CONSTRUCTION OF ORTHOGONAL F-SQUARES OF ORDER n = qk

BU-565-M W. T. Federer July 1975

Abstract

A method of constructing sets of orthogonal F-squares with two subsets of F-squares, i.e., two different numbers of symbols, is presented and illustrated with examples. The F-squares are of order n, and the cases for $n = 2k$, $n = 3k$, and $n = qk$ are discussed.

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1. Introduction

Orthogonality of latin squares and of F-squares contains many unexplored facets. Some of the explored facets have been discussed by Hedayat, Raghavarao, and Seiden [1973], by Mandeli [1975], and by Federer [1975 a,b]. The present paper presents results along the lines of these works. In particular, it is shown how to construct a set of mutually orthogonal F-squares. The set in no way forms a complete set, but it could form the basis for constructing additional F-squares by using the procedure given in Federer [1975b].

2. Construction for n = 2k

Let n = 2k, and if a set of t orthogonal latin squares of order n/2 exists, one may construct t orthogonal F-squares as follows:

$$\sum_{i=1}^{t} L_i(n/2) \underline{\bar{X}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = L_1(n/2) \underline{\bar{X}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + L_2(n/2) \underline{\bar{X}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \cdots + L_t(n/2) \underline{\bar{X}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

where $L_i(n/2)$ is a latin square of order n/2 and $\underline{\bar{X}}$ denotes Kronecker product. In addition, the F-square obtained by $J_{n/2 \times n/2} \underline{\bar{X}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is orthogonal to the above t F-squares.

To illustrate the above, let n = 10 and t = 4 to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix} \underline{\bar{X}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \\ 5 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{bmatrix} \underline{\bar{X}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix} \underline{\bar{X}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$
+\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix} \bar{\underline{X}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} .
$$

The following F-square with two symbols, i.e., $F(A_1^5, A_2^5)$, is orthogonal to the above four F-squares, $F(A_1^2, A_2^2, A_3^2, A_4^2, A_5^2)$, with

$$
\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \bar{\underline{X}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .
$$

3. Construction for n = 3k

If a set of t mutually orthogonal latin squares of order $k = n/3$, $k \neq$ to a multiple of 3, exists, then one may obtain a set of t mutually orthogonal F-squares with n/3 symbols, i.e., $F(A_1^3, A_2^3, \cdots A_{n/3}^3)$, as

$$
\sum_{i=1}^{t} L_i(n/3) \bar{\underline{X}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \sum_{i=1}^{t} L_i(n/3) \bar{\underline{X}} J_{3x3}
$$

where $L_i(n/3)$ is a latin square of order n/3 in the set of t orthogonal latin squares. In addition, the following square is orthogonal to the above t squares

$$
\begin{matrix} O_{3x3} & J_{3x3} & J_{3x3} \\ J_{3x3} & O_{3x3} & J_{3x3} \\ J_{3x3} & J_{3x3} & O_{3x3} \end{matrix}
$$

where $J_{3\times3}$ is a 3×3 matrix of ones and $O_{3\times3}$ is a 3×3 matrix of zeros. This forms an $F(A_1^{2n/3}, A_2^{n/3})$-square.

If k is a multiple of 3, say k = 3m, one may construct orthogonal F-squares in a simple manner. If a set of t orthogonal latin squares of order n/9 exist, then one may construct orthogonal F-squares as follows:

$$\sum_{i=1}^{t} L_i(n/9) \bar{\underline{X}} J_{9\times9}$$

where $J_{9\times9}$ is a matrix of order 9 with all elements equal to unity. The following F-square will be orthogonal to this set:

$$\begin{matrix} O_{9\times9} & J_{9\times9} & J_{9\times9} \\ J_{9\times9} & O_{9\times9} & J_{9\times9} \\ J_{9\times9} & J_{9\times9} & O_{9\times9} \end{matrix}$$

It will be of the type $F(A_1^{2n/3}, A_2^{n/3})$.

The procedure may be extended to the case where 3^p is a multiple of k in the manner described above. However, in investigations on complete sets of orthogonal F-squares, F-squares with a maximum number of symbols may be of more interest. The last procedure above results in the minimum number, $n/3^p$, of symbols after removing multiples of 3.

4. Construction for n = qk

Obviously, the above procedure of constructing F-square may be extended to the case where n = qk. If t orthogonal latin squares of order k exist, then a set of t orthogonal F-squares of order n may be constructed as follows.

$$\sum_{i=1}^{t} L_i(k)\bar{\underline{X}}J_{q\times q} \quad .$$

Further, suppose that ℓ latin squares of order q exist. Then, the following F-squares are orthogonal to the above F-squares and to each other:

$$\sum_{i=1}^{\ell} L_i(q)\bar{\underline{X}}J_{k\times k} \quad .$$

An example will illustrate the above; let n = 20, t = 4, ℓ = 3, and q = 4. Let $L_1(5)$, $L_2(5)$, $L_3(5)$, and $L_4(5)$ be the four orthogonal latin squares of order 5. Furthermore, let $L_1(4)$, $L_2(4)$, and $L_3(4)$ be the three orthogonal latin squares of order 4. Then the ijth $F(A_1^5,A_2^5,A_3^5,A_4^5)$ for j = 1, is constructed as follows:

1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4
1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4
1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4
1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4
1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4
2	2	2	2	2	1	1	1	1	1	4	4	4	4	4	3	3	3	3	3
2	2	2	2	2	1	1	1	1	1	4	4	4	4	4	3	3	3	3	3
2	2	2	2	2	1	1	1	1	1	4	4	4	4	4	3	3	3	3	3
2	2	2	2	2	1	1	1	1	1	4	4	4	4	4	3	3	3	3	3
2	2	2	2	2	1	1	1	1	1	4	4	4	4	4	3	3	3	3	3
3	3	3	3	3	4	4	4	4	4	1	1	1	1	1	2	2	2	2	2
3	3	3	3	3	4	4	4	4	4	1	1	1	1	1	2	2	2	2	2
3	3	3	3	3	4	4	4	4	4	1	1	1	1	1	2	2	2	2	2
3	3	3	3	3	4	4	4	4	4	1	1	1	1	1	2	2	2	2	2
3	3	3	3	3	4	4	4	4	4	1	1	1	1	1	2	2	2	2	2
4	4	4	4	4	3	3	3	3	3	2	2	2	2	2	1	1	1	1	1
4	4	4	4	4	3	3	3	3	3	2	2	2	2	2	1	1	1	1	1
4	4	4	4	4	3	3	3	3	3	2	2	2	2	2	1	1	1	1	1
4	4	4	4	4	3	3	3	3	3	2	2	2	2	2	1	1	1	1	1
4	4	4	4	4	3	3	3	3	3	2	2	2	2	2	1	1	1	1	1

Thus, there are four $F(A_1^4,A_2^4,A_3^4,A_4^4,A_5^4)$-squares plus three $F(A_1^5,A_2^5,A_3^5,A_4^5)$-squares which are mutually orthogonal.

5. Literature Cited

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