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Discrete Packing and Covering: An Annotated Bibliography

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1. Introduction

Suppose $G = (V,E)$ is a digraph with vertices V, edges E and with distinct $s,t \in V$. We denote by $A = [a_{ij}]$ the (edge-) incidence matrix of directed (s,t)-paths in G, whose rows index the (s,t)-dipaths of G and whose columns index the edges of G; thus $a_{ij} = 1$ when the jth edge appears in the ith (s,t)-dipath and otherwise $a_{ij} = 0$. Then for any nonnegative vector of edge capacities $c = (c(e)\colon e \in E)$ the maximum flow problem can be modeled as

$$\max\{1 \cdot y\colon yA \leq c,\ y \geq 0\}. \qquad P(A,c)$$

In the introductory pages of

[1] L.R. Ford, Jr. and D.R. Fulkerson, Flows in Networks (Princeton University Press, Princeton, New Jersey, 1962).

this path-edge formulation of the maximum flow problem is shown to be equivalent to the "standard" formulation based on the vertex-edge incidence matrix of G.

In the integer programming restriction of $P(A,c)$,

$$\max\{1 \cdot y\colon yA \leq c,\ y \geq 0,\ y \text{ integral}\}, \qquad P_I(A,c)$$

one seeks a maximum cardinality (integral) "packing" of the rows of A into the vector c. For any nonnegative matrix A and any nonnegative vector c we will refer to $P_I(A,c)$ as a discrete packing model. Thus the integral maximum flow problem with integral capacities is termed an (s,t)-dipath packing model. For discrete packing models it is generally the case that A is an integral matrix and one usually insures boundedness of objective value in $P_I(A,c)$ by assuming that A has no row all of whose entries are 0.

Also for any nonnegative A and c, the problem

$$\min\{1\cdot y\colon\ yA \geq c,\ y \geq 0,\ y \ \text{integral}\} \qquad C_I(A,c)$$

is termed a <u>discrete covering model</u>. Consider, for instance, the case in which A is the (vertex-) incidence matrix of stable sets in a simple graph G; $C_I(A,c)$ is then the weighted vertex coloring problem for G. In most covering applications A will again be an integral matrix and usually feasibility of $C_I(A,c)$ for all nonnegative vectors c is guaranteed by assuming that A has no 0-valued columns. The linear programming relaxation of $C_I(A,c)$, denoted $C(A,c)$, is obtained by deleting the integrality stipulation in $C_I(A,c)$.

In the sequel we categorize and briefly survey certain discrete packing and covering models. The models which we consider will be well-behaved in the sense that for a given matrix A the optimum <u>values</u> of $P(A,c)$ and $P_I(A,c)$, or the analogous <u>values</u> for covering models, will always be equal or nearly equal, i.e., for all nonnegative integral c. For brevity we omit explicit definitions of the combinatorial structures which give rise to these models; for such definitions the reader is referred to the references cited. For additional surveys of this and closely related topics, the reader should consult references [2] - [7] below, which we now briefly describe. Reference [2] provides the initial and fundamental survey of blocking theory and antiblocking theory. These are polyhedral duality theories treating, respectively, polyhedra of the forms given by the linear programming duals of P(A,c) and C(A,c). Important initial examples of discrete packing and covering models are presented here. In Chapter 2 of the dissertation [3] several combinatorial optimization models are discussed in a manner stressing their common algebraic features and a general survey of discrete packing and covering models is presented in Chapter 2 of the dissertation [4]. Reference [5] demonstrates that the ellipsoid method provides an important and powerful tool for the analysis of discrete packing and covering models. Finally, in [6] combinatorial min-max statements arising from discrete packing and covering models are surveyed and in [7] min-max statements are studied from the viewpoint of establishing such results using total dual integrality (see §2 below) arguments. These comprehensive surveys provide an indispensible overview of discrete packing and covering models.

[2] D.R. Fulkerson, "Blocking and Anti-Blocking Pairs of Polyhedra," Mathematical Programming 1 (1971) 168-194.

[3] S. Baum, "Integral Near-Optimal Solutions to Certain Classes of Linear Programming Problems," Ph.D. Thesis, Cornell University, School of Operations Research and Industrial Engineering, Technical Report 360 (Ithaca, New York, 1977).

[4] O. M.-C. Marcotte, "Topics in Combinatorial Packing and Covering," Ph.D. Thesis, School of Operations Research and Industrial Engineering, Cornell University (Ithaca, New York, 1983).

[5] M. Grötschel, L. Lovász and A. Schrijver, "The Ellipsoid Method and Its Consequences in Combinatorial Optimization," Combinatorica 1 (1981) 169-197.

[6] A. Schrijver, "Min-Max Results in Combinatorial Optimization," In: A. Bachem, M. Grötschel and B. Korte (Eds.), Mathematical Programming, Bonn 1982 - The State of the Art (Springer, Berlin, 1983) pp. 439-500.

[7] A. Schrijver, "Total Dual Integrality From Directed Graphs, Crossing Families, and Sub- and Supermodular Functions," Mimeographed Manuscript (1982).

2. Strong Integrality

Considering again the maximum flow problem we note that generally the dipath incidence matrix will not be totally unimodular. Nevertheless, when A is such a matrix the following (see [2]) strong max-min property is valid: for any nonnegative integral vector c, $P(A,c)$ has an integer-valued optimum solution vector. This is not difficult to show by interpreting $P(A,c)$ and its linear programming dual in light of the max-flow, min-cut theorem (see [1]). Analogously, when A is the (edge-) incidence matrix of vertex stars in a bipartite graph, it follows from the famous theorem of König (min vertex cover = max matching) that the following strong min-max property (defined in [2]) holds: $C(A,c)$ and $C_I(A,c)$ have the same optimum solution value for any nonnegative integral vector c. The terminology used here stems from the fact that the strong max-min and min-max properties give rise to discrete or combinatorial strengthenings of the usual max-min and min-max statements arising from linear programming duality considerations. In the present section we indicate several combinatorial instances for which such strong integrality results hold.

As noted above, the maximum flow problem provides a prototypical discrete packing model for which strong integrality holds. It follows from

[8] J. Edmonds, "Edge-Disjoint Branchings," In: R. Rustin (Ed.), Combinatorial Algorithms (Algorithmics Press, New York, 1972) pp. 91-96.

also an early and fundamental result in this area, that strong integrality holds for the incidence matrix of rooted spanning branchings in a digraph. For both of these examples, blocking duality (see [2]) suggests a "dual" instance of strong integrality. For the maximum flow case, the related family of positive parts of minimal (s,t)-cuts gives a discrete packing model for which strong integrality holds--see [2] and

[9] J.T. Robacker, "Min-Max Theorems on Shortest Chains and Disjunct Cuts of a Network," The RAND Corporation, Research Memorandum RM-1660-PR (Santa Monica, California, 1956).

With regard to rooted spanning branchings, strong integrality for positive parts of rooted cuts follows from

[10] J. Edmonds, "Optimum Branchings," in: G.B. Dantzig and A.F. Veinott, Jr. (Eds.), Mathematics of the Decision Sciences, Lectures in Applied Mathematics, Vol. 11 (Am. Math. Soc., Providence, Rhode Island, 1968) pp. 346-361.

and

[11] D.R. Fulkerson, "Packing Rooted Directed Cuts in a Weighted Directed Graph," Mathematical Programming 6 (1974) 1-14.

Note that the previous two examples arise from classes of cutsets in a digraph. A related, though apparently deeper, result of Lucchesi and Younger establishes strong integrality for the (edge-) incidence matrix of directed cutsets in a digraph.

[12] C.L. Lucchesi and D.H. Younger, "A Minimax Relation for Directed Graphs," Journal of the London Mathematical Society (2) 17 (1978) 369-374.

In contrast to the earlier examples, it has been shown by Schrijver in

[13] A. Schrijver, "A Counterexample to a Conjecture of Edmonds and Giles," Discrete Mathematics 32 (1980) 213-214.

that strong integrality does not generally hold for the blocking clutter (see [2]) of directed cutsets. For particular classes of digraphs, however, for which the latter model does have the strong integrality property, see

[14] A. Frank, "Kernel Systems of Directed Graphs," Acta Scientiarium Mathematicarum (Szeged) 41 (1979) 63-76.

and

[15] A. Schrijver, "Min-Max Relations for Directed Graphs," Annals of Discrete Mathematics 16 (1982) 261-280.

Several recent papers have considered generalizing certain of the above models in a manner so that strong integrality will still hold. Most notably, in

[16] P.D. Seymour, "The Matroids with the Max-Flow Min-Cut Property," Journal of Combinatorial Theory (B) 23 (1977) 189-222.

a natural matroid generalization of the maximum flow model is considered and a forbidden minor characterization is given for the class of matroids for which the associated discrete packing model has the strong integrality property. Relying on the work of Seymour in [16], Korach has given a characterization of those instances of $P(A,c)$ for which strong integrality will hold when A is the (edge)-incidence matrix of T-cuts in an undirected graph.

[17] E. Korach, "Packings of T-Cuts, and Other Aspects of Dual Integrality," Ph.D. Thesis, Department of Combinatorics and Optimization, University of Waterloo (Waterloo, Ontario, Canada, 1982).

In [14] Frank introduces the notion of a kernel system of a digraph and uses this combinatorial structure to generalize the maximum flow and rooted spanning branching models mentioned above. A generalization is also obtained for the respective blocking models, namely, the positive parts of minimal (s,t)-cuts and the positive parts of rooted cuts. In [15] Schrijver also has generalized these results using the concept of strong connectors for a digraph; the strong integrality result of Lucchesi and Younger [12] can also be deduced using the model of [15]--see [7]. The reader is especially referred to [7] in which Schrijver details the interrelationships among various combinatorial models, including many which give rise to strong integrality results. Finally, in

[18] A. Schrijver, "Packing and Covering of Crossing Families of Cuts," Mimeographed Manuscript (1983).

Schrijver characterizes certain crossing families which, when defined on the vertices of any digraph, give rise to strong integrality, both for the collection of cuts induced by the crossing family and for the (blocking) collection of covers of the crossing family. This setting subsumes many of the examples of strong integrality discussed in [15].

We now consider strong integrality results for discrete covering models. For covering models it is plain that when matrix A has integral entries, the strong min-max condition can hold only if A is a (0,1)-valued matrix. Thus we restrict attention to models for which A is (0,1)-valued and observe the well-known result that strong integrality holds here precisely when the (set-wise) maximal rows of A correspond to the maximal cliques in a perfect graph. The subject of perfect graphs is covered thoroughly by the following two recent references:

[19] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Academic Press, New York, New York, 1980).

and

[20] C. Berge and V. Chvatal (Eds.), Topics on Perfect Graphs (to appear in Annals of Discrete Mathematics).

Essentially two approaches have emerged for establishing strong integrality results such as those outlined above. The first we consider is algebraic in nature and is based on the concept of total dual integrality, first stated in full generality in

[21] J. Edmonds and F.R. Giles, "A Min-Max Relation for Submodular Functions on Graphs," Annals of Discrete Mathematics 1 (1977) 185-204.

For packing models total dual integrality of the linear system $\{Ax \geq 1, x \geq 0\}$ arising from the linear programming dual of $P(A,c)$ is a restatement of the strong max-min stipulation relating $P(A,c)$ and $P_I(A,c)$, and similarly for covering models and systems of the form $\{Ax \leq 1, x \geq 0\}$. The use of total dual integrality as a tool for establishing combinatorial max-min and min-max statements (and hence for establishing strong integrality properties) is surveyed extensively by Schrijver in [7]. Additional important references on the topic of total dual integrality are

[22] F.R. Giles and W.R. Pulleyblank, "Total Dual Integrality and Integer Polyhedra" Linear Algebra and Its Applications 25 (1979) 191-196.

where it is shown that any integral polyhedron can be represented by a totally dual integral system of the form $\{Ax \leq b\}$ with b integral and

[23] A. Schrijver, "On Total Dual Integrality," Linear Algebra and Its Applications 38 (1981) 27-32.

which establishes existence of a unique minimal totally dual integral system $\{Ax \leq b\}$ with A integral for representing a full dimensional rational polyhedron. In the latter case $\{x: Ax \leq b\}$ is integral if and only if b is integral.

One interesting and important consequence of total dual integrality of the system $\{Ax \geq 1, x \geq 0\}$ is that $\{x: Ax \geq 1, x \geq 0\}$ is an integral polyhedron (a similar statement holds for covering models). This was observed by Fulkerson in [2]; generalizations of this result are proved in

[24] A.J. Hoffman, "A Generalization of Max Flow-Min Cut," Mathematical Programming 6 (1974) 352-359.

and in [21].

A second approach for establishing strong integrality results is algorithmic. In many of the examples cited above, a polynomial-time algorithm is known for solving $P_I(A,c)$ or $C_I(A,c)$ which yields strong integrality as a by-product. For such algorithms the reader can refer to, e.g.,

[25] E.L. Lawler, Combinatorial Optimization: Networks and Matriods (Holt, Rinehart and Winston, New York, 1976). (for the max flow problem),

[26] L. Lovász, "On Two Minimax Theorems in Graph Theory," Journal of Combinatorial Theory (B) 21 (1976) 96-103. (for rooted spanning branchings and directed cutsets),

and to [2], [11], [14] and [15] for algorithmic discussions concerning, respectively, (s,t)-cut positive parts, rooted cut positive parts, kernel systems and strong connectors. Finally, a major contribution of [5] is the use of the ellipsoid algorithm to construct a polynomial-time algorithm for solving $C_I(A,c)$ when A is the (vertex-) incidence matrix of maximal cliques of a perfect graph.

It is important to point out that all the algorithms mentioned in the previous paragraph are polynomial-time in the input length required to describe the associated graph as opposed to the length required to describe the matrix A. The point here is that, for example in the case of a perfect graph G on n vertices, even though G may have exponentially (in n) many maximal cliques (rows of A), the algorithm of [5] for solving $C_I(A,c)$ runs in time which is a polynomial function of n and the length required to describe the vector c. If we do consider matrix A as the given data, however, then it has been shown in

[27] S. Baum and L.E. Trotter, Jr., "Finite Checkability for Integer Rounding Properties in Combinatorial Programming Problems," Mathematical Programming 22 (1982) 141-147.

that the optimal values of $P(A,c)$ and $P_I(A,c)$ are equal for all non-negative integral vectors c if and only if equality holds for a certain easily described finite set of nonnegative integral c, and similarly for covering. Hence strong integrality for a given matrix A can be verified in finite time. More generally, building on the algorithmic approach in

[28] R. Chandrasekaran, "Polynomial Algorithms for Totally Dual Integral Systems and Extensions" In: P. Hansen (Ed.), Studies on Graphs and Discrete Programming, Annals of Discrete Mathematics 11 (1981) 39-51.

Cook has shown that recognition of whether a given linear system is totally dual integral is a problem in co-NP. This result appears in

[29] W. Cook, "Recognition of Totally Dual Integral Systems," CORR Report 82-20, University of Waterloo (Waterloo, Ontario, Canada, 1982).

These computational complexity results extend naturally to the setting of integer rounding, which is the topic of the following section.

3. Integer Rounding

Suppose we are given a nonnegative matrix A and a nonnegative vector c for which the optimum value of $P(A,c)$ is not an integer. Then strong integrality fails for this discrete packing model, but it is reasonable to ask whether the optimum values of $P(A,c)$ and $P_I(A,c)$ remain "close". Thus it is said that a discrete packing model has the integer round down property when, for any nonnegative integral vector c, the optimum value of $P_I(A,c)$ is given by the largest integer which does not exceed the value of $P(A,c)$. An integer round up property is defined analogously for discrete covering models. In this section we indicate several packing and covering models which have these properties; we mention again that a survey of such models also appears in the dissertation [4].

Perhaps the most well-known integer rounding results arise when A is the incidence matrix of bases in a matroid. Then integer rounding holds for both packing and covering by the rows of A; this is a consequence of the work in

[30] J. Edmonds, "Minimum Partition of a Matroid into Independent Subsets," Journal of Research of the National Bureau of Standards 69B (1965) 67-72.

and

[31] J. Edmonds and D.R. Fulkerson, "Transversals and Matroid Partition," Journal of Research of the National Bureau of Standards 69B (1965) 147-153.

These results are extended in

[32] S. Baum and L.E. Trotter, Jr., "Integer Rounding for Polymatroid and Branching Optimization Problems," S.I.A.M. Journal on Algebraic and Discrete Methods 2 (1981) 416-425.

to the setting in which the rows of A correspond to the bases of an integral polymatroid. The approach in [32] is algebraic, using (local) total unimodularity to establish a form of polyhedral integral decomposition (see below), whereas in [30] a polynomial-time algorithm is given for covering the elements of a matroid by its bases. In [4] a min-max result for machine scheduling presented in

[33] T.C. Hu, "Parallel Sequencing and Assembly Line Problems," Operations Research 9 (1961) 841-848.

is derived from the integer round up property for matroid basis covering.

Next suppose we are given an integral supply-demand network (all supply, demand and capacity data are integral) and that the rows of A are the integral feasible (edge-) flows of this network. It is shown in

[34] D.R. Fulkerson and D.B. Weinberger, "Blocking Pairs of Polyhedra Arising From Network Flows," Journal of Combinatorial Theory 18 (1975) 265-283.

that integer round down holds for the associated discrete packing model. Note that the special case of 1 source, 1 sink with unit supply, demand and capacities corresponds to the maximum flow model considered in the previous section. Similiar results are obtained in [34] for uncapacitated integral supply-demand networks (using minimal integral feasible flows) and in

[35] D.B. Weinberger, "Network Flows, Minimum Coverings, and the Four-Color Conjecture," Operations Research 24 (1976) 272-290.

corresponding integer round up results are developed for the analogous covering models. These packing and covering results are extended in

[36] L.E. Trotter, Jr. and D.B. Weinberger, "Symmetric Blocking and Anti-Blocking Relations for Generalized Circulations," Mathematical Programming Study 8 (1978) 141-158.

to models defined by matrices whose rows consist of the integral solutions to linear systems of the form $\{Nx = 0,\ a \leq x \leq b\}$, where N is a totally unimodular matrix and $0 \leq a \leq b$ with vectors a and b integral. The results of these three references are established algebraically; in [4] and in

[37] R.E. Bixby, O. M.-C. Marcotte and L.E. Trotter, Jr., "Packing and Covering with Integral Feasible Flows of Integral Supply-Demand Networks" (to appear).

polynomial-time (in the size of the network data) algorithms are given which can be used to solve $P_I(A,c)$ and $C_I(A,c)$ in the network cases.

In special cases the incidence matrix of certain common independent sets for two matroids (defined on the same ground set) exhibits integer rounding properties. When the rows of A correspond to the maximum cardinality common independent sets of two strongly base orderable matroids, integer rounding results for packing and covering are obtained in

[38] C.J.H. McDiarmid, "On Pairs of Strongly-Base-Orderable Matroids," Cornell University, School of Operations Research and Industrial Engineering, Technical Report 283 (Ithaca, New York, 1976).

Integer round up results are also obtained in [38] for the case in which the rows of A give the incidence of (set-wise) maximal common independent sets of two strongly base orderable matroids. For general matroids these results fail (see [38]). In

[39] M.D. McDaniel, "Network Models for Linear Programming Problems with Integer Rounding Properties," M.S. Thesis, School of Operations Research and Industrial Engineering, Cornell University (Ithaca, New York, 1981).

it is shown that similar results for two gammoids (a class of matroids properly subsumed by strongly base orderable matroids) can be derived from the model of [34] by consideration of an appropriate supply-demand network, thus tracing these integrality results back, in an algebraic sense, to total unimodularity of the vertex-edge incidence matrix of a digraph. The approach of [38] is algorithmic, relying on earlier work in

[40] J. Davies and C.J.H. McDiarmid, "Disjoint Common Transversals and Exchange Structures," Journal of the London Mathematical Society 14 (1976) 55-62.

Branchings provide another "matroid intersection" example for which integer rounding properties hold. Integer round down for the family of maximum cardinality branchings in a digraph and integer round up for both this family and the family of maximal branchings are established in [32] using Edmonds' "edge-disjoint (rooted) branchings theorem" (see [8]). Integer round up for the case of rooted spanning branchings (the covering analogue of Edmonds' packing result in [8]) follows from min-max results in [14] and in

[41] K. Vidyasankar, "Covering the Edge Set of a Directed Graph with Trees," Discrete Mathematics 24 (1978) 79-85.

and

[42] A. Frank, "Covering Branchings," Acta Scientiarum Mathematicarum (Szeged) 41 (1979) 77-81.

This can also be deduced from the results in [32].

As a final example we mention that in [4] the integer round up property is shown to hold for certain classes of cutting stock problems. Note that the usual formulation of the cutting stock problem is as a discrete covering model for which the rows of matrix A are the integral feasible solutions to a knapsack problem.

The integer rounding properties for packing and covering models are equivalent to a type of integral decomposition of related polyhedra (see [32]). Thus integral decomposition provides a useful means for establishing integer rounding results. Several variations of the notion of integral decomposition, as well as an indication of certain combinatorial models for which these alternative refinements hold, are presented in

[43] C.J.H. McDiarmid, "Integral Decomposition in Polyhedra," Mathematical Programming 25 (1983) 183-198.

We mention again that the computational complexity results of [27], [28] and [29] indicated in the previous section remain valid in the integer rounding framework and we add that in

[44] J. Orlin, "A Polynomial Algorithm for Integer Programming Covering Problems Satisfying the Integer Round-Up Property," Mathematical Programming 22 (1982) 231-235.

co-NP recognition of the integer rounding properties for discrete packing and covering models was first established.

4. An Open Question

For certain combinatorial families of interest a slight weakening of the notion of integer rounding may hold. In this section we indicate such a possibility for the edge-coloring problem on an undirected graph. Suppose A is the (edge-) incidence matrix of matchings in a simple undirected graph G. Then $C_I(A,1)$ is the problem of determining a minimum coloring of the edges of G. One can verify that when G is, for example, the Peterson graph (see [45]), the values of $C(A,1)$ and $C_I(A,1)$ differ by unity.

[45] J.A. Bondy and U.S.R. Murty, Graph Theory With Applications (North Holland, New York, 1976).

Thus integer round up does not hold for this discrete covering problem. Nevertheless, Vizing's theorem (see [45]) states that for any simple graph G the minimum number of colors required to color the edges of G exceeds the maximum degree of a vertex in G by at most 1, which implies that the values of $C_I(A,1)$ and $C(A,1)$ differ by at most unity. The latter assertion follows because, for any nonnegative integral vector c,

$$\min\{1 \cdot y \colon yA \geq c,\ y \geq 0,\ y \text{ integral}\}$$

$$\geq \tag{1}$$

$$\min\{1 \cdot y \colon yA \geq c,\ y \geq 0\}$$

$$= \tag{2}$$

$$\max\{c \cdot x \colon Ax \leq 1,\ x \geq 0\}$$

$$\geq \tag{3}$$

$$\max\{c \cdot x \colon x \text{ is the incidence vector of a star in } G\},$$

where relation (1) is obvious, relation (2) follows from linear programming duality theory and (3) is valid because any incidence vector of the star of a vertex in G satisfies the linear system $\{Ax \leq 1,\ x \geq 0\}$. Vizing's theorem for simple graphs thus insures that all the above expressions differ by at most unity when $c = 1$.

To what extent is the preceding development valid for multigraphs, i.e., for arbitrary nonnegative integral c in the above expressions? Vizing's theorem for a multigraph G (see [45]) asserts that the difference between the size of a minimum edge coloring and that of the largest star in G is at most the largest multiplicity of an edge in G; it is easy to construct examples for which this maximum possible difference is achieved. Thus for general c the first and last expressions above may differ by as much as the largest component of c. In

[46] P.D. Seymour, "On Multi-Colourings of Cubic Graphs, and Conjectures of Fulkerson and Tutte," Proceedings of the London Mathematical Society 38 (1979) 423-460.

Seymour raises the question of whether for general c the difference between $C_I(A,c)$ and $C(A,c)$, i.e., the difference governing relation (1) above, remains at most unity. Resolving this question seems to be quite difficult, but were it to be settled in the affirmative, edge coloring would provide a combinatorial model for which a natural weakening of the integer round up property would hold. We conclude by recalling that an integer rounding result is often accompanied by a polynomial-time algorithm for solving the associated discrete packing or covering problem. In the present instance, however, Holyer has shown in

[47] I. Holyer, "The NP-Completeness of Edge-Colouring," S.I.A.M. Journal on Computing 10 (1981) 718-720.

that the edge coloring problem is NP-complete.