A Completeness Theorem for Kleene Algebras and the Algebra of Regular Events

Dexter Kozen*

TR 90-1123
May 1990

Department of Computer Science
Cornell University
Ithaca, NY 14853-7501

*Supported by NSF Grant CCR-8901061.

A Completeness Theorem for Kleene Algebras and the Algebra of Regular Events

Dexter Kozen*
Department of Computer Science
Cornell University
Ithaca, New York 14853

May 10, 1990

Abstract

We give a finite axiomatization of the algebra of regular events involving only universal Horn formulas. Unlike Salomaa's axiomatizations, ours is sound for all interpretations over Kleene algebras.

*Supported by NSF grant CCR-8901061.

1 Introduction

Kleene algebras are algebraic structures with operators $+$, $\cdot$, *, 0, and 1 satisfying certain properties. They arise in a variety of settings: relational algebra, semantics and logics of programs, automata theory, and the design and analysis of algorithms.

An important example of a Kleene algebra is $\mathbf{Reg}_\Sigma$, the family of regular sets over a finite alphabet Σ. The equational theory of this structure has been called the *algebra of regular events*. This theory was first studied by Kleene [8], who posed axiomatization as an open problem. Salomaa [15] gave two complete axiomatizations of the algebra of regular events in 1966, but these axiomatizations depend on rules of inference that are sound under the standard interpretation in $\mathbf{Reg}_\Sigma$ but not sound in general under other interpretations. Redko [14] proved in 1964 that no finite set of equational axioms could characterize the algebra of regular events. The algebra of regular events and its axiomatization is the subject of the extensive monograph of Conway [4]. In 1981, this author gave a sound and complete infinitary equational deductive system for the algebra of regular events that is sound over all *-continuous Kleene algebras [7]. A completeness theorem for relational algebras with * was given by Ng and Tarski [11,12], but this relies on the presence of a converse operator.

There is some disagreement regarding the proper definition of Kleene algebras [4,13,7]. In this paper we define a *Kleene algebra* to be any model of the equations and equational implications listed in §2. This definition is consistent with the philosophy espoused by Pratt [13] of adopting the most general finitary characterization that captures the desired equational theory; that the desired theory is indeed captured is the main result of this paper.

By general considerations of equational logic, the axioms of Kleene algebra listed in §2, along with the usual axioms for equality, instantiation, and rules for the introduction and elimination of implications, constitute a complete deductive system for the universal Horn theory of Kleene algebras (the set of universally quantified equational implications

$$\alpha_1 = \beta_1 \wedge \cdots \wedge \alpha_n = \beta_n \quad \rightarrow \quad \alpha = \beta \tag{1}$$

true in all Kleene algebras) [17].

The main result of this paper is that this deductive system is complete for the algebra of regular events. In other words, two regular expressions α, β over Σ denote the same regular set in $\mathbf{Reg}_\Sigma$ if and only if the equation $\alpha = \beta$ is a logical consequence of the axioms. Equivalently, $\mathbf{Reg}_\Sigma$ is the free Kleene algebra on generators Σ.

The proof of completeness is essentially an implementation of the following idea: we show that the classical results of the theory of finite automata (equivalence with regular expressions, determinization via the subset construction, elimination of ϵ-transitions, and state minimization) can be coded as theorems of Kleene algebra.

1.1 Examples of Kleene Algebras

Kleene algebras abound in computer science. We have already mentioned the regular sets $\mathbf{Reg}_\Sigma$.

In the area of relational algebra, the family of binary relations on a set with the operations of $\cup$ for $+$, relational composition

$$R \cdot S \;=\; \{(x,z) \mid \exists y\ (x,y) \in R,\ (y,z) \in S\}$$

for $\cdot$, the empty relation for 0, the identity relation for 1, and reflexive transitive closure for * constitute a Kleene algebra.

In semantics and logics of programs, Kleene algebras are used to model programs in Dynamic Logic and Dynamic Algebra [7,13].

In the design and analysis of algorithms, $n \times n$ Boolean matrices and matrices over the so-called min, + algebra are used to derive efficient algorithms for reachability and shortest paths in directed graphs [2,10]. These Kleene algebras appear in [2,10] in the guise of *closed semirings*, which are precisely the S-algebras of Conway [4]. Closed semirings and S-algebras are defined in terms of an infinitary summation operator $\sum$, whose sole purpose, it seems, is to define *.

1.2 Salomaa's Axiomatizations F_1 and F_2

Let R_Σ denote the interpretation of regular expressions over Σ in the Kleene algebra $\mathbf{Reg}_\Sigma$ in which

$$R_\Sigma(a) \;=\; \{a\}\ ,\quad a \in \Sigma\ .$$

This is called the *standard interpretation.*

Salomaa [15] presented two axiomatizations F_1 and F_2 for the algebra of regular events and proved their completeness. Aanderaa [1] independently presented a system similar to Salomaa's F_1. These systems are equational except for one rule of inference in each case that is sound under the standard interpretation R_Σ, but not sound in general for interpretations over other Kleene algebras.

In Salomaa's system F_1, let us say a regular expression possesses the *empty word property* (EWP) if the regular set it denotes under R_Σ contains the null string ϵ. The EWP can be characterized syntactically: a regular expression α has the EWP if either

- $\alpha = 1$;
- $\alpha = \beta^*$ for some β;
- α is a sum of regular expressions, at least one of which has the EWP; or

- α is a product of regular expressions, both of which have the EWP.

The system F_1 contains the rule

$$\frac{\gamma + \alpha\beta = \beta \ , \quad \alpha \text{ does not have the EWP}}{\alpha^*\gamma = \beta} \ . \tag{2}$$

This rule is sound under the interpretation R_Σ.

The proviso "α does not have the EWP" in the premise of (2) is not preserved under substititution, and consequently (2) is not valid under nonstandard interpretations. For example, if α, β, and γ are the single letters a, b and c respectively, then (2) holds; but it does not hold after the substitution

$$\begin{array}{rcl} a & \mapsto & 1 \\ b & \mapsto & 1 \\ c & \mapsto & 0 \ . \end{array}$$

Another way to say this is that (2) is not to be interpreted as a universal Horn formula.

To describe Salomaa's system F_2, say a regular expression β possesses the *null set property* (NSP) if the regular set it denotes under R_Σ is $\emptyset$. As with the EWP, the NSP can be characterized syntactically: a regular expression α has the NSP if either

- $\alpha = \emptyset$;
- α is a sum of regular expressions, both of which have the NSP; or
- α is a product of regular expressions, at least one of which has the NSP.

Salomaa's second axiomatization F_2 depends on the rule

$$\frac{\beta\gamma^n\delta \leq \alpha \ , \quad 0 \leq n \leq 2^{|\alpha|} + 2 \ , \quad \gamma \text{ does not have the NSP}}{\beta\gamma^*\delta \leq \alpha} \tag{3}$$

where $|\alpha|$ is the length of the expression α. This is a truncated version of the infinitary rule

$$\frac{\beta\gamma^n\delta \leq \alpha \ , \quad n \geq 0}{\beta\gamma^*\delta \leq \alpha}$$

used to prove that the equational theories of *-continuous Kleene algebras and the algebra of regular events coincide [7]. Rule (3) is not valid in nonstandard interpretations: if $\beta = \delta = 1$, γ is the single letter c, and α is the single letter a, then $|a| = 1$ and (3) becomes

$$\frac{c^n \leq a \ , \quad 0 \leq n \leq 4}{c^* \leq a}$$

which does not hold under the interpretation

$$\begin{aligned} a &\mapsto (1+c)^4 \\ c &\mapsto c\,. \end{aligned}$$

The axioms for Kleene algebra given in §2 below are all equations or equational implications in which the symbols are regarded as universally quantified, so substitution is allowed.

2 Axioms for Kleene Algebra

A *Kleene algebra* is an algebraic structure

$$\mathcal{K} \ = \ (K,\ +,\ \cdot,\ ^*,\ 0,\ 1)$$

satisfying the following equations and equational implications:

$$\begin{aligned}
a + (b + c) &= (a + b) + c && (4)\\
a + b &= b + a && (5)\\
a + 0 &= a && (6)\\
a + a &= a && (7)\\
a(bc) &= (ab)c && (8)\\
1a &= a && (9)\\
a1 &= a && (10)\\
a(b + c) &= ab + ac && (11)\\
(a + b)c &= ac + bc && (12)\\
0a &= 0 && (13)\\
a0 &= 0 && (14)\\
1 + aa^* &\leq a^* && (15)\\
1 + a^*a &\leq a^* && (16)
\end{aligned}$$

$$\begin{aligned}
b + ax \leq x &\rightarrow a^*b \leq x && (17)\\
b + xa \leq x &\rightarrow ba^* \leq x && (18)
\end{aligned}$$

where $\leq$ refers to the natural partial order on $\mathcal{K}$:

$$a \leq b \ \leftrightarrow \ a + b = b\ .$$

Instead of (17) and (18), we might take the equivalent axioms

$$\begin{aligned}
ax \leq x &\rightarrow a^*x \leq x && (19)\\
xa \leq x &\rightarrow xa^* \leq x\ . && (20)
\end{aligned}$$

Axioms (4–7) say that $(K,\ +,\ 0)$ is an idempotent commutative monoid. Axioms (8–10) say that $(K,\ \cdot,\ 1)$ is a monoid. Axioms (4–14) say that $(K,\ +,\ \cdot,\ 0,\ 1)$ is an idempotent semiring.

The remaining axioms (15–20) deal with *. They say essentially that * behaves like the Kleene star operator of formal language theory or the reflexive transitive closure operator of relational algebra. Using (15) and the distributivity axiom (12), we see that

$$b + aa^*b \;\leq\; a^*b \; ,$$

thus the left-hand-side of the implication (17) is satisfied when a^*b is substituted for x; moreover, (17) says that a^*b is the least element of $\mathcal{K}$ for which this is true.

Axioms (17–20) are studied by Pratt [13], who attributes (17) and (18) to Schröder and Dedekind. The equivalence of (17) and (19) (and, by symmetry, of (18) and (20)) are proved in [13].

All the structures mentioned in §1, in particular $\mathbf{Reg}_\Sigma$, are Kleene algebras.

2.1 Elementary consequences

In this section we derive some basic consequences of the Kleene algebra axioms. Many of these properties have been derived before in the literature; we refer the reader to [4] for a comprehensive introduction.

It is straightforward to verify that the relation $\leq$ is a partial order, and is *monotone* with respect to all the Kleene algebra operators in the sense that if $a \leq b$, then $ac \leq bc$, $ca \leq cb$, $a + c \leq b + c$, and $a^* \leq b^*$. With respect to $\leq$, $\mathcal{K}$ is an upper semilattice with join given by $+$ and minimum element 0.

Basic properties of * such as

$$\begin{aligned} 1 &\leq a^* \\ a &\leq a^* \\ a^*a^* &= a^* \\ a^{**} &= a^* \end{aligned}$$

are also easily derived. See [4] for formal proofs.

Lemma 1 *In any Kleene algebra, a^* is the unique element satisfying (15) and (17). It is also the unique element satisfying (16) and (18).*

Proof. By (15), a^* satisfies the inequality

$$1 + ax \;\leq\; x$$

when substituted for x. By (17), it is the least such element. Thus a^* is unique.

The second assertion is proved by a symmetric argument involving (16) and (18). $\square$

Proposition 2 *In any Kleene algebra, the inequalities (15) and (16) can be strengthened to equations:*

$$\begin{aligned} 1 + aa^* &= a^* \\ 1 + a^*a &= a^* . \end{aligned}$$

Proof. The inequality $1 + aa^* \leq a^*$ is given by (15). To show

$$a^* \leq 1 + aa^* ,$$

it suffices by (17) and (10) to show that

$$1 + a(1 + aa^*) \leq 1 + aa^* .$$

But this is immediate from (15) and the monotonicity of $\cdot$ and $+$.

The proof of $1 + a^*a = a^*$ is symmetric. □

Proposition 3 (Pratt [13]) *Under the assumptions (4–15), the implications (17) and (19) are equivalent. Under the assumptions (4–14) and (16), the implications (18) and (20) are equivalent.*

Proof. We prove the first statement; the second is symmetric. First assume (17) and the premise of (19). By assumption, $ax \leq x$, therefore $x + ax \leq x$. By (17), $a^*x \leq x$. Discharging the hypothesis, we obtain the implication (19).

Now assume (19) and the premise of (17). By assumption, $b + ax \leq x$, thus $b \leq x$ and $ax \leq x$. By (19), $a^*x \leq x$, and by monotonicity, $a^*b \leq a^*x$, therefore $a^*b \leq x$. Discharging the hypothesis, we obtain the implication (17). □

The following proposition is a key tool in the completeness proof of §5.

Proposition 4 *In all Kleene algebras,*

$$ax = xb \rightarrow a^*x = xb^* .$$

Proof. Suppose first that $ax \leq xb$. Then

$$axb^* \leq xbb^*$$

by monotonicity, and

$$x + xbb^* \leq xb^*$$

by (15) and distributivity, therefore by monotonicity,

$$\begin{aligned} x + axb^* &\leq x + xbb^* \\ &\leq xb^* . \end{aligned}$$

By (17),

$$a^*x \leq xb^* .$$

By a symmetric argument using (16) and (18),

$$xb \leq ax \rightarrow xb^* \leq a^*x .$$

The proposition follows from these two implications. □

Corollary 5 *In all Kleene algebras,*

$$(cd)^*c = c(dc)^* .$$

Proof. Substitute c for x, cd for a, and dc for b in Proposition 4. □

Corollary 6 *Let p be an invertible element of a Kleene algebra with inverse p^{-1}. Then*

$$p^{-1}a^*p = (p^{-1}ap)^* .$$

Proof. We have

$$\begin{aligned} a^*p &= (pp^{-1}a)^*p \\ &= p(p^{-1}ap)^* \end{aligned}$$

by Corollary 5. The result follows by multiplying on the left by p^{-1}. □

Proposition 7 *In all Kleene algebras,*

$$(a + b)^* = a^*(ba^*)^* .$$

Proof. To show

$$(a + b)^* \leq a^*(ba^*)^* , \tag{21}$$

observe that

$$\begin{aligned} 1 &\leq a^*(ba^*)^* \\ aa^*(ba^*)^* &\leq a^*(ba^*)^* \\ ba^*(ba^*)^* &\leq (ba^*)^* \\ &\leq a^*(ba^*)^* , \end{aligned}$$

therefore

$$\begin{aligned}1 + (a+b)a^*(ba^*)^* &\le 1 + aa^*(ba^*)^* + ba^*(ba^*)^* \\ &\le a^*(ba^*)^* \ .\end{aligned}$$

Then (21) follows from (17).

To show the reverse inequality, we use the monotonicity of all the operators:

$$\begin{aligned}a^*(ba^*)^* &\le (a+b)^*((a+b)(a+b)^*)^* \\ &\le (a+b)^*((a+b)^*)^* \\ &\le (a+b)^* \ .\end{aligned}$$

□

3 Matrices over a Kleene Algebra

In this section we show that under the appropriate definitions of the operators $+$, $\cdot$, *, 0, and 1, the family $\mathcal{M}(n,\mathcal{K})$ of $n \times n$ matrices over a Kleene algebra $\mathcal{K}$ forms a Kleene algebra. This result is proved for various related classes of algebras in [4], none of which are Kleene algebras according to our definition.

Define $+$ and $\cdot$ on $\mathcal{M}(n,\mathcal{K})$ to be the usual operations of matrix addition and multiplication, respectively, Z_n the $n \times n$ zero matrix, and I_n the $n \times n$ identity matrix. The partial order $\leq$ is defined on $\mathcal{M}(n,\mathcal{K})$ by

$$A \leq B \quad \leftrightarrow \quad A + B = B\ .$$

Under these definitions, it is routine to verify

Lemma 8 *The structure*

$$(\mathcal{M}(n,\mathcal{K}),\ +,\ \cdot,\ Z_n,\ I_n)$$

is an idempotent semiring; that is, the Kleene algebra axioms (4–14) are satisfied.

Proof. See [4]. □

To define the E^* for $E \in \mathcal{M}(n,\mathcal{K})$, we first consider the case $n = 2$. This construction will later be applied inductively.

Let

$$E \quad = \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\ .$$

Let

$$f \quad = \quad a + bd^*c$$

and define

$$E^* \quad = \quad \begin{bmatrix} f^* & f^*bd^* \\ d^*cf^* & d^* + d^*cf^*bd^* \end{bmatrix}\ . \tag{22}$$

This construction is motivated by a two-state finite automaton over the alphabet $\Sigma = \{a, b, c, d\}$ with states $\{s, t\}$ and transitions $s \xrightarrow{a} s$, $s \xrightarrow{b} t$, $t \xrightarrow{c} s$, $t \xrightarrow{d} t$. For each pair of states u, v, consider the set of input strings in Σ^* taking state u to state v in this automaton.

Each such set is regular and is represented by a regular expression corresponding to those derived for the matrix E^*:

$$\begin{array}{rcl} s \to s & : & (a + bd^*c)^* \\ s \to t & : & (a + bd^*c)^*bd^* \\ t \to s & : & d^*c(a + bd^*c)^* \\ t \to t & : & d^* + d^*c(a + bd^*c)^*bd^* \ . \end{array}$$

Lemma 9 *The matrix E^* defined in (22) satisfies the Kleene algebra axioms (15–18). That is,*

$$I + EE^* \ \leq \ E^* \tag{23}$$
$$I + E^*E \ \leq \ E^* \tag{24}$$

and for any X,

$$EX \ \leq \ X \ \rightarrow \ E^*X \ \leq \ X \tag{25}$$
$$XE \ \leq \ X \ \rightarrow \ XE^* \ \leq \ X \ . \tag{26}$$

Proof. We show (23) and (25). The arguments for (24) and (26) are symmetric.
The matrix inequality (23) reduces to the four inequalities

$$\begin{array}{rcl} 1 + af^* + bd^*cf^* & \leq & f^* \\ af^*bd^* + b(d^* + d^*cf^*bd^*) & \leq & f^*bd^* \\ cf^* + dd^*cf^* & \leq & d^*cf^* \\ 1 + cf^*bd^* + d(d^* + d^*cf^*bd^*) & \leq & d^* + d^*cf^*bd^* \end{array}$$

in $\mathcal{K}$. These are equivalent to the inequalities

$$\begin{array}{rcl} 1 + ff^* & \leq & f^* \\ (1 + ff^*)bd^* & \leq & f^*bd^* \\ (1 + dd^*)cf^* & \leq & d^*cf^* \\ (1 + dd^*)(1 + cf^*bd^*) & \leq & d^*(1 + cf^*bd^*) \end{array}$$

respectively, which follow from the axioms and basic properties of §2.

We now establish (25). We show that (25) holds for X an arbitrary column vector of length 2; then (25) for X any $2 \times n$ matrix follows by applying this result to the columns of X separately.

Let

$$X \ = \ \left[\begin{array}{c} x \\ y \end{array} \right] .$$

We need to show that under the assumptions

$$\begin{aligned} ax + by &\leq x && (27) \\ cx + dy &\leq y && (28) \end{aligned}$$

we can derive

$$\begin{aligned} f^*x + f^*bd^*y &\leq x && (29) \\ d^*cf^*x + (d^* + d^*cf^*bd^*)y &\leq y\ . && (30) \end{aligned}$$

We establish (29) and (30) in a sequence of steps. With each step, we identify the premises from which the conclusion follows by one of the axioms or basic properties of §2.

$$\begin{aligned}
ax &\leq x && (27) && (31) \\
by &\leq x && (27) && (32) \\
cx &\leq y && (28) && (33) \\
dy &\leq y && (28) && (34) \\
d^*y &\leq y && (34),\ (19) && (35) \\
bd^*y &\leq by && (35),\ \text{monotonicity} && (36) \\
bd^*y &\leq x && (32),\ (36) && (37) \\
bd^*cx &\leq bd^*y && (33),\ \text{monotonicity} && (38) \\
bd^*cx &\leq x && (37),\ (38) && (39) \\
fx &\leq x && (31),\ (39) && (40) \\
f^*x &\leq x && (40),\ (19) && (41) \\
f^*bd^*y &\leq f^*x && (37),\ \text{monotonicity} && (42) \\
f^*bd^*y &\leq x && (41),\ (42) && (43) \\
d^*cf^*x &\leq d^*cx && (41),\ \text{monotonicity} && (44) \\
d^*cx &\leq d^*y && (33),\ \text{monotonicity} && (45) \\
d^*cf^*x &\leq y && (35),\ (44),\ (45) && (46) \\
d^*cf^*bd^*y &\leq d^*cf^*x && (37),\ \text{monotonicity} && (47) \\
d^*cf^*bd^*y &\leq y && (46),\ (47) && (48)
\end{aligned}$$

The conclusion (29) now follows from (41) and (43) and (30) follows from (46), (35), and (48). □

We now apply this argument inductively.

Lemma 10 *Let $E \in \mathcal{M}(n,\mathcal{K})$. There is a unique matrix $E^* \in \mathcal{M}(n,\mathcal{K})$ satisfying the Kleene algebra axioms (15–18). That is,*

$$
\begin{aligned}
I + EE^* &\leq E^* && (49)\\
I + E^*E &\leq E^* && (50)
\end{aligned}
$$

and for any $n \times m$ matrix X over $\mathcal{K}$,

$$
\begin{aligned}
EX \leq X &\rightarrow E^*X \leq X && (51)\\
XE \leq X &\rightarrow XE^* \leq X\,. && (52)
\end{aligned}
$$

Proof. Partition E into submatrices A, B, C, and D such that A and D are square.

$$E = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \qquad (53)$$

By the induction hypothesis, D^* exists and is unique. Let $F = A + BD^*C$. Again by the induction hypothesis, F^* exists and is unique. We define

$$E^* = \left[\begin{array}{c|c} F^* & F^*BD^* \\ \hline D^*CF^* & D^* + D^*CF^*BD^* \end{array}\right]. \qquad (54)$$

The proof that E^* satisfies (15–18) is essentially identical to the proof of Lemma 9. We must check that the axioms and basic properties of §2 used in the proof of Lemma 9 still hold when the primitive symbols of regular espressions are interpreted as matrices of various dimensions, provided there is no type mismatch in the application of the operators.

The uniqueness of E^* follows from Lemma 1. □

Combining Lemmas 8 and 10, we obtain

Theorem 11 *The structure*

$$(\mathcal{M}(n,\mathcal{K}),\ +,\ \cdot,\ ^*,\ Z_n,\ I_n)$$

is a Kleene algebra.

We remark that the inductive definition (54) of E^* in Lemma 10 is independent of the partition of E chosen in (53). This is a consequence of Lemma 1, once we have established that the resulting structure is a Kleene algebra under some partition; cf. [4, Theorem 4, p. 27].

In the proof of Lemma 10, we must check that the basic axioms and properties of §2 still hold when the primitive letters of regular expressions are interpreted over matrices of various shapes, possibly nonsquare, provided there is no type mismatch in the application of operators; *e.g.*, one cannot add two matrices unless they are the same shape, one cannot form the matrix product AB unless the column dimension of A is the same as the row dimension of B, and one cannot form the matrix A^* unless A is square.

For example, the Kleene algebra theorem

$$ax \;=\; xb \;\rightarrow\; a^*x \;=\; xb^*$$

(Proposition 4) holds even when a is an $m \times m$ matrix, b is an $n \times n$ matrix, and x is an $m \times n$ matrix.

In order to formulate this property at an appropriate level of abstraction, we need to extend the notions of regular expression and Kleene algebra to allow types. This leads to a general notion of *typed regular expression* and *typed Kleene algebra* involving the typing rules

$$\frac{\alpha : s,t \quad \beta : s,t}{\alpha + \beta : s,t} \qquad \frac{\alpha : s,t \quad \beta : t,u}{\alpha\beta : s,u} \qquad \frac{\alpha : s,s}{\alpha^* : s,s}$$

$$1 : s,s \qquad 0 : s,t \tag{55}$$

$$\frac{\alpha : s,t \quad \beta : s,t}{\alpha = \beta : s,t}$$

which determine when an operator or relation symbol may be applied to a pair of typed regular expressions and the type of the resulting expression. The typing rules (55) determine a *most general typing* of a regular expression, and a general metatheorem can be proved to the effect that any theorem of Kleene algebra is a theorem of typed Kleene algebra under its most general typing.

A satisfactory development of this theory would constitute a major digression, so we forego it for the present and content ourselves with the following special case, which suffices for the purposes of this paper.

Theorem 12 *The axioms and basic properties of Kleene algebra listed in §2 hold when the basic letters are interpreted as possibly nonsquare matrices over a Kleene algebra, provided that there are no type conflicts in the application of operators as specified by the typing rules (55).*

Proof. A quick review of the axioms and basic properties of §2 in light of this more general interpretation should convince the reader of the truth of this statement. For example, consider the distributive law

$$a(b+c) \;=\; ab + ac\;.$$

Interpreting a, b, and c as matrices over a Kleene algebra $\mathcal{K}$, the typing rules (55) allow the formation of this equation provided the shapes of b and c are the same and the column dimension of a is the same as the row dimension of b and c. Other than that, there are no type constraints. It is easy to verify that the distributive law holds for any matrices a, b and c satisfying these constraints.

For a more involved example, consider the equational implication of Proposition 4:

$$ax = xb \quad \rightarrow \quad a^*x = xb^* \ .$$

The type constraints imposed by the typing rules (55) say that a and b must be square (say $s \times s$ and $t \times t$ respectively) and that x must be $s \times t$. Under this typing, all steps of the proof of Proposition 4 involve only well-typed expressions, thus the proof remains valid in the typed case. □

4 Finite Automata

Regular expressions and finite automata have traditionally been used as syntactic representations of the regular languages over an alphabet Σ. The relationship between these two formalisms forms the basis of a well-developed classical theory, but the classical treatment as found for example in [9,6] is generally combinatorial. Algebraic approaches involving formal power series over a free monoid, as found for example in [16,15,5,3] do not consider arbitrary Kleene algebras.

In this section we define the notion of an automaton over an arbitrary Kleene algebra. In subsequent sections, we will use this formalism to derive the classical results of the theory of finite automata (equivalence with regular expressions, determinization via the subset construction, elimination of ϵ-transitions, and state minimization) as consequences of the axioms of §2.

In the following, although we consider regular expressions and automata as "syntax", as a matter of convenience we will be reasoning modulo the axioms of Kleene algebra. Officially, we are considering regular expressions to be elements of $\mathcal{F}_\Sigma$, the free Kleene algebra over Σ. The Kleene algebra $\mathcal{F}_\Sigma$ is constructed by taking the quotient of the regular expressions modulo provable equivalence. The *canonical map* assigns to each regular expression its equivalence class in $\mathcal{F}_\Sigma$. Since we will be interpreting expressions only over Kleene algebras, and all interpretations factor through $\mathcal{F}_\Sigma$ via the canonical map, this usage is without loss of generality.

The following definition is closer to the algebraic definition used for example in [3,4] than to the combinatorial definition used in [9,6].

Definition 13 A *finite automaton* over $\mathcal{K}$ is a triple

$$\mathcal{A} \;=\; (u, A, v) \ ,$$

where $u, v \in \{0,1\}^n$ and $A \in \mathcal{M}(n, \mathcal{K})$ for some n.

The *states* are the row and column indices. The vector u determines the *start states* and the vector v determines the *final states*; a *start state* is an index i for which $u(i) = 1$ and a *final state* is one for which $v(i) = 1$. The $n \times n$ matrix A is called the *transition matrix*.

The *language accepted by* $\mathcal{A}$ is the element

$$u^T A^* v \;\in\; \mathcal{K} \ .$$

□

For automata over $\mathbf{Reg}_\Sigma$, this definition is essentially equivalent to the classical combinatorial definition as found in [9,6].

Example 14 Consider the two-state automaton in the sense of [9,6] with states $\{p, q\}$, start state p, final state q, and transitions

$$\begin{array}{ll} p \xrightarrow{a} p & q \xrightarrow{a} q \\ p \xrightarrow{b} q & q \xrightarrow{b} q \,. \end{array}$$

Classically, this automaton accepts the set of strings over $\Sigma = \{a, b\}$ containing at least one occurrence of b. In our formalism, this automaton is specified by the triple

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a & b \\ 0 & a+b \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Modulo the axioms of Kleene algebra, we have

$$\begin{aligned} \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ 0 & a+b \end{bmatrix}^* \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a^* & a^*b(a+b)^* \\ 0 & (a+b)^* \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= a^*b(a+b)^* \,. \end{aligned} \tag{56}$$

The language in $\mathbf{Reg}_\Sigma$ accepted by this automaton is the image under R_Σ of the expression (56). □

Definition 15 Let $\mathcal{A} = (u, A, v)$ be an automaton over $\mathcal{F}_\Sigma$, the free Kleene algebra on free generators Σ. The automaton $\mathcal{A}$ is said to be *simple* if A can be expressed as a sum

$$A = J + \sum_{a \in \Sigma} a \cdot A_a \tag{57}$$

where J and the A_a are 0-1 matrices. In addition, $\mathcal{A}$ is said to be *ϵ-free* if J is the zero matrix. Finally, $\mathcal{A}$ is said to be *deterministic* if it is simple and ϵ-free, and u and all rows of A_a have exactly one 1. □

In Definition 15, ϵ refers to the null string. The matrix A_a in (57) corresponds to the adjacency matrix of the graph consisting of edges labeled a in the combinatorial model of automata [6,9] or the image of a under a linear representation map in the algebraic approach of [16,3]. An automaton is deterministic according to this definition iff it is deterministic in the sense of [6,9].

The automaton of Example 14 is simple, ϵ-free, and deterministic.

5 Completeness

In this section we show the completeness of the axioms of §2 for the algebra of regular events. Another way of stating this is that $\mathbf{Reg}_\Sigma$ is the free Kleene algebra on generators Σ, and the homomorphism $R_\Sigma : \mathcal{F}_\Sigma \to \mathbf{Reg}_\Sigma$ is an isomorphism of Kleene algebras.

The first lemma asserts that Kleene's representation theorem [8] is a theorem of Kleene algebra.

Lemma 16 *For every regular expression α over Σ (or more accurately, its image in $\mathcal{F}_\Sigma$ under the canonical map), there is a simple automaton (u, A, v) over $\mathcal{F}_\Sigma$ such that*

$$\alpha \;=\; u^T A^* v \; .$$

Proof. The proof is by induction on the structure of the regular expression. For $a \in \Sigma$, the automaton

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

suffices, since

$$\begin{aligned} \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}^* \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= a \; . \end{aligned}$$

For the expression $\alpha + \beta$, let $\mathcal{A} = (u, A, v)$ and $\mathcal{B} = (s, B, t)$ be automata such that

$$\begin{aligned} \alpha &= u^T A^* v \\ \beta &= s^T B^* t \; . \end{aligned}$$

Consider the automaton with transition matrix

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right]$$

and start and final state vectors

$$\left[\begin{array}{c} u \\ \hline s \end{array} \right] \quad \text{and} \quad \left[\begin{array}{c} v \\ \hline t \end{array} \right] ,$$

respectively. This construction corresponds to the combinatorial construction of forming the disjoint union of the two sets of states, taking the start states to be the union of the start states of $\mathcal{A}$ and $\mathcal{B}$, and the final states to be the union of the final states of $\mathcal{A}$ and $\mathcal{B}$. Then

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]^* = \left[\begin{array}{c|c} A^* & 0 \\ \hline 0 & B^* \end{array}\right],$$

and

$$\begin{aligned}\left[\begin{array}{c|c} u^T & s^T \end{array}\right] \cdot \left[\begin{array}{c|c} A^* & 0 \\ \hline 0 & B^* \end{array}\right] \cdot \left[\begin{array}{c} v \\ \hline t \end{array}\right] &= u^T A^* v + s^T B^* t \\ &= \alpha + \beta\ .\end{aligned}$$

For the expression $\alpha\beta$, let $\mathcal{A} = (u, A, v)$ and $\mathcal{B} = (s, B, t)$ be automata such that

$$\begin{aligned}\alpha &= u^T A^* v \\ \beta &= s^T B^* t\ .\end{aligned}$$

Consider the automaton with transition matrix

$$\left[\begin{array}{c|c} A & vs^T \\ \hline 0 & B \end{array}\right]$$

and start and final state vectors

$$\left[\begin{array}{c} u \\ \hline 0 \end{array}\right] \text{ and } \left[\begin{array}{c} 0 \\ \hline t \end{array}\right],$$

respectively. This construction corresponds to the combinatorial construction of forming the disjoint union of the two sets of states, taking the start states to be the start states of $\mathcal{A}$, the final states to be the final states of $\mathcal{B}$, and connecting the final states of $\mathcal{A}$ with the start states of $\mathcal{B}$ by ϵ-transitions (this is the purpose of the vs^T in the upper right corner of the matrix). Then

$$\left[\begin{array}{c|c} A & vs^T \\ \hline 0 & B \end{array}\right]^* = \left[\begin{array}{c|c} A^* & A^* vs^T B^* \\ \hline 0 & B^* \end{array}\right],$$

and

$$\begin{aligned}\left[\begin{array}{c|c} u^T & 0 \end{array}\right] \cdot \left[\begin{array}{c|c} A^* & A^* vs^T B^* \\ \hline 0 & B^* \end{array}\right] \cdot \left[\begin{array}{c} 0 \\ \hline t \end{array}\right] &= u^T A^* vs^T B^* t \\ &= \alpha\beta\ .\end{aligned}$$

For the expression α^*, let $\mathcal{A} = (u, A, v)$ be an automaton such that

$$\alpha = u^T A^* v\ .$$

We first produce an automaton equivalent to the expression $\alpha\alpha^*$. Consider the automaton

$$(u,\ A + vu^T,\ v)\ .$$

This construction corresponds to the combinatorial construction of adding ϵ-transitions from the final states of $\mathcal{A}$ back to the start states. Then

$$\begin{aligned} u^T(A+vu^T)^*v &= u^TA^*(vu^TA^*)^*v && \text{by Proposition 7} \\ &= u^TA^*v(u^TA^*v)^* && \text{by Proposition 5} \\ &= \alpha\alpha^*\ . \end{aligned}$$

Once we have an automaton for $\alpha\alpha^*$, we can get an automaton for $\alpha^* = 1 + \alpha\alpha^*$ by the construction for $+$ given above, using a trivial one-state automaton for 1. □

Lemma 17 *For every simple automaton (u, A, v) over $\mathcal{F}_\Sigma$, there is a simple ϵ-free automaton (s, B, t) such that*

$$u^TA^*v \;=\; s^TB^*t\ .$$

Proof. By Definition 15, the matrix A can be written as a sum $A = J + A'$ where J is a 0-1 matrix and A' is ϵ-free. Then

$$\begin{aligned} u^TA^*v &= u^T(A'+J)^*v \\ &= u^TJ^*(A'J^*)^*v && \text{by Proposition 7}\ , \end{aligned}$$

so we can take

$$\begin{aligned} s^T &= u^TJ^* \\ B &= A'J^* \\ t &= v\ . \end{aligned}$$

Note that J^* is 0-1 and therefore B is ϵ-free. This construction models algebraically the combinatorial idea of computing the ϵ-closure of a state; see [6,9]. □

Lemma 18 *For every simple ϵ-free automaton (u, A, v) there is a deterministic automaton $(\hat{u}, \hat{A}, \hat{v})$ such that*

$$u^TA^*v \;=\; \hat{u}^T\hat{A}^*\hat{v}\ .$$

Proof. We model the "subset construction" [6,9] algebraically. Let (u, A, v) be a simple ϵ-free automaton with states Q. By Definition 15, A can be expressed

$$A \;=\; \sum_{a\in\Sigma} a \cdot A_a$$

where each A_a is a 0-1 matrix.

Let $\mathcal{P}(Q)$ denote the power set of Q. We identify elements of $\mathcal{P}(Q)$ with their characteristic vectors in $\{0,1\}^n$. For each $s \in \mathcal{P}(Q)$, let e_s be the $\mathcal{P}(Q) \times 1$ vector with 1 in position s and 0 elsewhere.

Let X be the $\mathcal{P}(Q) \times Q$ matrix whose s^{th} row is s^T; *i.e.*,

$$e_s^T X \;=\; s^T \;.$$

For each $a \in \Sigma$, let $\hat{A}_a$ be the $\mathcal{P}(Q) \times \mathcal{P}(Q)$ matrix whose s^{th} row is $e_{s^T A_a}$; in other words,

$$e_s^T \hat{A}_a \;=\; e_{s^T A_a} \;.$$

Let

$$\begin{aligned} \hat{A} &= \sum_{a \in \Sigma} a \cdot \hat{A}_a \\ \hat{u} &= e_u \\ \hat{v} &= Xv \;. \end{aligned}$$

The automaton $(\hat{u}, \hat{A}, \hat{v})$ is simple and deterministic.

The relationship between A and $\hat{A}$ is expressed succinctly by the equation

$$XA \;=\; \hat{A}X \;. \tag{58}$$

Intuitively, this says that the actions of the two automata in the two spaces $\mathcal{K}^Q$ and $\mathcal{K}^{\mathcal{P}(Q)}$ commute with the projection X. To prove (58), observe that for any $s \in \mathcal{P}(Q)$,

$$\begin{aligned} e_s^T XA &= s^T A \\ &= \sum_{a \in \Sigma} a \cdot s^T A_a \\ &= \sum_{a \in \Sigma} a \cdot e_{s^T A_a} X \\ &= \sum_{a \in \Sigma} a \cdot e_s^T \hat{A}_a X \\ &= e_s^T \hat{A} X \;. \end{aligned}$$

Now, by Proposition 4,

$$XA^* \;=\; \hat{A}^* X \;.$$

The theorem now follows:

$$\begin{aligned} \hat{u}^T \hat{A}^* \hat{v} &= e_u^T \hat{A}^* X v \\ &= e_u^T X A^* v \\ &= u^T A^* v \;. \end{aligned}$$

□

Lemma 19 *Let (u, A, v) be a simple deterministic automaton and let $(\overline{u}, \overline{A}, \overline{v})$ be the equivalent minimal deterministic automaton obtained from the classical state minimization procedure [6,9]. Then*

$$u^T A^* v \;=\; \overline{u}^T \overline{A}^* \overline{v}\ .$$

Proof. In the combinatorial approach [6,9], the unique minimal automaton is obtained as a quotient by a Myhill-Nerode equivalence relation after removing inaccessible states. We simulate this construction algebraically.

Let Q denote the set of states of (u, A, v). For $q \in Q$, let $e_q \in \{0,1\}^Q$ denote the vector with 1 in position q and 0 elsewhere. For $a \in \Sigma$, let A_a be the 0-1 matrix as given in Definition 15 (57). Then

$$A \;=\; \sum_{a\in\Sigma} a \cdot A_a\ .$$

For each $a \in \Sigma$ and $p \in Q$, let $\delta(p,a)$ be the unique state in Q such that the p^{th} row of A_a is $e^T_{\delta(p,a)}$; *i.e.*,

$$e_p^T A_a \;=\; e^T_{\delta(p,a)}\ .$$

The state $\delta(p,a)$ exists and is unique since the automaton is deterministic.

First we show how to get rid of unreachable states. A state q is *reachable* if

$$u^T A^* e_q \;\neq\; 0\ ,$$

otherwise it is *unreachable.* Let R be the set of reachable states and let $U = Q - R$ be the set of unreachable states. Partition A into four submatrices A_{RR}, A_{RU}, A_{UR}, and A_{UU} such that for $S, T \in \{R, U\}$, A_{ST} is the $S \times T$ submatrix of A. Then A_{RU} is the zero matrix, otherwise a state in U would be reachable. Similarly, partition the vectors u and v into u_R, u_U, v_R and v_U. The vector u_U is the zero vector, otherwise a state in U would be reachable. We have

$$\begin{aligned}
u^T A^* v &= \left[\, u_R^T \mid 0 \,\right] \cdot \left[\begin{array}{c|c} A_{RR} & 0 \\ \hline A_{UR} & A_{UU} \end{array}\right]^* \cdot \left[\begin{array}{c} v_R \\ \hline v_U \end{array}\right] \\
&= \left[\, u_R^T \mid 0 \,\right] \cdot \left[\begin{array}{c|c} A_{RR}^* & 0 \\ \hline A_{UU}^* A_{UR} A_{RR}^* & A_{UU}^* \end{array}\right] \cdot \left[\begin{array}{c} v_R \\ \hline v_U \end{array}\right] \\
&= u_R^T A_{RR}^* v_R\ .
\end{aligned}$$

Moreover, the automaton (u_R, A_{RR}, v_R) is simple and deterministic, and all states are reachable.

Assume now that (u, A, v) is simple and deterministic and all states are reachable. An equivalence relation $\equiv$ on Q is called *Myhill-Nerode* if $p \equiv q$ implies

$$\begin{aligned} \delta(p,a) &\equiv \delta(q,a)\ , \quad a \in \Sigma\ , && (59) \\ e_p^T v &= e_q^T v\ . && (60) \end{aligned}$$

(In combinatorial terms, $\equiv$ is *Myhill-Nerode* if it is respected by the action of the automaton under any input symbol $a \in \Sigma$, and the set of final states is a union of $\equiv$-classes.)

Let $\equiv$ be any Myhill-Nerode equivalence relation, and let

$$\begin{aligned} [p] &= \{q \in Q \mid q \equiv p\} \\ Q/{\equiv} &= \{[p] \mid p \in Q\}\ . \end{aligned}$$

For $[p] \in Q/{\equiv}$, let $e_{[p]} \in \{0,1\}^{Q/\equiv}$ denote the vector with 1 in position $[p]$ and 0 elsewhere. Let Y be the $Q \times Q/{\equiv}$ matrix whose $[p]^{\text{th}}$ column is the characteristic vector of $[p]$; *i.e.*,

$$e_p^T Y = e_{[p]}^T\ .$$

For each $a \in \Sigma$, let $\overline{A}_a$ be the $Q/{\equiv} \times Q/{\equiv}$ matrix whose $[p]^{\text{th}}$ row is $e_{[\delta(p,a)]}$; *i.e.*,

$$e_{[p]}^T \overline{A}_a = e_{[\delta(p,a)]}^T\ .$$

The matrix $\overline{A}_a$ is well-defined by (59). Let

$$\begin{aligned} \overline{A} &= \sum_{a\in\Sigma} a \cdot \overline{A}_a \\ \overline{u}^T &= u^T Y\ . \end{aligned}$$

Also, let $\overline{v} \in \{0,1\}^{Q/\equiv}$ be the vector such that

$$e_{[p]}^T \overline{v} = e_p^T v\ .$$

The vector $\overline{v}$ is well-defined by (60). Note also that

$$\begin{aligned} e_p^T Y \overline{v} &= e_{[p]}^T \overline{v} \\ &= e_p^T v\ , \end{aligned}$$

therefore

$$Y\overline{v} = v\ .$$

The automaton $(\overline{u}, \overline{A}, \overline{v})$ is simple and deterministic.

As in the proof of Lemma 18, the action of A and $\overline{A}$ commute with the linear projection Y:

$$AY = Y\overline{A} . \tag{61}$$

To prove (61), observe that for any $p \in Q$,

$$\begin{aligned} e_p^T AY &= \sum_{a\in\Sigma} a \cdot e_p^T A_a Y \\ &= \sum_{a\in\Sigma} a \cdot e_{\delta(p,a)}^T Y \\ &= \sum_{a\in\Sigma} a \cdot e_{[\delta(p,a)]}^T \\ &= \sum_{a\in\Sigma} a \cdot e_{[p]}^T \overline{A}_a \\ &= \sum_{a\in\Sigma} a \cdot e_p^T Y \overline{A}_a \\ &= e_p^T Y \overline{A} . \end{aligned}$$

Now by Proposition 4,

$$A^*Y = Y\overline{A}^* ,$$

therefore

$$\begin{aligned} \overline{u}^T \overline{A}^* \overline{v} &= u^T Y \overline{A}^* \overline{v} \\ &= u^T A^* Y \overline{v} \\ &= u^T A^* v . \end{aligned}$$

□

Theorem 20 (Completeness) *Let α and β be two regular expressions over Σ denoting the same regular set. Then $\alpha = \beta$ is a theorem of Kleene algebra.*

Proof. Let $\mathcal{A} = (s, A, t)$ and $\mathcal{B} = (u, B, v)$ be minimal deterministic finite automata over $\mathcal{F}_\Sigma$ such that

$$\begin{aligned} R_\Sigma(\alpha) &= R_\Sigma(s^T A^* t) \\ R_\Sigma(\beta) &= R_\Sigma(u^T B^* v) . \end{aligned}$$

By Lemmas 16, 18, and 19, we have

$$\begin{aligned} \alpha &= s^T A^* t \\ \beta &= u^T B^* v \end{aligned}$$

as theorems of Kleene algebra. Since

$$R_\Sigma(\alpha) \;=\; R_\Sigma(\beta) \;,$$

by the uniqueness of minimal automata, $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. Let P be a permutation matrix giving this isomorphism. Then

$$\begin{aligned} A &= P^T B P \\ s &= P^T u \\ t &= P^T v \;. \end{aligned}$$

Using Corollary 6, we have

$$\begin{aligned} \alpha &= s^T A^* t \\ &= (P^T u)^T (P^T B P)^* (P^T v) \\ &= u^T P (P^T B P)^* P^T v \\ &= u^T P P^T B^* P P^T v \\ &= u^T B^* v \\ &= \beta \;. \end{aligned}$$

□

Acknowledgments

I am grateful to Merrick Furst, Dana Scott, Vaughan Pratt, and Ross Willard for their valuable comments and criticism.

References

[1] S. Anderaa, "On the algebra of regular expressions," Appl. Math., Harvard U., January 1965, 1–18.

[2] Alfred V. Aho, John E. Hopcroft, and Jeffrey D. Ullman. *The Design and Analysis of Computer Algorithms.* Addison-Wesley, 1975.

[3] Jean Berstel and Christophe Reutenauer. *Rational Series and Their Languages.* Springer, Berlin, 1984.

[4] John Horton Conway. *Regular Algebra and Finite Machines.* Chapman and Hall, London, 1971.

[5] F. Gécseg and I. Peák. *Algebraic Theory of Automata.* Akadémiai Kiadó, Budapest, 1972.

[6] John E. Hopcroft, and Jeffrey D. Ullman. *Introduction to Automata Theory, Languages, and Computation.* Addison-Wesley, 1979.

[7] Dexter Kozen, "On induction vs. *-continuity," *Proc. Workshop on Logics of Programs 1981*, Spring-Verlag Lect. Notes in Comput. Sci. 131, ed. Kozen, 1981, 167-176.

[8] Stephen C. Kleene, "Representation of events in nerve nets and finite automata," in: *Automata Studies*, ed. Shannon and McCarthy, Princeton U. Press, 1956, 3–41.

[9] Harry Lewis and Christos Papadimitriou. *Elements of the Theory of Computation.* Prentice-Hall, 1981.

[10] Kurt Mehlhorn. *Data Structures and Algorithms 2: Graph Algorithms and NP-Completeness.* EATCS Monographs on Theoretical Computer Science, Springer-Verlag, 1984.

[11] K. C. Ng and A. Tarski, "Relation algebras with transitive closure," Abstract 742-02-09, *Notices Amer. Math. Soc.* 24 (1977), A29-A30.

[12] K. C. Ng. *Relation Algebras with Transitive Closure.* PhD Thesis, University of California, Berkeley, 1984.

[13] Vaughan Pratt, "Dynamic algebras as a well-behaved fragment of relation algebras," in: D. Pigozzi, ed., *Proc. Conf. on Algebra and Computer Science*, Ames, Iowa, June 2-4, 1988; Springer-Verlag Lecture Notes in Computer Science, to appear.

[14] V. N. Redko, "On defining relations for the algebra of regular events," *Ukrain. Mat. Z.* 16 (1964), 120–126 (in Russian).

[15] Arto Salomaa, "Two complete axiom systems for the algebra of regular events," *J. Assoc. Comput. Mach.* 13:1 (January, 1966), 158–169.

[16] Arto Salomaa and Matti Soittola. *Automata Theoretic Aspects of Formal Power Series.* Springer-Verlag, 1978.

[17] A. Selman, "Completeness of calculi for axiomatically defined classes of algebras," *Algebra Universalis* 2 (1972), 20–32.