DOUBLE INTEGRATION WITH RESPECT TO SYMMETRIC STABLE PROCESSES

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Abstract

We obtain analytic conditions on a nonrandom function f which are necessary and sufficient for the existence of the double integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,t)dX_s dX_t$, where X_t is the Lévy symmetric stable process of index α satisfying $1 < \alpha \leq 2$. The precise condition is that the integral operator having kernel f should define a completely summing map from $L^{\alpha'}(R)$ to $L^{\alpha}(R)$ $(1/\alpha + 1/\alpha' = 1)$. We also obtain bounds on the pth absolute moments of the integral for all $0 < p < \alpha/2$.

Central to our method and of independent interest is the following decoupling inequality for random bilinear forms: for each $1 < \alpha < 2$ and $1 < p < \alpha$ there is a constant $C(p,\alpha)$ such that for every $n \geq 2$ and bilinear form B on R^n we have

$$E\left|B(\underset{\sim}{X},\underset{\sim}{X})\right|^p \leq C(p,\alpha)E\left|B(\underset{\sim}{X},\underset{\sim}{Y})\right|^p,$$

where $\underset{\sim}{X} = (X_1,\ldots,X_n)$ is a random vector with i.i.d. symmetric α-stable components and $\underset{\sim}{Y}$ is an independent copy of $\underset{\sim}{X}$. Furthermore an analogue of this result is shown to hold for multilinear forms of each rank.

AMS 1980 Subject Classification. Primary 60H05; Secondary 60J30, 60B11

Key Words and Phrases: Multiple Wiener integral, stochastic integration, symmetric stable processes, random quadratic forms, p-summing maps, radonifying maps.

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Abstract

We obtain analytic conditions on a nonrandom function f which are necessary and sufficient for the existence of the double integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,t)dX_s dX_t$, where X_t is the Lévy symmetric stable process of index α satisfying $1 < \alpha \leq 2$. The precise condition is that the integral operator having kernel f should define a completely summing map from $L^{\alpha'}(R)$ to $L^{\alpha}(R)$ $(1/\alpha + 1/\alpha' = 1)$. We also obtain bounds on the pth absolute moments of the integral for all $0 < p < \alpha/2$.

Central to our method and of independent interest is the following decoupling inequality for random bilinear forms: for each $1 < \alpha < 2$ and $1 < p < \alpha$ there is a constant $C(p,\alpha)$ such that for every $n \geq 2$ and bilinear form B on R^n we have

$$E\left|B(\underset{\sim}{X},\underset{\sim}{X})\right|^p \leq C(p,\alpha)E\left|B(\underset{\sim}{X},\underset{\sim}{Y})\right|^p,$$

where $\underset{\sim}{X} = (X_1,\ldots,X_n)$ is a random vector with i.i.d. symmetric α-stable components and $\underset{\sim}{Y}$ is an independent copy of $\underset{\sim}{X}$. Furthermore an analogue of this result is shown to hold for multilinear forms of each rank.

1. Introduction and statement of main results

We investigate the integral

$$J(f) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(s,t)\,dX_s\,dX_t \tag{1.1}$$

where X_t is a symmetric stable process with index α, satisfying $1 < \alpha < 2$. We obtain conditions for the integral to exist based on summing properties of the integral operator having integral kernel f. These conditions are best possible in a sense to be made precise below. We also obtain bounds on moments of $J(f)$. A forthcoming paper will contain further information about the double integral as well as extensions to higher order multiple integrals.

The process $\{X_t, t \geq 0\}$ has stationary independent increments and the characteristic function of X_1 is $Ee^{iuX_1} = e^{-a|u|^\alpha}$. Without loss of generality we take $a = 1$. We extend X_t to all of R by choosing an independent copy $\{\tilde{X}_t, t \geq 0\}$ of $\{X_t, t \geq 0\}$ and setting $X_{-t} = -\tilde{X}_t$.

For $0 < \alpha \leq 2$, the single integral $\int_{-\infty}^{+\infty} f(t)dX_t$ exists if and only if $f \in L^\alpha(R^1)$ (see Schilder (1970)). When $\alpha = 2$, the process X_t is Brownian motion. In that case, one approach allows the double integral to be defined for any L^2 function as an L^2 limit of suitable simple functions (see Itô (1951)). Contrary to expectation, example 1 of Section 2 shows that for $1 < \alpha < 2$ the analogous condition $f \in L^\alpha(R^2)$ is not sufficient for the double integral to exist.

Our method is based on the following decoupling inequalities which may be of independent interest. They are established in Section 3.

Theorem 1.1 (Decoupling Inequalities). Let $M_1, M_2, \ldots$ be i.i.d. symmetric stable random variables with index α satisfying $1 < \alpha \leq 2$. For each $k = 1,2,\ldots$ let $\{M_i^k\}$ be an independent copy of the sequence $\{M_i\}$. Then for each integer $r \geq 1$, and for every p satisfying $1 < p < \infty$, there are constants $C(p,r)$ and $C'(p,r)$ such that for any Z_+^r-indexed family of real numbers $a_{i_1 i_2 \ldots i_r}$, all but finitely many of which are zero, we have

$$\begin{aligned}
&C(p,r)E\Big|\sum_{i_1<i_2<\ldots<i_r} a_{i_1 i_2 \ldots i_r} M_{i_1} M_{i_2} \ldots M_{i_r}\Big|^p \\
(1.2)\qquad &\leq E\Big|\sum_{i_1<i_2<\ldots<i_r} a_{i_1 i_2 \ldots i_r} M_{i_1}^1 \ldots M_{i_r}^r\Big|^p \\
&\leq C'(p,r)E\Big|\sum_{i_1<\ldots<i_r} a_{i_1 i_2 \ldots i_r} M_{i_1} \ldots M_{i_r}\Big|^p
\end{aligned}$$

Remark. It is possible to choose

$$C(p,r) = \left(\frac{18^2 p^3}{(p-1)^{3/2}}\, 2^{(r-1)p}\right)^r,$$

and, when $\alpha = 2$,

$$C(p,r) = \left(\frac{18^2 p^3}{(p-1)^{3/2}}\right)^{r-1}.$$

No attempt has been made to obtain the best constants. The expression for $C(p,r)$ is based in part on the constants appearing in Burkholder's square inequality (see for example Burkholder (1973), Theorem 3.2). Note that the inequality (1.2) holds trivially when $\alpha < 2$ and $p \geq \alpha$. Extensions of the decoupling inequalities to $\alpha \leq 1$ and to other infinitely divisible random variables will appear elsewhere.

We shall define the double integral $J(f)$ for functions f which satisfy $f(s,t) = f(t,s)$ and further conditions to be specified below. The symmetry assumption is for notational convenience only: for general f we may write $f = f_s + f_a$, where f_s is symmetric and f_a antisymmetric under interchange of arguments. The ordinary (nonstochastic) double integral of an antisymmetric integrable function is zero. Analogously, set $J(f) = J(f_s)$ whenever $J(f_s)$ and $J(|f_a|)$ are well-defined.

We are aware of three natural approaches to the problem of defining the double stochastic integral for symmetric integrands.

In the first approach one seeks a Banach space B of measurable functions and a constant $0 < p < \infty$ for which it is possible to prove an inequality of the form

$$(1.3) \qquad (E\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,t)dX_s dX_t\right|^p)^{1/p} \le C|f|_B,$$

for "simple" functions f. Here, and subsequently, $|\ |_B$ denotes the norm of B and C is a constant independent of f. If the functions f for which (1.3) holds are dense in B then the double integral may be defined by completion for all f in B. Spaces B suitable for a wide class of applications have been introduced by Surgailis (1981) in the case $1 < \alpha < 2$. Specifically, define $\sigma'_{\alpha,\varepsilon}$ by

$$(1.4) \quad \sigma'_{\alpha,\varepsilon}(f) = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |f(s,t)|^{\alpha+\varepsilon} ds\right)^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} dt + |f|_{L^{\alpha-\varepsilon}(R^2)} + |f|_{L^{\alpha+\varepsilon}(R^2)}$$

for $1 < \alpha < 2$ and $\varepsilon > 0$ such that $1 \le \alpha-\varepsilon < \alpha+\varepsilon \le 2$. Surgailis obtains an analogue of (1.3) for the space B of symmetric functions satisfying $\sigma'_{\alpha,\varepsilon}(f) < \infty$ for each such α and ε.

The second approach uses the fact that X_t is a semimartingale and the double integral is defined as an iterated Itô-type stochastic integral. While no-one seems to have applied this approach to the double integral in the stable non-Gaussian case, Kallenberg (1975) has developed an Itô integral for general Lévy processes. In the case of stable processes with index α in the range $1 < \alpha < 2$, the integral of Kallenberg is defined for the widest possible class of integrands.

The third approach has been developed by Neveu (1968) in the Gaussian case (see also the appendix of Mandelbaum and Taqqu (1984).) Briefly, one may define the integral directly for functions of the form $f(t,s) = g(t)h(s)$ and then for general f via a suitable orthogonal expansion of the form $f(t,s) = \sum_{i,j=1}^{\infty} c_{ij}\phi_i(s)\phi_j(t)$. In the non-$L^2$ case these methods are severely limited by the complicated geometry of L^α. Nevertheless, some success along these lines has been achieved by Szulga and Woyczinski (1983).

Our approach is a mixture of the first two. We define the double integral of a function vanishing on the diagonal as an iterated Itô integral using the work of Kallenberg (1975). The class of such integrands, Λ_α', is defined precisely below. We show that this class may be normed in such a way that it becomes a Banach space (Theorem 1.2); moreover the class of all step functions is dense in Λ_α' and an inequality having the form of (1.3) holds (Theorem 1.3).

Since we also obtain the reverse inequality to (1.3), our class of integrands is the largest Banach space of Lebesgue measurable functions for which it is possible to define $J(f)$ as a limit in mean of integrals of step functions as in approach (1).

At the same time, the class Λ_α' is the largest possible class to which Kallenberg's construction applies. Therefore our approach is also best possible amongst those of the second type.

Kallenberg proves that for any predictable process V_t satisfying

$$(1.5) \qquad P(\omega: \int_0^{\infty} |V_t(\omega)|^{\alpha} dt < \infty) = 1$$

there exists a sequence of simple predictable processes $V_n(t)$ such that

$$(1.6) \qquad \int_0^T |V_n(t) - V(t)|^{\alpha} dt \xrightarrow{P} 0, \quad 0 \leq T < \infty,$$

and such that for each $N > 0$

$$(1.7) \qquad \sup_{0 \leq T \leq N} \left| \int_0^T V_n(t) dX_t - \int_0^T V_m(t) dX_t \right| \to 0 \quad \text{a.s.},$$

as $m,n \to \infty$. Moreover (1.7) follows from (1.6). The integral $\int_0^T V(t)dX_t$ may be defined as the limit of the simple integrals. This definition agrees with the semimartingale definition whenever the latter is applicable, i.e., whenever V_t satisfies $P(\int_0^{\infty} V_t^2 < \infty) = 1$.

In order to use Kallenberg's result to define the double integral, we need to establish the existence of a predictable V_t.

<u>For the remainder of this paper we shall assume that the function</u> f <u>satisfies the following condition</u>:

$$(1.8) \qquad \int_{-\infty}^{+\infty} |f(s,t)|^{\alpha} ds < \infty, \quad \text{a.e.} \quad t.$$

Under this assumption the expression $\int_0^t f(s,t)dX_s$ defines a stable random variable for almost every t. We will show that there is a Lebesgue null set B such that the process

$$V_t^0 = 1_{B^c}(t) \int_0^t f(s,t)dX_s$$

has a version V_t which is predictable relative to the usual completed filtration of X_t. For notational convenience, we write

$$V_t = \int_0^t f(s,t)dX_s. \tag{1.9}$$

In particular, we may view V_t as a random element in $L^0[R_+]$, the metric space of equivalence classes of Lebesgue measurable functions, and we shall do so for the rest of the paper.

Let $\tilde{X}_t$ be an independent copy of X_t and recall that we have set $X_{-t} = -\tilde{X}_t$ in order to extend X_t to all of R. Let $\{V_t^{\pm}, t \geq 0\}$ and $\{\tilde{V}_t^{\pm}, t \geq 0\}$ denote, respectively, predictable versions of $1_{B^c}(t) \int_0^t f(s,\pm t)dX_s$ and $1_{B^c}(t) \int_0^t f(-s,\pm t)d\tilde{X}_s$, both of which exist under condition (1.8) by Theorem 1.2 below.

<u>Definition 1.1</u>. Let $\Lambda_\alpha^{'}$ denote the linear space of symmetric functions f on R^2 for which

$$P\left(\int_0^\infty |V_t^+|^\alpha dt + \int_0^\infty |V_t^-|^\alpha dt + \int_0^\infty |\tilde{V}_t^+|^\alpha dt + \int_0^\infty |\tilde{V}_t^-|^\alpha dt < \infty\right) = 1.$$

<u>Theorem 1.2</u>. Assume f satisfies (1.8). Then

(1) Predictable versions $V_t^{\pm}$ and $\tilde{V}_t^{\pm}$ exist.

(2) The condition $f \in \Lambda_\alpha^{'}$ is equivalent to

$$\left(E\left(\int_{-\infty}^{+\infty} \left|\int_0^t f(s,t)dX_s\right|^\alpha dt\right)^{p/\alpha}\right)^{1/p} < \infty \tag{1.10a}$$

and

$$\left(E\left(\int_{-\infty}^{+\infty} \left|\int_{-t}^0 f(s,t)dX_s\right|^\alpha dt\right)^{p/\alpha}\right)^{1/p} < \infty \tag{1.10b}$$

for some $0 < p < \alpha$. Moreover, if (1.10) holds for one such p then (1.10) holds for all $0 < p < \alpha$.

(3) The space Λ_α' is a Banach space with equivalent norms $\lambda_{\alpha,p}'$, $1 < p < \alpha$, given by the left side of (1.10). In fact, for all $0 < p' < p < \alpha$, there is a constant C depending on p' and p such that

$$\lambda_{\alpha,p'}(f) \leq \lambda_{\alpha,p}(f) \leq C\lambda_{\alpha,p'}(f).$$

Each function in Λ_α' is locally integrable.

This theorem is proved in Section 4. Using Kallenberg (1975), we can then define the integral $J'(f) = \int\int' f(s,t)dX_s dX_t$ as

(1.11) $$J'(f) = 2\int_0^\infty V_t^+ dX_t + 2\int_0^\infty V_t^- d\tilde{X}_t + 2\int_0^\infty \tilde{V}_t^+ dX_t + 2\int_0^\infty \tilde{V}_t^- d\tilde{X}_t$$

We now turn to the definition of the double integral $J(f)$. We shall define it as the sum of two terms. The first is precisely the expression in (1.11), and the second contains the contribution from the diagonal $\{s = t\}$. It has been traditional in the Gaussian theory $(\alpha = 2)$ not to include the diagonal term since this has the advantage of giving the isometry

$$EJ(f)^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty |f(s,t)|^2 ds dt.$$

This advantage disappears in the case $\alpha < 2$, and in its absence there is a compelling reason to include the diagonal term: call a function g dyadic if it has the form of a finite sum

$$g(s,t) = \sum_{i,j \leq N} a_{ij} 1_{I_i}(s) 1_{I_j}(t)$$

where the I_i are disjoint intervals with dyadic rational endpoints. For such g it seems natural to define

$$(1.12) \qquad \int_{-\infty}^{\infty}\int_{\infty}^{\infty} g(s,t)dX_s dX_t = \sum_{i,j\leq N} a_{ij}\Delta X(I_i)\Delta X(I_j),$$

where $\Delta X(I_i)$ denotes the increment of X_t over I_i. This is consistent with the definition of $J(g)$ we give below, but not consistent if the diagonal term is left out.

Let ν_t and $\tilde{\nu}_t$ denote respectively the canonical increasing processes in the Doob-Meyer decomposition of X_t^2 and $\tilde{X}_t^2$. The process ν_t may be realized as the quadratic variation process of X_t and it is well known (see, e.g., Greenwood (1969)) that ν_t is a stable subordinator of index $\frac{\alpha}{2}$. Similarly for $\tilde{\nu}_t$. Therefore, if $f(t,s)$ satisfies

$$(1.13) \qquad \int_{-\infty}^{\infty} |f(t,t)|^{\alpha/2} dt < \infty$$

we may define

$$(1.14) \qquad \int_{-\infty}^{+\infty} f(t,t)d\nu_t = \int_0^{\infty} f(-t,-t)d\tilde{\nu}_t + \int_0^{\infty} f(t,t)d\nu_t,$$

where $\nu_{-t} = -\tilde{\nu}_t$.

We can now introduce the class Λ_α of integrands f for which we define the double integral.

Definition 1.2. Let Λ_α denote the class of functions f which belong to Λ_α' and also satisfy (1.13).

For $f \in \Lambda_\alpha$, we define $J(f) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(s,t)dX_s dX_t$ as

$$J(f) = J'(f) + \int_{-\infty}^{+\infty} f(t,t)d\nu_t. \tag{1.15}$$

The integral with respect to ν_t embodies the contribution of the diagonal. The consistency of this definition with (1.12) is easily shown using Itô's formula.

The next theorem, which plays a key role in the sequel, shows that it is always possible to approximate the integrals $J'(f)$ and $J(f)$ by dyadic integrals. The bounds on the moments of $J'(f)$ are obtained by applying the decoupling inequalities (see Theorem 1.1). These bounds involve the $\lambda'_{\alpha,p}$ introduced above:

$$\begin{aligned}\lambda'_{\alpha,p}(f) &= \left|\left| \, \left|\left| \int_{-\infty}^{t} f(s,t)dX_s \right|\right|_{L^\alpha(R')} \right|\right|_{L^p(\Omega)} \\ &= \left(E\left(\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{t} f(s,t)dX_s\right|^\alpha dt\right)^{p/\alpha}\right)^{1/p},\end{aligned} \tag{1.16}$$

which by Theorem 1.2 is well-defined for $0 < p < \alpha$, whenever $f \in \Lambda'_\alpha$. (We have set $X_{-t} = -\tilde{X}_t$). The theorem also shows that dyadic functions are dense in Λ'_α.

Theorem 1.3.

(1) Suppose $f \in \Lambda'_\alpha$. Then there is a sequence h_n of dyadic functions satisfying $h_n(s,t) = 0$ on the squares that straddle the diagonal $\{s = t\}$, such that

$$\lim_{n\to\infty} \lambda'_{\alpha,p}(h_n - f) = 0$$

and

$$\lim_{n\to\infty} E\left|J'(h_n)-J'(f)\right|^p = 0$$

for each $0 < p < \alpha$. Moreover there are constants $c'_{\alpha,p}$ and $d'_{\alpha,p}$ such that

$$d'_{\alpha,p}\lambda_{\alpha,p}(f) \leq (E\left|J'(f)\right|^p)^{1/p} \leq c'_{\alpha,p}\lambda'_{\alpha,p}(f). \tag{1.17}$$

(2) Suppose $f \in \Lambda_\alpha$. Then there is a sequence g_n of dyadic functions such that

$$\lim_{n\to\infty} E\left|J(g_n)-J(f)\right|^p = 0$$

for each $0 < p < \alpha/2$. Moreover there are constants $c_{\alpha,p}$ and $d_{\alpha,p}$ such that

$$d_{\alpha,p}\{(\int_{-\infty}^{+\infty} \left|f(t,t)\right|^{\alpha/2}dt)^{2/\alpha} + \lambda'_{\alpha,p}(f)\} \leq (E\left|J(f)\right|^p)^{1/p}$$

$$\leq c_{\alpha,p}\{(\int_{-\infty}^{+\infty} \left|f(t,t)\right|^{\alpha/2}dt)^{2/\alpha} + \lambda'_{\alpha,p}(f)\}$$

Section 5 contains the proof of Theorem 1.3 and provides an explicit construction, based on f, of the dyadic functions h_n and g_n. The proof of Theorem 1.3 also establishes that a decoupling inequality holds for $J'(f)$, namely,

<u>Corollary 1.1</u>. Suppose $f \in \Lambda'_\alpha$ and let $\{X^i_t, -\infty < t < \infty\}$ for $i = 1,2$ be independent copies of $\{X_t, -\infty < t < \infty\}$. Let $1 < p < \alpha$ and let $C(p,2)$ and $C'(p,2)$ be the same constants as in Theorem 1.1. Then

$$C'(p,2)E\left|\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} f(s,t)dX_s^1 dX_t^2\right)\right|^p \le E|J'(f)|^p \le C(p,2)E\left|\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{t} f(s,t)dX_s^1\right)dX_t^2\right|^p.$$

Corollary 1.2. Relations (1.10) are equivalent to

$$(1.18) \qquad \left(E\left(\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{\infty} f(s,t)dX_s\right|^\alpha dt\right)^{p/\alpha}\right)^{1/p} < \infty.$$

The equivalence is a consequence of the decoupling inequalities and does not follow directly from the symmetry of f. Corollary 1.2 is proved at the end of Section 5.

The set Λ_α' of permissible integrands f has been described in terms of the realizations of the stochastic process X_t. In Theorem 1.4 below we give an equivalent analytic characterization. To state this result, it is necessary to introduce the notion of a completely summing operator.

Let B_1 and B_2 be Banach spaces and $A: B_1 \to B_2$ a continuous linear operator. We say that A is p-summing, $0 < p < \infty$, if there is a constant c such that for every collection $x_1, x_2, \ldots, x_n$ in B_1, we have

$$(1.19) \qquad \sum_{i=1}^{n} |Ax_i|_{B_2}^p \le c \sup_{\substack{x^* \in B_1^* \\ |x^*| \le 1}} \sum_{i=1}^{n} |x^*(x_i)|^p,$$

where B_1^* denotes the dual of B_1. The infimum of constants c that will do in (1.19) is denoted $|A|_{*p}^p$, and $|A|_{*p}$ is called the p-summing norm of A.

The facts below follow from the definition. (See e.g. Schwartz (1981).)

(A) $|A|_{*p} < \infty \Rightarrow |A|_{*q} < \infty$ for $q \ge p$.

(B) If B_2 is reflexive, then $|A|_{*p} < \infty$ for any p, implies A is compact.

(C) The space of p-summing operators A is a Banach space with norm $|A|_{*p}$.

The operator A is called completely summing if it is p-summing for all $p > 0$.

Theorem 1.4.

(1) A necessary and sufficient condition for $f \in \Lambda_\alpha'$ is that the operator A_f, defined for functions ϕ belonging to $L^{\alpha'}(R)$ by

$$(1.20) \qquad (A_f\phi)(t) = \int_{-\infty}^{+\infty} f(s,t)\phi(s)ds,$$

defines a continuous, completely summing linear mapping from $L^{\alpha'}(R)$ to $L^\alpha(R)$. (Here α' is defined by $1/\alpha' + 1/\alpha = 1$.) Moreover, the norms induced on Λ_α' by the $|\ |_{*q}$ are equivalent to the norms $\lambda_{\alpha,p}'$ for all $0 < p < \alpha$ and $0 < q < \infty$. In particular there are constants $c_{p,\alpha,q}'$ and $d_{p,\alpha,q}'$ such that

$$(1.21) \qquad d_{p,\alpha,q}'|A_f|_{*q} \le (E|J'(f)|^p)^{1/p} \le c_{p,\alpha,q}'|A_f|_{*q}.$$

(2) A necessary and sufficient condition for $f \in \Lambda_\alpha$ is $f \in \Lambda_\alpha'$ and $\int_{-\infty}^{+\infty} |f(t,t)|^{\alpha/2}dt < \infty$. Moreover, for any $0 < p < \alpha$ and $0 < q < \infty$, there are constants $c_{p,\alpha,q}$ and $d_{p,\alpha,q}$ such that

$$(1.22) \qquad d_{p,\alpha,q}\{(\int_{-\infty}^{+\infty} |f(t,t)|^{\alpha/2}dt)^{2/\alpha} + |A_f|_{*q}\} \le (E|J(f)|^p)^{1/p}$$

$$\le c_{p,\alpha,q}\{(\int_{-\infty}^{+\infty} |f(t,t)|^{\alpha/2}dt)^{2/\alpha} + |A_f|_{*q}\}$$

This theorem is proved in Section 6.

Remarks.

(1) By Remark (B) above we have that $f \in \Lambda_\alpha$ implies that the integral operator A_f with kernel f defined in (1.14) is compact as

a mapping from $L^{\alpha'}(R)$ to $L^{\alpha}(R)$. It is possible to give a direct proof of this fact without using the theory of p-summing operators. See McConnell (1984).

(2) In the case $\alpha = 2$ it is known that the class of completely summing integral kernel transformations of $L^2(R)$ coincides with the class of Hilbert-Schmidt operators (see Schwartz (1981).) Thus for $\alpha = 2$, Λ_2' coincides with $L^2(R^2)$.

(3) The double integral $J(f)$ exists when

$$f(s,t) = \sum_{i=1}^{N} g_i(s)h_i(t),$$

with each g_i and h_i belonging to $L^{\alpha}(-\infty,+\infty)$. One can verify this directly by checking that f belongs to Λ_α. One can also observe that the corresponding operator A_f has finite-dimensional range and hence is completely summing.

Basic facts and notation. The same letter c (or C) may denote different constants. The indicator function of a set A is denoted either $I(A)$ or 1_A, and I_j denotes the indicator function of an interval I_j.

A symmetric stable random variable Y with index $0 < \alpha \leq 2$ has characteristic function $Ee^{iuY} = e^{-a|u|^\alpha}$ and covariation norm $|Y|_\alpha = a^{1/\alpha}$. The random variable Y is standard if $|Y|_\alpha = 1$. Note that $|\ |_\alpha$ is a norm only when $\alpha \geq 1$. The terminology "covariation norm" is suggested by the term "covariation" introduced by Cambanis and Miller (1981) as an analogue of covariance. Convergence in covariation norm is equivalent to convergence in probability (Schilder 1970). In fact, there are constants $c_{p,\alpha}$ such that for $p > 0$ and $0 < \alpha \leq 2$,

$$(E|Y|^p)^{1/p} = c_{p,\alpha}|Y|_\alpha. \tag{1.23}$$

In order to verify (1.23) and identify $c_{p,\alpha}$, let X_t be a standard symmetric stable process with index α so that $|X_1|_\alpha = 1$. Then by the scaling property (self-similarity) of X_t,

$$E|Y|^p = E|X_a|^p = a^{p/\alpha}E|X_1|^p = (E|X_1|^p)|Y|_\alpha^p.$$

This proves (1.23) and shows that $c_{p,\alpha} = (E|X_1|^p)^{1/p}$.

We assume without loss of generality that f in (1.1) is symmetric. The double integral $J(f) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(s,t)dX_s dX_t$ involves integration over all of R^2, whereas $J'(f) = \int\int' f(s,t)dX_s dX_t$ "ignores" the contribution of the diagonal $\{s = t\}$ (the precise definition of $J(f)$ and $J(f)$ have been given earlier). The conditions $\{f \in \Lambda_\alpha\}$ refers to $J(f)$, whereas the conditions $\{f \in \Lambda_\alpha'\}$, $\lambda_{\alpha,p}'(f) < \infty$ and $\sigma_{\alpha,\varepsilon}'(f) < \infty$ refer to $J'(f)$.

Outline of the paper. Section 2 relates the various conditions on the integrand f. The decoupling inequalities (Theorem 1.1) are established in Section 3. They will be used in the proof of Theorem 1.3. Section 4 is devoted to the definition of the stochastic integral (Theorem 1.2) and Section 5 to the existence of dyadic approximations and moment estimates (Theorem 1.3 and Corollary 1.2). The analytic characterization of the spaces Λ_α' and Λ_α (Theorem 1.4) is established in Section 6.

2. Counterexamples

The following counterexamples relate the various conditions on the integrand f introduced in Section 1. The first shows that the condition $f \in L^{\alpha}(R^2)$ is not sufficient for the integral $J'(f) = \iint' f(s,t)dX_s dX_t$ to exist.

Example 1. ($L^{\alpha}(R^2)$ is not sufficient).

Consider the quadratic form version of $J'(f)$, namely $Q(f) = \lim_{n\to\infty} Q_n(f)$ a.s. where

$$Q_n(f) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} M_i M_j \quad \text{and} \quad M_1, M_2, \ldots, M_n \quad \text{are i.i.d.}$$

standard symmetric random variables with index α, for $0 < \alpha < 2$. The Q_n may be realized as $J'(f_n)$ for appropriate step functions f_n. Let

$$a_{ij} = \begin{cases} b_k & \text{when } i = 2k, \ j = i-1 \\ 0 & \text{otherwise,} \end{cases}$$

so that $Q_n(f) = \sum_{k=1}^{n} b_k M_{2k} M_{2k+1}$. We will choose b_k such that

(i) $\sum_{k=1}^{\infty} |b_k|^{\alpha} < \infty$

and such that

(ii) $\sum_{k=1}^{\infty} |b_k|^{\alpha} |M_{2k}|^{\alpha} = \infty$ a.s.

Since

$$Ee^{i\lambda Q_n(f)} = EE[e^{i\lambda Q_n(f)} | \{M_{2k}\}]$$

$$= Ee^{-|\lambda|^{\alpha} \sum_{k=1}^{n} |b_k|^{\alpha} |M_{2k}|^{\alpha}},$$

(ii) ensures that, as $n \to \infty$, $Q_n(f)$ diverges in distribution, and hence a.s.

To construct the sequence $\{b_k\}$ use the fact that the law of large numbers entails $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} |M_j|^\alpha = \infty$ a.s. Proceeding inductively, put $n_0 = 0$. For each $i \geq 1$ there is a number $n_i > n_{i-1}$ such that

$$P\left(\frac{1}{n_i} \sum_{j=1}^{n_i} |M_j|^\alpha > i^3\right) > \frac{1}{2} .$$

Now choose

$$b_k = \begin{cases} 0 & \text{if } k \leq n_1 \\ \dfrac{1}{(i^2 n_i)^{1/\alpha}} & \text{if } \sum_{j=1}^{i-1} n_j < k \leq \sum_{j=1}^{i} n_j \end{cases}$$

Then $\sum_{k=1}^{\infty} b_k^\alpha = \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$. But, if $N_i = \sum_{j=1}^{i} n_j$, then

$$\lim_{n\to\infty} \sum_{k=1}^{n} |b_k|^\alpha |M_{2k}|^\alpha = \lim_{I\to\infty} \sum_{i=1}^{I} \sum_{k=N_i+1}^{N_i+n_i} |b_k|^\alpha |M_{2k}|^\alpha$$

$$= \lim_{I\to\infty} \sum_{i=1}^{I} \frac{1}{i^2 n_i} \sum_{k=N_i+1}^{N_i+n_i} |M_{2k}|^\alpha$$

diverges a.s.

Example 2. ($\sigma'_{\alpha,\varepsilon}(f) < \infty$ is not necessary).

Choose $f(s,t) = g(s)g(t)I([0,1]^2)$ with

$$g(s) = \left(\frac{1}{s(1 + \log^2 s)}\right)^{1/\alpha}.$$

Then $J'(f)$ exists by remark (4) of Section 1 since $g \in L^\alpha(R')$. However $\sigma'_{\alpha,\varepsilon}(f)$ is infinite.

3. Proof of Theorem 1.1 (Decoupling Inequalities)

Throughout this section we let

$$A_p = (18p^{3/2}/(p-1))^{-1}$$

$$B_p = 18p^{3/2}/(p-1)^{1/2} \tag{3.1}$$

for $1 < p < \infty$. We shall need two preliminary lemmas. The first concerns symmetric Bernoulli random variables and it will be used to prove the theorem in the case $\alpha = 2$.

Lemma 3.1. Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables satisfying $P(\xi_i = 1) = 1 - P(\xi_i = -1) = \frac{1}{2}$, and let $\{\xi_i^k\}$ for $k = 1,2,3,\ldots$ be independent copies of this sequence. Then, for each $r \geq 1$, $1 < p < \infty$, and any Z_+^r-indexed family of real numbers $a_{i_1,i_2,\ldots,i_r}$, all but finitely many of which are zero, we have

$$\left(\frac{A_p}{B_p}\right)^{r-1} E\left|\sum_{i_1<\ldots<i_r} a_{i_1,\ldots,i_r}\, \xi_{i_1}^1 \cdots \xi_{i_r}^r\right|^p \tag{3.2}$$

$$\leq E\left|\sum_{i_1<\ldots<i_r} a_{i_1,\ldots,i_r}\, \xi_{i_1} \cdots \xi_{i_r}\right|^p$$

$$\leq \left(\frac{B_p}{A_p}\right)^{r-1} E\left|\sum_{i_1<\ldots<i_r} a_{i_1,\ldots,i_r}\, \xi_{i_1}^1 \cdots \xi_{i_r}^r\right|^p.$$

Proof. Let $S_r = E\left|\sum_{i_1<\ldots<i_2} a_{i_1,\ldots,i_r}\, \xi_{i_1} \cdots \xi_{i_r}\right|^p$ and set

$$S_\ell = E\left|\sum_{i_1<\ldots<i_r} a_{i_1,\ldots,i_r}\, \xi_{i_1} \cdots \xi_{i_{\ell-1}} \xi_{i_\ell} \xi_{i_{\ell+1}}^{\ell+1} \cdots \xi_{i_r}^r\right|^p$$

for $\ell = 0,1,\ldots,r-1$. It is sufficient to prove that

$$(3.3) \qquad \left(\frac{A_p}{B_p}\right)S_{\ell-1} \le S_\ell \le \left(\frac{B_p}{A_p}\right)S_{\ell-1}$$

for any $\ell = 2,\dots,r$. Fix such an ℓ. Let F be the σ-field generated by the independent random sequences $\{\xi_i^{\ell+1}\},\dots,\{\xi_i^r\}$, with F trivial when $\ell = r$. Let

$$e_{i_\ell} = \sum a_{i_1,\dots,i_r}\, \xi_{i_1}\cdots\xi_{i_{\ell-1}}\xi_{i_{\ell+1}}^{\ell+1}\cdots\xi_{i_r}^r$$

where the sum is over all $i_1,\dots,i_{\ell-1}, i_{\ell+1},\dots,i_r$ satisfying $i_1 < \dots < i_{\ell-1} < i_\ell < i_{\ell+1} < \dots < i_r$. Then

$$S_\ell = E\left|\sum_{i_\ell} e_{i_\ell}\xi_{i_\ell}\right|^p = EE\left(\left|\sum_i e_i\xi_i\right|^p \middle| F\right).$$

$\{e_i\xi_i\}$ forms a martingale difference sequence when conditioned on F. Applying Burkholder's square inequality (Burkholder (1973)), we obtain

$$(3.4) \quad A_pE\{(\sum e_i^2(\xi_i)^2)^{p/2}|F\} \le E\{|\sum e_i\xi_i|^p|F\} \le B_pE\{(\sum e_i^2(\xi_i)^2)^{p/2}|F\}$$

The observation

$$\begin{aligned} E\{(\sum e_i^2(\xi_i)^2)^{p/2}|F\} &= E\{(\sum e_i^2)^{p/2}|F\} \\ &= E\{(\sum e_i^2(\xi_i^i)^2)^{p/2}|F\}, \end{aligned}$$

and the fact that (3.4) holds with $\{\xi_i\}$ replaced by $\{\xi_i^i\}$, yield relation (3.3). This concludes the proof. $\square$

The next lemma will be used in the proof of the left side of the decoupling inequalities (1.2) in the case $\alpha < 2$. It will be applied iteratively and therefore it is important that no extraneous constants appear in the inequality (3.5) below.

Lemma 3.2. Let ξ_1 and ξ_2 be mean zero random variables and let Y be independent of ξ_1 and ξ_2 with $P(Y = 1) = 1 - P(Y = 0)$. Also let $\tilde{Y}$ be an independent copy of Y. Then, for every $1 \leq p < \infty$ and each real constant a, we have

$$E\left|a+\xi_1 Y+\xi_2 Y\right|^p \leq E\left|a+2\xi_1 Y+2\xi_2 \tilde{Y}\right|^p. \tag{3.5}$$

Proof. Let $\mu = P(Y = 1)$, $\sigma = 1-\mu$ and $\beta = \frac{2}{1+\mu} \leq 2$. We first show that

$$E\left|a+\xi_1 Y+\xi_2 Y\right|^p \leq E\left|a+\beta\xi_1 Y+\beta\xi_2 \tilde{Y}\right|^p. \tag{3.6}$$

Indeed, we have

$$E\left|a+\xi_1 Y+\xi_2 Y\right|^p = \sigma|a|^p + \mu E\left|a+\xi_1+\xi_2\right|^p$$

and

$$\begin{aligned} E\left|a+\beta\xi_1 Y+\beta\xi_2\tilde{Y}\right|^p &= \sigma^2|a|^p + \mu\sigma E\left|a+\beta\xi_1\right|^p + \mu\sigma E\left|a+\beta\xi_2\right|^p + \mu^2 E\left|a+\beta\xi_1+\beta\xi_2\right|^p \\ &= \{\sigma^2|a|^p + \tfrac{\mu\sigma}{2} E\left|a+\beta\xi_1\right|^p + \tfrac{\mu\sigma}{2} E\left|a+\beta\xi_2\right|^p\} \\ &\quad + \{\tfrac{\mu\sigma}{2} E\left|a+\beta\xi_1\right|^p + \tfrac{\mu\sigma}{2} E\left|a+\beta\xi_2\right|^p + \mu^2 E\left|a+\beta\xi_1+\beta\xi_2\right|^p\}. \end{aligned}$$

By convexity of $|x|^p$, $E\left|a+\beta\xi_1\right|^p \geq \left|E(a+\beta\xi_1)\right|^p = |a|^p$ since $E\xi_1 = 0$, so that the first bracket dominates $\sigma^2|a|^p + \mu\sigma|a|^p = \sigma|a|^p$. Consider now the second bracket. Factor one μ and note that $\mu + \frac{\sigma}{2} + \frac{\sigma}{2} = 1$. Again, by convexity of $|x|^p$, the expression in the second bracket dominates

$$\mu E\left|\frac{\sigma}{2}(a+\beta\xi_1) + \frac{\sigma}{2}(a+\beta\xi_2) + \mu(a+\beta\xi_1+\beta\xi_2)\right|^p = \mu E\left|a+\beta(\mu+\frac{\sigma}{2})\xi_1 + \beta(\mu+\frac{\sigma}{2})\xi_2\right|^p$$
$$= \mu E\left|a+\xi_1+\xi_2\right|^p$$

because $\beta(\mu+\frac{\sigma}{2}) = 1$ for the indicated choice of β. This proves (3.6).

To prove (3.5), it is sufficient to show that the right hand side of (3.6) is monotone increasing in β for $\beta > 0$. This is obviously true if $a = 0$. Now suppose that $a \neq 0$. The random variable $Z = \xi_1 Y + \xi_2\tilde{Y}$ has mean zero. The conclusion follows because the function $g(\beta) = E|a+\beta Z|^p$ is convex in β and satisfies $g'(0) = 0$. □

Remark. An extension of Lemma 2.2 is stated in (3.13) below.

Decoupling inequalities (left-hand side):

To simplify the notation we shall consider the case $r = 2$ only--it should be clear how the proof may be adapted to the general case. Consider first the Gaussian case ($\alpha = 2$). Then we are to prove the following: Let $\{M_i\}$ and $\{\tilde{M}_i\}$ be independent sequences of i.i.d. $N(0,1)$ random variables. Then, for $1 < p < \infty$, and for all matrices (a_{ij}) of real numbers with at most finitely many nonzero entries, we have

$$E\left|\sum_{i<j} a_{ij}M_iM_j\right|^p \leq \frac{B_p}{A_p} E\left|\sum_{i<j} a_{ij}M_i\tilde{M}_j\right|^p. \tag{3.7}$$

To see this, let $\{\xi_{ik}\}$ and $\{\tilde{\xi}_{ik}\}$ be independent matrices of i.i.d. symmetric Bernoulli random variables. Then

$$E\left|\sum_{i<j} a_{ij}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\xi_{ik}\right)\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\xi_{jk}\right)\right|^p$$
$$\leq \frac{B_p}{A_p} E\left|\sum_{i<j} a_{ij}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\xi_{ik}\right)\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\tilde{\xi}_{jk}\right)\right|^p$$

To verify this inequality, relabel the indices judiciously and apply Lemma 3.1 in the case $r = 2$. The desired inequality (3.7) now follows from the central limit theorem since $a_{ij} = 0$ for all but finitely many (i,j).

Consider now the case $1 < \alpha < 2$. Let $N(dx)$ be a Poisson random measure on R with intensity $EN(dx) = \frac{dx}{|x|^{1+\alpha}}$, let $\kappa > 1$ be arbitrary, and set

$$W_k = \int_{\{\kappa^k \leq |x| < \kappa^{k+1}\}} xN(dx), \quad k = \ldots,-1,0,1,\ldots .$$

The random variables $\ldots,W_{-1},W_0,W_1,\ldots$ are independent and have a symmetric distribution. It is well known that the series $\sum_{k=-\infty}^{+\infty} W_k$ converges almost surely, and in $L^p(\Omega)$ for $0 < p < \alpha$, to a symmetric stable random variable of index α.

Introduce independent copies $N_1,N_2,\ldots$ of $N(dx)$ and define variables W_{nk} in the same way as W_k, but with $N_n(dx)$ replacing $N(dx)$. For each integer $m > 0$ let

$$W(m) = \sum_{i<j} a_{ij}\Big(\sum_{k=-m}^{m} W_{ik}\Big)\Big(\sum_{k=-m}^{m} W_{jk}\Big)$$

and

$$\tilde{W}(m) = \sum_{i<j} a_{ij}\Big(\sum_{k=-m}^{m} W_{ik}\Big)\Big(\sum_{k=-m}^{m} \tilde{W}_{jk}\Big)$$

where the family $\{\tilde{W}_{jk}\}$ is an independent copy of the family $\{W_{jk}\}$. Let $W(\infty)$ and $\tilde{W}(\infty)$ denote the limits of these sequences, which exist almost surely, as m tends to infinity.

Let $1 < p < \alpha$, and let A_p and B_p be defined as in (3.1). Since $\kappa > 1$ is arbitrary, it is enough to prove that

$$E\left|W(\infty)\right|^p \leq 4^p\left(\kappa \frac{B_p}{A_p}\right)^2 E\left|\tilde{W}(\infty)\right|^p. \tag{3.8}$$

Now since the sums $\sum_{k=-\infty}^{m} W_{ik}$ converge in $L^p(\Omega)$, it follows that $\tilde{W}(m)$ converges to $\tilde{W}(\infty)$ in $L^p(\Omega)$. It suffices then to prove the inequalities

$$E\left|W(m)\right|^p \leq 4^p\left(\kappa \frac{B_p}{A_p}\right)^2 E\left|\tilde{W}(m)\right|^p,$$

because by Fatou's lemma,

$$\begin{aligned} E\left|W(\infty)\right|^p &\leq \liminf_{m\to\infty} E\left|W(m)\right|^p \\ &\leq 4^p\left(\kappa \frac{B_p}{A_p}\right)^2\left\{\lim_{m\to\infty} E\left|\tilde{W}(m)-\tilde{W}(\infty)\right|^p + E\left|\tilde{W}(\infty)\right|^p\right\}. \end{aligned}$$

We shall now simplify the expression

$$W(m) = \sum_{i<j} \sum_{k,\ell\in[-m,m]} a_{ij} W_{ik} W_{j\ell}$$

by relabeling the indices. Since k and ℓ take values in $[-m,m]$, the relabeling $s = 3mi+k$ and $t = 3mj+\ell$ ensures that to each (i,k) and (j,ℓ) there corresponds exactly one s and t, and moreover, if $i < j$ then $s < t$. Let then

$$b_{st} = \begin{cases} a_{ij} & \text{if } s = 3mi+k, \ t = 3mj+\ell \\ & \quad \text{for some } k,\ell \in [-m,m] \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_s = \begin{cases} W_{ik} & \text{if} \quad s = 3mi+k, \quad k \in [-m,m] \\ 0 & \text{otherwise,} \end{cases}$$

and define $\tilde{Z}_s$ with W_{ik} replaced by $\tilde{W}_{ik}$. Then we have

$$\begin{aligned} W(m) &= \sum_{i<j} \sum_{-m\leq k,\ell\leq m} a_{ij} W_{ik} W_{j\ell} \\ &= \sum_{s<t} b_{st} Z_s Z_t, \end{aligned}$$

and

$$\tilde{W}(m) = \sum_{s<t} b_{st} Z_s \tilde{Z}_t.$$

Clearly, then, it is enough to prove the following result.

Proposition 3.1. Let $N_\ell(dx)$ and $\tilde{N}_\ell(dx)$, $\ell = 1,2,...$ be independent Poisson random measures on R^1, each having intensity $|x|^{-1-\alpha}dx$. Let $\{\delta_i\}$ and $\{\gamma_i\}$ be sequences of non-negative real numbers such that for some constant $\kappa \geq 1$, we have $\delta_i < \gamma_i \leq \kappa\delta_i$. Set

$$Z_i = \int_{\{\delta_i \leq |x| < \gamma_i\}} x N_i(dx)$$

and

$$\tilde{Z}_i = \int_{\{\delta_i \leq |x| < \gamma_i\}} x \tilde{N}_i(dx).$$

Let $1 < p < \alpha$ and A_p and B_p be defined as above. Then for any matrix (b_{ij}) with at most finitely many non-zero entries, we have

$$E\Big|\sum_{i<j} b_{ij} Z_i Z_j\Big|^p \leq 4^p \left(\kappa \frac{B_p}{A_p}\right)^2 E\Big|\sum_{i<j} b_{ij} Z_i \tilde{Z}_j\Big|^p.$$

Remark. Theorem 1 follows from this proposition by identifying

$Z_s = \int_{\delta_s \leq |x| < \gamma_s} x N_s(dx)$ with $\int_{\kappa^k \leq |x| < \kappa^{k+1}} x N_i(dx)$ when $s = 3mi+k$.

Proof of the proposition. The proof is in 4 steps.

1) Since $\{Z_i\}$ is a sequence of independent compound Poisson random variables, it is convenient to write $Z_i = \sum_{k=1}^{\nu_i} u_{ik}$ where the random variable ν_i has a Poisson distribution with parameter $\lambda_i = \int_{\{\delta_i \le |x| < \gamma_i\}} EN(dx)$ and the u_{ik}'s are independent of ν_i, symmetric, independent, identically distributed for fixed i, and take values only in $\{x: \delta_i \le |x| < \gamma_i\}$. The random variable ν_i can be interpreted as the total number of jumps of a Poisson process in the interval $\{x: \delta_i \le |x| < \gamma_i\}$ and the u_{ik} are the jump sizes. Set $u_{ik} = \xi_{ik} |u_{ik}|$ where ξ_{ik} is the sign of the jump and $|u_{ik}|$ is its magnitude. The random variables ξ_{ik} are i.i.d., independent of ν_i, and they satisfy

$$P\{\xi_{ik} = 1\} = 1 - P\{\xi_{ik} = -1\} = 1/2$$

2) Set $U_i = \sum_{k=1}^{\nu_i} \delta_i \xi_{ik}$ and $\tilde{U}_j = \sum_{k=1}^{\nu_j} \delta_j \tilde{\xi}_{jk}$, where $\{\tilde{\xi}_{jk}\}$ is an independent copy of the sequence $\{\xi_{jk}\}$. Note that U_i and $\tilde{U}_i$ are not independent because of the presence of ν_i. We shall prove that for every $1 < p < \infty$, we have

$$(3.9) \quad \left(\frac{A_p}{B_p}\right)^2 E\Big|\sum_{i<j} b_{ij} U_i \tilde{U}_j\Big|^p \le E\Big|\sum_{i<j} b_{ij} Z_i Z_j\Big|^p \le \left(\kappa^p \frac{B_p}{A_p}\right)^2 E\Big|\sum_{i<j} b_{ij} U_i \tilde{U}_j\Big|^p .$$

By conditioning on the ν_i, it is enough to prove (3.9) under the assumption that the ν_i are non-random. Then $(\sum_{i:i<j} b_{ij} Z_i)\xi_{jk}|u_{jk}|$ is a martingale difference sequence indexed by $j = 1,2,\ldots,$ and $1 \le k \le \nu_j$. By applying Burkholder's square inequality and then the relations $\delta_j \le |u_{jk}| < \delta_j \kappa$ to

$$E\Big|\sum_{i<j} b_{ij}Z_iZ_j\Big|^p = E\Big|\sum_j \sum_{k=1}^{\nu_j} \big(\sum_{\substack{i\\ i<j}} b_{ij}Z_i\big)\xi_{jk}|u_{jk}|\Big|^p,$$

we get

$$A_pE\Big|\sum_j \sum_{k=1}^{\nu_j} \big(\sum_{\substack{i\\ i<j}} b_{ij}Z_i\big)^2\delta_j^2\Big|^{p/2} \leq E\Big|\sum_{i<j} b_{ij}Z_iZ_j\Big|^p$$

$$\leq \kappa^pB_pE\Big|\sum_j \sum_{k=1}^{\nu_j} \big(\sum_{\substack{i\\ i<j}} b_{ij}Z_i\big)^2\delta_j^2\Big|^{p/2}.$$

Since Burkholder's square inequality yields the same result (without the factor κ^p) when applied to $E|\sum_{i<j} b_{ij}Z_i\tilde{U}_j|^p$, we get

$$(3.10)\quad \Big(\frac{A_p}{B_p}\Big)\, E\Big|\sum_{i<j} b_{ij}Z_i\tilde{U}_j\Big|^p \leq E\Big|\sum_{i<j} b_{ij}Z_iZ_j\Big|^p \leq \Big(\kappa^p\,\frac{B_p}{A_p}\Big)\, E\Big|\sum_{i<j} b_{ij}Z_i\tilde{U}_j\Big|^p$$

Now, write $E|\sum_{i<j} b_{ij}Z_i\tilde{U}_j|^p = EE(|\sum_{i<j} b_{ij}Z_i\tilde{U}_j|^p|\{\tilde{U}_j\})$. Having conditioned on $\{\tilde{U}_j\}$, we can apply the same argument again, this time to the martingale difference sequence $(\sum_{j:j>1} b_{ij}\tilde{U}_j)\xi_{ik}|u_{ik}|$, $i = 1,2,\ldots$, $1 \leq k \leq \nu_i$, and get

$$(3.11)\quad \Big(\frac{A_p}{B_p}\Big)E\Big(E\Big|\sum_{i<j} b_{ij}U_i\tilde{U}_j\Big|^p\Big|\{\tilde{U}_j\}\Big) \leq E\Big(E\Big|\sum_{i<j} b_{ij}Z_i\tilde{U}_j\Big|^p\Big|\{\tilde{U}_j\}\Big)$$

$$\leq \Big(\kappa^p\,\frac{B_p}{A_p}\Big)E\Big(E\Big|\sum_{i<j} b_{ij}U_i\tilde{U}_j\Big|^p\Big|\{\tilde{U}_j\}\Big).$$

Relation (3.9) follows from (3.10) and (3.11).

3) We shall now approximate $\sum_{k=1}^{\nu_i} \xi_{ik}$ by $\sum_{k=1}^{n} Y_{ik}^{(n)}\xi_{ik}$ where for each n, $\{Y_{ik}^{(n)}\}$ is a family of i.i.d. Bernoulli random variables with $\frac{\lambda_i}{n} = P(Y_{ik}^{(n)} = 1) = 1 - P(Y_{ik}^{(n)} = 0)$. Recall that λ_i is the parameter of the Poisson distribution of ν_i. It is well known that the sequence $\sum_{k=1}^{n} Y_{ik}^{(n)}\xi_{ik}$

converges in distribution to $\sum_{k=1}^{\nu_i} \xi_{ik}$ as n tends to infinity. Moreover we have convergence of all moments. Thus, if we let $\{\tilde{Y}_{ik}^{(n)}\}$ denote an independent copy of the $\{Y_{ik}^{(n)}\}$, then it suffices to show that

$$(3.12) \qquad E\Big|\sum_{i<j} \delta_i\delta_j b_{ij}\Big(\sum_{\lambda=1}^{n} Y_{i\lambda}^{(n)}\xi_{i\lambda}\Big)\Big(\sum_{\eta=1}^{n} Y_{j\eta}^{(n)}\tilde{\xi}_{j\eta}\Big)\Big|^p$$

$$\leq 4^p E\Big|\sum_{i<j} \delta_i\delta_j b_{ij}\Big(\sum_{\lambda=1}^{n} Y_{i\lambda}^{(n)}\xi_{i\lambda}\Big)\Big(\sum_{\eta=1}^{n} \tilde{Y}_{j\eta}^{(n)}\tilde{\xi}_{j\eta}\Big)\Big|^p.$$

4) We shall now use Lemma 3.2 to establish (3.12).

We shall focus successively on all pairs (u,k), $u \in Z^1$, $1 \leq k \leq n$ for which $|b_{uj}| + |b_{iu}| > 0$ for some i and j. If $|b_{uj}| > 0$, then the random variable $Y_{uk}^{(n)}$ appears in the first factor of one of the summands of the left hand side of (3.12), and if $|b_{iu}| > 0$, then the random variable $Y_{uk}^{(n)}$ appears in the second factor of one of the summands. For each such pair (u,k), we apply Lemma 3.2 as follows. We set $Y = Y_{uk}^{(n)}$, we let ξ_1 (respectively ξ_2) be the sum of the coefficients of $Y_{uk}^{(n)}$ when $Y_{uk}^{(n)}$ appears in the first (respectively second) factor, and we let a denote the terms of the left-hand side of (3.12) that do not involve $Y_{uk}^{(n)}$. (Note that ξ_1 or ξ_2 may be zero, but not both). Thus

$$\xi_1 = \sum_{\eta=1}^{n} \sum_{j>u} \delta_u\delta_j b_{uj}\xi_{uk} Y_{j\eta}^{(n)}\tilde{\xi}_{j\eta},$$

$$\xi_2 = \sum_{\lambda=1}^{n} \sum_{i<u} \delta_i\delta_u b_{iu} Y_{i\lambda}^{(n)}\xi_{i\lambda}\tilde{\xi}_{uk},$$

and

$$a = \left\{\sum_{\substack{i\neq u\\ j\neq u\\ i<j}} \delta_i\delta_j b_{ij} Y_{ik}^{(n)}\xi_{ik} Y_{jk}^{(n)}\tilde{\xi}_{jk} + \sum_{i<j}\sum_{\substack{\lambda\neq k\\ \eta\neq k}} \delta_i\delta_j b_{ij} Y_{i\lambda}^{(n)}\xi_{i\lambda} Y_{j\eta}^{(n)}\tilde{\xi}_{j\eta}\right\}.$$

Thus

$$E\left|\sum_{i<j} \delta_i\delta_j b_{ij}\left(\sum_{\lambda=1}^{n} Y_{i\lambda}^{(n)}\xi_{i\lambda}\right)\left(\sum_{\eta=1}^{n} Y_{j\eta}^{(n)}\tilde{\xi}_{j\eta}\right)\right|^p = E\left|a+\xi_1 Y_{uk}^{(n)}+\xi_2 Y_{uk}^{(n)}\right|^p.$$

Let $\{\tilde{Y}_{j\eta}^{(n)}\}$ be an independent copy of $\{Y_{j\eta}^{(n)}\}$ and let G denote the σ-field generated by all random variables $Y_{j\eta}^{(n)}$, $\xi_{j\eta}$, $\tilde{Y}_{j\eta}$ and $\tilde{\xi}_{j\eta}$ with $(j,\eta) \neq (u,k)$. Then, by Lemma 2.2, we have

$$\begin{aligned} E\left|a+\xi_1 Y_{uk}^{(n)}+\xi_2 Y_{uk}^{(n)}\right|^p &= E(E\left|a+\xi_1 Y_{uk}^{(n)}+\xi_2 Y_{uk}^{(n)}\right|^p \big| G) \\ &\leq E(E\left|a+2\xi_1 Y_{uk}^{(n)}+2\xi_2 \tilde{Y}_{uk}^{(n)}\right|^p \big| G) \\ &= E\left|a+2\xi_1 Y_{uk}^{(n)}+2\xi_2 \tilde{Y}_{uk}^{(n)}\right|^p). \end{aligned}$$

We now apply the same reasoning to each such pair (u,k) in turn. (The definitions of ξ_1, ξ_2, and a must be slightly modified by changing $Y_{j\eta}^{(n)}$, if the previous argument had been already applied to the pair (j,η). In that case the variable $Y_{j\eta}^{(n)}$ should then be changed to $2Y_{j\eta}^{(n)}$ if it previously appeared in ξ_2, and it should be changed to $2\tilde{Y}_{j\eta}^{(n)}$ if it previously appeared in ξ_1.) Since the left hand side of (3.12) is a quadratic form in the Y's and since the previous reasoning is applied to each Y exactly once, we obtain (3.12). This completes the proof of the proposition. □

<u>Remark concerning the case $r > 2$</u>. The proof is similar to the case $r = 2$. To establish the equivalent of relation (3.9), proceed as in the proof of Lemma 3.1. The power 2 in (3.9) then becomes r. To establish the equivalent of relation (3.7), use the relation

$$(3.13)\qquad E\Big|a + (\sum_{i=1}^{r} \xi_i)Y\Big|^p \le E\Big|a+2\xi_1 Y^1+2^2\xi_2 Y^2 + \dots + 2^{r-2}\xi_{r-2}Y^{r-2} + 2^{r-1}(\xi_{r-1}Y^{r-1}+\xi_r Y^r)\Big|^p \le E\Big|a+2^{r-1}(\sum_{i=1}^{r} \xi_i Y^i)\Big|^p$$

instead of (3.5). Here $Y^1,\dots,Y^r$ are independent copies of Y and $\xi_1,\dots,\xi_r$ are independent random variables, independent of Y. To obtain (3.13), apply (3.5) many times while conditioning on the non-relevant random variables. The constant 4^p in (3.12) becomes $(2^{r-1})^{rp}$.

The constant in the left side of the decoupling inequalities (1.2) can thus be taken to be

$$\left(\frac{B_p}{A_p}\right)^r (2^{r-1})^{rp} = \left(\frac{18^2 p^3}{(p-1)^{3/2}}\, 2^{(r-1)p}\right)^r .$$

Decoupling inequalities (right-hand side):

Again we give details only in the case $r = 2$.

Lemma 3.3. Let $\{X_i\}$ be a sequence of i.i.d. symmetric random variables such that $E|X_i| < \infty$ and let $\{Y_i\}$ be an independent copy of $\{X_i\}$. Then for $1 \le p < \infty$ we have

$$E\Big|\sum_{i<j} a_{ij}X_iY_j\Big|^p \le E\Big|\sum_{i\ne j} a_{ij}X_iY_j\Big|^p \le 2^{1/p}E\Big|\sum_{i<j} a_{ij}X_iY_j\Big|^p$$

where the a_{ij} are symmetric and finitely many of them are non-zero.

Proof. The inequality on the right-hand side follows from the triangle inequality applied to $|\ \ |_{L^p(\Omega)}$. To prove the inequality on the left-hand side, note that by Jensen's inequality it is enough to show that

$$E\Big(\sum_{i>j} a_{ij}X_iY_j \Big| X_kY_\ell,\ k < \ell\Big) = 0$$

and, hence, that for each fixed $i > j$ we have $E(X_i Y_j \mid G) = 0$ where G is the σ-field generated by $X_1Y_j,\ldots,X_{j-1}Y_j,X_iY_{i+1},X_iY_{i+2},\ldots$. But $E(X_iY_j \mid G) = E(X_iE(Y_j \mid X_i,G) \mid G) = E(X_iE(Y_j \mid X_1Y_j,\ldots,X_{j-1}Y_j) \mid G)$. The last expression vanishes since Y_j has a symmetric conditional distribution given $X_1Y_j,\ldots,X_{j-1}Y_j$. □

Now fix an integer n so large that $a_{ij} = 0$ for $i > n$ or $j > n$ and assume from now on that the matrix a_{ij} is symmetic. Let B denote the symmetric bilinear form defined on R^n by

$$B(\underset{\sim}{x},\underset{\sim}{y}) = \sum_{i \neq j} a_{ij}x_iy_j$$

where $\underset{\sim}{x} = (x_1,\ldots,x_n)$ and $\underset{\sim}{y} = (y_1,\ldots,y_n)$. Let $\{M_j^1\}$ and $\{M_j^2\}$ be as in the statement of Theorem 1.1. In view of lemma 3.3 it is sufficient to prove that for $1 < p < \infty$

$$E\left|B(\underset{\sim}{X},\underset{\sim}{Y})\right|^p \leq 2^{p+2}E\left|B(\underset{\sim}{X},\underset{\sim}{X})\right|^p \tag{3.13}$$

where $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ denote, respectively, the n-dimensional random vectors with components M_j^1 $j = 1,\ldots,n$ and M_j^2, $j = 1,\ldots,n$. Inequality (3.13) follows easily from the polarization identity

$$B(\underset{\sim}{X},\underset{\sim}{Y}) = \frac{1}{2}\{B(\underset{\sim}{X}+\underset{\sim}{Y},\underset{\sim}{X}+\underset{\sim}{Y}) - B(\underset{\sim}{X},\underset{\sim}{X}) - B(\underset{\sim}{Y},\underset{\sim}{Y})\}$$

together with the observation that $\underset{\sim}{X}+\underset{\sim}{Y} \overset{d}{=} 2^{1/\alpha}\underset{\sim}{X}$. This concludes the proof of Theorem 1.1. □

<u>Remark</u>. Let $\{M_i\}$ be a sequence of i.i.d. symmetric α-stable random variables with $1 < \alpha \leq 2$. Then in view of the preceding results, the pth moments of

$$\sum_{i<j} a_{ij}M_iM_j, \quad \sum_{i \neq j} a_{ij}M_iM_j, \quad \sum_{i<j} a_{ij}M_i\tilde{M}_j, \quad \sum_{i \neq j} a_{ij}M_i\tilde{M}_j$$

are all comparable in size.

4. Proof of Theorem 1.2

We begin with a foundational lemma which establishes Part (1) of the theorem.

Lemma 4.1. Suppose $f(s,t)$ satisfies

$$\int_0^\infty |f(s,t)|^\alpha ds < \infty \quad \text{a.e.} \tag{4.1}$$

Then there is a Lebesgue null set B such that the process

$$V_t^0 = 1_{B^c}(t) \int_0^t f(s,t)dX_s$$

has a predictable version V_t.

Proof. It suffices to consider f supported in $[0,1]^2$. Consider a dyadic function $u(s,t)$ of the form

$$u(s,t) = \sum_{i,j\leq n} a_{ij}I_i(s)I_j(t)$$

where for some n we have $I_i = (2^{-n}(i-1), 2^{-n}i]$. (Recall that $I_i(s)$ denotes the indicator function of the interval I_i.) It is easy to see directly that the process $\int_0^t u(s,t)dX_s$ has a predictable version.

Now by (4.1) we may choose functions f_N having the form of the dyadic function u above (with possibly different n in each case) so that

$$\left|\{t: \int_0^1 |f(s,t) - f_N(s,t)|^\alpha ds > 2^{-N}\}\right| < 2^{-N}, \tag{4.2}$$

where $| \; |$ denotes the Lebesgue measure on $[0,1]$. This is easily done after first approximating f by a bounded measurable function. Let $V_N(t)$ be the predictable version of $\int_0^t f_N(s,t)dX_s$ as described above. By (4.2) and the Borel-Cantelli lemma there is a Lebesgue null set B so that

$$\int_0^1 \left|f_N(s,t) - f(s,t)\right|^\alpha ds \leq 2^{-N},$$

$\forall \; t \notin B$ and $N \geq N(t)$. Thus by (1.23) and the Borel-Cantelli lemma, we have

$$\lim_{N\to\infty} V_N(t) = V_t^0 \text{ a.s.}, \quad t \notin B. \tag{4.3}$$

Set $V_t = \overline{\lim}\, V_N(t)1_{B^c}(t)$. Then V_t is predictable, and by (4.3), we have $V_t = V_t^0$ a.s. Hence V_t is a version of V_t^0. □

As noted in the introduction we will henceforth use the abusive notation $V_t = \int_0^t f(s,t)dX_s$. We may view V_t as a random element of $L^0(R_+)$, the metric space of measurable functions on R_+.

<u>Proof of Theorem 1.2</u>.

(1) The first part of the theorem holds by Lemma 4.1.

(2) To prove the second part of the theorem, recall that $f \in \Lambda_\alpha'$ if and only if

$$P\left(\int_{-\infty}^{\infty} \left|\int_0^t f(s,t)dX_s\right|^\alpha dt < \infty\right) = 1 \tag{4.4}$$

together with the same condition in which $f(s,t)$ is replaced by $f(-s,t)$. Condition (4.4) implies that $\int_0^\infty f(s,t)dX_s$ is a well-defined $L^\alpha[0,\infty)$-valued stable random variable. By de Acosta (1975) we have

$E\left|\int_0^t f(s,t)dX_s\right|^p_{L^\alpha(R_+)} < \infty$ for $0 < p < \alpha$. The same considerations apply to $f(s,-t)$, establishing (1.10). Conversely, (1.10) implies (4.4), together with the corresponding statement with s replaced by $-s$.

(3) We now turn to the proof of the third part of the theorem. Let $0 < p < \alpha$ and let $\{f_n\}$ be a Cauchy sequence in $\lambda'_{\alpha,p}$. Then $\int_{-\infty}^{\infty}\left|\int_0^t f_n(s,t)dX_s\right|^\alpha dt$ converges in measure on the product space $\Omega\times R$. We conclude from Schilder (1970) that for almost every t, $f_n(\cdot,t)$ converges in $L^\alpha([0,t))$. Thus there is a Lebesgue measurable function $f(s,t)$ satisfying (1.8) such that

$$\int_0^t f(s,t)dX_s = \lim_{n\to\infty}\int_0^t f_n(s,t)dX_s$$

in probability, for almost every t. We conclude from Fatou's lemma and Fubini's theorem that $\lambda'_{\alpha,p}(f) < \infty$. By the same argument $\lambda'_{\alpha,p}(f_n-f) \leq \lim_{m\to\infty}\lambda'_{\alpha,p}(f_n-f_m)$. Hence f_n converges to f in $\lambda'_{\alpha,p}$ proving that Λ'_α is a Banach space.

In fact, the preceding argument shows also that Λ'_α is a complete metric space in both metrics $\lambda^{p'}_{\alpha,p'}$ and $\lambda^p_{\alpha,p}$ where $0 < p' < p < \alpha$. By Hölder's inequality $\lambda_{\alpha,p'}(f) < \lambda_{\alpha,p}(f)$. Thus, if τ' is the topology induced by the metric $\lambda^{p'}_{\alpha,p'}$ and if τ is the topology induced by the metric $\lambda^p_{\alpha,p}$, then $\tau' \subset \tau$ in the sense that τ is the finer topology. By a well-known consequence of the closed graph theorem, we have also $\tau \subset \tau'$. Using the homogeneity of $\lambda_{\alpha,p'}$ and of $\lambda_{\alpha,p}$, we conclude that there is a constant C such that $\lambda_{\alpha,p}(f) \leq C\lambda_{\alpha,p'}(f)$ and therefore

$$\lambda_{\alpha,p'}(f) \leq \lambda_{\alpha,p}(f) \leq C\lambda_{\alpha,p'}(f).$$

Finally, if f belongs to Λ_α' then it is locally integrable. Indeed, by Jensen's inequality and (1.23) for $1 < p < \alpha$,

$$\lambda_{\alpha,p}'(f) \geq (E(\int_0^1 \left| \int_0^t f(s,t)dX_s \right|^\alpha)^{p/\alpha}dt)^{1/p} \geq \int_0^1 E\left| \int_0^t f(s,t)dX_s \right|dt$$

$$(4.5) \qquad = c_{1,\alpha} \int_0^1 (\int_0^t |f(s,t)|^\alpha ds)^{1/\alpha}dt \geq c_{1,\alpha} \int_0^1 \int_0^t |f(s,t)|dsdt$$

$$= \frac{1}{2} c_{1,\alpha}|f|_{L^1([0,1]^2)}.$$

The same argument applies to the restriction of f to any unit square in R^2 with sides parallel to the axes, since X_t has stationary increments. The local integrability of f follows. □

5. Proof of Theorem 1.3 (Existence of dyadic approximations and moment estimates)

In what follows it is convenient to restrict the domain of the functions $f(s,t)$ to the unit square $[0,1]^2$. There is no loss in generality in doing so. Indeed, it is easy to pass from functions supported in $[0,1]^2$ to functions supported in R_+^2, and the considerations below apply to each of the 4 integrals defining $J'(f)$. The condition $f \in \Lambda_\alpha'$ becomes then

$$(5.1) \qquad P\left(\int_0^1 \left| \int_0^t f(s,t)dX_s \right|^\alpha dt < \infty\right) = 1.$$

and <u>it is assumed throughout this section</u>.

To prove Part (1) of Theorem 1.3, we construct a sequence of dyadic functions $h_n(s,t)$ that are identically 0 on the dyadic squares that straddle the diagonal, and which satisfy

$$(5.2) \qquad \lim_{n\to\infty} E\left(\int_0^1 \left| \int_0^t h_n(s,t)dX_s - \int_0^t f(s,t)dX_s \right|^\alpha dt\right)^{p/\alpha} = 0$$

for $0 < p < \alpha$. We shall need four preliminary results.

Let $\mathcal{P} = ([0,1],\mathcal{B},|\ |)$ be the unit interval probability space, with Borel σ-field $\mathcal{B}$ and Lebesgue measure $|\ |$. Let $\mathcal{G}_n$ be the σ-field of subsets of $[0,1]$ generated by the dyadic intervals

$$I_{n,i} = [(i-1)2^{-n}, i2^{-n}), \quad 1 \le i \le 2^n.$$

Put $\Delta X_{n,i} = X_{i2^{-n}} - X_{(i-1)2^{-n}}$ and let $A_n = \sigma\{\Delta X_{n,i},\ 1 \le i \le 2^n)$.

Lemma 5.1. Let $g \in L^{\alpha}[0,1]$. Then

$$(5.3) \qquad E\left(\int_0^1 g(s)dX_s \middle| A_n\right) = \int_0^1 E(g|G_n)(s)dX_s \quad \text{a.s.,}$$

where the expectation on the right is computed on the unit interval probability space P.

Proof. By Cambanis and Miller (1981), p. 45, we have for $(i-1)2^{-n} \leq s \leq i2^{-n}$,

$$(5.4) \qquad E(X_s - X_{(i-1)2^{-n}} | \Delta X_{n,i}) = 2^n(s - (i-1)2^{-n})\Delta X_{n,i}.$$

Since X_t has independent increments,

$$E\left(\int_0^1 g(s)dX_s \middle| A_n\right) = \sum_{i=1}^{2^n} E\left(\int_{(i-1)2^{-n}}^{i2^{-n}} g(s)dX_s \middle| \Delta X_{n,i}\right).$$

If g is dyadic, we can decompose the integral further, use (5.4), recollect terms and conclude that the last expression equals

$$\sum_{i=1}^{2^n} \left(2^n \int_{(i-1)2^{-n}}^{i2^{-n}} g(s)ds\right)\Delta X_{n,i} = \int_0^1 E(g|G_n)(s)dX_s.$$

Relation (5.3) follows for general g since dyadic functions are dense in $L^{\alpha}[0,1]$. □

Lemma 5.2. For any $0 < p < \alpha$ and f satisfying (5.1),

$$(5.5) \qquad \int_0^1 \left|E\left(\int_0^t f(s,t)dX_s \middle| G_n\right) - \int_0^t f(s,t)dX_s\right|^{\alpha} \to 0$$

as $n \to \infty$ in $L^p(\Omega)$. (The conditional expectation here is defined on the unit interval probability space P.)

Proof. By (5.1) and the martingale convergence theorem applied to P, the (dyadic) random functions $E(\int_0^t f(s,t)dX_s|\mathcal{G}_n)$ converge in $L^\alpha[0,1]$ to $\int_0^t f(s,t)dX_s$ a.s. Their $L^\alpha[0,1]$ norm converges in $L^p(\Omega)$ because we have uniform integrability. Indeed, by Jensen's inequality

$$\sup_n E\{\int_0^1 \left|E(\int_0^t f(s,t)dX_s|\mathcal{G}_n)\right|^\alpha dt\}^{p/\alpha} \leq E\{\int_0^1 \left|\int_0^t f(s,t)dX_s\right|^\alpha dt\}^{p/\alpha}$$

is finite because of (5.1) and Theorem 1.2. □

Lemma 5.3. Let ε_n be a sequence of positive numbers tending to zero. Put

$$f_n(s,t) = f(s,(t-\varepsilon_n)^+)I(s \leq t-\varepsilon_n).$$

where f satisfies (5.1). Then f_n satisfies (5.1) and, for $0 < p < \alpha$,

$$\lim_{n\to\infty} E\{\int_0^1 \left|\int_0^t f_n(s,t)dX_s - \int_0^t f(s,t)dX_s\right|^\alpha dt\}^{p/\alpha} = 0. \tag{5.6}$$

Proof. Define $T_n: L^\alpha[0,1] \to L^\alpha[0,1]$ by $T_n\Phi(t) = \Phi((t-\varepsilon_n)^+)$. Then

$$\left|T_n\Phi-\Phi\right|_{L^\alpha[0,1]} \to 0, \tag{5.7}$$

and, if $\Phi(0) = 0$, we have, after a change of variables,

$$\int_0^1 \left|T_n\Phi(t)\right|^\alpha dt = \int_0^{1-\varepsilon_n} \left|\Phi(t)\right|^\alpha dt. \tag{5.8}$$

Since $\int_0^1 \left|f_n(s,t)\right|^\alpha ds < \infty$ for almost every t, we conclude that $\int_0^t f_n(s,t)dX_s$ can be viewed as a predictable random element of $L^0[0,1]$, the space of Lebesgue measurable functions on $[0,1]$ (see Lemma 4.1.) To see that it also belongs to $L^\alpha[0,1]$, note that by (5.1) $\int_0^t f(s,t)dX_s$ belongs to that space and, for almost every $t \in [0,1]$, we have

$$T_n\left(\int_0^{\cdot} f(s,\cdot)dX_s\right)(t) = \int_0^t f_n(s,t)dX_s \quad \text{a.s.,} \tag{5.9}$$

with the left-hand side belonging to $L^\alpha[0,1]$. Hence f_n satisfies (5.1).

Moreover, by (5.7) and (5.9), we have

$$\lim_{n\to 0} \left(\int_0^1 \left| \int_0^t f_n(s,t)dX_s - \int_0^t f(s,t)dX_s \right|^\alpha dt\right)^{1/\alpha} = 0 \quad \text{a.s.}$$

The following estimate uses (5.8) and provides the uniform integrability needed to deduce (5.6) from the above relation:

$$E\left(\int_0^1 \left| \int_0^t f_n(s,t)dX_s \right|^\alpha\right)^{p/\alpha} = E\left(\int_0^{1-\varepsilon_n} \left| \int_0^t f(s,t)dX_s \right|^\alpha dt\right)^{p/\alpha}$$

$$\leq E\left(\int_0^1 \left| \int_0^t f(s,t)dX_s \right|^\alpha dt\right)^{p/\alpha}$$

which is finite for $0 < p < \alpha$. □

<u>Lemma 5.4</u>. Suppose that

$$P\left(\int_0^1 \left| \int_0^1 f(s,t)dX_s \right|^\alpha dt < \infty\right) = 1 \tag{5.10}$$

where f is not necessarily symmetric. Let $\phi \in L^{\alpha'}[0,1]$ where $1/\alpha + 1/\alpha' = 1$. Then

$$\int_0^1 \left(\int_0^1 f(s,t)\phi(s)ds\right)dX_t = \int_0^1 \phi(s)\left(\int_0^1 f(s,t)dX_t\right)ds. \tag{5.11}$$

Proof. The proof is in 3 steps.

1) We first prove

(5.12) $$\int_{E_n} \left(\int_0^1 f(s,t)\phi(s)ds\right)dX_t = \int_0^1 \phi(s)\left(\int_{E_n} f(s,t)dX_t\right)ds$$

for ϕ dyadic and

(5.13) $$E_n = \{t \in [0,1]: \int_0^1 |f(s,t)|^\alpha ds \leq n\}.$$

The left-hand side of (5.12) is well-defined because $\left|\left|\int_0^1 f(s,t)\phi(s)ds\right|\right|_{L^\alpha} \leq n^{1/\alpha} \left|\left|\phi\right|\right|_{L^{\alpha'}} < \infty$. To show that the right-hand side of (5.12) is well-defined, put $\xi = \int_{E_n} f(s,t)dX_t$ and $\eta = \int_{E_n^c} f(s,t)dX_t$. Relation (5.10) ensures that $\xi+\eta$ is a well-defined random element of $L^\alpha[0,1]$. Moreover, η is symmetric and ξ and η are independent. Thus $\xi-\eta$ has the same distribution as $\xi+\eta$, and therefore $\xi = \frac{1}{2}\{(\xi+\eta)+(\xi-\eta)\}$ is a random element of $L^\alpha[0,1]$. The right-hand side of (5.12) is thus a well-defined stable random variable.

To prove the equality in (5.12), choose dyadic functions $f_m(s,t)$ on $[0,1]^2$ satisfying

(5.14) $$\lim_{m\to\infty} \int_{E_n} \int_0^1 |f_m(s,t) - f(s,t)|^\alpha dsdt = 0.$$

Since the equality (5.12) clearly holds when f is replaced by f_m, it is sufficient to show that when f is replaced by f_m-f in (5.12) both sides converge a.s. to 0 along a subsequence.

For the left hand side, it is sufficient to show that the following covariation norm $|\ |_\alpha$ tends to 0 as $m \to \infty$:

$$\left|\int_{E_n} \left(\int_0^1 f_m(s,t)\phi(s)ds\right)dX_t - \int_{E_n} \left(\int_0^1 f(s,t)\phi(s)ds\right)dX_t\right|_\alpha^\alpha$$

$$= \int_{E_n} \left|\int_0^1 (f_m(s,t) - f(s,t))\phi(s)ds\right|^\alpha dt$$

$$\leq \|\phi\|^\alpha_{L^{\alpha'}} \int_{E_n} \int_0^1 |f_m(s,t) - f(s,t)|^\alpha dsdt$$

by Hölder's inequality. This tends to 0 as $m \to \infty$, by (5.14).

For the right hand side of (5.12), it is sufficient to show that the following $L^1(\Omega)$ norm tends to 0 as $m \to \infty$:

$$E\left|\int_0^1 \phi(s)\left(\int_{E_n} f_m(s,t)dX_t\right)ds - \int_0^1 \phi(s)\left(\int_{E_n} f(s,t)dX_t\right)ds\right|$$

$$\leq \int_0^1 |\phi(s)|\ E\left|\int_{E_n} (f_m(s,t) - f(s,t))dX_t\right|ds$$

$$= c_\alpha \int_0^1 |\phi(s)| \left(\int_{E_n} |f_m(s,t) - f(s,t)|^\alpha dt\right)^{1/\alpha} ds$$

$$\leq c'_\alpha \left(\int_0^1 \int_{E_n} |f_m(s,t) - f(s,t)|^\alpha dtds\right)^{1/\alpha}$$

by $\|\phi\|_\infty < \infty$ and Jensen's inequality. By (5.14), this tends to 0 as $m \to \infty$. This establishes (5.12).

2) Next we obtain (5.11) for dyadic ϕ. Therefore fix such a function ϕ and let us show that the right side of (5.12) converges in probability as n tends to infinity. For almost every s, we have that

$$\lim_{n\to\infty} \int_{E_n} f(s,t)dX_t = \int_0^1 f(s,t)dX_t \quad \text{a.s.}$$

Since ϕ is dyadic we need to deduce only

$$\lim_{n\to\infty} \int_I \int_{E_n} f(s,t)dX_t ds = \int_I \int_0^1 f(s,t)dX_t ds$$

for any fixed interval I. In order to interchange the limit and the s-integration it is enough to show that

$$\lim_{a\to\infty} P(\sup_n (\int_0^1 \left| \int_{E_n} f(s,t)dX_t \right|^\alpha ds)^{1/\alpha} > a) = 0 \tag{5.15}$$

since then the functions $\int_{E_n} f(s,t)dX_t$ are a.s. uniformly integrable on I.

Now by independence of ξ and η introduced above, Jensen's inequality and (5.10) we have for any $1 < p < \alpha$

$$E(\int_0^1 \left| \int_{E_n} f(s,t)dX_t \right|^\alpha ds)^{p/\alpha} \leq E(\int_0^1 \left| \int_0^1 f(s,t)dX_t \right|^\alpha ds)^{p/\alpha} < \infty. \tag{5.16}$$

For almost every fixed s the random variables $\int_{E_n} f(s,t)dX_t$ form a martingale. It then follows from (5.16) that the sequence

$$Y_n = (\int_0^1 \left| \int_{E_n} f(s,t)dX_t \right|^\alpha ds)^{1/\alpha}$$

forms a nonnegative submartingale. The desired result (5.15) now follows by Doob's inequality:

$$P(\sup_n Y_n > a) \leq \frac{1}{a^p} E(\sup_n Y_n)^p$$

$$\leq \frac{p}{(p-1)a^p} \sup_n E(Y_n)^p$$

$$< \infty$$

where the last inequality follows from (5.16).

By (5.12) the random variables $\int_{E_n} (\int_0^1 f(s,t)\phi(s)ds)dX_t$ converge in probability as n tends to infinity. Hence the integrands $1_{E_n}(t) \int_0^1 f(s,t)\phi(s)ds$ converge in L^α to $\int_0^1 f(s,t)\phi(s)ds$. In particular the left side of (5.11) is well-defined and equals the right side.

3) There remains to prove (5.11) for general $\phi \in L^{\alpha'}$. Given such a function ϕ choose dyadic ϕ_n with $|\phi_n - \phi|_{L^{\alpha'}} \to 0$. By (5.10) we have that the random variables $\int_0^1 \phi_n(s) \int_0^1 f(s,t)dX_t ds$ converge a.s. By step 2) of this proof the random variables $\int_0^1(\int_0^1 f(s,t)\phi_n(s)ds)dX_t$ also converge a.s. Thus the integrands $\int_0^1 f(s,t)\phi_n(s)ds$ converge in L^α. On the other hand, by choice of the ϕ_n,

$$\int_0^1 f(s,t)\phi_n(s)ds \to \int_0^1 f(s,t)\phi(s)ds$$

for almost every t. The latter function thus belongs to L^α, the left side of (5.11) is well-defined, and the two sides of (5.11) are equal. □

<u>Proof of Part (1) of Theorem 1.3</u>. Define $f_s(t) = f(s,t) = f_t(s)$ and let

$$E_1(f(s,t)|\mathcal{G}_n) = E(f_s|\mathcal{G}_n)(t)$$

and

$$E_2(f(s,t)|\mathcal{G}_n) = E(f_t|\mathcal{G}_n)(s)$$

Fix p satisfying $1 < p < \alpha$ and define f_n as in Lemma 5.3 with $\varepsilon_n = 2^{-n}$. For each $k = 1,2,\ldots$ choose n_k so that

$$(5.17)\qquad (E(\int_0^1 \left| \int_0^t f_n(s,t)dX_s - \int_0^t f(s,t)dX_s \right|^{\alpha} dt)^{p/\alpha})^{1/p} \leq 1/k$$

for $n \geq n_k$. By Lemma 5.1 and relation (5.17) the sequence

$$\int_0^1 E_2(f_{n_k}(s,t)|\mathcal{G}_{n_k+\ell})dX_s, \quad \ell = 1,2,\ldots,$$

is an $L^p(\Omega)$-bounded, $L^{\alpha}[0,1]$-valued martingale for $0 < p < \alpha$. Since $L^{\alpha}[0,1]$ has the Radon-Nikodym property the martingale convergence theorem for $L^{\alpha}[0,1]$-valued martingales (see, e.g. Diestel and Uhl (1977)) implies that

$$\lim_{\ell\to\infty} E(\int_0^1 \left| \int_0^t E_2(f_{n_k}(s,t)|\mathcal{G}_{n_k+\ell})dX_s - \int_0^t f_{n_k}(s,t)dX_s \right|^{\alpha} dt)^{p/\alpha} = 0.$$

Thus we may choose $n_k' > n_k$ such that

$$(5.18)\qquad \{E(\int_0^1 \left| \int_0^t E_2(f_{n_k}(s,t)|\mathcal{G}_{n_k'})dX_s - \int_0^t f_{n_k}(s,t)dX_s \right|^{\alpha} dt)^{p/\alpha}\}^{1/p} < 1/k.$$

For any $\ell = 1,2,\ldots$ we have that

$$E_1(\int_0^t E_2(f_{n_k}(s,t)|\mathcal{G}_{n_k'})dX_s|\mathcal{G}_{n_k'+\ell}) = \int_0^t E_1(E_2(f_{n_k}(s,t)|\mathcal{G}_{n_k'})|\mathcal{G}_{n_k'+\ell})dX_s,$$

the interchange of orders of integration being justified by Lemma 5.4 since f_{n_k} vanishes above the diagonal. By Lemma 5.2 we may find $n_k'' > n_k'$ with

(5.19)

$$\{E(\int_0^1 |\int_0^t E_1(E_2 f_{n_k}(s,t)|\mathcal{G}_{n_k'})|\mathcal{G}_{n_k''})dX_s - \int_0^t E_2(f_{n_k}(s,t)|\mathcal{G}_{n_k'})dX_s|^\alpha dt)^{p/\alpha}\}^{1/p} < 1/k.$$

Set

$$h_k(s,t) = E_1(E_2(f_{n_k}(s,t)|\mathcal{G}_{n_k'})|\mathcal{G}_{n_k''}). \tag{5.20}$$

The functions h_k are clearly dyadic and moreover satisfy $h_k(s,t) = 0$ for $s \geq t$ and on dyadic squares that straddle the diagonal. This is so because if I is a dyadic interval of length $2^{-n_k'}$, then f_{n_k} is identically zero on $I \times I$, hence both conditional expectations in (5.20) vanish on $I \times I$. Applying (5.17), (5.18) and (5.19) above with $1 \leq p < \alpha$, we have

$$(E(\int_0^1 |\int_0^t h_k(s,t)dX_s - \int_0^t f(s,t)dX_s|^\alpha dt)^{p/\alpha})^{1/p} \leq 3/k,$$

This establishes relation (5.2).

We can complete the construction of the h_k, by extending $h_k(s,t)$ to the region $s > t$ so as to be symmetric.

We now show that $J'(h_k) \overset{p}{\to} J'(f)$. For notational convenience reindex the h_k so that we may write

$$h_n(s,t) = \sum_{j=1}^{2^n} \sum_{i=1}^{j-1} a_{ij}^{(n)} I_{n,i}(s) I_{n,j}(t) \tag{5.21}$$

for $s < t$ where $I_{n,i}$ is the indicator function of the dyadic interval $[\frac{i-1}{2^n}, \frac{i}{2^n})$. Set $V_n(t) = \int_0^t h_n(s,t)dX_s$ and $V(t) = \int_0^t f(s,t)dX_s$. Then we have $\int_0^1 |V_n(t) - V(t)|^\alpha dt \overset{p}{\to} 0$. By Kallenberg (1975),

$$\int_0^1 V_n(t)dX_t \xrightarrow{p} \int_0^1 V(t)dX_t. \tag{5.22}$$

The $L^p(\Omega)$ convergence of $J'(h_n)$ to $J'(f)$ and the inequality in Part (1) of the theorem now follow easily. Set $1 < p < \alpha$. We have by (5.22), Theorem 1.1, relations (1.23) and (5.2),

$$\begin{aligned}
E\left|\int_0^1\int_0^t f(s,t)dX_s dX_t\right|^p &= \lim_{n\to\infty} E\left|\int_0^1\int_0^t h_n(s,t)dX_s dX_t\right|^p \\
&= \lim_{n\to\infty} E\left|\sum_{j=1}^{2^n}\sum_{i=1}^{j-1} a_{ij}^{(n)}\Delta X_{n,i}\Delta X_{n,j}\right|^p \\
&\le \lim_{n\to\infty} C(p,2)E\left|\sum_{j=1}^{2^n}\sum_{i=1}^{j-1} a_{ij}^{(n)}\Delta X_{n,i}\Delta\tilde{X}_{n,j}\right|^p \\
&= \lim_{n\to\infty} C'_{\alpha,p}\, E\Big(\sum_{j=1}^{2^n}\Big|\sum_{i=1}^{j-1} a_{ij}^{(n)}\Delta X_{n,i}\Big|^\alpha 2^{-n}\Big)^{p/\alpha} \\
&= \lim_{n\to\infty} C'_{\alpha,p}\, E\Big(\int_0^1\Big|\int_0^t h_n(s,t)dX_s\Big|^\alpha dt\Big)^{p/\alpha} \\
&= C'_{\alpha,p}\, E\Big(\int_0^1\Big|\int_0^t f(s,t)dX_s\Big|^\alpha dt\Big)^{p/\alpha}
\end{aligned}$$

where the $\Delta\tilde{X}_{n,j}$ are based on an independent copy $\tilde{X}_t$ of X_t. This establishes the right side of (1.17). The left side is proved similarly using the right side of (1.2), and the proof of Part (1) of theorem 1.3 is complete.

<u>Proof of Part (2) of Theorem 1.3</u>. We shall obtain the required dyadic functions g_n by modifying the h_n on the squares $I_{n,i} \times I_{n,i}$ that straddle the diagonal.

Let $J_{n,i} = [\sqrt{2}(i-1)2^{-n}, \sqrt{2}i2^{-n})$. Since step functions of the form $\sum_{i=1}^{2^n} b_i J_{n,i}$ are dense in $L^{\alpha/2}[0,\sqrt{2}]$ we may find constants $b_i^{(n)}$ such that

$$(5.23) \qquad \int_0^1 \left| \sum_{i=1}^{2^n} b_i^{(n)} I_{n,i}(t) - f(t,t) \right|^{\alpha/2} dt \to 0$$

as $n \to \infty$. We thus have

$$(5.24) \qquad \sum_{i=1}^{2^n} b_i^{(n)} \Delta\nu_{n,i} \to \int_0^1 f(t,t) d\nu_t$$

in L^p for each $0 < p < \alpha/2$ where $\Delta\nu_{n,i}$ is the increment of ν_t over $J_{n,i}$.

It follows from the results of Greenwood (1969) that for each fixed n and i we have

$$\Delta\nu_{n,i} = \lim_{m\to\infty} 2 \sum_{j=a}^{b} (\Delta X_{m,j})^2$$

in L^p as above, where $(i-1)2^{-n} = a2^{-m}$ and $i2^{-n} = b2^{-m}$. Using this and (5.24) it is then easy to find new constants $b_{M,i}$ such that

$$\sum_{i=1}^{2^M} 2b_{M,i}(\Delta X_{M,i})^2 \to \int_0^1 f(t,t) d\nu_t$$

in L^p for $0 < p < \alpha/2$.

Take for $g_n(s,t)$ the value $h_n(s,t)$ if the point (s,t) does not belong to one of the squares $I_{n,i} \times I_{n,i}$. On $I_{n,i} \times I_{n,i}$, take for $g_n(s,t)$ the value $2b_{n,i}$. The result of Part (2) follows. $\square$

Proof of Corollary 1.2. Write $A_n \approx B_n$ if there are positive constants C' and C such that $C'|A_n| \leq |B_n| \leq C|A_n|$ for all n. Restrict the domain of $f(s,t)$ to the unit square $[0,1]^2$.

We may view $\int_0^1 f(s,t)dX_s$ as a random element of $L^0[0,1]$. Indeed, by Lemma 4.1, $\int_0^t f(s,t)dX_s$ is a random element of $L^0[0,1]$ and so is $\int_t^1 f(s,t)dX_s$. (In the second case, Lemma 4.1 may be applied as stated if $f(s,t)$ is replaced by $f(1-s,1-t)$ and X_s is replaced by $X_{(1-s)+}$.)

Suppose first that $\lambda'_{\alpha,p}(f) < \infty$. Let h_n be the dyadic functions defined in (5.20) and (5.21) and set $h_n(s,t) = h_n(t,s)$. Then by Part 1) of Theorem 1.3, the decoupling inequalities, Lemma 3.3 and relation (1.23), we have

$$E\Big|\int_0^1 \Big| \int_0^t f(s,t)dX_s\Big|^\alpha dt)^{p/\alpha} \approx E|J'(f)|^p$$

$$\approx \lim_{n\to\infty} E|J'(h_n)|^p$$

$$\approx \lim_{n\to\infty} E\Big| \sum_{\substack{i<j \\ i,j=1,\dots,2^n}} a_{ij}^{(n)} \Delta X_{n,i} \Delta\tilde{X}_{n,j}\Big|^p$$

$$\approx \lim_{n\to\infty} E\Big| \sum_{\substack{i\neq j \\ i,j=1,\dots,2^n}} a_{ij}^{(n)} \Delta X_{n,i} \Delta\tilde{X}_{n,j}\Big|^p$$

$$= E(\int_0^1 \Big| \int_0^1 f(s,t)dX_s d\tilde{X}_t\Big|^p$$

$$= c_{p,\alpha}^p \; E(\int_0^1 \Big| \int_0^1 f(s,t)dX_s\Big|^\alpha dt)^{p/\alpha}$$

where the $\Delta\tilde{X}_{n,j}$ are based on an independent copy $\tilde{X}_t$ of X_t.

Suppose now that

$$E\left(\int_0^1 \left| \int_0^1 f(s,t)dX_s \right|^\alpha dt\right)^{p/\alpha} < \infty.$$

To show that $\lambda'_{\alpha,p}(f) < \infty$, construct dyadic functions H_k as in formula (5.20) with f_{n_k} replaced by f. The arguments using the martingale convergence theorems show that $(\int_0^1 | \int_0^1 H_k(s,t)dX_s|^\alpha)^{1/\alpha}$ converges to $(\int_0^1 | \int_0^1 f(s,t)dX_s|^\alpha)^{1/\alpha}$ in $L^p(\Omega)$. Then, by Lemma 3.3, $(\int_0^1 | \int_0^{t-2^{-k}} H_k(s,t)dX_s|^\alpha dt)^{1/\alpha}$ converges in $L^p(\Omega)$ and the limit may be identified with $(\int_0^1 | \int_0^t f(s,t)dX_s|^\alpha dt)^{1/\alpha}$. Thus $\lambda'_{\alpha,p}(f) < \infty$. □

6. Proof of Theorem 1.4 (Analytic characterization of Λ_α' and Λ_α)

As in Section 5, we restrict the domain of the function $f(s,t)$ to the unit square $[0,1]^2$ and we interpret the condition $f \in \Lambda_\alpha'$ to mean

$$(6.1) \qquad P\left(\int_0^1 \left| \int_0^1 f(s,t)dX_s \right|^\alpha dt < \infty\right) = 1.$$

This is justified by Corollary 1.2. We shall use results that are established in the Appendix.

Observe that while f may not belong to $L^\alpha([0,1]^2)$, the function $f(\cdot,t)$ belongs to $L^\alpha[0,1]$ for almost all t. Therefore, for any $\phi \in L^{\alpha'}[0,1]$, $1/\alpha' + 1/\alpha = 1$, the function

$$(A_f\phi)(t) = \int_0^1 f(s,t)\phi(s)ds$$

is defined for almost all t, and is such that we may change the order of integration in $\int_0^1 (A_f\phi)(t)dX_t$ (see Lemma 5.4 above.)

Lemma 6.1. Suppose that f satisfies (6.1). Then A_f is a linear continuous map from $L^{\alpha'}[0,1]$ to $L^\alpha[0,1]$.

Proof. Applying Lemma 5.4 and the definition of the norm of a linear functional, we get

$$\sup_{||\phi||_{L^{\alpha'}}=1} \left|\int_0^1 (A_f\phi)(t)dX_t\right| = \sup_{||\phi||_{L^{\alpha'}}=1} \left|\int_0^1 \phi(s)\left(\int_0^1 f(s,t)dX_t\right)ds\right|$$

$$= \left|\left|\int_0^1 f(s,t)dX_t\right|\right|_{L^\alpha}$$

which is a.s. finite by (6.1). There is therefore a constant M such that $||A_f\phi||_{L^\alpha} \leq M$ for all $||\phi||_{L^{\alpha'}} = 1$. Hence $||A_f|| < \infty$ and A_f is continuous. □

Let λ^{α} denote the standard symmetric stable cylinder probability of index α on $L^{\alpha'}[0,1]$. Although λ^{α} is not countably additive it induces a countably additive measure on each finite-dimensional quotient space of $L^{\alpha'}[0,1]$, which can be interpreted as the probability distribution of the random vector having components $\int_0^1 f_i(s)dX_s$, $i = 1,\ldots,\ell$ where the f_i are in $L^{\alpha}[0,1]$. (See the Appendix for more detail.)

A map A_f: $L^{\alpha'}[0,1] \to L^{\alpha}[0,1]$ is radonifying if $\lambda^{\alpha} \circ A_f^{-1}$ is a Radon measure on $L^{\alpha}[0,1]$. Since $L^{\alpha}[0,1]$ is separable and has a separable dual if $\alpha > 1$, this is equivalent to the statement that $\lambda^{\alpha} \circ A_f^{-1}$ extends to a countably additive Borel probability measure on L^{α}. In particular, $\lambda^{\alpha} \circ A_f^{-1}$ is tight by Ulam's theorem (Billingsley, 1968).

We prove in the Appendix (see Proposition A.1) that A_f is radonifying if f satisfies (6.1). Conversely (Proposition A.2), f satisfies (6.1) if A_f is continuous and radonifying.

The seminar of Maurey-Schwartz has investigated in depth the properties of radonifying maps and, in particular, the connection between summing properties and the radonifying property. We will use the following special case of Theorem 2, page V.4 of Maurey (1982). (Also see Schwartz (1981).)

Theorem (Maurey). A bounded linear operator A: $L^{\alpha'}([0,1]) \to L^{\alpha}([0,1])$ radonifies λ^{α} if and only if A is q-summing for some $0 < q < \alpha$. Moreover, if A is q-summing for one such q then A is q-summing for all $0 < q < \alpha$ and hence completely summing.

Proof of Theorem 1.4. The proof is in 4 parts.

1) If f satisfies (6.1), then the operator A_f is continuous (Lemma 6.1), radonifying (Proposition A.1 of the Appendix) and hence completely

summing (Maurey's theorem). Conversely, suppose A_f is a continuous completely summing map from $L^{\alpha'}[0,1]$ to $L^{\alpha}[0,1]$. It is radonifying (Maurey's theorem) and it follows from Proposition A.2 of the Appendix that (6.1) holds.

2) We begin by proving half of the equivalence of the norms $\lambda'_{\alpha,p}$ and $| \ |_{*q}$ directly:

(6.2) $$(E(\int_0^1 |\int_0^t f(s,t)dX_s|^{\alpha}dt)^{p/\alpha})^{1/p} \leq C_{p,\alpha,q}|A_f|_{*q}$$

for every $1 < p < \alpha$ and $1 < q < \alpha$.

Suppose no such inequality were true. Then we could find a sequence f_j of functions on $[0,1]^2$, each satisfying (6.1), with the following properties for fixed $1 < p < \alpha$ and $1 < q < \alpha$:

(6.3) $$|A_{f_j}|_{*q} \leq 2^{-j}$$

(6.4) $$(E\int_0^1 |\int_0^t f_j(s,t)dX_s|^{\alpha}dt)^{p/\alpha})^{1/p} \geq j$$

(6.5) $\text{support}(f_i(\cdot,t)) \cap \text{support}(f_j(\cdot,t')) = \emptyset$ for all t,t' and $i \neq j$.

In fact, since X_t scales, each square straddling the main diagonal has properties similar to $[0,1]^2$ and therefore it is possible to choose the support of each f_j to be a square straddling the main diagonal and such that the support of f_i and f_j are disjoint for $i \neq j$.

By (6.5), the f_j's have disjoint support, and therefore the function $f = \sum_{j=1}^{\infty} f_j$ and the map $A_f = \sum_{j=1}^{\infty} A_{f_j}$ are well-defined. Since $|A_f|_{*q} \leq \sum |A_{f_j}|_{*q} < \infty$, A_f is completely summing and so f satisfies (6.1).

Let $\xi_j(\cdot)$ be the independent random elements of $L^\alpha[0,1]$ with disjoint support defined by $\xi_j(t) = \int_0^t f_j(s,t)dX_s$ (independence and disjointness of the support follow from (6.5)). Set also $\xi(t) = \int_0^t f(s,t)dX_s$. Then

$$\xi(t) = \sum_{j=1}^{\infty} \xi_j(t) \tag{6.6}$$

almost surely, for each fixed t.

We now show that the two sides in (6.6) are equal as random elements in $L^\alpha[0,1]$. Note first that for almost all ω, we have

$$\left|\{t\colon \xi_i(t) \neq 0, \ \ \xi_j(t) \neq 0 \ \text{ for some } \ i \neq j\}\right| = 0,$$

so that $\sum \xi_j(t,\omega)$ converges a.s. in $L^0[0,1]$. Both $\xi(t,\omega)$ and $\sum \xi_j(t,\omega)$ are random elements of $L^0[0,1]$, there are almost surely equal for each fixed t, and therefore by Fubini's theorem they are equal as random elements of $L^0[0,1]$, and hence as random elements of $L^\alpha[0,1]$ since ξ satisfies (6.1). Since the ξ_j have disjoint support, we have

$$E|\xi_j|^p_{L^\alpha} = E\left(\sum_{j=1}^{\infty} |\xi_j|^\alpha_{L^\alpha}\right)^{p/\alpha} \geq E(|\xi_j|^p_{L^\alpha}) \geq j,$$

contradicting Theorem 1.2. Hence the inequalities (6.2) hold.

3) The space of functions Λ'_α is complete under the norm $\lambda'_{\alpha,p}$ (Theorem 1.2). We now prove that it is also complete under the norm $|\ |_{*q}$, where $|\ |_{*q}$ is viewed as a norm on the functions f in Λ'_α via $|f|_{*q} = |A_f|_{*q}$.

Let $\phi_1,\phi_2,\ldots$ be the standard Haar functions. They form a basis for L^α. For each continuous linear map $A\colon L^{\alpha'} \to L^\alpha$ there is an associated matrix a_{ij} of real numbers defined by $A\phi = \sum_{j=0}^\infty a_{ij}\phi_j$, $i = 0,1,\ldots,$ with the series convergent in L^α. Since the span of the ϕ_i is dense in $L^{\alpha'}$, the operator A is uniquely determined by the a_{ij}.

The space of p-summing operators A_f is a Banach space with norm $|A_f|_{*q}$. Take $f \in \Lambda_\alpha'$ such that $|A_{f_n} - A_{f_m}|_{*q} \to 0$ as $n,m \to \infty$ and let A be the q-summing operator such that $|A_{f_n} - A|_{*q} \to 0$. By (6.2) we have $\lambda_{\alpha,p}'(f_n - f_m) \to 0$ and therefore $\lambda_{\alpha,p}'(f_n - f) \to 0$ for some $f \in \Lambda_\alpha'$. Then we must have $A = A_f$ for the two operators have the same associated matrices:

$$\begin{aligned}
\int_0^1 \phi_j(t)(A\phi_i)(t)dt &= \lim_{n\to\infty} \int_0^1 \phi_j(t)(A_{f_n}\phi_i)(t)dt \\
&= \lim_{n\to\infty} \int_0^1\int_0^1 \phi_j(t)f_n(s,t)\phi_i(s)dsdt \\
&= \int_0^1\int_0^1 \phi_j(t)f(s,t)\phi_i(s)dsdt \\
&= \int_0^1 \phi_j(t)(A_f\phi_i)(t)dt.
\end{aligned}$$

The third equality holds since the Haar functions are bounded and $\lambda_{\alpha,p}'(f_n - f) \to 0$ implies $|f_n - f|_{L^1([0,1]^2)} \to 0$. (See (4.5).)

4) To complete the proof of the equivalence of the norms $\lambda'_{\alpha,p}$ and $|\ |_{*q}$, we use the conclusion of part 3) of this proof, relation (6.2) and the following fact. Let B be a Banach space which is complete under the norm $\|\ \|_1$ and $\|\ \|_2$. If $\|\ \|_1 \leq C\|\ \|_2$ then $\|\ \|_2 \leq C'\|\ \|_1$. (Proof: apply the closed graph theorem to the identity map $(B,\|\ \|_2) \to (B,\|\ \|_1)$.

APPENDIX

For the reader's convenience we present here a very brief discussion of stable cylinder probabilities. We also prove some results which are used in Section 6.

Let B be a Banach space. A finitely additive measure λ on B is called a symmetric stable cylinder probability of index α if λ induces a stable symmetric distribution on each finite-dimensional quotient space of B, i.e., if K is a subspace of B of finite codimension and $\pi: B \to B/K$ denotes the canonical projection, then $\lambda \circ \pi^{-1}$ defines a symmetric α-stable probability distribution on the finite dimensional vector space B/K.

We are interested in the case $B = L^{\alpha'}[0,1]$, $1/\alpha + 1/\alpha' = 1$, in which case we denote by λ^{α} the symmetric α stable cylinder probability defined in terms of the stable process X_t as follows:

Let $f_1,\ldots,f_\ell$ be linearly independent functions in $L^{\alpha}[0,1]$ and $K \subset L^{\alpha'}[0,1]$ be the annihilator of the f_i,

$$K = \{\phi \in L^{\alpha'}[0,1]: \langle\phi,f_i\rangle = 0, \quad i = 1,2,\ldots,\ell\}.$$

Then K has codimension ℓ. Let $\pi: L^{\alpha'} \to L^{\alpha'}/K$ be the quotient map and let λ_{π}^{α} be the stable distribution on $L^{\alpha'}/K$ with characteristic function given by

$$\text{(A.1)} \qquad \int_{L^{\alpha'}/K} \exp\Big(i \sum_{j=1}^{\ell} u_j\langle f_j,\overline{g}\rangle\Big)d(\lambda_{\pi}^{\alpha})(\overline{g}) = E\Big(\exp i \sum_{j=1}^{\ell} u_j \int_0^1 f_j(s)dX_s\Big).$$

(Note that the pairing $\langle f_j,\overline{g}\rangle$ is well-defined.) Let Π denote the collection of such projections obtained as the collection $\{f_i\}$ varies and put $C = \cup_{\pi\in\Pi} \pi^{-1}((\pi(L^{\alpha'}))$ where $B(\pi(L^{\alpha'}))$ denotes the family of Borel subsets of $\pi(L^{\alpha'})$. The family λ_π^α is consistent in the sense that if $\pi_i: L^\alpha \to L^\alpha/K_i$, $i = 1,2$, with $K_1 \subseteq K_2$, and $\bar{\pi}: L^{\alpha'}/K_1 \to L^{\alpha'}/K_1/K_2/K_1 \simeq L^{\alpha'}/K_2$ then $\lambda_{\pi_2}^\alpha = \lambda_{\pi_1}^\alpha \circ \bar{\pi}^{-1}$. Thus there is a well-defined finitely additive measure λ^α on C defined by $\lambda^\alpha(\pi^{-1}(E)) = \lambda_\pi^\alpha(E)$ for $E \in B(\pi(L^{\alpha'}))$. Intuitively, one may think of λ^α as the distribution of α-stable "noise", dX_t.

The measure λ^α is not a countably additive Borel probability measure on L^α. Recall that the map $A_f: L^{\alpha'} \to L^\alpha$ radonifies λ^α if $\lambda^\alpha \circ A_f^{-1}$ extends to a countably additive Borel probability measure on L^α (see Section 6). The following propositions give a necessary and sufficient condition for A_f to radonify λ^α.

Let $f(s,t)$ be a symmetric function satisfying (6.1) above and let

$$Y_t = \int_0^1 f(s,t)dX_s$$

which we may view as a random element of $L^\alpha[0,1]$. Let P denote the probability measure on the sample space Ω. We then prove

<u>Proposition A.1</u>. Suppose that f satisfies (6.1) and let $A_f: L^{\alpha'}[0,1] \to L^\alpha[0,1]$ be defined as in Section 6. Then

$$\lambda^\alpha \circ A_f^{-1} = P \circ Y^{-1}$$

as cylinder measures on $L^\alpha[0,1]$.

Proof: Choose $g_1, g_2, \ldots, g_\ell \in L^{\alpha'}$. Let $K \subseteq L^\alpha$ denote the annihilator of $g_1, \ldots, g_\ell$ and let $K' = A_f^{-1}(K)$. If $\pi: L^\alpha \to L^\alpha/K$ and $\pi': L^{\alpha'} \to L^{\alpha'}/K'$, then the map $A_f: L^{\alpha'} \to L^\alpha$ induces a map $\overline{A}_f$ from $L^{\alpha'}/K'$ to L^α/K so that $\pi \circ A_f = \overline{A}_f \circ \pi'$. Let $\overline{g}_i$ denote $\pi'(g_i)$. We then have

$$\text{(A.2)} \qquad \int_{L^\alpha/K} \exp\{i \sum_{j=1}^{\ell} u_j \langle \overline{\phi}, \overline{g}_j \rangle\} d(\lambda^\alpha \circ A_f^{-1})_\pi(\overline{\phi})$$

$$= \int_{L^\alpha/K} \exp\{i \sum_{j=1}^{\ell} u_j \langle \overline{\phi}, \overline{g}_j \rangle\} d(\lambda^\alpha_{\pi'} \circ \overline{A}_f^{-1})(\overline{\phi}).$$

After the change of variables $\overline{\phi} = \overline{A}_f(\overline{g})$ we obtain

$$\int_{L^{\alpha'}/K'} \exp\{i \sum_{j=1}^{\ell} u_j \langle \overline{A}_f(\overline{g}), \overline{g}_j \rangle\} d\lambda^\alpha_{\pi'}(\overline{g})$$

$$= \int_{L^{\alpha'}/K'} \exp\{i \sum_{j=1}^{\ell} u_j \langle \overline{A}_f(\overline{g}_j), \overline{g} \rangle\} d\lambda^\alpha_{\pi'}(\overline{g})$$

$$= E \exp\{i \sum_{j=1}^{\ell} u_j \int_0^1 A_f g_j(s) dX_s\}$$

where we used the symmetry of f in the first equality and (A.1) in the second.

By Lemma 5.4 we obtain

$$\text{(A.3)} \qquad E \exp\{i \sum_{j=1}^{\ell} u_j \int_0^1 A_f g_j(s) dX_s\} = E \exp\{i \sum_{j=1}^{\ell} u_j \int_0^1 g_j(t) Y_t dt\}.$$

Since the left hand side of (A.2) equals the right hand side of (A.3) the characteristic functional of $\lambda^\alpha \circ A_f^{-1}$ agrees with that of Y_t. The proof is complete.

Proposition A.2. Suppose A_f is continuous as a map from $L^{\alpha'}$ to L^{α} and that $\lambda^{\alpha} \circ A_f^{-1}$ is Radon on L^{α}. Then Y_t defines a random element of $L^{\alpha}[0,1]$, i.e., (6.1) holds.

Proof. Let Y_{ϕ} be the $L^{\alpha'}$-indexed stochastic process defined by

$$Y_{\phi} = \int_0^1 \left(\int_0^1 f(s,t)\phi(s)ds\right)dX_t,$$

$\phi \in L^{\alpha'}$. This is well defined in view of the continuity of A_f.

We begin by showing that the random linear functional Y_{ϕ} has a continuous version, $\overline{Y}_{\phi}$. That is, we construct $\overline{Y}_{\phi}$ so that

$$\text{(A.4)} \qquad \sup_{\|\phi\|_{\alpha'} \leq 1} |\overline{Y}_{\phi}| < \infty \quad \text{a.s.,}$$

and

$$\text{(A.5)} \qquad Y_{\phi} = \overline{Y}_{\phi} \quad \text{a.s.,} \quad \phi \in L^{\alpha'}.$$

To see this choose a linearly independent, dense sequence $\phi_1, \phi_2, \ldots$ of functions in the unit ball of $L^{\alpha'}$. Let $\pi_n \colon L^{\alpha} \to V_n := L^{\alpha}/K_n$ where $K_n \subseteq L^{\alpha}$ is the annihilator of $\phi_1, \phi_2, \ldots, \phi_n$. The calculation in Proposition A.1 (prior to the use of Lemma 5.4) shows that the random vector $(Y_{\phi_1}, \ldots, Y_{\phi_n})$ has probability distribution $(\lambda_{\alpha} \circ A_f^{-1})_{\pi_n}$, i.e., there are linear coordinate functionals $x_1, \ldots, x_n$ of norm at most 1 on V_n so that for any Borel subset E of R^n we have

$$\text{(A.6)} \quad (\lambda^{\alpha} \circ A_f^{-1})_{\pi_n}(\{v \in V_n \colon (x_1(v), \ldots, x_n(v)) \in E\}) = P((Y_{\phi_1}, \ldots, Y_{\phi_n}) \in E)$$

(Strictly speaking, the functionals x_i should be doubly indexed to indicate the dependence on V_n. However we shall suppress this to simplify the notation. This will cause no difficulties in the proof because of the consistency relations amongst the V_n.)

The fact that $\lambda^\alpha \circ A_f^{-1}$ is Radon implies that for each $\varepsilon > 0$ there is a large enough k so that

(A.7) $$P(\sup_{1\leq i\leq n} |Y_{\phi_i}| > k) < \varepsilon$$

uniformly in n.

To see this, first choose a compact $\Delta \subseteq L^\alpha$ so that $\lambda^\alpha \circ A_f^{-1}(\Delta^c) < \varepsilon$. Put $\Delta_n = \pi_n(\Delta)$. Each Δ_n is a compact subset of the corresponding V_n; moreover, it follows easily from the total-boundedness of Δ that there is N so large that for $n \geq N$ we have

(A.8) $$|x_i(v)| \leq \varepsilon,\ v \in \Delta_n, \quad i = N+1,\ldots,n.$$

Let $\beta_{n,k} \subseteq V_n$ be the "cube" defined by

$$\beta_{n,k} = \{v \in V_n : |x_i(v)| < k, \quad i = 1,2,\ldots,n\}.$$

Since $\Delta_1,\ldots,\Delta_N$ are compact we may choose $k > \varepsilon$ so large that $\Delta_i \subseteq \beta_{i,k}$ for $i = 1,2,\ldots,N$. It then follows from (A.8) that we have $\Delta_n \subseteq \beta_{n,k}$ for all n. Finally, by (A.6) and choice of Δ we have

$$P(\sup_{1\leq i\leq n} |Y_{\phi_i}| > k) = (\lambda \circ A_f^{-1})_{\pi_n}(\beta_{n,k}^c) \leq (\lambda \circ A_f^{-1})_{\pi_n}(\Delta_n^c)$$
$$\leq (\lambda \circ A_f^{-1})(\Delta^c) < \varepsilon,$$

for every n, establishing (A.7).

As an immediate consequence of (A.7) we have $\sup_n |Y_{\phi_n}| < \infty$ a.s. It follows that for almost every ω the restriction of the map $\phi \to Y_\phi(\omega)$

to $\{\phi_i\}$ has a unique linear continuous extension to all of $L^{\alpha'}$, denoted $\overline{Y}_\phi$. Property (A.5) follows from the continuity of A_f.

The next step is to establish the existence of a random element $Z_s(\omega)$ of $L^\alpha[0,1]$ such that

$$\text{(A.9)} \qquad \overline{Y}_\phi = \int_0^1 \phi(s)Z_s ds, \quad \text{a.s.,}$$

for each $\phi \in L^{\alpha'}$.

To see this, fix ω for which $\phi \to \overline{Y}_\phi(\omega)$ is continuous. By the Riesz representation theorem there is an L^α function Z_s^ω with $\overline{Y}_\phi(\omega) = \int_0^1 Z_s^\omega \phi(s)ds$, for all $\phi \in L^{\alpha'}$. Let $\psi_0, \psi_1, \psi_2, \ldots$ denote the Haar functions. Then we have

$$Z_s^\omega = \sum_{i=0}^{\infty} \left(\int_0^1 Z_u^\omega \psi_i(u)du\right)\psi_i(s)$$

$$= \sum_{i=0}^{\infty} \overline{Y}_{\psi_i}(\omega)\psi_i(s) \quad \text{a.e.(s).}$$

The last written expression defines the desired random element $Z_s(\omega)$.

The last step is to show that Z_s is a version of $\int_0^1 f(s,t)dX_s$.

Recall that $\mathcal{G}_m$ is the mth dyadic σ-field of $[0,1]$ and put $E_n = \{s \in [0,1]: \int_0^1 |f(s,t)|^\alpha dt \leq n\}$. We may view $1_{E_n}(s)f(s,\cdot)$ as an L^α-valued function on $[0,1]$. By the L^α-valued martingale convergence theorem we then have the following convergence in L^α for almost every s:

$$E(1_{E_n}(s)f(s,\cdot)|\mathcal{G}_m) \underset{m\to\infty}{\to} 1_{E_n}(s)f(s,\cdot).$$

Thus, by (1.23) we have for almost every s

$$\text{(A.10)} \qquad \int_0^1 E(1_{E_n}(s)f(s,t)|\mathcal{G}_m)dX_t \overset{p}{\underset{m\to\infty}{\to}} 1_{E_n}(s) \int_0^1 f(s,t)dX_t.$$

On the other hand, for every s

$$\int_0^1 E(1_{E_n}(\cdot)f(\cdot,t)|\mathcal{G}_m)(s)dX_t$$

$$= \int_0^1 \Big(\sum_{i=1}^{2^m} 2^m \int_{I_{m,i}} 1_{E_n}(u)f(u,t)du I_{m,i}(s)\Big)dX_t$$

$$= \sum_{i=1}^{2^m} \overline{Y}_{(2^m I_{E_n} I_{m,i})} I_{m,i}(s)$$

$$= \sum_{i=1}^{2^m} \Big(2^m \int_0^1 1_{E_n} I_{m,i} Z_u du\Big) I_{m,i}(s)$$

$$= E(1_{E_n}(\cdot)Z_{(\cdot)}|\mathcal{G}_m)(s),$$

each equality holding a.s. By the martingale convergence theorem, we have almost surely that

$$E(1_{E_n}(s)Z_s|\mathcal{G}_m) \underset{m\to\infty}{\to} 1_{E_n}(s)Z_s \quad \text{a.e.}(s).$$

Combining this observation with (A.10) shows that for almost every s we have

$$1_{E_n}(s) \int_0^1 f(s,t)dX_t = 1_{E_n}(s)Z_s$$

almost surely. Since n was arbitrary and Z_s is a random element of L^α the proof is complete. □

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