T}_\phi = 2$, *and* ν_2 *is moment determinate.*

(3) *If* $\phi^{\mathrm{t}} \in \mathcal{B}_{\mathcal{J}}$ *for all* $\mathrm{t} \in (0,1)$, *then*

$$\frac{2}{\beta_\phi} \leqslant \mathfrak{T}_\phi \leqslant \frac{2}{\delta_\phi}, \tag{3.8}$$

and hence, if additionally $\phi \in \mathcal{B}_{\asymp}$, *then*

$$\mathfrak{T}_\phi = \frac{2}{\beta_\phi}.$$

(4) *If there exists* $\vartheta \in \mathcal{B}$ *such that* $\frac{\phi}{\vartheta} \in \mathcal{B}$, *then* $\mathfrak{T}_\phi \leqslant \mathfrak{T}_\vartheta$. *In particular, if* $\vartheta^{\mathrm{t}} \in \mathcal{B}_{\mathcal{J}}$ *for all* $\mathrm{t} \in (0,1)$, *then*

$$\mathfrak{T}_\phi \leqslant \frac{2}{\delta_\vartheta}.$$

Remark 3.2.1. Note that all complete Bernstein functions satisfy the property $\phi^{\mathrm{t}} \in \mathcal{B}_{\mathcal{J}}$ for all $\mathrm{t} \in (0,1)$. Indeed, writing $\mathbb{H} = \{z \in \mathbb{C};\ \mathrm{Im}\, z > 0\}$ for the upper half-plane, we recall that a Bernstein function ϕ is said to be a complete if its Lévy measure μ has a completely monotone density, or equivalently if $\mathrm{Im}\,\phi(z) \geqslant 0$ for all $z \in \mathbb{H}$. Such functions are also sometimes called Pick or Nevanlinna functions in the complex analysis literature. If ϕ is a complete Bernstein function, then for $\mathrm{t} \in (0,1)$ and $z \in \mathbb{H}$,

$$\mathrm{Im}\,\phi^{\mathrm{t}}(z) = \mathrm{Im}\, e^{\mathrm{t}(\log|\phi(z)| + i\arg\phi(z))} = e^{\mathrm{t}\log|\phi(z)|}\,\mathrm{Im}\, e^{i\mathrm{t}\arg\phi(z)} \geqslant 0,$$

and hence ϕ^{t} is a complete Bernstein function, and in particular its Lévy measure has a non-increasing density. In particular $u \mapsto (u+\mathfrak{m})^\alpha$ is a complete Bernstein function, for any $\mathfrak{m} \geqslant 0$, $\alpha \in (0,1)$, and thus $u \mapsto (u+\mathfrak{m})^{\alpha\mathfrak{t}}$ is also a complete Bernstein function, for any $\mathfrak{t} \in (0,1)$. We refer to [109, Chapter 16] for abundant examples of complete Bernstein functions and to [109, Chapter 6] for further details on the theory of complete Bernstein functions; see also [54] for some interesting mappings related to complete Bernstein functions.

Remark 3.2.2. We mention that for Item (4) Patie and Savov, see [99, Proposition 4.4], have given sufficient conditions for the ratio of Bernstein functions to remain a Bernstein function, see also Proposition 3.4.1 below for another set of sufficient conditions.

This Theorem is proved in Section 3.4.4 and the proof makes use of several strategies that will be detailed throughout the rest of the paper. We proceed by offering some remarks regarding our results in relation to what has been proved in the literature.

First, Theorem 3.2.1(1) provides a generalization of the example provided in [16] for which the threshold function is infinite. Therein, the author considers the Bernstein function $u \mapsto \frac{u}{u+1}$, for which $\lim_{u\to\infty} \frac{u}{u+1} < \infty$ and therefore trivially $\beta_\phi = 0$. However, there exist $\phi \in \mathcal{B}$ such that $\phi(\infty) = \infty$ but $\beta_\phi = 0$, for example the function given, for $u \geqslant 0$ and any $\lambda > 0$, by

$$\phi(u) = \log\left(1 + \frac{u}{\lambda}\right) = \int_0^\infty (1 - e^{-ux})x^{-1}e^{-\lambda x}dx,$$

which we note is a specific instance of Example 3.3.2 below. This shows that a Berg-Urbanik semigroup may have unbounded support for all $t > 0$, see Theorem 3.2.4(1) below, but is still completely determinate. Furthermore, in Theorem 3.2.1(1) we provide a condition on ϕ that ensures that the lower bound in (3.8) is sharp, in the sense that $\nu_{\mathcal{T}_\phi}$ is moment determinate. It would be interesting to know what situations can occur when this condition is not fulfilled, in particular if it is possible that $\nu_{\mathcal{T}_\phi}$ is indeterminate.

In Theorem 3.2.1(2) we provide an exhaustive claim for the case when $\phi \in \mathcal{B}_d$, thereby generalizing Berg's result that the classical Urbanik semigroup $(e_t)_{t\geqslant 0}$ is moment determinate if and only if $t \leqslant 2$, which corresponds to the case $\phi(u) = u$. The proof relies on an application of Theorem 3.2.1(4) to yield the matching upper bound, which shows that $\mathcal{B}_{\mathcal{J}}$ can serve as a reference class for proving more general estimates. We borrow this idea of using reference objects from [99, Section 10] where the concept of reference semigroups was developed in the context of spectral theory of some non-self-adjoint operators. The fact that one can construct $\phi \in \mathcal{B}$ such that $0 \leqslant \delta_\phi < \beta_\phi < 1$ shows that the inequality in (3.8) may be far from optimal. Nevertheless, when $\phi \in \mathcal{B}_{\asymp}$, Theorem 3.2.1(3) allows one to classify the behavior of $\mathcal{T}_\phi$ entirely by the analytical exponent β_ϕ. Finally, as was suggested by an anonymous referee, it is worth emphasizing that for any $T \in (2, \infty)$ there exists a Bernstein function ϕ whose associated Berg-Urbanik semigroup has threshold index $\mathcal{T}_\phi = T$, see e.g. Example 3.3.1.

3.2.2 A related moment problem on infinitely divisible moment sequences

Before we proceed with developing results leading to the proof of Theorem 3.2.1, we briefly discuss a related moment problem, which requires us to introduce the notion of infinitely divisible moment sequences. A Stieltjes moment sequence $(\mathfrak{m}(n))_{n\geqslant 0}$ is said to be infinitely divisible if, for any $t > 0$, the sequence $(\mathfrak{m}^t(n))_{n\geqslant 0}$ is again a Stieltjes moment sequence, and this notion goes back to Tyan who introduced and studied infinitely divisible moment sequences in his thesis [120]. By definition, for each $t > 0$, there exists a random variable X_t with moments $(\mathfrak{m}^t(n))_{n\geqslant 0}$ and it is natural to ask how the moment determinacy of X_t (meaning the moment determinacy of its law) relates to the moment determinacy of X_1^t, as a function of t. This latter random variable X_1^t is the t^{th}-power of a random variable with moments $(\mathfrak{m}(n))_{n\geqslant 0}$, and it is straightforward that X_1^t has moments given by $(\mathfrak{m}(tn))_{n\geqslant 0}$. From Theorem 3.2.3 below it follows that, for any $\phi \in \mathcal{B}$, the moment sequence $(\mathcal{M}_{\nu_1}(n))_{n\geqslant 0}$ is infinitely divisible and hence Berg-Urbanik semigroups provide a natural setting in which to investigate this question. In what follows we let, for $\phi \in \mathcal{B}$, $X_t(\phi)$ denote the stochastic process whose law at time $t > 0$ is given by ν_t^ϕ and write simply $X(\phi) = X_1(\phi)$, suppressing the dependency on ϕ when this causes no confusion.

Theorem 3.2.2. *Let $\phi \in \mathcal{B}$.*

(1) The random variable X^t is moment determinate for $t < \frac{2}{\beta_\phi}$, and if $\beta_\phi > 0$ and $\overline{\lim}_{u\to\infty} u^{-\beta_\phi}\phi(u) < \infty$ then $X^{\frac{2}{\beta_\phi}}$ is moment determinate.

Moreover, the following hold.

(2) If $\phi \in \mathcal{B}_d$ then X^t is moment determinate if and only if $t \leqslant 2$.

(3) If $\phi \in \mathcal{B}_{\mathcal{J}}$ then X^t is moment indeterminate for $t > \frac{2}{\delta_\phi}$. If in addition $\delta_\phi = \beta_\phi$ and $\overline{\lim}_{u\to\infty} u^{-\beta_\phi}\phi(u) < \infty$ then X^t is moment indeterminate if and only if $t > \frac{2}{\beta_\phi}$.

(4) If there exists $\vartheta \in \mathcal{B}$ such that $\frac{\phi}{\vartheta} \in \mathcal{B}$ then, for any t such that $X^t(\vartheta)$ is moment indeterminate, the variable $X^t(\phi)$ is also moment indeterminate.

This Theorem is proved in Section 3.4.5. While Theorem 3.2.1 concerns the t-dependent moment determinacy of the process $(X_t)_{t\geqslant 0}$, Theorem 3.2.2 is the analogous result regarding the moment determinacy of X^t, or equivalently of the sequence $(\mathcal{M}_{\nu_1}(tn))_{n\geqslant 0}$. Note that the conditions in Theorem 3.2.2(3) are weaker than those in Theorem 3.2.1(3), which shows that the log-Lévy moment problem is the harder of the two moment problems. In [79] Lin stated the following conjecture regarding the moment determinacy of infinitely divisible moment sequences.

Conjecture (Conjecture 1 in [79]). *Let* $(X_t)_{t\geqslant 0}$ *be a stochastic process such that* $(\mathbb{E}[X_t^n])_{n\geqslant 0} = (\mathfrak{m}^t(n))_{n\geqslant 0}$, *i.e.* $(\mathfrak{m}(n))_{n\geqslant 0}$ *is an infinitely divisible moment sequence. Then* X_t *is moment determinate if and only if* X_1^t *is moment determinate.*

As a corollary of Theorems 3.2.1 and 3.2.2 we get an affirmative answer to Lin's conjecture for a subclass of Berg-Urbanik semigroups.

Corollary 3.2.1. *Let* $\phi \in \mathcal{B}$ *and suppose that any of the following conditions are satisfied:*

(i) $\beta_\phi = 0$,

(ii) $\phi \in \mathcal{B}_d$,

(iii) $\phi \in \mathcal{B}_{\asymp}$ *with* $\phi^{\mathrm{t}} \in \mathcal{B}_{\mathcal{J}}$ *for all* $\mathrm{t} \in (0,1)$, $\beta_\phi > 0$ *and* $\overline{\lim}_{u\to\infty} u^{-\beta_\phi}\phi(u) < \infty$.

Then Lin's conjecture holds.

We point out that recently Berg [18] proved a related conjecture by Lin (Conjecture 2 in [79]) concerning the moment sequence $(\Gamma(tn+1))_{n\geqslant 0}$, which among other things confirms Lin's conjecture (Conjecture 1) for this particular example. Note that the moment sequence $(\Gamma(tn+1))_{n\geqslant 0}$ corresponds to the Bernstein function $\phi(u) = u$, which falls under the assumption (ii) in Corollary 3.2.1.

3.2.3 A new Mellin transform representation in terms of Bernstein-Gamma functions

The proof of Theorem 3.2.1 relies on several intermediate results that are of independent interests. The first one is an alternative representation of $\mathcal{M}_{\nu_t}$. For $a \in \mathbb{R}$ we let

$\mathbb{C}_{(a,\infty)} = \{z \in \mathbb{C};\ \mathrm{Re}(z) > a\}$ and then write $\mathcal{A}_{(a,\infty)}$ for the set of analytic functions on $\mathbb{C}_{(a,\infty)}$. Recall that a function $f : i\mathbb{R} \to \mathbb{C}$ is said to be positive-definite if, for any $s_1, \ldots, s_n \in i\mathbb{R}$ and $z_1, \ldots, z_n \in \mathbb{C}$, $\sum_{i,j=1}^{n} f(s_i - s_j) z_i \overline{z}_j \geqslant 0$.

Next, for any $\phi \in \mathcal{B}$ we let $W_\phi : \mathbb{C}_{(0,\infty)} \to \mathbb{C}$ denote the so-called *Bernstein-Gamma function* associated to ϕ, which is given by

$$W_\phi(z) = \frac{e^{-\gamma_\phi z}}{\phi(z)} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)} z} \tag{3.9}$$

where the infinite product is absolutely convergent on at least $\mathbb{C}_{(0,\infty)}$, and

$$\gamma_\phi = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\phi'(k)}{\phi(k)} - \log \phi(n) \right) \in \left[-\log \phi(1), \frac{\phi'(1)}{\phi(1)} - \log \phi(1) \right].$$

This function, as defined in (3.9) on $\mathbb{R}_+$ was introduced and studied by Webster [124], and was extended (at least) to $\mathbb{C}_{(0,\infty)}$ by Patie and Savov who introduced the terminology and studied their analytical properties, such as uniform decay along imaginary lines, in the works [99, Chapter 6] and [98]. The product in (3.9) can be thought of as a generalized Weierstrass product, as it generalizes the classical Weierstrass product representation for the gamma function. Indeed, this case can be recovered by setting $\phi(u) = u$, in which case γ_ϕ boils down to the Euler-Mascheroni constant. Furthermore, W_ϕ is the unique positive-definite function that solves the functional equation

$$W_\phi(z+1) = \phi(z) W_\phi(z), \quad W_\phi(1) = 1,$$

valid for at least $z \in \mathbb{C}_{(0,\infty)}$, see [99, Theorem 6.1(3)]. Write Log for the branch of the complex logarithm that is analytic on the slit plane $\mathbb{C} \setminus (-\infty, 0]$ and satisfies $\mathrm{Log}\, 1 = 0$, commonly referred to as the principal branch. We use it to define, for $t > 0$ and $z \in \mathbb{C}_{(0,\infty)}$,

$$W_\phi^t(z) = e^{t \,\mathrm{Log}\, W_\phi(z)},$$

as well as $\phi^t(z) = e^{t\,\mathrm{Log}\,\phi(z)}$.

Theorem 3.2.3. *Let $\phi \in \mathcal{B}$ and let $(\nu_t)_{t\geqslant 0}$ be the corresponding Berg-Urbanik semigroup. Then, for $t > 0$,*

$$\mathcal{M}_{\nu_t}(z) = \int_0^\infty x^z \nu_t(dx) = W_\phi^t(z+1), \quad \mathrm{Re}(z) > -1, \tag{3.10}$$

where $W_\phi : \mathbb{C}_{(0,\infty)} \to \mathbb{C}$ is the Bernstein-Gamma function associated to ϕ. Moreover, $W_\phi^t \in \mathcal{A}_{(0,\infty)}$ and W_ϕ is the unique positive-definite function that solves, for all $t > 0$, the functional equation,

$$W_\phi^t(z+1) = \phi^t(z) W_\phi^t(z), \quad W_\phi^t(1) = 1, \tag{3.11}$$

valid for $z \in \mathbb{C}_{(0,\infty)}$.

Remark 3.2.3. Note that when $t = 1$, the equation (3.11) restricted to $\mathbb{R}_+$ was studied by Webster in [124], who showed that $W_\phi\big|_{\mathbb{R}_+}$ is the unique log-convex solution to the restricted functional equation.

Remark 3.2.4. We point out that in [98, Theorem 4.1] the authors proved that $W_\phi \in \mathcal{A}_{(\mathfrak{d}_\phi,\infty)}$, where

$$\mathfrak{d}_\phi = \sup\{u \leqslant 0; \phi(u) = -\infty \;\text{ or }\; \phi(u) = 0\} \in [-\infty, 0],$$

which is more than what we claim in Theorem 3.2.3 for $t = 1$. However, for $t \neq 1$, W_ϕ^t is only defined on the slit plane $\mathbb{C} \setminus (-\infty, 0]$ and hence it is not possible to extend the strip of analyticity of W_ϕ^t beyond $\mathbb{C}_{(0,\infty)}$.

This Theorem is proved in Section 3.4.1 . Our proof of (3.10) in Theorem 3.2.3 generalizes an argument given by Berg [16] for the case $W_\phi(z) = \Gamma(z)$, i.e. $\phi(u) = u$, which uses the (classical) Weierstrass product representation for the gamma function. We are able to readily adapt his argument to the generalized Weierstrass product for W_ϕ given by (3.9), which emphasizes the utility of such a product representation.

3.2.4 Existence, smoothness, and Mellin-Barnes representation of densities

In this section we obtain the existence of densities for subclasses of Berg-Urbanik semigroups, and quantify their regularities based on properties of the associated Bernstein function. We write $C_0(\mathbb{R}_+)$ for the set of continuous functions on $\mathbb{R}_+$ whose limit at infinity is zero. Then, for each $n \in \mathbb{N}$, we write $C_0^n(\mathbb{R}_+)$ for the set of n-times differentiable functions all of whose derivatives belong to $C_0(\mathbb{R}_+)$, and $C_0^\infty(\mathbb{R}_+)$ for the set of infinitely differentiable functions all of whose derivatives belong to $C_0(\mathbb{R}_+)$. Finally, for notational convenience, we write $\mu \in C_0^n(\mathbb{R}_+)$ to denote that a measure μ on $\mathbb{R}_+$ has a density, with respect to Lebesgue measure on $\mathbb{R}_+$, and that this density belongs to $C_0^n(\mathbb{R}_+)$.

To state our next result we need to consider some further subsets of $\mathcal{B}$. Following [98], we say that a Lévy measure μ satisfies *Condition-j* if $\mu(dy) = v(y)dy$ with $v(0^+) = \infty$, such that $v = v_1 + v_2$ for $v_1, v_2 \in \mathrm{L}^1(\mathbb{R}_+)$, and $v_1 \geqslant 0$ is non-increasing, while $\int_0^\infty v_2(y)dy \geqslant 0$ satisfies $|v_2(y)| \leqslant \left(\int_y^\infty v_1(r)dr\right) \vee C$, for some $C > 0$. Given this, we let

$$\mathcal{B}_j = \{\phi \in \mathcal{B};\ \mu \text{ satisfies Condition-}j\}$$

and note that $\mathcal{B}_{\mathcal{J}} \subset \mathcal{B}_j$.

Write $||v||_\infty = \sup_{y \geqslant 0} |v(y)|$ for the sup-norm of a function on $\mathbb{R}_+$, and set

$$\mathcal{B}_v = \{\phi \in \mathcal{B} \setminus \mathcal{B}_d;\ \mu(y) = v(y)dy \text{ with } ||v||_\infty < \infty\},$$

so that $\phi \in \mathcal{B}_v$ implies that $\phi(\infty) < \infty$. We define the quantity $\mathtt{N}_\phi$ as

$$\mathtt{N}_\phi = \begin{cases} \frac{v(0^+)}{\phi(\infty)} & \text{if } \phi \in \mathcal{B}_v, \\ \infty & \text{if } \phi \in \mathcal{B}_j \cup \mathcal{B}_d, \end{cases}$$

and set

$$\mathcal{B}_{\mathtt{N}} = \{\phi \in \mathcal{B}_j \cup \mathcal{B}_v \cup \mathcal{B}_d;\ \mathtt{N}_\phi > 0\}.$$

Next, let

$$\mathcal{B}_\Theta = \left\{\phi \in \mathcal{B};\ \Theta_\phi = \varliminf_{b\to\infty} \frac{1}{|b|}\int_0^b \arg\phi(1+iu)du > 0\right\},$$

and note that $\Theta_\phi \in [0, \frac{\pi}{2}]$ due to [98, Theorem 3.2(1)]. In fact, if $\phi \in \mathcal{B}_d$ then $\Theta_\phi = \frac{\pi}{2}$, while if $\lim_{u\to\infty}\phi(u)u^{-\alpha} = C_\alpha$, for $\alpha \in (0,1)$ and a constant $C_\alpha \in (0,\infty)$, then $\Theta_\phi = \alpha\frac{\pi}{2}$ (see [98, Theorem 3.3]). Furthermore, there is nothing special about the 1 in $\arg\phi(1+iu)$ as it can be replaced by any $a > 0$ without changing the value of Θ_ϕ, which follows from a combination of [99, Proposition 6.12] and [98, Theorem 3.1(1)]; in the definition of $\mathcal{B}_\Theta$ we simply choose to evaluate $\arg\phi$ along the imaginary line $\mathrm{Re}(z) = 1$ for convenience. For $\theta \in (0,\pi]$ let

$$\mathcal{A}(\theta) = \{f : \mathbb{C} \to \mathbb{C};\ f \text{ is analytic on the sector } |\arg z| < \theta\},$$

that is $\mathcal{A}(\pi)$ denotes the set of functions that are analytic on the slit plane $\mathbb{C} \setminus (-\infty, 0]$. Finally, we denote by $\mathrm{supp}(\mu)$ the support of a measure μ.

Theorem 3.2.4. *Let $(\nu_t)_{t\geqslant 0}$ be the Berg-Urbanik semigroup associated to $\phi \in \mathcal{B}$.*

(1) Assume that $\phi \not\equiv \mathsf{k}$ for $\mathsf{k} \geqslant 0$. If $\phi(\infty) < \infty$ then $\mathrm{supp}(\nu_t) = [0, \phi(\infty)^t]$, otherwise $\mathrm{supp}(\nu_t) = [0,\infty)$ for all $t > 0$.

(2) If $\phi \in \mathcal{B}_{\mathtt{N}}$ then, for any $t > \frac{1}{\mathtt{N}_\phi}$, $\nu_t \in C_0^{\mathfrak{n}(t)}(\mathbb{R}_+)$, i.e. $\nu_t(dx) = \nu_t(x)dx$, $x > 0$, where $\mathfrak{n}(t) = \lfloor \mathtt{N}_\phi t\rfloor - 1 \geqslant 0$. Furthermore, for each $n \leqslant \mathfrak{n}(t)$, the density $\nu_t(x)$, and its successive derivatives, admit the Mellin-Barnes representation

$$\nu_t^{(n)}(x) = \frac{(-1)^n}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-z-n}\frac{\Gamma(z+n)}{\Gamma(z)}W_\phi^t(z)dz,$$

for any $c, x > 0$.

(3) If $\phi \in \mathcal{B}_\Theta$, then, for any $0 < t < \frac{\pi}{\Theta_\phi}$, $\nu_t \in \mathcal{A}(\Theta_\phi t)$, and for any $t \geqslant \frac{\pi}{\Theta_\phi}$, $\nu_t \in \mathcal{A}(\pi)$.

Remark 3.2.5. From Theorem 3.2.4(1) it follows that the support of ν_t is bounded, pointwise in t, if and only if ϕ is a bounded function. Note that we exclude the case when $\phi \equiv \mathsf{k}$ as this corresponds to a Berg-Urbanik semigroup with degenerate support, i.e. $\text{supp}(\nu_t) = \delta_{\mathsf{k}^t}$.

This Theorem is proved in Section 3.4.2. A key ingredient in the proofs of Theorem 3.2.4(2) and Theorem 3.2.4(3) are estimates for Bernstein-Gamma functions along imaginary lines provided in [98, Theorem 4.2].

The main point of Theorem 3.2.4(2) is to quantify the differentiability of the Berg-Urbanik semigroup as a function of t and simple quantities associated to ϕ. In this sense our result complements and extends [99, Theorem 5.2], which deals with the differentiability at time 1. Finally, in Theorem 3.2.4(3) we describe the analyticity of ν_t both as a function of ϕ and t, and show that the sector of analyticity grows linearly in t. This gives rise to another kind of threshold phenomenon, whereby for large enough t we get that the density is analytic on $\mathbb{C} \setminus (-\infty, 0]$.

3.2.5 Asymptotics at infinity of densities and their successive derivatives

In this section we consider a subset of Berg-Urbanik semigroups admitting smooth densities, for all $t > 0$, for which we are able to obtain the exact large asymptotic behavior of the density, as well as for all of its successive derivatives, for all time $t > 0$. We write $f(x) \overset{\infty}{\sim} g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, and $f(x) \overset{\infty}{=} \mathrm{o}(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. The following theorem is the main result of this section, and one of the main results of this paper.

Theorem 3.2.5. *Let $\phi \in \mathcal{B}$ be such that $\phi(\infty) = \infty$ with $\phi^{\mathrm{t}} \in \mathcal{B}_{\mathcal{J}}$, for all $\mathrm{t} \in (0,1)$, and let $(\nu_t)_{t\geqslant 0}$ be the corresponding Berg-Urbanik semigroup. For any $t > 0$, $\nu_t \in C_0^\infty(\mathbb{R}_+)$, i.e. $\nu_t(dx) = \nu_t(x)dx$, $x > 0$, and the densities $\nu_t(x)$ satisfy the following large asymptotic behavior*

$$\nu_t\left(x^t\right) \stackrel{\infty}{\sim} \frac{C_\phi^t}{\sqrt{2\pi t}}\sqrt{x^{1-t}\varphi'(x)}\exp\left(-t\int_{\mathsf{k}}^x \frac{\varphi(r)}{r}dr\right) \tag{3.12}$$

where $C_\phi > 0$ is a constant depending only on ϕ, and $\varphi : [\mathsf{k},\infty) \to [0,\infty)$ is the continuous inverse of ϕ. Furthermore, for any $n \in \mathbb{N}$ and $t > 0$, the successive derivatives of the density satisfy

$$\nu_t^{(n)}\left(x^t\right) \stackrel{\infty}{\sim} (-1)^n x^{-nt}\varphi^n(x)\nu_t\left(x^t\right) \tag{3.13}$$

which can be specified as follows.

(1) If $\phi \in \mathcal{B}_d$ then

$$\nu_t\left(x^t\right) \stackrel{\infty}{\sim} \frac{\widetilde{C}_\phi^t}{\sqrt{2\pi t}}x^{\frac{\mathsf{d}+t(2\mathsf{k}-\mathsf{d})}{2\mathsf{d}}}\exp\left(-\frac{tx}{\mathsf{d}} + \frac{t}{\mathsf{d}}\int_{\mathsf{k}}^x \frac{E(r)}{r}dr\right)$$

where $\widetilde{C}_\phi > 0$ is a constant, and $E(u) \geqslant 0$ satisfies $E(u) \stackrel{\infty}{=} \mathrm{o}(u)$. Furthermore, for any $n \in \mathbb{N}$ and $t > 0$,

$$\nu_t^{(n)}\left(x^t\right) \stackrel{\infty}{\sim} (-1)^n \mathsf{d}^n x^{n(1-t)}\nu_t\left(x^t\right).$$

(2) If $\phi(u) \stackrel{\infty}{\sim} C_\alpha u^\alpha$, for a constant $C_\alpha > 0$ and $\alpha \in (0,1)$, then

$$\nu_t\left(x^t\right) \stackrel{\infty}{\sim} \frac{\overline{C}_\phi^t}{\sqrt{2\pi t}}x^{\frac{1-\alpha t}{2\alpha}}\exp\left(-t\alpha C_\alpha^{-\frac{1}{\alpha}}x^{\frac{1}{\alpha}} + t\int_k^x \frac{H(r)}{r}dr\right)$$

where $\overline{C}_\phi > 0$ is a constant, and $H(u^\alpha) \stackrel{\infty}{=} \mathrm{o}(u)$. Furthermore, for any $n \in \mathbb{N}$ and $t > 0$,

$$\nu_t^{(n)}\left(x^t\right) \stackrel{\infty}{\sim} (-1)^n C_\alpha^{-\frac{n}{\alpha}} x^{\frac{n}{\alpha}(1-\alpha t)}\nu_t\left(x^t\right).$$

Remark 3.2.6. Note the asymptotic (3.12) is a key ingredient in the proof of Theorem 3.2.1 regarding the moment determinacy of the Berg-Urbanik semigroups.

Remark 3.2.7. In the special case $\phi(u) = u$ the identity in (3.12) boils down to

$$e_t^{(n)}(x) \overset{\infty}{\sim} (-1)^n \frac{(2\pi)^{\frac{t-1}{2}}}{\sqrt{t}} x^{\frac{1-t}{2t}} x^{n(\frac{1}{t}-1)} e^{-tx^{\frac{1}{t}}} \tag{3.14}$$

where we recall that $(e_t)_{t>0}$ stands for the classical Urbanik semigroup, see (3.1). For $n = 0$ and $t > 0$ this asymptotic was proved by Berg and López in [20], see also Janson [68] for an independent proof. In both papers the authors apply a delicate saddle point argument hinging on special properties of the gamma function such as the Stirling's formula with Binet remainder for the gamma function as in [20]. Furthermore, Janson outlines how his saddle point argument can be applied to yield the asymptotics in (3.14) for arbitrary $n \in \mathbb{N}$, see [68, Remark 6.2]. It would be interesting to see if a saddle point approach could be applied for general Berg-Urbanik semigroups, using the Mellin transform representation we provide in Theorem 3.2.3 together with further study of Bernstein-Gamma functions.

This Theorem is proved in Section 3.4.3. There are three main steps in the proof of the asymptotics (3.12) and (3.13). The first one hinges on a non-classical Tauberian theorem whose version we use is due to Patie and Savov [99, Proposition 5.26] but originates from the work of Balkema [11, Theorem 4.4]. It enables us to get the large asymptotic behavior of the densities and of its successive derivatives at time $t = 1$, under the less stringent conditions $\phi \in \mathcal{B}_{\mathcal{J}}$. Since the conditions to invoke this non-classical Tauberian theorem are difficult to check, one can not follow this path for other times than 1. Instead, we combine the asymptotic at time 1 of the densities from [99, Theorem 5.5] together with assumption that $\phi^{\mathfrak{t}} \in \mathcal{B}_{\mathcal{J}}$, for all $\mathfrak{t} \in (0, 1)$, to obtain the asymptotic at time $\mathfrak{t}$. Lastly we adapt to our context a closure result due to Balkema et al. [10, Theorem 1.1], which states that the (additive) convolution of probabilities density with Gaussian tails also has a Gaussian tail, to extend the asymptotic from $\mathfrak{t} \in (0, 1)$ to all $t > 0$. Our application of this closure result is novel, since we use it not only for the densities (as it is stated in [10]) but also for their successive derivatives.

As a by-product of Theorem 3.2.5 we obtain the large asymptotic behavior of the density and its successive derivatives for the law of certain Lévy processes, which seems to be new in the Lévy literature. To state this we briefly recall that a (one-dimensional) Lévy process $(Y_t)_{t\geqslant 0}$ is a $\mathbb{R}$-valued stochastic process with stationary and independent increments, that is continuous in probability, and such that $Y_0 = 0$ a.s. For further information regarding Lévy processes we refer to the monograph [108]. Note that to each Berg-Urbanik semigroup there exists a corresponding Lévy process whose characteristic exponent is given by (3.3).

Corollary 3.2.2. *Let $\phi \in \mathcal{B}$ be such that $\phi(\infty) = \infty$ with $\phi^{\mathsf{t}} \in \mathcal{B}_{\mathcal{J}}$, for all $\mathsf{t} \in (0, 1)$, and let $(Y_t)_{t\geqslant 0}$ be a Lévy process whose characteristic exponent Ψ is given by* (3.3)*. Then, for $t > 0$, $\mathbb{P}(Y_t \in dy) = f_t(y)dy$, $y \in \mathbb{R}$ with $f_t \in C_0^\infty(\mathbb{R})$ and, for any $n \geqslant 0$,*

$$f_t^{(n)}(ty) \overset{\infty}{\sim} (-1)^n \frac{C_\phi^t}{\sqrt{2\pi t}} \varphi^n(e^y) \sqrt{e^{(1+t)y} \varphi'(e^y)} \exp\left(-t \int_{\mathsf{k}}^{e^y} \frac{\varphi(r)}{r} dr\right)$$

where $C_\phi > 0$ is a constant depending only on ϕ, and $\varphi : [\mathsf{k}, \infty) \to [0, \infty)$ is the continuous inverse of ϕ.

This corollary is obtained by combining (3.12) and (3.13) with the relation $f_t^{(n)}(y) \overset{\infty}{\sim} e^{(n+1)y} \nu_t^{(n)}(e^y)$, for any $n \geqslant 0$, which is established in the proof of Theorem 3.2.5. We are not aware of such a detailed description of the large asymptotic behavior for the law of a Lévy process, for all $t > 0$ as well as of its successive derivatives, having appeared in the Lévy literature before, except in some special cases.

3.3 Examples

In this section we consider two examples of Berg-Urbanik semigroups that illustrate the previous results.

Example 3.3.1. Let $\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}$ be the Bernstein function defined, for $u \geqslant 0$, by

$$\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}(u) = \frac{\Gamma(\alpha u + \mathfrak{a})}{\Gamma(\alpha u + \mathfrak{b})}$$

with $\alpha \in (0, 1]$ and $0 \leqslant \mathfrak{b} < \mathfrak{a} < \mathfrak{b} + 1$, where the fact that $\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}$ is a Bernstein functions follows from [73, Proposition 1 and Remark 1]. Next, let, for $\tau \in \mathbb{R}_+$, $G(z|\tau)$ denote the double gamma function, and recall that it satisfies the functional equation

$$G(z + 1|\tau) = \Gamma\left(\frac{z}{\tau}\right) G(z|\tau), \tag{3.15}$$

for $z \in \mathbb{C}_{(0,\infty)}$, with $G(1|\tau) = 1$. We claim that

$$W_{\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}}(z) = C_{\alpha,\mathfrak{a},\mathfrak{b}} \frac{G(z + \frac{\mathfrak{a}}{\alpha}|\frac{1}{\alpha})}{G(z + \frac{\mathfrak{b}}{\alpha}|\frac{1}{\alpha})}, \quad \text{where} \quad C_{\alpha,\mathfrak{a},\mathfrak{b}} = \frac{\Gamma(\mathfrak{b})G(\frac{\mathfrak{b}}{\alpha}|\frac{1}{\alpha})}{\Gamma(\mathfrak{a})G(\frac{\mathfrak{a}}{\alpha}|\frac{1}{\alpha})}. \tag{3.16}$$

Indeed, from (3.15) it follows that

$$\frac{G(z + 1 + \frac{\mathfrak{a}}{\alpha}|\frac{1}{\alpha})}{G(z + 1 + \frac{\mathfrak{b}}{\alpha}|\frac{1}{\alpha})} = \frac{\Gamma(\alpha z + \mathfrak{a})}{\Gamma(\alpha z + \mathfrak{b})} \frac{G(z + \frac{\mathfrak{a}}{\alpha}|\frac{1}{\alpha})}{G(z + \frac{\mathfrak{b}}{\alpha}|\frac{1}{\alpha})},$$

for $z \in \mathbb{C}_{(0,\infty)}$, and the choice of $C_{\alpha,\mathfrak{a},\mathfrak{b}}$ ensures the required normalization. Hence it remains to prove the uniqueness. To this end we note that, by a Malmsten-type representation for $G(z|\tau)$ due to [75], we have

$$\log\left(\frac{G(z + \frac{\mathfrak{a}}{\alpha}|\frac{1}{\alpha})}{G(z + \frac{\mathfrak{b}}{\alpha}|\frac{1}{\alpha})}\right) = -c - \kappa z + \int_0^\infty (e^{-zy} - 1 + zy) f_{\alpha,\mathfrak{a},\mathfrak{b}}(y)dy, \tag{3.17}$$

where c, κ are real-constants depending only on the underlying parameters, and

$$f_{\alpha,\mathfrak{a},\mathfrak{b}}(y) = \frac{(e^{-\frac{\mathfrak{b}}{\alpha}y} - e^{-\frac{\mathfrak{a}}{\alpha}y})}{y(1 - e^{-y})(1 - e^{-\frac{y}{\alpha}})},$$

see for instance [77, (2.15)]. Differentiating the right-hand side of (3.17) twice, which is justified by dominated convergence, shows that the ratio of double-gamma functions is log-convex. However, $W_{\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}}$ is the unique log-convex function on $\mathbb{R}_+$ solution to the functional equation, and thus the claim is proved.

Next, we note that $\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}$ is a complete Bernstein function. Indeed, $\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}$ is obtained by the dilation and translation of the argument of the function $\Phi_{\alpha,\mathfrak{m}}$ below, whose Lévy measure is easily seen via direct calculation to be completely monotone, and these operations preserve the property of being a complete Bernstein function, which can be seen by using the upper half-plane criterion as outlined in Remark 3.2.1. Moreover, the density of the Lévy measure of $\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}$ is necessarily infinite at 0, which follows from $\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}(\infty) = \infty$, and so $\Phi_{\alpha,\mathfrak{a},\mathfrak{b}} \in \mathcal{B}_j$, which gives by definition that $\mathtt{N}_{\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}} = \infty$. Thus, invoking Theorem 3.2.4(2) yields that, for all $t > 0$, $\nu_t \in C_0^\infty(\mathbb{R}_+)$ and since, by Stirling formula, recalled in (3.35) below, $\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}(u) \stackrel{\infty}{\sim} Cu^{\mathfrak{a}-\mathfrak{b}}$, for a constant $C > 0$ and with $\mathfrak{a}-\mathfrak{b} \in (0,1)$, these densities satisfy the large asymptotic behavior specified by Theorem 3.2.5(2). From [98, Theorem 3.3(2)] we get that $\Theta_{\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}} = \frac{(\mathfrak{a}-\mathfrak{b})\pi}{2}$, see the discussion prior to Theorem 3.2.4 for the definition, where we may apply this result since $\Phi_{\alpha,\mathfrak{a},\mathfrak{b}} \in \mathcal{B}_\alpha$ with $\ell \equiv 1$ in the notation therein. Hence invoking Theorem 3.2.4(3) gives that $\nu_t \in \mathcal{A}(\frac{(\mathfrak{a}-\mathfrak{b})\pi t}{2})$ for $t < \frac{2}{\mathfrak{a}-\mathfrak{b}}$ and $\nu_t \in \mathcal{A}(\pi)$ for $t \geqslant \frac{2}{\mathfrak{a}-\mathfrak{b}}$. Finally, the property $\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}(u) \stackrel{\infty}{\sim} Cu^{\mathfrak{a}-\mathfrak{b}}$ gives, by Theorem 3.2.1(3), that $\mathfrak{T}_{\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}} = \frac{2}{\mathfrak{a}-\mathfrak{b}}$ and, by Theorem 3.2.1(1), we also have that the semigroup is moment determinate at the threshold. As remarked earlier, this example reveals that for any $T \in (2,\infty)$ there exists a Bernstein function, namely $\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}$ with $\mathfrak{a}-\mathfrak{b} = \frac{2}{T}$ and any $\alpha \in (0,1]$, whose associated Berg-Urbanik semigroup has threshold index $\mathfrak{T}_{\Phi_{\alpha,\mathfrak{a},\mathfrak{b}}} = T$.

Now let us now mention that for the special case when $\mathfrak{a} = \alpha\mathfrak{m}+1$ and $\mathfrak{b} = \alpha\mathfrak{m}+1-\alpha$, where $\mathfrak{m} \in [1-\frac{1}{\alpha},\infty)$, so that $\mathfrak{a}-\mathfrak{b} = \alpha$, some expressions above simplify. Indeed, in this case, the Bernstein function takes the form

$$\begin{aligned}\Phi_{\alpha,\mathfrak{m}}(u) &= \frac{\Gamma(\alpha u + \alpha\mathfrak{m} + 1)}{\Gamma(\alpha u + \alpha\mathfrak{m} + 1 - \alpha)} \\ &= \frac{\Gamma(\alpha\mathfrak{m} + 1)}{\Gamma(\alpha\mathfrak{m} + 1 - \alpha)} + \int_0^\infty (1 - e^{-uy}) e^{-(\mathfrak{m}+\frac{1}{\alpha})y} (1 - e^{-\frac{y}{\alpha}})^{-\alpha-1} dy,\end{aligned}$$

and was studied in the context of the so-called Gauss-Laguerre semigroup in [97], see the computations on p.808 therein for the above equality. For $z \in \mathbb{C}_{(0,\infty)}$, the ratio of double gamma functions in (3.16) boils down to

$$W_{\Phi_{\alpha,\mathfrak{m}}}(z) = \frac{\Gamma(\alpha z + \alpha\mathfrak{m} + 1 - \alpha)}{\Gamma(\alpha\mathfrak{m} + 1)},$$

see e.g. [97, Lemma 3.1], and we also have

$$\nu_1(x) = \frac{x^{\mathfrak{m}+\frac{1}{\alpha}-1} e^{-x^{\frac{1}{\alpha}}}}{\Gamma(\alpha\mathfrak{m} + 1)}, \ x > 0,$$

see [99, Equation (3.10)] and more generally Section 3.3 of the aforementioned paper.

Example 3.3.2. Let $\phi \in \mathcal{B}$ and consider the function defined, on $\mathbb{R}^+$, by

$$\phi_\ell(u) = \log\left(\frac{\phi(u+1)}{\phi(1)}\right).$$

Observe that,

$$\phi'_\ell(u) = \log\left(\frac{\phi(u+1)}{\phi(1)}\right)' = \frac{\phi'(u+1)}{\phi(u+1)} = \int_0^\infty e^{-uy} e^{-y} \kappa(dy) = \int_0^\infty e^{-uy} \kappa_e(dy),$$

where we used that $\frac{\phi'(u)}{\phi(u)} = \int_0^\infty e^{-uy}\kappa(dy)$ and have set $\kappa_e(dy) = e^{-y}\kappa(dy)$. It means that ϕ'_ℓ is completely monotone and since ϕ_ℓ is plainly positive on $\mathbb{R}^+$, we deduce that $\phi_\ell \in \mathcal{B}$. Next, as a general result on Bernstein functions gives $\overline{\lim}_{u\to\infty} u^{-1}\phi(u) < \infty$, see for instance [99, Proposition 4.1(3)], it follows readily that for any $\beta > 0$, $\overline{\lim}_{u\to\infty} u^{-\beta}\phi_\ell(u) = 0$ and thus $\beta_{\phi_\ell} = 0$, see (3.7) for definition. Hence, the Berg-Urbanik semigroup associated to the Bernstein function ϕ_ℓ is completely determinate.

As an illustration, we choose, for $\lambda > 0$, $\phi(u) = 1 + \frac{u}{\lambda} \in \mathcal{B}$ and we have, writing $\phi_\ell = \phi_\lambda$, that

$$\phi_\lambda(u) = \log\left(1 + \frac{u}{\lambda}\right) = \int_0^\infty (1 - e^{-uy}) \frac{e^{-\lambda y}}{y} dy. \tag{3.18}$$

It follows plainly from the right-hand side of the equality (3.18) that the Lévy measure of ϕ_λ is completely monotone, and thus ϕ_λ is a complete Bernstein function. Furthermore, we have that $\mathtt{N}_{\phi_\lambda} = \infty$, since ϕ_λ satisfies Condition-j and $\phi_\lambda(\infty) = \infty$. Hence we get from Theorem 3.2.4(1) that $\mathrm{supp}(\nu_t) = [0, \infty)$ for all $t > 0$, and from Theorem 3.2.4(2) we conclude that for all $t > 0$, $\nu_t(dx) = \nu_t(x)dx$ with $\nu_t \in C_0^\infty(\mathbb{R}_+)$. A straightforward computation yields that the continuous inverse of ϕ_λ is given by $u \mapsto \lambda(e^u - 1)$. Hence, by Theorem 3.2.5, we have, for all $t > 0$, that

$$\nu_t\left(x^t\right) \overset{\infty}{\sim} \frac{C^t}{\sqrt{2\pi t}} x^{\frac{1-t(1+2\lambda)}{2}} \exp\left(-\lambda t \operatorname{Ei}(x) + \frac{x}{2}\right),$$

where $C > 0$ is a constant and $\operatorname{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt$ is the exponential integral, and we also used the well-known relation $\operatorname{Ei}(x) = \gamma + \log x + \int_0^x \frac{e^r - 1}{r} dr$, where γ is the Euler-Mascheroni constant.

3.4 Proofs of main results

Throughout the proofs we write $f(x) \overset{\infty}{=} \mathrm{O}(g(x))$ to denote that $\overline{\lim}_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| < \infty$, and recall that $f(x) \overset{\infty}{\sim} g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, and $f(x) \overset{\infty}{=} \mathrm{o}(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

3.4.1 Proof of Theorem 3.2.3

We begin with the proof of (3.10) and start by showing that the function $b \mapsto -\operatorname{Log} W_\phi(1 + ib)$ is a continuous negative-definite function, i.e. a continuous function f such that $f(0) \geqslant 0$ and $u \mapsto e^{-tf(u)}$ is positive-definite for all $t > 0$, see [109, Proposition 4.4]. As mention in the introduction, this fact has already been established by Berg [17] and independently by Hirsch and Yor [66] and we shall provide yet another proof utilizing the Weierstrass product representation for W_ϕ. We follow closely the arguments given by Berg for the proof of [16, Lemma 2.1]. First, from (3.9) we have, for $\mathrm{Re}(z) > 0$,

$$W_\phi(z) = \frac{e^{-\gamma_\phi z}}{\phi(z)} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)} z},$$

where $\gamma_\phi = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\phi'(k)}{\phi(k)} - \log\phi(n)\right) \in \left[-\log\phi(1), \frac{\phi'(1)}{\phi(1)} - \log\phi(1)\right]$. Hence,

$$-\operatorname{Log} W_\phi(1+ib) = \gamma_\phi(1+ib) + \operatorname{Log}\phi(1+ib) - \sum_{k=1}^{\infty} \left(\operatorname{Log}\left(\frac{\phi(k)}{\phi(k+1+ib)}\right) + (1+ib)\frac{\phi'(k)}{\phi(k)}\right).$$

Next, for $n \geqslant 1$, consider the truncated functions $L_{\phi,n}$ defined by

$$\begin{aligned} L_{\phi,n}(1+ib) &= \gamma_\phi(1+ib) + \operatorname{Log}\phi(1+ib) - \sum_{k=1}^{n} \left(\operatorname{Log}\left(\frac{\phi(k)}{\phi(k+1+ib)}\right) + (1+ib)\frac{\phi'(k)}{\phi(k)}\right) \\ &= L_{\phi,n}(1) + ib\left(\gamma_\phi - \sum_{k=1}^{n} \frac{\phi'(k)}{\phi(k)}\right) + \sum_{k=1}^{n+1} \operatorname{Log}\frac{\phi(k+ib)}{\phi(k)}, \end{aligned}$$

where

$$L_{\phi,n}(1) = \gamma_\phi - \sum_{k=1}^{n} \frac{\phi'(k)}{\phi(k)} + \log\phi(n+1) = \gamma_\phi - g(n),$$

and the last equality serves to define $g(n)$. We claim that $n \mapsto g(n)$ is non-decreasing with $\lim_{n\to\infty} g(n) = \gamma_\phi$. Indeed, we have from [99, Proposition 4.1(4)] that $\frac{1}{\phi}$ is completely monotone so that $\frac{\phi'}{\phi}$ is completely monotone, as the product of two completely monotone functions. Thus $u \mapsto \frac{\phi'(u)}{\phi(u)}$ is non-increasing, and we get that

$$\log\frac{\phi(n+2)}{\phi(n+1)} = \int_{n+1}^{n+2} \frac{\phi'(u)}{\phi(u)} du \leqslant \frac{\phi'(n+1)}{\phi(n+1)},$$

which yields

$$g(n+1) - g(n) = \frac{\phi'(n+1)}{\phi(n+1)} - \log\frac{\phi(n+2)}{\phi(n+1)} \geqslant 0.$$

Additionally, by [99, Proposition 4.1(6)]

$$\lim_{n\to\infty} \frac{\phi(n+1)}{\phi(n)} = 1,$$

so that

$$\begin{aligned} \lim_{n\to\infty} g(n) &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\phi'(k)}{\phi(k)} - \log\phi(n) + \log\phi(n) - \log\phi(n+1)\right) \\ &= \gamma_\phi - \lim_{n\to\infty} \log\frac{\phi(n+1)}{\phi(n)} = \gamma_\phi. \end{aligned}$$

Putting all of these observations together, we conclude that $L_{\phi,n}(1) \geqslant 0$. Furthermore, for any $a \in \mathbb{R}$ the function $b \mapsto iab$ is continuous negative-definite, and for any $1 \leqslant k \leqslant n+1$, $b \mapsto \operatorname{Log} \frac{\phi(k+ib)}{\phi(k)}$ is continuous negative-definite since $u \mapsto \log \frac{\phi(k+u)}{\phi(k)}$ is a Bernstein function, as the composition of two Bernstein functions, see [109, Corollary 3.8(iii)]. This shows that $L_{\phi,n}(1+ib)$ is a continuous negative-definite function, and since $\lim_{n\to\infty} L_{\phi,n}(1+ib) = -\operatorname{Log} W_\phi(1+ib)$ pointwise it follows that $b \mapsto -\operatorname{Log} W_\phi(1+ib)$ is a continuous negative-definite function.

Consequently, using the homeomorphism $x \mapsto e^x$ between $\mathbb{R}$ and $(0,\infty)$, we find that there exists a unique multiplicative convolution semigroup $(\mathcal{V}_t)_{t\geqslant 0}$ such that

$$\int_0^\infty y^{ib}\mathcal{V}_t(dy) = W_\phi^t(1+ib). \tag{3.19}$$

From [99, Theorem 6.1] we know that $W_\phi \in \mathcal{A}_{(0,\infty)}$ and hence $W_\phi^t \in \mathcal{A}_{(0,\infty)}$ for any $t > 0$. Thus the identity in (3.19) extends to

$$\int_0^\infty y^{z-1}\mathcal{V}_t(dy) = W_\phi^t(z),$$

for $z \in \mathbb{C}_{(0,\infty)}$. However, again from [99, Theorem 6.1], we have that $\mathcal{M}_{\nu_1}(z-1) = W_\phi(z)$ and thus $\mathcal{V}_1 = \nu_1$, since the Mellin transform uniquely characterizes a probability measure. By uniqueness of convolution semigroups it then follows that $\mathcal{V}_t = \nu_t$ for all $t \geqslant 0$, and thus (3.10) is established. Finally, from [99, Theorem 6.1] we have that $W_\phi : \mathbb{C}_{(0,\infty)} \to \mathbb{C}$ is the unique positive-definite function, i.e. the Mellin transform of a probability measure, that satisfies the functional equation

$$W_\phi(z+1) = \phi(z)W_\phi(z), \quad W_\phi(1) = 1,$$

for $z \in \mathbb{C}_{(0,\infty)}$, from which the last claim follows.

3.4.2 Proofs for Section 3.2.4

Proof of Theorem 3.2.4(1)

It is immediate from [19, Theorem 1.5] that $\phi(\infty) = \infty$ implies $\mathrm{supp}(\nu_t)$ is unbounded, and we also get from [99, Theorem 5.2(1)] that $\mathrm{supp}(\nu_1) = [0, \infty)$. By the homeomorphism $x \mapsto e^x$ between $\mathbb{R}$ and $(0, \infty)$ mentioned above, together with the fact that the boundedness from below of the support of the law of a Lévy process is time-independent, see [108, Theorem 24.7], we then conclude that $\mathrm{supp}(\nu_t) = [0, \infty)$ for all $t > 0$. Hence, we suppose that $\phi(\infty) \in (0, \infty)$. To prove the claim we will rely on the following auxiliary result: for any measure μ on $\mathbb{R}_+$, $\mathrm{supp}(\mu) \subseteq [0, c]$, for $c > 0$, if and only if $\int_0^\infty x^n \mu(dx) \overset{\infty}{=} \mathrm{O}(c^n)$, see [19, Lemma 2.9]. Since for any $\phi \in \mathcal{B}$ we have, by definition, that ϕ' is completely monotone it follows that all Bernstein functions are non-decreasing on $\mathbb{R}_+$. Thus we have, for any $n \geqslant 0$,

$$\mathcal{M}_{\nu_t}(n) = \left(\prod_{k=1}^{n} \phi(k)\right)^t \leqslant \phi(\infty)^{nt}.$$

By the quoted result, the above estimate implies that $\mathrm{supp}(\nu_t) \subseteq [0, \phi(\infty)^t]$. For the reverse inclusion, let $\varepsilon > 0$ be small and choose $N_{\varepsilon,\phi}$ large enough (depending on ε and ϕ) such that for $k \geqslant N_{\varepsilon,\phi} - 1$ we have $\phi(k) \geqslant \phi(\infty) - \varepsilon > 0$. Then, for $n \geqslant N_{\varepsilon,\phi}$ and again since ϕ is non-decreasing,

$$\mathcal{M}_{\nu_t}(n) = \left(\prod_{k=1}^{N_{\varepsilon,\phi}-1} \phi(k)\right)^t \left(\prod_{k=N_{\varepsilon,\phi}}^{n} \phi(k)\right)^t \geqslant C_{\varepsilon,\phi,t}(\phi(\infty) - \varepsilon)^{nt},$$

where

$$C_{\varepsilon,\phi,t} = \frac{\phi(1)^{(N_{\varepsilon,\phi}-1)t}}{(\phi(\infty) - \varepsilon)^{N_{\varepsilon,\phi} t}}$$

is a constant, which depends only on ε, ϕ, and t. Since $\varepsilon > 0$ is arbitrary this estimate shows that $\mathrm{supp}(\nu_t)$ cannot be contained in any sub-interval of $[0, \phi(\infty)^t]$. Thus we must either have that $\mathrm{supp}(\nu_t) = [0, \phi(\infty)^t]$ or $\mathrm{supp}(\nu_t) = \delta_{\phi(\infty)^t}$, a Dirac mass at the point $\phi(\infty)^t$. In the latter case,

$$\mathcal{M}_{\nu_t}(n) = \phi(\infty)^{nt} = \left(\prod_{k=1}^{n} \phi(k)\right)^t,$$

for all $n \geqslant 0$ and $t > 0$, from which it follows that ϕ must be constant.

Proof of Theorem 3.2.4(2)

We split the proof into two cases. First, suppose that $\mathtt{N}_\phi = \infty$, which implies that $\phi \in \mathcal{B}_d \cup \mathcal{B}_j$. Then one may invoke [98, Theorem 4.2(3)] to get that, for any $p \geqslant 0$ and $a > 0$,

$$\lim_{|b|\to\infty} |b|^p |W_\phi(a + ib)| = 0,$$

where $W_\phi : \mathbb{C}_{(0,\infty)} \to \mathbb{C}$ is the Bernstein-Gamma function associated to ϕ. Hence, for any $q \geqslant 0$ and $t > 0$ fixed,

$$\lim_{|b|\to\infty} |b|^q |W_\phi(a + ib)|^t = 0,$$

which yields the estimate

$$|W_\phi^t(a + ib)| \overset{\infty}{=} \mathrm{O}(|b|^{-q}),$$

uniformly on bounded a-intervals, i.e. uniformly on bounded intervals of $a \in (0, \infty)$. Indeed, the functions E_ϕ and R_ϕ in [98, Theorem 4.2] are uniformly bounded for all $a > 0$ and all $\phi \in \mathcal{B}$, while the function G_ϕ in [98, Theorem 4.2] depends only on a and $G_\phi(a) \leqslant a \log \phi(1 + a)$, so that G_ϕ is uniformly bounded on bounded a-intervals, see also [98, Remark 4.3]. By Theorem 3.2.3 we know that $\mathcal{M}_{\nu_t}(z - 1) = W_\phi^t(z)$, for $\mathrm{Re}(z) > 0$, so the estimate for W_ϕ^t established above, together with the fact that $W_\phi^t \in \mathcal{A}_{(0,\infty)}$, justifies the use of Mellin inversion, see e.g. [119], to conclude that, for any $c > 0$,

$$\nu_t(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-z} W_\phi^t(z) dz. \tag{3.20}$$

Note that the integrand in (3.20) is absolutely integrable for any $x > 0$, since $|x^{-(c+ib)}||W_\phi^t(c+ib)| = x^{-c}|W_\phi^t(c+ib)|$ and $|W_\phi^t(c+ib)| \overset{\infty}{=} \mathrm{O}(|b|^{-1})$, for $|b|$ large enough. Taking $\lim_{x\to\infty} \nu_t(x)$ in (3.20) and using the dominated convergence theorem to interchange the limit and the integral gives that $\nu_t \in C_0(\mathbb{R}_+)$. However, since for any $q \geqslant 0$ and $a > 0$, $|W_\phi^t(a+ib)| \overset{\infty}{=} \mathrm{O}(|b|^{-q})$, we deduce that, for any $n = 0, 1, 2, \ldots$, $z \mapsto z^n|W_\phi^t(z)|$ is absolutely integrable and uniformly decaying on a complex strip containing $c + n + i\mathbb{R}$, see e.g. [99, Section 1.7.4], and thus we get

$$\nu_t^{(n)}(x) = \frac{(-1)^n}{2\pi i} \int_{c+n-i\infty}^{c+n+i\infty} x^{-z} \frac{\Gamma(z)}{\Gamma(z-n)} W_\phi^t(z-n)dz.$$

By the change of variables $z \mapsto z + n$ then yields the claimed Mellin-Barnes representation,

$$\nu_t^{(n)}(x) = \frac{(-1)^n}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-z-n} \frac{\Gamma(z+n)}{\Gamma(z)} W_\phi^t(z)dz,$$

where we note that the integrand is absolutely integrable by Stirling's formula for the gamma function, see (3.35) below. Using the dominated convergence theorem once more to evaluate the limit at infinity yields that $\nu_t \in C_0^\infty(\mathbb{R}_+)$.

Next, suppose that $\phi \in \mathcal{B}_\nu$, i.e. $\mathtt{N}_\phi = \frac{\nu(0^+)}{\phi(\infty)} \in (0, \infty)$. Another application of [98, Theorem 4.2] yields that, for $a > 0$ fixed and any $\varepsilon > 0$,

$$\lim_{|b|\to\infty} |b|^{\mathtt{N}_\phi - \varepsilon}|W_\phi(a+ib)| = 0,$$

while

$$\lim_{|b|\to\infty} |b|^{\mathtt{N}_\phi + \varepsilon}|W_\phi(a+ib)| = \infty.$$

The first equality thus guarantees that, for $t > 0$ and any $\varepsilon > 0$,

$$\lim_{|b|\to\infty} |b|^{\mathtt{N}_\phi t - \varepsilon}|W_\phi(a+ib)|^t = 0. \tag{3.21}$$

Now let $t > \frac{1}{\mathtt{N}_\phi}$ and observe that $\mathfrak{n}(t) = \lfloor \mathtt{N}_\phi t \rfloor - 1 \geqslant 0$ and is the largest integer less than or equal to $\mathtt{N}_\phi t - 1$. Choose ε such that $\mathtt{N}_\phi t - 1 - \mathfrak{n}(t) > \varepsilon > 0$. Then, by (3.21), it follows that, uniformly on bounded a-intervals, and for $|b|$ large enough

$$|W_\phi(a+ib)|^t \leqslant C|b|^{-1-\mathfrak{n}(t)-\varepsilon},$$

for $C > 0$ a constant. Since the right-hand side is uniformly integrable and W_ϕ^t is analytic on $\mathbb{C}_{(0,\infty)}$, another application of the Mellin inversion formula and dominated convergence allows us to conclude that $\nu_t \in C_0^{\mathfrak{n}(t)}(\mathbb{R}_+)$. The Mellin-Barnes representation follows as in the previous case.

Proof of Theorem 3.2.4(3)

Since $\phi \in \mathcal{B}_\Theta$ we have, for any $\varepsilon > 0$ and $|b|$ large enough,

$$A_\phi(a+ib) \geqslant (\Theta_\phi - \varepsilon)|b|, \tag{3.22}$$

where $A_\phi(a+ib) = \int_0^b \arg\phi(a+iu)du$. Invoking [98, Theorem 4.2(1)] gives, for any $a > 0$,

$$|W_\phi(a+ib)|^t = C_{\phi,a,t}\left(\frac{\phi(a)}{|\phi(a+ib)|}\right)^{\frac{t}{2}} e^{-tA_\phi(a+ib)},$$

where $C_{\phi,a,t} > 0$ is a constant depending only on ϕ, a and t. Since [98, Proposition 3.1(9)] gives that $|\phi(a+ib)| \geqslant \phi(a)$, it follows from the estimate for A_ϕ in (3.22) that, for ε small enough such that $\Theta_\phi t - \varepsilon > 0$,

$$|W_\phi(a+ib)|^t \stackrel{\infty}{=} \mathrm{O}\left(e^{-(\Theta_\phi t-\varepsilon)|b|}\right), \tag{3.23}$$

where the big-O estimate holds pointwise in a, and thus uniformly on bounded a-intervals. By similar arguments as given in the proof of Theorem 3.2.4(2) above, it follows that $\nu_t \in C_0^\infty(\mathbb{R}_+)$, and hence we have the Mellin-Barnes representation for ν_t

$$\nu_t(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-z} W_\phi^t(z)dz, \tag{3.24}$$

for any $c > 0$. To show that ν_t is analytic on the claimed sector it suffices to analytically extend the right-hand side of (3.24), which amounts to replacing x by a suitable complex number. Let $\varepsilon > 0$ be fixed and consider $w \in \mathbb{C}$ such that $|\arg w| < \Theta_\phi t - \varepsilon$. From the estimate (3.23) it follows that, for any $c > 0$ and $b \in \mathbb{R}$,

$$|w^{-(c+ib)} W_\phi^t(c+ib)| \leqslant e^{|b||\arg w|} |W_\phi(c+ib)|^t \overset{\infty}{=} \mathrm{O}\left(e^{-(\Theta_\phi t - \varepsilon - |\arg w|)|b|}\right),$$

and by choice of w the right-hand side is integrable in b. Thus the integrand on the right-hand side of (3.24) is well-defined for $|\arg w| < \Theta_\phi t - \varepsilon$, which by uniqueness of the analytic extension gives that $\nu_t \in \mathcal{A}(\Theta_\phi t - \varepsilon)$. Since $\varepsilon > 0$ is arbitrary we get $\nu_t \in \mathcal{A}(\Theta_\phi t)$, and thus for $t > \frac{\pi}{\Theta_\phi}$ we have $\nu_t \in \mathcal{A}(\pi)$.

3.4.3 Proofs for Section 3.2.5

The proof of Theorem 3.2.5 combines ideas from several different areas. Hence we first state some definitions, and detail some lemmas and propositions that will be useful in the proof. We say that a function $s : (a, \infty) \to (0, \infty)$, for some $a \geqslant -\infty$, is *self-neglecting* if

$$\lim_{u \to \infty} \frac{s(u + ws(u))}{s(u)} = 1, \quad \text{locally uniformly in } w \in \mathbb{R}.$$

Furthermore, we say a function $G : (a, \infty) \to \mathbb{R}$ is *asymptotically parabolic* if it is twice differentiable with $G'' > 0$ on (a, ∞), and if its scale function $s_G(u) = (G''(u))^{-\frac{1}{2}}$ is self-neglecting. Denote the set of asymptotically parabolic functions by $\mathcal{A}$ and note that it is a convex cone. A function $h : (a, \infty) \to (0, \infty)$ is said to be *flat* with respect to G if

$$\lim_{u \to \infty} \frac{h(u + ws_G(u))}{h(u)} = 1, \quad \text{locally uniformly in } w \in \mathbb{R}, \tag{3.25}$$

where s_G is the scale function of G. In the following lemma we collect some properties of flat and asymptotically parabolic functions.

Lemma 3.4.1. *Let $G \in \mathcal{A}$ and h be flat with respect to G.*

(1) The function $u \mapsto 1/h(u)$ is flat with respect to G.

(2) For any $c > 0$, the function $u \mapsto h(cu)$ is flat with respect to G.

(3) The identity function is flat with respect to G and, for any $\alpha > 0$, the function $u \mapsto h^{\alpha}(u)$ is flat with respect to G. In particular, for any $n \geqslant 0$, the function $u \mapsto u^n$ is flat with respect to G.

(4) The function h satisfies

$$\lim_{u\to\infty} \frac{\log h(u)}{G(u)} = 0.$$

(5) For any $c > 0$, the function $u \mapsto cG\left(\frac{u}{c}\right) \in \mathcal{A}$.

Proof. The first claim is obvious from the definition in (3.25). Let $c > 0$ and consider the function h_c defined by $h_c(u) = h(cu)$. Then, writing $v = cu$,

$$\lim_{u\to\infty} \frac{h_c(u + ws_G(u))}{h_c(u)} = \lim_{u\to\infty} \frac{h(cu + cws_G(cu))}{h(cu)} = \lim_{v\to\infty} \frac{h(v + cws_G(v))}{h(v)} = 1,$$

where the last limit follows from fact that (3.25) holds locally uniformly for $w \in \mathbb{R}$. For the third claim, note that $s_G(u) \overset{\infty}{=} \mathrm{o}(u)$, see Lemma 2.3.1, so that, locally uniformly in $w \in \mathbb{R}$,

$$\lim_{u\to\infty} \frac{u + ws_G(u)}{u} = 1 + w \lim_{u\to\infty} \frac{s_G(u)}{u} = 1.$$

The fact that, for $\alpha > 0$, $u \mapsto h^{\alpha}(u)$ is flat follows trivially from the definition, and the proof of the fourth item is essentially known in the literature, see again Lemma 2.3.1. Finally, for the proof of the last claim, write $\widetilde{G}(u) = cG_c\left(\frac{u}{c}\right)$ and $s_{\widetilde{G}}$ for the corresponding scale function. Then $s_{\widetilde{G}}(u) = \sqrt{c}s_G\left(\frac{u}{c}\right)$ so that, for $w \in \mathbb{R}$,

$$\frac{s_{\widetilde{G}}(u + ws_{\widetilde{G}}(u))}{s_{\widetilde{G}}(u)} = \frac{s_G\left(\frac{u}{c} + \sqrt{c}ws_G\left(\frac{u}{c}\right)\right)}{s_G\left(\frac{u}{c}\right)}$$

and the self-neglecting property of s_G carries over readily to $s_{\widetilde{G}}$. □

In the next lemma we collect some properties about the specific asymptotically parabolic functions that will play a role in the proof of Theorem 3.2.5. To state it we recall that the Legendre transform of a convex function $\psi : \mathbb{R} \to \mathbb{R}$, which we denote as $\mathcal{L}_\psi$, is given by

$$\mathcal{L}_\psi(y) = \sup_{u\in\mathbb{R}}\{uy - \psi(u)\}.$$

If in addition $\psi \in C^1(\mathbb{R})$ then the above supremum is achieved at the unique point $u = \psi'^{-1}(y)$, and hence

$$\mathcal{L}_\psi(y) = y\psi'^{-1}(y) - \psi(\psi'^{-1}(y)).$$

The variables u and y obeying the relations $y = \psi'(u)$ and $u = \psi'^{-1}(y)$ are called conjugate variables.

Lemma 3.4.2. *Let $\phi \in \mathcal{B}_\mathcal{J}$ be such that $\phi(\infty) = \infty$. Then the function $s_G : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$s_G(u) = \sqrt{\frac{\phi(u)}{\phi'(u)}}$$

is self-neglecting, and consequently $G \in \mathcal{A}$, where $G : (1, \infty) \to \mathbb{R}$ is the function defined by

$$G(u) = \int_1^u \log\phi(r)dr + \log\phi(1). \tag{3.26}$$

The Legendre transform of G is given by

$$\mathcal{L}_G(y) = \int_{\mathsf{k}}^{e^y} \frac{\varphi(r)}{r}dr - \int_{\mathsf{k}}^{\phi(1)} \frac{\varphi(r)}{r}dr$$

where $\varphi : [\mathsf{k}, \infty) \to [0, \infty)$ is the continuous inverse of ϕ, and y and u are conjugate variables related by $y = \log\phi(u)$ and $u = \varphi(e^y)$. Furthermore, $\mathcal{L}_G \in \mathcal{A}$.

Proof. The fact that s_G is self-neglecting was proved in [99, Proposition 5.40] under the additional condition that $\mathsf{k} = \phi(0) > 0$. However, an inspection of the proof reveals that this property is not crucial for the self-neglecting property of s_G. Differentiating G twice shows that s_G is indeed the scale function of G, and hence $G \in \mathcal{A}$.

Taking derivatives in (3.26) we get $G'(u) = \log \phi(u)$ so that the conjugate variables are $y = \log \phi(u)$ and $u = \varphi(e^y)$. Also, by integration by parts we can rewrite G as

$$G(u) = u \log \phi(u) - \int_1^u \frac{r\phi'(r)}{\phi(r)} dr.$$

Hence,

$$\begin{aligned}\mathcal{L}_G(y) = y\varphi(e^y) - G(\varphi(e^y)) &= \int_1^{\varphi(e^y)} \frac{r\phi'(r)}{\phi(r)} dr \\ &= \int_{\phi(1)}^{e^y} \frac{\varphi(r)}{r} dr = \int_{\mathsf{k}}^{e^y} \frac{\varphi(r)}{r} dr - \int_{\mathsf{k}}^{\phi(1)} \frac{\varphi(r)}{r} dr\end{aligned}$$

where the third equality follows by the change of variables $r = \varphi(w)$. Finally, the fact that $\mathcal{L}_G \in \mathcal{A}$ follows from a closure property of $\mathcal{A}$ with respect to the Legendre transform, see [10, Theorem 5.3].

□

In the final lemma before the proof we collect some properties concerning additive convolution, especially a stability property for Gaussian tails under additive convolution. We write $*$ for the additive convolution of suitable functions $f, g : \mathbb{R} \to \mathbb{R}$, that is

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy = \int_{-\infty}^{\infty} f(y)g(x-y)dy,$$

with the additive convolution of measures being defined similarly. A probability density f is said to have a *Gaussian tail* if $f(y) \overset{\infty}{\sim} \eta(y)e^{-\psi(y)}$ for some $\psi \in \mathcal{A}$ and some η flat with respect to ψ.

Lemma 3.4.3.

(1) Let $(\nu_t)_{t\geqslant 0}$ be a multiplicative convolution semigroup and let, for each $t > 0$, f_t be the pushforward measure under the map $x \mapsto \log x$. Then $(f_t)_{t\geqslant 0}$ is an additive convolution semigroup, i.e. for $t, s \geqslant 0$, $f_t * f_s = f_{t+s}$.

(2) *Let* $f, g \in \mathrm{L}^1(\mathbb{R})$ *be such that* $f(y) \overset{\infty}{\sim} e^{-\psi_1(y)}$ *and* $g(y) \overset{\infty}{\sim} e^{-\psi_2(y)}$*, for some* ψ_1, ψ_2 *with* $\lim_{y\to\infty} \psi_1'(y) = \lim_{y\to\infty} \psi_2'(y) = \infty$*. Then* $(f * g)(y) \overset{\infty}{\sim} (e^{-\psi_1} * e^{-\psi_2})(y)$*.*

(3) *Let* f *and* g *be probability densities with Gaussian tails, that is* $f(y) \overset{\infty}{\sim} \eta_1(y)e^{-\psi_1(y)}$ *and* $g(y) \overset{\infty}{\sim} \eta_2(y)e^{-\psi_2(y)}$*, and suppose that we have* $\lim_{y\to\infty} \psi_1'(y) = \lim_{y\to\infty} \psi_2'(y) = \infty$*. Then* $f * g$ *has a Gaussian tail, i.e.* $(f * g)(y) \overset{\infty}{\sim} \eta_0(y)e^{-\psi_0(y)}$ *for some* $\psi_0 \in \mathcal{A}$ *and some* η_0 *flat with respect to* ψ_0*. Specifically, writing* $y(u) = q_1 + q_2 = \psi_1'^{-1}(u) + \psi_2'^{-1}(u)$*, we have*

$$\psi_0(y) = \psi_1(q_1) + \psi_2(q_2)$$
$$\eta_0(y) = \frac{\sqrt{2\pi} s_{\psi_1}(q_1)\eta_1(q_1)s_{\psi_2}(q_2)\eta_2(q_2)}{\sqrt{s_{\psi_1}^2(q_1) + s_{\psi_2}^2(q_2)}}.$$

In particular, for $d \geqslant 1$*, the* d*-fold convolution of* f *with itself* f^{*d} *satisfies*

$$f^{*d}(y) \overset{\infty}{\sim} \frac{1}{\sqrt{d}} \left(\frac{2\pi}{\psi_1''\left(\frac{y}{d}\right)} \right)^{\frac{d-1}{2}} f\left(\frac{y}{d}\right)^d .$$

Before giving the proof, we note that Item (2) of Lemma 3.4.3 gives conditions under which the asymptotics of the convolution of integrable functions can be identified from the asymptotics of the functions themselves. On the other hand, Item (3) states that Gaussian tails are closed under additive convolution and allows one to identify the asymptotic explicitly, this latter feature being particularly useful. The statement of Lemma 3.4.3(3) is the content of [10, Theorem 1.1 and (1.11)], and our aim, in incorporating it as an item of a lemma, is merely to improve the clarity and presentation of the proof of Theorem 3.2.5.

Proof. The first claim is straightforward. The proof of Item (2) is in the spirit of the proof of [10, Proposition 2.2]. Since f and g are asymptotic to positive functions it follows that they are themselves eventually positive. This, and the other properties of ψ_1 and ψ_2, allows us to choose $a > 0$ large enough such that: (1) both ψ_1 and ψ_2 are well-defined on (a,∞), (2) $\psi_1', \psi_2' > 0$ on (a,∞), (3) $\int_{-\infty}^{a} |g(y)|dy \neq 0$ and $\int_{-\infty}^{a} |f(y)|dy \neq 0$, and (4) $c_g = \int_{a+1}^{a+2} g(x)dx > 0$ and $c_f = \int_{a+1}^{a+2} f(x)dx > 0$. For $x > 2a$,

$$(f * g)(x) = \int_{a}^{x-a} f(x-y)g(y)dy + \int_{-\infty}^{a} f(x-y)g(y)dy + \int_{-\infty}^{a} f(y)g(x-y)dy,$$

so by symmetry it suffices to show that $\int_{-\infty}^{a} e^{-\psi_1(x-y)}g(y)dy$ is of order $\mathrm{o}\left(\int_{a}^{x-a} f(x-y)g(y)dy\right)$ at infinity. Since $\psi_1' > 0$ on (a,∞)

$$\left|\int_{-\infty}^{a} e^{-\psi_1(x-y)}g(y)dy\right| \leqslant Ce^{-\psi_1(x-a)}$$

with $C = \int_{-\infty}^{a} |g(y)|dy \neq 0$ a constant. By the mean value theorem,

$$\begin{aligned}\int_{a+1}^{a+2} e^{-\psi_1(x-y)}g(y)dy &\geqslant c_g e^{-\psi_1(x-a-1)} = c_g e^{-\psi_1(x-a)} e^{\psi_1'(x-z)} \\ &\geqslant \frac{c_g}{C} e^{\psi_1'(x-z)} \left|\int_{-\infty}^{a} e^{-\psi_1(x-y)}g(y)dy\right|,\end{aligned}$$

with $|z| \leqslant a+1$, and letting $x \to \infty$ finishes the proof of the second claim. Finally, Item (3) is the content of [10, Theorem 1.1 and (1.11)]. □

Proof of Theorem 3.2.5(1)

For convenience we write α in place of $\mathfrak{t}$ and thus our assumption is that ϕ is a Bernstein function such that $\phi(\infty) = \infty$ and $\phi^\alpha \in \mathcal{B}_{\mathcal{J}}$, for all $\alpha \in (0,1)$. We write $(\nu_t)_{t\geqslant 0}$ for the Berg-Urbanik semigroup associated to ϕ and, for any $\alpha \in (0,1)$, let $(\overline{\nu}_t)_{t>0}$ denote the Berg-Urbanik semigroup associated to ϕ^α. Then, for $n \geqslant 0$ and any $\alpha \in (0,1)$, we have by the moment determinacy of any Berg-Urbanik semigroup up to time 2 that

$$\mathcal{M}_{\overline{\nu}_1}(n) = \prod_{k=1}^{n} \phi^\alpha(k) = \left(\prod_{k=1}^{n} \phi(k)\right)^\alpha = \mathcal{M}_{\nu_\alpha}(n),$$

and applying [17, Theorem 2.2] then gives that $(\overline{\nu}_t)_{t\geqslant 0} = (\nu_{\alpha t})_{t\geqslant 0}$. Since $\phi^\alpha \in \mathcal{B}_{\mathcal{J}}$, for any $\alpha \in (0,1)$, and plainly $\phi(\infty) = \infty$ implies $\phi^\alpha(\infty) = \infty$, we conclude that $\mathtt{N}_{\phi^\alpha} = \infty$. Invoking Theorem 3.2.4(2) then yields, for any $t > 0$ and $\alpha \in (0,1)$, $\overline{\nu}_t \in C_0^\infty(\mathbb{R}_+)$, from which we deduce that $\nu_t \in C_0^\infty(\mathbb{R}_+)$, where $\nu_t(dx) = \nu_t(x)dx$, $x, t > 0$. Since $\phi^\alpha \in \mathcal{B}_\mathcal{J}$ with $\phi^\alpha(\infty) = \infty$ we may apply [99, Theorem 5.5] to obtain, for any $n \geqslant 0$, the asymptotic relation

$$\overline{\nu}_1^{(n)}(x) = \nu_\alpha^{(n)}(x) \overset{\infty}{\sim} (-1)^n \frac{C_{\phi,\alpha}}{\sqrt{2\pi}} x^{-n} \varphi_\alpha^n(x) \sqrt{\varphi_\alpha'(x)} e^{-\int_{\mathsf{k}^\alpha}^x \frac{\varphi_\alpha(y)}{y} dy}$$

where $C_{\phi,\alpha} > 0$ is a constant depending only on ϕ and α, $\varphi_\alpha : [\mathsf{k}^\alpha, \infty) \to [0, \infty)$ is the continuous inverse of the function $u \mapsto \phi^\alpha(u)$ and $\mathsf{k} = \phi(0)$. The constant $C_{\phi,\alpha}$ may be identified as C_ϕ^α, where $C_\phi > 0$ is a constant depending only on ϕ, cf. [99, Theorem 5.1(2)], and plainly $\varphi_\alpha(u) = \varphi(u^{\frac{1}{\alpha}})$, where $\varphi : [\mathsf{k}, \infty) \to [0, \infty)$ is the continuous inverse of ϕ. Thus, by some routine calculations, we conclude that

$$\nu_\alpha^{(n)}(x) \overset{\infty}{\sim} (-1)^n \frac{C_\phi^\alpha}{\sqrt{2\pi\alpha}} x^{-n-\frac{1}{2}} \varphi^n(x^{\frac{1}{\alpha}}) \sqrt{x^{\frac{1}{\alpha}} \varphi'(x^{\frac{1}{\alpha}})} e^{-\alpha \int_{\mathsf{k}}^{x^{\frac{1}{\alpha}}} \frac{\varphi(r)}{r} dr}. \tag{3.27}$$

Since $\alpha \in (0, 1)$ is arbitrary this proves the claimed asymptotic for any $n \geqslant 0$ and $t \in (0, 1)$.

We proceed by showing that for $n = 0$, i.e. for the density $\nu_t(x)$ itself, the claimed asymptotic holds for all $t > 0$, and then extend this to the case when $n \geqslant 1$. To this end we define, for $y \in \mathbb{R}$ and $t > 0$, $f_t(y) = e^y \nu_t(e^y)$ and set $f_0 = \delta_0$. Then by Lemma 3.4.3(1) $(f_t)_{t \geqslant 0}$ is an additive convolution semigroup of probability densities, and from (3.27) together with some simple algebra we get, for $\alpha \in (0, 1)$,

$$f_\alpha(y) \overset{\infty}{\sim} \frac{C_\phi^\alpha}{\sqrt{2\pi\alpha}} e^{\frac{y}{2}} \sqrt{e^{\frac{y}{\alpha}} \varphi'(e^{\frac{y}{\alpha}})} e^{-\alpha \int_{\mathsf{k}}^{e^{\frac{y}{\alpha}}} \frac{\varphi(r)}{r} dr}. \tag{3.28}$$

Let us write

$$\overline{\psi}(y) = \int_{\mathsf{k}}^{e^y} \frac{\varphi(r)}{r} dr = \mathcal{L}_G(y) + \int_{\mathsf{k}}^{\phi(1)} \frac{\varphi(r)}{r} dr$$

where $\mathcal{L}_G$ is the Legendre transform of the function G is defined in (3.26). From Lemma 3.4.2 we get that $\overline{\psi} \in \mathcal{A}$, and writing ψ for the function

$$\psi(y) = \alpha \int_{\mathsf{k}}^{e^{\frac{y}{\alpha}}} \frac{\varphi(r)}{r} dr = \alpha \overline{\psi}\left(\frac{y}{\alpha}\right), \tag{3.29}$$

we get from Lemma 3.4.1(5) that $\psi \in \mathcal{A}$. A straightforward calculation gives that its scale function s_ψ takes the form

$$s_\psi(y) = \sqrt{\frac{\alpha}{e^{\frac{y}{\alpha}}\varphi'(e^{\frac{y}{\alpha}})}},$$

so combining Items (1) and (2) of Lemma 3.4.1 we get that $\sqrt{e^{\frac{y}{\alpha}}\varphi'(e^{\frac{y}{\alpha}})}$ is flat with respect to ψ. Furthermore, as ϕ' is non-increasing positive, $\lim_{u\to\infty}\phi'(u) < \infty$ and thus we have

$$\lim_{y\to\infty} s_\psi(y) = \sqrt{\alpha}\lim_{y\to\infty}\frac{1}{\sqrt{e^{\frac{y}{\alpha}}\varphi'(e^{\frac{y}{\alpha}})}} = \sqrt{\alpha}\lim_{y\to\infty} e^{-\frac{y}{2\alpha}}\sqrt{\phi'(\varphi(e^{\frac{y}{\alpha}}))} = 0.$$

Hence

$$\lim_{y\to\infty}\exp\left(\frac{w s_\psi(y)}{2}\right) = 1, \quad \text{locally uniformly in } w \in \mathbb{R},$$

which shows that $e^{\frac{y}{2}}$ is flat with respect to ψ. Constants are trivially flat with respect to ψ, so that putting all of these observations together we get that all the terms in front of the exponential in (3.28) are flat with respect to ψ. Hence, for each $\alpha \in (0,1)$, f_α has a Gaussian tail.

Now we may invoke the second part of Lemma 3.4.3(3), which states that the property of having a Gaussian tail is stable under additive convolution, to obtain for any $d \in \mathbb{N}$

$$f_{d\alpha}(y) \overset{\infty}{\sim} \frac{1}{\sqrt{d}}\left(\frac{2\pi}{\psi''\left(\frac{y}{d}\right)}\right)^{\frac{d-1}{2}} f_\alpha\left(\frac{y}{d}\right)^d = \frac{1}{\sqrt{d}}\left(\frac{2\pi\alpha}{e^{\frac{y}{\alpha d}}\varphi'(e^{\frac{y}{\alpha d}})}\right)^{\frac{d-1}{2}} f_\alpha\left(\frac{y}{d}\right)^d.$$

Since for any $t > 0$ we can find $\alpha \in (0,1)$ and $d \in \mathbb{N}$ such that $t = \alpha d$ we get from the above relation the asymptotic of f_t for all $t > 0$. Hence, after performing some straightforward computations and changing variables again, we get that for any $t > 0$,

$$\nu_t(x) \overset{\infty}{\sim} \frac{C_\phi^t}{\sqrt{2\pi t}}\sqrt{x^{\frac{1-t}{t}}\varphi'(x^{\frac{1}{t}})}e^{-t\int_{\mathtt{k}}^{x^{\frac{1}{t}}}\frac{\varphi(r)}{r}dr}, \tag{3.30}$$

which proves the claim for $n = 0$.

Next, suppose that $n \geqslant 1$. A straightforward application of the chain rule gives that $f_\alpha^{(n)}(y) = (e^y\nu_\alpha(e^y))^{(n)}$ is a linear combination of terms of the form $e^{(k+1)y}\nu_\alpha^{(k)}(e^y)$, for

$0 \leqslant k \leqslant n$. However, from (3.27) we deduce that, for large y, the term $e^{(n+1)y}\nu_\alpha^{(n)}(e^y)$ grows faster than all terms of lower order. Therefore,

$$f_\alpha^{(n)}(y) \overset{\infty}{\sim} e^{(n+1)y}\nu_\alpha^{(n)}(e^y) \overset{\infty}{\sim} (-1)^n \frac{C_\phi^\alpha}{\sqrt{2\pi\alpha}} e^{\frac{y}{2}}\varphi^n(e^{\frac{y}{\alpha}})\sqrt{e^{\frac{y}{\alpha}}\varphi'(e^{\frac{y}{\alpha}})}e^{-\alpha\int_{\mathsf{k}}^{e^{\frac{y}{\alpha}}}\frac{\varphi(r)}{r}dr} \tag{3.31}$$

and the asymptotic on the right-hand side is obtained from the one in (3.27) after changing variables. From the right-hand side of (3.31) it is apparent that the mapping $y \mapsto (-1)^n f_\alpha^{(n)}(y)$ is eventually positive, so that there exists $a_n \in \mathbb{R}$ (depending on n) such that $\mathsf{f}_{\alpha,n}(y) = (-1)^n f_\alpha^{(n)}(y)\mathbb{I}_{\{y>a_n\}}$ is a positive function. Since $y \mapsto \varphi(e^{\frac{y}{\alpha}})$ is the derivative of ψ, which we recall from earlier denotes the function appearing within the exponential in (3.31), we have from [10, Proposition 5.8] that $y \mapsto \varphi(e^{\frac{y}{\alpha}})$ is flat with respect to ψ, and combined with Lemma 3.4.1(3) this gives that $y \mapsto \varphi^n(e^{\frac{y}{\alpha}})$ is flat with respect to ψ. Thus, once again all terms in front of the exponential in (3.31) are flat with respect to ψ. Let $\varepsilon \in (0, \alpha)$ so that, from Lemma 3.4.1(4) applied to (3.28), we deduce the estimate

$$\mathsf{f}_{\alpha,n}(y) \overset{\infty}{=} \mathrm{O}\left(e^{-(\alpha-\varepsilon)\int_{\mathsf{k}}^{e^{\frac{y}{\alpha}}}\frac{\varphi(r)}{r}dr}\right). \tag{3.32}$$

Then (3.28) allows us to identify the right-hand side of (3.32) as the dominant term in the asymptotic for the probability density $f_{\alpha-\varepsilon}(y) = e^y\nu_{\alpha-\varepsilon}(e^y)$, see (3.28). Indeed, the fact the function inside the big-O estimate of (3.32) term dominates all others in (3.28) is immediate, as the term in front of the exponential is increasing at infinity. Noting that dilating a function does not affect its integrability, we conclude that, for any $\alpha \in (0, 1)$ and $n \geqslant 1$, the function $\mathsf{f}_{\alpha,n}$ is integrable. In particular, for each $\alpha \in (0, 1)$ and $n \geqslant 1$ there exists a constant $c_{\alpha,n} > 0$ such that $c_{\alpha,n}\mathsf{f}_{\alpha,n}$ is a probability density.

Now, let us write $t = \alpha + \tau$, where $\alpha \in (0, 1)$ and $\tau > 0$. If, for any $n \geqslant 0$, $f_\alpha^{(n)} \in \mathrm{L}^2(\mathbb{R})$, and $f_\tau \in \mathrm{L}^2(\mathbb{R})$, then a standard result (see [52, Chapter 8, Ex. 8 & 9]) allows us to interchange differentiation and convolution to write that

$$f_t^{(n)}(y) = (f_\alpha^{(n)} * f_\tau)(y),\ y \in \mathbb{R}. \tag{3.33}$$

To this end, let $\boldsymbol{t} > 0$ and observe that

$$\int_{-\infty}^{\infty} \left(e^{(n+1)y} \nu_{\boldsymbol{t}}^{(n)}(e^y)\right)^2 dy = \int_0^{\infty} \left(x^{n+\frac{1}{2}} \nu_{\boldsymbol{t}}^{(n)}(x)\right)^2 dx$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\Gamma(1+n+ib)|^2}{|\Gamma(1+ib)|^2} \left|W_{\phi}^{\boldsymbol{t}}(1+ib)\right|^2 db, \tag{3.34}$$

where the first equality follows from a change of variables, and the second is a combination of the Parseval formula for the Mellin transform applied to the function $x \mapsto x^{n+\frac{1}{2}} \nu_{\boldsymbol{t}}^{(n)}(x)$ combined with Theorem 3.2.3. By [98, Theorem 4.2(3)(c)], the fact that $\phi(\infty) = \infty$ with $\phi^{\alpha} \in \mathcal{B}_{\mathcal{J}}$ implies that $b \mapsto |W_{\phi}^{\boldsymbol{t}}(1+ib)|$ decays faster than any polynomial along the real line. Next, we recall Stirling's formula for the gamma function, for any $a + ib$ with $a > 0$ fixed

$$|\Gamma(a+ib)| \overset{\infty}{\sim} C_a |b|^{a-\frac{1}{2}} e^{-\frac{\pi}{2}|b|} \tag{3.35}$$

for some constant $C_a > 0$. Hence, the term in (3.34) involving the ratio of gamma functions grows like $|b|^{2n+2}$, which by the aforementioned decay properties of $W_{\phi}^{\boldsymbol{t}}$ gives that the integral in (3.34) is finite. Since $f_{\tau}^{(n)}(y) = (e^y \nu_{\alpha}(e^y))^{(n)}$ is a linear combination of functions of the form $e^{(k+1)y} \nu_{\alpha}^{(k)}(e^y)$, for $k \leqslant n$, we get that $f_{\alpha}^{(n)} \in \mathrm{L}^2(\mathbb{R})$ for any $n \geqslant 0$, and that $f_{\tau} \in \mathrm{L}^2(\mathbb{R})$. Hence the equality in (3.33) is justified.

Next we aim to use a combination of Lemma 3.4.3(2) together with (3.33) in order to show that $f_t^{(n)}$ has a Gaussian tail. From (3.31) we have

$$(-1)^n f_{\alpha}^{(n)}(y) \overset{\infty}{\sim} h(y) e^{-\psi(y)}$$

where the function ψ is defined in (3.29), and h denotes the function consisting of all terms in front of the exponential of (3.31). Since h is flat with respect to ψ we know, by [10, Proposition 3.2], that there exists $\chi \in C^{\infty}(\mathbb{R})$ such that $\chi(y) \overset{\infty}{\sim} h(y)$ and $s_{\psi}(y)\chi'(y) \overset{\infty}{=} \mathrm{o}(\chi(y))$. Further, from Proposition 5.8 in the aforementioned paper $\lim_{y\to\infty} s_{\psi}(y)\psi'(y) = \infty$. Using these facts we get

$$\lim_{y\to\infty} \frac{(\log \chi(y))'}{\psi'(y)} = \lim_{y\to\infty} \frac{\chi'(y)}{\chi(y)\psi'(y)} = \lim_{y\to\infty} \frac{s_{\psi}(y)\chi'(y)}{\chi(y)} \frac{1}{s_{\psi}(y)\psi'(y)} = 0,$$

which is enough to show that $f_\alpha^{(n)}$ satisfies the assumptions of Lemma 3.4.3(2). Since the arguments for f_τ are similar we have, invoking Lemma 3.4.3(2), that

$$(-1)^n c_{\alpha,n} f_t^{(n)}(y) \overset{\infty}{\sim} (c_{\alpha,n}\mathsf{f}_{\alpha,n} * f_\tau)(y),$$

with both $c_{\alpha,n}\mathsf{f}_{\alpha,n}$ and f_τ having Gaussian tails. Applying Lemma 3.4.3(3) again we conclude that $c_{\alpha,n}\mathsf{f}_{\alpha,n} * f_\tau$ has a Gaussian tail, and hence $f_t^{(n)}(y) \overset{\infty}{\sim} (-1)^n \eta_0(y) e^{-\psi_0(y)}$, where $\psi_0 \in \mathcal{A}$ and η_0 is flat with respect to ψ_0.

To conclude the proof it remains to identify η_0 and ψ_0, which may be computed as described in Lemma 3.4.3(3), using a combination of (3.31) and, after changing variables, (3.30). As in the lemma, we write $y(u) = q_1(u) + q_2(u) = \alpha \log \phi(u) + \tau \log \phi(u) = t \log \phi(u)$, where the second equality serves as definition of q_1 and q_2, and the last equality defines the conjugate variables y and u. Using this notation it is straightforward to conclude that

$$\psi_0(y) = \alpha \int_{\mathsf{k}}^{e^{\frac{q_1}{\alpha}}} \frac{\varphi(r)}{r} dr + \tau \int_{\mathsf{k}}^{e^{\frac{q_2}{\tau}}} \frac{\varphi(r)}{r} dr = (\alpha + \tau) \int_{\mathsf{k}}^{\phi(u)} \frac{\varphi(r)}{r} dr = t \int_{\mathsf{k}}^{e^{\frac{y}{t}}} \frac{\varphi(r)}{r} dr.$$

The associated scale function s_{ψ_0} is then

$$s_{\psi_0}(y) = \sqrt{\frac{t}{e^{\frac{y}{t}} \varphi'(e^{\frac{y}{t}})}}.$$

Let η_1 and η_2 denote the flat terms, while ψ_1 and ψ_2 denote the asymptotically parabolic terms, in the Gaussian tails of $c_{\alpha,n}\mathsf{f}_{\alpha,n}$ and f_τ respectively. Then,

$$\eta_1(q_1(u)) = \frac{C_\phi^\alpha}{\sqrt{2\pi\alpha}} (\phi(u))^{\frac{\alpha}{2}} u^n \sqrt{\phi(u)\varphi'(\phi(u))},$$

and

$$\eta_2(q_2(u)) = \frac{C_\phi^\tau}{\sqrt{2\pi\tau}} (\phi(u))^{\frac{\tau}{2}} \sqrt{\phi(u)\varphi'(\phi(u))}.$$

Furthermore,

$$s_{\psi_1}(q_1(u)) = \sqrt{\frac{\alpha}{\phi(u)\varphi'(\phi(u))}} \quad \text{and} \quad s_{\psi_2}(q_2(u)) = \sqrt{\frac{\tau}{\phi(u)\varphi'(\phi(u))}}$$

where s_{ψ_1} and s_{ψ_2} are the scale functions of ψ_1 and ψ_2, respectively. Putting all of these observations together we get that η_0 can be written, after canceling like terms, as

$$\eta_0(y) = \frac{C_\phi^{(\alpha+\tau)}\sqrt{2\pi}}{\sqrt{2\pi\alpha}\sqrt{2\pi\tau}} e^{\frac{y}{2}}\sqrt{\alpha}\sqrt{\tau}\frac{\sqrt{e^{\frac{y}{t}}\varphi'(e^{\frac{y}{t}})}}{\sqrt{t}}\varphi^n(e^{\frac{y}{t}}) = \frac{C_\phi^t}{\sqrt{2\pi t}} e^{\frac{y}{2}}\varphi^n(e^{\frac{y}{t}})\sqrt{e^{\frac{y}{t}}\varphi'(e^{\frac{y}{t}})}.$$

This gives us $f_t^{(n)}(y) \overset{\infty}{\sim} (-1)^n\eta_0(y)e^{-\psi_0(y)} \overset{\infty}{\sim} e^{(n+1)y}\nu_t^{(n)}(e^y)$, and changing variables again, we finally obtain the claimed asymptotic

$$\nu_t^{(n)}(x) \overset{\infty}{\sim} (-1)^n\frac{C_\phi^t}{\sqrt{2\pi t}} x^{-n}\varphi^n(x^{\frac{1}{t}})\sqrt{x^{\frac{1-t}{t}}\varphi'(x^{\frac{1}{t}})}e^{-t\int_{\mathsf{k}}^{x^{\frac{1}{t}}}\frac{\varphi(r)}{r}dr},$$

for any $n \geqslant 0$ and $t > 0$, which completes the proof.

Proof of Theorem 3.2.5(1) and Theorem 3.2.5(2)

The proof is the same for [99, Theorem 5.5(1)] and [99, Theorem 5.5(2)], but we give the arguments for sake of completeness. Suppose that $\mathsf{d} > 0$, so that

$$\phi(u) = \mathsf{k} + \mathsf{d}u + u\int_0^\infty e^{-uy}\overline{\mu}(y)dy.$$

Then, invoking [99, Proposition 4.1(3)] we have $\phi(u) \overset{\infty}{\sim} \mathsf{d}u$ and hence $\varphi(u) \overset{\infty}{\sim} \mathsf{d}^{-1}u$. Furthermore, differentiating the identity $u = \phi(\varphi(u))$ gives $\varphi'(u) = \frac{1}{\phi'(\varphi(u))}$ and since, by the monotone density theorem, see [22, Theorem 1.7.2], $\phi'(u) \overset{\infty}{\sim} \mathsf{d}$, we get that $\varphi'(u) \overset{\infty}{\sim} \mathsf{d}^{-1}$. Next, as $u = \phi(\varphi(u))$ we have, on $[\mathsf{k}, \infty)$,

$$u = \mathsf{k} + \mathsf{d}\varphi(u) + \varphi(u)\int_0^\infty e^{-\varphi(u)y}\overline{\mu}(y)dy = \mathsf{k} + \mathsf{d}\varphi(u) + E(u)$$

where the last equality serves to define the function E. By dominated convergence we have that $\lim_{u\to\infty}\int_0^\infty e^{-\varphi(u)y}\overline{\mu}(y)dy = 0$ which, together with $\varphi(u) \overset{\infty}{\sim} \mathsf{d}^{-1}u$, shows that $E(u) = \mathrm{o}(u)$. Re-arranging, we obtain $\varphi(y) = \mathsf{d}^{-1}(u - \mathsf{k} - E(u))$, so that substituting all of these quantities into the identities (3.12) and (3.13) proves Item (1).

Next, assume that $\phi(u) \overset{\infty}{\sim} C_\alpha u^\alpha$, with $C_\alpha > 0$ a constant and $\alpha \in (0, 1)$. A standard result from regular variation theory gives that $\varphi(u) \overset{\infty}{\sim} C_\alpha^{-\frac{1}{\alpha}} u^{\frac{1}{\alpha}}$, see e.g. [22, Theorem 1.5.12]. This allows us to define $H(u) = C_\alpha^{-\frac{1}{\alpha}} u^{\frac{1}{\alpha}} - \varphi(u)$, so that $H(u) = \mathrm{o}(u^{\frac{1}{\alpha}})$. Next, the monotonicity of ϕ' allows us to again invoke the monotone density theorem to conclude that $\phi'(u) \overset{\infty}{\sim} C_\alpha \alpha u^{\alpha-1}$, see again [22, Theorem 1.7.2]. Combining these two statements with the identity $\varphi'(u) = \frac{1}{\phi'(\varphi(u))}$ yields the asymptotic $\varphi'(u) \overset{\infty}{\sim} \alpha^{-1} C_\alpha^{-\frac{1}{\alpha}} u^{\frac{1}{\alpha}-1}$. Finally, substituting these asymptotics proves the claim.

3.4.4 Proofs for Section 3.2.1

Before beginning with the proofs we state some preliminary results that will be used in the proof of Theorem 3.2.1(2) and Theorem 3.2.1(3).

Proposition 3.4.1. *For $\alpha \in (0, 1)$ and $\mathfrak{m} \geqslant 0$, let $\phi_{\alpha,\mathfrak{m}} : [0, \infty) \to [0, \infty)$ be defined by $\phi_{\alpha,\mathfrak{m}}(u) = (u + \mathfrak{m})^\alpha$.*

(1) For any $\alpha \in (0, 1)$ and $\mathfrak{m} \geqslant 0$, $\phi_{\alpha,\mathfrak{m}}$ is a complete Bernstein function.

(2) The potential measure of $\phi_{\alpha,\mathfrak{m}}$ admits a density, denoted by $U_{\alpha,\mathfrak{m}}$, given by

$$U_{\alpha,\mathfrak{m}}(y) = \frac{1}{\Gamma(\alpha)} e^{-\mathfrak{m}y} y^{\alpha-1}.$$

Furthermore, $U_{\alpha,\mathfrak{m}}$ is non-increasing, convex and solves, on $\mathbb{R}_+$, the differential equation

$$U'_{\alpha,\mathfrak{m}} = -U_{\alpha,\mathfrak{m}}(y) \left(\mathfrak{m} + \frac{1-\alpha}{y}\right).$$

(3) Let $\phi \in \mathcal{B}_d$, i.e. $\mathsf{d} > 0$. Then, for any $\alpha \in (0, 1)$,

$$y_\alpha = \inf\{y \geqslant 0; y\overline{\mu}(y) > \mathsf{d}(1-\alpha)\} \in (0, \infty],$$

and, for any $\mathfrak{m}$ such that $\mathsf{d}\mathfrak{m} \geqslant \overline{\mu}(\frac{y_\alpha}{2}) + \mathsf{k}$, we have that $\frac{\phi}{\phi_{\alpha,\mathfrak{m}}} \in \mathcal{B}$.

Proof. The fact that $\phi_{\alpha,\mathfrak{m}}$ is a complete Bernstein function is straightforward and was also mentioned in Remark 3.2.1. To show that $U_{\alpha,\mathfrak{m}}$ defined as above is the density of the potential measure of $\phi_{\alpha,\mathfrak{m}}$ we observe that

$$\frac{1}{u^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-uy} y^{\alpha-1} dy,$$

and then substitute $u + \mathfrak{m}$ for u. The claimed properties of $U_{\alpha,\mathfrak{m}}$ can then be verified by straightforward calculations. The proof of the last claim is, mutatis mutandis, the same as the one given for [99, Proposition 4.4(2)], so we omit it here. We note that the proof of [99, Proposition 4.4] does not explicitly use the fact that the Lévy measure of ϕ has a non-increasing density, and hence this restriction can be removed. Furthermore, we have modified y_α and the condition on $\mathfrak{m}$ to suit our potential measure $U_{\alpha,\mathfrak{m}}$. □

We write, for two functions f and g, $f(x) \overset{\infty}{\asymp} g(x)$ if $f(x) \overset{\infty}{=} \mathrm{O}(g(x))$ and $g(x) \overset{\infty}{=} \mathrm{O}(f(x))$. In the following theorem we rephrase, in the context of Berg-Urbanik semigroups, the Abelian type criterion for moment indeterminacy given in Theorem 2.1.2, which we use in the proof of Theorem 3.2.1(3).

Theorem 3.4.1. *Let $(\nu_t)_{t\geqslant 0}$ be a Berg-Urbanik semigroup and suppose that, for some $t > 0$, $\nu_t(dx) = \nu_t(x)dx$, $x > 0$, and*

$$\nu_t(x) \overset{\infty}{\asymp} e^{-G(\log x)},$$

with $G \in \mathcal{A}$ satisfying $\lim_{y\to\infty} G'(y)e^{-\frac{y}{2}} < \infty$. Then, writing γ for the inverse of the continuous, increasing function G',

$$\sum_{n=n_0}^{\infty} e^{-\frac{\gamma(n)}{2}} < \infty, \quad \textit{for some} \quad n_0 \geqslant 1 \qquad \Longleftrightarrow \qquad \nu_t \textit{ is moment indeterminate.}$$

Proof of Theorem 3.2.1(1)

First, invoking [99, Theorem 5.1(2)] we get

$$\mathcal{M}_\nu(n) = W_\phi(n+1) \overset{\infty}{\sim} C_\phi \sqrt{\phi(n)} e^{G(n)}$$

where $G(n) = \int_1^n \log \phi(r) dr$ and $C_\phi > 0$ is a constant depending only on ϕ. Integrating G by parts, for any $t > 0$ and $n \geqslant 1$, gives us

$$\frac{t}{2n} G(n) = \frac{t}{2} \log \phi(n) - \frac{t}{2n} \left(\log \phi(1) + \int_1^n u \frac{\phi'(u)}{\phi(u)} du \right).$$

Consequently, for some $C_1 > 0$ a constant, we have

$$\sum_{n=1}^{\infty} W_\phi^{-\frac{t}{2n}}(n+1) \geqslant$$
$$C_1 \sum_{n=1}^{\infty} \exp\left[-\frac{t}{2} \left(\log \phi(n) + \frac{1}{2n} \log \phi(n) \right) \right] \exp\left[\frac{t}{2n} \left(\log \phi(1) + \int_1^n u \frac{\phi'(u)}{\phi(u)} du \right) \right]. \tag{3.36}$$

The estimate $\phi(n) \overset{\infty}{=} \mathrm{O}(n)$, see e.g. [99, Proposition 4.1(3)], gives $\log \phi(n) \overset{\infty}{=} \mathrm{o}(n)$, which together with the positivity of the terms within the second exponential in (3.36) allows us to obtain, for $C_2 > 0$ a constant, the bound

$$\sum_{n=1}^{\infty} W_\phi^{-\frac{t}{2n}}(n+1) \geqslant C_1 e^{-C_2 t} \sum_{n=1}^{\infty} \phi^{-\frac{t}{2}}(n),$$

so to prove moment determinacy it suffices to show the divergence of this latter series. Let $\beta > \beta_\phi$. By definition of β_ϕ, $\phi(u) \overset{\infty}{=} \mathrm{O}(u^\beta)$, so that for some constant $C_3 > 0$

$$\sum_{n=1}^{\infty} \phi^{-\frac{t}{2}}(n) \geqslant C_3 \sum_{n=1}^{\infty} n^{-\frac{t\beta}{2}}.$$

The latter series diverges if and only if $t\beta \leqslant 2$, whence the moment determinacy of ν_t for any $t \leqslant \frac{2}{\beta} < \frac{2}{\beta_\phi}$. Since $\beta > \beta_\phi$ is arbitrary we conclude that $\mathfrak{T}_\phi \geqslant \frac{2}{\beta_\phi}$ if $\beta_\phi > 0$ and $\mathfrak{T}_\phi = \infty$ for $\beta_\phi = 0$. Finally, if $\overline{\lim}_{u \to \infty} u^{-\beta_\phi} \phi(u) < \infty$ then we may choose $\beta = \beta_\phi$ and apply the above argument to conclude that $\nu_{\mathfrak{T}_\phi}$ is moment determinate.

Proof of Theorem 3.2.1(3)

It suffices to treat the case when $\delta_\phi \in (0,1]$, since otherwise the claimed right-hand inequality in (3.8) is trivial. Therefore we assume also that $0 < \delta_\phi \leqslant \beta_\phi \leqslant 1$, and $\delta_\phi > 0$ is easily seen to imply that $\phi(\infty) = \infty$. Invoking Theorem 3.2.5 we get that, for any $t > 0$,

$$\nu_t(x) \overset{\infty}{\sim} \frac{C_\phi^t}{\sqrt{2\pi t}} x^{\frac{1-t}{2t}} \sqrt{\varphi'(x^{\frac{1}{t}})} e^{-t\int_{\mathsf{k}}^{x^{\frac{1}{t}}} \frac{\varphi(r)}{r} dr}.$$

Let $b(\log x)$ denote all the terms in front of the exponential and set $\overline{G}(\log x)$ for the function within the exponential on the right-hand of the above asymptotic relation. It was shown in the proof of Theorem 3.2.5 that b is flat with respect to $\overline{G}$, and thus, by Lemma 3.4.1(4) we have that $b(\log x) \overset{\infty}{=} \mathrm{o}(\overline{G}(\log x))$. Hence, for any $c \in (0,t)$ fixed we get that

$$\nu_t(x) \overset{\infty}{\asymp} e^{-G(\log x)}$$

where

$$G(\log x) = (t-c)\int_{\mathsf{k}}^{x^{\frac{1}{t}}} \frac{\varphi(r)}{r} dr.$$

From Lemma 3.4.2 it follows that $G \in \mathcal{A}$ and a simple calculation, after substituting $y = \log x$, gives that

$$G'(y) = \frac{(t-c)}{t}\varphi(e^{\frac{y}{t}}) = \mathrm{t}\varphi(e^{\frac{y}{t}})$$

where we write $\boldsymbol{t} = \frac{(t-c)}{t} \in (0,1)$ for ease of notation. Observe that, for any $\delta \in (0,\delta_\phi)$, the property $\underline{\lim}_{u\to\infty} u^{-\delta}\phi(u) > 0$ is equivalent to $\overline{\lim}_{u\to\infty} u^{-\frac{1}{\delta}}\varphi(u) < \infty$. Hence, for any $\delta > \delta_\phi$ and $t \geqslant \frac{2}{\delta}$ we have

$$\lim_{y\to\infty} G'(y)e^{-\frac{y}{2}} = \mathrm{t}\lim_{y\to\infty} \varphi(e^{\frac{y}{t}})e^{-\frac{y}{2}} = \boldsymbol{t}\lim_{y\to\infty} e^{-\frac{y}{\delta t}}\varphi(e^{\frac{y}{t}})e^{\left(\frac{1}{\delta t}-\frac{1}{2}\right)y} < \infty,$$

and thus all the assumptions of Theorem 3.4.1 are fulfilled for any $t > \frac{2}{\delta_\phi}$. The inverse of G' is easily identified as $\gamma(u) = t\log\phi(\boldsymbol{t}u)$ so that,

$$\sum_{n=1}^{\infty} e^{-\frac{\gamma(n)}{2}} = \sum_{n=1}^{\infty} \phi^{-\frac{t}{2}}(\boldsymbol{t}n). \tag{3.37}$$

Now, for any $\delta \in (0, \delta_\phi)$, there exists a constant $C > 0$ (depending only on t) such that, for n large enough,

$$\phi^{-\frac{t}{2}}(tn) \leqslant Cn^{-\frac{\delta t}{2}}.$$

Thus for any $t > \frac{2}{\delta}$ the series in (3.37) converges, so that ν_t is indeterminate. Since δ can be taken arbitrarily close to δ_ϕ this gives the indeterminacy of ν_t for any $t > \frac{2}{\delta_\phi}$.

Proof of Theorem 3.2.1(4)

Let $\frac{\phi}{\vartheta} \in \mathcal{B}$ and write $(\rho_t)_{t>0}$ for the Berg-Urbanik semigroup associated to ϑ. Since $\frac{\phi}{\vartheta} \in \mathcal{B}$ we may invoke [98, Theorem 4.7(3)] to get that, for any $t > 0$ and $n \geqslant 0$,

$$W_\phi^t(n+1) = W_{\frac{\phi}{\vartheta}}^t(n+1) W_\vartheta^t(n+1)$$

where each of the terms is a moment sequence. Applying [19, Lemma 2.2 and Remark 2.3] we conclude that whenever ρ_t is indeterminate then ν_t is indeterminate, i.e.

$$\{t > 0; \rho_t \text{ is indeterminate}\} \subseteq \{t > 0; \nu_t \text{ is indeterminate}\},$$

which implies that $\mathcal{T}_\phi \leqslant \mathcal{T}_\vartheta$. If $\vartheta^{\mathrm{t}} \in \mathcal{B}_\mathcal{J}$ for all $\mathrm{t} \in (0,1)$, then invoking Theorem 3.2.1(3) yields $\mathcal{T}_\phi \leqslant \frac{2}{\delta_\vartheta}$, which completes the proof.

Proof of Theorem 3.2.1(2)

First, by Proposition 3.4.1 and using the notation therein, we have for any $\alpha \in (0,1)$ and $\mathfrak{m} \geqslant \frac{\overline{\mu}(\frac{\mathrm{v}_\alpha}{2})+\mathrm{k}}{\mathrm{d}}$ that $\frac{\phi}{\phi_{\alpha,\mathfrak{m}}} \in \mathcal{B}$. Hence, by Theorem 3.2.1(4) it follows that $\mathcal{T}_\phi \leqslant \mathcal{T}_{\phi_{\alpha,\mathfrak{m}}}$. Proposition 3.4.1(1) gives that $\phi_{\alpha,\mathfrak{m}}$ is a complete Bernstein function so that $\phi_{\alpha,\mathfrak{m}}^{\mathrm{t}} \in \mathcal{B}_\mathcal{J}$ for all $\mathrm{t} \in (0,1)$, see e.g. Remark 3.2.1. Plainly $\phi_{\alpha,\mathfrak{m}} \overset{\infty}{\sim} u^\alpha$, which implies that $\delta_{\phi_{\alpha,\mathfrak{m}}} = \beta_{\phi_{\alpha,\mathfrak{m}}} = \alpha$. Invoking Theorem 3.2.1(3) we get that $\mathcal{T}_\phi \leqslant \frac{2}{\alpha}$. Since this inequality holds for any $\alpha \in (0,1)$ we get $\mathcal{T}_\phi \leqslant 2$, whence $\mathcal{T}_\phi = 2$. The claim that ν_2

is moment determinate follows from Theorem 3.2.1(1), since $\mathsf{d} > 0$ implies $\beta_\phi = 1$ and that $\overline{\lim}_{u\to\infty} u^{-1}\phi(u) = \lim_{u\to\infty} u^{-1}\phi(u) = \mathsf{d}$.

3.4.5 Proofs for Section 3.2.2

In the proofs below we write, for any $\phi \in \mathcal{B}$, $X(\phi) = X$ for the positive random variable whose law is ν_1^ϕ, and, for any $x, t > 0$, $\sigma_t(dx) = \mathbb{P}(X^t \in dx)$.

Proof of Theorem 3.2.2(1)

From [99, Theorem 5.1(2)] it follows that, for $t > 0$,

$$\mathbb{E}\left[\left(X^t\right)^n\right] = W_\phi(tn+1) \overset{\infty}{\sim} C_\phi \sqrt{\phi(tn)} e^{G(tn)}$$

where $G(tn) = \int_1^{tn} \log \phi(r) dr$ and $C_\phi > 0$ is a constant depending only on ϕ. By following similar arguments than the ones developed for the proof of Theorem 3.2.1(1) we obtain the estimate

$$\sum_{n=1}^{\infty} W_\phi^{-\frac{1}{2n}}(tn+1) \geqslant C \sum_{n=1}^{\infty} \phi^{-\frac{t}{2}}(tn),$$

for some constant $C > 0$. Now, for any $\beta > \beta_\phi$, $\phi(n) \overset{\infty}{=} \mathrm{O}(n^\beta)$ so that, for a constant $C_1 > 0$ depending on t and β,

$$\sum_{n=1}^{\infty} \phi^{-\frac{t}{2}}(tn) \geqslant C_1 \sum_{n=1}^{\infty} n^{-\frac{\beta t}{2}}.$$

This latter series diverges if and only if $t \leqslant \frac{2}{\beta} < \frac{2}{\beta_\phi}$, so that by Carleman's criterion X^t is moment determinate whenever $t < \frac{2}{\beta_\phi}$. When $\overline{\lim}_{u\to\infty} u^{-\beta_\phi}\phi(u) < \infty$ we may take $\beta = \beta_\phi$ and apply the above argument, which finishes the proof.

Proof of Theorem 3.2.2(3)

Observe that, for $z \in 1 + i\mathbb{R}$ and $t > 0$, we have

$$\mathcal{M}_{\sigma_t}(z-1) = \mathbb{E}[\left(X_1^t(\phi)\right)^{z-1}] = \mathbb{E}[X^{t(z-1)}] = W_\phi(tz - t + 1).$$

Since $W_\phi \in \mathcal{A}_{(0,\infty)}$ it follows that $\mathcal{M}_{\sigma_t}(z-1)$ can be analytically extended to $\mathrm{Re}(z) > 1 - \frac{1}{t}$, and we write $\mathcal{M}_{\sigma_t}$ for this analytical extension. Next, we may assume that $\delta_\phi > 0$, since the claim is trivial otherwise, from which it follows that $\phi(\infty) = \infty$. Combining this with the fact that $\phi \in \mathcal{B}_\mathcal{J}$ gives $\mathtt{N}_\phi = \infty$, where we refer to Section 3.2.4 for the definition of $\mathtt{N}_\phi$, and invoking [98, Theorem 4.2(3)] allows us to conclude that, for any $q \geqslant 0$ and $a > 0$,

$$|W_\phi(a + ib)| \stackrel{\infty}{=} \mathrm{O}(|b|^{-q})$$

uniformly on bounded a-intervals, so that for any $q \geqslant 0$ and $a > -\frac{1}{t}$

$$|\mathcal{M}_{\sigma_t}(a + ib)| \stackrel{\infty}{=} \mathrm{O}(|b|^{-q})$$

uniformly on bounded a-intervals. By Mellin inversion we get $\sigma(dx) = \sigma_t(x)dx$ for each $t > 0$ and, from similar arguments as given in the proof of Theorem 3.2.4(2), we get the Mellin-Barnes representation

$$\sigma_t(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-z} W_\phi(tz - t + 1)dz,$$

valid for any $c > 1 - \frac{1}{t}$. The change of variables $z \mapsto \frac{(z-1)}{t} + 1$ reveals that

$$\sigma_t(x) = \frac{1}{2\pi i t}\int_{c-i\infty}^{c+i\infty} x^{-\frac{(z-1)}{t}-1} W_\phi(z)dz, \tag{3.38}$$

for any $c > 0$, and using Theorem 3.2.4(2) to identify the right-hand side of (3.38) we establish, for all $t > 0$, the equality

$$\sigma_t(x) = \frac{1}{t} x^{\frac{1-t}{t}} \nu_1(x^{\frac{1}{t}})$$

where $\nu_1(dx) = \nu_1(x)dx$. This identity allows us to use the asymptotic behavior of ν_1 described in [99, Theorem 5.5] to get that

$$\sigma_t(x) \overset{\infty}{\sim} \frac{C_\phi}{t\sqrt{2\pi}} x^{\frac{1-t}{t}} \sqrt{\varphi'(x^{\frac{1}{t}})} \exp\left(-\int_{\mathsf{k}}^{x^{\frac{1}{t}}} \frac{\varphi(r)}{r} dr\right)$$

where $C_\phi > 0$ is a constant depending on ϕ and $\varphi : [\mathsf{k}, \infty) \to [0, \infty)$ is the continuous inverse of ϕ. Repeating, mutatis mutandis, the arguments from Theorem 3.2.1(3) we conclude that X^t is moment indeterminate for $t > \frac{2}{\delta}$, and the last claim is straightforward.

Proof of Theorem 3.2.2(4)

The proof is the same as the one of Theorem 3.2.1(4) after observing that the assumptions imply the factorization of moment sequences

$$W_\phi(tn + 1) = W_{\frac{\phi}{\vartheta}}(tn + 1)W_\vartheta(tn + 1)$$

valid for any $t > 0$ and $n \geqslant 0$.

Proof of Theorem 3.2.2(2)

When $\phi \in \mathcal{B}_d$ Proposition 3.4.1 guarantees, for any $\alpha \in (0, 1)$ and suitable $\mathfrak{m}$, that $\frac{\phi}{\phi_{\alpha,\mathfrak{m}}} \in \mathcal{B}$. Applying Theorem 3.2.2(4) it follows that $X^t(\phi)$ is indeterminate for any t such that $X^t(\phi_{\alpha,\mathfrak{m}})$ is indeterminate. However, $\phi_{\alpha,\mathfrak{m}}(u) = (u+\mathfrak{m})^\alpha$, so by a combination of Proposition 3.4.1 and some straightforward asymptotic analysis one gets that $\phi_{\alpha,\mathfrak{m}} \in \mathcal{B}_{\asymp}$ with $\beta_{\phi_{\alpha,\mathfrak{m}}} = \alpha > 0$ and $\overline{\lim}_{u\to\infty} u^{-\alpha}\phi_{\alpha,\mathfrak{m}}(u) < \infty$. From Proposition 3.4.1(1) we get that $\phi_{\alpha,\mathfrak{m}}$ is a complete Bernstein function and hence, in particular, $\phi_{\alpha,\mathfrak{m}} \in \mathcal{B}_{\mathcal{J}}$. Thus Theorem 3.2.2(3) gives that $X^t(\phi_{\alpha,\mathfrak{m}})$ is moment indeterminate if and only if $t > \frac{2}{\alpha}$, from which we conclude that $X^t(\phi)$ is indeterminate for $t > \frac{2}{\alpha}$. Since $\alpha \in (0, 1)$ is arbitrary we get that $X^t(\phi)$ is moment indeterminate if $t > 2$, and from Theorem 3.2.2(1) we get moment determinacy for $t \leqslant 2$, which finishes the proof.

CHAPTER 4

ON NON-LOCAL ERGODIC JACOBI SEMIGROUPS: SPECTRAL THEORY, CONVERGENCE-TO-EQUILIBRIUM AND CONTRACTIVITY

We introduce and study non-local Jacobi operators, which generalize the classical (local) Jacobi operator. We show that these operators extend to the generator of an ergodic Markov semigroup with a unique invariant probability measure and study its spectral and convergence properties. In particular, we give a series expansion of the semigroup in terms of explicitly defined polynomials, which are counterparts of the classical Jacobi orthogonal polynomials. In addition, we give a complete characterization of the spectrum of the non-self-adjoint generator and semigroup. We show that the variance decay of the semigroup is hypocoercive with explicit constants, which provides a natural generalization of the spectral gap estimate. After a random warm-up time the semigroup also decays exponentially in entropy and is both hypercontractive and ultracontractive. Our proofs hinge on the development of commutation identities, known as intertwining relations, between local and non-local Jacobi operators/semigroups, with the local Jacobi operator/semigroup serving as a reference object for transferring properties to the non-local ones.

4.1 Introduction

In this chapter we study the non-local Jacobi operators given for suitable functions f on $[0, 1]$ by

$$\mathbb{J} f(x) = \mathbf{J}_\mu f(x) - f' \diamond h(x), \tag{4.1}$$

where $\mathbf{J}_\mu$ is the classical Jacobi operator

$$\mathbf{J}_\mu f(x) = x(1-x)f''(x) - (\lambda_1 x - \mu)\, f'(x),$$

and $\diamond$ denotes the product convolution operator

$$f \diamond h(x) = \int_0^x f(r)h(xr^{-1})r^{-1}dr,$$

with λ_1, μ, and the function h satisfying Assumption 4.2.1 below. The classical Jacobi operator is a central object in the study of Markovian diffusions. For instance, it is a model candidate for testing functional inequalities such as the Sobolev and log-Sobolev inequalities, see for instance the papers by Bakry [8], Saloff-Coste [107], and Fontenas [53]. When $\mu = \frac{\lambda_1}{2} = n$, an integer, there exists a homeomorphism between this particular Jacobi operator and the radial part of the Laplace–Beltrami operator on the n-sphere, revealing connections to diffusions on higher-dimensional manifolds that, in particular, lead to a curvature-dimension inequality as described in Bakry et al. [9, Chapter 2.7]. From the spectral theory viewpoint, the Markov semigroup $\mathbf{Q}^{(\mu)} = (e^{t\mathbf{J}_\mu})_{t\geqslant 0}$ is diagonalizable with respect to an orthonormal, polynomial basis for $\mathrm{L}^2(\beta_\mu)$, where β_μ denotes its unique invariant probability measure. As a consequence of these facts the semigroup $\mathbf{Q}^{(\mu)}$ converges to equilibrium in various senses, such as in variance and in entropy, and is both hypercontractive and ultracontractive; see Section 4.5, where we review essential facts about the classical Jacobi operator, semigroup, and process. We mention that Jacobi processes have been popular in applications such as population genetics, under the name Wright-Fisher diffusion, see e.g. Ethier and Kurtz [50, Chapter 10] and the works by Griffiths et al. [58, 57], Huillet [67], and Pal [95], and in finance, see for instance Delbaen and Shirikawa [41] and Gourieroux and Jasiak [56].

Due to the non-local part of $\mathbb{J}$ and its non-self-adjointness as a densely defined and closed operator in $\mathrm{L}^2(\beta)$ with β denoting the invariant measure of the corresponding semigroup, a fact that is proved below, the traditional techniques that are used to study $\mathbf{J}_\mu$ seem out of reach. Nevertheless, our investigation of $\mathbb{J}$ yields generalizations of the classical and substantial results mentioned above. A central tool in our developments is the notion of an intertwining relation, which is a type of commutation relationship for linear operators. Fixing $\boldsymbol{\lambda}_1$ and for some parameters $\widetilde{\mu}$, $\overline{\mu}$ to be specified below, we develop identities of the form

$$\mathbb{J}\Lambda = \Lambda \mathbf{J}_{\widetilde{\mu}}, \quad \text{and} \quad V\mathbb{J} = \mathbf{J}_{\overline{\mu}} V,$$

on the space of polynomials, the first of which allows us to prove that $\mathbb{J}$ generates an ergodic Markov semigroup $\mathbb{Q} = (\mathbb{Q}_t)_{t\geqslant 0}$ with unique invariant probability measure β. We also establish, for $t \geqslant 0$,

$$\mathbb{Q}_t\Lambda = \Lambda \mathbf{Q}_t^{(\widetilde{\mu})} \quad \text{and} \quad V\mathbb{Q}_t = \mathbf{Q}_t^{(\overline{\mu})} V,$$

on $\mathrm{L}^2(\beta_{\widetilde{\mu}})$ and $\mathrm{L}^2(\beta)$, respectively, where $\Lambda : \mathrm{L}^2(\beta_{\widetilde{\mu}}) \to \mathrm{L}^2(\beta)$ and $V : \mathrm{L}^2(\beta) \to \mathrm{L}^2(\beta_{\overline{\mu}})$ are bounded linear operators. These latter identities are crucial for obtaining the spectral theory, convergence-to-equilibrium, hypercontractivity, and ultracontractivity estimates for $\mathbb{Q}$.

The chapter is organized as follows. We state our main results in Section 4.2. All proofs are given in Section 4.3 and a specific family of non-local Jacobi semigroups is considered in Section 4.4. Finally we collect known results on the classical Jacobi operator, semigroup, and process in Section 4.5.

4.2 Main results on non-local Jacobi operators and semigroups

4.2.1 Preliminaries and existence of Markov semigroup

In this section we state our main results concerning the non-local operator $\mathbb{J}$ defined in (4.1). We write $\mathbb{R}_+ = (0, \infty)$ and $\mathbf{1}$ for the indicator function, and throughout we shall operate under the following assumption.

Assumption 4.2.1. The function $h : (1, \infty) \to [0, \infty)$ is such that $\Pi(dr) = -(e^r h(e^r))' dr$ is a finite, non-negative Radon measure on $\mathbb{R}_+$, and $\hbar = \int_1^\infty h(r)dr < \infty$. Furthermore, if $h \not\equiv 0$,

$$\boldsymbol{\lambda}_1 > \mathbf{1}_{\{\mu<1+\hbar\}} + \mu \quad \text{and} \quad \mu > \hbar,$$

while otherwise $\boldsymbol{\lambda}_1 > \mu > 0$.

Anticipating the results of Theorem 4.2.1 below, we already mention that the càdlàg realization of the Markov semigroup $\mathbb{Q}$ has downward jumps from x to $e^{-r}x$, $r, x > 0$, which occur at a frequency given by the Lévy kernel $\Pi(dr)/x$, see Lemma 4.3.1 and (4.16) below. Note also that, for $h \not\equiv 0$, we have $\hbar > 0$ and thus $\boldsymbol{\lambda}_1 > 1$. Next, we consider the convex, twice differentiable and eventually increasing function $\Psi : [0, \infty) \to \mathbb{R}$ given by

$$\Psi(u) = u^2 + (\mu - \hbar - 1)u + u \int_1^\infty (1 - r^{-u})h(r)dr, \tag{4.2}$$

which is easily seen to always have 0 as a root, and has a root $\mathsf{r} > 0$ if and only if $\mu < 1 + \hbar$. Set

$$\mathsf{r}_0 = \mathsf{r}\mathbf{1}_{\{\mu<1+\hbar\}} \quad \text{and} \quad \mathsf{r}_1 = 1 - \mathsf{r}_0, \tag{4.3}$$

and define $\phi : [0, \infty) \to [0, \infty)$ to be the function given by

$$\phi(u) = \frac{\Psi(u)}{u - \mathsf{r}_0}. \tag{4.4}$$

For instance, when $\mathsf{r}_0 = 0$, then

$$\phi(u) = u + (\mu - \hbar - 1) + \int_1^\infty (1 - r^{-u})h(r)dr,$$

and we note that both ϕ and $\mathbb{J}$ are uniquely determined by $\boldsymbol{\lambda}_1$, μ, and h so that, for fixed $\boldsymbol{\lambda}_1$, there is a one-to-one correspondence between ϕ and $\mathbb{J}$. As we show in Lemma 4.3.2 ϕ is a Bernstein function, i.e. $\phi : [0, \infty) \to [0, \infty)$ is infinitely differentiable on $\mathbb{R}_+$ and $(-1)^{n+1} \frac{d^n}{du^n}\phi(u) \geqslant 0$, for all $n = 1, 2, \ldots$ and $u > 0$, see Bertoin [21] and Schilling et al. [109] for a thorough exposition on Bernstein functions and subordinators.

Any Bernstein function ϕ admits an analytic extension to the right half-plane $\{z \in \mathbb{C};\ \mathrm{Re}(z) > 0\}$, see e.g. Patie and Savov [99, Chapter 4], and we write W_ϕ for the unique solution, in the space of positive definite functions, to the functional equation

$$W_\phi(z+1) = \phi(z)W_\phi(z), \quad \mathrm{Re}(z) > 0,$$

with $W_\phi(1) = 1$, and we refer to Patie and Savov [98] for a thorough account on this set of functions that generalize the gamma function, which appears as a special case when $\phi(z) = z$. In particular, for any $n \in \mathbb{N}$,

$$W_\phi(n+1) = \prod_{k=1}^{n} \phi(k), \tag{4.5}$$

with the convention $\prod_{k=1}^{0} \phi(k) = 1$ and where throughout we write $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Let $C([0,1])$ denote the Banach space of continuous functions on $[0,1]$ equipped with the sup-norm $||\cdot||_\infty$, and let, for $k \in \mathbb{N}$, $C^k([0,1])$ denote the space of functions on $[0,1]$ admitting k continuous derivatives with $C^\infty([0,1]) = \cap_{k=0}^{\infty} C^k([0,1])$, $C^0([0,1]) = C([0,1])$. What we call a Markov semigroup on $C([0,1])$, $\mathbb{Q} = (\mathbb{Q}_t)_{t\geqslant 0}$, is a one parameter semigroup of operators such that, for all $t \geqslant 0$ and $f \in C([0,1])$, $\mathbb{Q}_t \mathbf{1}_{[0,1]} = \mathbf{1}_{[0,1]}$, $\mathbb{Q}_t f \geqslant 0$ when $f \geqslant 0$, $||\mathbb{Q}_t f||_\infty \leqslant ||f||_\infty$, and $\lim_{t\to 0} ||\mathbb{Q}_t f - f||_\infty = 0$. A probability measure β on $[0,1]$ is invariant for a Markov semigroup $\mathbb{Q}$ if, for all $f \in C([0,1])$ and $t \geqslant 0$,

$$\beta[\mathbb{Q}_t f] = \beta[f] = \int_0^1 f(y)\beta(dy),$$

where the last equality serves as a definition for the notation $\beta[f]$. It is then classical, see either Bakry et al. [9] or Da Prato [39], that given a Markov semigroup on $C([0,1])$ with invariant probability measure β one may extend it to a Markov semigroup on $\mathrm{L}^2(\beta)$, the weighted Hilbert space defined as

$$\mathrm{L}^2(\beta) = \left\{ f : [0,1] \to \mathbb{R} \text{ measurable with } \beta[f^2] < \infty \right\}.$$

Such a semigroup is said to be ergodic if, for every $f \in \mathrm{L}^2(\beta)$, $\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{Q}_t f dt = \beta[f]$ in the $\mathrm{L}^2(\beta)$-norm.

Next, for any $x \in [0, \infty)$ and $a \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$ we write $(a)_x$ to denote the Pochhammer symbol

$$(a)_x = \frac{\Gamma(a+x)}{\Gamma(a)}.$$

Writing $\mathcal{P}$ for the algebra of polynomials and letting $p_n(x) = x^n$ we define formally the following sequence, for any $n \in \mathbb{N}$,

$$\beta[p_n] = \frac{(\mathrm{r}_1)_n}{(\lambda_1)_n} \frac{W_\phi(n+1)}{n!}, \tag{4.6}$$

and note that in Lemma 4.3.2 we show that $\mathrm{r}_1 \in (0, 1]$. Recall that a sequence is said to be Stieltjes moment determinate if it is the moment sequence of a unique probability measure on $[0, \infty)$. Our first main result provides the existence of an ergodic Markov semigroup generated by the non-local Jacobi operator $\mathbb{J}$.

Theorem 4.2.1.

(1) The sequence $(\beta[p_n])_{n \geqslant 0}$ is a determinate Stieltjes moment sequence of an absolutely continuous probability measure β whose support is $[0, 1]$*, with a continuous density that is positive on* $(0, 1)$.

(2) The extension of $\mathbb{J}$ to an operator on $\mathrm{L}^2(\beta)$*, still denoted by $\mathbb{J}$, is the infinitesmal generator, having $\mathcal{P}$ as a core, of an ergodic Markov semigroup $\mathbb{Q} = (\mathbb{Q}_t)_{t \geqslant 0}$ on* $\mathrm{L}^2(\beta)$ *whose unique invariant measure is β.*

The proof of Item (2) makes use of an intertwining relation stated in Proposition 4.3.1, which is an original approach to showing that the assumptions of the Hille–Yosida–Ray Theorem are fulfilled; see Lemma 3.7 and its proof for more details. More generally, the idea of constructing a new Markov semigroup by intertwining with a known, reference Markov semigroup goes back to Dynkin [48] whose ideas were extended by Rogers and Pitman in [104]. More recently, Borodin and Olshanski [26] also used intertwining relations combined with a limiting argument to construct a Markov process on the Thoma cone.

We also point out that the invariant measure β is a natural extension of the beta distribution, which is recovered when $\phi(u) = u$, as in this case in (4.6) we get $W_\phi(n+1) = n!$. The condition in Assumption 4.2.1 that $\Pi(dr) = -(e^r h(e^r))' dr$ is a finite measure is necessary for the existence of an invariant probability measure for $\mathbb{Q}$. Indeed, as we illustrate in our proof of Theorem 4.2.1, any candidate for such a measure must have moments given by (4.6). If $\Pi(dr) = -(e^r h(e^r))' dr$ is not a finite measure, then estimates by Patie and Savov in [98, Theorem 3.3] imply that the analytical extension of (4.6) to $\{z \in \mathbb{C}; \operatorname{Re}(z) > \mathsf{r}_1\}$ is not bounded along imaginary lines, a necessary condition to be a probability measure.

4.2.2 Spectral theory of the Markov semigroup and generator

We proceed by developing the $\mathrm{L}^2(\beta)$-spectral theory for both the semigroup $\mathbb{Q}$ and the operator $\mathbb{J}$. Recalling that, for fixed $\boldsymbol{\lambda}_1$, there is a one-to-one correspondence between $\mathbb{J}$ and the Bernstein function ϕ in (4.4), we define, for $n \in \mathbb{N}$, the polynomial $\mathcal{P}_n^\phi : [0,1] \to \mathbb{R}$ as

$$\mathcal{P}_n^\phi(x) = \sqrt{\mathrm{C}_n(\mathsf{r}_1)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\boldsymbol{\lambda}_1 - 1)_{n+k}}{(\boldsymbol{\lambda}_1 - 1)_n} \frac{(\mathsf{r}_1)_n}{(\mathsf{r}_1)_k} \frac{x^k}{W_\phi(k+1)}, \tag{4.7}$$

where $\mathrm{C}_n(\mathsf{r}_1)$ is given by

$$\mathrm{C}_n(\mathsf{r}_1) = (2n + \boldsymbol{\lambda}_1 - 1) \frac{n!(\boldsymbol{\lambda}_1)_{n-1}}{(\mathsf{r}_1)_n (\boldsymbol{\lambda}_1 - \mathsf{r}_1)_n}.$$

Note that when $h \equiv 0$ then in (4.2) we get $\Psi(u) = u(u - (1-\mu))$ and the functions $(\mathcal{P}_n^\phi)_{n \geqslant 0}$ boil down to $(\mathcal{P}_n^{(\mu)})_{n \geqslant 0}$, the classical Jacobi orthogonal polynomials reviewed in

Section 4.5. Next, we write $\mathbf{R}_n$ for the following scaled Rodrigues operator,

$$\mathbf{R}_n f(x) = \frac{2^n}{n!}\frac{d^n}{dx^n}(x^n f(x)) \tag{4.8}$$

and set

$$\Delta = \boldsymbol{\lambda}_1 - \mathrm{r}_1 - (\mu - 1)\mathbf{1}_{\{\mu \geqslant 1+\hbar\}} - \hbar\mathbf{1}_{\{\mu < 1+\hbar\}}.$$

We write $\beta(dx) = \beta(x)dx$ for the density given in Theorem 4.2.1(1), and define, for every integer $n \geqslant 1$, the function $\beta_{\lambda_1+n,\lambda_1} : [0,1] \to [0,\infty)$ as

$$\beta_{\lambda_1+n,\lambda_1}(x) = \frac{(\boldsymbol{\lambda}_1)_n}{n!}x^{\lambda_1-1}(1-x)^{n-1}.$$

We denote by $\mathrm{L}^2([0,1])$ the usual Lebesgue space of square-integrable functions on $[0,1]$.

Proposition 4.2.1. *Let* $\mathcal{V}_0^\phi \equiv 1$ *and, for* $n = 1,2,\ldots$, *define* $\mathcal{V}_n^\phi : (0,1) \to \mathbb{R}$ *as*

$$\mathcal{V}_n^\phi(x) = \frac{1}{\beta(x)}\frac{(\boldsymbol{\lambda}_1 - \mathrm{r}_1)_n}{(\boldsymbol{\lambda}_1)_n}\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\,\mathbf{R}_n(\beta_{\lambda_1+n,\lambda_1} \diamond \beta)(x) = \frac{1}{\beta(x)}w_n(x). \tag{4.9}$$

Then $w_n \in C^\infty((0,1))$ *and, if* $\Delta > \frac{1}{2}$, *in addition,* $w_n \in \mathrm{L}^2([0,1])$. *If* $\lfloor\Delta\rfloor \geqslant 2$ *then* $\mathcal{V}_n^\phi \in C^{\lfloor\Delta\rfloor-1}((0,1))$.

Remark 4.2.1. The definition in (4.9) makes sense regardless of the differentiability of β, since $\beta_{\lambda_1+n,\lambda_1} \in C^\infty((0,1))$ and $\mathbf{R}_n(\beta_{\lambda_1+n,\lambda_1} \diamond \beta) = \mathbf{R}_n\beta_{\lambda_1+n,\lambda_1} \diamond \beta$. However, the differentiability of $\mathcal{V}_n^\phi$ is limited by the smoothness of β, which is quantified by the index $\lfloor\Delta\rfloor - 1$. Note that, when $h \equiv 0$ then $\beta = \beta_\mu$ and, by moment identification and determinacy, it is easily checked that (4.9) boils down to the Rodrigues representation of the classical Jacobi polynomials $\mathcal{P}_n^{(\mu)}$ given in (4.72). In this sense $(\mathcal{P}_n^\phi)_{n\geqslant 0}$ and $(\mathcal{V}_n^\phi)_{n\geqslant 0}$ both generalize $(\mathcal{P}_n^{(\mu)})_{n\geqslant 0}$ in different ways, coming from the different representations of these orthogonal polynomials.

We say that two sequences $(f_n)_{n\geqslant 0}, (g_m)_{m\geqslant 0} \in \mathrm{L}^2(\beta)$ are biorthogonal if $\beta[f_n g_m] = 1$, when $n = m$, and $\beta[f_n g_m] = 0$ otherwise, and then write $f_n \otimes g_n$ for the projection operator given by $f \mapsto \beta[f g_n] f_n$. Moreover, a sequence that admits a biorthogonal sequence will be called minimal and a sequence that is both minimal and complete, in the sense that its linear span is dense in $\mathrm{L}^2(\beta)$, will be called *exact*. It is easy to show that a sequence $(f_n)_{n\geqslant 0}$ is minimal if and only if none of its elements can be approximated by linear combinations of the others. If this is the case, then a biorthogonal sequence will be uniquely determined if and only if $(f_n)_{n\geqslant 0}$ is complete. Next, a sequence $(f_n)_{n\geqslant 0} \in \mathrm{L}^2(\beta)$ is said to be a Bessel sequence if there exists $B > 0$ such that, for all $f \in \mathrm{L}^2(\beta)$,

$$\sum_{n=0}^{\infty} \beta[f_n f]^2 \leqslant B\, \beta[f^2].$$

The quantity B is a Bessel bound of $(f_n)_{n\geqslant 0}$ and the smallest such B is called the optimal Bessel bound of $(f_n)_{n\geqslant 0}$, see the book by Christensen [37] for further information on these objects that play a central role in non-harmonic analysis.

We write $\sigma(\mathbb{Q}_t)$ for the spectrum of the operator $\mathbb{Q}_t$ in $\mathrm{L}^2(\beta)$ and $\sigma_p(\mathbb{Q}_t)$ for its point spectrum, and similarly define $\sigma(\mathbb{J})$ and $\sigma_p(\mathbb{J})$. For an isolated eigenvalue $\varrho \in \sigma_p(\mathbb{Q}_t)$ we write $\mathrm{M}_a(\varrho, \mathbb{Q}_t)$ and $\mathrm{M}_g(\varrho, \mathbb{Q}_t)$ for the algebraic and geometric multiplicity of ϱ, respectively. We also define, for $n \in \mathbb{N}$,

$$\lambda_n = n(n-1) + \boldsymbol{\lambda}_1 n = n^2 + (\boldsymbol{\lambda}_1 - 1)n, \tag{4.10}$$

noting that $\lambda_1 = \boldsymbol{\lambda}_1$, which explains our choice of notation, and recall that $\sigma(\mathbf{J}_\mu) = \sigma_p(\mathbf{J}_\mu) = \{-\lambda_n;\ n \in \mathbb{N}\}$, see Section 4.5. We write $\mathbb{Q}_t^*$ for the $\mathrm{L}^2(\beta)$-adjoint of $\mathbb{Q}_t$. We have the following spectral theorem for $\mathbb{Q}$.

Theorem 4.2.2. *Let $t > 0$.*

(1) Then, with equality holding in operator norm, we have

$$\mathbb{Q}_t = \sum_{n=0}^{\infty} e^{-\lambda_n t} \mathcal{P}_n^{\phi} \otimes \mathcal{V}_n^{\phi},$$

where the sum converges in operator norm and $(\mathcal{P}_n^{\phi})_{n\geqslant 0} \in \mathrm{L}^2(\beta)$ *is an exact Bessel sequence with optimal Bessel bound 1, and* $(\mathcal{V}_n^{\phi})_{n\geqslant 0} \in \mathrm{L}^2(\beta)$ *is its unique biorthogonal sequence, which is also exact. Moreover, for any* $n \in \mathbb{N}$, $\mathcal{P}_n^{\phi}$ *(resp.* $\mathcal{V}_n^{\phi}$*) is an eigenfunction for* $\mathbb{Q}_t$ *(resp.* $\mathbb{Q}_t^*$*) associated to the eigenvalue* $e^{-\lambda_n t}$.

(2) *The operator* $\mathbb{Q}_t$ *is compact, i.e. the semigroup* $\mathbb{Q}$ *is immediately compact.*

(3) *The following spectral mapping theorem holds*

$$\sigma(\mathbb{Q}_t) \setminus \{0\} = \sigma_p(\mathbb{Q}_t) \setminus \{0\} = e^{t\sigma_p(\mathbb{J})} = e^{t\sigma(\mathbb{J})} = \left\{e^{-\lambda_n t}; n \in \mathbb{N}\right\}.$$

Furthermore, $\sigma(\mathbb{Q}_t) = \sigma(\mathbb{Q}_t^*)$ *and, for any* $n \in \mathbb{N}$,

$$\mathrm{M}_a(e^{-\lambda_n t}, \mathbb{Q}_t) = \mathrm{M}_g(e^{-\lambda_n t}, \mathbb{Q}_t) = \mathrm{M}_a(e^{-\lambda_n t}, \mathbb{Q}_t^*) = \mathrm{M}_g(e^{-\lambda_n t}, \mathbb{Q}_t^*) = 1.$$

(4) *The operator* $\mathbb{Q}_t$ *is self-adjoint in* $\mathrm{L}^2(\beta)$ *if and only if* $h \equiv 0$.

The expansion in Theorem 4.2.2(1) is not valid for $t = 0$ as $(\mathcal{P}_n^{\phi})_{n\geqslant 0}$ is a Bessel sequence but not a Riesz sequence, as it is not the image of an orthogonal sequence by a bounded linear operator having a bounded inverse, see Proposition 4.3.5 below. The sequence of non-self-adjoint projections $\mathcal{P}_n^{\phi} \otimes \mathcal{V}_n^{\phi}$ is not uniformly bounded in n, see Remark 4.3.3, and, in contrast to the self-adjoint case, the eigenfunctions of $\mathbb{Q}_t$ and $\mathbb{Q}_t^*$ do not form a Riesz basis of $\mathrm{L}^2(\beta)$. Finally, we note that from Theorem 4.2.2(4) $\mathcal{P}_n^{\phi} \neq \mathcal{V}_n^{\phi}$ for all $n = 1, 2, \ldots$.

4.2.3 Convergence-to-equilibrium and contractivity properties

For an open interval $I \subseteq \mathbb{R}$, we say that a function $\Phi : I \to \mathbb{R}$ is admissible if

$$\Phi \in C^4(I) \text{ with both } \Phi \text{ and } -1/\Phi'' \text{ convex.} \tag{4.11}$$

Given an admissible function we write, for any $f : [0,1] \to I$ with $f, \Phi(f) \in \mathrm{L}^1(\beta)$,

$$\mathrm{Ent}_\beta^\Phi(f) = \beta[\Phi(f)] - \Phi(\beta[f]) \tag{4.12}$$

for the so-called Φ-entropy of f. An important case is when $\Phi(r) = r^2$, $I = \mathbb{R}$, so that (4.12) gives the variance $\mathrm{Var}_\beta(f)$ of $f \in \mathrm{L}^2(\beta)$. Recall that in the classical case, i.e. $h \equiv 0$, we have the following equivalence between the Poincaré inequality for $\mathbf{J}_\mu$ and the spectral gap inequality for $\mathbf{Q}^{(\mu)}$,

$$\boldsymbol{\lambda}_1 = \inf_f \frac{-\beta_\mu[f\mathbf{J}_\mu f]}{\mathrm{Var}_{\beta_\mu}(f)} \iff \mathrm{Var}_{\beta_\mu}(\mathbf{Q}_t^{(\mu)} f) \leqslant e^{-2\boldsymbol{\lambda}_1 t}\,\mathrm{Var}_{\beta_\mu}(f) \text{ for } f \in \mathrm{L}^2(\beta_\mu) \text{ and } t \geqslant 0$$

where the infimum is over all functions in the L^2-domain of $\mathbf{J}_\mu$, see for instance Bakry et al. [9, Chapter 4.2]. The above variance decay is optimal in the sense that the decay rate does not hold for any constant strictly greater than $2\boldsymbol{\lambda}_1$. Another important instance of (4.12) is when $\Phi(r) = r\log r$, $I = \mathbb{R}_+$, which recovers the classical notion of entropy for a non-negative function, written simply as $\mathrm{Ent}_\beta(f)$. Here the classical equivalence is between the log-Sobolev inequality and entropy decay,

$$\begin{aligned}&\boldsymbol{\lambda}_{\mathrm{logS}}^{(\mu)} = \inf_f \frac{-4\beta_\mu[f\mathbf{J}_\mu f]}{\mathrm{Ent}_{\beta_\mu}(f^2)} > 0 \iff \\ &\mathrm{Ent}_{\beta_\mu}(\mathbf{Q}_t^{(\mu)} f) \leqslant e^{-\boldsymbol{\lambda}_{\mathrm{logS}}^{(\mu)} t}\,\mathrm{Ent}_{\beta_\mu}(f) < \infty \text{ for } f \in \mathrm{L}^1(\beta) \text{ and } t \geqslant 0.\end{aligned}$$

Note that the optimal entropy decay rate is obtained only when $\mu = \frac{\boldsymbol{\lambda}_1}{2} > 1$, in which case $\boldsymbol{\lambda}_{\mathrm{logS}}^{(\mu)} = 2\boldsymbol{\lambda}_1$, while otherwise $\boldsymbol{\lambda}_{\mathrm{logS}}^{(\mu)} < 2\boldsymbol{\lambda}_1$, see, for instance, Fontenas [53] and Saloff-Coste [107]. We refer to the excellent article by Chafaï [33], the book by Ané et al. [3], the relevant sections of Bakry et al. [9], and also to Section 4.5 where we review these notions for the classical Jacobi semigroup. However, due to the non-self-adjointness and non-local properties of $\mathbb{J}$, it seems challenging to develop an approach based on the Poincaré or log-Sobolev inequalities. For this reason, we take an alternative route to tackling convergence to equilibrium by using concept of completely monotone intertwining relations recently introduced by Patie and Miclo in [87, Section 3.5] and [88].

Next, recalling that when $h \not\equiv 0$ we have $\boldsymbol{\lambda}_1 > 1$, we let $\rho : [0,\infty) \to [0,\infty)$ be defined as

$$\rho(u) = \sqrt{u + \frac{(\boldsymbol{\lambda}_1 - 1)^2}{4}} - \frac{\boldsymbol{\lambda}_1 - 1}{2}$$

and note that it is a Bernstein function, as it is obtained by translating and centering the well-known Bernstein function $u \mapsto \sqrt{u}$. In the literature ρ is known as the Laplace exponent of the so-called relativistic $1/2$-stable subordinator, see Bakry [7] and Bogdan et al. [25]. For any Bernstein function ϕ, we denote by

$$d_\phi = \inf\{u \geqslant 0;\ \phi(-u) = 0 \text{ or } \phi(-u) = \infty\} \in [0,\infty], \tag{4.13}$$

and we let, for any $\varepsilon \in (0, d_\phi] \cup \{d_\phi\}$,

$$\boldsymbol{d}_{\mathsf{r}_1,\varepsilon} = \mathsf{r}_1 \mathbf{1}_{\{\mu < 1+\hbar\}} + (d_\phi + 1 - \varepsilon)\mathbf{1}_{\{\mu \geqslant 1+\hbar\}} \tag{4.14}$$

noting that when $d_\phi = 0$ then $\varepsilon = 0$. We write, for any $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$ and $\varepsilon \in (0, d_\phi] \cup \{d_\phi\}$, τ for a random variable with Laplace transform

$$\mathbb{E}\left[e^{-u\tau}\right] = \frac{(\boldsymbol{d}_{\mathsf{r}_1,\varepsilon})_{\rho(u)}}{(\mathfrak{m})_{\rho(u)}} \frac{(\boldsymbol{\lambda}_1 - \mathfrak{m})_{\rho(u)}}{(\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathsf{r}_1,\varepsilon})_{\rho(u)}}, \quad u \geqslant 0, \tag{4.15}$$

and write $\mathbb{Q}_{t+\tau} = \int_0^\infty \mathbb{Q}_{t+s}\mathbb{P}(\tau \in ds)$.

Theorem 4.2.3. *Let $t \geqslant 0$. For any $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$ and $\varepsilon \in (0, d_\phi] \cup \{d_\phi\}$, we have the following.*

(1) For any $f \in \mathrm{L}^2(\beta)$

$$\mathrm{Var}_\beta(\mathbb{Q}_t f) \leqslant \frac{\mathfrak{m}(\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathsf{r}_1,\varepsilon})}{\boldsymbol{d}_{\mathsf{r}_1,\varepsilon}(\boldsymbol{\lambda}_1 - \mathfrak{m})} e^{-2\lambda_1 t}\, \mathrm{Var}_\beta(f),$$

with $\mathfrak{m}(\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathsf{r}_1,\varepsilon}) > \boldsymbol{d}_{\mathsf{r}_1,\varepsilon}(\boldsymbol{\lambda}_1 - \mathfrak{m})$.

(2) *The function* $\phi^{(\tau)} : u \mapsto -\log \mathbb{E}[e^{-u\tau}]$ *is a Bernstein function, which gives that* τ *is infinitely divisible and hence there exists a subordinator* $\boldsymbol{\tau} = (\tau_t)_{t\geqslant 0}$ *with* $\tau_1 \overset{(d)}{=} \tau$. *For any* $f \in \mathrm{L}^1(\beta)$ *with* $\mathrm{Ent}_\beta(f) < \infty$

$$\mathrm{Ent}_\beta(\mathbb{Q}_{t+\tau} f) \leqslant e^{-\boldsymbol{\lambda}^{(\mathfrak{m})}_{\mathrm{logS}} t} \,\mathrm{Ent}_\beta(f).$$

Furthermore, if $\boldsymbol{\lambda}_1 > 2(\mathbf{1}_{\{\mu<1+\hbar\}} + \mu)$ *then, with* $\mathfrak{m} = \boldsymbol{\lambda}_1/2$,

$$\mathrm{Ent}_\beta(\mathbb{Q}_{t+\tau} f) \leqslant e^{-2\lambda_1 t} \,\mathrm{Ent}_\beta(f).$$

Suppose, in addition, that $\mathbf{1}_{\{\mu<1+\hbar\}} + \mu < \boldsymbol{\lambda}_1/2 \in \mathbb{N}$, *and let* $\Phi : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, *be an admissible function, as in* (4.11). *Then, for any* $f : [0, 1] \to I$ *such that* $f, \Phi(f) \in \mathrm{L}^1(\beta)$ *and with* $\mathrm{Ent}^\Phi_\beta(f) < \infty$,

$$\mathrm{Ent}^\Phi_\beta(\mathbb{Q}_{t+\tau} f) \leqslant e^{-(\lambda_1-1)t} \,\mathrm{Ent}^\Phi_\beta(f).$$

Remark 4.2.2. Since $\frac{\mathfrak{m}(\boldsymbol{\lambda}_1-\boldsymbol{d}_{\mathrm{r}_1,\varepsilon})}{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}(\lambda_1-\mathfrak{m})} > 1$ the estimate in Theorem 4.2.3(1) gives the hypocoercivity, in the sense of Villani [123], for non-local Jacobi semigroups. This notion continues to attract research interests, especially in the area of kinetic Fokker-Planck equations, and we mention the works by Baudoin [12], Dolbeault et al. [43] and Mischler and Mouhout [89]. We are able to identify the hypocoercive constants, namely the exponential decay rate as twice the spectral gap, and the coefficient in front of the exponential, which is a measure of the deviation of the spectral projections from forming an orthogonal basis and is 1 in the case an orthogonal basis. Note that in general the hypocoercive constants may be difficult to identify, and may have little to do with the spectrum. Similar results have been obtained by Patie and Savov in [99] and Achleitner et al. in [1]. Our hypocoercive estimate is obtained via intertwining, which suggests that hypocoercivity may be studied purely from this viewpoint, an idea that is further investigated in the next chapter.

Remark 4.2.3. The second part of Theorem 4.2.3 gives the exponential decay in entropy of $\mathbb{Q}$ but after an independent random warm-up time. Note that, for $\boldsymbol{\lambda}_1 \leqslant 2(\mathbf{1}_{\{\mu<1+\hbar\}} + \mu)$

the entropy decay rate is the same as for $\mathbf{Q}^{(\mathfrak{m})}$ while under the mild assumption that $\lambda_1 > 2(\mathbf{1}_{\{\mu<1+\hbar\}} + \mu)$ we get the optimal rate for more than simply a fixed value of μ. The proof relies on developing so-called *completely monotone intertwining relations*, a concept which has been introduced and studied in the recent work by Miclo and Patie [88], where the classical Jacobi semigroup $\mathbf{Q}^{(\mathfrak{m})}$ serves as a reference object, see Proposition 4.3.6 below.

Remark 4.2.4. The additional condition $\lambda_1/2 \in \mathbb{N}$ for the Φ-entropic convergence in Theorem 4.2.3(2) ensures that we can invoke the known result in (4.79) for the classical Jacobi semigroup $\mathbf{Q}^{(\lambda_1/2)}$. However, our approach allows us to immediately transfer any improvement in (4.79) to the non-local Jacobi semigroup $\mathbb{Q}$.

Next, we recall the famous equivalence between entropy decay and hypercontractivity due to Gross [59], i.e. for any $t \geqslant 0$ and $f \in \mathrm{L}^1(\beta_\mu)$ such that $\mathrm{Ent}_{\beta_\mu}(f) < \infty$,

$$\mathrm{Ent}_{\beta_\mu}(\mathbf{Q}_t^{(\mathfrak{m})} f) \leqslant e^{-\lambda_{\log S}^{(\mathfrak{m})} t}\, \mathrm{Ent}_{\beta_\mathfrak{m}}(f) \iff ||\mathbf{Q}_t^{(\mathfrak{m})}||_{2\to q} \leqslant 1 \text{ where } 2 \leqslant q \leqslant 1 + e^{\lambda_{\log S}^{(\mathfrak{m})} t},$$

where we use the shorthand $||\cdot||_{p\to q} = ||\cdot||_{\mathrm{L}^p(\beta_\mathfrak{m})\to\mathrm{L}^q(\beta_\mathfrak{m})}$ for $1 \leqslant p, q \leqslant \infty$. To state our next result we write, when $\lambda_1 - \mathfrak{m} > 1$, $c_\mathfrak{m} > 0$ for the Sobolev constant of $\mathbf{J}_\mathfrak{m}$ of order $\frac{2(\lambda_1-\mathfrak{m})}{(\lambda_1-\mathfrak{m}-1)}$, and recall that as a result of the Sobolev inequality for $\mathbf{J}_\mathfrak{m}$ one gets that $||\mathbf{Q}_t^{(\mathfrak{m})}||_{1\to\infty} \leqslant c_\mathfrak{m} t^{-\frac{\lambda_1-\mathfrak{m}}{\lambda_1-\mathfrak{m}-1}}$, for $0 < t \leqslant 1$, which implies that $\mathbf{Q}^{(\mathfrak{m})}$ is ultracontractive, i.e. $||\mathbf{Q}_t^{(\mathfrak{m})}||_{1\to\infty} < \infty$ for all $t > 0$, see Section 4.5 for a review of these concepts. We have the following concerning the contractivity of $\mathbb{Q}$.

Theorem 4.2.4. *For any $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \lambda_1)$ and $\varepsilon \in (0, d_\phi] \cup \{d_\phi\}$, the following holds:*

(1) For $t \geqslant 0$, we have the hypercontractivity estimate

$$||\mathbb{Q}_{t+\tau}||_{2\to q} \leqslant 1, \quad \textit{where} \quad 2 \leqslant q \leqslant 1 + e^{\lambda_{\log S}^{(\mathfrak{m})} t},$$

and furthermore, if $\boldsymbol{\lambda}_1 > 2(\mathbf{1}_{\{\mu<1+\hbar\}} + \mu)$*, then, with* $\mathfrak{m} = \frac{\boldsymbol{\lambda}_1}{2}$*,*

$$||\mathbb{Q}_{t+\tau}||_{2\to q} \leqslant 1, \quad \textit{where} \quad 2 \leqslant q \leqslant 1 + e^{2\boldsymbol{\lambda}_1 t}.$$

(2) If in addition $\boldsymbol{\lambda}_1 - \mathfrak{m} > 1$ *then, for* $0 < t \leqslant 1$*, we have the ultracontractivity estimate*

$$||\mathbb{Q}_{t+\tau}||_{1\to\infty} \leqslant c_{\mathfrak{m}} t^{-\frac{\boldsymbol{\lambda}_1-\mathfrak{m}}{\boldsymbol{\lambda}_1-\mathfrak{m}-1}}$$

where, as soon as $\boldsymbol{\lambda}_1 > 2$*, one can choose* $\mathfrak{m} = \frac{\boldsymbol{\lambda}_1}{2}$ *giving* $c_{\frac{\boldsymbol{\lambda}_1}{2}} = \frac{4}{\boldsymbol{\lambda}_1(\boldsymbol{\lambda}_1-2)}$*.*

4.2.4 Bochner subordination of the semigroup

We write $\mathbb{Q}^{\boldsymbol{\tau}} = (\mathbb{Q}^{\boldsymbol{\tau}}_t)_{t\geqslant 0}$ for the semigroup subordinated, in the sense of Bochner, with respect to the subordinator $\boldsymbol{\tau} = (\tau_t)_{t\geqslant 0}$ whose existence is provided by Theorem 4.2.3 (2), i.e.

$$\mathbb{Q}^{\boldsymbol{\tau}}_t = \int_0^\infty \mathbb{Q}_s \mathbb{P}(\tau_t \in ds),$$

so that $\mathbb{Q}^{\boldsymbol{\tau}}_1 = \mathbb{Q}_\tau$. Note that $\mathbb{Q}^{\boldsymbol{\tau}}$ is also an ergodic Markov semigroup in $\mathrm{L}^2(\beta)$ with β as an invariant measure, and its generator is given by $-\phi^{(\tau)}(-\mathbb{J}) = \log \mathbb{Q}_\tau$, see Sato [108, Chapter 6]. We have the following results concerning the subordinated semigroup.

Theorem 4.2.5. *For any* $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$ *and* $\varepsilon \in (0, d_\phi] \cup \{d_\phi\}$ *the statement of Theorem 4.2.2 holds for* $\mathbb{Q}^{\boldsymbol{\tau}}$ *upon replacing* $(\lambda_n)_{n\geqslant 0}$ *by* $(\log \frac{(\mathfrak{m})_n(\boldsymbol{\lambda}_1-\boldsymbol{d}_{\mathrm{r}_1,\varepsilon})_n}{(\boldsymbol{d}_{\mathrm{r}_1,\varepsilon})_n(\boldsymbol{\lambda}_1-\mathfrak{m})_n})_{n\geqslant 0}$ *for* $t \geqslant 1$*, and the statements of Theorem 4.2.3(2) and Theorem 4.2.4(1) hold for* $\mathbb{Q}^{\boldsymbol{\tau}}$ *upon replacing* $\boldsymbol{\lambda}_1$ *by* $\log \frac{\mathfrak{m}(\boldsymbol{\lambda}_1-\boldsymbol{d}_{\mathrm{r}_1,\varepsilon})}{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}(\boldsymbol{\lambda}_1-\mathfrak{m})}$ *and* τ *by 1. Moreover, for any* $\mathfrak{m}$ *and* ε *such that* $1 < \boldsymbol{\lambda}_1 - \mathfrak{m} < (\mathfrak{m} - \boldsymbol{d}_{\mathrm{r}_1,\varepsilon})(\boldsymbol{\lambda}_1 - \mathfrak{m} - 1)$, $\mathbb{Q}^{\boldsymbol{\tau}}_t f(x) = \int_0^1 f(y) q^{(\boldsymbol{\tau})}_t(x,y)\beta(dy)$ *for any* $f \in \mathrm{L}^2(\beta)$ *and* $t > 2$*, where the heat kernel satisfies the following estimate*

$$|q^{(\boldsymbol{\tau})}_t(x,y) - 1| \leqslant c_{\mathfrak{m}}(\mathbb{E}[\tau^{-\frac{\boldsymbol{\lambda}_1-\mathfrak{m}}{\boldsymbol{\lambda}_1-\mathfrak{m}-1}}] + 1)\left(\frac{\mathfrak{m}(\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathrm{r}_1,\varepsilon})}{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}(\boldsymbol{\lambda}_1 - \mathfrak{m})}\right)^{\frac{1-2t}{2}} < \infty,$$

for Lebesgue a.e. $(x, y) \in [0,1]^2$. *As above, as soon as* $\boldsymbol{\lambda}_1 > 2$, *one can choose* $\mathfrak{m} = \frac{\lambda_1}{2}$ *giving* $c_{\frac{\lambda_1}{2}} = \frac{4}{\lambda_1(\lambda_1-2)}$.

We point out that the Markov process which is the realization of $\mathbb{Q}$ (resp. $\mathbb{Q}^{\boldsymbol{\tau}}$) has non-symmetric and spectrally negative (resp. two-sided) jumps and can easily be shown to be a polynomial process on $[0, 1]$ in the sense of Cuchiero et al. [38]. We emphasize that what also belongs to this class are the realizations of Markov semigroups obtained by subordinating $\mathbb{Q}$ with respect to any conservative subordinator $\widetilde{\boldsymbol{\tau}} = (\widetilde{\tau}_t)_{t \geqslant 0}$ with Laplace exponent $\phi^{(\widetilde{\tau})}$ (growing fast enough at infinity, e.g. logarithmically) and we obtain, from Theorem 4.2.2, the spectral expansion for the subordinated semigroup by replacing $(\lambda_n)_{n \geqslant 0}$ with $(\phi^{(\widetilde{\tau})}(\lambda_n))_{n \geqslant 0}$. Note that in the aforementioned paper the authors investigate the martingale problem for general polynomial operators on the unit simplex, of which $\mathbb{J}$ and $-\phi^{(\tau)}(-\mathbb{J})$ are specific instances. In particular, $\mathbb{J}$ is a Lévy type operator with affine jumps of Type 2, in the sense of [38], and for such operators they prove the existence and uniqueness for the martingale problem under the weaker condition $\boldsymbol{\lambda}_1 \geqslant \mu$. However, the conditions in Assumption 4.2.1 allow us to obtain the existence and uniqueness of an invariant probability measure.

4.3 Proofs

4.3.1 Preliminaries

We state and prove some preliminary results that will be useful throughout the paper. We start by giving an alternative form of the operator $\mathbb{J}$, which will make some later proofs more transparent.

Lemma 4.3.1. *Recall that* $\Pi(dr) = -(e^r h(e^r))' dr$*,* $r > 0$*. Then,* Π *is a finite, non-negative Radon measure on* $(0, \infty)$ *with* $\int_0^\infty r\Pi(dr) = \hbar < \infty$*, and the operator* $\mathbb{J}$ *defined in* (4.1) *may be written, for suitable* f*, as*

$$\begin{aligned}\mathbb{J} f(x) &= x(1-x) f''(x) - (\lambda_1 x - \mu + \hbar) f'(x) \\ &\quad + \int_0^\infty (f(e^{-r}x) - f(x) + xrf'(x)) \frac{\Pi(dr)}{x}, \quad x \in [0,1].\end{aligned} \tag{4.16}$$

Proof. Since

$$\hbar = \int_1^\infty h(r)dr = \int_0^\infty e^r h(e^r) dr < \infty$$

it follows that $\lim_{r\to\infty} e^r h(e^r) = 0$. Consequently, for any $y > 0$,

$$\overline{\Pi}(y) = \int_y^\infty \Pi(dr) = -\int_y^\infty (e^r h(e^r))' dr = e^y h(e^y) - \lim_{r\to\infty} e^r h(e^r) = e^y h(e^y).$$

Thus, by a change of variables and integration by parts, one gets

$$\int_0^\infty r\Pi(dr) = \int_0^\infty \overline{\Pi}(r)dr = \int_1^\infty h(r)dr = \hbar < \infty.$$

Next, we again use $\hbar < \infty$ to get that

$$\int_0^\infty (f(e^{-r}x) - f(x) + xrf'(x)) \frac{\Pi(dr)}{x} = \hbar f'(x) + \int_0^\infty \frac{f(e^{-r}x) - f(x)}{x} \Pi(dr).$$

Integrating the right-hand side by parts, and noting that the boundary terms evaluate to zero, yields

$$\begin{aligned}\int_0^\infty \frac{f(e^{-r}x) - f(x)}{x} \Pi(dr) &= -\int_0^\infty e^{-r} f'(e^{-r}x) \overline{\Pi}(r) dr \\ &= -\int_0^\infty f'(e^{-r}x) h(e^r) dr = -f' \diamond h(x)\end{aligned}$$

where the last equality follows from a straightforward change of variables, and uses the definition of product convolution. □

In the sequel we keep the notation $\Pi(dr) = -(e^r h(e^r))' dr$, $r > 0$ and $\overline{\Pi}(y) = e^y h(e^y)$, $y > 0$. Let $\phi^{\checkmark}_{\mathrm{r}_1} : [0, \infty) \to [0, \infty)$ be the function given by

$$\phi^{\checkmark}_{\mathrm{r}_1}(u) = \frac{u + \mathrm{r}_1}{u + 1} \phi(u + 1). \tag{4.17}$$

The following result collects some useful properties of the functions ϕ and $\phi_{\mathrm{r}_1}^{\checkmark}$.

Lemma 4.3.2. *Let ϕ be given by* (4.4).

(1) ϕ is a Bernstein function and satisfies $\lim_{u\to\infty} \frac{\phi(u)}{u} = 1$.

(2) We have $\mathrm{r}_1 \in (0,1]$, with $\mathrm{r}_1 = 1$ if and only if $\mu \geqslant 1+\hbar$ where we recall that r_1 is defined in (4.3). *Additionally, if $\mu \geqslant 1+\hbar$ then $\phi(0) = \mu - \hbar - 1$ while if $\mu < 1+\hbar$ then $\phi(0) = 0$.*

(3) Suppose $\mu < 1+\hbar$. Then $\phi_{\mathrm{r}_1}^{\checkmark}$ defined in (4.17) *is a Bernstein function that is in correspondence with the non-local Jacobi operator $\mathbb{J}_{\phi_{\mathrm{r}_1}^{\checkmark}}$ with parameters $\boldsymbol{\lambda}_1$, $\mu_{\phi_{\mathrm{r}_1}^{\checkmark}} = 1+\mu$, and the non-negative function $h_{\phi_{\mathrm{r}_1}^{\checkmark}}(r) = r^{-1}\overline{\Pi}_{\phi_{\mathrm{r}_1}^{\checkmark}}(\log r), r > 1$, where $\Pi_{\phi_{\mathrm{r}_1}^{\checkmark}}$ is the finite non-negative Radon measure given by*

$$\Pi_{\phi_{\mathrm{r}_1}^{\checkmark}}(dr) = e^{-r}\left(\Pi(dr) + \overline{\Pi}(r)dr\right), \ r > 0.$$

Furthermore, writing $\hbar_{\phi_{\mathrm{r}_1}^{\checkmark}} = \int_1^\infty h_{\phi_{\mathrm{r}_1}^{\checkmark}}(r)dr$, we have $\hbar_{\phi_{\mathrm{r}_1}^{\checkmark}} < \infty$ with $\mu_{\phi_{\mathrm{r}_1}^{\checkmark}} \geqslant 1 + \hbar_{\phi_{\mathrm{r}_1}^{\checkmark}}$ and $\boldsymbol{\lambda}_1 > \mu_{\phi_{\mathrm{r}_1}^{\checkmark}}$.

Proof. First we rewrite (4.2) using a straightforward integration by parts to get, for any $u \geqslant 0$,

$$\begin{aligned}\Psi(u) &= u^2 + (\mu - \hbar - 1)u + u\int_1^\infty (1 - r^{-u})h(r)dr \\ &= u^2 + (\mu - \hbar - 1)u + \int_0^\infty (e^{-ur} + 1 - ur)\Pi(dr).\end{aligned} \tag{4.18}$$

Since, by Lemma 4.3.1 we have $\int_0^\infty r\Pi(dr) < \infty$, we recognize Ψ as the Laplace exponent of a spectrally negative Lévy process with a finite mean given by $\Psi'(0^+) = \mu - \hbar - 1$. In particular, on $[0,\infty)$, Ψ is a convex, eventually increasing, twice differentiable function which is always zero at 0 and hence it has a strictly positive root r_0 if and only if $\mu < 1+\hbar$. By the Wiener-Hopf factorization of Lévy processes, see e.g. [74, Chapter 6.4], we get, when $\Psi'(0^+) \geqslant 0$ (resp. $\Psi'(0^+) < 0$) that $\Psi(u) = u\phi(u)$ (resp. $\Psi(u) = (u - \mathrm{r}_0)\phi(u)$)

for a Bernstein function ϕ. The limit then follows from the well-known result that $\lim_{u\to\infty} u^{-2}\Psi(u) = 1$, which can be obtained by dominated convergence since Π is a finite measure, and this completes the proof of the first item. Next, we will show that $\Psi(1) > 0$, which, by the convexity of Ψ is equivalent to $\mathrm{r}_0 \in [0, 1)$. Indeed, from (4.18) and an application of Fubini's theorem we get

$$\Psi(1) = \mu - \hbar + \int_0^\infty (1 - e^{-r})\overline{\Pi}(r)dr > 0,$$

where we used the assumption that $\mu > \hbar$ and the positivity of $\overline{\Pi}$. Next, if $\mu \geqslant 1 + \hbar$ then, as $\mathrm{r}_0 = 0$ in this case, we get, from (4.18), that

$$\phi(u) = u + (\mu - \hbar - 1) + \int_0^\infty (e^{-ur} + 1 - ur)\Pi(dr),$$

and the expression for $\phi(0)$ readily follows. On the other hand if $\mathrm{r}_0 > 0$, then the fact that $\Psi(0) = -\mathrm{r}_0\phi(0) = 0$ forces $\phi(0) = 0$, which completes the proof of the second item. Next, write $\Psi_1(u) = \frac{u}{u+1}\Psi(u+1)$ so that, according to [34, Proposition 2.2], we get that Ψ_1 is also the Laplace exponent of a spectrally negative Lévy process whose Gaussian component is 1, mean is $\mu_{\phi^{\checkmark}_{\mathrm{r}_1}}$, and Lévy measure is $\Pi_{\phi^{\checkmark}_{\mathrm{r}_1}}$. Observe that $\Psi_1'(0^+) = \Psi(1) > 0$ and

$$\Psi_1(u) = \frac{u}{u+1}(u + 1 - \mathrm{r}_0)\phi(u+1) = u\frac{u + \mathrm{r}_1}{u+1}\phi(u+1) = u\phi^{\checkmark}_{\mathrm{r}_1}(u),$$

so, by the Wiener-Hopf factorization of Ψ_1, it follows that $\phi^{\checkmark}_{\mathrm{r}_1}$ is a Bernstein function. Moreover, integration by parts of $\Pi_{\phi^{\checkmark}_{\mathrm{r}_1}}$ gives

$$\hbar_{\phi^{\checkmark}_{\mathrm{r}_1}} = \int_0^\infty \overline{\Pi}_{\phi^{\checkmark}_{\mathrm{r}_1}}(r)dr = \int_0^\infty e^{-r}\overline{\Pi}(r)dr \leqslant \hbar < \infty,$$

where the boundary terms are easily seen to evaluate to 0. Finally, using the assumption that $\mu > \hbar$ we get that $\mu_{\phi^{\checkmark}_{\mathrm{r}_1}} = 1 + \mu - \hbar_{\phi^{\checkmark}_{\mathrm{r}_1}} + \hbar_{\phi^{\checkmark}_{\mathrm{r}_1}} \geqslant 1 + \mu - \hbar + \hbar_{\phi^{\checkmark}_{\mathrm{r}_1}} > 1 + \hbar_{\phi^{\checkmark}_{\mathrm{r}_1}}$, while the condition $\boldsymbol{\lambda}_1 > \mu_{\phi^{\checkmark}_{\mathrm{r}_1}}$ follows from the assumption that $\boldsymbol{\lambda}_1 > \mathbf{1}_{\{\mu<1+\hbar\}} + \mu = 1 + \mu = \mu_{\phi^{\checkmark}_{\mathrm{r}_1}}$. □

4.3.2 Proof of Theorem 4.2.1(1)

Before we begin we provide an analytical result, which will allow us to show that the support of β is $[0, 1]$ and will also be used in subsequent proofs. We say that a linear operator Λ is a Markov multiplicative kernel if $\Lambda f(x) = \mathbb{E}[f(xI)]$ for some random variable I. With the definition of d_ϕ in (4.13), we let, for any $\varepsilon \in (0, d_\phi] \cup \{d_\phi\}$,

$$\boldsymbol{d}_{1,\varepsilon} = \mathbf{1}_{\{\mu<1+\hbar\}} + (d_\phi + 1 - \varepsilon)\mathbf{1}_{\{\mu\geqslant 1+\hbar\}}, \tag{4.19}$$

recalling that when $d_\phi = 0$ then $\varepsilon = 0$, so that at least $\boldsymbol{d}_{1,\varepsilon} \geqslant 1$. Note that $\boldsymbol{d}_{1,\varepsilon} = \boldsymbol{d}_{\mathsf{r}_1,\varepsilon}$ when $\mathsf{r}_1 = 1$ explaining the notation. By [99, Lemma 10.3], the mapping

$$u \mapsto \phi_{\boldsymbol{d}_{1,\varepsilon}}(u) = \frac{u}{u + \boldsymbol{d}_{1,\varepsilon} - 1}\phi(u) \tag{4.20}$$

is a Bernstein function, writing simply $\phi_1 = \phi$, and by Proposition 4.4(1) of the same paper we also have that, for any $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$, the mapping

$$u \mapsto \phi^*_{\mathfrak{m}}(u) = \frac{\phi(u)}{u + \mathfrak{m} - 1} \tag{4.21}$$

is a Bernstein function. We define the following linear operators acting on the space of polynomials $\mathcal{P}$, recalling that for $n \in \mathbb{N}$, $p_n(x) = x^n$,

$$\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} p_n(x) = \frac{(\boldsymbol{d}_{1,\varepsilon})_n}{W_\phi(n+1)} p_n(x), \tag{4.22}$$

$$\mathrm{V}_{\phi^*_{\mathfrak{m}}} p_n(x) = \frac{W_\phi(n+1)}{(\mathfrak{m})_n} p_n(x), \tag{4.23}$$

and

$$\mathrm{U}_{\phi^{\checkmark}_{\mathsf{r}_1}} p_n(x) = \frac{\phi^{\checkmark}_{\mathsf{r}_1}(0)}{\phi^{\checkmark}_{\mathsf{r}_1}(n)} p_n(x), \tag{4.24}$$

where $\mathrm{V}_{\phi^*_{\mathfrak{m}}}$ is defined for any $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$, and $\phi^{\checkmark}_{\mathsf{r}_1}$ was defined in (4.17). We write $\mathcal{B}(C([0, 1]))$ for the unital Banach algebra of bounded linear operators on $C([0, 1])$ and say that a linear operator between two Banach spaces is a quasi-affinity if it has trivial kernel and dense range.

Lemma 4.3.3. *The operators* $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}$, $\mathrm{V}_{\phi^*_{\mathfrak{m}}}$ *and* $\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}$ *defined in* (4.22) *are Markov multiplicative kernels associated to random variables* $X_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}$, $X_{\phi^*_{\mathfrak{m}}}$ *and* $X_{\phi^{\checkmark}_{\mathrm{r}_1}}$, *respectively, valued in* $[0,1]$, *and hence moment determinate. Furthermore, all operators belong to* $\mathcal{B}(C([0,1]))$, *and* $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}$ *is a quasi-affinity on* $C([0,1])$ *while* $\mathrm{V}_{\phi^*_{\mathfrak{m}}}$ *and* $\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}$ *have dense range in* $C([0,1])$.

Proof. The claims regarding the operators $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}$ and $\mathrm{V}_{\phi^*_{\mathfrak{m}}}$, and their respective random variables, have been proved in [99], see e.g. Proposition 6.7(1), Theorem 5.2, and Section 7.1 therein. Let $W : [0,\infty) \to [0,\infty)$ be the function characterized by its Laplace transform via

$$\int_0^\infty e^{-ux} W(x)dx = \frac{1}{\Psi(u)}, \quad u > 0,$$

and note that W is increasing and, since Ψ has a Gaussian component, it is at least continuously differentiable, see e.g. [74, Section 8.2]. Then $X_{\phi^{\checkmark}_{\mathrm{r}_1}}$ is the random variable whose law is given by

$$\mathbb{P}(X_{\phi^{\checkmark}_{\mathrm{r}_1}} \in dx) = \phi^{\checkmark}_{\mathrm{r}_1}(0) W'(-\log x)dx, \quad x \in [0,1],$$

which is clearly supported on $[0,1]$, and the claims concerning $\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}$ were shown in [100, Lemma 4.2], where we note that $W(0) = 0$ since Ψ has a Gaussian component. $\square$

Now, suppose $\mu \geqslant 1 + \hbar$ so that, by Lemma 4.3.2, $\mathrm{r}_1 = 1$. Then, for all $n \in \mathbb{N}$, (4.6) reduces to

$$\beta[p_n] = \frac{W_\phi(n+1)}{(\boldsymbol{\lambda}_1)_n}.$$

Since $\boldsymbol{\lambda}_1 > \mu \geqslant 1$, we get that $\phi^*_{\boldsymbol{\lambda}_1}$ as in (4.21) is a Bernstein function. Indeed, in the case when $\mu = 1$ we clearly must have $\hbar = 0$, and the function $u \mapsto \frac{u}{u+\boldsymbol{\lambda}_1-1}$ is Bernstein since $\boldsymbol{\lambda}_1 > 1$, see e.g. [109, Chapter 16], while on the other hand the same Proposition

4.4(1) guarantees that $\phi^*_{\lambda_1}$ is a Bernstein function. Thus, one straightforwardly checks that, for all $n \in \mathbb{N}$,

$$\beta[p_n] = W_{\phi^*_{\lambda_1}}(n+1)$$

which implies from [19] that, in this case, $(\beta[p_n])_{n \geqslant 0}$ is indeed a determinate Stieltjes moment sequence of a probability measure β, and its absolute continuity follows from [96, Proposition 2.4]. Now suppose $\mu < 1 + \hbar$ so that $\lambda_1 > 1 + \mu > 1$ and observe that (4.6) factorizes as

$$\beta[p_n] = \frac{W_\phi(n+1)}{(\lambda_1)_n} \frac{(\mathrm{r}_1)_n}{n!},$$

where the first term in the product is a Stieltjes moment sequence by the above arguments, and the second term is the moment sequence of a beta distribution, see e.g. (4.68). Consequently, in this case one also has that $(\beta[p_n])_{n \geqslant 0}$ is a Stieltjes moment sequence, and we temporarily postpone the proof of its moment determinacy, and its absolute continuity, to after the proof of Lemma 4.3.4. For our next result we write $(\beta_{\phi^{\checkmark}_{\mathrm{r}_1}}[p_n])_{n \geqslant 0}$ for the sequence obtained from (4.6) by replacing ϕ with $\phi^{\checkmark}_{\mathrm{r}_1}$ defined in (4.17), and with the same λ_1.

Lemma 4.3.4. *With $\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}$ as in* (4.14)*, the following factorization of operators holds on the space $\mathcal{P}$,*

$$\beta \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} = \beta_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}, \quad \beta_{\mathfrak{m}} \mathrm{V}_{\phi^*_{\mathfrak{m}}} = \beta, \quad \textit{and} \quad \beta_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} = \beta, \tag{4.25}$$

where the second identity holds for $\mu \geqslant 1 + \hbar$, while the third holds for $\mu < 1 + \hbar$.

Remark 4.3.1. Once we establish the moment determinacy of β for $\mu < 1 + \hbar$, then the factorizations of operators in Lemma 4.3.4 extends to the space of bounded measurable functions. Indeed, (4.25) implies

$$B_\phi \times X_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} \stackrel{(d)}{=} B_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}$$

where B_ϕ and $B_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}$ are random variables with laws β and $\beta_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}$, respectively, and $\times$ denotes the product of independent random variables.

Proof. Observe, from (4.22), that for any $n \in \mathbb{N}$,

$$\beta[\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} p_n] = \frac{(\boldsymbol{d}_{1,\varepsilon})_n}{W_\phi(n+1)} \beta[p_n] = \frac{(\boldsymbol{d}_{1,\varepsilon})_n}{W_\phi(n+1)} \frac{(\mathsf{r}_1)_n}{(\boldsymbol{\lambda}_1)_n} \frac{W_\phi(n+1)}{n!} = \frac{(\boldsymbol{d}_{1,\varepsilon})_n}{n!} \frac{(\mathsf{r}_1)_n}{(\boldsymbol{\lambda}_1)_n}.$$

By considering the cases $\mathsf{r}_1 = 1$ and $\mathsf{r}_1 < 1$ separately we obtain the desired right-hand side, noting that $\beta_{\boldsymbol{d}_{1,\varepsilon}}$ is well-defined, i.e. $\boldsymbol{\lambda}_1 > d_\phi + 1$, due to $\boldsymbol{\lambda}_1 > \mu = (\mu - \hbar) + \hbar$ and [99, Proposition 4.4(1)]. For the second claim we get that for any $n \in \mathbb{N}$ and since, by Lemma 4.3.2(3), $\mu \geqslant 1 + \hbar$ if and only if $\mathsf{r}_1 = 1$,

$$\beta_{\mathfrak{m}}[\mathrm{V}_{\phi_{\mathfrak{m}}^*} p_n] = \frac{W_\phi(n+1)}{(\mathfrak{m})_n} \beta_{\mathfrak{m}}[p_n] = \frac{W_\phi(n+1)}{(\mathfrak{m})_n} \frac{(\mathfrak{m})_n}{(\boldsymbol{\lambda}_1)_n} = \frac{W_\phi(n+1)}{(\boldsymbol{\lambda}_1)_n} = \beta[p_n],$$

which, by linearity, completes the proof. For the last claim we have, by Lemma 4.3.2(3) and using the notation therein, that $\mu_{\phi_{\mathsf{r}_1}^{\checkmark}} \geqslant 1 + \hbar_{\phi_{\mathsf{r}_1}^{\checkmark}}$ and thus 0 is the only non-negative root of $u \mapsto u\phi_{\mathsf{r}_1}^{\checkmark}(u)$. Consequently

$$\beta_{\phi_{\mathsf{r}_1}^{\checkmark}}[p_n] = \frac{W_{\phi_{\mathsf{r}_1}^{\checkmark}}(n+1)}{(\boldsymbol{\lambda}_1)_n}.$$

Some straightforward computations give that, for any $n \in \mathbb{N}$,

$$W_{\phi_{\mathsf{r}_1}^{\checkmark}}(n+1) = \frac{(\mathsf{r}_1+1)_n}{(n+1)!} \frac{W_\phi(n+2)}{\phi(1)}, \quad \text{and} \quad \mathrm{U}_{\phi_{\mathsf{r}_1}^{\checkmark}} p_n(x) = \frac{\phi_{\mathsf{r}_1}^{\checkmark}(0)}{\phi_{\mathsf{r}_1}^{\checkmark}(n)} = \frac{\mathsf{r}_1 \phi(1)(n+1)}{(n+\mathsf{r}_1)\phi(n+1)} p_n(x).$$

Putting these observations together yields

$$\beta_{\phi_{\mathsf{r}_1}^{\checkmark}}[\mathrm{U}_{\phi_{\mathsf{r}_1}^{\checkmark}} p_n] = \frac{1}{(\boldsymbol{\lambda}_1)_n} \frac{\mathsf{r}_1(\mathsf{r}_1+1)_n}{(n+\mathsf{r}_1)} \frac{(n+1)}{(n+1)!} \frac{W_\phi(n+2)}{\phi(n+1)} = \frac{1}{(\boldsymbol{\lambda}_1)_n}(\mathsf{r}_1)_n \frac{1}{n!} W_\phi(n+1) = \beta[p_n],$$

where we repeatedly used the recurrence relations for both the gamma function and the function W_ϕ, see e.g. (4.5). □

Now suppose that, when $\mu < 1 + \hbar$, the measure β is moment indeterminate. Then, as the sequence $\left(\frac{(\boldsymbol{d}_{1,\varepsilon})_n}{W_\phi(n+1)}\right)_{n \geqslant 0}$ is a non-vanishing Stieltjes moment sequence, it follows, by (4.25) and invoking [19, Lemma 2.2], that the beta distribution $\beta_{\boldsymbol{d}_{\mathsf{r}_1,\varepsilon}}$ is moment indeterminate, which is a contradiction. Therefore we conclude that, in all cases, β is moment determinate and consequently we have the extended factorization of operators as described in Remark 4.3.1. To get the absolute continuity of β in the case $\mu < 1+\hbar$ we note that the factorization $\beta[p_n] = \frac{W_\phi(n+1)}{(\lambda_1)_n}\frac{(\mathrm{r}_1)_n}{n!}$ implies, by moment determinacy, that β is the product convolution of two absolutely continuous measures. Next, take $\varepsilon = d_\phi$ so that $\boldsymbol{d}_{\mathrm{r}_1,\varepsilon} = \mathrm{r}_1$, see (4.14). As in the proof Lemma 4.3.3, the distribution of X_ϕ, denoted by ι, satisfies $\mathrm{supp}(\iota) = [0,1]$, where $\mathrm{supp}(\iota)$ denotes the support of the measure ι. Consequently, since $\mathrm{supp}(\beta_{\mathrm{r}_1}) = [0,1]$, it follows from (4.25) that $\mathrm{supp}(\beta) = [a,b]$ for some $0 \leqslant a < b \leqslant 1$, which may be deduced from the corresponding factorization of random variables, see again Remark 4.3.1. To show that, in fact $\mathrm{supp}(\beta) = [0,1]$, we suppose that $b < 1$. Then, by (4.25) we have

$$0 < \beta_{\mathrm{r}_1}[\mathbf{1}_{(b,1]}] = \int_b^1 \beta[\mathbf{1}_{(b/y,1]}]\iota(dy) \leqslant \beta[\mathbf{1}_{(b,1]}] = 0,$$

which is a contradiction. If $\mu \geqslant 1+\hbar$ then, since $\mathrm{supp}(\beta_{\mathfrak{m}}) = [0,1]$ and $\mathrm{supp}(\beta) = [a,1]$, we deduce from (4.25) and similar arguments as above, that the distribution of $X_{\phi^*_{\mathfrak{m}}}$, say $\nu_{\mathfrak{m}}$, satisfies $\mathrm{supp}(\nu_{\mathfrak{m}}) = [c,1]$, for some$c \in [a,1)$. Assume $a > 0$. Then, from (4.25) we get that

$$0 = \beta[\mathbf{1}_{[0,a)}] = \int_c^1 \beta_{\mathfrak{m}}[\mathbf{1}_{[0,a/y)}]\nu_{\mathfrak{m}}(dy) \geqslant \beta_{\mathfrak{m}}[\mathbf{1}_{[0,a)}] > 0,$$

which is a contradiction. Therefore, $a = 0$, and we conclude that $\mathrm{supp}(\beta) = [0,1]$ in this case. The case when $\mu < 1+\hbar$ follows by similar arguments, with $\beta_{\mathfrak{m}}$ and $X_{\phi^*_{\mathfrak{m}}}$ replaced by $\beta_{\phi^{\checkmark}_{\mathrm{r}_1}}$ and $X_{\phi^{\checkmark}_{\mathrm{r}_1}}$, respectively, where we note that $\mathrm{supp}(\beta_{\phi^{\checkmark}_{\mathrm{r}_1}}) = [0,1]$ since $\mu_{\phi^{\checkmark}_{\mathrm{r}_1}} \geqslant 1 + h_{\phi^{\checkmark}_{\mathrm{r}_1}}$. This completes the proof of Theorem 4.2.1(1). □

4.3.3 Proof of Theorem 4.2.1(2)

We start by stating and proving the following more general intertwining that will be useful in subsequent proofs, recalling the definition of $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}$ in (4.22).

Proposition 4.3.1. *With $\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}$ and $\boldsymbol{d}_{1,\varepsilon}$ as in* (4.14) *and* (4.19)*, respectively, we have, for any $\varepsilon \in (0, d_\phi] \cup \{d_\phi\}$,*

$$\mathbb{J}\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} = \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} \mathbf{J}_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}, \quad on \quad \mathcal{P}. \tag{4.26}$$

Remark 4.3.2. Note that $\boldsymbol{\lambda}_1$ is the common parameter of the Jacobi type operators in (4.26) while the constant part of the affine drift, as well as the non-local components are different. The commonality of $\boldsymbol{\lambda}_1$ is what ensures the isospectrality of these operators, as their spectrum depends only on $\boldsymbol{\lambda}_1$, see Theorem 4.2.2(2) and (4.73).

We split the proof of Proposition 4.3.1 into two lemmas and, among other things, our proof hinges on the interesting observation that intertwining relations are stable under perturbation with an operator that commutes with the intertwining operator, see Lemma 4.3.6 below. Let $\mathbf{L}_\mu$ be the operator defined as

$$\mathbf{L}_\mu f(x) = x f''(x) + \mu f'(x) \tag{4.27}$$

and write $\mathbb{I}_h f(x) = -f' \diamond h(x)$ where h is as in Assumption 4.2.1, and set $\mathbb{L} = \mathbf{L}_\mu + \mathbb{I}_h$.

Lemma 4.3.5. *With the notation of Proposition 4.3.1 the following holds on $\mathcal{P}$,*

$$\mathbb{L}\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} = \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} \mathbf{L}_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}. \tag{4.28}$$

Proof. Using that $\hbar = \int_1^\infty h(r)dr$ and the symmetry of $\diamond$ we get, by straightforward calculation, that, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{L}p_n(x) &= n(n-1)p_{n-1}(x) + \mu n p_{n-1}(x) - n p_{n-1}(x)\int_1^\infty r^{-(n-1)} h(r) r^{-1} dr \\
&= n^2 p_{n-1}(x) + (\mu - \hbar - 1) n p_{n-1}(x) - n p_{n-1}(x)\int_1^\infty (1 - r^{-n}) h(r) dr \\
&= (n - \mathrm{r}_0)\phi(n) p_{n-1}(x).
\end{aligned}$$

Thus, combining this with (4.22) one obtains, for any $n \in \mathbb{N}$,

$$\mathbb{L}\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} p_n(x) = \frac{(\boldsymbol{d}_{1,\varepsilon})_n}{W_\phi(n+1)}(n - \mathrm{r}_0)\phi(n) p_{n-1}(x) = \frac{(\boldsymbol{d}_{1,\varepsilon})_n}{W_\phi(n)}(n - \mathrm{r}_0) p_{n-1}(x),$$

while on the other hand,

$$\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}\mathbf{L}_{\boldsymbol{d}_{\mathsf{r}_1,\varepsilon}}p_n(x) = n(n+\boldsymbol{d}_{\mathsf{r}_1,\varepsilon}-1)\frac{(\boldsymbol{d}_{1,\varepsilon})_{n-1}}{W_\phi(n)}p_{n-1}(x) = (n-\mathsf{r}_0)\frac{(\boldsymbol{d}_{1,\varepsilon})_n}{W_\phi(n)}p_{n-1}(x),$$

where the second equality follows by considering the cases $\mathsf{r}_1 = 1$ and $\mathsf{r}_1 < 1$ separately. The linearity of the involved operators completes the proof. $\square$

The next lemma allows us to identify a family of operators commuting with the Markov operators defined above, although, more generally, it is a statement on commuting operators and intertwinings. Denote by $\mathbf{D}_n$ the operator acting via $\mathbf{D}_n f(x) = x^n \frac{d^n}{x^n} f(x)$ and write $\mathrm{d}_y f(x) = f(yx)$, $y > 0$ for the dilation operator.

Lemma 4.3.6. *Let $\Lambda_\eta f(x) = \int_0^1 f(xy)\eta(dy)$, where η is any signed measure on $[0,1]$ endowed with the Borel sigma-algebra. Suppose for a linear operator $\mathbf{A}$ on $C([0,1])$ and any f in its domain we have*

$$\eta\mathbf{A}f = \mathbf{A}\eta f \quad \textit{and} \quad \mathrm{d}_y\mathbf{A}f = \mathbf{A}\mathrm{d}_y f, \quad \forall y > 0.$$

Then, for such functions,

$$\mathbf{A}\Lambda_\eta f = \Lambda_\eta \mathbf{A} f.$$

In particular, suppose that $\int_0^1 y^n|\eta|(dy) < \infty$, for all $n \in \mathbb{N}$, where $|\eta|$ stands for the total variation of the measure η. Then, for any $n \in \mathbb{N}$ we have for $f \in C^\infty([0,1])$

$$\mathbf{D}_n\Lambda_\eta f = \Lambda_\eta \mathbf{D}_n f.$$

Proof. Since

$$\Lambda_\eta f(x) = \int_0^1 f(xy)\eta(dy) = \int_0^1 \mathrm{d}_x f(y)\eta(dy) = \eta \mathrm{d}_x f$$

it follows that any operator $\mathbf{A}$ commuting with η and with d_x, for any $x > 0$, commutes with η, for suitable functions f. Next, the assumption on the measure η allows us to invoke Fubini's theorem and conclude that $\eta \mathbf{R}_n = \mathbf{R}_n \eta$. Finally, observing that, for any $n \in \mathbb{N}$ and $x, y > 0$,

$$\mathrm{d}_y \mathbf{D}_n f = y^n x^n f^{(n)}(yx) = \mathbf{D}_n \mathrm{d}_y f$$

completes the proof. □

Proof of Proposition 4.3.1. It is now an easy exercise to complete the proof of Proposition 4.3.1. Let us write

$$\mathbf{A} = \mathbf{D}_2 + \lambda_1 \mathbf{D}_1.$$

Then, for any $f \in \mathcal{P}$, we get by combining Lemma 4.3.5 and Lemma 4.3.6, that

$$\mathbb{J}\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} f = (\mathbb{L} - \mathbf{A})\,\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} f = \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} \left(\mathbf{L}_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}} - \mathbf{A}\right) f = \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} \mathbf{J}_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}} f,$$

where we also use the linearity of the involved operators. □

Having established the necessary intertwining relation we are now able to show that $\mathbb{J}$ extends to the generator of a Markov semigroup.

Lemma 4.3.7. *The operator* $(\mathbb{J}, \mathcal{P})$ *is closable in* $C([0,1])$*, and its closure is the infinitesimal generator of a Markov semigroup* $\mathbb{Q} = (\mathbb{Q}_t)_{t \geqslant 0}$ *on* $C([0,1])$.

Proof. We aim at invoking the Hille–Yosida–Ray Theorem for Markov generators, see [28, Theorem 1.30], which requires that both $\mathcal{P}$ and, for some (or all) $q > 0$, $(q - \mathbb{J})(\mathcal{P})$ are dense in $C([0,1])$, and that $\mathbb{J}$ satisfies the positive maximum principle on $\mathcal{P}$. Since the density of $\mathcal{P}$ in $C([0,1])$ follows from the Stone–Weierstrass Theorem, we focus on showing that $(q - \mathbb{J})(\mathcal{P})$ is dense in $C([0,1])$. To this end, set $\varepsilon = d_\phi$, and note, by Lemma 4.3.3, that Λ_ϕ is injective and bounded on $C([0,1])$, which gives that its inverse Λ_ϕ^{-1} is a closed, densely defined, linear operator on $\Lambda_\phi(\mathcal{P})$. Furthermore, since Λ_ϕ is a Markov multiplicative kernel it follows that it preserves the set of polynomials, i.e. $\Lambda_\phi(\mathcal{P}) = \mathcal{P}$, and consequently by injectivity we get $\Lambda_\phi^{-1}(\mathcal{P}) = \mathcal{P}$. Putting these observations together we deduce, from the first intertwining in Proposition 4.3.1, that

$$\mathbb{J} = \Lambda_\phi \mathbf{J}_{\mathsf{r}_1} \Lambda_\phi^{-1} \quad \text{on} \quad \mathcal{P},$$

and hence, for any $q > 0$,

$$(q - \mathbb{J})(\mathcal{P}) = (q - \Lambda_\phi \mathbf{J}_{\mathsf{r}_1} \Lambda_\phi^{-1})(\mathcal{P}) = \Lambda_\phi(q - \mathbf{J}_{\mathsf{r}_1})\Lambda_\phi^{-1}(\mathcal{P}) = \Lambda_\phi(q - \mathbf{J}_{\mathsf{r}_1})(\mathcal{P}), \tag{4.29}$$

where we use the trivial commutation of Λ_ϕ with q. Next, the assumption on $\boldsymbol{\lambda}_1$ guarantees that $\boldsymbol{\lambda}_1 > \mathsf{r}_1$, since we always have $\boldsymbol{\lambda}_1 > 1$ and $\mathsf{r}_1 = 1 - \mathsf{r}_0 \in (0, 1]$. Thus it follows that $\mathcal{P}$ belongs to the domain of $\mathbf{J}_{\mathsf{r}_1}$, which is explicitly described in (4.67), and as $\mathcal{P}$ is an invariant subspace for the classical Jacobi semigroup $\mathbf{Q}^{(\mathsf{r}_1)}$ we get that $\mathcal{P}$ is a core for $\mathbf{J}_{\mathsf{r}_1}$, see [28, Lemma 1.34]. Hence, by the converse of the Hille–Yosida–Ray Theorem, we get that $(q - \mathbf{J}_{\mathsf{r}_1})(\mathcal{P})$ is dense in $C([0,1])$ for any $q > 0$. It is a straightforward exercise to show that the image of a dense subset under a bounded operator with dense range is also dense in the codomain. Thus it follows that $\Lambda_\phi(q - \mathbf{J}_{\mathsf{r}_1})(\mathcal{P})$, and from (4.29) we get that $(q - \mathbb{J})(\mathcal{P})$ is dense in $C([0,1])$ for any $q > 0$. Next, let $f \in \mathcal{P}$, set $f(x_0) = \sup_{x \in [0,1]} f(x)$, and observe that

$$f(ax_0) - f(x_0) \leqslant 0 \text{ for any } a \in [0,1]. \tag{4.30}$$

Using Lemma 4.3.1 we can write $\mathbb{J} f(x_0)$ as

$$\mathbb{J} f(x_0) = x_0(1 - x_0) f''(x_0) - (\boldsymbol{\lambda}_1 x_0 - \mu) f'(x_0) + \int_0^\infty (f(e^{-r}x_0) - f(x_0)) \frac{\Pi(dr)}{x_0}, \tag{4.31}$$

where we note that since $\hbar = \int_0^\infty r\Pi(dr)$ these two terms cancel. Then, from (4.30) it follows that, for $x_0 \in [0,1]$,

$$\int_0^\infty (f(e^{-r}x_0) - f(x_0)) \frac{\Pi(dr)}{x_0} \leqslant 0.$$

Now suppose that $x_0 \in (0,1]$. From the previous equation it suffices, in this case, to only consider the terms involving derivatives in (4.31). When $x \in (0,1)$ then $f''(x_0) \leqslant 0$ and

$f'(x_0) = 0$, and thus plainly $\mathbb{J}f(x_0) \leqslant 0$. On the other hand, if $x_0 = 1$ then we must have $f'(1) \geqslant 0$ and so $\mathbb{J}f(1) \leqslant -(\boldsymbol{\lambda}_1 - \mu) f'(1) \leqslant 0$, where the latter follows trivially from $\boldsymbol{\lambda}_1 > \mu$. Finally assume that $x_0 = 0$, so that then $f'(0) \leqslant 0$. For x small we have $\frac{f(e^{-r}x)-f(x)}{x} = e^{-r}f'(0) + R(x)$, where the function R satisfies $\overline{\lim}_{x\to 0} \frac{|R(x)|}{x} < \infty$, from which it follows that $\mathbb{J}f(0) \leqslant (\mu + \int_0^\infty e^{-r}\Pi(dr))f'(0) \leqslant 0$, since both μ and $\int_0^\infty e^{-r}\Pi(dr)$ are clearly positive. Thus $\mathbb{J}$ satisfies the maximum principle (and in particular the positive maximum principle) on $\mathcal{P}$, which gives that $\mathbb{J}$ extends to the generator of a Feller semigroup $\mathbb{Q} = (\mathbb{Q}_t)_{t\geqslant 0}$, in the sense of [28, Theorem 1.30]. However, the fact that $\mathbb{Q}$ is conservative, i.e. $\mathbb{Q}_t\mathbf{1}_{[0,1]} = \mathbf{1}_{[0,1]}$, follows from $\mathbb{J}\mathbf{1}_{[0,1]} = 0$, since

$$\mathbb{Q}_t\mathbf{1}_{[0,1]} - \mathbf{1}_{[0,1]} = \int_0^t \mathbb{Q}_s\mathbb{J}\mathbf{1}_{[0,1]}\,ds = 0,$$

see e.g. [28, Lemma 1.26]. □

Proof of Theorem 4.2.1(2). To complete the proof it suffices to establish the claims concerning the invariant measure. For $f \in \mathcal{P}$ we have,

$$\beta[\mathbb{J}\Lambda_\phi f] = \beta[\Lambda_\phi \mathbf{J}_{\mathrm{r}_1} f] = \beta_{\mathrm{r}_1}[\mathbf{J}_{\mathrm{r}_1} f] = 0, \tag{4.32}$$

where successively we have used Proposition 4.3.1 (setting $\varepsilon = d_\phi$), Lemma 4.3.4, and the fact that β_{r_1} is the invariant measure of $\mathbf{J}_{\mathrm{r}_1}$. The fact that (4.32) holds on the dense subset $\Lambda_\phi(\mathcal{P}) = \mathcal{P}$ of $C([0,1])$ implies that β is an invariant measure for $\mathbb{Q}$, see for instance [9, Section 1.4.1]. To show uniqueness, we note that any other invariant measure $\widetilde{\beta}$ for $\mathbb{J}$ must first, have all positive moments finite, and also satisfy

$$\widetilde{\beta}[\mathbb{J}\Lambda_\phi f] = \widetilde{\beta}[\Lambda_\phi \mathbf{J}_{\mathrm{r}_1} f] = 0,$$

for any $f \in \mathcal{P}$, where we used that $\Lambda_\phi(\mathcal{P}) = \mathcal{P}$. By uniqueness of the invariant measure for $\mathbf{J}_{\mathrm{r}_1}$ we then get the factorization of operators $\widetilde{\beta}\Lambda_\phi = \beta_{\mathrm{r}_1}$, on $\mathcal{P}$, and the moment determinacy of β then forces $\widetilde{\beta} = \beta$. Finally the extension of $\mathbb{Q}$ to a Markov semigroup on $\mathrm{L}^2(\beta)$ is classical, see for instance the remarks before the theorem, and it is well-known that if $\mathbb{Q}$ has a unique invariant measure then it is an ergodic Markov semigroup, see e.g. [39, Theorem 5.16]. $\square$

4.3.4 Proof of Proposition 4.2.1

Before giving the proof of Proposition 4.2.1 we state and prove two auxiliary results, the first of which characterizes w_n in a distributional sense. To this end we recall that the Mellin transform of a finite measure ν, resp. of an integrable function f, on $\mathbb{R}_+$ is given by

$$\mathcal{M}_\nu(z) = \nu[p_{z-1}] = \int_0^\infty x^{z-1}\nu(dx), \quad \text{resp. } \mathcal{M}_f(z) = \int_0^\infty x^{z-1} f(x)dx,$$

which is valid for at least $z \in 1 + i\mathbb{R}$. We denote by $\mathrm{E}_{p,q}$ (resp. $\mathrm{E}'_{p,q}$), with $p < q$ reals, the linear space of functions $f \in C^\infty(\mathbb{R}_+)$ such that there exist $c, c' > 0$ for which, for all $k \in \mathbb{N}$,

$$\lim_{x\to 0}\left| x^{k+1-p-c}\frac{d^k}{dx^k}f(x)\right| = 0 \quad \text{and} \quad \lim_{x\to\infty}\left| x^{k+1+c'-q}\frac{d^k}{dx^k}f(x)\right| = 0,$$

(resp. the linear space of continuous linear functionals on $\mathrm{E}_{p,q}$ endowed with a structure of a countably multinormed space as described in [90, p. 231]). Next, we write, for any $n \in \mathbb{N}$ and $x \in [0,1]$,

$$\mathrm{p}_n^{(\mathrm{r}_1)}(x) = \beta_{\mathrm{r}_1}(x)\mathcal{P}_n^{(\mathrm{r}_1)}(x) = \frac{(\lambda_1 - \mathrm{r}_1)_n}{(\lambda_1)_n}\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\mathbf{R}_n\beta_{\lambda_1+n,\mathrm{r}_1}(x),$$

where $\mathbf{R}_n$ denotes the Rodrigues operator defined in (4.8) and the last identity follows from (4.72). For suitable a we also extend the Pochhammer notation $(a)_z$ to any $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and, for the remainder of the proofs, we shall write $\langle\cdot,\cdot\rangle_\beta$ for the $\mathrm{L}^2(\beta)$-inner product, adopting the same notation for other weighted Hilbert spaces.

Proposition 4.3.2. *For any $n \in \mathbb{N}$, the Mellin convolution equation*

$$\widehat{\Lambda}_\phi \hat{f}(x) = \mathrm{p}_n^{(\mathrm{r}_1)}(x) \tag{4.33}$$

has a unique solution, in the sense of distributions, given by

$$w_n(x) = \frac{(\lambda_1 - \mathrm{r}_1)_n}{(\lambda_1)_n}\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\, \mathbf{R}_n(\beta_{\lambda_1+n,\lambda_1} \diamond \beta)(x) \in \mathrm{E} = \cup_{q>\mathrm{r}_0}\mathrm{E}_{\mathrm{r}_0,q}. \tag{4.34}$$

Its Mellin transform is given, for any $z \in \mathbb{C}$ with $\mathrm{Re}(z) > \mathrm{r}_0$, *by*

$$\mathcal{M}_{w_n}(z) = \frac{(-2)^n}{n!}\frac{(\lambda_1 - \mathrm{r}_1)_n}{(\lambda_1)_n}\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\frac{\Gamma(z)}{\Gamma(z-n)}\mathcal{M}_{\beta_{\lambda_1+n,\lambda_1}}(z)\mathcal{M}_\beta(z). \tag{4.35}$$

Proof. The proof is an adaptation of the proof of [99, Lemma 8.5] to the current setting. We write $\iota^*(y) = \iota(1/y)1/y$ where ι is the density of X_ϕ, which is well-known to exist, and let Λ_ϕ^* be the operator characterized, for any $f \in \mathrm{L}^2(\beta)$, by

$$\Lambda_\phi^* f(x) = \frac{1}{\beta_{\mathrm{r}_1}(x)}\int_0^1 f(xy)\beta(xy)\iota^*(y)dy = \frac{1}{\beta_{\mathrm{r}_1}(x)}\widehat{\Lambda}_\phi(f\beta)(x)$$

where $\widehat{\Lambda}_\phi f(x) = \int_0^1 f(xy)\iota^*(y)dy$ and $\beta(x)$ is the density of the invariant measure β. Then, for any non-negative functions $f \in \mathrm{L}^2(\beta_{\mathrm{r}_1})$ and $g \in \mathrm{L}^2(\beta)$, we get

$$\begin{aligned}
\langle \Lambda_\phi f, g\rangle_\beta &= \int_0^\infty \left(\int_0^\infty f(xy)\iota(y)dy\right) g(x)\beta(x)dx \\
&= \int_0^\infty f(r)\beta_{\mathrm{r}_1}{}^{-1}(r)\left(\int_0^\infty \iota(r/x)g(x)\beta(x)/xdx\right)\beta_{\mathrm{r}_1}(r)dr \\
&= \int_0^\infty f(r)\beta_{\mathrm{r}_1}{}^{-1}(r)\left(\int_0^\infty g(rv)\beta(rv)\iota^*(v)dv\right)\beta_{\mathrm{r}_1}(r)dr \\
&= \langle f, \tfrac{1}{\beta_{\mathrm{r}_1}}\widehat{\Lambda}_\phi g\beta\rangle_{\beta_{\mathrm{r}_1}}.
\end{aligned}$$

However, $f \in \mathrm{L}^2(\beta)$ implies that $|f| \in \mathrm{L}^2(\beta)$, so we conclude that the above holds for any $f \in \mathrm{L}^2(\beta)$ and $g \in \mathrm{L}^2(\beta_{\mathrm{r}_1})$. Thus Λ_ϕ^* is the $\mathrm{L}^2(\beta)$-adjoint of the Markov multiplicative kernel Λ_ϕ which justifies the notation and, by Lemma 4.3.10, we have $\Lambda_\phi^* \in \mathscr{B}(\mathrm{L}^2(\beta), \mathrm{L}^2(\beta_{\mathrm{r}_1}))$. Next, since the mapping $z \mapsto \mathcal{M}_\iota(z) = \mathcal{M}_{\Lambda_\phi}(z) = \mathcal{M}_{\iota^*}(1-z)$ is analytic on $\mathrm{Re}(z) > 0$ and $|\mathcal{M}_{\Lambda_\phi}(z)| \leqslant \mathcal{M}_{\Lambda_\phi}(\mathrm{Re}(z)) < \infty$, for any $\mathrm{Re}(z) > 0$, see for instance [99, Proposition 6.8], we deduce from [90, Theorem 11.10.1] that $\iota \in \mathrm{E}'_{0,q}$, for every $q > 0$ and $\iota^* \in \mathrm{E}'_{p,1}$ for every $p < 1$. Consequently, since for any $f \in \mathrm{E}_{0,q}$, $q > 0$,

$$\Lambda_\phi f(x) = \int_0^1 f(xy)\iota(y)dy = \langle \iota, f(x.)\rangle_{\mathrm{E}'_{0,q},\mathrm{E}_{0,q}},$$

we have, for any $w \in \mathrm{E}'_{0,q}$, with $q > 0$,

$$\langle \widehat{\Lambda}_\phi w, f\rangle_{\mathrm{E}'_{0,q},\mathrm{E}_{0,q}} = \langle w \surd\, \iota, f\rangle_{\mathrm{E}'_{0,q},\mathrm{E}_{0,q}} = \langle w, \Lambda_\phi f\rangle_{\mathrm{E}'_{0,q},\mathrm{E}_{0,q}},\ \forall f \in \mathrm{E}_{0,q},$$

where we recall that the last relation is a definition given in [90, 11.11.1], and where we used the notation $\widehat{\Lambda}_\phi w := w \surd\, \iota$ with $w \surd\, \iota$ being the Mellin convolution operator in the space of distributions, see [90, Chapter 11.11] for definitions and notation. Here also note that for $w \in \mathrm{L}^1(\iota^*)$, we have the identities $w \surd\, \iota(x) = \int_0^\infty w(x/y)\iota(y)dy/y = \int_0^\infty w(xy)\iota^*(y)dy = \widehat{\Lambda}_\phi w(x)$, which justifies the notation above. Next, recalling that $\widehat{\Lambda}_\phi w = w \surd\, \iota$ and taking $w \in \mathrm{E}'_{0,q}$, $q > 0$, and, with $0 < \mathrm{Re}(z) < q$, $p_z(x) = x^z \in \mathrm{E}_{0,q}$, we have

$$\mathcal{M}_{\widehat{\Lambda}_\phi w}(z) = \langle w \surd \iota, p_{z-1}\rangle_{\mathrm{E}'_{0,q},\mathrm{E}_{0,q}} = \langle w, \Lambda_\phi p_{z-1}\rangle_{\mathrm{E}'_{0,q},\mathrm{E}_{0,q}} = \mathcal{M}_{\Lambda_\phi}(z)\mathcal{M}_w(z),$$

where we used that $\Lambda_\phi p_{z-1}(x) = p_{z-1}(x)\mathcal{M}_{\Lambda_\phi}(z)$. On the other hand, for any $n \in \mathbb{N}$, we get, from [90, 11.7.7] and a simple computation,

$$\mathcal{M}_{\mathrm{p}_n^{(\mathrm{r}_1)}}(z) = \frac{(-2)^n}{n!}(\lambda_1 - \mathrm{r}_1)_n\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\frac{\Gamma(z)}{\Gamma(z-n)}\frac{(\mathrm{r}_1)_{z-1}}{(\lambda_1)_{z+n-1}}.$$

Putting pieces together, we deduce that the Mellin transform of a solution to (4.33) takes the form

$$\begin{aligned}
\mathcal{M}_{\hat{f}}(z) = \frac{\mathcal{M}_{\mathrm{p}_n^{(\mathrm{r}_1)}}(z)}{\mathcal{M}_{\Lambda_\phi}(z)} &= \frac{(-2)^n}{n!}\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\frac{\Gamma(z)}{\Gamma(z-n)}(\lambda_1 - \mathrm{r}_1)_n\frac{(\mathrm{r}_1)_{z-1}}{(\lambda_1)_{z+n-1}}\frac{W_\phi(z)}{\Gamma(z)} \\
&= \frac{(-2)^n}{n!}\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\frac{\Gamma(z)}{\Gamma(z-n)}\frac{(\lambda_1 - \mathrm{r}_1)_n}{(\lambda_1)_n}\frac{(\lambda_1)_{z-1}}{(\lambda_1 + n)_{z-1}}\mathcal{M}_\beta(z) \\
&= \frac{(-2)^n}{n!}\frac{(\lambda_1 - \mathrm{r}_1)_n}{(\lambda_1)_n}\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\frac{\Gamma(z)}{\Gamma(z-n)}\mathcal{M}_{\beta_{\lambda_1+n,\lambda_1}}(z)\mathcal{M}_\beta(z).
\end{aligned}$$

Next, we have that for $\mathrm{Re}(z) > \mathrm{r}_0$, $z \mapsto \mathcal{M}_\beta(z)$ is analytical with $|\mathcal{M}_\beta(z)| \leqslant \mathcal{M}_\beta(\mathrm{Re}(z)) < \infty$, so we deduce, from [90, Theorem 11.10.1] that $\beta \in \mathrm{E}'_{\mathrm{r}_0,q}$, for any $q > \mathrm{r}_0$. Hence, by means of [90, 11.7.7], we have that $\hat{f} \in \mathrm{E}'_{\mathrm{r}_0,q}$ with $\hat{f} = w_n$ is a solution to (4.33), and the uniqueness of the solution follows from the uniqueness of Mellin transforms in the distributional sense. □

Lemma 4.3.8. *For $a > \mathrm{r}_0$ fixed and $b \in \mathbb{R}$, we have the estimate*

$$\left|\mathcal{M}_\beta(a+ib)\right| \leqslant C|b|^{-\Delta},$$

which holds uniformly on bounded a-intervals and for $|b|$ large enough, where $C > 0$ is a constant depending on ϕ and a.

Proof. By uniqueness of W_ϕ in the space of positive-definite functions, the Mellin transform of β is given by

$$\mathcal{M}_\beta(z) = \frac{(\mathrm{r}_1)_{z-1}}{(\lambda_1)_{z-1}} \frac{W_\phi(z)}{\Gamma(z)},$$

where $z = a + ib$, with $a > \mathrm{r}_0 \geqslant 0$. Invoking [98, Equation (6.20)] we get the following estimate, which holds uniformly on bounded a-intervals and for $|b|$ large enough,

$$\left|\frac{W_\phi(a+ib)}{\Gamma(a+ib)}\right| \leqslant C_\phi |b|^{\phi(0)+\overline{\nu}(0)}, \tag{4.36}$$

with $C_\phi > 0$ a constant depending on ϕ, and where, for any $y > 0$, $\overline{\nu}(y) = \int_y^\infty \nu(ds)$ with ν denoting the Lévy measure of ϕ. Lemma 4.3.2(3) gives in all cases the expression of $\phi(0)$ and when $\mu \geqslant 1 + \hbar$, $\nu(dy) = \overline{\overline{\Pi}}(y)dy$ follows from (4.4). Thus to utilize the estimate in (4.36) we need to identify $\overline{\nu}(0)$ when $\mu < 1 + \hbar$, which we do as follows. First, let us write $\Psi(u) = (u - \mathrm{r}_0)\phi(u) = (u - \mathrm{r}_0)\phi_{\mathrm{r}_0}(u - \mathrm{r}_0)$, where $\phi_{\mathrm{r}_0}(u) = \phi(u + \mathrm{r}_0)$. From the fact that $\Psi(\mathrm{r}_0) = 0$ we conclude that $\Psi(u + \mathrm{r}_0) = u\phi_{\mathrm{r}_0}(u)$ is itself a function of the form (4.18), which gives $\nu_{\mathrm{r}_0}(dy) = \overline{\overline{\Pi}}_{\mathrm{r}_0}(y)dr$, $y > 0$, where Π_{r_0} is the Lévy measure of $\Psi(u + \mathrm{r}_0)$ obtained via (4.18) and ν_{r_0} denotes the Lévy measure of ϕ_{r_0}. As ϕ_{r_0} is a

Bernstein function it is given, for $u \geqslant -\mathsf{r}_0$, by

$$\phi_{\mathsf{r}_0}(u) = \kappa + u + u \int_0^\infty e^{-uy}\overline{\rho}_{\mathsf{r}_0}(y)dy,$$

for some $\kappa \geqslant \mathsf{r}_0$. Thus, for $u \geqslant 0$,

$$\begin{aligned}\phi(u) = \phi_{\mathsf{r}_0}(u-\mathsf{r}_0) &= \kappa + (u-\mathsf{r}_0) + (u-\mathsf{r}_0)\int_0^\infty e^{-(u-\mathsf{r}_0)y}\overline{\nu}_{\mathsf{r}_0}(y)dy \\ &= (\kappa-\mathsf{r}_0) + u + u\int_0^\infty e^{-uy}e^{\mathsf{r}_0 y}\overline{\nu}_{\mathsf{r}_0}(y)dy - \mathsf{r}_0\int_0^\infty e^{-uy}e^{\mathsf{r}_0 y}\overline{\nu}_{\mathsf{r}_0}(y)dy \\ &= (\kappa-\mathsf{r}_0) + u + u\int_0^\infty e^{-uy}e^{\mathsf{r}_0 y}\overline{\nu}_{\mathsf{r}_0}(y)dy \\ &\quad - \mathsf{r}_0 u\int_0^\infty e^{-uy}\int_0^y e^{\mathsf{r}_0 s}\overline{\nu}_{\mathsf{r}_0}(s)dsdy \\ &= (\kappa-\mathsf{r}_0) + u + u\int_0^\infty e^{-uy}\left(e^{\mathsf{r}_0 r}\overline{\nu}_{\mathsf{r}_0}(y) - \mathsf{r}_0\int_0^y e^{\mathsf{r}_0 s}\overline{\nu}_{\mathsf{r}_0}(s)ds\right)dy.\end{aligned}$$

The third equality follows from Fubini's theorem, justified as all integrands therein are non-negative, and using $e^{-uy} = \int_y^\infty ue^{-us}ds$. Thus we deduce

$$\overline{\nu}(y) = e^{\mathsf{r}_0 y}\overline{\nu}_{\mathsf{r}_0}(y) - \mathsf{r}_0\int_0^y e^{\mathsf{r}_0 s}\overline{\nu}_{\mathsf{r}_0}(s)ds = \int_y^\infty e^{\mathsf{r}_0 s}\nu_{\mathsf{r}_0}(ds),$$

where the latter follows by some straightforward integration by parts and shows that ν is indeed the Lévy measure of ϕ. Next, an application of [99, Proposition 4.1(9)] together with another integration by parts yields $\int_0^\infty e^{-\mathsf{r}_0 y}\overline{\overline{\Pi}}(y)dy \leqslant \int_0^\infty \overline{\overline{\Pi}}(y)dy = \hbar$. Putting pieces together we get $\overline{\nu}(0) = \overline{\nu}_{\mathsf{r}_0}(0) \leqslant \hbar$, so that in all cases $\overline{\nu}(0) \leqslant \hbar$. Therefore from the estimate in (4.36) we deduce

$$\left|\frac{W_\phi(a+ib)}{\Gamma(a+ib)}\right| \leqslant C_\phi|b|^{\phi(0)+\hbar}, \tag{4.37}$$

which, as before, holds uniformly on bounded a-intervals and for $|b|$ large enough. Next, we recall the following classical estimate for the gamma function,

$$\lim_{|b|\to\infty} C_a|b|^{\frac{1}{2}-a}e^{\frac{\pi}{2}|b|}\,|\Gamma(a+ib)| = 1, \tag{4.38}$$

where $C_a > 0$ is a constant depending on a. Combining this estimate with the one in (4.37) we thus get, uniformly on bounded a-intervals and for $|b|$ large enough,

$$\left|\mathcal{M}_\beta(z)\right| \leqslant C|b|^{-\lambda_1+\mathsf{r}_1+\phi(0)+\hbar},$$

for a constant $C > 0$. Since C is a function of C_ϕ and the constants in the estimate for the Γ-function, it follows that it only depends on ϕ and a. Finally, the fact that $\Delta = \boldsymbol{\lambda}_1 - \mathrm{r}_1 - \phi(0) + \hbar$ follows by Lemma 4.3.2(3). $\square$

Proof of Proposition 4.2.1. Note that $\mathbf{R}_n \beta_{\lambda_1+n,\lambda_1} \in C^\infty((0,1))$ and, trivially, $\beta \in \mathrm{L}^1([0,1])$. Then, well-known properties of convolution give $\mathbf{R}_n\left(\beta_{\lambda_1+n,\lambda_1} \diamond \beta\right) = \mathbf{R}_n \beta_{\lambda_1+n,\lambda_1} \diamond \beta$, and that w_n is a well-defined $C^\infty((0,1))$-function, which completes the proof of this claim. To show that $\Delta > \frac{1}{2}$ implies $w_n \in \mathrm{L}^2([0,1])$ we note that the classical estimate for the gamma function given in (4.38) yields that, for $z = a + ib$ with $a > n$ fixed,

$$\lim_{|b|\to\infty} \left| \frac{\Gamma(z)}{\Gamma(z-n)} \mathcal{M}_{\beta_{\lambda_1+n,\lambda_1}}(z) \right| = \lim_{|b|\to\infty} (\boldsymbol{\lambda}_1)_n \left| \frac{\Gamma(z)}{\Gamma(z-n)} \frac{\Gamma(z+\boldsymbol{\lambda}_1-1)}{\Gamma(z+\boldsymbol{\lambda}_1+n-1)} \right| = C,$$

where C is a positive constant depending only on a, $\boldsymbol{\lambda}_1$, and n. Thus, from (4.34) we get that $\mathcal{M}_{w_n}$ has the same rate of decay along imaginary lines as $\mathcal{M}_\beta$, and combining Lemma 4.3.8 together with Parseval's identity for Mellin transforms shows that $w_n \in \mathrm{L}^2([0,1])$. Finally, since $w_n \in C^\infty((0,1))$, it follows that the differentiability of $\mathcal{V}_n^\phi$ is determined by the differentiability of β. Invoking Lemma 4.3.8 we get, for $a > \mathrm{r}_0$ and $|b|$ large enough that

$$\left|(a+ib)^n \mathcal{M}_\beta(a+ib)\right| \leqslant C|b|^{n-\Delta},$$

uniformly on bounded a-intervals and with $C > 0$ a constant, so that, for any $n \leqslant \lfloor \Delta \rfloor - 1$, the right-hand side is integrable in b. A classical Mellin inversion argument then gives $\beta \in C^{\mathrm{n}}((0,1))$. $\square$

4.3.5 Proof of Theorem 4.2.2

To prove this result we shall need to develop further intertwinings for $\mathbb{J}$, and then will lift these to the level of semigroups. We write $\mathbb{J}_{\phi_{\mathrm{r}_1}^{\checkmark}}$ for the non-local Jacobi operator with parameters λ_1, $\mu_{\phi^{\checkmark}_{\mathrm{r}_1}}$ and $h_{\phi^{\checkmark}_{\mathrm{r}_1}}$, as in Lemma 4.3.2, which is in one-to-one correspondence with the Bernstein function $\phi^{\checkmark}_{\mathrm{r}_1}$ defined in (4.17).

Lemma 4.3.9. *For any* $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \lambda_1)$*, the following identities hold on* $\mathcal{P}$*,*

$$\mathbf{J}_{\mathfrak{m}} \mathrm{V}_{\phi^*_{\mathfrak{m}}} = \mathrm{V}_{\phi^*_{\mathfrak{m}}} \mathbb{J}, \quad \textit{and} \quad \mathbb{J}_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} = \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathbb{J}, \tag{4.39}$$

in the cases $\mu \geqslant 1 + \hbar$ *and* $\mu < 1 + \hbar$*, respectively.*

Proof. It suffices to prove that $\mathbf{L}_{\mathfrak{m}} \mathrm{V}_{\phi^*_{\mathfrak{m}}} = \mathrm{V}_{\phi^*_{\mathfrak{m}}} \mathbb{L}$ and $\mathbb{L}_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} = \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathbb{L}$ hold on $\mathcal{P}$, where we write $\mathbb{L}_{\phi^{\checkmark}_{\mathrm{r}_1}} = \mathbf{L}_{\mu_{\phi^{\checkmark}_{\mathrm{r}_1}}} + \mathbb{I}_{h_{\phi^{\checkmark}_{\mathrm{r}_1}}}$ and refer to (4.27) and subsequent discussion for the definitions, as then the same arguments for the proof of Proposition 4.3.1 will go through. In the case $\mu \geqslant 1 + \hbar$, we have, for any $n \in \mathbb{N}$ and using the recurrence relation of the gamma function,

$$\begin{aligned}
\mathbf{L}_{\mathfrak{m}} \mathrm{V}_{\phi^*_{\mathfrak{m}}} p_n(x) &= \frac{W_\phi(n+1)}{(\mathfrak{m})_n} \mathbf{L}_{\mathfrak{m}} p_n(x) \\
&= \frac{W_\phi(n+1)}{(\mathfrak{m})_n} n(n+\mathfrak{m}-1) p_{n-1}(x) \\
&= \frac{W_\phi(n+1)}{(\mathfrak{m})_{n-1}} n p_{n-1}(x).
\end{aligned}$$

On the other hand, since $W_\phi(n+1) = \phi(n) W_\phi(n)$ and $\mathrm{r}_1 = 1$,

$$\mathrm{V}_{\phi^*_{\mathfrak{m}}} \mathbb{L} p_n(x) = \frac{W_\phi(n)}{(\mathfrak{m})_{n-1}} n \phi(n) p_{n-1}(x) = \frac{W_\phi(n+1)}{(\mathfrak{m})_{n-1}} n p_{n-1}(x),$$

which proves this claim in this case. Finally,

$$\mathbb{L}_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} p_n(x) = \frac{\phi^{\checkmark}_{\mathrm{r}_1}(0)}{\phi^{\checkmark}_{\mathrm{r}_1}(n)} \mathbb{L}_{\phi^{\checkmark}_{\mathrm{r}_1}} p_n(x) = \frac{\phi^{\checkmark}_{\mathrm{r}_1}(0)}{\phi^{\checkmark}_{\mathrm{r}_1}(n)} n \phi^{\checkmark}_{\mathrm{r}_1}(n) p_{n-1}(x) = \phi^{\checkmark}_{\mathrm{r}_1}(0) n p_{n-1}(x),$$

while on the other hand, using the definition of $\phi^{\checkmark}_{\mathrm{r}_1}$ in (4.17),

$$\begin{aligned}
\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathbb{L} p_n(x) &= (n - \mathrm{r}_0)\phi(n) \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} p_{n-1}(x) \\
&= (n - \mathrm{r}_0)\phi(n) \frac{\phi^{\checkmark}_{\mathrm{r}_1}(0)}{\phi^{\checkmark}_{\mathrm{r}_1}(n-1)} p_{n-1}(x) \\
&= \phi^{\checkmark}_{\mathrm{r}_1}(0) n p_{n-1}(x),
\end{aligned}$$

which completes the proof, by linearity. $\square$

The following result lifts the intertwinings in Proposition 4.3.1 and Lemma 4.3.9 to the level of semigroups. We write here $\mathbb{Q} = \mathbb{Q}^{\phi} = (\mathbb{Q}_t^{\phi})_{t\geqslant 0}$ to emphasize the one-to-one correspondence, given fixed $\boldsymbol{\lambda}_1$, between ϕ and $\mathbb{Q}$.

Proposition 4.3.3. *Let $\varepsilon \in (0, d_\phi] \cup \{d_\phi\}$ and $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$. Then, with $\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}$ as in* (4.14)*, the following identities hold for all $t \geqslant 0$ on the appropriate* L^2*-spaces,*

$$\mathbb{Q}_t^{\phi}\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} = \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}\mathbf{Q}_t^{(\boldsymbol{d}_{\mathrm{r}_1,\varepsilon})}, \quad \mathbf{Q}_t^{(\mathfrak{m})}\mathrm{V}_{\phi_{\mathfrak{m}}^*} = \mathrm{V}_{\phi_{\mathfrak{m}}^*}\mathbb{Q}_t^{\phi}, \quad \textit{and} \quad \mathbb{Q}_t^{\phi_{\mathrm{r}_1}^{\checkmark}}\mathrm{U}_{\phi_{\mathrm{r}_1}^{\checkmark}} = \mathrm{U}_{\phi_{\mathrm{r}_1}^{\checkmark}}\mathbb{Q}_t^{\phi}, \tag{4.40}$$

with the latter two holding when $\mu \geqslant 1 + \hbar$, and $\mu < 1 + \hbar$, respectively.

We shall need an auxiliary result concerning the corresponding intertwining operators, which extends their boundedness from $C([0, 1])$ to the corresponding weighted Hilbert spaces. For two Banach spaces B and $\widetilde{B}$ we write $\mathcal{B}(B, \widetilde{B})$ for the space of bounded linear operators from B to $\widetilde{B}$.

Lemma 4.3.10. *Under the assumptions above, the operators $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}$, $\mathrm{V}_{\phi_{\mathfrak{m}}^*}$, and $\mathrm{U}_{\phi_{\mathrm{r}_1}^{\checkmark}}$ belong to $\mathcal{B}(\mathrm{L}^p(\beta_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}), \mathrm{L}^p(\beta))$, $\mathcal{B}(\mathrm{L}^p(\beta), \mathrm{L}^p(\beta_{\mathfrak{m}}))$, and $\mathcal{B}(\mathrm{L}^p(\beta), \mathrm{L}^p(\beta_{\phi_{\mathrm{r}_1}^{\checkmark}}))$, respectively, for any $p \in \{1, 2, \ldots, \infty\}$; in all cases, and for all p, the Markov multiplicative kernels have operator norm 1.*

Proof. Let $f \in \mathcal{P}$ with $p < \infty$. Then, applying Jensen's inequality to the Markov multiplicative kernel $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}$ together with Lemma 4.3.4 gives

$$\begin{aligned} \beta\left[\left(\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} f\right)^p\right] &= \int_0^1 \left(\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} f(x)\right)^p \beta(dx) \\ &\leqslant \int_0^1 \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} f^p(x)\beta(dx) \\ &= \beta[\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} f^p] = \beta_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}[f^p], \end{aligned}$$

where we used that $f^p \in \mathcal{P}$. Since $\beta_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}$ is a probability measure on the compact set $[0, 1]$ it follows that $\mathcal{P}$ is a dense subset of $\mathrm{L}^p(\beta_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}})$, see e.g. [44, Corollary 22.10], so by density we conclude that $\mathcal{B}(\mathrm{L}^p(\beta_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}), \mathrm{L}^p(\beta))$ with operator norm less than or equal to 1, and equality then follows from $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}\mathbf{1}_{[0,1]} = \mathbf{1}_{[0,1]}$. The case when $p = \infty$ is a straightforward consequence of $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}$ being a Markov multiplicative kernel and the claims regarding the other operators are proved similarly, by invoking the remaining items of Lemma 4.3.4. □

Next, since $\mathbb{J}$ and $\mathbf{J}_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}$ are generators of $C([0,1])$-Markov semigroups, it follows that their resolvent operators, given for $q > 0$, by

$$\mathbb{R}_q = (q - \mathbb{J})^{-1}, \quad \text{and} \quad \mathrm{R}_q = (q - \mathbf{J}_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}})^{-1}$$

are bounded, linear operators on $C([0,1])$. We write $\mathrm{R}_q^{\mathfrak{m}}$ (resp. $\mathbb{R}_q^{\phi_{\mathrm{r}_1}^{\checkmark}}$) for the resolvent associated to $\mathbf{J}_{\mathfrak{m}}$ (resp. $\mathbb{J}_{\phi_{\mathrm{r}_1}^{\checkmark}}$).

Lemma 4.3.11. *Let $q > 0$. Under the assumptions in Proposition 4.3.3, the following identities hold on $\mathcal{P}$*

$$\mathbb{R}_q\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} = \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}\mathrm{R}_q, \quad \mathrm{V}_{\phi_{\mathfrak{m}}^*}\mathbb{R}_q = \mathrm{R}_q^{\mathfrak{m}}\mathrm{V}_{\phi_{\mathfrak{m}}^*}, \quad \textit{and} \quad \mathrm{U}_{\phi_{\mathrm{r}_1}^{\checkmark}}\mathbb{R}_q = \mathbb{R}_q^{\phi_{\mathrm{r}_1}^{\checkmark}}\mathrm{U}_{\phi_{\mathrm{r}_1}^{\checkmark}}. \tag{4.41}$$

Proof. We shall only provide the proof of the first claim, which relies on the intertwining in Proposition 4.3.1, as the other claims follow by invoking Lemma 4.3.9 and involve the same arguments, mutatis mutandis. First, suppose that $\mathbb{R}_q(\mathcal{P}) \subseteq \mathcal{P}$ and $\mathrm{R}_q(\mathcal{P}) \subseteq \mathcal{P}$, and let $f \in \mathcal{P}$ so that there exists $g \in \mathcal{P}$ such that $(q - \mathbf{J}_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}})g = f$. Applying $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}$ to both sides of this equality gives that

$$\begin{aligned}\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}f &= \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}(q - \mathbf{J}_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}})g = (\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}q - \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}\mathbf{J}_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}})g \\ &= (q\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} - \mathbb{J}\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}})g = (q - \mathbb{J})\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}g,\end{aligned}$$

where in the third equality we have invoked Proposition 4.3.1, which is justified as $g \in \mathcal{P}$. This equality may be rewritten as $\mathbb{R}_q\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}f = \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}g$ and consequently, for any $f \in \mathcal{P}$, we get

$$\mathbb{R}_q \Lambda_{\phi_{d_{1,\varepsilon}}} f = \Lambda_{\phi_{d_{1,\varepsilon}}} g = \Lambda_{\phi_{d_{1,\varepsilon}}} \mathrm{R}_q f.$$

Thus it remains to show the inclusions $\mathbb{R}_q(\mathcal{P}) \subseteq \mathcal{P}$ and $\mathrm{R}_q(\mathcal{P}) \subseteq \mathcal{P}$ for which we recall, from the proof of Proposition 4.3.1, that $\mathbb{J} = \mathbb{L} - \mathbf{A}$ with $\mathbb{L}p_n = (n - \mathsf{r}_0)\phi(n)p_{n-1}$, for any $n \geqslant 1$. A straightforward computation gives that $\mathbf{A}p_n = (\mathbf{D}_2 + \boldsymbol{\lambda}_1 \mathbf{D}_1)p_n = (n(n-1) + \boldsymbol{\lambda}_1 n)p_n$ and hence

$$(q - \mathbb{J})p_n = (q + n(n-1) + \boldsymbol{\lambda}_1 n)p_n - (n - \mathsf{r}_0)\phi(n)p_{n-1},$$

from which it follows, by the injectivity of $\mathbb{R}_q$ on $\mathcal{P} \subset C([0,1])$, that

$$\mathbb{R}_q \left((q + n(n-1) + \boldsymbol{\lambda}_1 n)p_n - (n - \mathsf{r}_0)\phi(n)p_{n-1}\right) = p_n.$$

Rearranging the above yields the equation

$$\mathbb{R}_q p_n = \frac{1}{(q + n(n-1) + \boldsymbol{\lambda}_1 n)} p_n + \frac{(n - \mathsf{r}_0)\phi(n)}{(q + n(n-1) + \boldsymbol{\lambda}_1 n)} \mathbb{R}_q p_{n-1}, \tag{4.42}$$

which is justified as, for any $q > 0$, both roots of the quadratic equation $n^2 + (\boldsymbol{\lambda}_1 - 1)n + q = 0$ are always negative. Note that $\mathbb{R}_q p_0 = q^{-1}$ so by iteratively using the equality in (4.42) we conclude that, for any $n \in \mathbb{N}$, $\mathbb{R}_q p_n \in \mathcal{P}$, and by linearity $\mathbb{R}_q(\mathcal{P}) \subseteq \mathcal{P}$ follows. Similar arguments applied to R_q then allow us to also conclude that $\mathrm{R}_q(\mathcal{P}) \subseteq \mathcal{P}$, which completes the proof. □

Proof of Proposition 4.3.3. We are now able to complete the proof of Proposition 4.3.3. As was shown in the proof of Lemma 4.3.11 above and using the notation therein, $\mathbb{R}_q(\mathcal{P}) \subseteq \mathcal{P}$ and $\mathrm{R}_q(\mathcal{P}) \subseteq \mathcal{P}$, so that on $\mathcal{P} \subset C([0,1])$ we have

$$\mathbb{R}_q^2 \Lambda_{\phi_{d_{1,\varepsilon}}} = \mathbb{R}_q \mathbb{R}_q \Lambda_{\phi_{d_{1,\varepsilon}}} = \mathbb{R}_q \Lambda_{\phi_{d_{1,\varepsilon}}} \mathrm{R}_q = \Lambda_{\phi_{d_{1,\varepsilon}}} \mathrm{R}_q \mathrm{R}_q = \Lambda_{\phi_{d_{1,\varepsilon}}} \mathrm{R}_q^2,$$

and, by induction, for any $n \in \mathbb{N}$,

$$\mathbb{R}_q^n \Lambda_{\phi_{d_{1,\varepsilon}}} = \Lambda_{\phi_{d_{1,\varepsilon}}} \mathrm{R}_q^n.$$

In particular, for any $f \in \mathcal{P}$ and $t > 0$,

$$(n/t)\mathbb{R}^n_{n/t}\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} f = \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}(n/t)\mathrm{R}^n_{n/t} f.$$

Now, taking the strong limit in $C([0,1])$ as $n \to \infty$ of the above yields, by the exponential formula [101, Theorem 8.3] and the continuity of the involved operators guaranteed by Lemma 4.3.3, for any $f \in \mathcal{P}$ and $t \geqslant 0$,

$$\mathbb{Q}_t \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} f = \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} \mathbf{Q}_t^{(\boldsymbol{d}_{\mathrm{r}_1,\varepsilon})} f, \tag{4.43}$$

where $\mathbf{Q}^{(\boldsymbol{d}_{\mathrm{r}_1,\varepsilon})} = (\mathbf{Q}_t^{(\boldsymbol{d}_{\mathrm{r}_1,\varepsilon})})_{t \geqslant 0}$ is the classical Jacobi semigroup on $C([0,1])$ with parameters $\boldsymbol{\lambda}_1$ and $\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}$. By density of $\mathcal{P}$ in $\mathrm{L}^2(\beta_{\mathrm{r}_1})$ and since Lemma 4.3.10 with $p = 2$ gives $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} \in \mathcal{B}(\mathrm{L}^2(\beta_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}), \mathrm{L}^2(\beta))$ it follows that the identity in (4.43) extends to $\mathrm{L}^2(\beta_{\mathrm{r}_1})$, which completes the proof of the first item. The remaining items follow by similar arguments and so the proof is omitted. □

For $\boldsymbol{\lambda}_1 > s \geqslant 1$ we define, for $n \in \mathbb{N}$, the quantity $\mathfrak{c}_n(s)$ as

$$\mathfrak{c}_n(s) = \frac{(s)_n}{n!}\sqrt{\frac{\mathrm{C}_n(s)}{\mathrm{C}_n(1)}} = \sqrt{\frac{(s)_n}{n!}\frac{(\boldsymbol{\lambda}_1 - 1)_n}{(\boldsymbol{\lambda}_1 - s)_n}}, \tag{4.44}$$

where the first equality comes from some straightforward algebra given the definition of $\mathrm{C}_n(s)$ in (4.70). Note that, with $s = 1$ we get $\mathfrak{c}_n(1) = 1$, for all n. We shall need the following result.

Lemma 4.3.12. *For any $\boldsymbol{\lambda}_1 > s > r \geqslant 1$ the mapping $n \mapsto \frac{\mathfrak{c}_n(s)}{\mathfrak{c}_n(r)}$ is strictly increasing on $\mathbb{N}$ with*

$$\lim_{n\to\infty} \frac{\mathfrak{c}_n(s)}{n^s} = \sqrt{\frac{\Gamma(\boldsymbol{\lambda}_1 - s)}{\Gamma(s)\Gamma(\boldsymbol{\lambda}_1 - 1)}}. \tag{4.45}$$

Proof. Using the definition in (4.44) we get that

$$\frac{\mathfrak{c}_n^2(s)}{\mathfrak{c}_n^2(r)} = \prod_{j=0}^{n-1} \frac{(s+j)(\boldsymbol{\lambda}_1 - r + j)}{(r+j)(\boldsymbol{\lambda}_1 - s + j)},$$

Since $s > r$ each term in the product is strictly greater than 1 and together with Stirling's formula for the gamma function this completes the proof. $\square$

Next we write $\mathrm{V}^*_{\phi^*_{\mathfrak{m}}} : \mathrm{L}^2(\beta_{\mathfrak{m}}) \to \mathrm{L}^2(\beta)$ and $\mathrm{U}^*_{\phi^{\checkmark}_{\mathfrak{r}_1}} : \mathrm{L}^2(\beta_{\phi^{\checkmark}_{\mathfrak{r}_1}}) \to \mathrm{L}^2(\beta)$ for the Hilbertian adjoints of the operators $\mathrm{V}_{\phi^*_{\mathfrak{m}}}$ and $\mathrm{U}_{\phi^{\checkmark}_{\mathfrak{r}_1}}$, respectively.

Proposition 4.3.4. *Let* $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$ *and* $\varepsilon \in (0, d_\phi] \cup \{d_\phi\}$. *Then, with* $\boldsymbol{d}_{1,\varepsilon}$ *as in* (4.19), *the sequence* $\left(\mathfrak{c}_n(\boldsymbol{d}_{1,\varepsilon})\, \mathcal{P}^\phi_n\right)_{n\geqslant 0}$ *is a complete, Bessel sequence in* $\mathrm{L}^2(\beta)$, *with Bessel bound 1. Furthermore, for any* $n \in \mathbb{N}$, *we have, when* $\mu \geqslant 1 + \hbar$, *that*

$$\mathcal{V}^\phi_n = \mathfrak{c}_n(\mathfrak{m})\mathrm{V}^*_{\phi^*_{\mathfrak{m}}}\mathcal{P}^{(\mathfrak{m})}_n, \tag{4.46}$$

while otherwise

$$\mathcal{V}^\phi_n = \frac{\mathfrak{c}_n(\mathfrak{m})}{\mathfrak{c}_n(\mathfrak{r}_1)}\mathrm{U}^*_{\phi^{\checkmark}_{\mathfrak{r}_1}}\mathrm{V}^*_{\phi^*_{\mathfrak{m}}}\mathcal{P}^{(\mathfrak{m})}_n. \tag{4.47}$$

and $(\mathcal{V}^\phi_n)_{n\geqslant 0}$ *is the unique biorthogonal sequence to* $(\mathcal{P}^\phi_n)_{n\geqslant 0}$ *in* $\mathrm{L}^2(\beta)$, *which is equivalent to* $\mathcal{V}^\phi_n$ *being the unique* $\mathrm{L}^2(\beta)$-*solution to* $\Lambda^*_\phi g = \mathcal{P}^{(\mathfrak{r}_1)}_n$, *for any* $n \in \mathbb{N}$. *In all cases* $\left(\frac{\mathfrak{c}_n(\mathfrak{r}_1)}{\mathfrak{c}_n(\mathfrak{m})}\mathcal{V}^\phi_n\right)_{n\geqslant 0}$ *is a complete, Bessel sequence in* $\mathrm{L}^2(\beta)$ *with Bessel bound 1.*

Remark 4.3.3. Note that Proposition 4.3.4 yields norm bounds in $\mathrm{L}^2(\beta)$ for the functions $\mathcal{P}^\phi_n$ and $\mathcal{V}^\phi_n$ for any $n \in \mathbb{N}$. Indeed, writing $||\cdot||_\beta$ for the $\mathrm{L}^2(\beta)$-norm we get, from the boundedness claims of Lemma 4.3.10, for any $\varepsilon \in (0, d_\phi] \cup \{d_\phi\}$ and any $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$,

$$||\mathcal{P}^\phi_n||_\beta \leqslant \frac{1}{\mathfrak{c}_n(\boldsymbol{d}_{1,\varepsilon})} \leqslant Cn^{-\boldsymbol{d}_{1,\varepsilon}}, \quad \text{and} \quad ||\mathcal{V}^\phi_n||_\beta \leqslant \frac{\mathfrak{c}_n(\mathfrak{m})}{\mathfrak{c}_n(\mathfrak{r}_1)} \leqslant Cn^{\mathfrak{m}-\mathfrak{r}_1}$$

where $C > 0$ and we used for the two estimates Lemma 4.3.12. We show in the proof below that

$$\frac{\mathfrak{c}_n(\mathfrak{m})}{\mathfrak{c}_n(\boldsymbol{d}_{1,\varepsilon})\mathfrak{c}_n(\mathfrak{r}_1)} = \frac{\mathfrak{c}_n(\mathfrak{m})}{\mathfrak{c}_n(\boldsymbol{d}_{\mathfrak{r}_1,\varepsilon})},$$

and since $\mathfrak{m} > \boldsymbol{d}_{\mathfrak{r}_1,\varepsilon}$, invoking again Lemma 4.3.12, we have that the above ratio grows with n.

Proof. Since, for all $n \in \mathbb{N}$, $\mathcal{P}_n^{(\mathsf{r}_1)} \in \mathrm{L}^2(\beta_{\mathsf{r}_1})$ we get from the intertwining in (4.40) and the linearity of Λ_ϕ that

$$\Lambda_\phi \mathcal{P}_n^{(\mathsf{r}_1)}(x) = \sqrt{\mathrm{C}_n(\mathsf{r}_1)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\boldsymbol{\lambda}_1 - 1)_{n+k}}{(\boldsymbol{\lambda}_1 - 1)_n} \frac{(\mathsf{r}_1)_n}{(\mathsf{r}_1)_k} \frac{k!}{W_\phi(k+1)} \frac{x^k}{k!} = \mathcal{P}_n^\phi(x). \tag{4.48}$$

Recall that the sequence $(\mathcal{P}_n^{(\mathsf{r}_1)})_{n \geqslant 0}$ forms an orthonormal basis of $\mathrm{L}^2(\beta_{\mathsf{r}_1})$ and thus, as the image under a bounded operator of an orthonormal basis, we get that $(\mathcal{P}_n^\phi)_{n \geqslant 0}$ is a Bessel sequence in $\mathrm{L}^2(\beta)$ with Bessel bound given by the operator norm of Λ_ϕ, which by Lemma 4.3.10 is 1. When $\mathsf{r}_1 > 1$ we have $\mathfrak{c}_n(\boldsymbol{d}_{1,\varepsilon}) = \mathfrak{c}_n(1) = 1$, so that the first claim is proved in this case. In the case when $\mathsf{r}_1 = 1$ we suppose, without loss of generality, that $d_\phi > 0$ and $\varepsilon \in (0, d_\phi)$. Then $\mathcal{P}_n^\phi$ reduces to

$$\mathcal{P}_n^\phi(x) = \sqrt{\mathrm{C}_n(1)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\boldsymbol{\lambda}_1 - 1)_{n+k}}{(\boldsymbol{\lambda}_1 - 1)_n} \frac{n!}{k!} \frac{x^k}{W_\phi(k+1)}$$

and from the intertwining (4.40) we get

$$\begin{aligned}
\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} \mathcal{P}_n^{(\boldsymbol{d}_{1,\varepsilon})}(x) &= \sqrt{\mathrm{C}_n(\boldsymbol{d}_{1,\varepsilon})} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\boldsymbol{\lambda}_1 - 1)_{n+k}}{(\boldsymbol{\lambda}_1 - 1)_n} \frac{(\boldsymbol{d}_{1,\varepsilon})_n}{(\boldsymbol{d}_{1,\varepsilon})_k} \frac{(\boldsymbol{d}_{1,\varepsilon})_k}{W_\phi(k+1)} x^k \\
&= \mathfrak{c}_n(\boldsymbol{d}_{1,\varepsilon}) \mathcal{P}_n^\phi(x).
\end{aligned}$$

By Lemma 4.3.10 $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} \in \mathcal{B}(\mathrm{L}^2(\beta_{\boldsymbol{d}_{1,\varepsilon}}), \mathrm{L}^2(\beta))$ with operator norm 1 and thus, by similar arguments as above, we deduce that $(\mathfrak{c}_n(\boldsymbol{d}_{1,\varepsilon}) \mathcal{P}_n^\phi)_{n \geqslant 0}$ is also a Bessel sequence in $\mathrm{L}^2(\beta)$ with Bessel bound 1. We continue with the claims regarding $\mathcal{V}_n^\phi$, starting again with the case when $\mathsf{r}_1 = 1$. Following similar arguments as in the proof of Proposition 4.3.2, we get that, for any $f \in \mathrm{L}^2(\beta_{\mathfrak{m}})$

$$\mathrm{V}^*_{\phi^*_{\mathfrak{m}}} f(x) = \frac{1}{\beta(x)} \widehat{\mathrm{V}}_{\phi^*_{\mathfrak{m}}} (\beta_{\mathfrak{m}} f)(x),$$

where $\widehat{\mathrm{V}}_{\phi^*_{\mathfrak{m}}} f(x) = \int_0^1 f(xy) \nu^*_{\mathfrak{m}}(y) dy$ with $\nu^*_{\mathfrak{m}}(y) = \nu_{\mathfrak{m}}(1/y)/y$, and where $\nu_{\mathfrak{m}}$ denotes the density of the random variable $V_{\phi^*_{\mathfrak{m}}}$, whose existence is due to [96, Proposition 2.4]. Thus it suffices to show that, for all $n \in \mathbb{N}$,

$$w_n(x) = \mathfrak{c}_n(\mathfrak{m}) \widehat{\mathrm{V}}_{\phi^*_{\mathfrak{m}}} (\beta_{\mathfrak{m}} \mathcal{P}_n^{(\mathfrak{m})})(x) = \mathfrak{c}_n(\mathfrak{m}) \widehat{\mathrm{V}}_{\phi^*_{\mathfrak{m}}} \mathrm{p}_n^{(\mathfrak{m})}(x).$$

To this end, taking the Mellin transform of the right-hand side yields, for $\mathrm{Re}(z) > \mathrm{r}_0$,

$$
\begin{aligned}
\mathcal{M}_{\widehat{\mathrm{V}}_{\phi^*_{\mathfrak{m}}}\mathrm{p}_n^{(\mathfrak{m})}}(z) &= \mathcal{M}_{V_{\phi^*_{\mathfrak{m}}}}(z)\mathcal{M}_{\mathrm{p}^{(\mathfrak{m})}}(z) \\
&= \frac{(-2)^n}{n!}\frac{(\lambda_1-\mathfrak{m})_n}{(\lambda_1)_n}\sqrt{\mathrm{C}_n(\mathfrak{m})}\frac{W_\phi(z)}{(\mathfrak{m})_{z-1}}\frac{\Gamma(z)}{\Gamma(z-n)}\mathcal{M}_{\beta_{\lambda_1+n,\mathfrak{m}}}(z) \\
&= \frac{(-2)^n}{n!}\frac{(\lambda_1-\mathfrak{m})_n}{(\lambda_1)_n}\sqrt{\mathrm{C}_n(\mathfrak{m})}\frac{\Gamma(z)}{\Gamma(z-n)}\mathcal{M}_{\beta_{\lambda_1+n,\lambda_1}}(z)\mathcal{M}_\beta(z).
\end{aligned}
$$

After substituting the definitions of $\mathfrak{c}_n(\mathfrak{m})$, $\mathrm{C}_n(\mathfrak{m})$ in (4.44) and (4.70), respectively, we get, by some straightforward algebra,

$$
\mathfrak{c}_n(\mathfrak{m})2^n\frac{(\lambda_1-\mathfrak{m})_n}{(\lambda_1)_n}\frac{\sqrt{\mathrm{C}_n(\mathfrak{m})}}{n!} = 2^n\frac{(\lambda_1-\mathfrak{m})_n}{(\lambda_1)_n}\frac{(\mathfrak{m})_n}{n!}\frac{\mathrm{C}_n(\mathfrak{m})}{n!\sqrt{\mathrm{C}_n(1)}} = 2^n\frac{\sqrt{\mathrm{C}_n(1)}}{n!}\frac{(\lambda_1-1)_n}{(\lambda_1)_n},
$$

and the right-hand side is the constant in front of the definition of w_n in (4.34) when $\mathrm{r}_1 = 1$. Invoking the uniqueness claim in Proposition 4.3.2 yields (4.46), as desired. The case when $\mathrm{r}_1 < 1$ follows by similar arguments, albeit with more tedious algebra, and its proof is omitted. Next, using the second intertwining relation (4.40) we get that

$$
\begin{aligned}
\mathrm{V}_{\phi^*_{\mathfrak{m}}}\mathcal{P}_n^\phi(x) &= \sqrt{\mathrm{C}_n(\lambda_1)}\sum_{k=0}^n\frac{(-1)^{n+k}}{(n-k)!}\frac{(\lambda_1-1)_{n+k}}{(\lambda_1-1)_n}\frac{n!}{k!}\frac{W_\phi(k+1)}{(\mathfrak{m})_k}\frac{x^k}{W_\phi(k+1)} \\
&= \mathfrak{c}_n^{-1}(\mathfrak{m})\mathcal{P}_n^{(\mathfrak{m})}(x).
\end{aligned}
$$

As $(\mathcal{P}_n^{(\mathfrak{m})})_{n\geqslant 0}$ is an orthonormal sequence in $\mathrm{L}^2(\beta_{\mathfrak{m}})$, we have for any $n, p \in \mathbb{N}$,

$$
\delta_{np} = \langle\mathcal{P}_n^{(\mathfrak{m})},\mathcal{P}_p^{(\mathfrak{m})}\rangle_{\beta_{\mathfrak{m}}} = \mathfrak{c}_n(\mathfrak{m})\langle\mathrm{V}_{\phi^*_{\mathfrak{m}}}\mathcal{P}_n^\phi,\mathcal{P}_p^{(\mathfrak{m})}\rangle_{\beta_{\mathfrak{m}}} = \mathfrak{c}_n(\mathfrak{m})\langle\mathcal{P}_n^\phi,\mathrm{V}^*_{\phi^*_{\mathfrak{m}}}\mathcal{P}_p^{(\mathfrak{m})}\rangle_\beta,
$$

and thus we get that $(\mathcal{V}_n^\phi)_{n\geqslant 0}$ is a biorthogonal sequence in $\mathrm{L}^2(\beta)$ of $(\mathcal{P}_n^\phi)_{n\geqslant 0}$. As before, the continuity of $\mathrm{V}^*_{\phi^*_{\mathfrak{m}}}$ given by Lemma 4.3.10 combined with the fact that $(\mathcal{P}_n^{(\mathfrak{m})})_{n\geqslant 0}$ forms an orthonormal basis for $\mathrm{L}^2(\beta_{\mathfrak{m}})$ implies that $(\mathfrak{c}_n^{-1}(\mathfrak{m})\mathcal{V}_n^\phi)_{n\geqslant 0}$ is a Bessel sequence in $\mathrm{L}^2(\beta)$ with Bessel bound 1. To show uniqueness, we first observe that any sequence $(g_n)_{n\geqslant 0} \in \mathrm{L}^2(\beta)$ biorthogonal to $(\mathcal{P}_n^\phi)_{n\geqslant 0}$ must satisfy

$$
\delta_{np} = \langle\mathcal{P}_n^\phi, g_p\rangle_\beta = \langle\mathcal{P}_n^{(\mathrm{r}_1)}, \Lambda^*_\phi g_p\rangle_{\beta_{\mathrm{r}_1}}
$$

that is $(\Lambda_\phi^* g_n)_{n\geqslant 0}$ must be biorthogonal to $(\mathcal{P}_n^{(\mathsf{r}_1)})_{n\geqslant 0}$. However, since $(\mathcal{P}_n^{(\mathsf{r}_1)})_{n\geqslant 0}$ is an orthonormal basis for $\mathrm{L}^2(\beta_\mu)$ the only sequence in $\mathrm{L}^2(\beta_\mu)$ biorthogonal to it is itself. Thus, if there exists another sequence $(g_n)_{n\geqslant 0} \in \mathrm{L}^2(\beta)$ biorthogonal to $(\mathcal{P}_n^\phi)_{n\geqslant 0}$ it follows that, for all $n \in \mathbb{N}$,

$$\Lambda_\phi^* \mathcal{V}_n^\phi = \mathcal{P}_n^{(\mathsf{r}_1)} = \Lambda_\phi^* g_n \implies \Lambda_\phi^* \left(\mathcal{V}_n^\phi - g_n\right) = 0.$$

Since Lemma 4.3.3 gives that $\mathrm{Ran}(\Lambda_\phi)$ is dense in $\mathrm{L}^2(\beta)$ it follows that $\mathrm{Ker}(\Lambda_\phi^*) = \{0\}$ and we conclude that $(\mathcal{V}_n^\phi)_{n\geqslant 0}$ is the unique sequence in $\mathrm{L}^2(\beta)$ biorthogonal to $(\mathcal{P}_n^\phi)_{n\geqslant 0}$. Finally, assume now that $\mathsf{r}_1 < 1$. Then, using the definition of $\phi_{\mathsf{r}_1}^{\checkmark}$ in (4.17) we get that

$$\mathcal{P}_n^{\phi_{\mathsf{r}_1}^{\checkmark}}(x) = \sqrt{\mathrm{C}_n(\mathsf{r}_1)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\lambda_1 - 1)_{n+k}}{(\lambda_1 - 1)_n} \frac{n!(k+1)}{(\mathsf{r}_1 + 1)_k} \frac{\phi(1)x^k}{W_\phi(k+2)}.$$

On the other hand, since $\mathrm{U}_{\phi_{\mathsf{r}_1}^{\checkmark}} p_n = \frac{\phi_{\mathsf{r}_1}^{\checkmark}(0)}{\phi_{\mathsf{r}_1}^{\checkmark}(n)} p_n$, see (4.24), simple algebra yields that

$$\begin{aligned}\mathrm{U}_{\phi_{\mathsf{r}_1}^{\checkmark}} \mathcal{P}_n^\phi(x) &= \frac{(\mathsf{r}_1)_n}{n!} \sqrt{\mathrm{C}_n(\mathsf{r}_1)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\lambda_1 - 1)_{n+k}}{(\lambda_1 - 1)_n} \frac{n!(k+1)}{(\mathsf{r}_1 + 1)_k} \frac{\phi(1)x^k}{W_\phi(k+2)} \\ &= \mathfrak{c}_n(\mathsf{r}_1) \mathcal{P}_n^{\phi_{\mathsf{r}_1}^{\checkmark}}(x).\end{aligned} \tag{4.49}$$

We know that, since $\lambda_1 > \mathfrak{m} > 1 + \mu = \mu_{\phi_{\mathsf{r}_1}^{\checkmark}}$, $(\mathcal{V}_n^{\phi_{\mathsf{r}_1}^{\checkmark}})_{n\geqslant 0} = (\mathfrak{c}_n(\mathfrak{m}) \mathrm{V}_{\phi_{\mathfrak{m}}^*}^* \mathcal{P}_n^{(\mathfrak{m})})_{n\geqslant 0}$ is the unique sequence biorthogonal to $(\mathcal{P}_n^{\phi_{\mathsf{r}_1}^{\checkmark}})_{n\geqslant 0}$, and combining this with (4.49) gives the biorthogonality of $(\mathcal{V}_n^\phi)_{n\geqslant 0}$ in $\mathrm{L}^2(\beta)$ as well as uniqueness, using similar arguments as above. Finally, the completeness of $(\mathcal{V}_n^\phi)_{n\geqslant 0}$ is a consequence of the fact that $\mathcal{V}_n^\phi$ is, in all cases and by Lemmas 4.3.3 and 4.3.10, the image under a continuous operator with dense range of the sequence $\left(\frac{\mathfrak{c}_n(\mathfrak{m})}{\mathfrak{c}_n(\mathsf{r}_1)} \mathcal{P}_n^{(\mathfrak{m})}\right)_{n\geqslant 0}$, which is itself easily seen to be complete. □

Proof of Theorem 4.2.2. We are now able to give the proof of all items of Theorem 4.2.2, which we tackle sequentially. Setting $\varepsilon = d_\phi$ in (4.14) we get, by the first intertwining in Proposition 4.3.3 and the spectral expansion of the self-adjoint semigroup $\mathbf{Q}^{(\mathsf{r}_1)}$ in (4.74), that for any $f \in \mathrm{L}^2(\beta_{\mathfrak{r}_1})$ and $t \geqslant 0$,

$$\mathbb{Q}_t \Lambda_\phi f = \Lambda_\phi \mathbf{Q}_t^{(\mathfrak{r}_1)} f = \sum_{n=0}^{\infty} e^{-\lambda_n t} \langle f, \mathcal{P}_n^{(\mathfrak{r}_1)} \rangle_{\beta_{\mathfrak{r}_1}} \mathcal{P}_n^{\phi} = \sum_{n=0}^{\infty} e^{-\lambda_n t} \langle \Lambda_\phi f, \mathcal{V}_n^{\phi} \rangle_{\beta} \mathcal{P}_n^{\phi},$$

where the second identity is justified by $(\langle f, \mathcal{P}_n^{(\mathfrak{r}_1)} \rangle_{\beta_{\mathfrak{r}_1}})_{n \geqslant 0} \in \ell^2(\mathbb{N})$ and the fact that $(\mathcal{P}_n^{\phi})_{n \geqslant 0}$ is a Bessel sequence in $\mathrm{L}^2(\beta)$, see [37, Theorem 3.1.3], and the last identity uses the fact that, by Proposition 4.3.4, $\mathcal{V}_n^{\phi}$ is the unique $\mathrm{L}^2(\beta)$-solution to the equation $\Lambda_\phi^* \mathcal{V}_n^{\phi} = \mathcal{P}_n^{(\mathfrak{r}_1)}$. Next, from the first intertwining in (4.40) and the fact that, for any $n \in \mathbb{N}$, $\mathbf{Q}_t^{(\mathfrak{r}_1)} \mathcal{P}_n^{(\mathfrak{r}_1)} = e^{-\lambda_n t} \mathcal{P}_n^{(\mathfrak{r}_1)}$, see (4.74), we get that $\mathcal{P}_n^{\phi}$ is an eigenfunction for $\mathbb{Q}_t$ with eigenvalue $e^{-\lambda_n t}$. Taking the adjoint of the first identity in (4.40) and using the self-adjointness of $\mathbf{Q}_t^{(\mathfrak{r}_1)}$ on $\mathrm{L}^2(\beta_{\mathfrak{r}_1})$ yields $\Lambda_\phi^* \mathbb{Q}_t^* = \mathbf{Q}_t^{(\mathfrak{r}_1)} \Lambda_\phi^*$ and thus, for any $n \in \mathbb{N}$ and $t \geqslant 0$,

$$\Lambda_\phi^* \mathbb{Q}_t{}^* \mathcal{V}_n^{\phi} = \mathbf{Q}_t^{(\mathfrak{r}_1)} \Lambda_\phi^* \mathcal{V}_n^{\phi} = \mathbf{Q}_t^{(\mathfrak{r}_1)} \mathcal{P}_n^{(\mathfrak{r}_1)} = e^{-\lambda_n t} \mathcal{P}_n^{(\mathfrak{r}_1)} = e^{-\lambda_n t} \Lambda_\phi^* \mathcal{V}_n^{\phi},$$

and since $\mathrm{Ker}(\Lambda_\phi^*) = \{0\}$ we deduce $\mathbb{Q}_t{}^* \mathcal{V}_n^{\phi} = e^{-\lambda_n t} \mathcal{V}_n^{\phi}$. Next, let S_t be the linear operator on $\mathrm{L}^2(\beta)$ defined by

$$S_t f = \sum_{n=0}^{\infty} \langle \mathbb{Q}_t f, \mathcal{V}_n^{\phi} \rangle_{\beta} \mathcal{P}_n^{\phi}$$

so that, by the above observations,

$$S_t f = \sum_{n=0}^{\infty} \langle \mathbb{Q}_t f, \mathcal{V}_n^{\phi} \rangle_{\beta} \mathcal{P}_n^{\phi} = \sum_{n=0}^{\infty} e^{-\lambda_n t} \langle f, \mathcal{V}_n^{\phi} \rangle_{\beta} \mathcal{P}_n^{\phi}.$$

For convenience, we set $\boldsymbol{\mathcal{V}}_n^{\phi} = \frac{\mathfrak{c}_n(\mathfrak{r}_1)}{\mathfrak{c}_n(\mathfrak{m})} \mathcal{V}_n^{\phi}$, $n \in \mathbb{N}$. Then, for any $t > 0$ and $f \in \mathrm{L}^2(\beta)$ we have, for $C > 0$ a constant independent of n,

$$\begin{aligned} \sum_{n=0}^{\infty} e^{-2\lambda_n t} \left| \left\langle f, \mathcal{V}_n^{\phi} \right\rangle_{\beta} \right|^2 &= \sum_{n=0}^{\infty} e^{-2\lambda_n t} \frac{\mathfrak{c}_n^2(\mathfrak{m})}{\mathfrak{c}_n^2(\mathfrak{r}_1)} \left| \left\langle f, \boldsymbol{\mathcal{V}}_n^{\phi} \right\rangle_{\beta} \right|^2 \\ &\leqslant C \sum_{n=0}^{\infty} \left| \left\langle f, \boldsymbol{\mathcal{V}}_n^{\phi} \right\rangle_{\beta} \right|^2 \leqslant C \beta[f^2] < \infty, \end{aligned}$$

where the first inequality follows from the asymptotic in (4.45) combined with the decay of the sequence $(e^{-2\lambda_n t})_{n \geqslant 0}$, $t > 0$, and the second inequality follows from the

Bessel property of $(\mathcal{V}_n^\phi)_{n\geqslant 0}$ guaranteed by Proposition 4.3.4. Hence we deduce that $\left(e^{-\lambda_n t}\langle f, \mathcal{V}_n^\phi\rangle_\beta\right)_{n\geqslant 0} \in \ell^2(\mathbb{N})$ and, as $(\mathcal{P}_n^\phi)_{n\geqslant 0}$ is a Bessel sequence, it follows that S_t defines a bounded linear operator on $\mathrm{L}^2(\beta)$ for any $t > 0$, again by [37, Theorem 3.1.3]. However, $S_t = \mathbb{Q}_t$ on $\mathrm{Ran}(\Lambda_\phi)$, a dense subset of $\mathrm{L}^2(\beta)$. Therefore, by the bounded linear extension theorem, we have $S_t = \mathbb{Q}_t$ on $\mathrm{L}^2(\beta)$ for any $t > 0$. Note that, by similar Bessel sequence arguments as above, for any $N \geqslant 1$,

$$\left\| \mathbb{Q}_t f - \sum_{n=0}^{N} e^{-\lambda_n t} \left\langle f, \mathcal{V}_n^\phi \right\rangle_\beta \mathcal{P}_n^\phi \right\|_\beta^2 \leqslant \beta[f^2] \sup_{n\geqslant N+1} e^{-2\lambda_n t} \frac{\mathfrak{c}_n^2(\mathfrak{m})}{\mathfrak{c}_n^2(\mathfrak{r}_1)}.$$

Since the supremum on the right-hand side is decreasing in n, for any $t > 0$, we get that in the operator norm topology

$$\mathbb{Q}_t = \lim_{N\to\infty} \sum_{n=0}^{N} e^{-\lambda_n t} \mathcal{P}_n^\phi \otimes \mathcal{V}_n^\phi,$$

where each $\sum_{n=0}^{N} e^{-\lambda_n t} \mathcal{P}_n^\phi \otimes \mathcal{V}_n^\phi$ is of finite rank. This completes the proof of Item (1) and also shows that $\mathbb{Q}_t$ is a compact operator for any $t > 0$, which completes the proof of Item (2). Next, the intertwining identity (4.40) and the completeness of $(\mathcal{P}_n^\phi)_{n\geqslant 0}$ and $(\mathcal{V}_n^\phi)_{n\geqslant 0}$ enable us to invoke [99, Proposition 11.4] to obtain the equalities for algebraic and geometric multiplicities in Item (3), and also to conclude that

$$\sigma_p(\mathbb{Q}_t) = \sigma_p(\mathbb{Q}_t^*) = \sigma_p(\mathbf{Q}_t^{(\mathfrak{r}_1)}) = \{e^{-\lambda_n t};\ n \in \mathbb{N}\}.$$

Since $\mathbb{Q}_t$ is compact we get that $\mathbb{Q}_t^*$ is compact, and thus for both of these operators their spectrum is equal to their point spectrum. To establish the remaining equalities we use the immediate compactness of $\mathbb{Q}$ to invoke [49, Corollary 3.12] and obtain $\sigma(\mathbb{Q}_t) \setminus \{0\} = e^{t\sigma(\mathbb{J})}$, while we also have from [49, Theorem 3.7] that, $\sigma_p(\mathbb{Q}_t) \setminus \{0\} = e^{t\sigma_p(\mathbb{J})}$. Putting all of these together completes the proof of Item (3). Finally it remains to prove the last item concerning the self-adjointness of $\mathbb{Q}$. Clearly if $h \equiv 0$ then $\mathbb{Q}$ is self-adjoint, as in this case β reduces to β_μ and $\mathbb{Q}$ reduces to the classical Jacobi semigroup $\mathbf{Q}^{(\mu)}$, which is self-adjoint on $\mathrm{L}^2(\beta_\mu)$. Now suppose that $\mathbb{Q}$ is self-adjoint on $\mathrm{L}^2(\beta)$, that is $\mathbb{Q}_t = \mathbb{Q}_t^*$ for all $t \geqslant 0$. By differentiating in t the identity, for any $n, m \in \mathbb{N}$,

$$\langle \mathbb{Q}_t p_n, p_m \rangle_\beta = \langle p_n, \mathbb{Q}_t p_m \rangle_\beta$$

we deduce, by a simple application of Fubini's Theorem using the finiteness of the measure β, that

$$\langle \mathbb{J} p_n, p_m \rangle_\beta = \langle p_n, \mathbb{J} p_m \rangle_\beta \,. \tag{4.50}$$

Note that (4.50) holds trivially if either $n = 0$ or $m = 0$, or if $n = m$, so we may suppose that $n \neq m$; all together we take, without loss of generality, $n > m > 0$. Now, for any $n \geqslant 1$, a straightforward calculation shows that

$$\mathbb{J} p_n(x) = \Psi(n) p_{n-1}(x) - \lambda_n p_n(x), \tag{4.51}$$

where we recall from (4.4) that $\Psi(n) = (n - \mathsf{r}_0)\phi(n)$ and from (4.10) that $\lambda_n = n^2 + (\boldsymbol{\lambda}_1 - 1)n$. Using (4.51) on both sides of (4.50) and rearranging gives

$$(\lambda_n - \lambda_m)\, \beta p_{n+m} = (\Psi(n) - \Psi(m))\, \beta p_{n+m-1}. \tag{4.52}$$

By (4.6) and the recurrence relations for W_ϕ and the gamma function, the ratio $\beta[p_{n+m}]/\beta[p_{n+m-1}]$ evaluates to

$$\frac{\beta[p_{n+m}]}{\beta[p_{n+m-1}]} = \frac{(n + m + \mathsf{r}_0)}{(n + m + \boldsymbol{\lambda}_1 - 1)} \frac{\phi(n + m)}{(n + m)} = \frac{\Psi(n + m)}{\lambda_{n+m}},$$

so that substituting into (4.52) shows that the following must be satisfied

$$\Psi(n + m)\, (\lambda_n - \lambda_m) = \lambda_{n+m}\, (\Psi(n) - \Psi(m))\,. \tag{4.53}$$

Next, we write Ψ as

$$\Psi(n) = n^2 + (\mu - \hbar - 1)n + n \int_1^\infty (1 - r^{-n}) h(r) dr = n^2 + (\mu - 1)n + n \int_1^\infty r^{-n} h(r) dr,$$

where the first equality is simply the definition of Ψ in (4.2) and the second follows from the assumption that $\hbar = \int_1^\infty h(r)dr < \infty$. Let us write $G(n) = n^2 + (\mu - 1)n$ and $H(n) = n\int_1^\infty r^{-n}h(r)dr$. By direct verification we get

$$\begin{aligned} G(n+m)\,(\lambda_n - \lambda_m) &= (n-m)\left[(n+m)^3 + (\boldsymbol{\lambda}_1 + \mu - 2)(n+m)^2(\boldsymbol{\lambda}_1 - 1)(\mu - 1)(n+m)\right] \\ &= \lambda_{n+m}\,(G(n) - G(m)), \end{aligned}$$

so that (4.53) is equivalent to

$$H(n+m)\,(\lambda_n - \lambda_m) = \lambda_{n+m}\,(H(n) - H(m))\,. \tag{4.54}$$

Observe that

$$H(n+m)\,(\lambda_n - \lambda_m) = (n-m)(n+m)\,(n+m+\boldsymbol{\lambda}_1 - 1)\int_1^\infty r^{-(n+m)}h(r)dr,$$

while

$$\lambda_{n+m}\,(H(n) - H(m)) = (n+m)(n+m+\boldsymbol{\lambda}_1 - 1)\left(n\int_1^\infty r^{-n}h(r)dr - m\int_1^\infty r^{-m}h(r)dr\right).$$

Hence canceling $(n+m)(n+m+\boldsymbol{\lambda}_1 - 1)$ on both sides of (4.54), then dividing by nm and rearranging the resulting equation yields

$$\int_1^\infty r^{-m}h(r)dr = \int_1^\infty r^{-n}h(r)dr + \left(\frac{1}{n} - \frac{1}{m}\right)\int_1^\infty r^{-(n+m)}h(r)dr.$$

Applying the dominated convergence theorem when taking the limit as $n \to \infty$ of the right-hand side we find that, for all $m > 0$ with $m \neq n$,

$$\int_1^\infty r^{-m}h(r)dr = 0,$$

which implies that $h \equiv 0$. This completes the proof of Item (4) and thus the proof of the theorem. $\square$

To conclude this section we give a result concerning the intertwining operators in Proposition 4.3.3 which illustrates that, except in the self-adjoint case of $h \equiv 0$ and

$\mu \leqslant 1$, none of these operators admit bounded inverses. This latter fact combined with the relation (4.48) imply that $(\mathcal{P}_n^{\phi})_{n\geqslant 0}$ is a not a Riesz sequence in $\mathrm{L}^2(\beta)$, as it is not the image of an orthogonal sequence by an invertible bounded operator, see [37]. Recall that a quasi-affinity is a linear operator between two Banach spaces with trivial kernel and dense range.

Proposition 4.3.5. *Let* $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$ *and* $\varepsilon \in (0, d_\phi] \cup \{d_\phi\}$.

(1) The operators $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} : \mathrm{L}^2(\beta_{\boldsymbol{d}_{1,\varepsilon}}) \to \mathrm{L}^2(\beta)$, $\mathrm{V}_{\phi^*_{\mathfrak{m}}} : \mathrm{L}^2(\beta) \to \mathrm{L}^2(\beta_{\mathfrak{m}})$, *and* $\mathrm{U}_{\phi^{\checkmark}_{\mathfrak{r}_1}} : \mathrm{L}^2(\beta_{\phi^{\checkmark}_{\mathfrak{r}_1}}) \to \mathrm{L}^2(\beta)$ *are all quasi-affinities.*

(2) The operator $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}$ *admits a bounded inverse if and only if* $h \equiv 0$ *and* $\mu \leqslant 1$ *when* $\boldsymbol{d}_{1,\varepsilon} = 1$, *where* $\boldsymbol{d}_{1,\varepsilon}$ *was defined in* (4.19). *In all cases* $\mathrm{V}_{\phi^*_{\mathfrak{m}}}$ *and* $\mathrm{U}_{\phi^{\checkmark}_{\mathfrak{r}_1}}$ *do not admit bounded inverses.*

Proof. Since polynomials belong to the L^2-range of the operators $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}$, $\mathrm{V}_{\phi^*_{\mathfrak{m}}}$, and $\mathrm{U}_{\phi^{\checkmark}_{\mathfrak{r}_1}}$, we get, by moment determinacy, that each of these has dense range in their respective codomains. For the remaining claims we proceed sequentially by considering each operator individually, starting with $\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}$. Proposition 4.3.4 gives that, for any $n \in \mathbb{N}$

$$\mathcal{P}_n^{\phi} = \frac{1}{\mathfrak{c}_n(\boldsymbol{d}_{1,\varepsilon})} \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} \mathcal{P}_n^{(\boldsymbol{d}_{1,\varepsilon})},$$

and also that $(\mathcal{P}_n^{\phi})_{n\geqslant 0}$ and $(\mathcal{V}_n^{\phi})_{n\geqslant 0}$ are biorthogonal. Consequently,

$$\delta_{np} = \left\langle \mathcal{P}_n^{\phi}, \mathcal{V}_p^{\phi} \right\rangle_{\beta} = \left\langle \frac{1}{\mathfrak{c}_n(\boldsymbol{d}_{1,\varepsilon})} \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} \mathcal{P}_n^{(\boldsymbol{d}_{1,\varepsilon})}, \mathcal{V}_p^{\phi} \right\rangle_{\beta} = \frac{1}{\mathfrak{c}_n(\boldsymbol{d}_{1,\varepsilon})} \left\langle \mathcal{P}_n^{(\boldsymbol{d}_{1,\varepsilon})}, \Lambda^*_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} \mathcal{V}_p^{\phi} \right\rangle_{\beta_{\boldsymbol{d}_{1,\varepsilon}}}.$$

However, as $(\mathcal{P}_n^{(\boldsymbol{d}_{1,\varepsilon})})_{n\geqslant 0}$ forms an orthonormal basis for $\mathrm{L}^2(\beta_{\boldsymbol{d}_{1,\varepsilon}})$ it must be its own unique biorthogonal sequence, which forces

$$\frac{1}{\mathfrak{c}_n(\boldsymbol{d}_{1,\varepsilon})} \Lambda^*_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} \mathcal{V}_n^{\phi} = \mathcal{P}_n^{(\boldsymbol{d}_{1,\varepsilon})},$$

for all $n \in \mathbb{N}$. Thus we conclude that $\mathcal{P} \subset \mathrm{Ran}(\Lambda^*_{\phi_{d_{1,\varepsilon}}})$, so that by moment determinacy of $(\beta_{d_{1,\varepsilon}})$, we get that $\mathrm{Ker}(\Lambda_{\phi_{d_{1,\varepsilon}}}) = \{0\}$. Next, by straightforward computation we have, for any $n \in \mathbb{N}$,

$$\|p_n\|^{-2}_{\beta_{d_{1,\varepsilon}}} \left\|\Lambda_{\phi_{d_{1,\varepsilon}}} p_n\right\|^2_{\beta} = \frac{W_\phi(2n+1)}{W^2_\phi(n+1)} \frac{(\boldsymbol{d}_{1,\varepsilon})^2_n}{(\boldsymbol{d}_{1,\varepsilon})_{2n}} = \frac{W_{\phi_{d_{1,\varepsilon}}}(2n+1)}{W^2_{\phi_{d_{1,\varepsilon}}}(n+1)} \frac{(n!)^2}{(2n)!}, \tag{4.55}$$

where the second equality follows by using the definition of $\phi_{d_{1,\varepsilon}}$, see (4.20), together with the recurrence relation for $W_{\phi_{d_{1,\varepsilon}}}$. Now, the same arguments as in the proof of [99, Theorem 7.1(2)] may be applied, see e.g. Section 7.3 therein, to get that the ratio in (4.55) tends to 0 as $n \to \infty$ if and only if $\phi_{d_{1,\varepsilon}}(0) = 0$ and $\Pi \equiv 0 \iff h \equiv 0$. This is because, with the notation of the aforementioned paper, the expression for $\frac{\psi(u)}{u^2}$ is equal to $\frac{\phi_{d_{1,\varepsilon}}(u)}{u}$ in our notation, and we have $\sigma^2 = 1$ from $\lim_{u\to\infty} \frac{\phi_{d_{1,\varepsilon}}(u)}{u} = 1$. From the definition of $\phi_{d_{1,\varepsilon}}$ in (4.20) we find that, if $\boldsymbol{d}_{1,\varepsilon} = 1$, then $\phi_{d_{1,\varepsilon}}(0) = \phi(0) = 0$ and from Lemma 4.3.2(3) we get that $\phi(0) = \mu - 1 - \hbar$ if $\mu \geqslant 1 + \hbar$ while $\phi(0)$ is always zero when $\mu < 1 + \hbar$, which shows that if $\boldsymbol{d}_{1,\varepsilon} = 1$ then $\phi(0) = 0 \iff \mu \leqslant 1$. On the other hand, from (4.20), it is clear that if $\boldsymbol{d}_{1,\varepsilon} > 1$ then always $\phi_{d_{1,\varepsilon}}(0) = 0$. This completes the proof of the claims regarding $\Lambda_{\phi_{d_{1,\varepsilon}}}$. Next, by Proposition 4.3.4, $\mathcal{V}^\phi_n \in \mathrm{Ran}(\mathrm{V}^*_{\phi^*_{\mathfrak{m}}})$, for each $n \in \mathbb{N}$, and as proved in Proposition 4.3.3, the sequence $(\mathcal{V}^\phi_n)_{n\geqslant 0}$ is complete. Thus $\mathrm{Ran}(\mathrm{V}^*_{\phi^*_{\mathfrak{m}}})$ is dense in $\mathrm{L}^2(\beta_{\mathfrak{m}})$, or equivalently $\mathrm{Ker}(\mathrm{V}_{\phi^*_{\mathfrak{m}}}) = \{0\}$. By direct calculation we get that,

$$\|p_n\|^{-2}_{\beta} \left\|\mathrm{V}_{\phi^*_{\mathfrak{m}}} p_n\right\|^2_{\beta_{\mathfrak{m}}} = \frac{W^2_\phi(n+1)}{W_\phi(2n+1)} \frac{(\mathfrak{m})_{2n}}{(\mathfrak{m})^2_n} = \prod_{k=1}^{n} \frac{\phi^*_{\mathfrak{m}}(k)}{\phi^*_{\mathfrak{m}}(k+n)}, \tag{4.56}$$

where $\phi^*_{\mathfrak{m}}$ was defined in (4.21). Now the fact that $\lim_{u\to\infty} \frac{\phi(u)}{u} = 1$ allow us to deduce $\lim_{u\to\infty} \phi^*_{\mathfrak{m}}(u) = 1$ and, as noted earlier, $\phi^*_{\mathfrak{m}}$ is a Bernstein function and hence non-decreasing. As the case $\phi^*_{\mathfrak{m}} \equiv 1$ is excluded by the assumption on $\mathfrak{m}$, we get that, as $n \to \infty$, the ratio in (4.56) tends to 0. Next, by taking the adjoint of (4.40) we get

$$\mathrm{U}^*_{\phi^{\checkmark}_{\mathfrak{r}_1}} \mathbb{Q}^{\phi^{\checkmark}_{\mathfrak{r}_1}*}_t = \mathbb{Q}^{\phi^*}_t \mathrm{U}^*_{\phi^{\checkmark}_{\mathfrak{r}_1}}$$

and using this identity we get that $\mathrm{U}^*_{\phi^{\checkmark}_{\mathrm{r}_1}}\mathcal{V}^{\phi}_n$ is an eigenfunction for $\mathbb{Q}_t^{\phi^{\checkmark}_{\mathrm{r}_1}{}^*}$ associated to the eigenvalue $e^{-\lambda_n t}$. Then, Theorem 4.2.2(3) forces $\mathrm{U}^*_{\phi^{\checkmark}_{\mathrm{r}_1}}\mathcal{V}^{\phi}_n = \mathcal{V}^{\phi^{\checkmark}_{\mathrm{r}_1}}_n$, and the latter is a complete sequence, whence $\mathrm{Ker}(\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}) = \{0\}$. Finally, another straightforward calculation gives that

$$\|p_n\|^{-2}_{\beta_\phi}\left\|\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}p_n\right\|^2_{\beta_{\phi^{\checkmark}_{\mathrm{r}_1}}} = \frac{\phi^{\checkmark 2}_{\mathrm{r}_1}(0)}{\phi^{\checkmark 2}_{\mathrm{r}_1}(n)}\frac{\phi(2n+1)}{\mathrm{r}_1\phi(1)}\frac{2n+\mathrm{r}_1}{2n+1} = \phi^{\checkmark}_{\mathrm{r}_1}(0)\frac{2n+\mathrm{r}_1}{(n+\mathrm{r}_1)^2}\left(\frac{n+!}{\phi(n+1)}\right)^2\frac{\phi(2n+1)}{2n+1},$$

and using the fact that $\lim_{u\to\infty}\frac{\phi(u)}{u} = 1$ we conclude that the right-hand side tends to 0 as $n \to \infty$. $\square$

4.3.6 Proof of Theorem 4.2.3(1)

Theorem 4.2.2 gives, for any $f \in \mathrm{L}^2(\beta)$ and $t > 0$,

$$\mathbb{Q}_t f = \sum_{n=0}^{\infty} e^{-\lambda_n t}\langle f, \mathcal{V}^{\phi}_n\rangle_{\beta}\mathcal{P}^{\phi}_n$$

so that, since $\lambda_0 = 0$ and $\mathcal{P}^{\phi}_0 \equiv 1 \equiv \mathcal{V}^{\phi}_0$,

$$\mathbb{Q}_t f - \beta f = \sum_{n=1}^{\infty} e^{-\lambda_n t}\langle f, \mathcal{V}^{\phi}_n\rangle_{\beta}\mathcal{P}^{\phi}_n. \tag{4.57}$$

Next, we note that

$$\begin{aligned}\sup_{n\geqslant 1} e^{-2n\lambda_1 t}\frac{\mathfrak{c}^2_n(\mathfrak{m})}{\mathfrak{c}^2_n(\boldsymbol{d}_{\mathrm{r}_1,\varepsilon})} \leqslant e^{-2\lambda_1 t}\frac{\mathfrak{c}^2_1(\mathfrak{m})}{\mathfrak{c}^2_1(\boldsymbol{d}_{\mathrm{r}_1,\varepsilon})} \iff \\ 2\lambda_1 t \geqslant \log\left(\frac{(\mathfrak{m}+1)(\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathrm{r}_1,\varepsilon}+1)}{(\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}+1)(\boldsymbol{\lambda}_1 - \mathfrak{m}+1)}\right),\end{aligned} \tag{4.58}$$

since

$$e^{-2(n-1)\lambda_1 t}\frac{\mathfrak{c}^2_n(\mathfrak{m})}{\mathfrak{c}^2_n(\boldsymbol{d}_{\mathrm{r}_1,\varepsilon})}\frac{\mathfrak{c}^2_1(\boldsymbol{d}_{\mathrm{r}_1,\varepsilon})}{\mathfrak{c}^2_1(\mathfrak{m})} = \prod_{j=1}^{n-1} e^{-2\lambda_1 t}\frac{(\mathfrak{m}+j)(\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathrm{r}_1,\varepsilon}+j)}{(\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}+j)(\boldsymbol{\lambda}_1 - \mathfrak{m}+j)},$$

and $\mathfrak{m} > \boldsymbol{d}_{\mathrm{r}_1,\varepsilon}$, which is trivial when $\mathrm{r}_1 < 1$, as then $\mathfrak{m} > 1 > \boldsymbol{d}_{\mathrm{r}_1,\varepsilon} = \mathrm{r}_1$, while if $\mathrm{r}_1 = 1$ we have $\mathfrak{m} - 1 > d_\phi > \boldsymbol{d}_{\mathrm{r}_1,\varepsilon} - 1$ from [99, Proposition 4.4(1)]. Now, we claim that the following computation is valid, writing $||\cdot||_\beta$ again for the $\mathrm{L}^2(\beta)$-norm and $\mathcal{V}_n^\phi = \frac{\mathfrak{c}_n(\mathfrak{r}_1)}{\mathfrak{c}_n(\mathfrak{m})}\mathcal{V}_n^\phi$,

$$\begin{aligned}
\|\mathbb{Q}_t f - \beta f\|^2 &\leqslant \sum_{n=1}^{\infty} \frac{1}{\mathfrak{c}_n^2(\boldsymbol{d}_{1,\varepsilon})}\left|\langle \mathbb{Q}_t f, \mathcal{V}_n^\phi\rangle_\beta\right|^2 = \sum_{n=1}^{\infty} e^{-2\lambda_n t}\frac{\mathfrak{c}_n^2(\mathfrak{m})}{\mathfrak{c}_n^2(\boldsymbol{d}_{\mathfrak{r}_1,\varepsilon})}\left|\left\langle f, \mathcal{V}_n^\phi\right\rangle_\beta\right|^2 \\
&\leqslant \frac{\mathfrak{m}(\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathfrak{r}_1,\varepsilon})}{\boldsymbol{d}_{\mathfrak{r}_1,\varepsilon}(\boldsymbol{\lambda}_1 - \mathfrak{m})} e^{-2\lambda_1 t}\sum_{n=1}^{\infty}\left|\left\langle f, \mathcal{V}_n^\phi\right\rangle_\beta\right|^2 \\
&= \frac{\mathfrak{m}(\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathfrak{r}_1,\varepsilon})}{\boldsymbol{d}_{\mathfrak{r}_1,\varepsilon}(\boldsymbol{\lambda}_1 - \mathfrak{m})} e^{-2\lambda_1 t}\sum_{n=1}^{\infty}\left|\left\langle f - \beta f, \mathcal{V}_n^\phi\right\rangle_\beta\right|^2 \\
&\leqslant \frac{\mathfrak{m}(\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathfrak{r}_1,\varepsilon})}{\boldsymbol{d}_{\mathfrak{r}_1,\varepsilon}(\boldsymbol{\lambda}_1 - \mathfrak{m})} e^{-2\lambda_1 t}\,\|f - \beta f\|_\beta^2 .
\end{aligned}$$

To justify this we start by observing that the first inequality follows from (4.57) together with $(\mathfrak{c}_n(\boldsymbol{d}_{1,\varepsilon})\mathcal{P}_n^\phi)_{n\geqslant 0}$ being a Bessel sequence with Bessel bound 1, which was proved in Proposition 4.3.4. Next we use the fact that $\mathcal{V}_n^\phi$ is an eigenfunction for $\mathbb{Q}_t^*$ associated to the eigenvalue $e^{-\lambda_n t}$, and then the identity

$$\mathfrak{c}_n(\mathfrak{r}_1)\mathfrak{c}_n(\boldsymbol{d}_{1,\varepsilon}) = \mathfrak{c}_n(\boldsymbol{d}_{\mathfrak{r}_1,\varepsilon}),$$

which follows by considering the cases $\mathfrak{r}_1 = 1$ and $\mathfrak{r}_1 < 1$ separately. Indeed, when $\mathfrak{r}_1 = 1$ then $\boldsymbol{d}_{\mathfrak{r}_1,\varepsilon} = \boldsymbol{d}_{1,\varepsilon}$ and $\mathfrak{c}_n^2(\mathfrak{r}_1) = 1$, while otherwise $\boldsymbol{d}_{1,\varepsilon} = 1$ so that $\boldsymbol{d}_{\mathfrak{r}_1,\varepsilon} = \mathfrak{r}_1$ and $\mathfrak{c}_n^2(\boldsymbol{d}_{1,\varepsilon}) = 1$. The second inequality follows from (4.58) and then we use the biorthogonality of $(\mathcal{P}_n^\phi)_{n\geqslant 0}$ and $(\mathcal{V}_n^\phi)_{n\geqslant 0}$, given by Proposition 4.3.4, which implies that for any $c \in \mathbb{R}$, $\langle c\mathbf{1}_{[0,1]}, \mathcal{V}_n^\phi\rangle_\beta = 0$ if $n \neq 0$. The last inequality follows from the fact that $(\mathcal{V}_n^\phi)_{n\geqslant 0}$ is a Bessel sequence with Bessel bound 1, again due to Proposition 4.3.4. Next, when $0 \leqslant 2\boldsymbol{\lambda}_1 t < \log\left(\frac{(1+\mathfrak{m})(1+\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathfrak{r}_1,\varepsilon})}{(1+\boldsymbol{d}_{\mathfrak{r}_1,\varepsilon})(1+\boldsymbol{\lambda}_1-\mathfrak{m})}\right)$ and since $\mathfrak{m} > \boldsymbol{d}_{\mathfrak{r}_1,\varepsilon}$, we get

$$\frac{\mathfrak{m}(\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathfrak{r}_1,\varepsilon})}{\boldsymbol{d}_{\mathfrak{r}_1,\varepsilon}(\boldsymbol{\lambda}_1 - \mathfrak{m})} e^{-2\lambda_1 t} \geqslant \frac{\mathfrak{m}}{\mathfrak{m}+1}\,\frac{\boldsymbol{d}_{\mathfrak{r}_1,\varepsilon}+1}{\boldsymbol{d}_{\mathfrak{r}_1,\varepsilon}}\,\frac{\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathfrak{r}_1,\varepsilon}}{\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathfrak{r}_1,\varepsilon}+1}\,\frac{\boldsymbol{\lambda}_1 - \mathfrak{m}+1}{\boldsymbol{\lambda}_1 - \mathfrak{m}} \geqslant 1,$$

so that the contractivity of the semigroup $\mathbb{Q}$ yields, for $f \in \mathrm{L}^2(\beta)$ and any $t > 0$,

$$\|\mathbb{Q}_t f - \beta f\|_\beta^2 \leqslant e^{-2\boldsymbol{\lambda}_1 t}\,\|f - \beta f\|_\beta^2 .$$

Finally, since β is an invariant probability measure,

$$\begin{aligned}||\mathbb{Q}_t f - \beta f||_\beta^2 &= \beta[(\mathbb{Q}_t f - \beta f)^2] \\ &= \beta[(\mathbb{Q}_t f)^2] - 2\beta[f]\beta[\mathbb{Q}_t f] + (\beta[f])^2 \\ &= \beta[(\mathbb{Q}_t f)^2] - (\beta[f])^2 = \mathrm{Var}_\beta(\mathbb{Q}_t f),\end{aligned}$$

which completes the proof. □

4.3.7 Proof of Theorem 4.2.3(2)

We first give a result that strengthens the intertwining relations in Proposition 4.3.3 and falls into the framework of the work by Miclo and Patie [88]. Write $\mathrm{V}_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}$ for the Markov multiplicative kernel associated to a random variable with law $\beta_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}$, which, by the same arguments as in the proof of Lemma 4.3.10, satisfies $\mathrm{V}_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}} \in \mathcal{B}(\mathrm{L}^2(\beta_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}), \mathrm{L}^2(\beta_{\mathfrak{m}}))$. We write $\overline{\mathrm{V}}_\phi = \Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}} \mathrm{V}^*_{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}}$ and, for $\mu \geqslant 1+\hbar$, let $\widetilde{\mathrm{V}}_\phi = \mathrm{V}_{\phi^*_{\mathfrak{m}}}$ and otherwise let $\widetilde{\mathrm{V}}_\phi = \mathrm{V}_{\phi^*_{\mathfrak{m}}} \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}$. Recall that a function $F : \mathbb{R}_+ \to [0,\infty)$ is said to be completely monotone if $F \in C^\infty(\mathbb{R}_+)$ and $(-1)^n \frac{d^n}{dx^n} F(u) \geqslant 0$, for $u > 0$ and $n \in \mathbb{N}$. By Bernstein's theorem, any completely monotone function F is the Laplace transform of a positive measure on $[0,\infty)$, and if $\lim_{u\to 0} F(u) < \infty$ (resp. $\lim_{u\to 0} F(u) = 1$) then F is the Laplace transform of finite (resp. probability) measure on $\mathbb{R}_+$, see e.g. [109, Chapter 1].

Proposition 4.3.6. *Under the assumptions of the theorem, we have a completely monotone intertwining relationship between* $\mathbb{Q}$ *and* $\mathbf{Q}^{(\mathfrak{m})}$*, in the sense of [88], that is for* $t \geqslant 0$ *and on the respective* L^2*-spaces*

$$\mathbb{Q}^\phi_t \overline{\mathrm{V}}_\phi = \overline{\mathrm{V}}_\phi \mathbf{Q}^{(\mathfrak{m})}_t \quad \textit{and} \quad \widetilde{\mathrm{V}}_\phi \mathbb{Q}^\phi_t = \mathbf{Q}^{(\mathfrak{m})}_t \widetilde{\mathrm{V}}_\phi \quad \textit{with} \quad \widetilde{\mathrm{V}}_\phi \overline{\mathrm{V}}_\phi = F_\phi(-\mathbf{J}_{\mathfrak{m}}), \tag{4.59}$$

where $-\log F_\phi$ *is a Bernstein function with* $F_\phi : [0,\infty) \to [0,\infty)$ *being the completely* *monotone function given by*

$$F_\phi(u) = \frac{(\boldsymbol{d}_{\mathrm{r}_1,\varepsilon})_{\rho(u)}}{(\mathfrak{m})_{\rho(u)}} \frac{(\boldsymbol{\lambda}_1 - \mathfrak{m})_{\rho(u)}}{(\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathrm{r}_1,\varepsilon})_{\rho(u)}}, \quad u \geqslant 0.$$

Proof. We give the proof only in the case $\mu \geqslant 1 + \hbar$, so that $\boldsymbol{d}_{\mathrm{r}_1,\varepsilon} = \boldsymbol{d}_{1,\varepsilon}$, as the other case follows by similar arguments. From Proposition 4.3.3 we get, with $\mathbb{J} = \mathbf{J}_{\boldsymbol{d}_{1,\varepsilon}}$,

$$\mathbf{Q}_t^{(\mathfrak{m})} \mathrm{V}_{\boldsymbol{d}_{1,\varepsilon}} = \mathrm{V}_{\boldsymbol{d}_{1,\varepsilon}} \mathbf{Q}_t^{(\boldsymbol{d}_{1,\varepsilon})},$$

and taking the adjoint and using that both $\mathbf{Q}^{(\mathfrak{m})}$ and $\mathbf{Q}^{(\boldsymbol{d}_{1,\varepsilon})}$ are self-adjoint on $\mathrm{L}^2(\beta_{\mathfrak{m}})$ and $\mathrm{L}^2(\beta_{\boldsymbol{d}_{1,\varepsilon}})$, respectively, we get that

$$\mathbf{Q}_t^{(\boldsymbol{d}_{1,\varepsilon})} \mathrm{V}^*_{\boldsymbol{d}_{1,\varepsilon}} = \mathrm{V}^*_{\boldsymbol{d}_{1,\varepsilon}} \mathbf{Q}_t^{(\mathfrak{m})}.$$

Combining this with the first intertwining relation in Proposition 4.3.3 then yields

$$\mathbb{Q}_t \overline{\mathrm{V}}_\phi = \overline{\mathrm{V}}_\phi \mathbf{Q}_t^{(\mathfrak{m})},$$

and, together with second intertwining relation in Proposition 4.3.1, we conclude that

$$\mathbf{Q}_t^{(\mathfrak{m})} \widetilde{\mathrm{V}}_\phi \overline{\mathrm{V}}_\phi = \widetilde{\mathrm{V}}_\phi \mathbb{Q}_t \overline{\mathrm{V}}_\phi = \widetilde{\mathrm{V}}_\phi \overline{\mathrm{V}}_\phi \mathbf{Q}_t^{(\mathfrak{m})}. \tag{4.60}$$

As $\mathbf{Q}_t^{(\mathfrak{m})}$ is self-adjoint with simple spectrum the commutation identity (4.60) implies, by the Borel functional calculus, see e.g. [105], that $\widetilde{\mathrm{V}}_\phi \overline{\mathrm{V}}_\phi = F(\mathbf{J}_{\mathfrak{m}})$ for some bounded Borelian function F, and to identify F it suffices to identify the spectrum of $\widetilde{\mathrm{V}}_\phi \overline{\mathrm{V}}_\phi$. To this end we observe that, for any $g \in \mathrm{L}^2(\beta_{\boldsymbol{d}_{1,\varepsilon}})$,

$$\begin{aligned}
\langle \mathrm{V}^*_{\boldsymbol{d}_{1,\varepsilon}} \mathcal{P}_n^{(\mathfrak{m})}, g \rangle_{\beta_{\boldsymbol{d}_{1,\varepsilon}}} &= \langle \mathcal{P}_n^{(\mathfrak{m})}, \mathrm{V}_{\boldsymbol{d}_{1,\varepsilon}} g \rangle_{\beta_{\mathfrak{m}}} \\
&= \sum_{m=0}^{\infty} \langle g, \mathcal{P}_m^{(\boldsymbol{d}_{1,\varepsilon})} \rangle_{\beta_{\boldsymbol{d}_{1,\varepsilon}}} \langle \mathcal{P}_n^{(\mathfrak{m})}, \mathrm{V}_{\boldsymbol{d}_{1,\varepsilon}} \mathcal{P}_m^{(\boldsymbol{d}_{1,\varepsilon})} \rangle_{\beta_{\mathfrak{m}}} \\
&= \frac{\mathfrak{c}_n(\boldsymbol{d}_{1,\varepsilon})}{\mathfrak{c}_n(\mathfrak{m})} \langle \mathcal{P}_n^{(\boldsymbol{d}_{1,\varepsilon})}, g \rangle_{\beta_{\boldsymbol{d}_{1,\varepsilon}}},
\end{aligned}$$

where we used that $(\mathcal{P}_n^{(\boldsymbol{d}_{1,\varepsilon})})_{n\geqslant 0}$ forms an orthonormal basis for $\mathrm{L}^2(\beta_{\boldsymbol{d}_{1,\varepsilon}})$ and the identity $\mathrm{V}_{\boldsymbol{d}_{1,\varepsilon}}\mathcal{P}_m^{(\boldsymbol{d}_{1,\varepsilon})} = \mathfrak{c}_m(\boldsymbol{d}_{1,\varepsilon})\mathcal{P}_m^{(\mathfrak{m})}/\mathfrak{c}_m(\mathfrak{m})$ follows by a straightforward, albeit tedious, computation. Consequently, for any $n \in \mathbb{N}$,

$$\widetilde{\mathrm{V}}_\phi\overline{\mathrm{V}}_\phi\mathcal{P}_n^{(\mathfrak{m})} = \frac{\mathfrak{c}_n(\boldsymbol{d}_{1,\varepsilon})}{\mathfrak{c}_n(\mathfrak{m})}\mathrm{V}_{\phi^*_{\mathfrak{m}}}\Lambda_{\phi_{\boldsymbol{d}_{1,\varepsilon}}}\mathcal{P}_n^{(\boldsymbol{d}_{1,\varepsilon})} = \frac{\mathfrak{c}_n^2(\boldsymbol{d}_{1,\varepsilon})}{\mathfrak{c}_n(\mathfrak{m})}\mathrm{V}_{\phi^*_{\mathfrak{m}}}\mathcal{P}_n^{\phi} = \frac{\mathfrak{c}_n^2(\boldsymbol{d}_{1,\varepsilon})}{\mathfrak{c}_n^2(\mathfrak{m})}\mathcal{P}_n^{(\mathfrak{m})},$$

where the second and third equalities follow from calculations that were detailed in the proof of Proposition 4.3.4. Using the definition of $\mathfrak{c}_n$ in (4.44) we thus get that, for $n \in \mathbb{N}$,

$$F(\lambda_n) = \frac{\mathfrak{c}_n^2(\boldsymbol{d}_{1,\varepsilon})}{\mathfrak{c}_n^2(\mathfrak{m})} = \frac{(\boldsymbol{d}_{1,\varepsilon})_n}{(\mathfrak{m})_n}\frac{(\boldsymbol{\lambda}_1 - \mathfrak{m})_n}{(\boldsymbol{\lambda}_1 - \boldsymbol{d}_{1,\varepsilon})_n}$$

recalling from (4.10) that $(\lambda_n)_{n\geqslant 0}$ are the eigenvalues of $-\mathbf{J}_{\mathfrak{m}}$, which proves that $F_\phi = F$. Next, one readily computes that the non-negative inverse of the mapping $n \mapsto \lambda_n$ is given by the function ρ defined prior to the statement of the theorem, which was remarked to be a Bernstein function. For another short proof of this fact, observe that, for $u \geqslant 0$,

$$\rho'(u) = \left((\boldsymbol{\lambda}_1 - 1)^2 + 4u\right)^{-\frac{1}{2}},$$

which is completely monotone. Since $u \mapsto F_\phi(u^2 + (\boldsymbol{\lambda}_1 - 1)u)$ is the Laplace transform of the product convolution of the beta distributions $\beta_{\boldsymbol{d}_{1,\varepsilon}}$ and $\beta_{\mathfrak{m}}$ we may invoke [109, Theorem 3.7] to conclude F_ϕ is completely monotone. Finally, to show that $-\log F_\phi$ is a Bernstein function we note that, for any $a, b > 0$, the function $u \mapsto \log(a+b)_u - \log(a)_u$ is a Bernstein function, see e.g. Example 88 in [109, Chapter 16]. Since

$$-\log F_\phi(u) = \log\frac{(\mathfrak{m})_{\rho(u)}}{(\boldsymbol{d}_{1,\varepsilon})_{\rho(u)}} + \log\frac{(\boldsymbol{\lambda}_1 - \boldsymbol{d}_{1,\varepsilon})_{\rho(u)}}{(\boldsymbol{\lambda}_1 - \mathfrak{m})_{\rho(u)}},$$

with $\boldsymbol{d}_{1,\varepsilon} < \mathfrak{m}$, and the composition of Bernstein functions remains Bernstein together with the fact that the set of Bernstein functions is a convex cone, see e.g. [109, Corollary 3.8] for both of these claims, it follows that $-\log F_\phi$ is a Bernstein function. □

Proof of Theorem 4.2.3(2). Since $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$ we may apply Proposition 4.3.6 to conclude that $\widetilde{\mathrm{V}}_\phi\overline{\mathrm{V}}_\phi = F_\phi(-\mathbf{J}_{\mathfrak{m}})$ and a straightforward substitution gives $\mathbb{E}\left[e^{-u\tau}\right] =$

$F_\phi(u)$, $u \geqslant 0$, with $-\log F_\phi$ a Bernstein function. From the Borel functional calculus we get, since $\mathbf{Q}_t^{(\mathfrak{m})}$ is self-adjoint on $\mathrm{L}^2(\beta_{\mathfrak{m}})$, that

$$\mathbf{Q}_\tau^{(\mathfrak{m})} = \int_0^\infty \mathbf{Q}_t^{(\mathfrak{m})} \mathbb{P}(\tau \in dt) = \int_0^\infty e^{t\mathbf{J}_{\mathfrak{m}}} \mathbb{P}(\tau \in dt) = F_\phi(-\mathbf{J}_{\mathfrak{m}}) = \widetilde{\mathrm{V}}_\phi \overline{\mathrm{V}}_\phi.$$

Combining this identity with (4.59) yields, for non-negative $f \in \mathrm{L}^2(\beta)$,

$$\widetilde{\mathrm{V}}_\phi \overline{\mathrm{V}}_\phi \widetilde{\mathrm{V}}_\phi f = \int_0^\infty \mathbf{Q}_t^{(\mathfrak{m})} \widetilde{\mathrm{V}}_\phi f \mathbb{P}(\tau \in dt) = \int_0^\infty \widetilde{\mathrm{V}}_\phi \mathbb{Q}_t f \mathbb{P}(\tau \in dt) = \widetilde{\mathrm{V}}_\phi \int_0^\infty \mathbb{Q}_t f \mathbb{P}(\tau \in dt),$$

and the general case follows by linearity and by decomposing f into the difference of non-negative functions. By Proposition 4.3.5 $\widetilde{\mathrm{V}}_\phi$ has trivial kernel on $\mathrm{L}^2(\beta)$ so we deduce

$$\overline{\mathrm{V}}_\phi \widetilde{\mathrm{V}}_\phi = \int_0^\infty \mathbb{Q}_t \mathbb{P}(\tau \in dt) = \mathbb{Q}_\tau, \tag{4.61}$$

and thus $\mathbb{Q}$ satisfies a completely monotone intertwining relation with $\mathbf{Q}^{(\mathfrak{m})}$, in the sense of [88]. Consequently we may invoke [88, Theorems 7, 24] to transfer the entropy decay and Φ-entropy decay of $\mathbf{Q}^{(\mathfrak{m})}$, reviewed in Section 4.5, to the semigroup $\mathbb{Q}$ but after a time shift of the independent random variable τ. Note that, when $\boldsymbol{\lambda}_1 > 2(\mathbf{1}_{\{\mu<1+\hbar\}} + \mu)$, we may take $\mathfrak{m} = \frac{\lambda_1}{2}$ so that the reference semigroup is $\mathbf{Q}^{(\lambda_1/2)}$, which has optimal entropy decay rate. □

4.3.8 Proof of Theorem 4.2.4

The proof of Theorem 4.2.4(1) follows by using Equation (4.61) above to invoke [88, Theorem 8]. Next, by Equation (4.61) and using Proposition 4.3.6 we get

$$||\mathbb{Q}_{t+\tau}||_{1\to\infty} = ||\mathbb{Q}_t \overline{\mathrm{V}}_\phi \widetilde{\mathrm{V}}_\phi||_{1\to\infty} = ||\overline{\mathrm{V}}_\phi \mathbf{Q}_t^{(\mathfrak{m})} \widetilde{\mathrm{V}}_\phi||_{1\to\infty} \leqslant ||\mathbf{Q}_t^{(\mathfrak{m})}||_{1\to\infty},$$

where the last inequality follows by applying Lemma 4.3.10 twice, once in the case $p = \infty$ for $\overline{\mathrm{V}}_\phi$ and once with $p = 1$ for $\widetilde{\mathrm{V}}_\phi$. The claim now follows from the corresponding ultracontractivity estimate for $\mathbf{Q}^{(\mathfrak{m})}$. □

4.3.9 Proof of Theorem 4.2.5

The following arguments are taken from the proof of [87, Proposition 5]. We denote by $\mathbf{Q}^{(\mathfrak{m},\boldsymbol{\tau})}$ for the classical Jacobi semigroup $\mathbf{Q}^{(\mathfrak{m})}$ subordinated with respect to $\boldsymbol{\tau} = (\tau_t)_{t\geqslant 0}$. By [88, Theorem 3] we obtain, from Proposition 4.3.6, a completely monotone intertwining relationship between the subordinate semigroups, i.e. writing $\overline{\mathrm{V}}_\phi$ and $\widetilde{\mathrm{V}}_\phi$ as above, we have, for any $t \geqslant 0$ and on the appropriate L^2-spaces,

$$\mathbb{Q}_t^{\boldsymbol{\tau}}\overline{\mathrm{V}}_\phi = \overline{\mathrm{V}}_\phi\mathbf{Q}_t^{(\mathfrak{m},\boldsymbol{\tau})} \quad \text{and} \quad \widetilde{\mathrm{V}}_\phi\mathbb{Q}_t^{\boldsymbol{\tau}} = \mathbf{Q}_t^{(\mathfrak{m},\boldsymbol{\tau})}\widetilde{\mathrm{V}}_\phi \quad \text{with} \quad \overline{\mathrm{V}}_\phi\widetilde{\mathrm{V}}_\phi = \mathbb{Q}_1^{\boldsymbol{\tau}}. \tag{4.62}$$

Using this we get, for any $f \in \mathrm{L}^2(\beta)$ and $t \geqslant 1$,

$$\begin{aligned}
\mathbb{Q}_t^{\boldsymbol{\tau}} f = \mathbb{Q}_{t-1}^{\boldsymbol{\tau}}\overline{\mathrm{V}}_\phi\widetilde{\mathrm{V}}_\phi f = \overline{\mathrm{V}}_\phi\mathbf{Q}_{t-1}^{(\mathfrak{m},\boldsymbol{\tau})}\widetilde{\mathrm{V}}_\phi f &= \sum_{n=0}^{\infty}\mathbb{E}\left[e^{-\lambda_n\tau_{t-1}}\right]\langle\widetilde{\mathrm{V}}_\phi f, \mathcal{P}_n^{(\mathfrak{m})}\rangle_{\beta_\mathfrak{m}}\overline{\mathrm{V}}_\phi\mathcal{P}_n^{(\mathfrak{m})} \\
&= \sum_{n=0}^{\infty}\mathbb{E}\left[e^{-\lambda_n\tau_{t-1}}\right]\frac{\mathfrak{c}_n^2(\boldsymbol{d}_{\mathfrak{r}_1,\varepsilon})}{\mathfrak{c}_n^2(\mathfrak{m})}\langle f, \mathcal{V}_n^\phi\rangle_\beta\mathcal{P}_n^\phi \\
&= \sum_{n=0}^{\infty}\mathbb{E}\left[e^{-\lambda_n\tau_t}\right]\langle f, \mathcal{V}_n^\phi\rangle_{\beta_\mathfrak{m}}\mathcal{P}_n^\phi,
\end{aligned}$$

where in the second equality we used the boundedness of $\overline{\mathrm{V}}_\phi$ together the expansion for the subordinated classical Jacobi semigroup which follows from (4.74) and standard arguments, then the properties of $\widetilde{\mathrm{V}}_\phi$ and $\overline{\mathrm{V}}_\phi$ detailed in previous sections, and finally the expression for $\mathbb{E}[e^{-\lambda_n\tau}]$ in (4.15). All but the last claim of the corollary then follow from [88, Theorems 7, 24] applied to (4.62). Next, we establish the ultracontractive bound $||\mathbb{Q}_t^{\boldsymbol{\tau}}||_{1\to\infty} \leqslant c_\mathfrak{m}(\mathbb{E}[\tau^{-p}] + 1)$ for $t > 2$. From (4.15) we get, by applying Stirling's formula for the gamma function together with $\lim_{u\to\infty} u^{-1/2}\rho(u) = 1$, that $\lim_{u\to\infty} u^{(\mathfrak{m}-\boldsymbol{d}_{\mathfrak{r}_1,\varepsilon})}\mathbb{E}[e^{-u\tau}] = 1$. Writing for convenience $p = \frac{\lambda_1-\mathfrak{m}}{\lambda_1-\mathfrak{m}-1} > 0$, we get by assumption on the parameters that $p < \mathfrak{m} - \boldsymbol{d}_{\mathfrak{r}_1,\varepsilon}$ so that the previous asymptotic yields, for $t \geqslant 1$,

$$\mathbb{E}[\tau_t^{-p}] = \frac{1}{\Gamma(p)}\int_0^\infty \mathbb{E}[e^{-u\tau_t}]u^{p-1}du \leqslant \frac{1}{\Gamma(p)}\int_0^\infty \mathbb{E}[e^{-u\tau}]u^{p-1}du = \mathbb{E}[\tau^{-p}] < \infty,$$

where the two equalities follow by applying Tonelli's theorem together with a change of variables, and the inequality follows from the fact that, for all $u \geqslant 0$, $t \mapsto \mathbb{E}[e^{-u\tau_t}]$ is non-increasing, recalling the notation $\tau_1 \stackrel{(d)}{=} \tau$. Hence, from the ultracontractive bound $||\mathbf{Q}_s^{(\mathfrak{m})}||_{1\to\infty} \leqslant c_{\mathfrak{m}} \max(1, s^{-p})$, valid for all $s > 0$, we deduce that for $t \geqslant 1$

$$\begin{aligned}
||\mathbf{Q}_t^{(\mathfrak{m},\tau)}||_{1\to\infty} &\leqslant \int_0^\infty ||\mathbf{Q}_s^{(\mathfrak{m})}||_{1\to\infty}\mathbb{P}(\tau_t \in ds) \\
&\leqslant c_{\mathfrak{m}} \left(\int_0^1 s^{-p}\mathbb{P}(\tau_t \in ds) + \int_1^\infty \mathbb{P}(\tau_t \in ds) \right) \\
&\leqslant c_{\mathfrak{m}}(\mathbb{E}[\tau^{-p}] + 1).
\end{aligned}$$

Consequently from (4.62) we get that, for $t > 2$,

$$||\mathbb{Q}_t^\tau||_{1\to\infty} = ||\mathbb{Q}_{t-1}^\tau \overline{\mathrm{V}}_\phi \widetilde{\mathrm{V}}_\phi||_{1\to\infty} = ||\overline{\mathrm{V}}_\phi \mathbf{Q}_{t-1}^{(\mathfrak{m},\tau)} \widetilde{\mathrm{V}}_\phi||_{1\to\infty} \leqslant ||\mathbf{Q}_{t-1}^{(\mathfrak{m},\tau)}||_{1\to\infty} \leqslant c_{\mathfrak{m}}(\mathbb{E}[\tau^{-p}]+1).$$

Then it is easy to complete the proof of the last claim by following similar arguments as in the proof of [9, Proposition 6.3.4], noting that the required variance decay estimate therein, namely

$$\mathrm{Var}_\beta(\mathbb{Q}_t^\tau f) \leqslant \left(\frac{\mathfrak{m}(\boldsymbol{\lambda}_1 - \boldsymbol{d}_{\mathrm{r}_1,\varepsilon})}{\boldsymbol{d}_{\mathrm{r}_1,\varepsilon}(\boldsymbol{\lambda}_1 - \mathfrak{m})} \right)^{1-2t} \mathrm{Var}_\beta(f)$$

valid for all $t \geqslant 0$ and $f \in \mathrm{L}^2(\beta)$, follows trivially from Theorem 4.2.3(1) via subordination. □

4.4 Examples

In this section we consider a parametric family of non-local Jacobi operators for which h is a power function. More specifically, let $\boldsymbol{\delta} \geqslant 1$ and consider the integro-differential operator $\mathbb{J}_\delta$ given by

$$\mathbb{J}_\delta f(x) = x(1-x)f''(x) - (\boldsymbol{\lambda}_1 x - \boldsymbol{\delta} - 1)f'(x) - x^{-(\delta+1)} \int_0^x f'(r)r^\delta dr$$

Then $\mathbb{J}_{\delta}$ is a non-local Jacobi operator with $\mu = \delta + 1$ and $h(r) = r^{-\delta-1}$, $r > 1$, or one easily gets that equivalently $\overline{\Pi}(r) = e^{-\delta r}$, $r > 0$. One readily computes that $\hbar = \int_1^\infty h(r)dr = \delta^{-1}$ and thus the condition $\mu \geqslant 1 + \hbar$ is always satisfied, which implies that $\mathsf{r}_1 = 1$. Writing ϕ_δ for the Bernstein function in one-to-one correspondence with $\mathbb{J}_{\delta}$, we have that for $u \geqslant 0$,

$$\phi_\delta(u) = u + \frac{\delta^2 - 1}{\delta} + \int_1^\infty (1 - r^{-u}) r^{-\delta-1} dr = \frac{(u + \delta + 1)(u + \delta - 1)}{u + \delta}. \tag{4.63}$$

From the right-hand side of (4.63) we easily see that $d_{\phi_\delta} = \delta - 1$. Now, we assume that $\lambda_1 > \delta + 2 > 3$ and, for sake of simplicity, take $\lambda_1 - \delta \notin \mathbb{N}$. The following result characterizes all the spectral objects for these non-local Jacobi operators.

Proposition 4.4.1.

(1) The density of the unique invariant measure of the Markov semigroup associated to $\mathbb{J}_{\delta}$ is given by

$$\beta(x) = \frac{((\lambda_1 - \delta - 2)x + 1)}{(\delta + 1)(1 - x)} \beta_\delta(x), \ x \in (0, 1).$$

(2) We have that $\mathcal{P}_0^{\phi_\delta} \equiv 1$ and, for $n \geqslant 1$,

$$\mathcal{P}_n^{\phi_\delta}(x) = \frac{n!}{(\delta + 2)_n} \sqrt{\mathrm{C}_n(1)} \left(\frac{\mathcal{P}_n^{(\lambda_1, \delta+2)}(x)}{\sqrt{\mathrm{C}_n(\delta + 2)}} + \frac{x}{\delta} \frac{\mathcal{P}_{n-1}^{(\lambda_1+1, \delta+3)}(x)}{\sqrt{\widetilde{\mathrm{C}}_{n-1}(\delta + 3)}} \right), \ x \in [0, 1].$$

making explicit the dependence on the two parameters for the classical Jacobi polynomials, see (4.69)*, and where* $\widetilde{\mathrm{C}}_n(\delta + 3) = n!(2n + \lambda_1)(\lambda_1 + 1)_n / (\delta + 3)_n (\lambda_1 - \delta - 2)_n$.

(3) For any $n \in \mathbb{N}$ the function $\mathcal{V}_n^{\phi_\delta}$ is given by

$$\mathcal{V}_n^{\phi_\delta}(x) = \frac{w_n(x)}{\beta(x)}, \ x \in (0, 1),$$

where w_n has the so-called Barnes integral representation, see e.g. [30], for any $a > 0$,

$$\begin{aligned} w_n(x) &= -C_{\lambda_1,\delta,n}\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty}\frac{\Gamma(\delta+2-z)\Gamma(-z)\Gamma(\delta-z)}{\Gamma(\delta+1-z)\Gamma(-n-z)\Gamma(z+\lambda_1+n)}x^z dz, \\ &= C_{\lambda_1,\delta,n}\frac{\sin(\pi(\delta-\lambda_1))}{\pi}\sum_{k=0}^{\infty}\frac{(\delta+1)_{k+n}}{(\delta+1)_k}\frac{\Gamma(k+\delta-n-\lambda_1+1)}{k!}(k-1)x^{k+\delta}, \end{aligned}$$

with $|x| < 1$ and $C_{\lambda_1,\delta,n} = \delta(\lambda_1-1)\Gamma(\lambda_1+n-1)\sqrt{\mathrm{C}_n(1)}(-2)^n/(n!\Gamma(\delta+2))$.

Proof. First, from (4.63) and (4.5) we get that, for any $n \in \mathbb{N}$,

$$W_{\phi_\delta}(n+1) = \frac{\delta}{n+\delta}(\delta+2)_n \tag{4.64}$$

so that from (4.6) we deduce that

$$\beta[p_n] = \frac{W_{\phi_\delta}(n+1)}{(\lambda_1)_n} = \frac{\delta}{n+\delta}\frac{(\delta+2)_n}{(\lambda_1)_n}. \tag{4.65}$$

The first term on the right of (4.65) is the n^{th}-moment of the probability density $f_\delta(x) = \delta x^{\delta-1}$ on $[0,1]$ while the second term is the n^{th}-moment of a $\beta_{\delta+2}$ density. Thus, by moment identification and determinacy, we conclude that $\beta(x) = f_\delta \diamond \beta_{\delta+2}(x)$ and after some easy algebra we get, for $x \in (0,1)$, that

$$\beta(x) = \frac{\Gamma(\lambda_1)\delta x^{\delta-1}}{\Gamma(\delta+2)\Gamma(\lambda_1-\delta-2)}\int_x^1 y(1-y)^{\lambda_1-\delta-3}dy = \frac{((\lambda_1-\delta-2)x+1)}{(\delta+1)(1-x)}\beta_\delta(x),$$

which completes the proof of the first item. Next, substituting (4.64) in (4.7), gives $\mathcal{P}_0^\delta \equiv 1$, and for $n = 1, 2, \ldots$,

$$\begin{aligned} \mathcal{P}_n^{\phi_\delta}(x) &= \sqrt{\mathrm{C}_n(1)}\left(\sum_{k=0}^{n}\frac{(-1)^{n+k}}{(n-k)!}\frac{(\lambda_1-1)_{n+k}}{(\lambda_1-1)_n}\frac{n!}{k!}\frac{x^k}{(\delta+2)_k}\right. \\ &\qquad\left.+\sum_{k=0}^{n}\frac{(-1)^{n+k}}{(n-k)!}\frac{(\lambda_1-1)_{n+k}}{(\lambda_1-1)_n}\frac{n!}{k!}\frac{k}{\delta}\frac{x^k}{(\delta+2)_k}\right) \\ &= \frac{n!}{(\delta+2)_n}\sqrt{\mathrm{C}_n(1)}\left(\frac{\mathcal{P}_n^{(\delta+2)}(x)}{\sqrt{\mathrm{C}_n(\delta+2)}} + \frac{x}{\delta}\frac{\mathcal{P}_{n-1}^{(\lambda_1+1,\delta+3)}(x)}{\sqrt{\widetilde{\mathrm{C}}_{n-1}(\delta+3)}}\right), \end{aligned}$$

where, to compute the second equality we made a change of variables and used the recurrence relation of the gamma function, and the definition of the classical Jacobi polynomials, see Section 4.5 and also [116]. This completes the proof of Item (2). To prove Item (3) we recall from (4.9) that, for any $n \in \mathbb{N}$, $\mathcal{V}_n^{\phi_\delta}(x) = \frac{1}{\beta(x)} w_n(x)$, where, by (4.35), the Mellin transform of w_n is given, for any $\mathrm{Re}(z) > 0$, as

$$\mathcal{M}_{w_n}(z) = C_{\lambda_1,\delta,n}(z+\delta+1)\frac{\Gamma(z)}{\Gamma(z-n)}\frac{\Gamma(z+\delta)}{\Gamma(z+\lambda_1+n)},$$

used twice the functional equation for the gamma function and the definition of the constant $C_{\lambda_1,\delta,n}$ in the statement. Next, writing $z = a + ib$ for any $b \in \mathbb{R}$ and $a > 0$, we recall from (4.38) that there exists a constant $C_a > 0$ such that

$$\lim_{|b|\to\infty} C_a |b|^{\lambda_1+n-1} \left|(z+\delta+1)\frac{\Gamma(z)}{\Gamma(z-n)}\frac{\Gamma(z+\delta)}{\Gamma(z+\lambda_1+n)}\right| = 1, \tag{4.66}$$

where we recall that $\lambda_1 > \delta + 2 > 3$ and $n \geqslant 0$. Hence, since $z \mapsto \mathcal{M}_{w_n}(z)$ is analytic on the right half-plane, by Mellin's inversion formula, see e.g. [90, Chapter 11], one gets for any $a > 0$,

$$w_n(x) = C_{\lambda_1,\delta,n}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}(z+\delta+1)\frac{\Gamma(z)}{\Gamma(z-n)}\frac{\Gamma(z+\delta)}{\Gamma(z+\lambda_1+n)}x^{-z}dz,$$

where the integral is absolutely convergent for any $x > 0$. Note that this is a Barnes-integral since we can write, again using the functional equation for the gamma function,

$$w_n(x) = -C_{\lambda_1,\delta,n}\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty}\frac{\Gamma(\delta+2-z)}{\Gamma(\delta+1-z)}\frac{\Gamma(-z)}{\Gamma(-z-n)}\frac{\Gamma(\delta-z)}{\Gamma(z+\lambda_1+n)}x^{z}dz,$$

see for instance [30]. Next, since $(z+\delta+1)\frac{\Gamma(z)}{\Gamma(z-n)} = (z+\delta+1)(z-n)\cdots(z-1)$, it follows that the function $z \mapsto (z+\delta+1)\frac{\Gamma(z)}{\Gamma(z-n)}$ does not have any poles, while the function $z \mapsto \frac{\Gamma(z+\delta)}{\Gamma(z+\lambda_1+n)}$ has simple poles at $z = -k-\delta$ for all $k \in \mathbb{N}$. Consequently, by Cauchy's residue theorem we have, for any $|x| < 1$,

$$w_n(x) = C_{\lambda_1,\delta,n}\sum_{k=0}^{\infty}\frac{(1-k)\Gamma(-k-\delta)}{\Gamma(-k-\delta-n)}\frac{(-1)^k}{k!}\frac{x^{k+\delta}}{\Gamma(-k-\delta+\lambda_1+n)},$$

where we used that the integrals along the two horizontal segments of any closed contour vanish, as by (4.66) they go to 0 when $|b| \to \infty$. We justify the radius of convergence of the series as follows. Since $\lambda_1 - \delta \notin \mathbb{N}$, using Euler's reflection formula for the gamma function, i.e. $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, $z \notin \mathbb{Z}$, we conclude that

$$w_n(x) = C_{\lambda_1,\delta,n} \frac{\sin(\pi(\delta - \lambda_1))}{\pi} \sum_{k=0}^{\infty} \frac{(\delta+1)_{k+n}}{(\delta+1)_k} \frac{\Gamma(k+\delta-n-\lambda_1+1)}{k!} (k-1) x^{k+\delta},$$

where we used that $\sin(x + k\pi) = (-1)^k \sin(x)$ for $k \in \mathbb{N}$. Using the recurrence relation of the gamma function we deduce that the radius of convergence of this series is 1, which completes the proof. □

4.5 Classical Jacobi operator and semigroup

4.5.1 Introduction and boundary classification

Before we begin reviewing the classical Jacobi operator, semigroup, and process we clarify the notational convention that is used for these objects throughout the paper. Namely, with λ_1 being fixed, instead of writing $\mathbf{J}_{\lambda_1,\mu}$ we suppress the dependency on λ_1 and write simply $\mathbf{J}_\mu$, and similarly for the beta distribution, Jacobi semigroup, and polynomials. The exception is when these objects depend in a not-straightforward way on λ_1, in which case we highlight the dependency explicitly. Now, let $\lambda_1 > \mu > 0$ and let $\mathbf{Q}^{(\mu)} = (\mathbf{Q}_t^{(\mu)})_{t\geqslant 0}$ be the classical Jacobi semigroup whose càdlàg realization is the Jacobi process $(Y_t)_{t\geqslant 0}$ on $[0,1]$, i.e. for bounded measurable functions f

$$\mathbf{Q}_t^{(\mu)} f(x) = \mathbb{E}_x\left[f(Y_t)\right], \quad x \in [0,1].$$

Then $\mathbf{Q}^{(\mu)}$ is a Feller semigroup and its infinitesimal generator $\mathbf{J}_\mu$ has, for any $f \in C^2([0,1])$, the following form

$$\mathbf{J}_\mu f(x) = x(1-x)f''(x) - (\lambda_1 x - \mu)f'(x), \quad x \in [0,1].$$

Note that when the state space of the Jacobi process is taken to be $[-1,1]$ then the associated infinitesmal generator $\widetilde{\mathbf{J}}_\mu$ is given by

$$\widetilde{\mathbf{J}}_\mu f(x) = (1-x^2)f''(x) + (2\mu - \lambda_1 - \lambda_1 x)f'(x),$$

and setting $g(x) = \frac{x+1}{2}$ yields

$$\widetilde{\mathbf{J}}_\mu (f \circ g)(g^{-1}(x)) = x(1-x)f''(x) - (\lambda_1 x - \mu)f'(x) = \mathbf{J}_\mu f(x).$$

Since the operator $\mathbf{J}_\mu$ is degenerate at the boundaries $\{0,1\}$, it is important to specify how the process behaves at these points. After some straightforward computations, as outlined in [27, Chapter 2] and using the notation therein, we get the boundaries are classified as follows,

$$0 \text{ is } \begin{cases} \text{exit-not-entrance for} & \mu \leqslant 0, \\ \text{regular for} & 0 < \mu < 1, \\ \text{entrance-not-exit for} & \mu \geqslant 1, \end{cases}$$

and

$$1 \text{ is } \begin{cases} \text{exit-not-entrance for} & \lambda_1 \leqslant \mu, \\ \text{regular for} & 0 < \lambda_1 < 1 + \mu, \\ \text{entrance-not-exit for} & \lambda_1 \geqslant 1 + \mu. \end{cases}$$

Thus assumptions on λ_1 and μ guarantee that both 0 and 1 are at least entrance, and may be regular or entrance-not-exit depending on the particular values of λ_1 and μ. Let us write $\mathcal{D}_F(\mathbf{J}_\mu)$ for the domain of the generator $\mathbf{J}_\mu$ of the Feller semigroup, and to specify it we recall that the so-called scale function s of $\mathbf{J}_\mu$ satisfies

$$s'(x) = x^{-\lambda_1}(1-x)^{-(\lambda_1 - \mu)}, \quad x \in (0,1).$$

Let f^+ and f^- denote the right and left derivatives of a function f with respect to s, i.e.

$$f^+(x) = \lim_{h\downarrow 0} \frac{f(x+h) - f(x)}{s(x+h) - s(x)}, \quad \text{and} \quad f^-(x) = \lim_{h\downarrow 0} \frac{f(x) - f(x-h)}{s(x) - s(x-h)}.$$

Then,

$$\mathcal{D}_F(\mathbf{J}_\mu) = \left\{ f \in C^2([0,1]);\ f^+(0^+) = f^-(1^-) = 0 \right\}, \tag{4.67}$$

and in particular, $\mathcal{P} \subset \mathcal{D}_F(\mathbf{J}_\mu)$, since for any $f \in \mathcal{P}$ we have

$$f^+(0^+) = \lim_{x\downarrow 0} x^{\lambda_1} f'(x) = 0 \quad \text{and} \quad f^-(1^-) = \lim_{x\uparrow 1} (1-x)^{\lambda_1 - \mu} f'(x) = 0.$$

From the boundary conditions in (4.67) we get that if any point in $\{0,1\}$ is regular then it is necessarily a reflecting boundary for the Jacobi process with $\lambda_1 > \mu > 0$.

4.5.2 Invariant measure and L^2-properties

The classical Jacobi semigroup $\mathbf{Q}^{(\mu)} = (\mathbf{Q}_t^{(\mu)})_{t\geqslant 0}$ has a unique invariant measure β_μ, which is the distribution of a beta random variable on $[0,1]$, i.e.

$$\beta_\mu(dx) = \beta_\mu(x)dx = \frac{\Gamma(\lambda_1)}{\Gamma(\mu)\Gamma(\lambda_1 - \mu)} x^{\mu-1}(1-x)^{\lambda_1-\mu-1}dx, \quad x \in (0,1),$$

and we recall that, for any $n \in \mathbb{N}$,

$$\beta_\mu[p_n] = \int_0^1 x^n \beta_\mu(dx) = \frac{(\mu)_n}{(\lambda_1)_n}. \tag{4.68}$$

Since β_μ is invariant for $\mathbf{Q}^{(\mu)}$ we get that $\mathbf{Q}^{(\mu)}$ extends to a contraction semigroup on $\mathrm{L}^2(\beta_\mu)$ and, moreover, the stochastic continuity of Y ensures that this extension is strongly continuous in $\mathrm{L}^2(\beta_\mu)$ and thus we obtain a Markov semigroup in $\mathrm{L}^2(\beta_\mu)$, which we still denote by $\mathbf{Q}^{(\mu)} = (\mathbf{Q}_t^{(\mu)})_{t\geqslant 0}$. The eigenfunctions of $\mathbf{J}_\mu$ are the Jacobi polynomials given, for any $n \in \mathbb{N}$ and $x \in [0,1]$, by

$$\mathcal{P}_n^{(\mu)}(x) = \sqrt{\mathrm{C}_n(\mu)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\lambda_1 - 1)_{n+k}}{(\lambda_1 - 1)_n} \frac{(\mu)_n}{(\mu)_k} \frac{x^k}{k!}, \tag{4.69}$$

where we have set

$$\mathrm{C}_n(\mu) = (2n + \boldsymbol{\lambda}_1 - 1)\frac{n!(\boldsymbol{\lambda}_1)_{n-1}}{(\mu)_n(\boldsymbol{\lambda}_1 - \mu)_n}. \tag{4.70}$$

In particular, when $\mu = 1$ then, we get, for any $n \in \mathbb{N}$,

$$\mathcal{P}_n^{(1)}(x) = \sqrt{\mathrm{C}_n(1)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\boldsymbol{\lambda}_1 - 1)_{n+k}}{(\boldsymbol{\lambda}_1 - 1)_n} \frac{n!}{k!} \frac{x^k}{k!}, \tag{4.71}$$

where we note that $\mathrm{C}_n(1) = \frac{\Gamma(\boldsymbol{\lambda}_1 - 1)}{\Gamma(\boldsymbol{\lambda}_1)}(2n + \boldsymbol{\lambda}_1 - 1)$. These polynomials are the orthogonal polynomials with respect to the measure β_μ and, by choice of $\mathrm{C}_n(\mu)$, satisfy the normalization condition

$$\int_0^1 \mathcal{P}_n^{(\mu)}(x)\mathcal{P}_m^{(\mu)}(x)\beta_\mu(dx) = \langle \mathcal{P}_n^{(\mu)}, \mathcal{P}_m^{(\mu)} \rangle_{\beta_\mu} = \delta_{nm},$$

and thus form an orthonormal basis for $\mathrm{L}^2(\beta_\mu)$. Furthermore we have, for $n \in \mathbb{N}$, the following formula

$$\begin{aligned}\mathcal{P}_n^{(\mu)}(x) &= \frac{2^n}{n!}\sqrt{\mathrm{C}_n(\mu)}\frac{1}{\beta_\mu(x)}\frac{d^n}{dx^n}\left(x^n(1-x)^n\beta_\mu(x)\right) \\ &= \frac{1}{\beta_\mu(x)}\beta_{\boldsymbol{\lambda}_1-\mu}[p_n]\sqrt{\mathrm{C}_n(\mu)}\mathbf{R}_n\beta_{\boldsymbol{\lambda}_1+n,\mu}(x)\end{aligned} \tag{4.72}$$

where we recall the definition in (4.8) of $\mathbf{R}_n$. All of these relations follow, by the change of variables $x \mapsto 2x - 1$ and simple algebra, from the corresponding relations for the polynomials $P_n^{(\mu-1,\boldsymbol{\lambda}_1-\mu-1)}$, defined in [72, Section 0.1], which are orthogonal for the weight $(1-x)^{\mu-1}(1+x)^{\boldsymbol{\lambda}_1-\mu-1}$, and are also called Jacobi polynomials in the literature. Indeed, the relationship between $\mathcal{P}_n^{(\mu)}$ and $P_n^{(\mu-1,\boldsymbol{\lambda}_1-\mu-1)}$ is given by

$$\mathcal{P}_n^{(\mu)}(x) = (-1)^n\sqrt{\frac{(2n + \boldsymbol{\lambda}_1 - 1)n!(\boldsymbol{\lambda}_1)_{n-1}}{(\mu)_n(\boldsymbol{\lambda}_1 - \mu)_n}}P_n^{(\mu-1,\boldsymbol{\lambda}_1-\mu-1)}(1-2x).$$

Next, the eigenvalue associated to the eigenfunction $\mathcal{P}_n^{(\mu)}(x)$ is, for $n \in \mathbb{N}$,

$$-\lambda_n = -n^2 - (\boldsymbol{\lambda}_1 - 1)n = -n(n-1) - \boldsymbol{\lambda}_1 n. \tag{4.73}$$

Observe that when $n = 1$ (4.73) reduces to $-\boldsymbol{\lambda}_1$ and that $\lambda_0 = 0$, so that $-\boldsymbol{\lambda}_1$ denotes the largest, non-zero eigenvalue of $\mathbf{J}_\mu$, which is also called the spectral gap. The semigroup

$\mathbf{Q}^{(\mu)}$ then admits the spectral decomposition given, for any $f \in \mathrm{L}^2(\beta_\mu)$ and $t \geqslant 0$, by

$$\mathbf{Q}_t^{(\mu)} = \sum_{n=0}^{\infty} e^{-\lambda_n t} \langle \cdot, \mathcal{P}_n^{(\mu)} \rangle_{\beta_\mu} \mathcal{P}_n^{(\mu)} = \sum_{n=0}^{\infty} e^{-\lambda_n t} \mathcal{P}_n^{(\mu)} \otimes \mathcal{P}_n^{(\mu)} \tag{4.74}$$

where the equality holds in the $\mathrm{L}^2(\beta_\mu)$-sense and in operator norm, and the sums converge in the operator norm. The domain of $\mathbf{J}_\mu$, the generator of the Markov semigroup in $\mathrm{L}^2(\beta_\mu)$, which we write as $\mathcal{D}_{\mathrm{L}^2}(\mathbf{J}_\mu)$, can then be identified as

$$\mathcal{D}_{\mathrm{L}^2}(\mathbf{J}_\mu) = \left\{ f \in \mathrm{L}^2(\beta_\mu);\ \sum_{n=0}^{\infty} n^4 \left| \langle f, \mathcal{P}_n^{(\mu)} \rangle_{\beta_\mu} \right|^2 < \infty \right\}.$$

4.5.3 Variance and entropy decay; hypercontractivity and ultracontractivity

As mentioned in the introduction, the fact that $\mathbf{Q}^{(\mu)}$ has nice spectral properties and satisfies certain functional inequalities gives quantitative rates of convergence to the equilibrium measure β_μ. For instance, from (4.74) one gets the following variance decay estimate, valid for any $f \in \mathrm{L}^2(\beta_\mu)$ and $t \geqslant 0$,

$$\mathrm{Var}_{\beta_\mu}(\mathbf{Q}_t^{(\mu)} f) \leqslant e^{-2\lambda_1 t}\, \mathrm{Var}_{\beta_\mu}(f),$$

which may also be deduced directly from the Poincaré inequality for $\mathbf{J}_\mu$, see [9, Chapter 4.2]. This convergence is optimal in the sense that the decay rate does not hold for any constant greater than $2\lambda_1$. Next, let us write $\lambda_{\mathrm{logS}}^{(\mu)}$ for the log-Sobolev constant of $\mathbf{J}_\mu$ defined as

$$\lambda_{\mathrm{logS}}^{(\mu)} = \inf_{f \in \mathcal{D}_{\mathrm{L}^2}(\mathbf{J}_\mu)} \left\{ \frac{-4\beta_\mu[f \mathbf{J}_\mu f]}{\mathrm{Ent}_{\beta_\mu}(f^2)};\ \mathrm{Ent}_{\beta_\mu}(f^2) \neq 0 \right\}. \tag{4.75}$$

Note that always $\lambda_{\mathrm{logS}}^{(\mu)} \leqslant 2\lambda_1$, and in the case of the symmetric Jacobi operator, i.e. $\mu = \frac{\lambda_1}{2} > 1$, we get

$$\lambda_{\mathrm{logS}}^{(\frac{\lambda_1}{2})} = 2\lambda_1, \tag{4.76}$$

while otherwise $\lambda^{(\mu)}_{\mathrm{logS}} < 2\lambda_1$, see e.g. [53], although the equality for the symmetric case goes back to [102]. As a consequence of (4.75) we have on the one hand the convergence in entropy, for any $t \geqslant 0$ and $f \in \mathrm{L}^1(\beta_\mu)$ such that $\mathrm{Ent}_{\beta_\mu}(f) < \infty$,

$$\mathrm{Ent}_{\beta_\mu}(\mathbf{Q}^{(\mu)}_t f) \leqslant e^{-\lambda^{(\mu)}_{\mathrm{logS}} t}\, \mathrm{Ent}_{\beta_\mu}(f), \tag{4.77}$$

and on the other hand from Gross [59] the hypercontractivity estimate, that is for all $t \geqslant 0$,

$$||\mathbf{Q}^{(\mu)}_t||_{2\to q} \leqslant 1 \text{ where } 2 \leqslant q \leqslant 1 + e^{\lambda^{(\mu)}_{\mathrm{logS}} t}. \tag{4.78}$$

From (4.76) we thus get that the symmetric Jacobi semigroup attains the optimal entropic decay and hypercontractivity rate. Further, when $\frac{\lambda_1}{2} = n \in \mathbb{N}$ there exists a homeomorphism between $\mathbf{J}_\mu$ and the radial part of the Laplace-Beltrami operator on the n-sphere, which leads to the curvature-dimension condition $CD(\lambda_1 - 1, \lambda_1)$, see [9] for the definition. Thus for any admissible function $\Phi : I \to \mathbb{R}$, we get

$$\mathrm{Ent}^{\Phi}_{\beta_{\lambda_1/2}}(\mathbf{Q}^{(\lambda_1/2)}_t f) \leqslant e^{-(\lambda_1-1)t}\, \mathrm{Ent}^{\Phi}_{\beta_{\lambda_1/2}}(f) \tag{4.79}$$

for any $t \geqslant 0$ and $f : [0,1] \to I$ such that $f, \Phi(f) \in \mathrm{L}^1(\beta_{\lambda_1/2})$.When $\lambda_1 - \mu > 1$ the operator $\mathbf{J}_\mu$ also satisfies a Sobolev inequality, see e.g. [8], and thus we get by [9, Theorem 6.3.1] that, for $0 < t \leqslant 1$,

$$||\mathbf{Q}^{(\mu)}_t||_{1\to\infty} \leqslant c_\mu t^{-\frac{\lambda_1-\mu}{\lambda_1-\mu-1}}$$

where c_μ is the Sobolev constant for $\mathbf{Q}^{(\mu)}$ of exponent $p = \frac{2(\lambda_1-\mu)}{(\lambda_1-\mu-1)}$, i.e.

$$c_\mu = \inf_{f\in\mathcal{D}_{\mathrm{L}^2}(\mathbf{J})} \left\{ \frac{||f||_2^2 - ||f||_p^2}{\beta_\mu[f\mathbf{J}_\mu f]};\ f, \mathbf{J}_\mu f \neq 0 \right\}.$$

The fact that $\mathbf{Q}^{(\mu)}$ is a contraction on $\mathrm{L}^1(\beta_\mu)$ together with the above ultracontractive bound yields the estimate $||\mathbf{Q}^{(\mu)}_t||_{1\to\infty} \leqslant c_\mu$, for any $t \geqslant 1$. Finally, we mention that $c_{\frac{\lambda_1}{2}} = \frac{4}{\lambda_1(\lambda_1-2)}$ and upper and lower bounds are known in the general case, see again [8].

CHAPTER 5

A SPECTRAL APPROACH FOR HYPOCOERCIVITY APPLIED TO SOME DEGENERATE HYPOELLIPTIC, AND NON-LOCAL OPERATORS

The aim of this chapter is to offer an original and comprehensive spectral theoretical approach to the study of convergence to equilibrium, and in particular of the hypocoercivity phenomenon, for contraction semigroups in Hilbert spaces. Our approach rests on a commutation relationship for linear operators known as intertwining, and we utilize this identity to transfer spectral information from a known, reference semigroup $\widetilde{P} = (e^{-t\widetilde{\mathbf{A}}})_{t\geqslant 0}$ to a target semigroup P which is the object of study. This allows us to obtain conditions under which P satisfies a hypocoercive estimate with exponential decay rate given by the spectral gap of $\widetilde{\mathbf{A}}$. Along the way we also develop a functional calculus involving the non-self-adjoint resolution of identity induced by the intertwining relations. We apply these results in a general Hilbert space setting to two cases: degenerate, hypoelliptic Ornstein-Uhlenbeck semigroups on $\mathbb{R}^d$, and non-local Jacobi semigroups on $[0,1]^d$, which have been introduced and studied for $d = 1$ in the previous chapter. In both cases we obtain hypocoercive estimates and are able to explicitly identify the hypocoercive constants.

5.1 Introduction

When a system has a steady-state, one is naturally interested in how quickly the dynamics convergence to this equilibrium. We think of such a system as being described by a contraction semigroup $P = (P_t)_{t\geqslant 0} = (e^{-t\mathbf{A}})_{t\geqslant 0}$ acting on a Hilbert space $\mathcal{H}$, and the equilibrium consisting of P-invariant vectors given by $\{f \in \mathcal{H};\ P_t f = f,\ \forall t \geqslant 0\}$ with corresponding projection P_∞. Of particular interest is an estimate of the form

$$\|P_t f - P_\infty f\|_{\mathcal{H}} \leqslant C e^{-\gamma t} \|f - P_\infty f\|_{\mathcal{H}},$$

where $C \geqslant 1$ and $\gamma > 0$ are constants, which is said to be hypocoercive. The literature on this topic is very rich and active, and several elegant techniques have been developed; we mention, without aiming to be exhaustive, generalizations of the Γ-calculus by Baudoin [12] and Monmarché [91], entropy functional techniques by Dolbeault et al. [43] and Arnold [6], the shrinkage/enlargenment approach by Gualdini et al. [62], Bouin et al. [29] and Mischler and Mouhot [89], generalized quadratic and Dirichlet form approaches by Ottobre et al. [93] and Grothaus and Stilgenbauer [60], respectively, a weak Poincaré inequality approach by Grothaus and Wang [61], a direct spectral approach for some toy models by Gadat and Miclo [55], and a spectral approach combined with techniques from non-harmonic analysis by Patie and Savov [99] and Patie et al. [100]. We also mention the fundamental memoir by Villani [123], noting that the techniques developed therein were inspired by the work of Talay [117]. Now, when $C = 1$ and P is self-adjoint in $\mathcal{H}$, the constant γ can be identified as the spectral gap of the operator $\mathbf{A}$ and thus there is a clear connection with the spectral theory of the underlying generator; however, outside of this situation a description of the constants C and γ is, first, difficult to obtain, and, second, is often not connected to the spectrum of $\mathbf{A}$. The aim of this work is to offer a new and spectral approach to the hypocoercivity phenomenon. Our approach rests on investigating the commutation relationship, known as intertwining, given, for any $t \geqslant 0$, by

$$P_t \Lambda = \Lambda \widetilde{P}_t,$$

where $\Lambda : \widetilde{\mathcal{H}} \to \mathcal{H}$ is a bounded, linear operator and $\widetilde{P} = (e^{-t\widetilde{\mathbf{A}}})_{t \geqslant 0}$ is a reference contraction semigroup on another Hilbert space $\widetilde{\mathcal{H}}$. Our main results in this context assume that $\widetilde{\mathbf{A}}$ is a normal operator with a spectral gap $\boldsymbol{\gamma}_1$, and we are able to show, under some conditions, that P satisfies a hypocoercive estimate with exponential rate $\boldsymbol{\gamma}_1$, the spectral gap of the reference operator $\widetilde{\mathbf{A}}$. As applications of these results we obtain hypocoercive estimates for degenerate, hypoelliptic Ornstein-Uhlenbeck semigroups on

$\mathbb{R}^d$, and for non-local Jacobi semigroups on $[0,1]^d$. In both cases we make explicit the two hypocoercive constants in terms of the initial data.

This chapter is organized as follows. In the remainder of this section we consider a motivating example and some preliminaries. In Section 5.2 we state our main results in a general Hilbert space setting and in Section 5.3 we present our application of these general results to degenerate, hypoelliptic Ornstein-Uhlenbeck semigroups and non-local Jacobi semigroups. Finally, in Section 5.4 we provide the proofs.

5.1.1 A motivating example

We present a motivating example from [99], which served as an inspiration for this work. Denote by $P = (e^{-t\mathbf{G}})_{t\geqslant 0}$ and $\widetilde{P} = (e^{-t\widetilde{\mathbf{G}}})_{t\geqslant 0}$ the generalized and classical Laguerre semigroup, which are contraction semigroups on the spaces $\mathrm{L}^2(\nu)$ and $\mathrm{L}^2(\varepsilon)$, respectively, where $\varepsilon(x) = e^{-x}$, $x > 0$, and ν is the unique invariant probability density on $(0,\infty)$ for P, see [99, Theorem 1.6(2)]. The operator $-\mathbf{G}$ acts on suitable functions f via

$$-\mathbf{G}f(x) = a^2 x f''(x) + (\mathsf{k} + a^2 - x)f'(x) + \int_0^\infty (f(e^{-y}x) - f(y) + yxf'(x))\,\Pi(x,dy),$$

where, in what follows, we consider $a^2 > 0$, $\mathsf{k} \geqslant 0$ and $\Pi(x,dy) = x^{-1}\Pi(dy)$ with Π a finite Radon measure on $(0,\infty)$ satisfying $\int_0^\infty (y^2 \wedge y)\Pi(dy) < \infty$. Note that $-\widetilde{\mathbf{G}}$ is given from the above formula by setting $a^2 = 1$, $\mathsf{k} = 0$, and $\Pi \equiv 0$, and that in [99] the authors treat a much wider class of parameters. For each generalized Laguerre semigroup P, there exists a bounded linear operator $\Lambda : \mathrm{L}^2(\varepsilon) \to \mathrm{L}^2(\nu)$ with dense range such that, for all $t \geqslant 0$ and on $\mathrm{L}^2(\varepsilon)$,

$$P_t\Lambda = \Lambda\widetilde{P}_t. \tag{5.1}$$

Recall that $\widetilde{P}$, as a self-adjoint and compact semigroup on $\mathrm{L}^2(\varepsilon)$, is diagonalized by an orthonormal basis $(\mathcal{L}_n)_{n\geqslant 0}$ of $\mathrm{L}^2(\varepsilon)$ formed of Laguerre polynomials, i.e. for $f \in \mathrm{L}^2(\varepsilon)$

and $t \geqslant 0$, we have $\widetilde{P}_t f = \sum_{n=0}^{\infty} e^{-nt} \langle f, \mathcal{L}_n \rangle_{\mathrm{L}^2(\varepsilon)} \mathcal{L}_n$. This fact, together with (5.1), gives, for $t \geqslant 0$ and on the dense subspace $\mathrm{Ran}(\Lambda)$,

$$P_t f = \sum_{n=0}^{\infty} e^{-nt} \langle \Lambda^{\dagger} f, \mathcal{L}_n \rangle_{\mathrm{L}^2(\nu)} \mathcal{P}_n, \tag{5.2}$$

where $\Lambda^{\dagger}$ denotes the pseudo-inverse of Λ, and $\mathcal{P}_n = \Lambda \mathcal{L}_n$ is a Bessel sequence, i.e. for $f \in \mathrm{L}^2(\nu)$ we have $\sum_{n=0}^{\infty} |\langle f, \mathcal{P}_n \rangle_{\mathrm{L}^2(\nu)}|^2 \leqslant \|f\|^2_{\mathrm{L}^2(\nu)}$ (Λ has operator norm 1). For this subclass of generalized Laguerre semigroups there exists $(\mathcal{V}_n)_{n \geqslant 0} \in \mathrm{L}^2(\nu)$ solving the equation $\Lambda^* \mathcal{V}_n = \mathcal{L}_n$, where Λ^* denotes the Hilbertian adjoint of Λ, see Section 8 of the aforementioned paper. It follows that $(\mathcal{P}_n)_{n \geqslant 0}$ and $(\mathcal{V}_n)_{n \geqslant 0}$ are biorthogonal, i.e. $\langle \mathcal{P}_n, \mathcal{V}_k \rangle_{\mathrm{L}^2(\nu)} = 1$ if $n = k$ and 0 otherwise, but as $(\mathcal{V}_n)_{n \geqslant 0}$ is not itself a Bessel sequence we cannot substitute $\Lambda^* \mathcal{V}_n$ for $\mathcal{L}_n$ in (5.2). Nevertheless, the multiplier sequence given by $m_n^2 = \frac{\Gamma(n+1)\Gamma(\mathsf{m}+1)}{\Gamma(n+\mathsf{m}+1)}$, where $\mathsf{m} = a^{-2}\left(\mathsf{k} + \int_0^{\infty} y \Pi(dy)\right) < \infty$, is such that $(m_n \mathcal{V}_n)_{n \geqslant 0}$ becomes a Bessel sequence, and consequently $\mathrm{L}^2(\nu) \ni f \mapsto \sum_{n=0}^{\infty} \langle f, m_n \mathcal{V}_n \rangle_{\mathrm{L}^2(\nu)} \mathcal{P}_n$ defines a bounded linear operator. Furthermore, there exists a constant $T_m > 0$ such that, for $t > T_m$, $\sup_{n \geqslant 1} (m_n e^{nt})^{-1} \leqslant \sqrt{\mathsf{m}+1}\, e^{-t}$ and for any $f \in \mathrm{L}^2(\nu)$,

$$P_t f = \sum_{n=0}^{\infty} e^{-nt} \langle f, \mathcal{V}_n \rangle_{\mathrm{L}^2(\nu)} \mathcal{P}_n.$$

A consequence of the above spectral expansion for P is the hypocoercive estimate

$$\|P_t f - \nu[f]\|_{\mathrm{L}^2(\nu)} \leqslant \sqrt{\mathsf{m}+1}\, e^{-t} \, \|f - \nu[f]\|_{\mathrm{L}^2(\nu)},$$

which holds for all $t > 0$ and any $f \in \mathrm{L}^2(\nu)$, noting that, as the only P-invariant functions are constant, $P_\infty f = \nu[f] = \int_0^{\infty} f(x)\nu(x)dx$. In this paper we provide a comprehensive framework that generalizes this approach, wherein the reference semigroup admits merely a spectral gap and does not necessarily have a discrete point spectrum, neither is necessarily compact.

5.1.2 Preliminaries

For a (real or complex) separable Hilbert space $\mathcal{H}$ we write $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ for the inner product and norm, respectively. Given two Hilbert spaces $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ we write $\mathcal{B}(\mathcal{H},\widetilde{\mathcal{H}})$ for the space of bounded linear operators from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$, with norm $\|\cdot\|_{\mathcal{H}\to\widetilde{\mathcal{H}}}$, writing simply $\mathcal{B}(\mathcal{H})$ for the unital Banach algebra of bounded linear operators on $\mathcal{H}$. Next, recall that a mapping $P : [0,\infty) \to \mathcal{B}(\mathcal{H})$ is said to be a *strongly continuous contraction semigroup*, or simply *contraction semigroup* for short, if

(1) $P_0 = I$, where I is the identity on $\mathcal{H}$,

(2) $P_{t+s} = P_t P_s$ for any $t, s \geqslant 0$,

(3) $\|P_t\|_{\mathcal{H}\to\mathcal{H}} \leqslant 1$ for all $t \geqslant 0$,

(4) and $\lim_{t\to 0} \|P_t f - f\|_{\mathcal{H}}$ for all $f \in \mathcal{H}$.

For a contraction semigroup P let

$$\mathcal{D}(-\mathbf{A}) = \left\{ f \in \mathcal{H};\ \lim_{t\to 0} \frac{P_t f - f}{t} \text{ exists} \right\}, \quad \text{and} \quad -\mathbf{A}f = \lim_{t\to 0} \frac{P_t f - f}{t}, \ \ \forall f \in \mathcal{H}.$$

The operator $(-\mathbf{A}, \mathcal{D}(-\mathbf{A}))$ is generator of the semigroup P, which justifies writing $P = (e^{-t\mathbf{A}})_{t\geqslant 0}$, and we adopt this convention in order to have, by the Hille-Yosida Theorem, that the spectrum of $\mathbf{A}$ is contained in $\{z \in \mathbb{C};\ \mathrm{Re}(z) \geqslant 0\}$. When $\mathbf{A}$ is a normal operator then $P = (e^{-t\mathbf{A}})_{t\geqslant 0}$ also holds in the sense of the Borel functional calculus for $\mathbf{A}$, see e.g. [106, 14]. We refer to the excellent monographs [40, 49] for further aspects on the theory of one-parameter semigroups. Next, recall that P_∞ denotes the orthogonal projection onto the closed subspace $\{f \in \mathcal{H};\ P_t f = f,\ \forall t \geqslant 0\}$ of P-invariant vectors.

Definition 5.1.1. We say that P *converges to equilibrium with rate* $r(t)$ if, for all $f \in \mathcal{H}$ and t large enough,

$$\|P_t f - P_\infty f\|_{\mathcal{H}} \leqslant r(t)\, \|f - P_\infty f\|_{\mathcal{H}}, \tag{5.3}$$

where $\lim_{t\to\infty} r(t) = 0$. In the case when, for some $C \geqslant 1$ and $\gamma > 0$,

$$\|P_t f - P_\infty f\|_{\mathcal{H}} \leqslant Ce^{-\gamma t}\, \|f - P_\infty f\|_{\mathcal{H}} \tag{5.4}$$

then we say that P satisfies a *hypocoercive* estimate.

Note that our definition of hypocoercivity for a contraction semigroup $P = (e^{-t\mathbf{A}})_{t\geqslant 0}$ agrees with the definition (on the semigroup level) given by Villani in [123, Chapter 3], when $\mathrm{Ran}(P_\infty) = \mathrm{Ker}(\mathbf{A})$. However, for our purposes, it is useful to maintain a definition of convergence to equilibrium, and of hypocoercivity, purely on the semigroup level. When $P = (e^{-t\mathbf{A}})_{t\geqslant 0}$ is a normal semigroup and satisfies a hypocoercive estimate with $C = 1$ and $\gamma > 0$ then γ is a gap in the spectrum of $\mathbf{A}$, in which case (5.4) is also known as the spectral gap inequality see [9, Section 4.2]. Indeed, for the converse assertion, assuming that P is normal and that $\mathbf{A}$ admits a spectral gap $\boldsymbol{\gamma}_1 > 0$, one gets that, for any $f \in \mathcal{H}$ with $P_\infty f = 0$ and $t \geqslant 0$,

$$||P_t f||^2_{\mathcal{H}} = \int_{\sigma(\mathbf{A})} e^{-2\,\mathrm{Re}(\gamma)t} d\langle \mathtt{E}_\gamma f, f\rangle_{\mathcal{H}} = \int_{\{\mathrm{Re}(\gamma)\geqslant \boldsymbol{\gamma}_1\}} e^{-2\,\mathrm{Re}(\gamma)t} d\langle \mathtt{E}_\gamma f, f\rangle_{\mathcal{H}} \leqslant e^{-2\boldsymbol{\gamma}_1 t}\, \|f\|^2_{\mathcal{H}}$$

where $\mathtt{E}$ is the unique resolution of identity associated to $\mathbf{A}$, see the proof of Lemma 5.4.2 below where we recall this classical argument in more detail. We mention that Miclo in [86] gives a sufficient condition for a self-adjoint operator to admit a spectral gap. However, in general, the constants C and γ in (5.4) may have little to do with the spectrum of $\mathbf{A}$, and one of the purposes of our work is to elucidate their role. Finally, we now state our definition of intertwining.

Definition 5.1.2. Two contraction semigroups P and $\widetilde{P}$ on $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, respectively, are said to *intertwine* if there exists $\Lambda \in \mathcal{B}(\widetilde{\mathcal{H}}, \mathcal{H})$ such that, for all $t \geqslant 0$ and on $\widetilde{\mathcal{H}}$,

$$P_t \Lambda = \Lambda \widetilde{P}_t.$$

The operator Λ is called the *intertwining operator* and we use the shorthand $P \overset{\Lambda}{\hookrightarrow} \widetilde{P}$.

5.2 Main Results

5.2.1 The similarity case

Throughout this section P and $\widetilde{P} = (e^{-t\widetilde{\mathbf{A}}})_{t \geqslant 0}$ shall denote contraction semigroups on Hilbert spaces $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, respectively. We think of P as the object of interest and $\widetilde{P}$ as a reference semigroup so that the intertwining $P \overset{\Lambda}{\hookrightarrow} \widetilde{P}$ allows us to transfer properties from the reference to the target. The relation $\hookrightarrow$ between contraction semigroups on Hilbert spaces is trivially reflexive and transitive but is, in general, not an equivalence relation due to the lack of symmetry. There are several ways that one can symmetrize this relation, one that involves further assumptions on the intertwining operator and another that is more structural. First, if $P \overset{\Lambda}{\hookrightarrow} \widetilde{P}$ and the intertwining operator Λ is a bijection then it is straightforward that $\hookrightarrow$ defines an equivalence relation among contraction semigroups on Hilbert spaces. We denote the equivalence class of $\widetilde{P}$ by $\mathcal{S}(\widetilde{P})$, which we call the *similarity orbit of* $\widetilde{P}$. Hence,

$$P \in \mathcal{S}(\widetilde{P}) \iff \exists \Lambda \in \mathcal{B}(\widetilde{\mathcal{H}}, \mathcal{H}) \text{ a bijection s.t. } P_t = \Lambda \widetilde{P}_t \Lambda^{-1}, \ \forall t \geqslant 0.$$

For a bijective operator $\Lambda \in \mathcal{B}(\widetilde{\mathcal{H}}, \mathcal{H})$ we denote its condition number by $\kappa(\Lambda) = \|\Lambda\|_{\widetilde{\mathcal{H}} \to \mathcal{H}} \left\|\Lambda^{-1}\right\|_{\mathcal{H} \to \widetilde{\mathcal{H}}} \geqslant 1$. Next, we write $\sigma(\widetilde{\mathbf{A}}) \subset \mathbb{C}$ for the spectrum of $\widetilde{\mathbf{A}}$ and $\mathsf{B}(\mathbb{C})$ for the Borel subsets of the complex plane. Recall that a densely defined operator $\widetilde{\mathbf{A}}$ on $\widetilde{\mathcal{H}}$ is normal if $\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^* = \widetilde{\mathbf{A}}^*\widetilde{\mathbf{A}}$, where $\widetilde{\mathbf{A}}^*$ denotes its adjoint in $\widetilde{\mathcal{H}}$. To every normal operator

$\widetilde{\mathbf{A}}$ on $\widetilde{\mathcal{H}}$ there exists a unique (self-adjoint) resolution of identity $\mathtt{E} : \mathsf{B}(\mathbb{C}) \to \mathcal{B}(\widetilde{\mathcal{H}})$ such that, by the Borel functional calculus for $\widetilde{\mathbf{A}}$,

$$\widetilde{P}_t = \int_{\sigma(\widetilde{\mathbf{A}})} e^{-\gamma t} d\mathtt{E}_\gamma,$$

where we recall that, for each $\Omega \in \mathsf{B}(\mathbb{C})$, $\mathtt{E}_\Omega$ is a self-adjoint projection and that, for $(f, g) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}}$, $\gamma \mapsto d\langle \mathtt{E}_\gamma f, g\rangle$ defines a complex valued measure on $\sigma(\widetilde{\mathbf{A}})$, see e.g. [106, 14]. Let $\mathcal{L}(\mathcal{H})$ be the space of linear (not necessarily continuous) operators on $\mathcal{H}$ and write $D \subset_d \mathcal{H}$ if D is a dense subset of $\mathcal{H}$. Then, we say that $\mathtt{F} : \mathsf{B}(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$ is a non-self-adjoint (nsa) resolution of identity if

(1) there exists $D \subset_d \mathcal{H}$ such that for each $\Omega \in \mathsf{B}(\mathbb{C})$, $\mathtt{F}_\Omega$ is a closed, linear operator with domain D,

(2) for each $\Omega \in \mathsf{B}(\mathbb{C})$, $\mathtt{F}_\Omega \neq \mathtt{F}^*_\Omega$,

(3) $\mathtt{F}_\emptyset = 0$, $\mathtt{F}_\mathbb{C} = I$, and, for any subsets $\Omega_1, \Omega_2 \in \mathcal{B}(\mathbb{C})$, $\mathtt{F}_{\Omega_1}\mathtt{F}_{\Omega_2} = \mathtt{F}_{\Omega_2}\mathtt{F}_{\Omega_1} = \mathtt{F}_{\Omega_1 \cap \Omega_2}$,

(4) for a countable collection of pairwise disjoint subsets $(\Omega_i)_{i=1}^\infty$ we have, in the strong operator topology,

$$\mathtt{F}_{\cup_{i=1}^\infty \Omega_i} = \sum_{i=1}^\infty \mathtt{F}_{\Omega_i}.$$

We shall always write $\mathtt{F}$ for a nsa resolution of identity, keeping the notation $\mathtt{E}$ exclusively for a self-adjoint resolution of identity, and this notion has been studied, with $\mathbb{C}$ replaced by $\mathbb{R}$, by Burnap and Zwiefel [31]. A semigroup P is a spectral operator in the sense of Dunford [45, 46] if there is a uniformly bounded nsa resolution of identity $\mathtt{F}$ commuting with P, and is of scalar type if, for all $t \geqslant 0$,

$$P_t = \int_{\sigma(\mathbf{A})} e^{-\gamma t} d\mathtt{F}_\gamma.$$

We refer to [47] for more on the theory of scalar and spectral operators. The following result, proved in Section 5.4.2, highlights a first connection between intertwining and convergence to equilibrium.

Proposition 5.2.1. *Suppose that $P \in \mathcal{S}(\widetilde{P})$. If $\widetilde{P}$ converges to equilibrium with rate $r(t)$ then P converges to equilibrium with rate $\kappa(\Lambda)r(t)$. In particular if $\widetilde{P}$ satisfies a hypocoercive estimate with constants $C \geqslant 1$ and $\lambda > 0$, as in* (5.4)*, then P satisfies a hypocoercive estimate with constants $C\kappa(\Lambda)$ and λ. Furthermore, if $\widetilde{P}$ is a normal semigroup then P is a scalar, spectral operator in the sense of Dunford.*

The idea of classifying and studying contraction semigroups via their similarity orbit has been used, in the context of transition semigroups of Markov chains, in [35, 36] where the authors study more than simply convergence to equilibrium. As a concrete example to which the above proposition applies, one can take $\widetilde{P}$ to be a normal, contraction semigroup on $\mathbb{R}^d$, $d \geqslant 1$, and let $\Lambda f(x) = f(Vx)$, where V is an invertible, d-dimensional matrix. Then the semigroup P defined via $P_t = \Lambda \widetilde{P}_t \Lambda^{-1}$, $t \geqslant 0$, is a scalar, spectral operator.

5.2.2 Beyond the similarity case

In this section we go beyond the case when P is a scalar, spectral operator in sense of Dunford, and when the intertwining operator is a bijection. To this end we need the following notion.

Definition 5.2.1 (Proper intertwining)**.** Let $P \overset{\Lambda}{\hookrightarrow} \widetilde{P} = (e^{-t\widetilde{\mathbf{A}}})_{t\geqslant 0}$, where $\widetilde{\mathbf{A}}$ is a normal operator. We say that Λ is a *proper* intertwining operator if $\mathrm{Ran}(\Lambda) \subset_d \mathcal{H}$ and if, for any $\Omega \in \mathsf{B}(\mathbb{C})$,

$$\mathtt{E}_\Omega\left(\overline{\mathrm{Ran}(\Lambda^*)}\right) \subseteq \overline{\mathrm{Ran}(\Lambda^*)},$$

where $\mathtt{E} : \mathsf{B}(\mathbb{C}) \to \mathcal{B}(\widetilde{\mathcal{H}})$ is the unique resolution of identity associated to $\widetilde{\mathbf{A}}$, and $\overline{\mathrm{Ran}(\Lambda^*)}$ denotes the closure of $\mathrm{Ran}(\Lambda^*)$. In such case we say that P intertwines with $\widetilde{P}$ *properly*, or $P \overset{\Lambda}{\hookrightarrow} \widetilde{P}$ properly, for short.

We note that the second property of the definition holds trivially, and independently of $\mathtt{E}$, when $\mathrm{Ker}(\Lambda) = \{0\}$. An operator $\Lambda \in \mathcal{B}(\widetilde{\mathcal{H}}, \mathcal{H})$ with $\mathrm{Ker}(\Lambda) = \{0\}$ and $\mathrm{Ran}(\Lambda) \subset_d \mathcal{H}$ is said to be a *quasi-affinity*, and two semigroups P and $\widetilde{P}$ are said to be *quasi-similar* if $P \overset{\Lambda}{\hookrightarrow} \widetilde{P} \overset{\widetilde{\Lambda}}{\hookrightarrow} P$, with Λ and $\widetilde{\Lambda}$ being quasi-affinities. The study of quasi-similarities of contraction operators on Hilbert spaces was initiated by Sz. Nagy and Foias, see [115]. This notion yields another symmetrization of the relation $\hookrightarrow$, and the results presented below may be viewed as extending the quasi-similar framework. We also mention that Antoine and Trapani [4] have studied quasi-similarity applied to pseudo-Hermitian quantum mechanics. Given $\Lambda \in \mathcal{B}(\widetilde{\mathcal{H}}, \mathcal{H})$ we write $\Lambda^\dagger$ for its pseudo-inverse, which is well-defined as Λ is a closed, densely-defined linear operator, see e.g. [13, Chapter 9]. As a stepping stone towards convergence to equilibrium we establish the following.

Proposition 5.2.2. *Suppose that* $P \overset{\Lambda}{\hookrightarrow} \widetilde{P} = (e^{-t\widetilde{\mathbf{A}}})_{t\geqslant 0}$ *properly and that* $\widetilde{\mathbf{A}}$ *is a normal operator with unique resolution of identity* $\mathtt{E} : \mathsf{B}(\mathbb{C}) \to \mathcal{B}(\widetilde{\mathcal{H}})$*. Then the intertwining induces a nsa resolution of identity* $\mathtt{F} : \mathsf{B}(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$ *with domain* $\mathrm{Ran}(\Lambda)$ *via*

$$\mathtt{F}_\Omega = \Lambda \mathtt{E}_\Omega \Lambda^\dagger.$$

Furthermore, for each $(f, g) \in \mathrm{Ran}(\Lambda) \times \mathcal{H}$, $\gamma \mapsto \langle \mathtt{F}_\gamma f, g\rangle$ *defines a complex-valued measure, and for all* $t \geqslant 0$,

$$P_t = \int_{\sigma(\widetilde{\mathbf{A}})} e^{-\gamma t} d\mathtt{F}_\gamma$$

on $\mathrm{Ran}(\Lambda)$*, in the sense that* $\langle P_t f, g\rangle_{\mathcal{H}} = \int_{\sigma(\widetilde{\mathbf{A}})} e^{-\gamma t} d\langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}}$.

This result is proved in Section 5.4.3. Note that the intertwining $P \overset{\Lambda}{\hookrightarrow} \widetilde{P}$ allows P_t to be expressed as a spectral integral, with respect to the nsa resolution of identity induced by Λ, over the spectrum of $\widetilde{\mathbf{A}}$, i.e.

$$e^{-t\mathbf{A}} = \int_{\sigma(\widetilde{\mathbf{A}})} e^{-\gamma t} d\mathtt{F}_\gamma, \quad \text{on} \quad \mathrm{Ran}(\Lambda)\,.$$

As we show in Lemma 5.4.3, the function $\gamma \mapsto e^{-\gamma t}$ may be replaced more generally by any bounded measurable function on $\sigma(\widetilde{\mathbf{A}})$ and thus we get a Borel functional calculus for $\mathbf{A}$, even though $\mathbf{A}$ itself is not necessarily normal. Let us mention that such a spectral integral with respect to an nsa resolution of identity has also been shown in Patie et al. [100] in the context of Krein's spectral theory of strings, see Corollary 2.6 therein.

Next, we say that a normal operator $\widetilde{\mathbf{A}}$ on $\widetilde{\mathcal{H}}$ with $\sigma(\widetilde{\mathbf{A}}) \subseteq \{z \in \mathbb{C};\ \mathrm{Re}(z) \geqslant 0\}$ has a *spectral gap*, denoted by $\boldsymbol{\gamma}_1$, if

$$\boldsymbol{\gamma}_1 = \inf\left\{\mathrm{Re}(\gamma); \mathrm{Re}(\gamma) > 0,\ \gamma \in \sigma(\widetilde{\mathbf{A}})\right\} = \inf\left\{\frac{\mathrm{Re}\langle \widetilde{\mathbf{A}} f, f\rangle_{\widetilde{\mathcal{H}}}}{\|f\|^2_{\widetilde{\mathcal{H}}}}; 0 \neq f \in \mathcal{D}(\widetilde{\mathbf{A}})\right\} > 0.$$

We write $\mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$ for the space of complex-valued, bounded Borelian functions on $\sigma(\widetilde{\mathbf{A}})$ equipped with the uniform norm $\|\cdot\|_\infty$ and, for any complex valued measure μ we denote its total variation by $|\mu|$. The following is one of the main results of this work.

Theorem 5.2.1. *Let $P \overset{\Lambda}{\hookrightarrow} \widetilde{P} = (e^{-t\widetilde{\mathbf{A}}})_{t\geqslant 0}$ properly, and suppose that $\widetilde{\mathbf{A}}$ is normal with spectral gap $\boldsymbol{\gamma}_1$. Assume that there exists a function $m \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$ such that*

(a) for $(f, g) \in \mathrm{Ran}(\Lambda) \times \mathcal{H}$,

$$\int_{\sigma(\widetilde{\mathbf{A}})} |m(\gamma)| d|\langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}}| \leqslant \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}},$$

where $\mathtt{F}$ is the nsa resolution of identity induced by the intertwining,

(b) and for $t > T_m > 0$, with T_m a constant,

$$\gamma \mapsto \frac{e^{-\gamma t}}{m(\gamma)} \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}})).$$

Then, we have the following.

(1) For $t > T_m$, $\int_{\sigma(\widetilde{\mathbf{A}})} e^{-\gamma t} d\mathtt{F}_\gamma$ extends to a bounded, linear operator on $\mathcal{H}$.

(2) *Let* $M_t^{(\gamma_1)} \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$ *be given by* $M_t^{(\gamma_1)}(\gamma) = \frac{e^{-\gamma t}}{m(\gamma)}\mathbf{1}_{\{\mathrm{Re}(\gamma)\geqslant\boldsymbol{\gamma}_1\}}$. *Then, for all* $f \in \mathcal{H}$ *and* $t > T_m$,

$$\|P_t f - P_\infty f\|_{\mathcal{H}} \leqslant \|M_t^{(\gamma_1)}\|_\infty \|f - P_\infty f\|_{\mathcal{H}} .$$

If $M_t^{(\gamma_1)}$ *attains its supremum at* $\boldsymbol{\gamma}_1$ *then, for all* $f \in \mathcal{H}$ *and* $t > T_m$,

$$\|P_t f - P_\infty f\|_{\mathcal{H}} \leqslant \frac{1}{|m(\boldsymbol{\gamma}_1)|} e^{-\boldsymbol{\gamma}_1 t} \|f - P_\infty f\|_{\mathcal{H}} .$$

This theorem is proved in Section 5.4.4 and in Theorem 5.2.2 we provide a sufficient condition for Item (a) of the theorem to be fulfilled. Note that, except in the case when $\Lambda^{-1} \in \mathcal{B}(\mathcal{H}, \widetilde{\mathcal{H}})$, the function m must be decreasing as $|\gamma| \to \infty$. Indeed, supposing that $|m(\gamma)| \geqslant c > 0$ for all $\gamma \in \sigma(\widetilde{\mathbf{A}})$, the condition in Theorem 5.2.1(a) yields

$$c \int_{\sigma(\widetilde{\mathbf{A}})} d|\langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}}| \leqslant \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}} .$$

However, as we show in Lemma 5.4.3, the measure $\gamma \mapsto \langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}}$ has total variation no greater than $\left\|\Lambda^\dagger f\right\|_{\mathcal{H}} \|\Lambda\|_{\widetilde{\mathcal{H}}\to\mathcal{H}} \|g\|_{\mathcal{H}}$ and thus, for a finite constant K, we deduce that $\left\|\Lambda^\dagger f\right\|_{\mathcal{H}} \leqslant K \|f\|_{\mathcal{H}}$. Similarly, the condition in Item (b) cannot hold for $t = 0$ except in the case when Λ admits a bounded inverse. In this sense the function m indicates the departure of $\mathtt{F}$ from being a uniformly bounded nsa resolution of identity, and the rate of convergence in Theorem 5.2.1(2) is given by the norm of an operator that measures this departure.

The second part of Theorem 5.2.1(2) provides a simple condition under which P satisfies a hypocoercive estimate with a rate equal to the spectral gap of the normal operator $\widetilde{\mathbf{A}}$ associated to the reference semigroup $\widetilde{P}$. As mentioned earlier, this is not surprising given that intertwining transfers spectral information from the reference to the target semigroup. In Section 5.3 below we will give examples of functions m satisfying the conditions of Theorem 5.2.1 and for which $M_t^{(\gamma_1)}$ attains its supremum at the spectral gap $\boldsymbol{\gamma}_1$. Finally, the fact that the small-time behavior for the rate of convergence may be different from exponential has been observed in the context of some toy models by Gadat and Miclo [55], for degenerate, hypoelliptic Ornstein-Uhlenbeck semigroups by Monmarché [91], see also Theorem 5.3.1 below, and in the context of some non-reversible Markov chains by Patie and Choi [35, 36]. This suggest that studying hypocoercivity only for $t > T_m$ may be natural.

For the next result, we recall that a normal operator $\widetilde{\mathbf{A}}$ is said to have simple spectrum if there exists a vector $v \in \widetilde{\mathcal{H}}$ such that, for all non-negative integers k, l, $v \in \mathcal{D}(\widetilde{\mathbf{A}}^{*^k}\widetilde{\mathbf{A}}^l)$ and $\widetilde{\mathcal{H}}$ is the closed linear span of $\{\widetilde{\mathbf{A}}^{*^k}\widetilde{\mathbf{A}}^l;\ k, l \geqslant 0\}$.

Theorem 5.2.2. *Let $P \overset{\Lambda}{\hookrightarrow} \widetilde{P} = (e^{-t\widetilde{\mathbf{A}}})_{t\geqslant 0} \overset{\widetilde{\Lambda}}{\hookrightarrow} P$ properly, and suppose that $\widetilde{\mathbf{A}}$ is normal with spectral gap $\boldsymbol{\gamma}_1$. If there exists $m \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$ such that*

$$m(\widetilde{\mathbf{A}}) = \widetilde{\Lambda}\Lambda,$$

then the condition in Theorem 5.2.1(a) is fulfilled with the normalized function $m/||\Lambda||_{\widetilde{\mathcal{H}}\to\mathcal{H}}||\widetilde{\Lambda}||_{\mathcal{H}\to\widetilde{\mathcal{H}}}$. In particular, if $\widetilde{\mathbf{A}}$ has simple spectrum then there exists $m \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$ such that $m(\widetilde{\mathbf{A}}) = \widetilde{\Lambda}\Lambda$. If such a function m also satisfies the condition in Item (b) then the conclusions of Theorem 5.2.1 hold.

This theorem is proved in Section 5.4.5. The observation that the composition of intertwining operators can equal a function of the generator has been made before, and has been used recently in [87, 88] as well as in the previous chapter. In particular, in [88] the authors introduce and study the notion of *completely monotone intertwining relationships*, which corresponds to m in Theorem 5.2.2 being a completely monotone function, and obtain, among other things, entropic convergence and hypercontractivity in this manner.

We have the following corollary of Theorem 5.2.2, which follows at once from the observation that, if $P \overset{\Lambda}{\hookrightarrow} \widetilde{P} \overset{\widetilde{\Lambda}}{\hookrightarrow} P$ with Λ and $\widetilde{\Lambda}$ quasi-affinities, then $P^* \overset{\widetilde{\Lambda}^*}{\hookrightarrow} \widetilde{P}^* \overset{\Lambda^*}{\hookrightarrow} P^*$

with $\widetilde{\Lambda}^*$ and Λ^* being quasi-affinities.

Corollary 5.2.1. *Under the assumptions of Theorem 5.2.2, suppose that the intertwining operators Λ and $\widetilde{\Lambda}$ are quasi-affinities, and that the function $m \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$ satisfies the condition in Theorem 5.2.1(b). Then the conclusions of Theorem 5.2.1 hold upon replacing P by its adjoint semigroup $P^* = (P_t^*)_{t \geqslant 0}$, and by replacing $\mathtt{F}$ by $\widetilde{\mathtt{F}}$, the nsa resolution of identity induced by the intertwining $P^* \overset{\widetilde{\Lambda}^*}{\hookrightarrow} \widetilde{P}^*$.*

This result gives that, under a mild strengthening of the hypothesis in Theorem 5.2.2, the adjoint semigroup may be also expressed as an integral over the spectrum of $\widetilde{\mathbf{A}}^*$ with respect to another nsa resolution of identity.

5.3 Applications

5.3.1 Hypoelliptic Ornstein-Uhlenbeck semigroups

In this section we apply the results from the previous section to hypoelliptic Ornstein-Uhlenbeck semigroups on $\mathbb{R}^d$, $d \geqslant 1$. Without aiming to be exhaustive, we mention that [82] and the series of papers [83, 84, 85] have been important works on the Ornstein-Uhlenbeck semigroup, as well as the Ornstein-Uhlenbeck operator, and the main findings are collected nicely in [81, Chapter 9]; the recent survey [24], which presents a thorough account on the state-of-the-art for Ornstein-Uhlenbeck semigroups, shows that these objects continue to be active areas of research.

Let B be a matrix such that $\sigma(B) \subseteq \{z \in \mathbb{C};\ \mathrm{Re}(z) > 0\}$ and suppose Q is a positive semi-definite matrix such that, with

$$Q_t = \int_0^t e^{-tB} Q e^{-tB^*} ds,$$

we have $\det Q_t > 0$, for all $t > 0$. In particular, this holds when Q is invertible, which we call the non-degenerate case, although it can happen that $\det Q_t > 0$, for all $t > 0$, with $\det Q = 0$, which we call the degenerate case. Under these assumptions on (Q, B), the hypoelliptic Ornstein-Uhlenbeck semigroup P associated to (Q, B) admits the representation

$$P_t f(x) = \frac{1}{(2\pi)^{d/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^d} f(e^{-tB}x - y)e^{-\langle Q_t^{-1}y,y\rangle/2}dy,$$

where f is a bounded measurable function and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^d$, and also extends to a contraction semigroup on the weighted Hilbert space $\mathrm{L}^2(\rho_\infty)$, which plays the role of $\mathcal{H}$ from the previous section, where

$$\rho_\infty(x) = \frac{1}{(2\pi)^{d/2}(\det Q_\infty)^{1/2}} e^{-\langle Q_\infty^{-1}x,x\rangle/2},$$

with $Q_\infty = \int_0^\infty e^{-tB}Qe^{-tB^*}ds$, and

$$\mathrm{L}^2(\rho_\infty) = \left\{ f : \mathbb{R}^d \to \mathbb{C} \text{ measurable; } \|f\|^2_{\mathrm{L}^2(\rho_\infty)} = \int_{\mathbb{R}^d} |f(x)|^2 \rho_\infty(x)dx < \infty \right\}.$$

In fact ρ_∞ is the unique invariant measure of P in the sense that, for any $f \in \mathrm{L}^2(\rho_\infty)$ and $t \geqslant 0$, $\int_{\mathbb{R}^d} P_t f(x)\rho_\infty(x)dx = \int_{\mathbb{R}^d} f(x)\rho_\infty(x)dx$, and, since the only P-invariant functions are constants, we get that the projection P_∞ is given by $P_\infty f(x) = \rho_\infty[f] = \int_{\mathbb{R}^d} f(x)\rho_\infty(x)dx$. The generator of the Ornstein-Uhlenbeck semigroup $P = (e^{-t\mathbf{A}})_{t\geqslant 0}$ acts on suitable functions $f : \mathbb{R}^d \to \mathbb{R}$ via

$$-\mathbf{A}f(x) = \frac{1}{2}\sum_{i,j=1}^{d} q_{ij}\partial_i\partial_j f(x) - \sum_{i,j=1}^{d} b_{ij}x_j\partial_j f(x) = \frac{1}{2}\operatorname{Tr}(Q\nabla^2)f(x) - \langle Bx, \nabla\rangle f(x),$$

and the condition $\det Q_t > 0$, for all $t > 0$, is equivalent to the hypoellipticity of $\frac{\partial}{\partial t} + \mathbf{A}$ in the $d + 1$ variables $(t, x_1, \dots, x_d)$, hence the terminology. In [84, Theorem 3.4] it was shown that the spectrum of $\mathbf{A}$ in $\mathrm{L}^2(\rho_\infty)$ is entirely determined by the matrix B, specifically that, writing $\mathbb{N} = \{0, 1, 2, \dots\}$, $\sigma(\mathbf{A}) = \left\{\sum_{i=1}^r k_i\lambda_i;\ k_i \in \mathbb{N}\right\}$, where $\lambda_1, \dots, \lambda_r$ are the distinct eigenvalues of B. Hence, in particular, the spectral gap $\boldsymbol{\gamma}_1$ of $\mathbf{A}$ is given by the smallest eigenvalue of $\frac{1}{2}(B + B^*)$. Recall that $\kappa(V)$ denotes the condition number of any invertible matrix V, and note that if V is positive-definite then $\kappa(V) = v_{\max}/v_{\min}$, where $v_{\max}, v_{\min} > 0$ are the largest and smallest eigenvalues of V, respectively. The following is the main result of this section, and one of the main results of this paper.

Theorem 5.3.1. *Let P be an Ornstein-Uhlenbeck semigroup associated to (Q, B) such that $\mathrm{Ker}(Q)$ does not contain any invariant subspace of B^*. Suppose that B is diagonalizable with similarity matrix V, and that $\sigma(B) \subseteq (0, \infty)$. Then, there exists a non-degenerate, hypoelliptic Ornstein-Uhlenbeck semigroup $\widetilde{P}$, self-adjoint on $\mathrm{L}^2(\widetilde{\rho}_\infty)$, such that $P \overset{\Lambda}{\hookrightarrow} \widetilde{P} \overset{\widetilde{\Lambda}}{\hookrightarrow} P$, where Λ and $\widetilde{\Lambda}$ are quasi-affinities. Furthermore, setting $\boldsymbol{t} = \frac{1}{\gamma_1} \log \kappa(VQ_\infty V^*)$, we have*

$$\widetilde{\Lambda}\Lambda = \widetilde{P}_{\boldsymbol{t}}.$$

Consequently, for any $f \in \mathrm{L}^2(\rho_\infty)$ and $t \geqslant 0$,

$$\|P_t f - \rho_\infty[f]\|_{\mathrm{L}^2(\rho_\infty)} \leqslant \kappa(VQ_\infty V^*) e^{-\gamma_1 t} \|f - \rho_\infty[f]\|_{\mathrm{L}^2(\rho_\infty)}.$$

This result is proved in Section 5.4.6 and we proceed by offering some remarks. First, we emphasize that our result covers the case when Q is degenerate, which has attracted a lot of research interest and seen several elegant techniques developed, see e.g. [55, 76, 93, 60, 5]. The difficulty in dealing with the degenerate case stems, in part, from the fact that a degenerate Ornstein-Uhlenbeck semigroup can never be normal on $\mathrm{L}^2(\rho_\infty)$, cf. [93, Lemma 3.3]. We mention that Arnold and Erb [6] have already shown hypocoercivity, under our assumptions, with exponential rate given by the spectral gap $\boldsymbol{\gamma}_1$ and that Arnold et al. [5] and Monmarché [91] have proved hypocoercivity with exponential rate $\boldsymbol{\gamma}_1$ without assuming that B is diagonalizable. However, in contrast to these existing results, we are able to explicitly identify the constant in front of the exponential, i.e. $\kappa(VQ_\infty V^*)$, in terms of the initial data Q and B. In particular, if B is symmetric then V is unitary and $\kappa(VQ_\infty V^*) = \kappa(Q_\infty)$. Similar results have been obtained by Achleitner et al. [1], and by Patie and Savov [99] in the context of generalized Laguerre semigroups, as well as in Chapter 4 in the context of non-local Jacobi semigroups. Let us mention that the restriction $\sigma(B) \subseteq (0,\infty)$ was made only to simplify the computations involving the composition $\widetilde{\Lambda}\Lambda$, and we believe that with some additional effort Theorem 5.3.1 holds for all diagonalizable matrices B with $\sigma(B) \subseteq \{z \in \mathbb{C};\ \mathrm{Re}(z) > 0\}$. Finally, we note that the intertwinings in Theorem 5.3.1 yield a completely monotone intertwining relationship, in the sense of [88], between P and $\widetilde{P}$, and this stronger type of intertwining will be exploited to investigate, among other things, the hypercontractivity of Ornstein-Uhlenbeck semigroups.

5.3.2 Non-local Jacobi semigroups

In this section we consider the non-local Jacobi semigroup on $[0,1]^d$, whose generator is a non-local perturbation of the classical (local) Jacobi operator on $[0,1]^d$. Given $\boldsymbol{\gamma}_1 > 0$ and $\mu \in \mathbb{R}^d$ such that $\boldsymbol{\gamma}_1 > \mu_i > 0$, for all $i = 1,\dots,d$, we recall that the classical Jacobi operator $-\widetilde{\mathbf{A}}_\mu$ acts on smooth functions $f : [0,1]^d \to \mathbb{R}$ such that $f(x) = f_1(x_1)\cdots f_d(x_d)$ via

$$\widetilde{\mathbf{A}}_\mu f(x) = -\sum_{i=1}^{d} x_i(1-x_i)\partial_i^2 f(x) + \sum_{i=1}^{d}(\boldsymbol{\gamma}_1 x_i - \mu_i)\partial_i f(x).$$

It generates the Jacobi semigroup $\widetilde{P}^{(\mu)} = (e^{-t\widetilde{\mathbf{A}}_\mu})_{t\geqslant 0}$, which is a self-adjoint contraction semigroup on the weighted Hilbert space $\mathrm{L}^2(\beta_\mu)$, where β_μ is the unique invariant measure of $\widetilde{P}^{(\mu)}$ consisting of the product of beta densities on $[0,1]$. Moreover, $\boldsymbol{\gamma}_1$ is the spectral gap of $\widetilde{\mathbf{A}}_\mu$ – hence the notation – and the spectral gap uniquely determines the spectrum of $\widetilde{\mathbf{A}}_\mu$ in $\mathrm{L}^2(\beta_\mu)$, which is given by $\sigma(\widetilde{\mathbf{A}}_\mu) = \{n(n-1)+\boldsymbol{\gamma}_1 n;\ n \in \mathbb{N}\}$. We refer to [9, Section 2.7.4], as well as Section 4.5, for a review of these objects.

The non-local Jacobi semigroup $P = (e^{-t\mathbf{A}})_{t\geqslant 0}$ on $[0,1]^d$ is the tensor product of the one-dimensional non-local Jacobi semigroups that have been recently introduced and studied in Chapter 4. The generator $-\mathbf{A}$ acts on suitable product functions $f : [0,1]^d \to \mathbb{R}$, $f(x) = f_1(x_1)\cdots f_d(x_d)$, via

$$\mathbf{A}f(x) = \widetilde{\mathbf{A}}_\mu f(x) + \sum_{i=1}^d \int_0^1 f_i(y) h_i(x_i y^{-1}) y^{-1} dy,$$

where we assume that, for each $i = 1, \ldots, d$, $h_i : (1,\infty) \to [0,\infty)$ is such that

$$\begin{aligned} &-(e^y h_i(e^y))' \text{ is a finite non-negative Radon measure on } (0,\infty) \\ &\text{with } \boldsymbol{\gamma}_1 > \mu_i > 1 + \int_1^\infty h_i(y)dy. \end{aligned} \tag{5.5}$$

Note that the condition $\int_1^\infty h_i(y)dy < \infty$ is implied by the previous requirement. This operator generates a contraction semigroup on the weighted Hilbert space $\mathrm{L}^2(\beta) = \mathrm{L}^2(\beta_1) \otimes \cdots \otimes \mathrm{L}^2(\beta_d)$, where each β_i is the probability density on $[0,1]$ uniquely determined, for $n \geqslant 1$, by its moments

$$\int_0^1 x^n \beta_i(x)dx = \prod_{k=1}^n \frac{\phi_i(k)}{k + \boldsymbol{\gamma}_1 - 1}, \quad \text{where} \quad \phi_i(u) = (\mu - 1) + u - \int_1^\infty y^{-u} h_i(y)dy,$$

and $\phi_i : [0,\infty) \to [0,\infty)$ is a Bernstein function, i.e. $\phi \in \mathrm{C}^\infty(\mathbb{R}_+)$, the space of infinitely differentiable functions on $\mathbb{R}^d$, with ϕ' a completely monotone function, see [109]. The invariant measure $\beta(x)dx$ of P is unique, and again we have that $P_\infty f = \beta[f] = \int_{[0,1]^d} f(x)\beta(x)dx$, however, except in the trivial case $h \equiv 0$, the semigroup P is non-self-adjoint on $\mathrm{L}^2(\beta)$. We refer to Section 4.2.1 of Chapter 4 for detailed information, specifically Theorem 4.2.2 regarding the last two claims. In the following we use the notation $(a)_x = \Gamma(x+a)/\Gamma(a)$ for $a > 0$ and $x \geqslant 0$.

Theorem 5.3.2. *Let P be a non-local Jacobi semigroup with parameters $\boldsymbol{\gamma}_1$, μ and $h_1, \ldots, h_d$ satisfying the conditions in* (5.5). *Then, for each $\mathfrak{m} \in (\max \mu_i, \boldsymbol{\gamma}_1)$ there exists a local Jacobi semigroup $\widetilde{P}^{(\mathfrak{m})} = (e^{-\widetilde{\mathbf{A}}_\mathfrak{m}})_{t\geqslant 0}$ on $[0,1]^d$, with spectral gap $\boldsymbol{\gamma}_1$ and drift vector $(\mathfrak{m}, \ldots, \mathfrak{m})$, such that $P \overset{\Lambda}{\hookrightarrow} \widetilde{P}^{(\mathfrak{m})} \overset{\widetilde{\Lambda}}{\hookrightarrow} P$, where Λ and $\widetilde{\Lambda}$ are quasi-affinities* *satisfying*

$$\widetilde{\Lambda}\Lambda = F_{\mathfrak{m}}(\widetilde{\mathbf{A}}_{\mathfrak{m}}), \quad \textit{where} \quad \gamma \mapsto F_{\mathfrak{m}}(\gamma) = \frac{(1)_\gamma}{(\mathfrak{m})_\gamma} \frac{(\boldsymbol{\gamma}_1 - \mathfrak{m})_\gamma}{(\boldsymbol{\gamma}_1 - 1)_\gamma} \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}_{\mathfrak{m}})).$$

Consequently, for any $\mathfrak{m} \in (\max \mu_i, \boldsymbol{\gamma}_1)$*,* $f \in \mathrm{L}^2(\beta)$ *and* $t \geqslant 0$*,*

$$\|P_t f - \beta[f]\|_{\mathrm{L}^2(\beta)} \leqslant \mathfrak{m} \frac{(\boldsymbol{\gamma}_1 - 1)}{(\boldsymbol{\gamma}_1 - \mathfrak{m})} e^{-\boldsymbol{\gamma}_1 t} \|f - \beta[f]\|_{\mathrm{L}^2(\beta)}.$$

This result is proved in Section 5.4.7. Note that the non-local components of P may be different in each coordinate. Any homeomorphism $\mathrm{H} : [0,1]^d \to E \subset \mathbb{R}^d$ induces a non-local Jacobi semigroup P^{H} with state space E and invariant measure β^{H}, the image of β under H, and also a unitary operator $\Lambda_{\mathrm{H}} \in \mathcal{B}(\mathrm{L}^2(\beta), \mathrm{L}^2(\beta^{\mathrm{H}}))$ such that $P^{\mathrm{H}} \overset{\Lambda_{\mathrm{H}}}{\hookrightarrow} P$. Consequently, by a combination of Theorem 5.3.2 and Proposition 5.2.1 we deduce that P^{H} satisfies the same kind of hypocoercive estimate as P. In this way one may construct non-local dynamics on compact state spaces which are guaranteed to be hypocoercive. As a concrete example one may take $\mathrm{H} : [0,1]^d \to S^d$ to be the homeomorphism from $[0,1]^d$ to $S^d = \{x \in \mathbb{R}^d;\ x_1 + \cdots + x_d \leqslant 1,\ x_i \geqslant 0,\ i = 1, \ldots, d\}$, the standard simplex in d-dimensions.

5.4 Proofs

5.4.1 Preliminaries

Before giving the proofs of the main theorems we state and prove some preliminary results. Recall that an idempotent Π is any operator satisfying $\Pi^2 = \Pi$. The following simple result concerns the robustness of the convergence to equilibrium condition in (5.3) when considering bounded idempotents different from P_∞.

Lemma 5.4.1. *Let P be a contraction semigroup on a Hilbert space $\mathcal{H}$. If there exists an idempotent $\Pi \in \mathcal{B}(\mathcal{H})$ such that, for all $f \in \mathcal{H}$ and t large enough,*

$$\|P_t f - \Pi f\|_{\mathcal{H}} \leqslant r(t)\,\|f - \Pi f\|_{\mathcal{H}},$$

with $\lim_{t\to\infty} r(t) = 0$, then $\Pi = P_\infty$, and hence P converges to equilibrium with rate $r(t)$.

Proof. Suppose that $f \in \mathrm{Ran}(P_\infty)$. Then, by the convergence assumption and as f is P-invariant, it follows that for t large enough

$$\|f - \Pi f\|_{\mathcal{H}} = \|P_t f - \Pi f\|_{\mathcal{H}} \leqslant r(t)\,\|f - \Pi f\|_{\mathcal{H}},$$

and hence $f = \Pi f$, i.e $f \in \mathrm{Ran}(\Pi)$. On the other hand if $f \in \mathrm{Ran}(\Pi)$ then, for any $s \geqslant 0$ fixed and $t \geqslant 0$,

$$\begin{aligned}\|P_s\Pi f - \Pi f\|_{\mathcal{H}} &\leqslant \|P_{s+t} f - \Pi f\|_{\mathcal{H}} + \|P_{s+t} f - P_s \Pi f\|_{\mathcal{H}} \\ &\leqslant \|P_{s+t} f - \Pi f\|_{\mathcal{H}} + \|P_t f - \Pi f\|_{\mathcal{H}},\end{aligned}$$

where the second inequality uses that $\|P_s\|_{\mathcal{H}\to\mathcal{H}} \leqslant 1$. Taking the limit as $t \to \infty$ yields, by the convergence assumption, that $\Pi f = P_s \Pi f$, and thus $f \in \mathrm{Ran}(P_\infty)$. To finish the proof we observe that for any $f \in \mathcal{H}$,

$$\|\Pi f\|_{\mathcal{H}} \leqslant \|P_t f\|_{\mathcal{H}} + \|P_t f - \Pi f\|_{\mathcal{H}} \leqslant \|f\|_{\mathcal{H}} + \|P_t f - \Pi f\|_{\mathcal{H}},$$

and taking $t \to \infty$ yields $\|\Pi f\|_{\mathcal{H}} \leqslant \|f\|_{\mathcal{H}}$. This gives that $\|\Pi\|_{\mathcal{H}\to\mathcal{H}} \leqslant 1$, however, any idempotent satisfies $\|\Pi\|_{\mathcal{H}\to\mathcal{H}} \geqslant 1$, and thus we deduce $\|\Pi\|_{\mathcal{H}\to\mathcal{H}} = 1$. Consequently Π must be an orthogonal projection, and since orthogonal projections are uniquely characterized by their range we get $\Pi = P_\infty$. □

Lemma 5.4.1 allows us to prove the norm convergence of P_t to any bounded idempotent, a strategy we will use in the sequel. Using it we can establish the following classical result, which will also be used in the proofs below, and we provide its proof for sake of completeness.

Lemma 5.4.2. *Let $\widetilde{P} = (e^{-t\widetilde{\mathbf{A}}})_{t \geqslant 0}$ be a contraction semigroup on a Hilbert space $\widetilde{\mathcal{H}}$ and suppose $\widetilde{\mathbf{A}}$ is normal with spectral gap $\boldsymbol{\gamma}_1 > 0$. Let Ω be either $\{0\}$ or $i\mathbb{R}$. Then, for any $f \in \widetilde{\mathcal{H}}$ and $t \geqslant 0$,*

$$\left\|\widetilde{P}_t - \mathtt{E}_\Omega f\right\|_{\widetilde{\mathcal{H}}} \leqslant e^{-\boldsymbol{\gamma}_1 t} \left\| f - \mathtt{E}_\Omega f \right\|_{\widetilde{\mathcal{H}}},$$

where $\mathtt{E} : \mathsf{B}(\mathbb{C}) \to \mathcal{B}(\widetilde{\mathcal{H}})$ is the unique resolution of identity associated to $\widetilde{\mathbf{A}}$. Consequently, $\mathtt{E}_{\{0\}} = \mathtt{E}_{i\mathbb{R}} = \widetilde{P}_\infty$.

Proof. By the Borel functional calculus for $\mathbf{A}$ we have, for any $t \geqslant 0$ and $f \in \widetilde{\mathcal{H}}$, writing $\overline{f} = f - \mathtt{E}_{\{0\}} f$,

$$\begin{aligned}
\left\|\widetilde{P}_t \overline{f}\right\|^2_{\widetilde{\mathcal{H}}} &= \left\|\mathtt{E}_{\{0\}} \overline{f}\right\|^2_{\widetilde{\mathcal{H}}} + \int_{\sigma(\widetilde{\mathbf{A}})\setminus\{0\}} e^{-2\operatorname{Re}(\gamma)t} d\langle \mathtt{E}_\gamma \overline{f}, \overline{f}\rangle_{\widetilde{\mathcal{H}}} \\
&= \int_{\sigma(\widetilde{\mathbf{A}})\setminus\{0\}} e^{-2\operatorname{Re}(\gamma)t} d\langle \mathtt{E}_\gamma \overline{f}, \overline{f}\rangle_{\widetilde{\mathcal{H}}} \\
&\leqslant e^{-2\boldsymbol{\gamma}_1 t} \left\|\overline{f}\right\|^2_{\widetilde{\mathcal{H}}}
\end{aligned}$$

where the inequality uses the fact that $\widetilde{\mathbf{A}}$ has spectral gap $\boldsymbol{\gamma}_1$. Next, by the spectral mapping theorem, see e.g. [106], we get that $\mathtt{E}_{\{0\}} = \widetilde{P}_\infty$, and this may also be deduced from the Borel functional calculus for $\widetilde{\mathbf{A}}$ via

$$\left\|(\widetilde{P}_t - \mathtt{E}_{\{0\}})\mathtt{E}_{\{0\}} f\right\|^2_{\widetilde{\mathcal{H}}} = \int_{\sigma(\widetilde{\mathbf{A}})} |e^{-\gamma t} - 1|^2 d\langle \mathtt{E}_\gamma \mathtt{E}_{\{0\}} f, f\rangle_{\widetilde{\mathcal{H}}} = \int_{\{0\}} |e^{-\gamma t} - 1|^2 d\langle \mathtt{E}_\gamma f, f\rangle_{\widetilde{\mathcal{H}}}.$$

Thus invoking Lemma 5.4.1 we conclude that $\widetilde{P}$ satisfies the spectral gap inequality and, in particular, converges to equilibrium. Hence it remains to show that $\mathtt{E}_{\{0\}} = \mathtt{E}_{i\mathbb{R}}$. To this end, for any $f \in \widetilde{\mathcal{H}}$ and $t \geqslant 0$, we have

$$\left\|\widetilde{P}_t f\right\|^2_{\widetilde{\mathcal{H}}} = \int_{\sigma(\mathbf{A})} e^{-2\operatorname{Re}(\gamma)t} d\langle \mathtt{E}_\gamma f, f\rangle_{\widetilde{\mathcal{H}}} = \|\mathtt{E}_{i\mathbb{R}} f\|^2_{\widetilde{\mathcal{H}}} + \int_{\sigma(\mathbf{A})\setminus i\mathbb{R}} e^{-2\operatorname{Re}(\gamma)t} d\langle \mathtt{E}_\gamma f, f\rangle_{\widetilde{\mathcal{H}}}.$$

Taking $f - \mathtt{E}_{\{0\}} f = f - \widetilde{P}_\infty f$ in the above identity yields

$$\begin{aligned}
\left\|\widetilde{P}_t f - \widetilde{P}_\infty f\right\|^2_{\widetilde{\mathcal{H}}} &= \left\|\widetilde{P}_t (f - \widetilde{P}_\infty f)\right\|^2_{\widetilde{\mathcal{H}}} \\
&= \left\|\mathtt{E}_{i\mathbb{R}}(f - \mathtt{E}_{\{0\}} f)\right\|^2_{\widetilde{\mathcal{H}}} + \int_{\sigma(\mathbf{A})\setminus i\mathbb{R}} e^{-2\operatorname{Re}(\gamma)t} d\langle \mathtt{E}_\gamma (f - \mathtt{E}_{\{0\}} f), f - \mathtt{E}_{\{0\}} f\rangle_{\widetilde{\mathcal{H}}}.
\end{aligned}$$

The left-hand side converges to zero as $t \to \infty$ since $\widetilde{P}$ converges to equilibrium, while the integral on the right-hand side is also easily seen to convergence to zero as $t \to \infty$. Hence, by orthogonality of $\mathtt{E}$, we get

$$0 = \left\|\mathtt{E}_{i\mathbb{R}}(f - \mathtt{E}_{\{0\}}f)\right\|_{\widetilde{\mathcal{H}}}^2 = \left\|\mathtt{E}_{i\mathbb{R}}f - \mathtt{E}_{\{0\}\cap i\mathbb{R}}f\right\|_{\widetilde{\mathcal{H}}}^2 = \left\|\mathtt{E}_{i\mathbb{R}}f - \mathtt{E}_{\{0\}}f\right\|_{\widetilde{\mathcal{H}}}^2,$$

and since $f \in \widetilde{\mathcal{H}}$ was arbitrary we get $\mathtt{E}_{\{0\}} = \mathtt{E}_{i\mathbb{R}}$ as desired. □

5.4.2 Proof of Proposition 5.2.1

Since $\Lambda \in \mathcal{B}(\widetilde{\mathcal{H}}, \mathcal{H})$ is a bijection we get that its inverse satisfies $\Lambda^{-1} \in \mathcal{B}(\mathcal{H}, \widetilde{\mathcal{H}})$. Set

$$\Pi = \Lambda \widetilde{P}_\infty \Lambda^{-1},$$

where $\widetilde{P}_\infty$ is the projection onto the set of $\widetilde{P}$-invariant vectors. Then, as the composition of bounded operators we get that $\Pi \in \mathcal{B}(\mathcal{H})$, and it is straightforward to check that $\Pi^2 = \Pi$, i.e. Π is a bounded idempotent. If $\widetilde{P}$ converges to equilibrium with rate $r(t)$ then

$$\begin{aligned}
\|P_t f - \Pi f\|_{\mathcal{H}} &= \left\|\Lambda \widetilde{P}_t \Lambda^{-1} f - \Lambda \widetilde{P}_\infty \Lambda^{-1} f\right\|_{\mathcal{H}} \\
&\leqslant \|\Lambda\|_{\widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}} \left\|\widetilde{P}_t \Lambda^{-1} f - \widetilde{P}_\infty \Lambda^{-1} f\right\|_{\mathcal{H}} \\
&\leqslant \kappa(\Lambda) r(t) \left\|f - \widetilde{P}_\infty f\right\|_{\mathcal{H}},
\end{aligned}$$

and hence P converges to equilibrium by Lemma 5.4.1. The proof of the last claim is straightforward and hence omitted. □

5.4.3 Proof of Proposition 5.2.2

To begin we establish the following lemma, and we say that $\Lambda \in \mathcal{B}(\widetilde{\mathcal{H}}, \mathcal{H})$ is proper with respect to a self-adjoint resolution of identity $\mathtt{E}$ if the conditions in Definition 5.2.1 are fulfilled.

Lemma 5.4.3. *Let $\widetilde{\mathbf{A}}$ be a normal operator on $\widetilde{\mathcal{H}}$ with unique self-adjoint resolution of identity $\mathtt{E} : \mathsf{B}(\mathbb{C}) \to \mathcal{B}(\widetilde{\mathcal{H}})$. Suppose $\Lambda : \widetilde{\mathcal{H}} \to \mathcal{H}$ is a proper linear operator with respect to $\mathtt{E}$, and define $\mathtt{F} : \mathsf{B}(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$ via*

$$\mathtt{F}_\Omega = \Lambda \mathtt{E}_\Omega \Lambda^\dagger.$$

Then $\mathtt{F} : \mathsf{B}(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$ is a nsa resolution of identity with domain $\mathrm{Ran}(\Lambda)$ and, for $(f, g) \in \mathrm{Ran}(\Lambda) \times \mathcal{H}$, we have the following properties.

(1) The measure $\gamma \mapsto \langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}}$ is of bounded variation, and $\gamma \mapsto \langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}} = \langle \mathtt{E}_\gamma \Lambda^\dagger f, \Lambda^* g\rangle_{\widetilde{\mathcal{H}}}$.

(2) For each $m(\widetilde{\mathbf{A}}) \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$ there exists a unique closed, densely-defined linear operator

$$\int_{\sigma(\widetilde{\mathbf{A}})} m(\gamma) d\mathtt{F}_\gamma$$

with domain $\mathrm{Ran}(\Lambda)$, which satisfies

$$\langle \Lambda m(\widetilde{\mathbf{A}}) \Lambda^\dagger f, g\rangle_{\mathcal{H}} = \int_{\sigma(\widetilde{\mathbf{A}})} m(\gamma) d\langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}}.$$

(3) For any $m \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$,

$$\int_{\sigma(\widetilde{\mathbf{A}})} |m(\gamma)| d|\langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}}| = \int_{\sigma(\widetilde{\mathbf{A}})} d\left|\langle \mathtt{F}_\gamma \Lambda m(\widetilde{\mathbf{A}}) \Lambda^\dagger f, g\rangle_{\mathcal{H}}\right|.$$

(4) For $m_1, m_2 \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$ we have, on $\mathrm{Ran}(\Lambda)$, the multiplicative property

$$\left(\int_{\sigma(\widetilde{\mathbf{A}})} m_1(\gamma) d\mathtt{F}_\gamma\right)\left(\int_{\sigma(\widetilde{\mathbf{A}})} m_2(\gamma) d\mathtt{F}_\gamma\right) = \int_{\sigma(\widetilde{\mathbf{A}})} (m_1 m_2)(\gamma) d\mathtt{F}_\gamma.$$

Proof. First we note that all the properties of the pseudo-inverse $\Lambda^\dagger$ used below are given in [13, Theorem 9.2], starting with the fact that $\mathcal{D}(\Lambda^\dagger) = \mathrm{Ran}(\Lambda)$. Then, by assumption on $\mathrm{Ran}(\Lambda)$, we get that, for each $\Omega \in \mathsf{B}(\mathbb{C})$, $\mathtt{F}_\Omega$ is densely-defined, and the linearity of $\mathtt{F}_\Omega$ follows from the linearity of each of the factors in the definition. Since $\Lambda^\dagger$ is a closed operator, and both Λ and $\mathtt{E}_\Omega$ are bounded, it follows that $\mathtt{F}_\Omega$ is closed. Next, let $\Omega_1, \Omega_2 \in \mathsf{B}(\mathbb{C})$, so that

$$\mathtt{F}_{\Omega_1}\mathtt{F}_{\Omega_2} = \Lambda\mathtt{E}_{\Omega_1}\Lambda^\dagger\Lambda\mathtt{E}_{\Omega_2}\Lambda^\dagger,$$

where we used the fact that $\mathtt{F}_\Omega(\mathrm{Ran}(\Lambda)) \subseteq \mathrm{Ran}(\Lambda)$. Since $\Lambda^\dagger\Lambda$ is the projection onto the closed subspace $\overline{\mathrm{Ran}(\Lambda^*)}$, the assumption that Λ is proper with respect to $\mathtt{E}$ then gives that

$$\Lambda\mathtt{E}_{\Omega_1}\Lambda^\dagger\Lambda\mathtt{E}_{\Omega_2}\Lambda^\dagger = \Lambda\mathtt{E}_{\Omega_1}\mathtt{E}_{\Omega_2}\Lambda^\dagger = \Lambda\mathtt{E}_{\Omega_1\cap\Omega_2}\Lambda^\dagger = \mathtt{F}_{\Omega_1\cap\Omega_2}.$$

Finally, we suppose that $(\Omega_i)_{i=1}^\infty \in \mathsf{B}(\mathbb{C})$ is a countable collection of pairwise disjoint subsets. Then, by continuity of the inner product we get, for $(f, g) \in \mathrm{Ran}(\Lambda) \times \mathcal{H}$,

$$\langle \mathtt{F}_{(\Omega_i)_{i=1}^\infty} f, g\rangle_{\mathcal{H}} = \sum_{i=1}^\infty \langle \mathtt{F}_{\Omega_i} f, g\rangle_{\mathcal{H}} = \sum_{i=1}^\infty \langle \mathtt{E}_{\Omega_i}\Lambda^\dagger f, \Lambda^* g\rangle_{\widetilde{\mathcal{H}}},$$

and the countable additivity of $\mathtt{F}$ follows from the same property for $\mathtt{E}$, which completes the proof that $\mathtt{F} : \mathsf{B}(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$ defines a nsa resolution of identity. As shown above, we have, for $\Omega \in \mathsf{B}(\mathbb{C})$,

$$\langle \mathtt{F}_\Omega f, g\rangle_{\mathcal{H}} = \langle \Lambda\mathtt{E}_\Omega\Lambda^\dagger f, g\rangle_{\mathcal{H}} = \langle \mathtt{E}_\Omega\Lambda^\dagger f, \Lambda^* g\rangle_{\widetilde{\mathcal{H}}},$$

and since $\gamma \mapsto \langle \mathtt{E}_\gamma\Lambda^\dagger f, \Lambda^* g\rangle_{\widetilde{\mathcal{H}}}$ has total variation $\left\|\Lambda^\dagger f\right\|_{\mathcal{H}} \|\Lambda\|_{\widetilde{\mathcal{H}}\to\mathcal{H}} \|g\|_{\mathcal{H}}$, we complete the proof of Item (1). Next, let $s \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$ be a simple function, i.e. for $k \geqslant 1$,

$$s(\gamma) = \sum_{i=1}^k \alpha_i \mathbf{1}_{\Omega_i}(\gamma),$$

where $\Omega_1, \ldots, \Omega_k \in \mathsf{B}(\mathbb{C})$ are disjoint subsets and $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$, so that by the Borel functional calculus for $\widetilde{\mathbf{A}}$,

$$s(\widetilde{\mathbf{A}}) = \sum_{i=1}^k \alpha_i \mathtt{E}_{\Omega_i}.$$

Then, with $(f, g) \in \mathrm{Ran}(\Lambda) \in \mathcal{H}$, we have that

$$\langle \Lambda s(\widetilde{\mathbf{A}})\Lambda^\dagger f, g\rangle_{\mathcal{H}} = \sum_{i=1}^{k} \alpha_i \langle \Lambda \mathtt{E}_{\Omega_i}\Lambda^\dagger f, g\rangle_{\mathcal{H}} = \sum_{i=1}^{k} \alpha_i \langle \mathtt{F}_{\Omega_i} f, g\rangle_{\mathcal{H}} = \int_{\sigma(\widetilde{\mathbf{A}})} s(\gamma) d\langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}}.$$

Now let $\varepsilon > 0$, $m \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$ and choose a simple function s such that $\|m - s\|_\infty < \varepsilon$. Then, using the above representation for s we get that

$$\begin{aligned}
&\left|\langle \Lambda m(\widetilde{\mathbf{A}})\Lambda^\dagger f, g\rangle_{\mathcal{H}} - \int_{\sigma(\widetilde{\mathbf{A}})} m(\gamma) d\langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}}\right| \\
&\leqslant \left|\langle \Lambda m(\mathbf{A})\Lambda^\dagger f, g\rangle_{\mathcal{H}} - \langle \Lambda s(\mathbf{A})\Lambda^\dagger f, g\rangle_{\mathcal{H}}\right| + \left|\int_{\sigma(\widetilde{\mathbf{A}})} (m - s)(\gamma) d\langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}}\right| \\
&\leqslant \|\Lambda\|_{\widetilde{\mathcal{H}}\to\mathcal{H}} \|m - s\|_\infty \left\|\Lambda^\dagger f\right\|_{\mathcal{H}} \|g\|_{\mathcal{H}} \\
&\quad + \int_{\sigma(\widetilde{\mathbf{A}})} |m(\gamma) - s(\gamma)| d|\langle \mathtt{E}_\gamma \Lambda^\dagger f, \Lambda^* g\rangle_{\widetilde{\mathcal{H}}}| \\
&\leqslant 2\varepsilon \|\Lambda\|_{\widetilde{\mathcal{H}}\to\mathcal{H}} \left\|\Lambda^\dagger f\right\|_{\mathcal{H}} \|g\|_{\mathcal{H}}.
\end{aligned}$$

Since ε was arbitrary it follows that

$$\Lambda m(\widetilde{\mathbf{A}})\Lambda^\dagger = \int_{\sigma(\widetilde{\mathbf{A}})} m(\gamma) d\mathtt{F}_\gamma$$

on $\mathrm{Ran}(\Lambda)$, and the fact that $\Lambda m(\widetilde{\mathbf{A}})\Lambda^\dagger$ is closed follows immediately from the closedness of $\Lambda^\dagger$, which completes the proof of Item (2). For the proof of Item (3) let again $s \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$ be a simple function. Then, for $(f, g) \in \mathrm{Ran}(\Lambda) \times \mathcal{H}$,

$$\int_{\sigma(\widetilde{\mathbf{A}})} |s(\gamma)| d|\langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}}| = \int_{\sigma(\mathbf{A})} \sum_{i=1}^{k} |\alpha_i| \mathbf{1}_{\Omega_i}(\gamma) d|\langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}}| = \sum_{i=1}^{k} |\alpha_i| |\langle \mathtt{F}_{\Omega_i} f, g\rangle_{\mathcal{H}}|,$$

while on the other hand, since the measure $\gamma \mapsto \langle \mathtt{F}_\gamma \Lambda s(\widetilde{\mathbf{A}})\Lambda^\dagger f, g\rangle_{\mathcal{H}}$ is the sum of Dirac masses,

$$\int_{\sigma(\widetilde{\mathbf{A}})} d|\langle \mathtt{F}_\gamma \Lambda s(\widetilde{\mathbf{A}})\Lambda^\dagger f, g\rangle_{\mathcal{H}} = \sum_{i=1}^{k} |\langle \alpha_i \mathtt{F}_{\Omega_i} f, g\rangle_{\mathcal{H}}| = \sum_{i=1}^{k} |\alpha_i| |\langle \mathtt{F}_{\Omega_i} f, g\rangle_{\mathcal{H}}|.$$

For general $m \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$ and given $\varepsilon > 0$, let s be a simple function such that $\|m - s\|_\infty < \infty$. Write μ_m for the measure $\gamma \mapsto \langle F_\gamma \Lambda m(\mathbf{A})\Lambda^\dagger f, g\rangle_{\mathcal{H}}$, and similarly for

μ_s. Then, using what we just proved for simple functions, we get

$$\begin{aligned}
&\left|\int_{\sigma(\widetilde{\mathbf{A}})} |m(\gamma)| d|\langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}}| - \int_{\sigma(\widetilde{\mathbf{A}})} d|\langle \mathtt{F}_\gamma \Lambda m(\mathbf{A})\Lambda^\dagger f, g\rangle_{\mathcal{H}}|\right| \\
&\leqslant \int_{\sigma(\widetilde{\mathbf{A}})} \big||m(\gamma)| - |s(\gamma)|\big| \, d|\langle \mathtt{F}_\gamma f, g\rangle_{\mathcal{H}}| + \left||\mu_m|(\sigma(\widetilde{\mathbf{A}})) - |\mu_s|(\sigma(\widetilde{\mathbf{A}}))\right| \\
&\leqslant \|\Lambda\|_{\widetilde{\mathcal{H}}\to\mathcal{H}} \|m - s\|_\infty \left\|\Lambda^\dagger f\right\|_{\mathcal{H}} \|g\|_{\mathcal{H}} + \int_{\sigma(\widetilde{\mathbf{A}})} d|\langle \mathtt{F}_\gamma \Lambda \left(m(\widetilde{\mathbf{A}}) - s(\widetilde{\mathbf{A}})\right) \Lambda^\dagger f, g\rangle_{\mathcal{H}}| \\
&\leqslant 2\varepsilon \|\Lambda\|_{\widetilde{\mathcal{H}}\to\mathcal{H}} \left\|\Lambda^\dagger f\right\|_{\mathcal{H}} \|g\|_{\mathcal{H}},
\end{aligned}$$

where in the second inequality we used the reverse triangle inequality for the sup-norm, while in the last inequality we used the reverse triangle inequality for the total variation norm together with linearity of the inner product in the first variable. This completes the proof of Item (3). Finally, for the multiplicative property of the integrals we observe that, by the multiplicative property for $\mathtt{E}$, $\Lambda m_1(\widetilde{\mathbf{A}})\Lambda^\dagger \Lambda m_2(\widetilde{\mathbf{A}})\Lambda^\dagger = \Lambda m_1(\widetilde{\mathbf{A}}) m_2(\widetilde{\mathbf{A}})\Lambda^\dagger = \Lambda(m_1 m_2)(\widetilde{\mathbf{A}})\Lambda^\dagger$, where we again used that Λ is proper with respect to $\mathtt{E}$. □

Proof of Proposition 5.2.2. Applying $\Lambda^\dagger$ to both sides of the intertwining $P \overset{\Lambda}{\hookrightarrow} \widetilde{P}$ gives

$$P_t \Lambda\Lambda^\dagger = \Lambda \widetilde{P}_t \Lambda^\dagger.$$

By [13, Theorem 9.2(e)], we have that $\Lambda\Lambda^\dagger$ is the projection onto $\overline{\mathrm{Ran}(\Lambda)}$, and from $\mathrm{Ran}(\Lambda) \subset_d \mathcal{H}$, we deduce that this projection is the identity on $\mathrm{Ran}(\Lambda)$. Thus, together with Lemma 5.4.3(3) and the fact that $\widetilde{P} = (e^{-t\widetilde{\mathbf{A}}})_{t\geqslant 0}$ we conclude that, on $\mathrm{Ran}(\Lambda)$,

$$P_t = \Lambda \widetilde{P}_t \Lambda^\dagger = \int_{\sigma(\widetilde{\mathbf{A}})} e^{-\gamma t} d\mathtt{F}_\gamma,$$

and the remaining claims were proved in Lemma 5.4.3. □

5.4.4 Proof of Theorem 5.2.1

Let $M_t : \sigma(\mathbf{A}) \to \mathbb{C}$ be the function defined by

$$M_t(\gamma) = \frac{e^{-\gamma t}}{m(\gamma)},$$

which, for $t > T_m$, belongs to $\mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$ by assumption. From the condition in Item (a) we deduce that, for $(f, g) \in \mathrm{Ran}(\Lambda) \times \mathcal{H}$ and $t > T_m$,

$$\begin{aligned}
\left| \langle \int_{\sigma(\widetilde{\mathbf{A}})} e^{-\gamma t} d\mathtt{F}_\gamma f, g \rangle_{\mathcal{H}} \right| &\leqslant \int_{\sigma(\widetilde{\mathbf{A}})} |e^{-\gamma t}| d|\langle \mathtt{F}_\gamma f, g \rangle_{\mathcal{H}} \\
&\leqslant \int_{\sigma(\widetilde{\mathbf{A}})} \left| \frac{e^{-\gamma t}}{m(\gamma)} \right| |m(\gamma)| d|\langle \mathtt{F}_\gamma f, g \rangle_{\mathcal{H}}| \\
&\leqslant \|M_t\|_\infty \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}} .
\end{aligned}$$

Thus we conclude that, for $t > T_m$, the operator $\int_{\sigma(\widetilde{\mathbf{A}})} e^{-\gamma t} d\mathtt{F}_\gamma$ is bounded on $\mathrm{Ran}(\Lambda) \subset_d \mathcal{H}$, so that by invoking the bounded linear extension theorem we obtain a unique, continuous linear extension to all of $\mathcal{H}$. Next, let us write simply $\mathtt{E}_0$ and $\mathtt{F}_0$ in place of $\mathtt{E}_{\{0\}}$ and $\mathtt{F}_{\{0\}}$, respectively. Then, by evaluating the assumption in Item (b) at $\gamma = 0$ we get that the idempotent $\mathtt{F}_0 = \Lambda \mathtt{E}_0 \Lambda^{-1}$ satisfies, for all $(f, g) \in \mathrm{Ran}(\Lambda) \times \mathcal{H}$,

$$\begin{aligned}
|\langle \mathtt{F}_0 f, g \rangle_{\mathcal{H}}| &\leqslant \int_{\{0\}} \frac{1}{|m(\gamma)|} |m(\gamma)| d|\langle \mathtt{F}_\gamma f, g \rangle_{\mathcal{H}}| \\
&\leqslant \frac{1}{|m(0)|} \int_{\sigma(\mathbf{A})} |m(\gamma)| d|\langle \mathtt{F}_\gamma f, g \rangle_{\mathcal{H}}| \\
&\leqslant \frac{1}{|m(0)|} \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}} ,
\end{aligned}$$

and thus we deduce that $\mathtt{F}_0$ is bounded on $\mathrm{Ran}(\Lambda)$. Since Lemma 5.4.2 gives that $\mathtt{E}_0 = \widetilde{P}_\infty$ we get

$$P_t \mathtt{F}_0 = \Lambda \widetilde{P}_t \Lambda \Lambda^\dagger \mathtt{E}_0 \Lambda^\dagger = \Lambda \widetilde{P}_t \mathtt{E}_0 \Lambda^\dagger = \mathtt{F}_0,$$

so that $\mathtt{F}_0$ is invariant for P, and in particular

$$(P_t - \mathtt{F}_0)(I - \mathtt{F}_0) = P_t - P_t \mathtt{F}_0 - \mathtt{F}_0 + \mathtt{F}_0^2 = P_t - \mathtt{F}_0 - \mathtt{F}_0 + \mathtt{F}_0 = P_t - \mathtt{F}_0, \tag{5.6}$$

where both of these equalities hold on $\mathrm{Ran}(\Lambda)$. Putting all of these observations together we get that, for $t > T_m$ and any $f \in \mathrm{Ran}(\Lambda)$,

$$
\begin{aligned}
\|P_t f - \mathtt{F}_0 f\|_{\mathcal{H}}^2 &= \left| \int_{\sigma(\widetilde{\mathbf{A}})\setminus\{0\}} e^{-\gamma t} d\langle \mathtt{F}_\gamma (f - \mathtt{F}_0 f), P_t f - \mathtt{F}_0 f \rangle_{\mathcal{H}} \right| \\
&\leqslant \int_{\sigma(\widetilde{\mathbf{A}})\setminus\{0\}} |e^{-\gamma t}| d|\langle \mathtt{F}_\gamma (f - \mathtt{F}_0 f), P_t f - \mathtt{F}_0 f \rangle_{\mathcal{H}}| \\
&= \int_{\mathrm{Re}(\gamma)\geqslant \boldsymbol{\gamma}_1} |e^{-\gamma t}| d|\langle \mathtt{F}_\gamma (f - \mathtt{F}_0 f), P_t f - \mathtt{F}_0 f \rangle_{\mathcal{H}}| \\
&\leqslant \left\| M_t^{(\boldsymbol{\gamma}_1)} \right\|_\infty \int_{\mathrm{Re}(\gamma)\geqslant \boldsymbol{\gamma}_1} |m(\gamma)| d|\langle \mathtt{F}_\gamma (f - \mathtt{F}_0 f), P_t f - \mathtt{F}_0 f \rangle_{\mathcal{H}}| \\
&\leqslant \left\| M_t^{(\boldsymbol{\gamma}_1)} \right\|_\infty \|f - \mathtt{F}_0 f\|_{\mathcal{H}} \, \|P_t f - \mathtt{F}_0 f\|_{\mathcal{H}} \qquad (5.7)
\end{aligned}
$$

where, in order, we have used (5.6), the representation for P_t as a spectral integral, the fact that $\widetilde{\mathbf{A}}$ admits a spectral gap $\boldsymbol{\gamma}_1 > 0$ and finally the assumptions on the function m. Canceling $\|P_t f - \mathtt{F}_0 F\|$ from both sides of the inequality in (5.7) yields, for $t > T_m$ and $f \in \mathrm{Ran}(\Lambda)$,

$$
\|P_t f - \mathtt{F}_0 f\|_{\mathcal{H}} \leqslant \left\| M_t^{(\boldsymbol{\gamma}_1)} \right\|_\infty \|f - \mathtt{F}_0 f\|_{\mathcal{H}},
$$

which extends by density, and the continuity of the involved operators, to all of $\mathcal{H}$. Then, invoking Lemma 5.4.1 completes the proof that P converges to equilibrium, since plainly $\lim_{t\to\infty} \left\| M_t^{(\boldsymbol{\gamma}_1)} \right\|_\infty = 0$. □

5.4.5 Proof of Theorem 5.2.2

We shall provide two proofs of Theorem 5.2.2, one that invokes Theorem 5.2.1 and hence is based on properties of nsa resolutions of the identity, and another that makes use of the Borel functional calculus for $\widetilde{\mathbf{A}}$. We need a preliminary results regarding commuting operators. We say that an operator $M \in \mathcal{B}(\widetilde{\mathcal{H}})$ commutes with a closed, densely-defined operator $\widetilde{\mathbf{A}}$ on $\widetilde{\mathcal{H}}$ if for some $z \in \rho(\widetilde{\mathbf{A}}) = \mathbb{C} \setminus \sigma(\widetilde{\mathbf{A}})$, the resolvent set of A, we have

$$
M R_z(\widetilde{\mathbf{A}}) = R_z(\widetilde{\mathbf{A}}) M,
$$

where $R_z(\widetilde{\mathbf{A}})$ denotes the resolvent operator. The following two results are adapted from [118], and we refer to [118, Chapter 3] for the appropriate definitions.

Lemma 5.4.4. *Let $\widetilde{P} = (e^{-t\widetilde{\mathbf{A}}})_{t\geqslant 0}$ be a contraction semigroup on $\widetilde{\mathcal{H}}$ and suppose that $\widetilde{\mathbf{A}}$ is normal and has simple spectrum. Then for operator $M \in B(\widetilde{\mathcal{H}})$*

$$M\widetilde{P}_t = \widetilde{P}_t M, \ \forall t \geqslant 0 \iff M = m(\widetilde{\mathbf{A}}), \text{ for some } m \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}})).$$

Proof. First we will show that

$$M\widetilde{P}_t = \widetilde{P}_t M, \ \forall t \geqslant 0 \iff MR_z(\widetilde{\mathbf{A}}) = R_z(\widetilde{\mathbf{A}})M. \tag{5.8}$$

For the only if direction, if $\widetilde{\mathbf{A}}$ is normal and commutes with a bounded operator M, then $Mf(\widetilde{A}) = f(\widetilde{\mathbf{A}})M$ for any $f \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$, which includes the exponential function $\gamma \mapsto e^{-\gamma t}$, for any $t \geqslant 0$. In the other direction we use the fact that, for $z \in \rho(\widetilde{\mathbf{A}})$,

$$R_z(\widetilde{\mathbf{A}}) = \int_0^\infty e^{-zt}\widetilde{P}_t dt.$$

Then, $M\widetilde{P}_t = \widetilde{P}_t M$, for all $t \geqslant 0$, implies that

$$R_z(\widetilde{\mathbf{A}})M = \int_0^\infty e^{-zt}\widetilde{P}_t M dt = \int_0^\infty e^{-zt} M\widetilde{P}_t dt = MR_z(\widetilde{\mathbf{A}}).$$

Having established (5.8), the claim follows by similar arguments as in the proof of [118, Theorem 4.8]. □

First proof of Theorem 5.2.2. By assumption we have, for $\Lambda : \widetilde{\mathcal{H}} \to \mathcal{H}$ and $\widetilde{\Lambda} : \mathcal{H} \to \widetilde{\mathcal{H}}$ proper intertwining operators,

$$P_t\Lambda = \Lambda\widetilde{P}_t \quad \text{and} \quad \widetilde{\Lambda}P_t = \widetilde{P}_t\widetilde{\Lambda},$$

and by combining these two identities we deduce that $\widetilde{P}_t\widetilde{\Lambda}\Lambda = \widetilde{\Lambda}\Lambda\widetilde{P}_t$. Now, if $\widetilde{\mathbf{A}}$ has simple spectrum then we may invoke Lemma 5.4.4 to get that there exists $m \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}))$ such that

$$m(\widetilde{\mathbf{A}}) = \widetilde{\Lambda}\Lambda. \tag{5.9}$$

Next, by Lemma 5.4.3(3) and for $(f, g) \in \operatorname{Ran}(\Lambda) \times \mathcal{H}$, we have

$$\int_{\sigma(\widetilde{\mathbf{A}})} |m(\gamma)| d|\langle \mathrm{F}_\gamma f, g\rangle_{\mathcal{H}}| = \int_{\sigma(\widetilde{\mathbf{A}})} d|\langle \mathrm{F}_\gamma \Lambda m(\widetilde{\mathbf{A}})\Lambda^\dagger f, g\rangle_{\mathcal{H}}|.$$

Together with (5.9) and Lemma 5.4.3(1), this gives

$$\begin{aligned}
\int_{\sigma(\mathbf{A})} |m(\gamma)| d|\langle \mathrm{F}_\gamma f, g\rangle_{\mathcal{H}}| &= \int_{\sigma(\widetilde{\mathbf{A}})} d|\langle \mathrm{F}_\gamma \Lambda \widetilde{\Lambda} f, g\rangle_{\mathcal{H}}| \\
&\leqslant \|\Lambda\|_{\widetilde{\mathcal{H}}\to\mathcal{H}} \left\|\Lambda^\dagger \Lambda \widetilde{\Lambda} f\right\|_{\mathcal{H}} \|g\|_{\mathcal{H}} \\
&\leqslant \|\Lambda\|_{\widetilde{\mathcal{H}}\to\mathcal{H}} \left\|\widetilde{\Lambda}\right\|_{\mathcal{H}\to\widetilde{\mathcal{H}}} \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}},
\end{aligned}$$

where we also used that $\Lambda\Lambda^\dagger$ is the identity and that $\Lambda^\dagger\Lambda$ is a projection and thus a bounded operator with norm 1, see [13, Theorem 9.2] for both of these claims. Invoking Theorem 5.2.1 then completes the proof. □

Second proof of Theorem 5.2.2. By assumption on the function m, there exists $T_m > 0$ such that for $t > T_m$,

$$\gamma \mapsto M_t(\gamma) = \frac{e^{-\gamma t}}{m(\gamma)} \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}})), \tag{5.10}$$

and by the Borel functional calculus for $\widetilde{\mathbf{A}}$ it follows that $M_t(\widetilde{\mathbf{A}})$ is a bounded operator for $t > T_m$. Next, by the intertwining $P \overset{\Lambda}{\hookrightarrow} \widetilde{P}$ and by the multiplicative property of the Borel functional calculus for $\widetilde{\mathbf{A}}$, we get, for $t > T_m$,

$$\Lambda M_t(\widetilde{\mathbf{A}})\widetilde{\Lambda}\Lambda = \Lambda M_t(\widetilde{\mathbf{A}}) m(\widetilde{\mathbf{A}}) = \Lambda e^{-t\widetilde{\mathbf{A}}} = P_t \Lambda,$$

and thus we deduce that

$$P_t = \Lambda M_t(\widetilde{\mathbf{A}})\widetilde{\Lambda}$$

where the equality holds on $\operatorname{Ran}(\Lambda)$. However, as the right-hand side is a bounded linear operator we get, by the Bounded Linear Extension Theorem, that for $t > T_m$,

$$P_t = \Lambda M_t(\widetilde{\mathbf{A}})\widetilde{\Lambda}, \quad \text{on} \quad \mathcal{H}. \tag{5.11}$$

Next, since $\|M_t\|_\infty < \infty$ for $t > T_m$ and taking $\gamma = 0$ in the definition of M_t in (5.10), we get that

$$\gamma \mapsto m_0(\gamma) = \frac{1}{m(\gamma)}\mathbf{1}_{\{\gamma=0\}} \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}})),$$

and also note that $m_0(\widetilde{\mathbf{A}})\mathtt{E}_{\{0\}} = \mathtt{E}_{\{0\}}m_0(\widetilde{\mathbf{A}}) = m_0(\widetilde{\mathbf{A}})$. Let $\Pi \in \mathcal{B}(\mathcal{H})$ be the operator defined by $\Pi = \Lambda m_0(\widetilde{\mathbf{A}})\widetilde{\Lambda}$. Using $m(\widetilde{\mathbf{A}}) = \widetilde{\Lambda}\Lambda$, the previous observation, and the multiplicative property of the Borel functional calculus for $\widetilde{\mathbf{A}}$ we get

$$\begin{aligned}\Pi^2 &= \left(\Lambda m_0(\widetilde{\mathbf{A}})\widetilde{\Lambda}\right)\left(\Lambda m_0(\widetilde{\mathbf{A}})\widetilde{\Lambda}\right) = \Lambda m_0(\widetilde{\mathbf{A}})m(\widetilde{\mathbf{A}})m_0(\widetilde{\mathbf{A}})\widetilde{\Lambda} \\ &= \Lambda m_0(\widetilde{\mathbf{A}})\mathtt{E}_{\{0\}}\widetilde{\Lambda} = \Lambda m_0(\widetilde{\mathbf{A}})\widetilde{\Lambda} = \Pi,\end{aligned}$$

and we conclude that Π is a bounded idempotent. Now, by (5.11) we get, for $t > T_m$,

$$\begin{aligned}P_t\Pi &= \left(\Lambda M_t(\widetilde{\mathbf{A}})\widetilde{\Lambda}\right)\left(\Lambda m_0(\widetilde{\mathbf{A}})\widetilde{\Lambda}\right) = \Lambda M_t(\widetilde{\mathbf{A}})m(\widetilde{\mathbf{A}})m_0(\widetilde{\mathbf{A}})\widetilde{\Lambda} \\ &= \Lambda M_t(\widetilde{\mathbf{A}})\mathtt{E}_{\{0\}}\widetilde{\Lambda} = \Lambda m_0(\widetilde{\mathbf{A}})\widetilde{\Lambda} = \Pi f,\end{aligned}$$

and thus it follows that

$$(P_t - \Pi)(I - \Pi) = P_t - P_t\Pi - \Pi + \Pi^2 = P_t - \Pi - \Pi + \Pi = P_t - \Pi.$$

Recall that, for $t > T_m$, $M_t^{(\gamma_1)} : \sigma(\widetilde{\mathbf{A}}) \to \mathbb{C}$ is given by $M_t^{(\gamma_1)}(\gamma) = \frac{e^{-\gamma t}}{m(\gamma)}\mathbf{1}_{\{\mathrm{Re}(\gamma)\geqslant\gamma_1\}}$, and observe that

$$\|M_t - m_0\|_\infty = \sup_{\gamma\in\sigma(\widetilde{\mathbf{A}})}\left|\frac{e^{-\gamma t}}{m(\gamma)}\mathbf{1}_{\{\gamma\neq0\}}\right| = \sup_{0\neq\gamma\in\sigma(\widetilde{\mathbf{A}})}\left|\frac{e^{-\gamma t}}{m(\gamma)}\right| = \sup_{\mathrm{Re}(\gamma)>0}\left|\frac{e^{-\gamma t}}{m(\gamma)}\right| = \left\|M_t^{(\gamma_1)}\right\|_\infty,$$

where in the third equality we used the fact that $\mathtt{E}_{i\mathbb{R}} = \mathtt{E}_0$, which is a consequence of Lemma 5.4.2, and for the last equality used that $\widetilde{\mathbf{A}}$ admits a spectral gap $\boldsymbol{\gamma}_1$. Then, we conclude that, for any $f \in \mathcal{H}$ and $t > T_m$,

$$\begin{aligned}\|(P_t - \Pi)(f - \Pi f)\|_{\mathcal{H}} &= \left\|\left(\Lambda M_t(\widetilde{\mathbf{A}})\widetilde{\Lambda} - \Lambda m_0(\widetilde{\mathbf{A}})\widetilde{\Lambda}\right)(f - \Pi f)\right\|_{\mathcal{H}} \\ &= \left\|\Lambda\left(M_t(\widetilde{\mathbf{A}}) - m_0(\widetilde{\mathbf{A}})\right)\widetilde{\Lambda}(f - \Pi f)\right\|_{\mathcal{H}} \\ &\leqslant \|\Lambda\|_{\widetilde{\mathcal{H}}\to\mathcal{H}}\left\|\widetilde{\Lambda}\right\|_{\mathcal{H}\to\widetilde{\mathcal{H}}}\|M_t - m_0\|_\infty\|f - \Pi f\|_{\mathcal{H}} \\ &= \|\Lambda\|_{\widetilde{\mathcal{H}}\to\mathcal{H}}\left\|\widetilde{\Lambda}\right\|_{\mathcal{H}\to\widetilde{\mathcal{H}}}\left\|M_t^{(\gamma_1)}\right\|_\infty\|f - \Pi f\|_{\mathcal{H}}.\end{aligned}$$

□

5.4.6 Proof of Theorem 5.3.1

To give the proof of Theorem 5.3.1 we state and prove some auxiliary results that may be of independent interests. We write $\mathcal{F}_f$ for the Fourier transform of a suitable function f, which for a function $f \in \mathrm{L}^1(\mathbb{R}^d)$ and any $\xi \in \mathbb{R}^d$, can be represented by

$$\mathcal{F}_f(\xi) = \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} f(x)dx,$$

and use, when needed, the notation $e_{i\xi} : x \mapsto e^{i\langle \xi, x\rangle}$. We also write $\prec$ for the Löwner ordering of positive-definite matrices, that is, for two symmetric matrices X and Y, $X \prec Y$ if and only if $X - Y$ is positive-definite. Hence $X \succ 0$ is shorthand for saying that X is positive-definite.

Proposition 5.4.1. *Let P and $\widetilde{P}$ be two Ornstein-Uhlenbeck semigroups associated to (Q, B) and $(\widetilde{Q}, B)$, respectively, and suppose that $0 \prec Q_\infty \prec \widetilde{Q}_\infty$. Then the operator $\Lambda_{Q\to\widetilde{Q}} = \Lambda : \mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d)$ defined, for $f \in \mathrm{L}^2(\mathbb{R}^d)$, by*

$$\mathcal{F}_{\Lambda f}(\xi) = e^{-\langle(\widetilde{Q}_\infty - Q_\infty)\xi,\xi\rangle/2}\mathcal{F}_f(\xi), \quad \xi \in \mathbb{R}^d,$$

belongs to $\mathcal{B}(\mathrm{L}^2(\mathbb{R}^d))$. Moreover, $\Lambda \in \mathcal{B}(\mathrm{L}^2(\widetilde{\rho}_\infty), \mathrm{L}^2(\rho_\infty))$ is a quasi-affinity with $\|\Lambda\|_{\mathrm{L}^2(\widetilde{\rho}_\infty)\to\mathrm{L}^2(\rho_\infty)} = 1$, and we have the intertwining $P \overset{\Lambda}{\hookrightarrow} \widetilde{P}$.

For the proof of this proposition we shall need the following lemma, where we recall that any Ornstein-Uhlenbeck semigroup P extends to a contraction semigroup from $\mathrm{L}^p(\mathbb{R}^d)$, $p \geqslant 1$, to itself, see e.g. [81, Proposition 9.4.1].

Lemma 5.4.5. *Let P be the Ornstein-Uhlenbeck semigroup associated to (Q, B). Then, for any $f \in \mathrm{L}^2(\mathbb{R}^d)$ and $t \geqslant 0$,*

$$\mathcal{F}_{P_t f}(e^{-tB^*}\xi) = \frac{1}{|\det(e^{tB})|} e^{-\langle Q_t\xi,\xi\rangle/2}\mathcal{F}_f(\xi), \quad \xi \in \mathbb{R}^d.$$

Proof. By a change of variables we have that

$$P_t f(x) = \int_{\mathbb{R}^d} f(y)\rho_t(y - e^{-tB}x)dy.$$

Then, since for any $f \in \mathrm{L}^2(\mathbb{R}^d) \cap \mathrm{L}^1(\mathbb{R}^d)$, $P_t f \in \mathrm{L}^2(\mathbb{R}^d) \cap \mathrm{L}^1(\mathbb{R}^d)$, we get using Fubini's Theorem

$$\begin{aligned}
\mathcal{F}_{P_t f}(\xi) = \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} P_t f(x)dx &= \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} \int_{\mathbb{R}^d} f(y)\rho_t(y - e^{-tB}x)dydx \\
&= \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} \rho_t(y - e^{-tB}x)dxdy \\
&= \frac{1}{|\det(e^{tB})|} \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} e^{-i\langle e^{tB^*}\xi, x\rangle} \rho_t(y + x)dxdy \\
&= \frac{1}{|\det(e^{tB})|} e^{-\langle Q_t e^{tB^*}\xi, e^{tB^*}\xi\rangle/2} \int_{\mathbb{R}^d} e^{i\langle e^{tB^*}\xi, y\rangle} f(y)dy,
\end{aligned}$$

and the claim follows from $\mathrm{L}^2(\mathbb{R}^d) \cap \mathrm{L}^1(\mathbb{R}^d) \subset_d \mathrm{L}^2(\mathbb{R}^d)$ together with the continuity of the Fourier transform. □

Proof of Proposition 5.4.1. First, we note that Λ is a Fourier multiplier operator whose multiplier is, by the assumption $\widetilde{Q}_\infty - Q_\infty \succ 0$, a bounded measurable function; thus $\Lambda \in \mathcal{B}(\mathrm{L}^2(\mathbb{R}^d))$. By identifying the multiplier we deduce that Λ is the convolution operator associated to the Gaussian measure with covariance matrix $(\widetilde{Q}_\infty - Q_\infty)^{-1}$, i.e. writing ρ_Λ for this Gaussian measure, $\Lambda f(x) = f * \rho_\Lambda(x)$, with $*$ denoting the additive convolution operator. Clearly for any $f \in \mathrm{L}^2(\widetilde{\rho}_\infty)$, Λf makes sense, and hence it remains to show the boundedness and that Λ is a quasi-affinity. To this end we observe that the following factorization of measures holds

$$\rho_\infty * \rho_\Lambda = \widetilde{\rho}_\infty,$$

which follows from the identity,

$$\mathcal{F}_{\rho_\infty}(\xi) \cdot \mathcal{F}_{\rho_\Lambda}(\xi) = e^{-\langle Q_\infty \xi, \xi\rangle/2} e^{-\langle (\widetilde{Q}_\infty - Q_\infty)\xi, \xi\rangle/2} = e^{-\langle \widetilde{Q}_\infty \xi, \xi\rangle/2} = \mathcal{F}_{\widetilde{\rho}_\infty}(\xi), \quad \xi \in \mathbb{R}^d,$$

together with the fact that the Fourier transform uniquely characterizes probability measures. Using this factorization we get, for any $f \in \mathrm{L}^2(\widetilde{\rho}_\infty)$, and by appealing to Jensen's inequality

$$\int_{\mathbb{R}^d} |\Lambda f(x)|^2 \rho_\infty(x)dx \leqslant \int_{\mathbb{R}^d} \Lambda |f|^2(x)\rho_\infty(x)dx = \int_{\mathbb{R}^d} |f(x)|^2 \widetilde{\rho}_\infty(x)dx,$$

and thus $\Lambda \in \mathcal{B}(\mathrm{L}^2(\widetilde{\rho}_\infty), \mathrm{L}^2(\rho_\infty))$ with $\|\Lambda\|_{\mathrm{L}^2(\widetilde{\rho}_\infty)\to\mathrm{L}^2(\rho_\infty)} \leqslant 1$; however, equality is achieved by the constant function $\mathbf{1} \in \mathrm{L}^2(\widetilde{\rho}_\infty)$. Next we observe that, for any $\xi \in \mathbb{R}^d$, $\Lambda e_{i\xi}(x) = e^{-\langle(\widetilde{Q}_\infty - Q_\infty)\xi,\xi\rangle/2} e_{i\xi}(x)$. Thus $\mathrm{Ran}(\Lambda)$ contains the linear span of the set $\{e_{i\xi};\ \xi \in \mathbb{R}^d\}$ and to show that $\mathrm{Ran}(\Lambda) \subset_d \mathrm{L}^2(\rho_\infty)$ it then suffices to show that this linear span is dense in $\mathrm{L}^2(\rho_\infty)$. To this end, we suppose there exists $g \in \mathrm{L}^2(\rho_\infty)$ such that $\langle e_{i\xi}, g\rangle_{\mathrm{L}^2(\rho_\infty)} = 0$, for all $\xi \in \mathbb{R}^d$. Then, by standard application of Jensen's inequality, $g \in \mathrm{L}^2(\rho_\infty)$ implies that $g \in \mathrm{L}^1(\rho_\infty)$ and hence $(g\rho_\infty) \in \mathrm{L}^1(\mathbb{R}^d)$. Thus $(g\rho_\infty)$ is an integrable function with vanishing Fourier transform, and we conclude that $g = 0 \in \mathrm{L}^2(\rho_\infty)$. Now, for any $f \in \mathrm{Ker}(\Lambda)$ we have that $f * \rho_\Lambda(x) = 0$ a.e., which forces $f(x) = 0$ a.e. and thus $f = 0$ in $\mathrm{L}^2(\widetilde{\rho}_\infty)$, which gives that Λ is a quasi-affinity. Finally, it remains to show that the intertwining relation $P \overset{\Lambda}{\hookrightarrow} \widetilde{P}$ holds. Using the fact that $\Lambda \in \mathcal{B}(\mathrm{L}^2(\mathbb{R}^d))$, we have on the one hand, for $f \in \mathrm{L}^2(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
\mathcal{F}_{P_t\Lambda f}(e^{-tB^*}\xi) &= \frac{1}{|\det(e^{tB})|} e^{-\langle Q_t\xi,\xi\rangle/2}\mathcal{F}_{\Lambda f}(\xi) \\
&= \frac{1}{|\det(e^{tB})|} e^{-\langle Q_t\xi,\xi\rangle/2} e^{-\langle(\widetilde{Q}_\infty - Q_\infty)\xi,\xi\rangle/2}\mathcal{F}_f(\xi) \\
&= \frac{1}{|\det(e^{tB})|} e^{\langle(Q_\infty - Q_t)\xi,\xi\rangle/2} e^{-\langle\widetilde{Q}_\infty\xi,\xi\rangle/2},
\end{aligned}$$

while on the other hand,

$$\begin{aligned}
\mathcal{F}_{\Lambda\widetilde{P}_t f}(e^{-tB^*}\xi) &= \lambda(e^{-tB^*}\xi)\mathcal{F}_{\widetilde{P}_t f}(e^{-tB^*}\xi) \\
&= \frac{1}{|\det(e^{tB})|} e^{-\langle(\widetilde{Q}_\infty - Q_\infty)e^{-tB^*}\xi, e^{-tB^*}\xi\rangle/2} e^{-\langle\widetilde{Q}_t\xi,\xi\rangle/2}\mathcal{F}_f(\xi),
\end{aligned}$$

where we used twice Lemma 5.4.5. Then, since

$$e^{-tB}Q_\infty e^{-tB^*} = \int_0^\infty e^{-(t+s)B}Qe^{-(t+s)B^*}ds = \int_t^\infty e^{-sB}Qe^{-sB^*}ds = Q_\infty - Q_t$$

we get that

$$e^{-\langle(\widetilde{Q}_\infty - Q_\infty)e^{-tB^*}\xi, e^{-tB^*}\xi\rangle/2} e^{-\langle\widetilde{Q}_t\xi,\xi\rangle/2} = e^{-\langle(\widetilde{Q}_\infty-\widetilde{Q}_t)\xi,\xi\rangle/2} e^{\langle(Q_\infty-Q_t)\xi,\xi\rangle/2} e^{-\langle\widetilde{Q}_t\xi,\xi\rangle/2}$$
$$= e^{\langle(Q_\infty-Q_t)\xi,\xi\rangle/2} e^{-\langle\widetilde{Q}_\infty\xi,\xi\rangle/2},$$

and thus we conclude that, for any $f \in \mathrm{L}^2(\mathbb{R}^d)$ and $t \geqslant 0$,

$$\mathcal{F}_{P_t\Lambda f}(e^{-tB^*}\xi) = \mathcal{F}_{\Lambda\widetilde{P}_t f}(e^{-tB^*}\xi).$$

By the L^2-isomorphism of the Fourier transform we then deduce that, for any $f \in \mathrm{L}^2(\mathbb{R}^d)$ and $t \geqslant 0$,

$$P_t\Lambda f = \Lambda\widetilde{P}_t f.$$

In particular, this holds for f belonging to $C_c^\infty(\mathbb{R}^d)$, the space of smooth, compactly supported functions on $\mathbb{R}^d$ and we have the inclusions $C_c^\infty(\mathbb{R}^d) \subset_d \mathrm{L}^2(\rho_\infty)$ and $C_c^\infty(\mathbb{R}^d) \subset_d \mathrm{L}^2(\widetilde{\rho}_\infty)$. Hence, by density and the continuity of all involved operators, the claimed intertwining also holds on $\mathrm{L}^2(\widetilde{\rho})$. □

In the following we write, for a vector $\alpha \in \mathbb{R}^d$, D_α for the diagonal matrix with diagonal entries given by α. For $\alpha, \delta \in \mathbb{R}^d$ we denote by $D_{\alpha\delta} = \mathrm{diag}(\alpha_1\delta_1, \ldots, \alpha_d\delta_d) = D_\alpha D_\delta$.

Proposition 5.4.2. *Let P be an Ornstein-Uhlenbeck semigroup associated to (Q, B), and suppose that $B = D_b$, with $b_i > 0$ for all i. For any $i \in \{1, \ldots, d\}$ set*

$$\alpha_i = q_{\infty,\min}\left(\frac{q_{\infty,\max}}{q_{\infty,\min}}\right)^{b_i/b_{\min}} \quad \text{and} \quad \delta_i = q_{\infty,\min},$$

where $q_{\infty,\min}$ and $q_{\infty,\min}$ are the largest and smallest eigenvalues of Q_∞, respectively, and $b_{\min}$ is the smallest eigenvalue of B.

(1) We have $D_\alpha \succ Q_\infty \succ D_\delta$, and there exist matrices $Q^{(\alpha)}, Q^{(\delta)} \succ 0$ such that

$$D_\alpha = \int_0^\infty e^{-sB}Q^{(\alpha)}e^{-sB}ds, \quad \text{and} \quad D_\delta = \int_0^\infty e^{-sB}Q^{(\delta)}e^{-sB}ds.$$

(2) *The Ornstein-Uhlenbeck semigroups* $P^{(\alpha)}$ *and* $P^{(\delta)}$ *associated to* $(Q^{(\alpha)}, B)$ *and* $(Q^{(\delta)}, B)$*, respectively, are self-adjoint and* $P \overset{\Lambda_\alpha}{\hookrightarrow} P^{(\alpha)}$, $P^{(\delta)} \overset{\Lambda_\delta}{\hookrightarrow} P$*, and* $P^{(\delta)} \overset{\Lambda_{\delta,\alpha}}{\hookrightarrow} P^{(\alpha)}$ *where, in the notation of Proposition 5.4.1,* $\Lambda_\alpha = \Lambda_{Q \to Q^{(\alpha)}}$, $\Lambda_\delta = \Lambda_{Q^{(\delta)} \to Q}$*, and* $\Lambda_{\delta,\alpha} = \Lambda_{Q^{(\delta)} \to Q^{(\alpha)}}$*. Hence,*

$$P \overset{\Lambda_\alpha}{\hookrightarrow} P^{(\alpha)} \overset{\Lambda^*_{\delta,\alpha}\Lambda_\delta}{\hookrightarrow} P \quad \textit{and} \quad P^{(\alpha)} \overset{\Lambda^*_{\delta,\alpha}\Lambda_\delta\Lambda_\alpha}{\hookrightarrow} P^{(\alpha)}.$$

(3) *For* $x, \xi \in \mathbb{R}^d$

$$\Lambda^*_{\delta,\alpha}\Lambda_\delta\Lambda_\alpha e_{i\xi}(x) = e^{-\langle D_{(\alpha^2-\delta^2)/\alpha}\xi,\xi\rangle/2} e_{i\xi}(D_{\frac{\delta}{\alpha}} x).$$

Consequently, with $\boldsymbol{t} = b_{\min}^{-1} \log \frac{q_{\infty,\max}}{q_{\infty,\min}}$,

$$\Lambda^*_{\delta,\alpha}\Lambda_\delta\Lambda_\alpha = P^{(\alpha)}_{\boldsymbol{t}}.$$

Proof. Writing I_d for the d-dimensional identity matrix we recall that, for the Löwner ordering of symmetric positive-definite matrices, $q_{\infty,\max} I_d \succ Q_\infty \succ q_{\infty,\min} I_d$. By definition of α it follows that the smallest eigenvalue of the diagonal matrix D_α is $q_{\infty,\max}$, from which we conclude that $D_\alpha \succ q_{\infty,\max} I_d \succ Q_\infty$. Next we recall that $\int_0^\infty e^{-s2B} ds = (2B)^{-1}$. Since B, D_α, and e^{-sB}, for any $s \geqslant 0$, are diagonal matrices it follows that they commute. Setting $Q^{(\alpha)} = D_\alpha + 2B = D_{\alpha+2b} \succ 0$ we get that

$$\int_0^\infty e^{-sB} Q^{(\alpha)} e^{-sB} ds = D_\alpha (2B) \int_0^\infty e^{-s2B} ds = D_\alpha.$$

Similarly, setting $Q^{(\delta)} = q_{\infty,\min} 2B$ we get that $\int_0^\infty e^{-sB} Q^{(\delta)} e^{-sB} ds = D_\delta$, which proves the first claim. The intertwinings $P \overset{\Lambda_\alpha}{\hookrightarrow} P^{(\alpha)}$, $P^\delta \overset{\Lambda_\delta}{\hookrightarrow} P$, and $P^\delta \overset{\Lambda_{\delta,\alpha}}{\hookrightarrow} P^{(\alpha)}$ then follow from Proposition 5.4.1 and the fact that $D_\alpha \succ Q_\infty \succ D_\delta$. The self-adjointness of $P^{(\alpha)}$ and $P^{(\delta)}$ is equivalent to the commutation identities $Q^{(\alpha)} B = B Q^{(\alpha)}$ and $Q^{(\delta)} B = B Q^{(\delta)}$, respectively, see e.g. [81, Proposition 9.3.10]. Taking the adjoint of the identity $P^{(\delta)} \overset{\Lambda_{\alpha,\delta}}{\hookrightarrow} P^{(\alpha)}$ and using the self-adjointness of $P^{(\alpha)}$ and $P^{(\delta)}$ then yields $P^{(\alpha)} \overset{\Lambda^*_{\alpha,\delta}}{\hookrightarrow} P^{(\delta)}$, which combined with the aforementioned intertwinings finishes the proof of the second claim. For the last claim we get, from the definition of Λ_α and Λ_δ that, for all $x, \xi \in \mathbb{R}^d$,

$$\begin{aligned}\Lambda_\delta \Lambda_\alpha e_{i\xi}(x) &= e^{-\langle (D_\alpha - Q_\infty)\xi,\xi\rangle/2} \Lambda_\delta e_{i\xi}(x) e^{-\langle (D_\alpha - Q_\infty)\xi,\xi\rangle/2} e^{-\langle (Q_\infty - D_\delta)\xi,\xi\rangle/2} e_{i\xi}(x) \\ &= e^{-\langle D_{\alpha-\delta}\xi,\xi\rangle/2} e_{i\xi}(x).\end{aligned}$$

Next, we shall characterize the adjoint operator $\Lambda^*_{\delta,\alpha}$. To this end we note that, since $D_\alpha \succ D_\delta \succ 0$, the invariant measures of $P^{(\alpha)}$ and $P^{(\delta)}$ admit Gaussian densities, which we denote by $\rho^{(\alpha)}_\infty$ and $\rho^{(\delta)}_\infty$, respectively. Let us formally define, for $f \in \mathrm{L}^2(\rho^{(\delta)}_\infty)$, the operator $\Lambda^*_{\delta,\alpha} : \mathrm{L}^2(\rho^{(\delta)}_\infty) \to \mathrm{L}^2(\rho^{(\alpha)}_\infty)$ by

$$\Lambda^*_{\delta,\alpha} f(x) = \frac{1}{\rho^{(\alpha)}_\infty(x)} (f\rho^{(\delta)}_\infty) * \rho_{\delta,\alpha}(x), \quad x \in \mathbb{R}^d, \tag{5.12}$$

where $\rho_{\delta,\alpha}$ is the Gaussian density satisfying $\mathcal{F}_{\rho_{\delta,\alpha}}(\xi) = e^{-\langle (D_\alpha - D_\delta)\xi,\xi\rangle/2} = e^{-\langle D_{\alpha-\delta}\xi,\xi\rangle/2}$, which is well-defined due to $D_\alpha - D_\delta = D_{\alpha-\delta} \succ 0$. Then, for non-negative functions $f \in \mathrm{L}^2(\rho^{(\delta)}_\infty)$ and $g \in \mathrm{L}^2(\rho^{(\alpha)}_\infty)$,

$$\begin{aligned}\langle \Lambda^*_{\delta,\alpha} f, g\rangle_{\mathrm{L}^2(\rho^{(\alpha)}_\infty)} &= \langle (f\rho^{(\delta)}_\infty) * \rho_{\delta,\alpha}, g\rangle_{\mathrm{L}^2(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(y)\rho^{(\delta)}_\infty(y)\rho_{\delta,\alpha}(x-y)dy \right) g(x)dx \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} g(x)\rho_{\delta,\alpha}(y-x)dx \right) f(y)\rho^{(\delta)}_\infty(y)dy \\ &= \langle f, \Lambda_{\delta,\alpha} g\rangle_{\mathrm{L}^2(\rho^{(\delta)}_\infty)}\end{aligned}$$

where we used Fubini's theorem and the symmetry of the density $\rho_{\delta,\alpha}$. By decomposing any $f \in \mathrm{L}^2(\rho^{(\delta)}_\infty)$ and $g \in \mathrm{L}^2(\rho^{(\alpha)}_\infty)$ into the difference of non-negative functions it follows that the above holds for all $f \in \mathrm{L}^2(\rho^{(\delta)}_\infty)$ and $g \in \mathrm{L}^2(\rho^{(\alpha)}_\infty)$, so that indeed $\Lambda^*_{\delta,\alpha}$ is the $\mathrm{L}^2(\rho^{(\delta)}_\infty)$ adjoint of the operator $\Lambda_{\delta,\alpha}$. By substituting the expression for the densities in

(5.12) we find that

$$
\begin{aligned}
\Lambda^*_{\delta,\alpha} e_{i\xi}(x) &= (2\pi)^{-d/2}\sqrt{\frac{\det D_\alpha}{\det D_{\delta(\alpha-\delta)}}}\, e^{\langle D_\alpha^{-1}x,x\rangle/2} \\
&\quad \times \int_{\mathbb{R}^d} e^{i\langle \xi, y\rangle} e^{-\langle D_\delta^{-1}y,y\rangle/2} e^{-\langle (D_{\alpha-\delta})^{-1}(x-y),(x-y)\rangle/2} dy \\
&= \frac{(2\pi)^{-d/2}}{(\det D_{\frac{\delta(\alpha-\delta)}{\alpha}})^{1/2}} e^{\langle (D_{\frac{1}{\alpha}} - (D_{\alpha-\delta})^{-1})x,x\rangle/2} \\
&\quad \times \int_{\mathbb{R}^d} e^{\langle (D_{\alpha-\delta})^{-1}x + i\xi, y\rangle} e^{-\langle (D_\delta^{-1} + (D_{\alpha-\delta})^{-1})y,y\rangle/2} dy \\
&= \frac{(2\pi)^{-d/2}}{(\det D_{\frac{\delta(\alpha-\delta)}{\alpha}})^{1/2}} e^{-\langle D_{\frac{\delta}{\alpha(\alpha-\delta)}}x,x\rangle/2} \int_{\mathbb{R}^d} e^{\langle D_{\frac{1}{\alpha-\delta}}x + i\xi, y\rangle} e^{-\langle D^{-1}_{\frac{\delta(\alpha-\delta)}{\alpha}}y,y\rangle/2} dy \\
&= e^{-\langle D_{\frac{\delta}{\alpha(\alpha-\delta)}}x,x\rangle/2} e^{\langle D_{\frac{\delta(\alpha-\delta)}{\alpha}}(D_{\frac{1}{\alpha-\delta}}x + i\xi), D_{\frac{1}{\alpha-\delta}}x + i\xi\rangle/2} \\
&= e^{-\langle D_{\frac{\delta}{\alpha(\alpha-\delta)}}x,x\rangle/2} e^{\langle D_{\frac{\delta}{\alpha(\alpha-\delta)}}x,x\rangle/2} e^{i\langle \xi, D_{\frac{\delta}{\alpha}}x\rangle} e^{-\langle D_{\frac{\delta(\alpha-\delta)}{\alpha}}\xi,\xi\rangle/2} \\
&= e^{i\langle \xi, D_{\frac{\delta}{\alpha}}x\rangle} e^{-\langle D_{\frac{\delta(\alpha-\delta)}{\alpha}}\xi,\xi\rangle/2},
\end{aligned}
$$

where in the second equality we expanded the quadratic form, and we repeatedly used some standard properties of diagonal matrices. Putting things together, we deduce that

$$
\Lambda^*_{\delta,\alpha}\Lambda_\delta\Lambda_\alpha e_{i\xi}(x) = e^{i\langle \xi, D_{\frac{\delta}{\alpha}}x\rangle} e^{-\langle (D_{\alpha-\delta} + D_{\frac{\delta(\alpha-\delta)}{\alpha}})\xi,\xi\rangle/2} = e^{i\langle \xi, D_{\frac{\delta}{\alpha}}x\rangle} e^{-\langle D_{(\alpha^2-\delta^2)/\alpha}\xi,\xi\rangle/2}.
$$

Next, we note that

$$
P_{\boldsymbol{t}}^{(\alpha)} e_{i\xi}(x) = e^{i\langle \xi, e^{-\boldsymbol{t}B}x\rangle} e^{-\langle Q_{\boldsymbol{t}}^{\alpha}\xi,\xi\rangle/2}.
$$

Since $B = D_b$ we get, by definition of α, δ and $\boldsymbol{t}$, that the identity $e^{-\boldsymbol{t}D_b} = D_{\frac{\delta}{\alpha}}$ is satisfied. Using the identity $Q_{\boldsymbol{t}}^{(\alpha)} = Q_\infty^{(\alpha)} - e^{-\boldsymbol{t}D_b} Q_\infty^{(\alpha)} e^{-\boldsymbol{t}D_b} = D_\alpha - e^{-\boldsymbol{t}D_b} D_\alpha e^{-\boldsymbol{t}D_b}$ and substituting $e^{-\boldsymbol{t}D_b} = D_{\frac{\delta}{\alpha}}$ we find that

$$
Q_{\boldsymbol{t}}^{(\alpha)} = D_\alpha - D_{\frac{\delta}{\alpha}} D_\alpha D_{\frac{\delta}{\alpha}} = D_\alpha - D_{\delta^2/\alpha} = D_{(\alpha^2-\delta^2)/\alpha}.
$$

This gives that $\Lambda^*_{\delta,\alpha}\Lambda_\delta\Lambda_\alpha e_{i\xi}(x) = P_{\boldsymbol{t}}^{(\alpha)} e_{i\xi}(x)$ and, as the Fourier transform uniquely characterizes probability measures, the proof is complete. □

We are now able to give the proof of Theorem 5.3.1.

Proof of Theorem 5.3.1. Since B is diagonalizable with similarity matrix V we have that $VBV^{-1} = D_b$, where $b \in \mathbb{R}^d$ is the vector of eigenvalues of B with $b_i > 0$ for all i. Under this change of coordinates, (Q, B) gets mapped to (VQV^*, D_b) and a simple calculation shows that Q_∞ then gets mapped to $VQ_\infty V^*$. The change of coordinates map $\Phi_V f(x) = f(V^{-1}x)$ is a unitary operator from $\mathrm{L}^2(\rho_\infty)$ to $\mathrm{L}^2(\rho_\infty^{\Phi_V})$, where $\rho_\infty^{\Phi_V}$ denotes the image density of ρ_∞ under Φ_V, i.e. for $x \in \mathbb{R}^d$, $\rho_\infty^{\Phi_V}(x) = \frac{1}{|\det V|}\rho_\infty(\Phi_V(x))$. Hence if we prove the desired result for the Ornstein-Uhlenbeck semigroup $\overline{P}$ associated to (VQV^*, D_b) then, since $P_t = \Phi_V{}^{-1}\overline{P}_t\Phi_V$ we get, by Proposition 5.2.1 and the unitarity of Φ_V, that the claims hold for the Ornstein-Uhlenbeck semigroup P associated to (Q, B). Thus, we suppose that $B = D_b$ with $b_i > 0$ for all i. We aim to invoke Theorem 5.2.2 and to this end, since B is diagonal, Proposition 5.4.2(2) furnishes the intertwinings $P \overset{\Lambda^\alpha}{\hookrightarrow} P^{(\alpha)} \overset{\Lambda^*_{\delta,\alpha}\Lambda_\delta}{\hookrightarrow} P$, where $P^{(\alpha)}$ is the non-degenerate Ornstein-Uhlenbeck semigroup associated to $(Q^{(\alpha)}, B)$, see the notation therein. From Proposition 5.4.1 we have that Λ_α, Λ_δ, and $\Lambda_{\delta,\alpha}$ are quasi-affinities, and hence $\Lambda^*_{\delta,\alpha}\Lambda_\delta$ is also a quasi-affinity which proves that the intertwinings are proper. Next, Proposition 5.4.2(3) gives that the function $m : \sigma(\mathbf{A}^{(\alpha)}) \to \mathbb{C}$, in the notation of Theorem 5.2.2 and where $-\mathbf{A}^{(\alpha)}$ is the generator of $P^{(\alpha)}$, is given by

$$m(\gamma) = e^{-\gamma \boldsymbol{t}},$$

with $\boldsymbol{t} = \frac{1}{b_{\min}} \log \frac{q_{\infty,\max}}{q_{\infty,\min}} = \frac{1}{b_{\min}} \log \kappa(Q_\infty)$. However, from [84, Theorem 3.4] we get that $\boldsymbol{\gamma}_1 = b_{\min}$, and thus $\boldsymbol{t} = \frac{1}{\boldsymbol{\gamma}_1} \log \kappa(Q_\infty)$ as claimed. Now, for any $t > \boldsymbol{t}$,

$$\gamma \mapsto \frac{e^{-\gamma t}}{m(\gamma)} = e^{-\gamma(t-\boldsymbol{t})} \in \mathrm{L}^\infty(\sigma(\mathbf{A}^{(\alpha)})),$$

and plainly

$$\sup_{\mathrm{Re}(\gamma) \geqslant \boldsymbol{\gamma}_1} e^{-\gamma(t-\boldsymbol{t})} = e^{-\boldsymbol{\gamma}_1(t-\boldsymbol{t})}.$$

Next, Proposition 5.4.1 gives that $\|\Lambda_\alpha\|_{\mathrm{L}^2(\rho_\infty^{(\alpha)}) \to \mathrm{L}^2(\rho_\infty)} = 1$ and $\left\|\Lambda^*_{\delta,\alpha}\Lambda_\delta\right\|_{\mathrm{L}^2(\rho_\infty) \to \mathrm{L}^2(\rho_\infty^{(\alpha)})} \leqslant 1$. To deduce that $\left\|\Lambda^*_{\delta,\alpha}\Lambda_\delta\right\|_{\mathrm{L}^2(\rho_\infty) \to \mathrm{L}^2(\rho_\infty^{(\alpha)})} = 1$ it suffices to observe that $\Lambda^*_{\delta,\alpha}\mathbf{1} = \mathbf{1}$, which follows from Equation (5.12) and the identity $\rho_\infty^{(\delta)} * \rho_{\delta,\alpha} = \rho_\infty^{(\alpha)}$, with the notation therein. Consequently, invoking Theorem 5.2.2 we deduce that, for any $f \in \mathrm{L}^2(\rho_\infty)$ and $t > \boldsymbol{t}$,

$$\|P_t f - P_\infty f\|_{\mathrm{L}^2(\rho_\infty)} \leqslant e^{-\gamma_1(t-\boldsymbol{t})} \|f - P_\infty f\|_{\mathrm{L}^2(\rho_\infty)} = \kappa(Q_\infty) e^{-\gamma_1 t} \|f - P_\infty f\|_{\mathrm{L}^2(\rho_\infty)}.$$

However, for $0 \leqslant t \leqslant \boldsymbol{t}$ we have that $\kappa(Q_\infty)e^{-\gamma_1 t} \geqslant 1$ and thus, by the contractivity of P on $\mathrm{L}^2(\rho_\infty)$, it follows that the hypocoercive estimate holds for all $t \geqslant 0$. Finally, as remarked before the theorem, we have that $P_\infty f(x) = \rho_\infty[f] = \int_{\mathbb{R}^d} f(x)\rho_\infty(x)dx$. $\square$

5.4.7 Proof of Theorem 5.3.2

In this proof we use standard properties of tensor products of semigroups and generators, see for instance [9, Section 1.15.3]. Let us write $P^{(i)}$ for the one-dimensional factors of the product semigroup P. By Proposition 4.3.6 we get that, for each $i = 1, \dots, d$, there exists a one-dimensional classical Jacobi semigroup $\widetilde{P}^{(\mathfrak{m},i)} = (e^{-t\widetilde{\mathbf{A}}_{\mathfrak{m},i}})_{t \geqslant 0}$ on $\mathrm{L}^2(\beta_{\mathfrak{m}})$ such that $P^{(i)} \overset{\Lambda_{\mathfrak{m},i}}{\hookrightarrow} \widetilde{P}^{(\mathfrak{m},i)} \overset{\widetilde{\Lambda}_{\mathfrak{m},i}}{\hookrightarrow} P^{(i)}$, where $\Lambda_{\mathfrak{m},i} \in \mathcal{B}(\mathrm{L}^2(\beta_{\mathfrak{m}}), \mathrm{L}^2(\beta_i))$ and $\widetilde{\Lambda}_{\mathfrak{m},i} \in \mathcal{B}(\mathrm{L}^2(\beta_i), \mathrm{L}^2(\beta_{\mathfrak{m}}))$ are quasi-affinities with operator norm 1, such that

$$\widetilde{\Lambda}_{\mathfrak{m},i}\Lambda_{\mathfrak{m},i} = F_\phi(\widetilde{\mathbf{A}}_{\mathfrak{m},i}),$$

see Proposition 3.5 and Lemma 3.10, respectively, of the same paper, and where we note that the quantity $\boldsymbol{d}$, in the notation therein, may be taken to be 1. Since the parameter $\mathfrak{m}$ is common to all factors of the product semigroup we get, by tensorization, the intertwinings $P \overset{\Lambda_{\mathfrak{m}}}{\hookrightarrow} \widetilde{P}^{(\mathfrak{m})} \overset{\widetilde{\Lambda}_{\mathfrak{m}}}{\hookrightarrow} P$, where $\Lambda_{\mathfrak{m}}$ acts on $f \in \mathrm{L}^2(\beta)$ via $\Lambda_{\mathfrak{m}} f(x) = \Lambda_{\mathfrak{m},1} f_1(x_1) \cdots \Lambda_{\mathfrak{m},d} f_d(x_d)$, and similarly for $\widetilde{\Lambda}_{\mathfrak{m}}$, and plainly both $\Lambda_{\mathfrak{m}}$ and $\widetilde{\Lambda}_{\mathfrak{m}}$ are also quasi-affinities, hence proper linear operators. Next, the fact that $F_{\mathfrak{m}} \in \mathrm{L}^\infty(\sigma(\widetilde{\mathbf{A}}_{\mathfrak{m}}))$ follows from $F_{\mathfrak{m}}$ being the Laplace transform of a probability measure on $[0, \infty)$, see Section 3.7 in the same paper, so that $|F_{\mathfrak{m}}(\gamma)| < \infty$ for any $\mathrm{Re}(\gamma) \geqslant 0$. Recall that $\sigma(\widetilde{\mathbf{A}}_{\mathfrak{m}}) = \{\gamma_n; n \in \mathbb{N}\}$ where

$\gamma_n = n(n-1) + \boldsymbol{\gamma}_1 n$ and thus, for $\boldsymbol{\gamma}_1 t \geqslant \log \frac{\boldsymbol{\gamma}_1(\mathfrak{m}+1)}{2(\boldsymbol{\gamma}_1-\mathfrak{m}+1)}$,

$$\sup_{\gamma\geqslant\boldsymbol{\gamma}_1} \frac{e^{-\gamma t}}{F_{\mathfrak{m}}(\gamma)} = \sup_{n\geqslant 1} \frac{e^{-\gamma_n t}}{F_{\mathfrak{m}}(\gamma_n)} \leqslant \sup_{n\geqslant 1} \frac{e^{-n\boldsymbol{\gamma}_1 t}}{F_{\mathfrak{m}}(\gamma_n)} \leqslant \mathfrak{m}\frac{(\boldsymbol{\gamma}_1 - 1)}{(\boldsymbol{\gamma}_1 - \mathfrak{m})} e^{-\boldsymbol{\gamma}_1 t},$$

see (4.59). Hence, by Theorem 5.2.2, we deduce the convergence to equilibrium estimate

$$\|P_t f - P_\infty f\|_{\mathrm{L}^2(\beta)} \leqslant \mathfrak{m}\frac{(\boldsymbol{\gamma}_1 - 1)}{(\boldsymbol{\gamma}_1 - \mathfrak{m})} e^{-\boldsymbol{\gamma}_1 t} \, \|f - P_\infty f\|_{\mathrm{L}^2(\beta)}, \tag{5.13}$$

which is valid for all $f \in \mathrm{L}^2(\beta)$ and $t \geqslant \frac{1}{\boldsymbol{\gamma}_1} \log \frac{\boldsymbol{\gamma}_1(\mathfrak{m}+1)}{2(\boldsymbol{\gamma}_1-\mathfrak{m}+1)}$. However, for any $0 \leqslant t < \frac{1}{\boldsymbol{\gamma}_1} \log \frac{\boldsymbol{\gamma}_1(\mathfrak{m}+1)}{2(\boldsymbol{\gamma}_1-\mathfrak{m}+1)}$ it is straightforward to check that the constant in front of the exponential in (5.13) is strictly greater than 1 so that, by the contractivity of P, the estimate (5.13) holds for all $f \in \mathrm{L}^2(\beta)$ and $t \geqslant 0$. Recalling that $P_\infty f = \beta[f] = \int_{[0,1]^d} f(x)\beta(x)dx$ we complete the proof of the claimed hypocoercive estimate.

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