THE STABILITY OF PARAMETRICALLY EXCITED SYSTEMS: COEXISTENCE AND TRIGONOMETRIFICATION

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This dissertation addresses questions regarding the stability of two degree of freedom oscillating systems. The systems being discussed fall into three classes.

The first class we discuss has the property that one of the non-linear normal modes (NNM) has a harmonic solution, $x(t) = A \cos t$. For this class, the equation governing the stability of the system will be a second order differential equation with parametric excitation. Mathieu's equation (1), or more generally Ince's equation (2), are standard examples of such systems.

$$\ddot{x} + (\delta + \epsilon \cos t)x = 0 \tag{1}$$

$$(1 + a \cos t)\ \ddot{x} + (b \sin t)\ \dot{x} + (\delta + c \cos t)\, x = 0 \tag{2}$$

For Ince's equation we know that the stability portraits have tongues of instability defined by two transition curves. When these two transition curves overlap, the unstable region disappears and we say that the hidden tongue is coexistent. In this thesis we obtain sufficient conditions for coexistence to occur in stability equations of the form

$$\begin{aligned}
&(1 + a_1\ \cos t + a_2\ \cos 2t + ... + a_n\ \cos nt)\, \ddot{v} \\
&\quad +(b_1\ \sin t + b_2\ \sin 2t + ... + b_n\ \sin nt)\, \dot{v} \\
&+(\delta + c_1\ \cos t + c_2\ \cos 2t + ... + c_n\ \cos nt)\, v \quad = 0
\end{aligned}$$

Ince's equation has no damping. For the second class of systems, we seek to understand how dissipation affects coexistence. Here the analysis focuses on the behavior of coexistence as damping (μ) is added. Our analysis indicates coexistence is not possible in a damped Ince equation (3).

$$(1 + a\,\epsilon \cos t)\ \ddot{x} + (\mu + b\,\epsilon \sin t)\ \dot{x} + (\delta + c\,\epsilon \cos t)\, x = 0 \tag{3}$$

The previous two classes address systems with a harmonic NNM. The third class of systems treated in this thesis involve two degree of freedom systems that have a periodic NNM, not in general harmonic. To accomplish this we rescale time such that the periodic solution to the NNM is transformed into the form $x(\tau) = A_0 + A_1 \cos \tau$. We call this procedure of rescaling time trigonometrification. The power of trigonometrification is that it is exact, requiring no approximations and produces a stability equation in new time (τ) of the form

$$\begin{aligned}
(1 + a_1\ \cos\tau + a_2\ \cos 2\tau + \ldots + a_n\ \cos n\tau)\, v'' & \\
+(b_1\ \sin\tau + b_2\ \sin 2\tau + \ldots + b_n\ \sin n\tau)\, v' & \\
+(\delta + c_1\ \cos\tau + c_2\ \cos 2\tau + \ldots + c_n\ \cos n\tau)\, v & = 0
\end{aligned}$$

Trigonometrification can be used to study any system property that is invariant under a time transformation.

BIOGRAPHICAL SKETCH

Geoffrey D. Recktenwald was born on May 8th, 1980. He grew up in Michigan and enjoyed soccer, downhill skiing and his church youth group. He graduated from high school in the spring of 1998.

In the fall of 1998 he began studies at Cedarville College, now Cedarville University. He pursued two degrees, one in mechanical engineering and one in mathematics. He soon realized that he enjoyed the physical sciences and changed his second major in mathematics to a major in physics. He was privileged to be a part of the Cedarville honors program called 'Making of the Modern Mind' which let him engage his mind with courses in history, philosophy, and culture.

While at Cedarville Geoff was advised by Dr. Larry Zavodney. Geoff's first contact with Dr. Zavodney as an instructor came during a sophomore dynamics class. It was in this class that he first showed an aptitude for dynamics. Dr. Zavodney challenged Geoff to take his vibrations course the following fall. He reluctantly agreed, but soon found he really enjoyed the subject.

Upon completion of the class Geoff and several students persuaded Dr. Zavodney to teach a second more advanced vibrations course. They also persuaded Dr. Gollmer of the physics department to teach a course in acoustics.

Geoff enjoyed his work in vibrations and submitted a proposal for senior design to create a vibrations laboratory to augment the vibrations course. Geoff spent part of his time on the project studying the non-linear behavior of the tri-filar pendulum. He presented his findings on this system during a physics colloquium his senior year. Geoff was awarded Outstanding Mechanical Engineering Senior for his work on the project.

He graduated from Cedarville with a B.S. in Mechanical Engineering and a B.A. in Physics in 2002. He completed both the Honors program and the engineering honors program.

In the fall of 2002 Geoff began working on his Ph.D. at Cornell University. He arrived as a fellow in the IGERT program for interdisciplinary interaction in nonlinear studies. He worked on several projects while in the program including slip-fault modeling, genetic algorithms, and evolutionary robotics.

During his second year at Cornell, Geoff started working with Dr. Richard Rand. He worked on coexistence as part of an NSF grant before branching off to his own related project on time transformation methods for coupled ODEs. The results of both projects appear in journal articles and are fully realized in this dissertation.

I would like to dedicated this dissertation to my parents, David and Janet Recktenwald, for all of their love and support.

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I begin these acknowledgments by affirming my reliance and dependence on my Savior and Lord, Jesus, throughout this endeavor. My life and accomplishments are gifts from Him and I hope to glorify Him with them.

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Throughout my life I have had the support of a wonderful family. I thank my father Dave who challenged me to succeed and provided guidance and wisdom; my mother Jan who has always been there to support and encourage me; my brother Chris who is a mentor, role model and friend; and my loving sister Jackie who shares and encourages my passion for research and teaching.

I am very grateful for my friends and the support that they have been over the years. I want to thank Erin Peterson for blessing she has been to me, for her wisdom and peacefulness; William Leukhardt for his encouragement and passion for learning; and Bronson Hokuf and John Compton for their friendship. I also want to thank my GCF friends for their fellowship. Ray for his zeal; Chris for his enthusiasm; Joe for his calmness; Beth for her craziness; Matt for his thoughtfulness; and the Chock's for their friendship.

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To these people and many more, thank you.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

The field of non-linear dynamics is rich with problems that exhibit parametric excitation. These problems are fascinating to study and provide answers to important design questions. However, solving them can be a challenge. Most of the problems encountered in this thesis have no analytic solution. Numerical techniques can be used, but they don't provide a true understanding of the system. Perturbation techniques are used as a compromise between exact analytic analysis and numerical approximations. The answer they provide is an approximation, but accurately describes how the system's behavior changes with respect to its parameters. Perturbation techniques can also be used to find specific system properties (i.e. stability) without solving the entire system.

These analytical approximations become particularly important if one is interested in behavior that is structurally unstable (ie small changes in the system drastically change its behavior). Such behavior is extremely difficult to capture numerically since the process of numerical integration changes the system. We encounter such problems when studying the stability of parametrically excited systems (see chapters 3 and 4).

Analytic techniques also have their limitations. They require some knowledge of the system and solutions to simplified cases. If there is no such simplified case or if the solution is unwieldy the techniques lose their effectiveness and may fail all together. Often however these difficulties are merely superficial, the product of looking at the problem through the wrong frame of reference or variables. By wisely changing a variable or reference frame difficult problems can be simplified.

Such a procedure is described in chapter 5 of this thesis.

1.1 Origins

The work in this thesis began with a problem posed by Yang and Rosenberg [20], [21] called the particle in the plane. The particle in the plane is interesting because the stability diagram generated by the stability equation (1.1).

$$\frac{d^2v}{dt^2} + \left(\frac{\delta - A^2\cos^2 t}{1 - A^2\cos^2 t}\right) v = 0 \tag{1.1}$$

appears to be missing a tongue of instability at $\delta = 1$. This phenomena, called coexistence, occurs when a pair of transition curves overlap, effectively destroying the tongue. This example will be described in detail in chapter 3.

The standard system for studying coexistence is Ince's equation (1.2)

$$(1 + a\cos t)\ \ddot{x} + (b\sin t)\ \dot{x} + (\delta + c\cos t)\, x = 0 \tag{1.2}$$

Coexistence in Ince's equation has been studied by [4]. Also, the question of coexistence in a non-linear version of Ince's equation has been studied by [7].

In this thesis we study coexistence in the following generalization of Ince's equation [14]:

$$\begin{aligned} (1 + a_1\ \cos t + a_2\ \cos 2t + ... + a_n\ \cos nt)\,\ddot{v} \\ +(b_1\ \sin t + b_2\ \sin 2t + ... + b_n\ \sin nt)\,\dot{v} \\ +(\delta + c_1\ \cos t + c_2\ \cos 2t + ... + c_n\ \cos nt)\,v \quad &= 0 \end{aligned} \tag{1.3}$$

We obtain sufficient conditions for coexistence to occur in the case of eq 1.3. We also study the effect of damping on coexistence.

The analytical techniques we used for eq 1.3 required parametric excitation to be introduced by a cosine solution to a non-linear normal mode (NNM). This requirement always produces a stability equation with trigonometric parametric terms (1.3).

During the process of this research it became apparent that an entire class of systems, a generalization Ince's equation with non-trigonometric parametric excitation, could be studied. An example of such a system is Lame's equation (1.4). Lame's equation describes the stability of a two-degree of freedom system with a NNM in the form of a Jacobi-elliptic function, $x(t) = \mathrm{cn}(\alpha t, k)$ [9]. The result is the stability equation

$$\ddot{v} + (\omega_2^2 + A^2 \,\mathrm{cn}^2(\alpha t, k)) \quad v \quad = 0 \tag{1.4}$$

where the parametric forcer is periodic, but not trigonometric. Magnus and Winkler tackled eq 1.4 in [4]. They solved for the stability by rescaling time to turn $A\mathrm{cn}(\alpha t, k) \rightarrow A\cos(\tau)$.

As a natural expansion of their work we have developed a process to transform time so that any periodic parametric excitation becomes exactly trigonometric. We do this by choosing a time transformation that turns the harmonic solution to the NNM into a cosine solution. We call this process trigonometrification. Given $f(t) = f(t+T)$, the trigonometrification process looks like

$$x(t) = f(t) \xrightarrow{\boxed{\text{trigonometrification}}} x(\tau) = A_0 + A_1 \cos 2\tau \tag{1.5}$$

The application of trigonometrification extends beyond our previous focus on coexistence since it can be used on any system property that remains invariant under a time transformation. We conclude the work here and suggest ideas for using trigonometrification in other studies.

1.2 Organization of the Thesis

The following provides a summary of how the chapters in this dissertation are organized.

Chapter 2 contains an introduction to the stability of nonlinear two degree of freedom systems. It covers the history of the problems and discusses several methods for analyzing them. Topics include both analytic methods (harmonic balance and two-variable expansion) and numerical methods (stability algorithms based on Floquet theory). **Chapter 3** makes use of these methods, primarily harmonic balance, to analyze the stability of a class of coupled ODEs whose stability equation takes the form of a generalization of Ince's equation. The goal is to understand the characteristics of systems that exhibit coexistence, or the collapse of a tongue of instability. In **Chapter 4** we look at these coexistent systems when damping is present.

In **Chapter 5** we look at two-degree of freedom stability problems that cannot be studied by harmonic balance or other techniques that depend on the properties of trigonometric functions. Such methods require perturbation off a trigonometric function, which enters the problem as a solution to one of the nonlinear normal modes (NNM) of the system. We develop a method of time transformations that transforms any periodic solution to a independent NNM into a trigonometric function. The method can be also used for NNM equations that have no known or tabulated solution. We coined the neologism trigonometrification to describe this process. This process, though motivated by our discussion of coexistence, extends far beyond coexistence to questions of general stability.

Each chapter ends with a self contained summary of the research in that chapter. We conclude in **Chapter 6** by looking at ideas for future work.

Chapter 2

An Introduction to Tongues of Instability

2.1 Mathieu, Ince, and Hill

In 1868 the mathematician Emile Mathieu developed what is now known as the Mathieu equation while working on the vibrations of elliptical drum heads. The equation, shown below (2.1), has become the predominant example of parametric excitation.

$$\ddot{x} + (\delta + \epsilon \cos t)x = 0 \tag{2.1}$$

There is no analytical solution to Mathieu's Equation. However, we can study the stability of the system for different parameter values. The $\delta - \epsilon$ stability portrait maps out parameter values for which the system is stable or unstable. Figure 2.2 shows the stability portrait for Mathieu's equation. The regions of instability (marked U in the figure) are often referred to as tongues of instability.

The particular parametric phenomena we seek to study is coexistence. Coexistence occurs when the tongues of instability cross or overlap (effectively closing the unstable region). In 1922 Edward Ince proved that coexistence is not possible in Mathieu's equation. In 1926 he published a proof that any equation of the Hill type will fail to support coexistence. His proof was shown to have an error in 1943 when Klotter and Kotowski demonstrated coexistence in the equation

$$\ddot{x} + (\delta + \gamma_1 \cos t + \gamma_2 \cos 2t)\, x = 0 \tag{2.2}$$

This equation (2.2) can be transformed into

$$\ddot{u} - 4q \sin 2t\dot{u} + \left(\delta + 2q^2 + 4(m-1)q \cos 2t\right) u = 0 \tag{2.3}$$

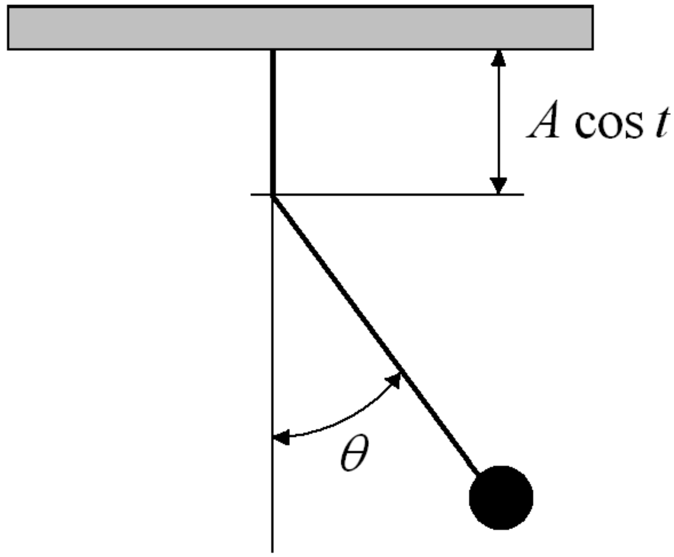

Figure 2.1: *The vertically driven pendulum is one of the simplest physical systems described by Mathieu's equation*

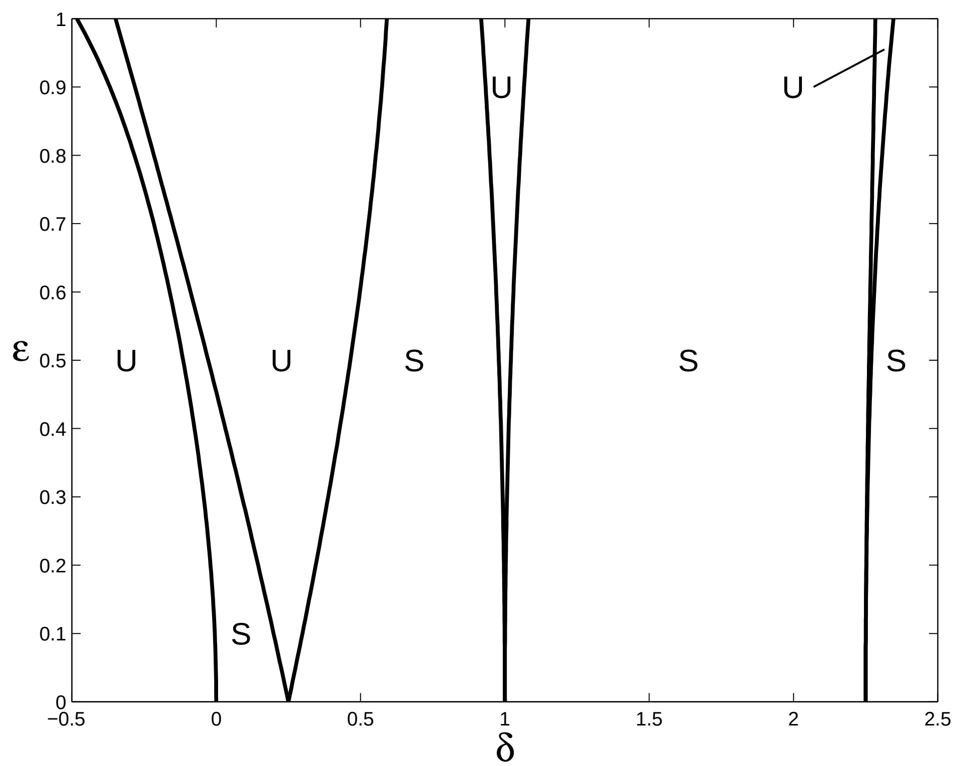

Figure 2.2: *$\delta - \epsilon$ stability portrait for the Mathieu equation* (2.1): S *represents stable regions and* U *represents unstable regions*

by letting $\gamma_1 = 2q^2$, $\gamma_2 = 4mq$, and making the substitution

$$x = u \exp(q \cos 2t) \tag{2.4}$$

Magnus and Winker proved [4] eq 2.3 is coexistent if and only if m is an integer. Eq 2.3 is in the form of Ince's equation.

$$(1 + a \cos t)\ \ddot{x} + (b \sin t)\ \dot{x} + (\delta + c \cos t)\, x = 0 \tag{1.2}$$

The error in Ince's proof is shown in [7].

For this thesis we will start by looking at coexistence in Ince's equation and generalizations of Ince's equation. In particular, we are looking at the type of coexistence where the tongue is closed and the region of instability is hidden (figure 2.3), waiting for only a small parameter change to open up (figure 2.4) with possible catastrophic consequences in an engineering application.

The importance of understanding coexistence is quite fundamental. Consider a design whose system parameters lie on the coexistent curve (dotted line) of figure 2.3. The design parameters seem to be ideal, situating the system in the 'middle' of a stable region. In fact, the system is precariously located on a tongue of instability that could open (figure 2.4) at the slightest parameter change.

In figure 2.3 the coexistent curve is clearly marked. However, the curve does not show up in a normal stability analysis and therefore poses a danger for designers. We seek to classify systems that exhibit coexistence and show when and where coexistence will occur.

At this point in the discussion, it would be fruitful to demonstrate several methods to find the tongues of instability. We will begin with a generic perturbation technique called two-variable expansion (TVE) and then get into the more specialized method of harmonic balance (HB). We will conclude with a simple numerical

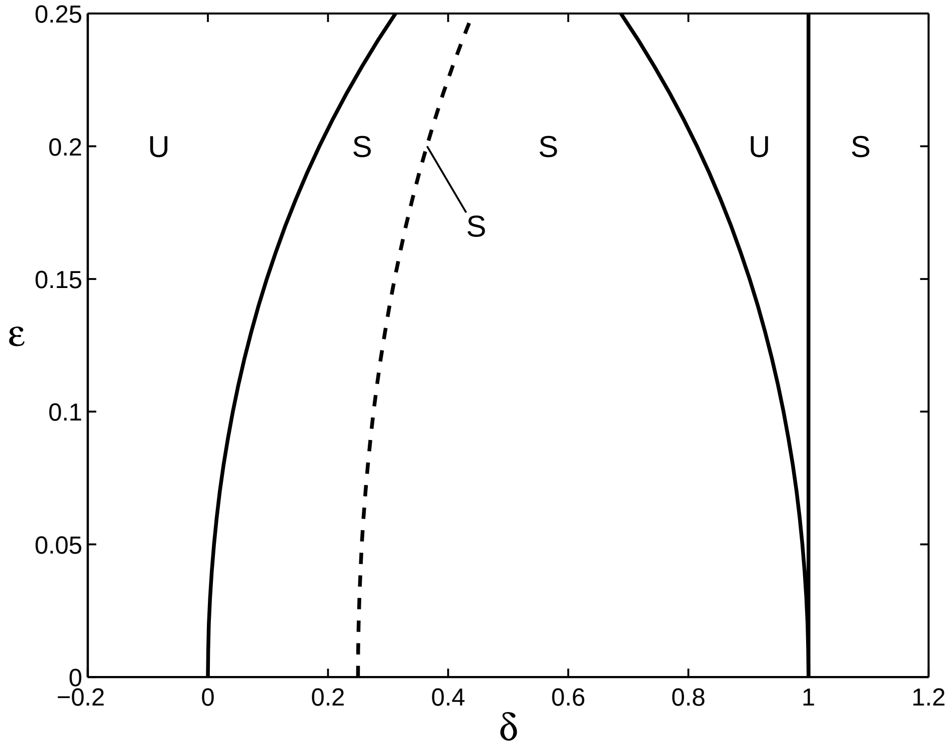

Figure 2.3: *$\delta - \epsilon$ stability portrait for the coexistent Ince equation* (1.2) *with $a = 4\epsilon$, $b = 2\epsilon$, and $c = 2\epsilon$:* S *represents stable regions and* U *represents unstable regions*

routine that allows us to double check our results.

2.2 Two-Variable Expansion (Two Time Scales)

Parts of this section have been adapted from Rand [12].

Two-variable expansion allows us to look at our system on two different time scales. The goal is to bypass the short time scale behavior and characterize the long term behavior of the system by looking at the long time scale. We will give an example of the procedure using Mathieu's equation (2.1).

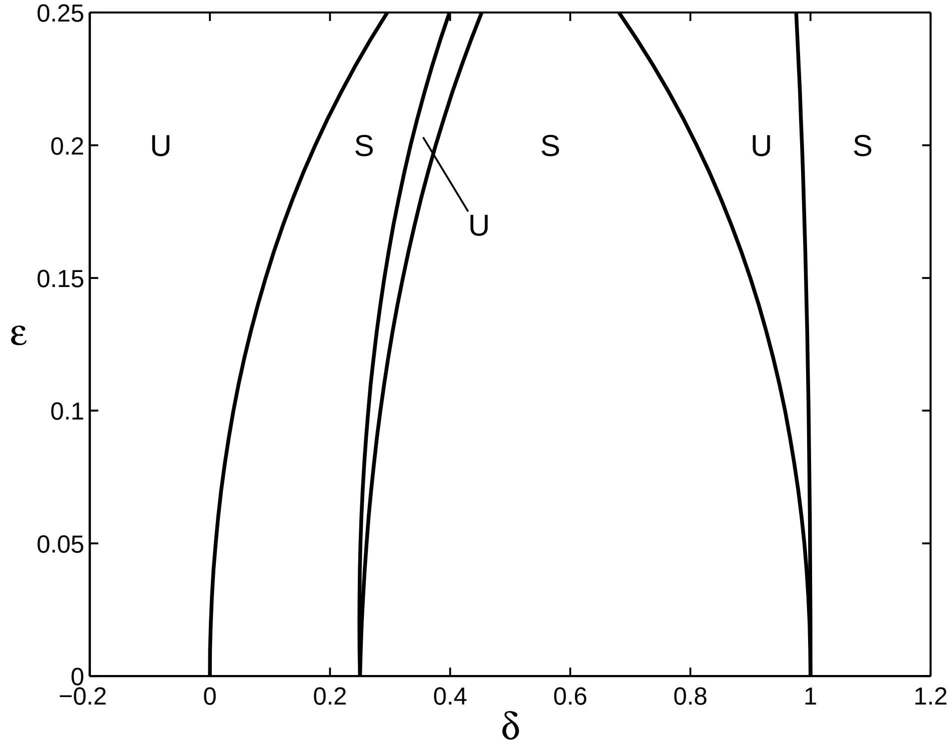

Figure 2.4: *$\delta-\epsilon$ stability portrait for Ince's equation* (1.2) *showing the coexistent tongue opening up for parameters $a = 4\epsilon$, $b = 2\epsilon$, and $c = 1.8\epsilon$:* S *represents stable regions and* U *represents unstable regions*

Starting with Mathieu's equation

$$\ddot{x} + (\delta + \epsilon \cos t)x = 0 \tag{2.1}$$

and assume that there are two time variables in the system ($\xi = t$ and $\eta = \epsilon t$). [1] Under this transformation the derivatives of x take the form

$$\dot{x} = \frac{\partial x}{\partial \xi} + \epsilon \frac{\partial x}{\partial \eta} \tag{2.5}$$

$$\ddot{x} = \frac{\partial^2 x}{\partial \xi^2} + 2\epsilon \frac{\partial^2 x}{\partial \eta \partial \xi} + \epsilon^2 \frac{\partial^2 x}{\partial \eta^2} \tag{2.6}$$

[1]We do not have to use a new time that is a function of the displacement ($\xi = \xi(x,t)$) since the system is already linear.

Thus, eq 2.1 becomes

$$\frac{\partial^2 x}{\partial \xi^2} + 2\epsilon \frac{\partial^2 x}{\partial \eta \partial \xi} + \epsilon^2 \frac{\partial^2 x}{\partial \eta^2} + (\delta + \epsilon \cos \xi) x = 0 \tag{2.7}$$

The next step is to expand x in a power series.

$$x(t) = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ... + \epsilon^n x_n \tag{2.8}$$

We are looking for a transition curve in the $\delta - \epsilon$ plane so we want δ to be a function of ϵ. We will assume that δ is a power series in ϵ.

$$\delta = \delta_0 + \epsilon \delta_1 + \epsilon^2 \delta_2 + ... + \epsilon^n \delta_n \tag{2.9}$$

After substituting eqs 2.8 and 2.9 into eq 2.7 and collecting terms in orders of epsilon up to $O(\epsilon^2)$ we find

$$\epsilon^0 : \qquad \frac{\partial^2 x_0}{\partial \xi^2} + \frac{1}{4} x_0 = 0 \tag{2.10}$$

$$\epsilon^1 : \qquad \frac{\partial^2 x_1}{\partial \xi^2} + \frac{1}{4} x_1 = -2 \frac{\partial^2 x_0}{\partial \xi \partial \eta} - x_0 \cos \xi - \delta_1 x_0 \tag{2.11}$$

$$\epsilon^2 : \qquad \frac{\partial^2 x_2}{\partial \xi^2} + \frac{1}{4} x_2 = -2 \frac{\partial^2 x_1}{\partial \xi \partial \eta} - x_1 \cos \xi - \delta_1 x_1 - \frac{\partial^2 x_2}{\partial \eta^2} - \delta_2 x_0 \tag{2.12}$$

The solution to eq 2.10 is

$$x_0 = A_0(\eta) \cos \frac{\xi}{2} + B_0(\eta) \sin \frac{\xi}{2} \tag{2.13}$$

We take this solution and substitute it into eq 2.11 to obtain

$$\begin{aligned} \frac{\partial^2 x_1}{\partial \xi^2} + \frac{1}{4} x_1 \quad = \quad & \frac{dA_0}{d\eta} \sin \frac{\xi}{2} - \frac{A_0(\eta)}{2} \left(\cos \frac{\xi}{2} + \cos \frac{3\xi}{2} + \delta_1 \cos \frac{\xi}{2} \right) \\ & - \frac{dB_0}{d\eta} \cos \frac{\xi}{2} + \frac{B_0(\eta)}{2} \left(\sin \frac{\xi}{2} - \sin \frac{3\xi}{2} - \delta_1 \sin \frac{\xi}{2} \right) \end{aligned} \tag{2.14}$$

The left side of the equation has the homogeneous solution

$$x_1 = A_1(\eta) \cos \frac{\xi}{2} + B_1(\eta) \sin \frac{\xi}{2} \tag{2.15}$$

which, apart from subscripts is identical to eq 2.13. We note that the right hand side of eq 2.14 has resonant terms that need to be removed so there are no secular terms in the solution. To remove these terms we set their coefficients equal to zero. The result is two coupled differential equations containing only the slow time variables. We will call these differential equations the slow flow (2.16).

$$\frac{dA_0}{d\eta} = \left(\delta_1 - \frac{1}{2}\right) B_0 \qquad \frac{dB_0}{d\eta} = -\left(\delta_1 + \frac{1}{2}\right) A_0 \tag{2.16}$$

The solution to eq 2.16 is

$$A_0 = C_1 e^{\eta\sqrt{\delta_1^2 - 1/4}} + C_2 e^{-\eta\sqrt{\delta_1^2 - 1/4}} \tag{2.17}$$

Which has exponential growth when $\delta_1^2 - 1/4 > 0$. This means the edge of unstable region is $\delta_1^2 - 1/4 = 0$ or $\delta_1 = \pm 1/2$. Therefore our transition curves are

$$\delta_l = \frac{1}{4} - \frac{\epsilon}{2} + O(\epsilon^2) \quad \text{and} \quad \delta_{\mathrm{r}} = \frac{1}{4} + \frac{\epsilon}{2} + \mathrm{O}(\epsilon^2) \tag{2.18}$$

To solve for the next term, we choose a branch to follow. We will chose δ_r and $\delta = 1/2$. This makes $A_0(\eta) = a_0$ and $B_0(\eta) = -a_0\eta + b_0$. The term $-a_0\eta$ in the solution has linear growth in the slow time η which is unacceptable. To rectify this we make $a_0 = 0$. This makes the slow flow neutrally stable and our differential equation (2.14) becomes

$$\frac{\partial^2 x_1}{\partial \xi^2} + \frac{1}{4} x_1 = -\frac{b_0}{2} \sin \frac{3\xi}{2} \tag{2.19}$$

which can be solved to find

$$x_1 = A_1(\eta) \cos \frac{\xi}{2} + B_1(\eta) \sin \frac{\xi}{2} + \frac{1}{4} b_0 \sin \frac{3\xi}{2} \tag{2.20}$$

We continue the process by plugging the solutions (2.13,2.20) into the next order equation (2.12) and looking for resonant terms. Removing the resonant terms produces the slow flow

$$\frac{dA_1}{d\eta} = \left(\frac{1}{8} + \delta_2\right) b_0 \qquad \frac{dB_1}{d\eta} = -A_1 \tag{2.21}$$

To make the slow flow equation (2.21) neutrally stable, we e set $A_1 = 0$ and $\delta_2 = -1/8$. We can repeat this procedure for the left transition curve ($\delta_1 = -1/2$) to find that $\delta_2 = -1/8$. The transition curves are thus

$$\delta_l = \frac{1}{4} - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + O(\epsilon^3) \quad \text{and} \quad \delta_{\mathrm{r}} = \frac{1}{4} + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \mathrm{O}(\epsilon^3) \tag{2.22}$$

It is not always the case that δ_2 for the left transition curve is the same as δ_2 for the right transition curve.

The process can be repeated for the ϵ^3 and higher order terms. This method has the advantage of giving us a solution to the system on the transition curve (or anywhere in the space if we change δ_0). However, it has the disadvantage of being very cumbersome for finding the higher order terms and will never yield an exact solution. The next method we will look at, harmonic balance, does not provide a solution to the ODE, but quickly produces higher order terms and can give their exact solution.

2.3 Floquet Theory

A discussion of harmonic balance necessitates an introduction to Floquet theory. The idea behind Floquet theory is to take a periodic n-dimensional system (x, $\ddot{x}$,...,$x^{(n)}$, and t) and reduce it to a (n-1)-dimensional system by removing any explicit reference to time. We can use Floquet theory on any equation of form (2.23).

$$\dot{x} = A(t)x \tag{2.23}$$

where $A(t)$ is periodic with period T $(A(t) = A(t+T))$. Since A is periodic, if $X(t)$ is a fundamental solution matrix to eq 2.23 then $X(t+T)$ is also a solution and is related to $X(t)$ by a constant matrix.

$$X(t+T) = X(t)\,C \tag{2.24}$$

We note then that at time $(t = 0)$ the fundamental solution matrix becomes

$$X(0+T) = X(0)\,C \quad \rightarrow \quad X(T) = C \tag{2.25}$$

Which means we can find C by numerically integrating eq 2.23 from 0 to T.

We now wish to find a solution to eq 2.24. To do this, we let $Y(t) = X(t)R$ with the result

$$Y(t+T) = Y(t)RCR^{-1} \tag{2.26}$$

Assuming that C is non-singular, an appropriate choice for R will separate the equations. The separated equations become

$$y_n(t+T) = \lambda_n y_n(t) \tag{2.27}$$

where λ_n is the n^{th} eigenvalue of C. We will now assume eq 2.27 has solutions of the form

$$y_n(t) = \lambda_n^{kt} p_n(t) \tag{2.28}$$

Substituting (2.28) into (2.27) results in

$$\lambda_n^{kt+kT} p_n(t+T) = \lambda_n \lambda_n^{kt} p_n(t) \tag{2.29}$$

where we assume $p_n(t)$ is periodic in time T $(p_n(t+T) = p_n(t))$. Eq 2.29 becomes

$$\lambda_n^{kT} p_n(t) = \lambda_n p_n(t) \tag{2.30}$$

Picking $k = 1/T$ satisfies the system (2.27) making y_n

$$y_n(t) = \lambda_n^{t/T} p_n(t) = e^{\frac{t}{T}\ln\lambda_n} p_n(t) \tag{2.31}$$

At time $t = 0, T, 2T, .., mT$ eq 2.31 becomes

$$\begin{aligned}
t = 0 \qquad & y_n(0) && = p_n(0) && (2.32)\\
t = T \qquad & y_n(T) && = \lambda_n p_n(T) && (2.33)\\
t = 2T \qquad & y_n(2T) && = \lambda_n^2 p_n(2T) && (2.34)\\
\ldots \qquad & \ldots && \ldots && \\
t = 3T \qquad & y_n(mT) && = \lambda_n^m p_n(mT) && (2.35)
\end{aligned}$$

The periodicity of p_n and y_n means that we can rewrite eq 2.35 as

$$y_n(mT) = \lambda_n^m p_n(0) \tag{2.36}$$

Substituting (2.32) into (2.36) produces

$$y_n(mT) = \lambda_n^m y(0) \tag{2.37}$$

More generically, eq 2.37 can be written as

$$y_n(t + mT) = \lambda_n^m y(t) \tag{2.38}$$

Eq 2.38 proves that the stability of the system at period m depends exclusively on λ_n. If for all n, $|\lambda_n| < 1$, y_n is stable. However, if for any p $|\lambda_p| > 1$ then the system is unstable since it has an unbounded solution.

To find solutions along transition curves, we look for system parameters $(A(t))$ such that the there is at least one value of k such that

$$|\lambda_k| \begin{cases} = 1 & n = k \\ \leq 1 & n \neq k \end{cases} \tag{2.39}$$

2.3.1 Applications to Hill's Equation

We now seek to apply Floquet theory to a generalization of Mathieu's equation called Hill's equation.

$$\ddot{x} + f(t)\, x = 0 \tag{2.40}$$

First we rewrite Hill's equation as two coupled first order systems (2.41)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -f(t) & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{2.41}$$

We will create a fundamental solution matrix out of two solution vectors X_i and X_j. X_i and X_j have initial conditions

$$\begin{bmatrix} x_{i1}(0) \\ x_{i2}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_{j1}(0) \\ x_{j2}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.42}$$

According to eq 2.25 the fundamental solution matrix (C) has the form

$$C = \begin{bmatrix} x_{i1}(T) & x_{j1}(T) \\ x_{i2}(T) & x_{j2}(T) \end{bmatrix} \tag{2.43}$$

The next step is to solve for the eigenvalues of C. The characteristic equation for matrix C is

$$\lambda^2 - \operatorname{tr} C\, \lambda + \det C = 0 \tag{2.44}$$

We now show that the derivative of $\det C = 0$.

$$\det C = x_{i1}(T) \cdot x_{j2}(T) - x_{j1}(T) \cdot x_{i2}(T) \tag{2.45}$$

$$\frac{d}{dt}[\det C] = \dot{x}_{i1}(T) \cdot x_{j2}(T) + x_{i1}(T) \cdot \dot{x}_{j2}(T) - \dot{x}_{j1}(T) \cdot x_{i2}(T) - x_{j1}(T) \cdot \dot{x}_{i2}(T) \tag{2.46}$$

From eq 2.41 we note that $\dot{x}_1 = x_2$ and $\dot{x}_2 = -f(t)x_1$. Substituting these into (2.46) produces

$$\begin{aligned} \frac{d}{dt}[\det C] &= x_{i2}(T) \cdot x_{j2}(T) + x_{i1}(T) \cdot (-f(t))x_{j1}(T) \\ &\quad - x_{j2}(T) \cdot x_{i2}(T) - x_{j1}(T) \cdot (-f(t))x_{i1}(T) = 0 \end{aligned} \tag{2.47}$$

Since $\det C$ is a constant, we can find a valid value for $\det C$ by evaluating eq 2.43 at any time. The easiest time is at $t = 0$. Plugging our ICs (2.42) into eq 2.43 gives us $C = 1$. Our characteristic equation (2.44) becomes

$$\lambda^2 - \operatorname{tr} C\lambda + 1 = 0 \tag{2.48}$$

Next, we solve the characteristic equation (2.48) to find λ_1 and λ_2.

$$\lambda_{1,2} = \frac{\operatorname{tr} C \pm \sqrt{(\operatorname{tr} C)^2 - 4}}{2} \tag{2.49}$$

Since the determinate of a matrix can be expressed as the product of the eigenvalues, we know that

$$\lambda_1 \lambda_2 = \det C = 1 \tag{2.50}$$

There are three regimes for $\operatorname{tr} C$, namely

$$|\operatorname{tr} C| < 2 \quad \rightarrow \quad \lambda_{1,2} = x \pm yi \text{ or } -x \pm yi \quad \text{where } x < 1 \tag{2.51}$$

$$|\operatorname{tr} C| = 2 \quad \rightarrow \quad \lambda_{1,2} = 1, 1 \text{ or } -1, -1 \tag{2.52}$$

$$|\operatorname{tr} C| > 2 \quad \rightarrow \quad \lambda_{1,2} = x_1, x_2 \text{ or } -x_1, -x_2 \quad \text{where } x_1 < 1 < x_2 \tag{2.53}$$

From eq 2.38 any $|\lambda_n| > 1$ indicates instability. Thus our three regimes are

$$\begin{aligned} &|\operatorname{tr} C| < 2 \qquad \text{stable periodic motion} \\ &|\operatorname{tr} C| = 2 \qquad \text{neutrally stable motion} \\ &|\operatorname{tr} C| > 2 \qquad \text{unstable motion} \end{aligned} \tag{2.54}$$

When $\operatorname{tr} C = 2$, $\lambda = 1, 1$, the system is behaving with periodic motion of period T (see eq 2.38). When $\operatorname{tr} C = -2$, $\lambda = 1, 1$, the system is behaving with periodic motion of period $2T$ (see eq 2.38).

Thus we come to our main conclusion from Floquet theory. Namely, that for systems of the Hill type "on the transition curves in parameter space between stable and unstable regions, there exist periodic motions of period T or $2T$." [12]

2.3.2 Locations of Tongues of Instability

One important use of Floquet theory is to find where the tongues of instability start (or emanate from the $\epsilon = 0$ axis). According to Floquet theory the tongues of instability occur when the period of the parametric forcer is at T or $2T$ where T is the natural frequency of the system. Thus for Mathieu's equation

$$\ddot{x} + (\delta + \epsilon \cos t)x = 0 \tag{2.55}$$

The system period (T_s) is $2\pi/\sqrt{\delta}$. However, that is only the lowest period. Actually, the system has an infinite number of periods described by nT_s where $n \in \mathbb{Z}$. The period of the forcer (T_f) is 2π. Thus, resonance occurs when

$$nT_s = T_f \text{ or } 2\mathrm{T_f} \quad \rightarrow \quad \mathrm{n}\frac{2\pi}{\sqrt{\delta}} = \mathrm{m}2\pi \tag{2.56}$$

where $m = 1$ or 2. Solving for δ we find

$$\delta = \left(\frac{n}{m}\right)^2 \quad \rightarrow \quad \delta = n^2 \text{ or } \frac{n^2}{4} \tag{2.57}$$

which gives us $\delta = n^2$ for $n \in \mathbb{Z}$ from resonances due to $nT_s = T_f$ and $\delta = \frac{n^2}{4}$ for odd $n \in \mathbb{Z}$ from resonances due to $nT_s = 2T_f$. Thus, the tongues occur at

$$\delta = 0,\ \frac{1}{4},\ 1,\ \frac{9}{4},\ 4,\ \frac{25}{4},\ 16,\ ...,\ \frac{n^2}{4} \quad n \in \mathbb{Z} \tag{2.58}$$

2.4 Harmonic Balance

2.4.1 Theoretical Development

At this point we want to use the results of Floquet theory (section 2.3) to find the transition curves analytically, through a method called harmonic balance (HB). Our example for this section will be Mathieu's equation (2.1).

$$\ddot{x} + (\delta + \epsilon \cos t)x = 0 \tag{2.1}$$

For HB we make use of the fact that on transition curves there exist periodic solutions of period T or $2T$. For Mathieu's equation $T_f = 2\pi$. Thus, solutions to eq 2.1 take the form of the Fourier series:

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{2}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{2}\right) \tag{2.59}$$

which has a period of $T = 4\pi$. We will substitute eq 2.59 into eq 2.1 to find

$$\sum_{n=0}^{\infty}\left(-\frac{n^4}{4} + \delta + \epsilon\cos t\right) A_n \cos\frac{nt}{2} + \sum_{n=1}^{\infty}\left(-\frac{n^4}{4} + \delta + \epsilon\cos t\right) B_n \sin\frac{nt}{2} = 0 \tag{2.60}$$

Since the trigonometric functions are orthogonal, the coefficient for any A_n or B_n must sum to zero. Thus we have an infinite number of coupled equations on the A_n and B_n terms. We see from (2.60) that equations with A_n coefficients are decoupled from those with B_n coefficients. (Note that the left half of the equation is even and the right half is odd).

$$\sum_{n=0}^{\infty}\left(-\frac{n^4}{4} + \delta + \epsilon\cos t\right) A_n \cos\frac{nt}{2} = 0 \tag{2.61}$$

$$\sum_{n=1}^{\infty}\left(-\frac{n^4}{4} + \delta + \epsilon\cos t\right) B_n \sin\frac{nt}{2} = 0 \tag{2.62}$$

After the appropriate trig substitutions eq 2.62 becomes

$$\left[B_1\left(\delta - \frac{1}{4} - \frac{\epsilon}{2}\right) + B_3\frac{\epsilon}{2}\right]\sin\frac{t}{2} \tag{2.63}$$

$$+\left[B_2(\delta - 1) + B_4\frac{\epsilon}{2}\right]\sin t \tag{2.64}$$

$$+ \ldots$$

$$+\left[B_{n-2}\frac{\epsilon}{2} + B_n\left(\delta - \frac{n^2}{4}\right) + B_{n+2}\frac{\epsilon}{2}\right]\sin\frac{nt}{2} = 0 \tag{2.65}$$

In eq 2.65 the n^{th} term contains only B_n coefficients of the same parity. This means that we can further decouple the equations into odd and even n terms. This makes sense since the even terms correspond to the negative Floquet multiplier ($\lambda = -1$)

and the odd terms to the positive Floquet multiplier ($\lambda = 1$). A similar expansion of eq 2.61 shows that we can decouple the A_n terms into odd and even n terms as well. The result is four sets of coupled equations. We will call them the A-even, A-odd, B-even, and B-odd equations.

A-even

$$\begin{bmatrix} 1 \\ \cos 2t \\ \cos 4t \\ \cos 6t \\ \vdots \end{bmatrix}^T \begin{bmatrix} \delta & \frac{\epsilon}{2} & 0 & 0 & \ldots \\ \epsilon & \delta - 1 & \frac{\epsilon}{2} & 0 & \ldots \\ 0 & \frac{\epsilon}{2} & \delta - 4 & \frac{\epsilon}{2} & \ldots \\ 0 & 0 & \frac{\epsilon}{2} & \delta - 9 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} A_0 \\ A_2 \\ A_4 \\ A_6 \\ \vdots \end{bmatrix} = 0 \tag{2.66}$$

B-even

$$\begin{bmatrix} \sin 2t \\ \sin 4t \\ \sin 6t \\ \sin 8t \\ \vdots \end{bmatrix}^T \begin{bmatrix} \delta - 1 & \frac{\epsilon}{2} & 0 & 0 & \ldots \\ \frac{\epsilon}{2} & \delta - 4 & \frac{\epsilon}{2} & 0 & \ldots \\ 0 & \frac{\epsilon}{2} & \delta - 9 & \frac{\epsilon}{2} & \ldots \\ 0 & 0 & \frac{\epsilon}{2} & \delta - 16 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} B_2 \\ B_4 \\ B_6 \\ B_8 \\ \vdots \end{bmatrix} = 0 \tag{2.67}$$

A-odd

$$\begin{bmatrix} \cos t \\ \cos 3t \\ \cos 5t \\ \cos 7t \\ \vdots \end{bmatrix}^T \begin{bmatrix} \delta - \frac{1}{4} + \frac{\epsilon}{2} & \frac{\epsilon}{2} & 0 & 0 & \ldots \\ \frac{\epsilon}{2} & \delta - \frac{9}{4} & \frac{\epsilon}{2} & 0 & \ldots \\ 0 & \frac{\epsilon}{2} & \delta - \frac{25}{4} & \frac{\epsilon}{2} & \ldots \\ 0 & 0 & \frac{\epsilon}{2} & \delta - \frac{49}{4} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \\ A_5 \\ A_7 \\ \vdots \end{bmatrix} = 0 \tag{2.68}$$

B-odd

$$\begin{bmatrix} \sin t \\ \sin 3t \\ \sin 5t \\ \sin 7t \\ \vdots \end{bmatrix}^T \begin{bmatrix} \delta - \frac{1}{4} + \frac{\epsilon}{2} & \frac{\epsilon}{2} & 0 & 0 & \ldots \\ \frac{\epsilon}{2} & \delta - \frac{9}{4} & \frac{\epsilon}{2} & 0 & \ldots \\ 0 & \frac{\epsilon}{2} & \delta - \frac{25}{4} & \frac{\epsilon}{2} & \ldots \\ 0 & 0 & \frac{\epsilon}{2} & \delta - \frac{49}{4} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} B_1 \\ B_3 \\ B_5 \\ B_7 \\ \vdots \end{bmatrix} = 0 \qquad (2.69)$$

2.4.2 Finding the Transition Curves

We will now demonstrate how to find the transition curves from these matrices (2.66),(2.67),(2.68),(2.69). We will begin by looking at the A-odd matrix.

The A-odd system of equations can be solved to find the A_n for the solution along the transition curve. However, to find a non-trivial solution for the A_n the determinate of the A-odd matrix must be zero. Since we don't care about the actual solution on the transition curve we can ignore the actual A_n values and set

$$\det \begin{vmatrix} \delta - \frac{1}{4} + \frac{\epsilon}{2} & \frac{\epsilon}{2} & 0 & 0 & \ldots \\ \frac{\epsilon}{2} & \delta - \frac{9}{4} & \frac{\epsilon}{2} & 0 & \ldots \\ 0 & \frac{\epsilon}{2} & \delta - \frac{25}{4} & \frac{\epsilon}{2} & \ldots \\ 0 & 0 & \frac{\epsilon}{2} & \delta - \frac{49}{4} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0 \qquad (2.70)$$

Obviously we can't take the determinant of an infinite matrix. However, if we truncate the matrix and take the determinant, we can get an approximation for the curves. Here we will take the determinate of a 3x3 truncation of the A-odd matrix

$$\delta^3 + \left(\frac{1}{2}\epsilon - \frac{35}{4}\right)\delta^2 - \left(\frac{1}{2}\epsilon^2 + \frac{17}{4}\epsilon - \frac{259}{16}\right)\delta - \left(\frac{1}{8}\epsilon^3 - \frac{13}{8}\epsilon^2 - \frac{225}{32}\epsilon + \frac{225}{64}\right) = 0 \tag{2.71}$$

Solving eq 2.71 produces an equation for $\delta(\epsilon)$, however depending on the size of the truncation, this method becomes quickly prohibitive. We recall that the method of two-variable expansion (section 2.2) produced an equation for δ in terms of a power series in ϵ (2.22). Since we are approximating the determinate it makes sense that we should assume δ takes the form

$$\delta = \delta_0 + \delta_1\epsilon + \delta_2\epsilon^2 + \delta_3\epsilon^3 + ... + \delta_n\epsilon^n \tag{2.72}$$

Substituting [2.72] into [2.71] and collecting in terms of ϵ produces

$$\begin{aligned}
&\frac{1}{64}(4\delta_0 - 25)(4\delta_0 - 9)(4\delta_0 - 1) \\
&+ \left(\frac{225}{32} + 3\delta_0^2\delta_1 - \frac{17}{4}\delta_0 + \frac{259}{16}\delta_1 - \frac{35}{2}\delta_0\delta_1 + \frac{1}{2}\delta_0^2\right)\epsilon \\
&+ \left(\frac{13}{8} + 3\delta_0\delta_1^2 + 3\delta_0^2\delta_2 - \frac{35}{4}\delta_1^2 - \frac{1}{2}\delta_0 - \frac{17}{4}\delta_1 + \frac{259}{16}\delta_2 + \delta_0\delta_1 - \frac{35}{2}\delta_0\delta_2\right)\epsilon^2 = 0
\end{aligned} \tag{2.73}$$

We see from the ϵ^0 term that $\delta_0 = \frac{1}{4}$, $\frac{9}{4}$, or $\frac{25}{4}$. We note that this corresponds to our known δ values for where the tongues emanate from the $\epsilon = 0$ axis. For the sake of comparison with two-variable expansion we will chose $\delta_0 = \frac{1}{4}$ and solve for the higher order terms. The result is that eq 2.73 becomes

$$\epsilon^0 : \quad \tfrac{1}{64}(4\delta_0 - 25)(4\delta_0 - 9)(4\delta_0 - 1) = 0 \quad \rightarrow \delta_0 = \frac{1}{4} \tag{2.74}$$

$$\epsilon^1 : \quad 6 + 12\delta_1 = 0 \quad \rightarrow \delta_1 = -\frac{1}{2} \tag{2.75}$$

$$\epsilon^2 : \quad \tfrac{3}{2} + 12\delta_2 = 0 \quad \rightarrow \delta_2 = -\frac{1}{8} \tag{2.76}$$

Thus,

$$\delta = \frac{1}{4} - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + O(\epsilon^3) \tag{2.77}$$

which is exactly what we found for δ_l from two-variable expansion. Using this process we can go to any order truncation to find the transition curves. If we go to a higher order truncation we find

$$\delta = \frac{1}{4} - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \frac{\epsilon^3}{32} - \frac{\epsilon^4}{384} - \frac{11\epsilon^5}{4608} + O(\epsilon^6) \tag{2.78}$$

where the first three terms have not changed. Thus we see that we can get arbitrary accurate equations for the transition curves. If we could solve the infinite determinate problem we could find the equations exactly.

2.5 Numerical Techniques

2.5.1 Numerical Applications of Floquet Theory

Numerical routines for finding transition curves rely on Floquet theory as well. The basic approach is numerically find the fundamental solution matrix. In section 2.3 we created a fundamental solution matrix from two solution vectors X_i and X_j. X_i and X_j have initial conditions

$$\begin{bmatrix} x_{i1}(0) \\ x_{i2}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_{j1}(0) \\ x_{j2}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.42}$$

From (2.25) the fundamental solution matrix (C) has the form

$$C = \begin{bmatrix} x_{i1}(T) & x_{j1}(T) \\ x_{i2}(T) & x_{j2}(T) \end{bmatrix} \tag{2.43}$$

In section 2.3 we used the fundamental solution matrix analytically. However, it is also possible to analyze it numerically. We start by integrating the system for one full period (T) using the initial conditions from eq 2.42. We use the values at $t = T$ to create the fundamental solution matrix and then perform an eigenvalue analysis using the stability criteria in section 2.3 (2.39). A sample Matlab program is provided as an example. Combining this algorithm with root finding techniques enables us to find the transition curves numerically.

2.5.2 FILE: stability.m

```
function [isstable]=stability(T,A,B,C,D)
global a b c delta
a=A; b=B; c=C; delta=D;  %Assign Ince Equation coefficients
% This program analyzes the stability of an ince type system
% for a given set of parameters (a,b,c,delta) which are
% provided as inputs.  It outputs a binary answer.
%Example: To run this program use the command:
%  'output=stability(2*pi,0.5,-3,4,1)'


%Integrate the system for the given ICs from t=0..T.
   [Ti,Xi]=ode45('ode',[0 T], [1 0]);
   [Tj,Xj]=ode45('ode',[0 T], [0 1]);
%Create the fundamental solution matrix (FSM)
   C  =([[Xi(end,1)    Xj(end,1)] ;
         [Xi(end,2)    Xj(end,2)]]);
%find the eigenvalues of the FSM
   [EigVec EigVal]=eig(C);
%determine the stability
   Eig=[abs(real(EigVal(1,1))) abs(real(EigVal(2,2)))];
```

```
   if max(Eig) >= 1.0,  isstable=0;  %false
    else,               isstable=1;  %true
   end
```

2.5.3 FILE: ode.m

```
function xdot=ode(t,x)
global a b c delta


%This file is called by ode45 and returns Ince's Equation


xdot=[x(2); - b*sin(t)/(1+a*cos(t))*x(2)...
      -(delta+c*cos(t))/(1+a*cos(t))*x(1)];
%the period of this system is T=2 pi
```

2.6 Conclusions

The theory and techniques discussed in this chapter will be referred to throughout the thesis. While the bulk of the analysis presented is analytical, numerical integration provides confirmation of the results.

Chapter 3

Coexistence of a Generalized Inces Equation

3.1 Introduction

3.1.1 Introductory Example: The Particle in the Plane

This chapter concerns the stability of nonlinear normal modes in two degree of freedom systems. Instabilities in such cases are due to autoparametric excitation [5], that is, parametric excitation which is caused by the system itself, rather than by an external periodic driver. The investigation of stability involves the solution of a system of linear differential equations with periodic coefficients (see Floquet theory in section 2.3). The typical behavior of such a system involves tongues of instability representing parametric resonances (Mathieu's equation for example). Coexistence phenomenon refers to the circumstance in which some of these tongues of instability have closed up and disappeared. Their absence cloaks hidden instabilities which may emerge due to small changes in the system. This effect is important because it occurs in various mechanical systems.

We begin by illustrating the phenomenon with a physical example. This example, called "the particle in the plane" by Yang and Rosenberg [20], [21] who first studied it, involves a unit mass which is constrained to move in the x-y plane, and is restrained by two linear springs, each with spring constant of $k = \frac{1}{2}$. The anchor points of the two springs are located on the x axis at $x = 1$ and $x = -1$. Each of the two springs has unstretched length L (figure 3.1).

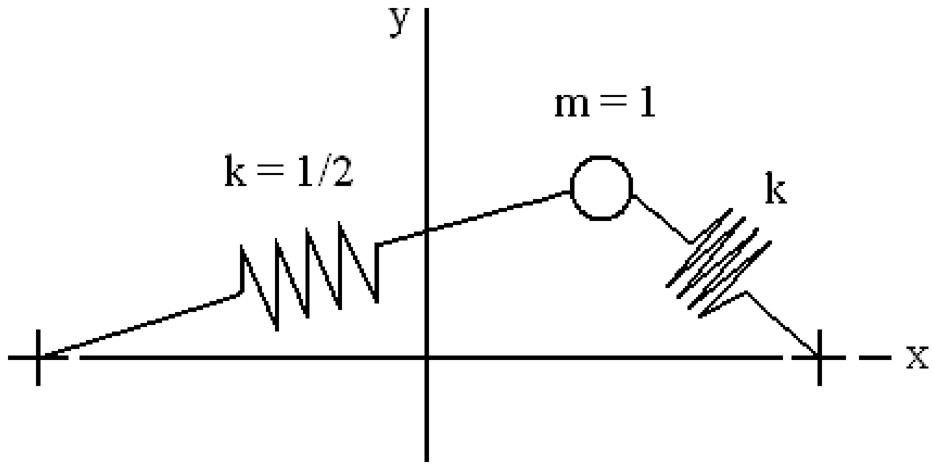

Figure 3.1: *The Particle in the Plane.*

This autonomous two degree of freedom system has the following equations of motion [20]:

$$\ddot{x} + (x+1)f_1(x,y) + (x-1)f_2(x,y) = 0 \tag{3.1}$$

$$\ddot{y} + yf_1(x,y) + yf_2(x,y) = 0 \tag{3.2}$$

where

$$f_1(x,y) = \frac{1}{2}\left(1 - \frac{L}{\sqrt{(1+x)^2 + y^2}}\right) \tag{3.3}$$

$$f_2(x,y) = \frac{1}{2}\left(1 - \frac{L}{\sqrt{(1-x)^2 + y^2}}\right) \tag{3.4}$$

This system exhibits an exact solution corresponding to a mode of vibration in which the particle moves along the x axis (the x-mode):

$$x = A\cos t, \qquad y = 0 \tag{3.5}$$

In order to determine the stability of this motion, one must substitute $x = A\cos t + u, \;\; y = 0 + v$ into the equations of motion (3.1),(3.2) where u and v are small deviations from the motion (3.5), and then linearize in u and v. The result is two

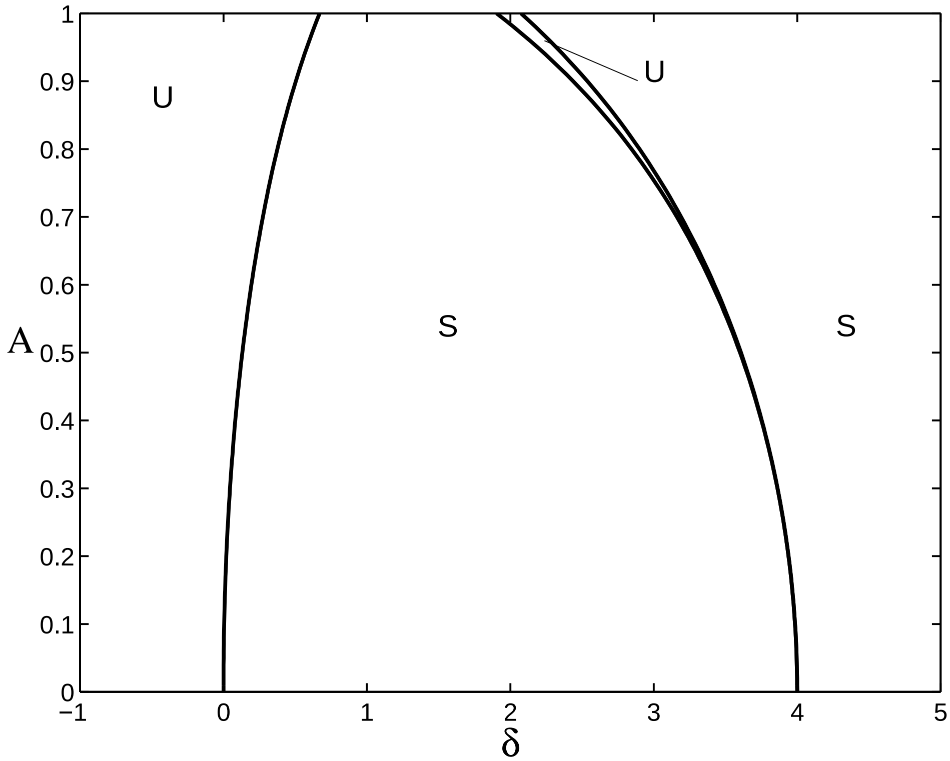

Figure 3.2: *Stability chart for eq 3.6. S=stable, U=unstable. Curves obtained by perturbation analysis.*

linear differential equations on u and v. The u equation turns out to be the simple harmonic oscillator, and cannot produce instability. The v equation is:

$$\frac{d^2v}{dt^2} + \left(\frac{\delta - A^2\cos^2 t}{1 - A^2\cos^2 t}\right) v = 0 \tag{3.6}$$

where $\delta = 1 - L$. For a particular pair of parameters (A, δ), eq 3.6 is said to be stable if all solutions to eq 3.6 are bounded, and unstable if an unbounded solution exists. A stability chart for eq 3.6 may be obtained by using either perturbation theory or numerical integration together with Floquet theory (see section 2.3,5 or [12] for examples). See figure 3.2. Note that although this equation (3.6) exhibits an infinite number of tongues of instability, only one of them (emanating from the point $\delta = 4$, $A = 0$) is displayed, for convenience. (The tongues of instability emanate from $\delta = 4n^2$, $A = 0$ for $n = 1, 2, 3, \cdots$, and becomes progressively nar-

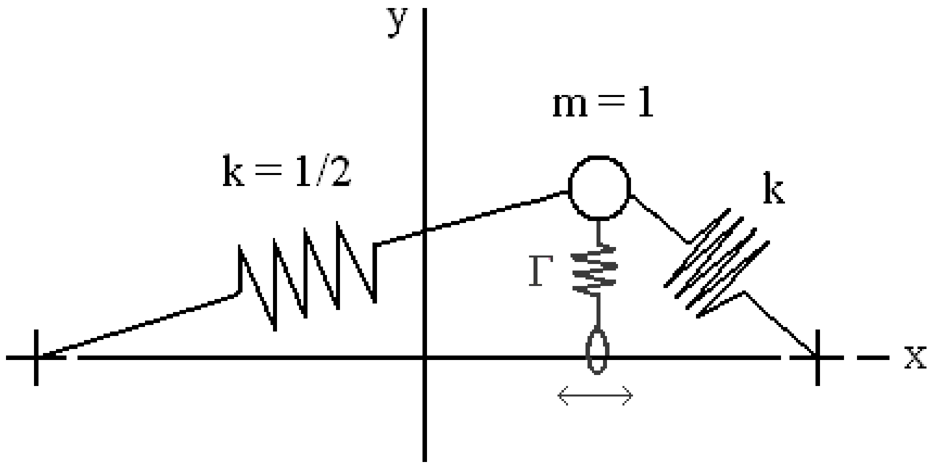

Figure 3.3: *The Particle in the Plane with a vertical spring added.*

rower for increasing n.) Since the unstretched spring length $L > 0$, the parameter $\delta = 1 - L < 1$. Thus the only tongue of instability for eq 3.6 which has physical significance is the one which emanates from $\delta = 0$ (figure 3.2).

Now we wish to compare the behavior of this system with a slightly perturbed system in which some extra stiffness is added. We add a spring which gives a force $-\Gamma y$ in the y-direction (see figure 3.3). This adds a term $+\Gamma y$ to the left hand side of eq 3.2. The new system still exhibits the periodic solution (3.5), and its stability turns out to be governed by the O.D.E.

$$\frac{d^2 v}{dt^2} + \left(\frac{\delta + \Gamma - (1 + \Gamma) A^2 \cos^2 t}{1 - A^2 \cos^2 t} \right) v = 0 \tag{3.7}$$

Note that eq 3.7 reduces to eq 3.6 for $\Gamma = 0$. Figure 3.4 shows the stability chart for eq 3.7.

Comparison of figures 3.2 and 3.4 shows that a new region of instability has occurred due to the small change made in the system. If an engineering design was based on figure 3.2, and if the actual engineering system involved slight departures from the model of eq 3.6, the appearance of such an unexpected region of instability could cause disastrous consequences. For this reason we investigate the possibility

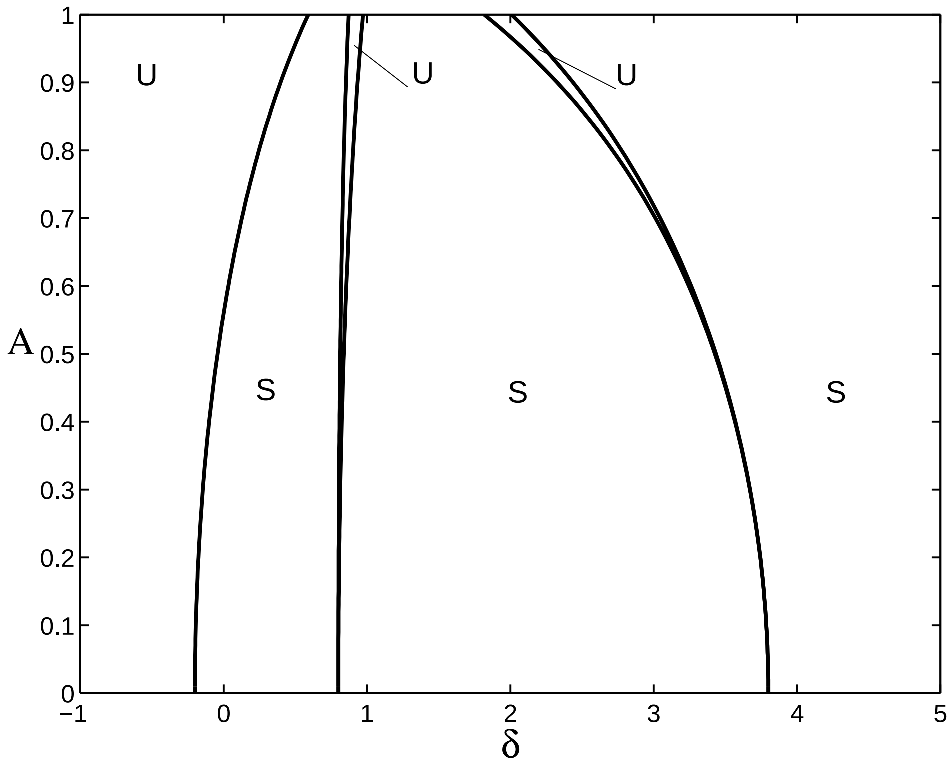

Figure 3.4: *Stability chart for eq 3.7 for* $\Gamma = 0.2$*. S=stable, U=unstable. Note the presence of an additional tongue of instability compared to figure 3.2. See text.*

of the occurrence of such hidden instabilities in a class of two degree of freedom systems.

3.1.2 Coexistence Phenomenon

The appearance of an unexpected instability region in the foregoing example may be explained by stating that eq 3.6 had buried in it an instability region of zero thickness [13]. This is shown in figure 3.5, which is a replot of figure 3.2 with the zero-thickness instability region displayed as a dashed line. This curve, which happens to have the simple equation $\delta = 1$, is characterized by the *coexistence* of two linearly independent periodic solutions of period 2π. This condition is singular and so we are not surprised to find that nearly any perturbation of the original system

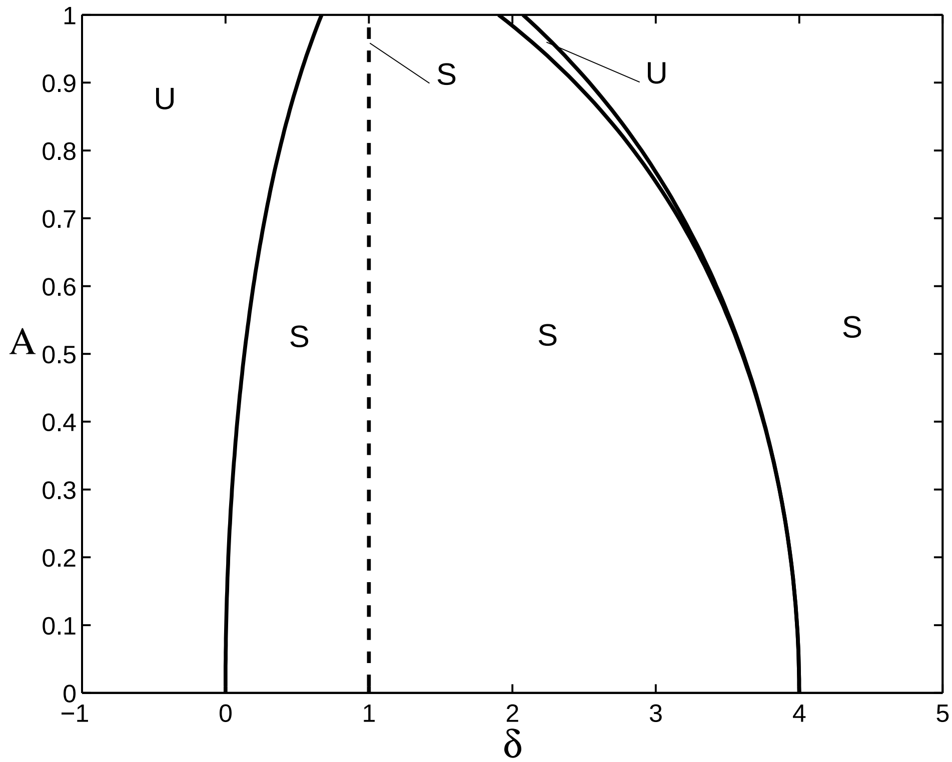

Figure 3.5: *Stability chart for eq 3.6, or equivalently eq 3.7 when* $\Gamma = 0$*, showing coexistence curve as a dashed line (here* $\delta = 1$*). Note that although the coexistence curve is itself stable, it may give rise to a tongue of instability if the system is perturbed.*

(3.6), such as the reassignment of spring stiffnesses in (3.7), will produce an opening up of the zero-thickness instability region. It should be mentioned that there are various other physical systems which are known to exhibit coexistence. These include a simplified model of a vibrating elastica [8], the elastic pendulum [12], rain-wind induced vibrations [17], Josephson junctions [3] and coupled nonlinear oscillators [9].

Coexistence phenomenon has been treated from a theoretical point of view in [4], and more recently in [12] and [7]. In this chapter we use perturbation methods to rederive and extend the results given in [4], [12] and [7]. In particular, we address the question of finding conditions under which a class of linear O.D.E.'s with periodic coefficients will exhibit coexistence phenomenon.

3.2 Motivating Application

We wish to study autoparametric excitation in a class of systems which on the one hand have the following very general expressions for kinetic energy T and potential energy V:

$$T = \beta_1(x,y)\dot{x}^2 + \beta_2(x,y)\dot{x}\dot{y} + \beta_3(x,y)\dot{y}^2 \tag{3.8}$$

$$\begin{aligned} V = &\frac{1}{2}\omega_1^2 x^2 + \frac{1}{2}\omega_2^2 y^2 \\ &+\alpha_{40}x^4 + \alpha_{31}x^3y + \alpha_{22}x^2y^2 + \alpha_{13}xy^3 + \alpha_{04}y^4 \end{aligned} \tag{3.9}$$

and on the other hand generalize the particle in the plane example by exhibiting an x-mode of the form of eq 3.5:

$$x = A\cos t, \qquad y = 0 \tag{3.10}$$

Writing Lagrange's equations for the system (3.8),(3.9), we find that in order for eq 3.10 to be a solution, we must have $\alpha_{40} = 0$, $\alpha_{31} = 0$, $\beta_2 = 0$ and $\beta_1 = \omega_1^2/2$. Choosing $\omega_1 = 1$ without loss of generality, we obtain the following expressions for T and V:

$$T = \frac{1}{2}\dot{x}^2 + \beta_3(x,y)\dot{y}^2 \tag{3.11}$$

$$V = \frac{1}{2}x^2 + \frac{1}{2}\omega_2^2 y^2 + \alpha_{22}x^2y^2 + \alpha_{13}xy^3 + \alpha_{04}y^4 \tag{3.12}$$

We further assume that $\beta_3(x,y)$ has the following form:

$$\beta_3(x,y) = \beta_{00} + \beta_{01}x + \beta_{10}y + \beta_{02}x^2 + \beta_{11}xy + \beta_{20}y^2 \tag{3.13}$$

Now we investigate the linear stability of the x-mode (3.10). We set $x = A\cos t + u, \;\; y = 0 + v$ in Lagrange's equations and then linearize in u and v. This gives the u equation as $\ddot{u} + u = 0$ and the v equation as:

$$\begin{aligned} (2\beta_{00} + A^2\beta_{02} + 2A\beta_{01}\cos t + A^2\beta_{02}\cos 2t) \quad & \ddot{v} \\ +(-2A\beta_{01}\sin t - 2A^2\beta_{02}\sin 2t) \quad & \dot{v} \\ +(\omega_2^2 + A^2\alpha_{22} + A^2\alpha_{22}\cos 2t) \quad & v \quad = 0 \end{aligned} \tag{3.14}$$

This leads us to consider the following abbreviated form of eq 3.14:

$$\begin{aligned} (1 + a_1 \;\; \cos t + a_2 \;\; \cos 2t) \quad & \ddot{v} \\ +(b_1 \;\; \sin t + b_2 \;\; \sin 2t) \quad & \dot{v} \\ +(\delta + c_1 \;\; \cos t + c_2 \;\; \cos 2t) \quad & v \quad = 0 \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} a_1 &= \frac{2A\beta_{01}}{2\beta_{00} + A^2\beta_{02}} \\ a_2 &= \frac{A^2\beta_{02}}{2\beta_{00} + A^2\beta_{02}} \\ b_1 &= \frac{-2A\beta_{01}}{2\beta_{00} + A^2\beta_{02}} \quad = -a_1 \\ b_2 &= \frac{-2A^2\beta_{02}}{2\beta_{00} + A^2\beta_{02}} \quad = -2a_2 \\ \delta &= \frac{\omega_2^2 + A^2\alpha_{22}}{2\beta_{00} + A^2\beta_{02}} \\ c_1 &= 0 \\ c_2 &= \frac{A^2\alpha_{22}}{2\beta_{00} + A^2\beta_{02}} \end{aligned} \tag{3.16}$$

3.3 Generalized Ince's Equation

We come now to the main content of this paper, namely a study of the coexistence phenomenon in the O.D.E. (3.15):

$$\begin{aligned} (1 + a_1 \ \cos t + a_2 \ \cos 2t) & \quad \ddot{v} \\ +(b_1 \ \sin t + b_2 \ \sin 2t) & \quad \dot{v} \\ +(\delta + c_1 \ \cos t + c_2 \ \cos 2t) & \quad v \ = 0 \end{aligned} \tag{3.17}$$

In the case that $a_2 = 0$, $b_2 = 0$ and $c_2 = 0$, eq 3.17 reduces to a well-known O.D.E. called Ince's equation. Coexistence in Ince's equation has been studied in [4], [12] and [7]. In the rest of this paper, we generalize the previously obtained results for Ince's equation to apply to the generalized Ince's equation (3.17).

Eq(3.17) is a linear O.D.E. with periodic coefficients having period 2π. From Floquet theory (section 2.3) we know that the transition curves separating regions of stability from regions of instability are defined by sets of parameter values that allow periodic solutions of period 2π or 4π. These curves can be found by using the method of harmonic balance (section 2.4). Periodicity enables the solution to be written in the form of a Fourier series:

$$v(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{nt}{2} + \sum_{n=1}^{\infty} B_n \sin \frac{nt}{2} \tag{3.18}$$

Substituting (3.18) into (3.17) and trigonometrically reducing and collecting terms gives an infinite set of coupled equations. These uncouple into four sets of equations on even and odd cosine (A_n) and sine (B_n) coefficients. For example, the A-even coefficients satisfy the following equations:

A-even

$$\begin{bmatrix} \delta & -\frac{1}{2}\,a_1-\frac{1}{2}\,b_1+\frac{1}{2}\,c_1 & -2\,a_2-b_2+\frac{1}{2}\,c_2 & \ldots & \\ c_1 & \delta-1 & -\frac{1}{2}\,a_2-\frac{1}{2}\,b_2+\frac{1}{2}\,c_2 & \ldots & \\ c_2 & -\frac{1}{2}\,a_1+\frac{1}{2}\,b_1+\frac{1}{2}\,c_1 & \delta-4 & \ldots & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} A_0 \\ A_2 \\ A_4 \\ \vdots \end{bmatrix} = 0 \tag{3.19}$$

To simplify the notation, we introduce the following substitutions:

$$T(n) = \delta - \left(\tfrac{n}{2}\right)^2 \tag{3.20}$$

$$M(n) = \tfrac{1}{2}\left(-\left(\tfrac{n}{2}\right)^2\, a_1 + \tfrac{n}{2}\, b_1 + c_1\right) \tag{3.21}$$

$$P(n) = \tfrac{1}{2}\left(-\left(\tfrac{n}{2}\right)^2\, a_2 + \tfrac{n}{2}\, b_2 + c_2\right) \tag{3.22}$$

The four sets of penta-diagonal matrix equations may then be written:

A-even

$$\begin{bmatrix} T(0) & M(-2) & P(-4) & 0 & 0 & \ldots \\ 2M(0) & T(2)+P(-2) & M(-4) & P(-6) & 0 & \ldots \\ 2P(0) & M(2) & T(4) & M(-6) & P(-8) & \ldots \\ 0 & P(2) & M(4) & T(6) & M(-8) & \ldots \\ 0 & 0 & P(4) & M(6) & T(8) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} A_0 \\ A_2 \\ A_4 \\ A_6 \\ A_8 \\ \vdots \end{bmatrix} = 0 \tag{3.23}$$

B-even

$$\begin{bmatrix} T(2)-P(-2) & M(-4) & P(-6) & 0 & 0 & \dots \\ M(2) & T(4) & M(-6) & P(-8) & 0 & \dots \\ P(2) & M(4) & T(6) & M(-8) & P(-10) & \dots \\ 0 & P(4) & M(6) & T(8) & M(-10) & \dots \\ 0 & 0 & P(6) & M(8) & T(10) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} B_2 \\ B_4 \\ B_6 \\ B_8 \\ B_{10} \\ \vdots \end{bmatrix} = 0 \tag{3.24}$$

A-odd

$$\begin{bmatrix} T(1)+M(-1) & M(-3)+P(-3) & P(-5) & 0 & \dots \\ M(1)+P(-1) & T(3) & M(-5) & P(-7) & \dots \\ P(1) & M(3) & T(5) & M(-7) & \dots \\ 0 & P(3) & M(5) & T(7) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \\ A_5 \\ A_7 \\ \vdots \end{bmatrix} = 0 \tag{3.25}$$

B-odd

$$\begin{bmatrix} T(1)-M(-1) & M(-3)-P(-3) & P(-5) & 0 & \dots \\ M(1)-P(-1) & T(3) & M(-5) & P(-7) & \dots \\ P(1) & M(3) & T(5) & M(-7) & \dots \\ 0 & P(3) & M(5) & T(7) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} B_1 \\ B_3 \\ B_5 \\ B_7 \\ \vdots \end{bmatrix} = 0 \tag{3.26}$$

Each of the four above sets of equations is homogeneous and of infinite order, so for a nontrivial solution the determinants must vanish. Note that the resulting determinants for A-odd and B-odd are identical except for the first row and the first column. A comparable similarity exists between the determinants for A-even and B-even. Although generally the vanishing of, say, the A-odd determinant will give a completely different result than that of the B-odd determinant, nevertheless there may exist a special relationship between the coefficients such that the two results will give infinitely many identical branches, that is, infinitely many of the transition curves will be identical, in which case the associated instability regions will disappear (or rather will have zero width). On such transition curves we will have both an odd and an even periodic motion, that is, two linearly independent periodic motions will **coexist.** In order to derive conditions for coexistence, we write any one of the above infinite penta-diagonal determinants in the form:

$$\begin{vmatrix} R & R & R & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ R & R & R & R & 0 & 0 & 0 & 0 & 0 & \cdots \\ R & R & R & R & Y & 0 & 0 & 0 & 0 & \cdots \\ 0 & R & R & R & Y & Y & 0 & 0 & 0 & \cdots \\ 0 & 0 & X & X & S & S & S & 0 & 0 & \cdots \\ 0 & 0 & 0 & X & S & S & S & S & 0 & \cdots \\ 0 & 0 & 0 & 0 & S & S & S & S & S & \cdots \\ 0 & 0 & 0 & 0 & 0 & S & S & S & S & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & S & S & S & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0 \qquad (3.27)$$

If all three of the X terms vanish, or if all three of the Y terms vanish, the determinant will decompose into two determinants, one involving only the R terms, and the other involving only the S terms. Since the A-odd and B-odd determinants are identical except for the upper left hand corner, the corresponding determinant of (3.27) involving only the S terms will be the same for both A-odd and B-odd, and we will have coexistence. The vanishing of the three X terms or of the three Y terms turns out to give the following conditions:

$$P(n-2) = 0, \quad M(n) = 0, \quad P(n) = 0 \tag{3.28}$$

where n can be any integer,

$$n = \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$$

From our definitions (3.21,3.22) of M and P, we are left with the following conditions for coexistence in the generalized Ince's equation (3.17):

$$\begin{aligned} c_1 &= \left(\tfrac{n}{2}\right)^2 a_1 - \tfrac{n}{2}\, b_1 \\ b_2 &= (n-1)\, a_2 \\ c_2 &= \left(\tfrac{n}{2}\right)^2 a_2 - \tfrac{n}{2}\, b_2 \end{aligned} \tag{3.29}$$

Thus coexistence will occur in the generalized Ince equation (3.17) if eqs 3.29 hold for any integer n, positive, negative or zero.

Note that in the special case $a_2 = b_2 = c_2 = 0$, eq 3.17 becomes Ince's equation:

$$(1 + a_1 \ \cos t)\, \ddot{v} + (b_1 \ \sin t)\, \dot{v} + (\delta + c_1 \ \cos t)\, v = 0 \tag{3.30}$$

In this case the matrices (3.23)-(3.26) become tri-diagonal (instead of penta-diagonal) and the condition for coexistence reduces to just a single equation [4],[12]:

$$c_1 = \left(\frac{n}{2}\right)^2 a_1 - \frac{n}{2}\, b_1 \tag{3.31}$$

Note also that in the parallel case $a_1 = b_1 = c_1 = 0$, eq 3.17 again becomes a version of Ince's equation:

$$(1 + a_2 \ \cos 2t)\, \ddot{v} + (b_2 \ \sin 2t)\, \dot{v} + (\delta + c_2 \ \cos 2t)\, v = 0 \tag{3.32}$$

In this case we set $\tau = 2t$ giving

$$(1 + a_2 \ \cos \tau)\, \ddot{v} + (\frac{b_2}{2} \ \sin \tau)\, \dot{v} + (\delta^* + \frac{c_2}{4} \ \cos \tau)\, v = 0 \tag{3.33}$$

which is of the form of eq 3.30 with $a_1 = a_2$, $b_1 = b_2/2$, $c_1 = c_2/4$ and $\delta^* = \delta/4$, whereupon the condition (3.31) for coexistence becomes:

$$c_2 = n^2\, a_2 - n\, b_2 \tag{3.34}$$

It can be shown that even more complicated versions of Ince's equation cannot be shown to support coexistence. For example, the equation

$$\begin{aligned} (1 + a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t) \quad & \ddot{v} \\ +(b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t) \quad & \dot{v} \\ +(\delta + c_1 \cos t + c_2 \cos 2t + c_3 \cos 3t) \quad & v \quad = 0 \end{aligned} \tag{3.35}$$

gives rise to four 7-diagonal determinants (cf. eqs 3.23-3.26) and requires 6 conditions to be met in order for coexistence to occur (cf. eqs 3.28). These conditions turn out to be self-contradictory, so our analysis indicates that eq 3.35 cannot support coexistence (unless some of the coefficients are zero, thereby reducing it to the form of eq 3.17).

Note that the coexistence conditions (3.29) do not involve the parameter δ in eq 3.17. Once the parameters of the system have been chosen to satisfy the coexistence conditions (3.29), the vanishing of the associated determinant (3.27) will relate δ to the other parameters of the system.

3.4 Application to Stability of Motion

Earlier in this chapter we showed that the stability of the x-mode, eq 3.10, in the system (3.11),(3.12),(3.13) was governed by the generalized Ince's equation (3.17) with coefficients given by eq 3.16. From eq 3.16 we substitute $c_1 = 0$ and $b_1 = -a_1$ into the first of the coexistence conditions (3.29) with the result:

$$0 = \left(\frac{n}{2}\right)^2 a_1 - \frac{n}{2}(-a_1) \tag{3.36}$$

which is satisfied by either $n = -2$ or $n = 0$ or $a_1 = 0$.

Next, from eq 3.16 we substitute $b_2 = -2a_2$ into the second of the coexistence conditions (3.29) with the result:

$$-2a_2 = (n-1)a_2 \tag{3.37}$$

which is satisfied by either $n = -1$ or $a_2 = 0$.

Thus we see that if both a_1 and a_2 are non-zero, then coexistence cannot occur in the general system defined by eqs 3.11, 3.12, 3.13, since there is no integer n which can satisfy the conditions (3.29). From the definitions (3.16) of a_1 and a_2, this assumes that both β_{01} and β_{02} are nonzero (assuming $A > 0$). (Recall that the β_{ij} coefficients occur in the kinetic energy T, see eqs 3.11,3.13).

Note that if β_{01}=0 but β_{02} does not vanish, then coexistence is possible. However in this case eq 3.17 reduces to Ince's equation, which is well-known to support coexistence [4],[12].

3.5 Another Application

In this section we extend the foregoing work by considering systems in which the x-mode satisfies the nonlinear ODE:

$$\ddot{x} + x + x^3 = 0 \tag{3.38}$$

which has a solution in terms of the Jacobian elliptic function cn:

$$x = A \,\mathrm{cn}(\alpha t, k) \tag{3.39}$$

where ([11], p.80)

$$\alpha = \sqrt{A^2 + 1}, \qquad k = \frac{A}{\sqrt{2(A^2 + 1)}} \tag{3.40}$$

This requires that we relax the condition that $\alpha_{40} = 0$ (cf. eqs 3.9 and 3.12), and we take:

$$T = \frac{1}{2}\dot{x}^2 + \beta_3(x, y)\dot{y}^2 \tag{3.41}$$

$$V = \frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}\omega_2^2 y^2 + \alpha_{22}x^2y^2 + \alpha_{13}xy^3 + \alpha_{04}y^4 \tag{3.42}$$

$$\beta_3(x, y) = \beta_{00} + \beta_{01}x + \beta_{10}y + \beta_{02}x^2 + \beta_{11}xy + \beta_{20}y^2 \tag{3.43}$$

We set $x = A \,\mathrm{cn}(\alpha t, k) + u, \quad y = 0 + v$ in Lagrange's equations and then linearize in u and v. This gives the v equation as

$$\begin{aligned} &2(\beta_{02}A^2 \,\mathrm{cn}^2(\alpha t, k) + \beta_{01}A \,\mathrm{cn}(\alpha t, k) + \beta_{00})\,\ddot{v} \\ &-\alpha \,\mathrm{dn}(\alpha t, k)\,\mathrm{sn}(\alpha t, k)(2\beta_{01}A + 4\beta_{02}A^2 \,\mathrm{cn}(\alpha t, k))\,\dot{v} \\ &+(2\alpha_{22}A^2 \,\mathrm{cn}^2(\alpha t, k) + \omega_2^2)\,v \;= 0 \end{aligned} \tag{3.44}$$

Although eq 3.44 has coefficients involving Jacobian elliptic functions, we may transform it to a generalized Ince equation by utilizing a transformation given in

[4]. We begin by replacing t with a new time variable $T = \alpha t$, so that $\mathrm{cn}(\alpha t, k) = \mathrm{cn}(T, k)$. Then we replace T by τ, where

$$dT = \frac{d\tau}{\sqrt{1 - k^2 \sin^2 \tau}} \tag{3.45}$$

This turns out to convert the Jacobian elliptic functions to trig functions [2] as follows:

$$\begin{aligned} \mathrm{sn}(T, k) &= \sin \tau \\ \mathrm{cn}(T, k) &= \cos \tau \\ \mathrm{dn}(T, k) &= \sqrt{1 - k^2 \sin^2 \tau} \end{aligned} \tag{3.46}$$

The result of these transformations is to replace eq.(3.44) by the following generalized Ince equation:

$$\begin{aligned} &(1 + a_1 \cos \tau + a_2 \cos 2\tau + a_3 \cos 3\tau + a_4 \cos 4\tau)\ v'' \\ &\quad +(b_1 \sin \tau + b_2 \sin 2\tau + b_3 \sin 3\tau + b_4 \sin 4\tau)\ v' \\ &+(\delta + c_1 \cos \tau + c_2 \cos 2\tau + c_3 \cos 3\tau + c_4 \cos 4\tau)\ v\ = 0 \end{aligned}$$

where the coefficients a_i, b_i and c_i are given as follows:

$$a_1 = \frac{2A\beta_{01}(1 - \frac{1}{4}k^2)}{a_0} \tag{3.47}$$

$$a_2 = \frac{\beta_{00}k^2 + \beta_{02}A^2}{a_0} \tag{3.48}$$

$$a_3 = \frac{\frac{1}{2}\beta_{01}Ak^2}{a_0} \tag{3.49}$$

$$a_4 = \frac{\frac{1}{4}A^2k^2\beta_{02}}{a_0} \tag{3.50}$$

$$b_1 = \frac{-\beta_{01}A(2-k^2)}{a_0} \tag{3.51}$$

$$b_2 = \frac{-2\beta_{02}A^2(1-\frac{1}{4}k^2) - \beta_{00}k^2}{a_0} \tag{3.52}$$

$$b_3 = \frac{-\beta_{01}Ak^2}{a_0} \tag{3.53}$$

$$b_4 = \frac{-\frac{3}{4}A^2k^2\beta_{02}}{a_0} \tag{3.54}$$

$$\delta = \frac{\omega_2^2 + \alpha_{22}A^2}{a_0\alpha^2} \tag{3.55}$$

$$c_1 = 0 \tag{3.56}$$

$$c_2 = \frac{\alpha_{22}A^2}{a_0\alpha^2} \tag{3.57}$$

$$c_3 = 0 \tag{3.58}$$

$$c_4 = 0 \tag{3.59}$$

where

$$a_0 = \beta_{00}(2-k^2) + \beta_{02}A^2(1-\frac{1}{4}k^2) \tag{3.60}$$

As mentioned in connection with eq 3.35 above, eq 3.47 cannot in general support coexistence. However, if $\beta_{01} = 0$, the trigonometric terms in eq 3.47 with arguments of τ and 3τ will vanish, leaving an equation which can easily be transformed into the generalized Ince eq 3.17 by replacing τ by $z = 2\tau$. Once this transformation is completed, conditions for coexistence in the resulting equation will be given by eqs 3.29. Carrying out this plan yields three equations corresponding to eqs 3.29. The equation which corresponds to the second of eqs 3.29 turns out to be:

$$(n+1/2)\alpha^2\beta_{02}A^2k^2 = 0 \tag{3.61}$$

which requires that $n = -1/2$ and thus cannot be satisfied by any integer value of n. However, eq 3.61 as well as the other two eqs. coming from eqs 3.29 can be satisfied by taking $\beta_{02} = 0$.

So we conclude that in order for coexistence to occur in eq 3.44, both β_{01} and β_{02} must be taken equal to zero. This simplifies eq 3.47 to the following:

$$(1 + a_2 \cos 2\tau)\, v'' + (b_2 \sin 2\tau)\, v' + (\delta + c_2 \cos 2\tau)\, v = 0 \tag{3.62}$$

This is of the form of eq 3.32 and as was discussed above, involves a single condition (3.34) for coexistence:

$$c_2 = n^2\, a_2 - n\, b_2 \tag{3.63}$$

Using eqs 3.47-3.59, eq 3.63 becomes:

$$(-\alpha^2 \beta_{00} k^2) n^2 + (-\alpha^2 \beta_{00} k^2) n + \alpha_{22} A^2 = 0 \tag{3.64}$$

which becomes simplified by using eqs 3.40:

$$n^2 + n - \frac{2\alpha_{22}}{\beta_{00}} = 0 \tag{3.65}$$

The condition for coexistence therefore becomes simply:

$$\frac{\alpha_{22}}{\beta_{00}} = \frac{n(n+1)}{2} \tag{3.66}$$

where n is an integer, positive, negative or zero.

3.6 Example: Lame's Equation

As an example, we may take $\beta_{00} = 1/2$ and $\alpha_{22} = 1/2$, which from eq 3.66 corresponds to $n = 1$ and $n = -2$. Eqs.(3.41),(3.42) become:

$$T = \frac{1}{2}\dot{x}^2 + \left(\frac{1}{2} + \beta_{10} x + \beta_{20} x^2 + \beta_{11} x y\right) \dot{y}^2 \tag{3.67}$$

$$V = \frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}\omega_2^2 y^2 + \frac{1}{2}x^2 y^2 + \alpha_{13} x y^3 + \alpha_{04} y^4 \tag{3.68}$$

In order to consider the simplest possible such example, we take $\beta_{10} = \beta_{20} = \beta_{11} = \alpha_{13} = \alpha_{04} = 0$, for which case Lagrange's equations become:

$$\ddot{x} + x + x^3 + xy^2 = 0 \tag{3.69}$$

$$\ddot{y} + \omega_2^2 y + x^2 y = 0 \tag{3.70}$$

This system exhibits the exact solution (the x-mode):

$$x = A\ \mathrm{cn}(\alpha t, k), \qquad y = 0 \tag{3.71}$$

where α and k are given by eq 3.40. The stability of the x-mode depends upon the two parameters ω_2 and A, and is governed by the ODE (3.44), which becomes:

$$\ddot{v} + (\omega_2^2 + A^2\,\mathrm{cn}^2(\alpha t, k)) \quad v \quad = 0 \tag{3.72}$$

The stability chart corresponding to eq 3.72 consists of transition curves which maybe displayed in the ω_2^2-A^2 plane. Since the period of the variable coefficient $\mathrm{cn}^2(\alpha t, k))$ approaches π as A approaches zero, we may expect instability tongues to emanate from the ω_2^2 axis at each of the points $\omega_2^2 = n^2$, where $n = 1, 2, 3, \cdots$. However, because α_{22} and β_{00} have been chosen to satisfy the coexistence condition (3.66) for $n = 1$ and $n = -2$, there are no even tongues and only one odd tongue, which emanates from the point $\omega_2^2 = 1$, $A^2 = 0$ [12]. See figure 3.6, which shows this single instability tongue as well as a coexistence curve emanating from $\omega_2^2 = 4$, $A^2 = 0$. Figure 3.6 was obtained as follows:

Eq 3.72 is a version of Lame's equation [1]. Following the procedure given in eqs 3.45,3.46, it can be transformed to:

$$\begin{aligned}(3A^2 + 4 + A^2\cos 2\tau)\, v'' - A^2 \sin 2\tau\, v' & \\ + (4\omega_2^2 + 2A^2 + 2A^2 \cos 2\tau)\, v &= 0\end{aligned} \tag{3.73}$$

Note that eq 3.73 has the exact solution $v = \cos\tau$ corresponding to the parameter $\omega_2^2 = 1$. Therefore the straight line $\omega_2^2 = 1$ is a transition curve as shown in figure 3.6. Similarly, eq 3.73 has the exact solution $v = \sin\tau$ corresponding to the parameter $\omega_2^2 = 1 + A^2/2$, which also plots as a straight line in figure 3.6.

In order to obtain an expression for the coexistence curves, we may use a regular perturbation method [18]. We expand

$$\omega_2^2 = n^2 + k_1 A^2 + k_2 A^4 + \cdots \tag{3.74}$$

$$v = \left\{ \begin{array}{c} \sin n\tau \\ \\ \cos n\tau \end{array} \right\} + v_1 A^2 + v_2 A^4 + \cdots \tag{3.75}$$

We substitute eqs 3.74,3.75 into eq 3.73, collect terms, and choose the values of the coefficients k_i to eliminate secular terms at each order of A^2, as usual in regular perturbations [18]. Doing this for $n = 2$ we obtain the same result for both sin and cos choices in eq 3.75, signifying coexistence. The resulting curve is displayed in figure 3.6 and has the equation (obtained by using macsyma to do the computer algebra):

$$\begin{aligned} \omega_2^2 = 4 + \frac{5\,A^2}{2} - \frac{5\,A^4}{96} + \frac{5\,A^6}{128} - \frac{26665\,A^8}{884736} \\ + \frac{9385\,A^{10}}{393216} - \frac{19720235\,A^{12}}{1019215872} + \cdots \end{aligned} \tag{3.76}$$

3.7 Conclusions

We have obtained conditions (3.29) for coexistence to occur in the generalized Ince equation (3.17). These conditions are more numerous and thus more difficult to meet than the comparable condition for Ince's equation:

$$(1 + a_1\, \cos t)\, \ddot{v} + b_1\, \sin t\;\, \dot{v} + (\delta + c_1\, \cos t)\, v = 0 \tag{1.2}$$

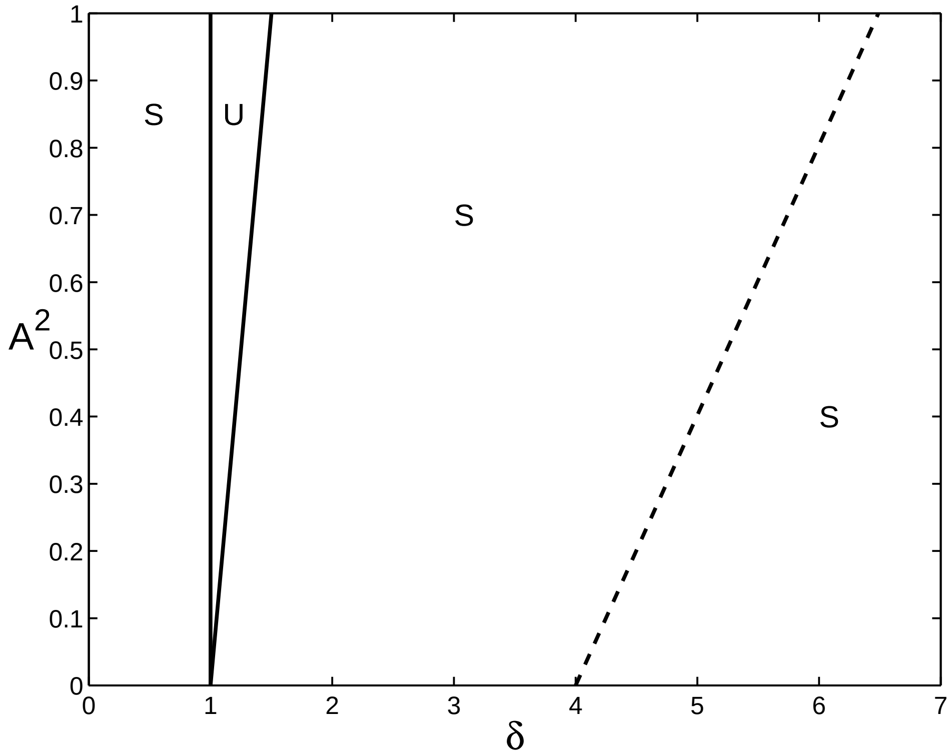

Figure 3.6: *Stability chart for eq 3.72. S=stable, U=unstable. Curves obtained by perturbation analysis. The dashed line is a coexistence curve, which is stable.*

The necessary and sufficient condition for coexistence to occur in (1.2) has been obtained in [4] and can be written in the form:

$$M(n) = \frac{1}{2}\left(-\left(\frac{n}{2}\right)^2 a_1 + \frac{n}{2} b_1 + c_1\right) = 0 \tag{3.77}$$

where n can be any integer,

$$n = \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$$

That is, coexistence will occur in (1.2) iff condition (3.77) is satisfied for any integer value of n.

In applications to the stability of the x-mode in the class of two degree of freedom systems (3.8),(3.9) considered in this paper, we have shown that in general coexistence will not occur if the system is sufficiently complicated, i.e. if both of the coefficients β_{01} and β_{02} occurring in eq 3.13 are non-zero. The reason for this is that the equation governing stability is the generalized Ince's equation (3.17), and the conditions for coexistence to occur in this equation are more difficult to meet than for Ince's equation (1.2).

We have also shown that the same general procedure can be used on problems in which the x-mode satisfies a nonlinear ODE, eq 3.38.

Chapter 4

Coexistent Systems with Damping

The systems explored in chapter 3 are energy conserving. It is logical to continue the analysis by exploring the effect of damping on a coexistent system. It is a practical question since real systems have some measure of damping and coexistent behavior is structurally unstable. Does the presence of minute amounts of damping change the system in such a way as to make coexistence impossible?

The addition of damping to a system like the particle in the plane fundamentally changes the structure of the stability equation (1.2), putting it outside the scope of our previous analysis (chapter 3). In chapter 2, the even-odd (4.1) structure of the equation allowed us to separate the A_n and B_n matrices (section 2.4).

$$(\,\text{even}\,)\;\ddot{x} + (\,\text{odd}\,)\;\dot{x} + (\,\text{even}\,)\,x = 0 \tag{4.1}$$

In this chapter, the even-odd symmetry is broken by the presence of the even dissipative term μ in the $\dot{x}$ coefficient (4.2).

$$(1 + a\cos t)\;\ddot{x} + (\mu + b\sin t)\;\dot{x} + (\delta + c\cos t)\,x = 0 \tag{4.2}$$

4.1 Insight from Mathieu's equation

Mathieu's equation with damping has been studied previously [[12]] and provides some insight to our question. We know that for the damped Mathieu equation

$$\ddot{x} + \mu\dot{x} + (\delta + \epsilon\cos t)\,x = 0 \tag{4.3}$$

the damping shrinks the unstable tongue and lifts the bottom of the tongue off the $\epsilon = 0$ axis (figure 4.1).

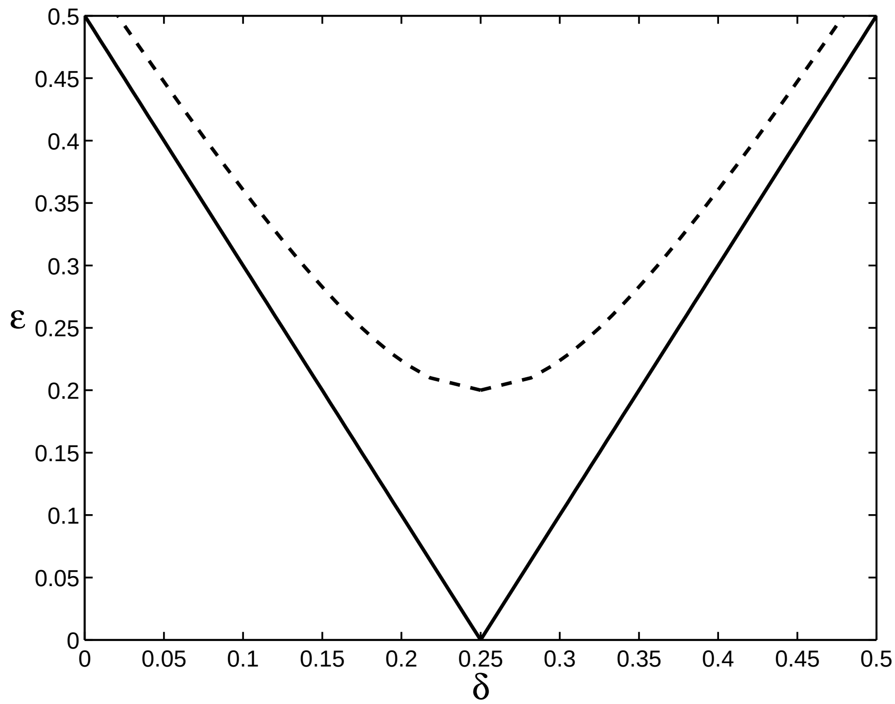

Figure 4.1: *A comparison of the* $\delta = \frac{1}{4}$ *tongue for the undamped (solid) and damped (dashed) Mathieu equation.*

We see similar behavior for damping in Ince's equation. However, for the coexistent Ince equation we don't know if damping simply lifts the bottom of the coexistent tongue off the axis to some finite height or if it pushes it out to infinity (effectively destroying the instability). The zero thickness of the coexistent tongue makes this question impossible to answer based on numerical integration.

Approaching the problem by adding damping to a coexistent system is unfruitful because of the singular nature of coexistence. Instead we introduce an opening parameter k, to open the coexistent tongue (figure 4.2). Then damping is added to the system and we look at what happens to the tongue as k approaches zero (4.4).

$$(1 + a\epsilon \cos t)\ \ddot{x} + (\mu + b\epsilon \sin t)\ \dot{x} + (\delta + (c + k)\, \epsilon \cos t)\, x = 0 \tag{4.4}$$

At this point it is useful to mention the competitive behavior of μ and k.

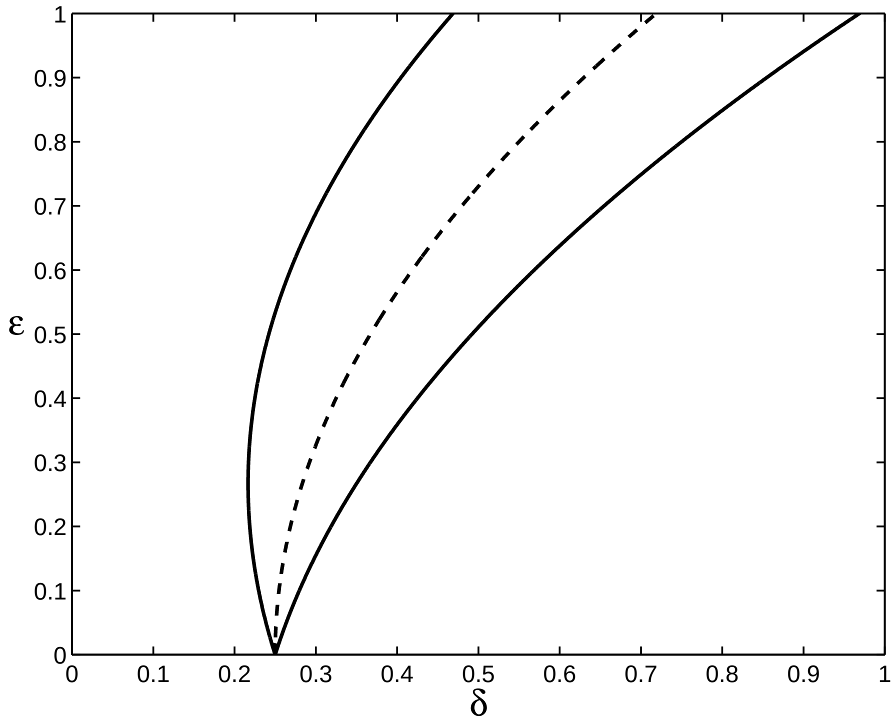

Figure 4.2: *The opening of the coexistent tongue $\delta = \frac{1}{4}$ for $k = 0.5$ (solid) ($k = 0$ dashed)*

The damping parameter, μ, tends to shrink the unstable region while the opening parameter, k, adds to the instability. There is the possibility that μ and k could balance and coexistence could persist for non-zero k. For our analysis, we start by seeking a balance between k and μ that allows coexistence. We will then look at limit as the opening parameter approaches zero, $k \to 0$, to see what happens to a coexistent tongue with damping.

4.2 Methodology

The analysis poses some difficulty since damping breaks the even-odd symmetry of Ince's equation. The periodic solution on a transition curve is no longer odd or even. Instead the solution has both sine and cosine terms. This means that it is inconvenient to use harmonic balance as we did in Chapter 3 since it is difficult to decouple the system of equations into separate sine and cosine matrices (2.66-2.69).

Instead of harmonic balance, we will use two-variable expansion to explore the behavior of the transition curve as μ and k vary. We will then validate and expand this model by numerically integrating the system.

4.3 Two-Variable Expansion

In section 2.2 we discussed the method of two-variable expansion. Here we will apply it to the equation

$$\ddot{x} + (M\epsilon + 2\epsilon \sin t)\, \dot{x} + (\delta + (1+k)\,\epsilon \cos t)\, x = 0 \tag{4.5}$$

where $\mu = M\epsilon$. The coefficients of eq 4.5 are chosen such that when $k = M = 0$ the system will be coexistent for all tongues emanating from $\delta = n^2/4$ where n is odd. The equation is chosen with the numerical analysis in mind since it doesn't have the singularities inherent in the Ince equation (see eq 4.2 for $a > 1$).

As before (Chapter 2) we begin by setting $\xi = t$ and $\eta = \epsilon t$. x and δ are expanded in power series:

$$x(t) = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ... + \epsilon^n x_n \tag{4.6}$$

$$\delta = \delta_0 + \epsilon\delta_1 + \epsilon^2\delta_2 + ... + \epsilon^n\delta_n \tag{4.7}$$

We are interested in the tongue at $\delta = 1/4$. Therefore we set $\delta_0 = 1/4$. Collecting in orders of ϵ we find

$$\epsilon^0: \quad \frac{\partial^2 x_0}{\partial \xi^2} + \frac{1}{4}x_0 \quad = 0 \tag{4.8}$$

$$\epsilon^1: \quad \frac{\partial^2 x_1}{\partial \xi^2} + \frac{1}{4}x_1 \quad = -2x_{0\eta\xi} - x_0(1+k)\cos\xi - 2x_{0\xi}\sin\xi - \delta_1 x_0 - Mx_{0\xi} \tag{4.9}$$

$$\begin{aligned} \epsilon^2: \quad \frac{\partial^2 x_2}{\partial \xi^2} + \frac{1}{4}x_2 \quad &= -2x_{1\eta\xi} - x_1(1+k)\cos\xi - 2x_{1\xi}\sin\xi - \delta_1 x_1 - Mx_{1\xi} \\ &- x_{0\eta\eta} - 2x_{0\eta}\sin\xi - Mx_{0\eta} - \delta_2 x_0 \end{aligned} \tag{4.10}$$

which is solved recursively. The solution to eq 4.8

$$x_0 = F_1(\eta)\sin\frac{t}{2} + F_2(\eta)\cos\frac{t}{2} \tag{4.11}$$

is substituted into eq 4.9. Resonant terms are removed to find the first order slow flow equations (4.12). Here, the resonant terms are those that have a fundamental frequency of 1/2. The first order slow flow equations are

$$\begin{aligned} \frac{d}{d\eta}F_1 = \quad & -\tfrac{1}{2}MF_1 - \left(\delta_1 + \tfrac{1}{2}k\right)F_2 \\ \frac{d}{d\eta}F_2 = \quad & -\tfrac{1}{2}MF_2 + \left(\delta_1 - \tfrac{1}{2}k\right)F_1 \end{aligned} \tag{4.12}$$

Or equivalently

$$\begin{bmatrix} F_1' \\ F_2' \end{bmatrix} = \begin{bmatrix} -\frac{M}{2} & -\delta_1 - \frac{1}{2}k \\ \delta_1 - \frac{1}{2}k & -\frac{M}{2} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{4.13}$$

The eigenvalues of eq 4.13 are

$$\lambda_{1,2} = -1/2\left(M \pm \sqrt{k^2 - 4\delta_1^2}\right)\eta \tag{4.14}$$

Which correspond to the exponential solution

$$\begin{aligned} F1(\eta) = \quad & \Big(-C_1\exp\left(-\frac{1}{2}(M - \sqrt{-4\delta_1^2 + k^2})\eta\right) \\ & +C_2\exp\left(-\frac{1}{2}(M + \sqrt{-4\delta_1^2 + k^2})\eta\right)\Big) \end{aligned} \tag{4.15}$$

$$F2(\eta) = \quad \frac{\sqrt{-4\delta_1^2 + k^2}}{2\delta_1 + k}F1(\eta) \tag{4.16}$$

Exponential functions with positive real arguments are unstable while those with negative real arguments decay. The transition curves themselves are neutrally stable (ie no growth or decay). To stay on the transition curves we set the exponential arguments equal to zero. To do this we let $\delta_1 = \pm\frac{1}{2}\sqrt{k^2 - M^2}$ and $C_2 = 0$. The two transition curves at $\delta = 1/4$ are, to first order,

$$\delta_l = \frac{1}{4} - \frac{\epsilon}{2}\sqrt{k^2 - M^2} + O(\epsilon^2) \quad \text{and} \quad \delta_r = \frac{1}{4} + \frac{\epsilon}{2}\sqrt{k^2 - M^2} + O(\epsilon^2) \tag{4.17}$$

At this point we wish to remove the ϵ dependence of M. To this end we substitute $M = \mu/\epsilon$ with the result that eq 4.17 becomes

$$\delta_l = \frac{1}{4} - \frac{1}{2}\sqrt{k^2\epsilon^2 - \mu^2} \quad \text{and} \quad \delta_r = \frac{1}{4} + \frac{1}{2}\sqrt{k^2\epsilon^2 - \mu^2} \tag{4.18}$$

Here we see that for real values of δ the minimum value for ϵ is

$$\epsilon_{min} = \frac{\mu}{k} \tag{4.19}$$

This means that for any amount of damping the tongue leaves the $\epsilon = 0$ axis. We also see that as the opening parameter, k, approaches zero, the $\epsilon_{min} \to \infty$. We must mention here that this is a perturbation analysis for small ϵ and therefore is not authoritative for large ϵ. To explore the system when ϵ is large we turn to numeral integration.

4.4 Numerical analysis

Since two-variable expansion is valid for small ϵ we will look at the system by numerical integration. We can analyze the tongues using floquet theory, the matlab routine from Chapter 2, and several root finding techniques [10]. Figures 4.3 and 4.4 show the result of fixed damping on the system (4.5) as the opening parameter closes.

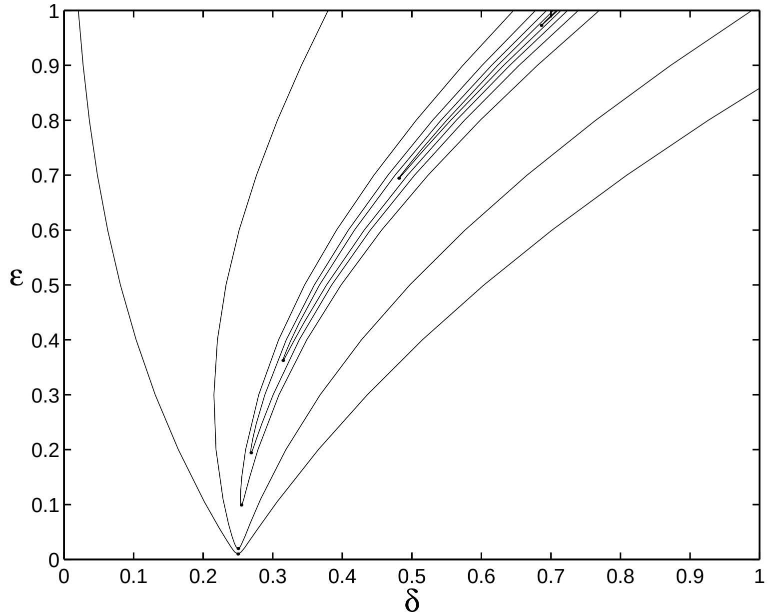

Figure 4.3: *Tongues of instability disappear as k decreases. Damping is fixed at $\mu = 0.01$. k values plotted are 1, 0.5, 0.1, 0.05, 0.025, 0.01, 0.005. Results obtained by numerical integration of eq 4.5.*

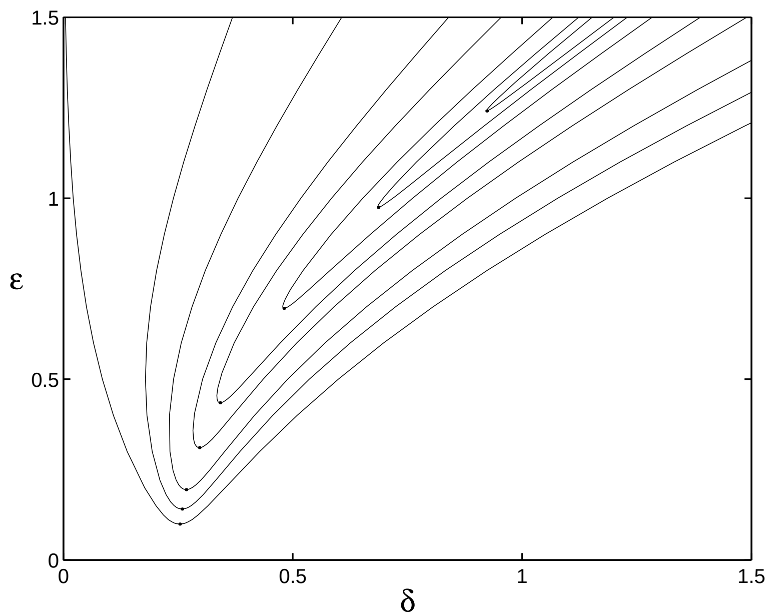

Figure 4.4: *Tongues of instability disappear as k decreases. Damping is fixed at $\mu = 0.1$. k values plotted are 1, 0.7, 0.5, 0.3, 0.2, 0.1, 0.05, 0.025. Results obtained by numerical integration of eq 4.5.*

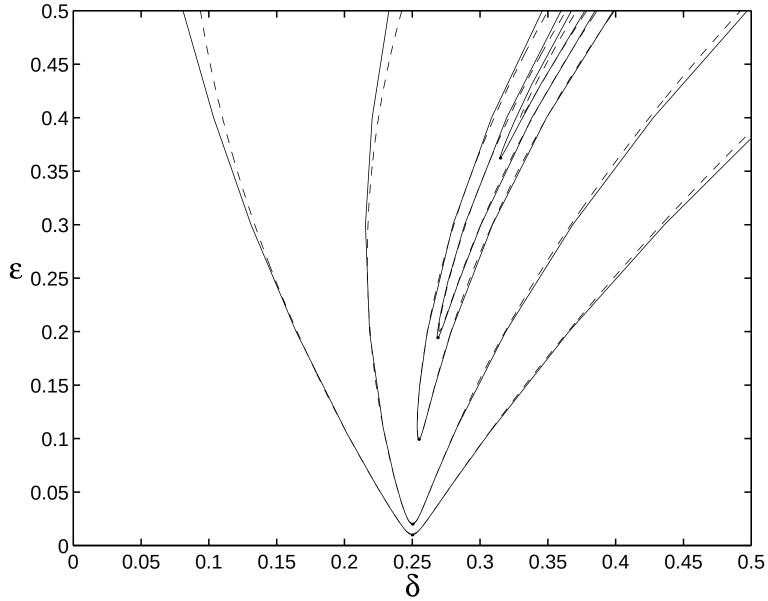

Figure 4.5: *A comparison of the numerical integration results of figure 4.3 (solid) with the perturbation predictions (dashed) for eq 4.5. Damping is fixed at* $\mu = 0.01$*. k values plotted are 1, 0.5, 0.1, 0.05, 0.025.*

The bottom of the tongues are marked with a point on each of these figures. We will refer to this point as ϵ_{min} since it the lowest value for ϵ the tongue can reach. We are interested in what happens to ϵ_{min} as $k \to 0$. Our perturbation results (4.19) indicate that $\epsilon_{min} \to \infty$. In figures 4.5, 4.6 we compare the numerical results to our perturbation predictions.

We see that the perturbation method accurately predicts the shape of the unstable tongues for small ϵ. As ϵ increases the prediction becomes less reliable. The numerical integration shows that ϵ_{min} increases as k decreases, but at a slower rate than we predicted.

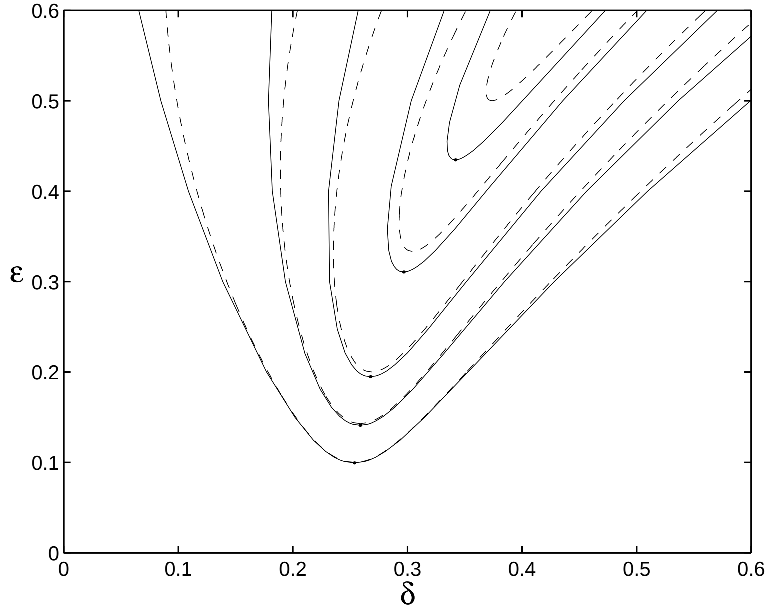

Figure 4.6: *A comparison of the numerical integration results of figure 4.4 (solid) with the perturbation predictions (dashed) for eq 4.5. Damping is fixed at* $\mu = 0.1$. *k values plotted are 1, 0.7, 0.5, 0.3, 0.2.*

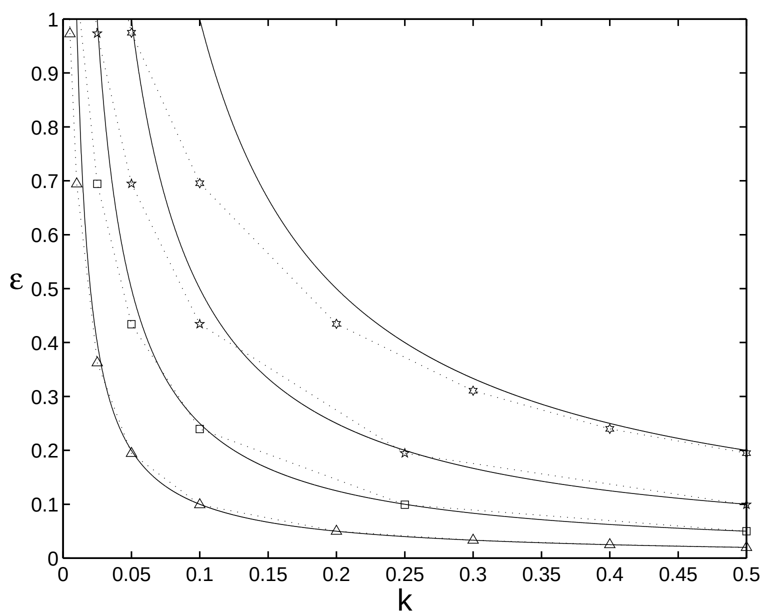

Figure 4.7: *Finding the minimum* ϵ *for the tongue of instability for* $\mu = 0.01$, $0.025, 0.05, 0.1$ *(bottom to top - dotted lines). First order perturbation predictions (4.19) included (solid lines).*

4.5 Conclusions

The numerical and analytical methods agree that damping will not allow coexistent tongues to emanate from the $\epsilon = 0$ axis. The numerical analysis also indicated that the tongues disappear, i.e. ϵ_{min} goes to infinity, before they have a chance to close up. The stability which damping brings to the system cannot be countered by an opening parameter in such a way as to create coexistence. Figure 4.7 also implies that damping destroys coexistence since as $k \to 0$, ϵ_{min} appears unbounded. Although we have not conclusively proven that damping precludes coexistence, the numerical analysis indicates an extended validity of the perturbation result, namely that $\epsilon_{min} = \frac{\mu}{k}$ does not allow for coexistence.

Chapter 5

Time Transformations - Trigonometrification

The final example of chapter 3 (section 3.6) demonstrates that it is possible to deal with certain classes of systems where the NNM is not trigonometric. Lame's equation (3.72) describes the stability of a two-degree of freedom system with a NNM in the form of a Jacobi-elliptic function, $x(t) = \mathrm{cn}(\alpha t, k)$. Magnus and Winkler tackled this problem in [4] (see section 3.6). They solved for the stability by rescaling time to turn $A\mathrm{cn}(\alpha t, k) \rightarrow A\cos(\tau)$.

This chapter is a natural expansion of their work. We develop a process to transform time so that any periodic NNM (not just $\cos t$ or $\mathrm{cn}(\alpha t, k)$) can be transformed into $x(\tau) = A_0 + A_1 \cos 2\tau$. We call this process trigonometrification [16]. The chapter begins with a second look at Lame's equation.

5.1 Introduction

It is well-known that the nonlinear oscillator given by the ODE

$$\frac{d^2x}{dt^2} + x + x^3 = 0 \tag{5.1}$$

has a solution which can be written in terms of the Jacobian elliptic function cn [12],[11]:

$$x(t) = A\ \mathrm{cn}(\alpha t, k) \tag{5.2}$$

where the constants α and k are related to the amplitude A as follows:

$$\alpha = \sqrt{1 + A^2}, \qquad k = \frac{A}{\sqrt{2(1 + A^2)}} \tag{5.3}$$

It is also well-known that a transformation of time from t to τ permits the solution (5.2) to be written in a simplified form, namely [14]

$$x(\tau) = A \cos \tau \tag{5.4}$$

where t and τ are related by the equation [4]:

$$dt = \frac{d\tau}{\alpha\sqrt{1 - k^2 \sin^2 \tau}} \tag{5.5}$$

For applications which involve manipulations of the solution to eq 5.1, it is naturally more convenient to use the form (5.4) than the form (5.2). As an example, consider the question of the stability of a nonlinear normal mode (NNM) in a two degree of freedom system which is defined by the following expressions for kinetic T and potential V energies [14]:

$$T = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 \tag{5.6}$$

$$V = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 \tag{5.7}$$

Lagrange's equations for this system are:

$$\ddot{x} + x + x^3 + xy^2 = 0 \tag{5.8}$$

$$\ddot{y} + y + x^2y = 0 \tag{5.9}$$

where dots represent differentiation with respect to t. This system exhibits the exact solution (the x-mode):

$$x = A\ \mathrm{cn}(\alpha t, k), \qquad y = 0 \tag{5.10}$$

where α and k are given by eqs 5.3. To investigate the stability of this mode, we set

$$x = A\,\mathrm{cn}(\alpha t, k) + u(t)$$

$$y = v(t)$$

Substituting (5.11) into (5.8),(5.9) and linearizing in $u(t)$ and $v(t)$ results in

$$\begin{aligned} \ddot{u} + u + 3A^2 \,\mathrm{cn}^2(\alpha t, k)\; u &= 0 \\ \ddot{v} + v + A^2 \mathrm{cn}^2(\alpha t, k)\; v &= 0 \end{aligned} \tag{5.11}$$

The first of eqs 5.11 determines the stability of the motion (5.10) in the invariant manifold $y = 0$, that is, in the x-$\dot{x}$ phase plane. This is well-known to be Liapunov unstable due to phase shear, that is, due to the change in period associated with a change in amplitude, but is orbitally stable [19]. This effect is well understood and is of no interest to us here.

We are rather interested in the boundedness of solutions to the second of eqs 5.11, the v-equation, which determines the stability of the invariant manifold $y = 0$. The NNM (5.10) will be said to be stable if all solutions of the v-equation are bounded, and unstable if an unbounded solution exists.

The presence of the elliptic function coefficient in the v-equation makes the analysis of this equation difficult. However, the v-equation can be simplified by using the transformation (5.5), replacing t by τ as independent variable. This results in the new v-equation [14]:

$$(3A^2 + 4 + A^2 \cos 2\tau)\, v'' - A^2 \sin 2\tau\, v' + (4 + 2A^2 + 2A^2 \cos 2\tau)\, v = 0 \tag{5.12}$$

where primes denote differentiation with respect to τ. **Note that eq 5.12 is exact, i.e., no assumption of small amplitude A has been made.** The boundedness of solutions in eq 5.12 can be investigated by using the method of harmonic balance [18],[14], i.e. by expanding v in a Fourier series.

To summarize, the stability analysis of the NNM (5.11) has been simplified by using the transformation (5.5) of time from t to τ, which replaced the elliptic cn function in the v-equation (5.11), by trig functions in eq 5.12.

In this paper we generalize this idea, replacing eq 5.1 by a conservative nonlinear oscillator equation of the form:

$$\frac{d^2x}{dt^2} + f(x) = 0 \tag{5.13}$$

where $f(x)$ is an analytic function of x. Of course an equation of the form (5.13) will not in general have an elliptic integral solution. Nevertheless we show how to produce a time transformation from t to new time τ which allows the periodic solution of (5.13) to be expressed in terms of a cosine function. We will refer to this process of trigonometric simplification by the neologism *trigonometrification.*

5.2 Trigonometrification

In this section we derive the transformation (5.5) which trigonometrifies eq 5.1 *without* using the fact that the solution to (5.1) involves the elliptic function cn. The procedure we use here will be shown later in this paper to be applicable to a general class of nonlinear oscillator equations.

Using the form of eq 5.5 as a model, we assume a time transformation of the form

$$dt = \frac{d\tau}{\sqrt{g(\tau)}} \tag{5.14}$$

where $g(\tau)$ is to be found. Using eq 5.14 to transform eq 5.1 results in

$$x''g + \frac{1}{2}x'g' + x + x^3 = 0 \tag{5.15}$$

where primes denote derivatives with respect to τ. We can turn this into an equation on g:

$$g' + \frac{2x''}{x'}g + \frac{2(x + x^3)}{x'} = 0 \tag{5.16}$$

We want the time transformation to give us $x(\tau) = A\cos\tau$, so we assume this solution for x. We substitute $x(\tau) = A\cos\tau$ into eq 5.16 and obtain a first order linear ODE on $g(\tau)$:

$$g' + \frac{2}{\tan\tau}g + \frac{-2}{A\sin\tau}(A\cos\tau + A^3\cos^3\tau) = 0 \tag{5.17}$$

The homogeneous part of eq 5.17

$$g' + \frac{2}{\tan\tau}g = 0 \tag{5.18}$$

has the solution

$$g(\tau) = \frac{K}{\sin^2\tau} \tag{5.19}$$

where K is an arbitrary constant. Using variation of parameters, we seek a solution to eq 5.17 in the form

$$g(\tau) = \frac{K(\tau)}{\sin^2\tau} \tag{5.20}$$

Plugging (5.20) into eq 5.17 and solving for $K'(\tau)$ yields

$$K'(\tau) = 2\sin\tau(\cos\tau + A^2\cos^3\tau) \tag{5.21}$$

Integrating, we obtain

$$K(\tau) = \int 2\sin\tau(\cos\tau + A^2\cos^3\tau)d\tau \tag{5.22}$$

We solve the integral using the substitution of $u = \cos\tau$ and find

$$K(\tau) = -(\cos^2\tau + \frac{1}{2}A^2\cos^4\tau) + C \tag{5.23}$$

where C is an arbitrary constant. This gives $g(\tau)$ in the form

$$g(\tau) = \frac{-1}{\sin^2\tau}\left(\cos^2\tau + \frac{1}{2}A^2\cos^4\tau - C\right) \tag{5.24}$$

We note that $g(\tau)$ has singularities at $\tau = 0$ and π. These singularities are undesirable, so we choose C appropriately to remove them. To do this, we let $\cos^2\tau = 1 - \sin^2\tau$ and simplify.

$$g(\tau) = \frac{-(1 + \frac{1}{2}A^2 - C)}{\sin^2\tau} + (1 + A^2) - \frac{1}{2}A^2 \sin^2\tau \tag{5.25}$$

Setting $C = 1 + \frac{1}{2}A^2$ removes the singularities at $\tau = 0$ and π and we are left with

$$g(\tau) = (1 + A^2) - \frac{1}{2}A^2 \sin^2\tau \tag{5.26}$$

Substituting this back into our original ansatz (5.14), we find

$$dt = \frac{d\tau}{\sqrt{(1 + A^2) - \frac{1}{2}A^2 \sin^2\tau}} \tag{5.27}$$

Using the expressions for α and k given in eq 5.3, we obtain

$$dt = \frac{d\tau}{\alpha\sqrt{(1 - k^2 \sin^2\tau)}} \tag{5.28}$$

which is the same as eq 5.5.

5.3 Generalization

In this section we generalize the trigonometrification process to apply to equations of the form:

$$\ddot{x} + f(x) = 0 \tag{5.29}$$

where we assume f is odd, $f(-x) = -f(x)$. We seek to stretch the time in eq 5.29 so that the transformed equation has the solution $x(\tau) = A\cos(\tau)$. As in the previous section, we assume a time transformation of the form

$$dt = \frac{d\tau}{\sqrt{g(\tau)}} \tag{5.30}$$

where $g(\tau)$ is to be found. Eq 5.30 turns eq 5.29 into

$$x''g + \frac{1}{2}x'g' + f(x) = 0 \tag{5.31}$$

We want $x(\tau)$ to have a solution in the form $x(\tau) = A\cos\tau$. Thus, plugging $x(\tau) = A\cos\tau$ into eq 5.31 yields

$$g' + \frac{2}{\tan\tau}g + \frac{-2}{A\sin\tau}f(A\cos\tau) = 0 \tag{5.32}$$

As in the previous section, we look for a solution to eq 5.32 in the form of eq 5.20

$$g(\tau) = \frac{K(\tau)}{\sin^2\tau} \tag{5.33}$$

Plugging this into eq 5.32 and solving for $K'(\tau)$ we find

$$K'(\tau) = \frac{2}{A}\sin\tau\, f(A\cos\tau) \tag{5.34}$$

Integrating, we obtain

$$K(\tau) = \int \frac{2}{A}\sin\tau\, f(A\cos\tau)d\tau \tag{5.35}$$

We evaluate this integral by using the trig substitution $u = \cos\tau$ and find

$$K(\tau) = -\frac{2}{A^2}F(A\cos\tau) + C \tag{5.36}$$

where F is defined by $F'(x) = f(x)$. Our equation for g, eq 5.33, then becomes

$$g(\tau) = \frac{1}{\sin^2\tau}\left(-\frac{2}{A^2}F(A\cos\tau) + C\right) \tag{5.37}$$

We wish to choose C such that $g(\tau)$ has no singularities at $\tau = 0$ or π. We note that

$$F(A\cos\tau)|_{\tau=0} = F(A) \qquad \text{and} \qquad \mathrm{F(A\cos\tau)}|_{\tau=\pi} = \mathrm{F(-A)} \tag{5.38}$$

Our assumption that $f(x)$ is odd means $F(A)$ is even, thus $F(A) = F(-A)$. We thus choose $C = 2F(A)/A^2$ to remove the singularities. The expression for the time transformation becomes:

$$g(\tau) = \frac{-2}{A^2\sin^2\tau}\left(F(A\cos\tau) - F(A)\right) \tag{5.39}$$

5.4 Example 1

As an example of the application of the previous formula (5.39), we consider the following system, which has no known closed form solution:

$$\ddot{x} + x + x^5 = 0 \tag{5.40}$$

We begin by computing $F(x)$ as the antiderivative of $f(x)=x + x^5$:

$$F(x) = \frac{x^2}{2} + \frac{x^6}{6} \tag{5.41}$$

Substituting eq 5.41 into eq 5.39 gives the following expression for $g(\tau)$:

$$g(\tau) = \frac{-2}{A^2 \sin^2 \tau} \left(\frac{1}{2} A^2 \cos^2 \tau + \frac{1}{6} A^6 \cos^6 \tau - (\frac{1}{2} A^2 + \frac{1}{6} A^6) \right) \tag{5.42}$$

which reduces to

$$g(\tau) = 1 + A^4 \left(1 - \sin^2 \tau + \frac{1}{3} \sin^4 \tau \right) \tag{5.43}$$

resulting in the time transformation

$$dt = \frac{d\tau}{\sqrt{1 + A^4 \left(1 - \sin^2 \tau + \frac{1}{3} \sin^4 \tau \right)}} \tag{5.44}$$

As a check, the transformation (5.44) applied to eq 5.40 gives:

$$g(\tau) x'' + \frac{1}{2} g'(\tau) x' + x + x^5 = 0 \tag{5.45}$$

which becomes, using eq 5.43,

$$\begin{aligned} &\left(1 + A^4 \left(1 - \sin^2 \tau + \tfrac{1}{3} \sin^4 \tau\right)\right) x'' \\ &+ \tfrac{1}{2} A^4 \cos \tau \left(-2 \sin \tau + \tfrac{4}{3} \sin^3 \tau\right) x' + x + x^5 \ = 0 \end{aligned} \tag{5.46}$$

which turns out to have the exact solution $x(\tau) = A \cos \tau$ as desired.

5.5 Example 2

In this section we consider an example for which $f(x)$ in eq 5.13 is not a polynomial. We select the familiar example of the pendulum:

$$\ddot{x} + \sin x = 0 \tag{5.47}$$

In this case $f(x)$=$\sin x$ giving that $F(x)$=-$\cos x$. The associated expression for $g(\tau)$ becomes, from (5.39):

$$g(\tau) = \frac{-2}{A^2 \sin^2 \tau} \left(\cos A - \cos\left(A \cos \tau\right)\right) \tag{5.48}$$

which has the limit of $\sin(A)/A$ as τ goes to 0 or π.

The resulting time transformation is

$$dt = \frac{d\tau}{\sqrt{\frac{-2}{A^2 \sin^2 \tau} \left(\cos A - \cos\left(A \cos \tau\right)\right)}} \tag{5.49}$$

Thus the trigonometrified version of the pendulum eq 5.47 has the exact solution $x(\tau)$=$A \cos \tau$:

$$g(\tau)x'' + \frac{1}{2}g'(\tau)x' + \sin x = 0 \tag{5.50}$$

5.6 Does Trigonometrification Only Work for Odd $f(x)$?

We now return to Magnus and Winker's original transformation (5.5). We recall that this transformation is valid for any Jacobi Elliptic function. In our example we used it in the case when $f(x)$ was odd. Our generalization of the method required $f(x)$ to be odd as well since it was not possible to choose a value for the arbitrary constant, C, to remove the singularities otherwise. In this section we seek to find an example of a system that can be trigonometrified when $f(x)$ is not odd.

We know that the solution to

$$\ddot{x} + x + x^2 = 0 \tag{5.51}$$

has a solution of the form $x(t) = B_0 - B_1 \operatorname{sn}^2(\alpha t, k)$ where B_0, B_1, α, and k are related by

$$B_0 = \frac{1}{2}B_1 + \frac{1}{6}\sqrt{-3B_1^2 + 9} - \frac{1}{2} \tag{5.52}$$

$$\alpha^2 = \frac{1}{12}(B_1 + \sqrt{-3B_1^2 + 9}) \tag{5.53}$$

$$k^2 = \frac{2B_1}{B_1 + \sqrt{-3B_1^2 + 9}} \tag{5.54}$$

We also know that the time transformation (5.5)

$$dt = \frac{d\tau}{\alpha\sqrt{(1 - k^2 \sin^2(\tau))}} \tag{5.5}$$

is valid for transforming all Jacobi-elliptic functions into trigonometric functions. We apply the transformation to the solution of eq 5.51, $x(t) = B_0 - B_1 \operatorname{sn}^2(\alpha t, k)$. The transformation turns the $\operatorname{sn}(\alpha t, k)^2$ into $\sin(\tau)^2$. After the transformation we find

$$x(\tau) = A_0 + A_1 \cos(2\tau) \tag{5.55}$$

where $A_0 = B_0 - \frac{1}{2}B_1$ and $A_1 = \frac{1}{2}B_1$.

We see from this example that it is possible to apply the time transformation method to functions where $f(x)$ is non-odd. The existence of a second unrestricted variable A_0 gives the flexibility needed to avoid singularities in $g(\tau)$.

We now go through a procedure similar to section 5.2 to derive the time transformation for a non-odd system. We use the following coupled ODEs as an example:

$$\begin{aligned} &\ddot{x} + x + x^2 + 2a_{22}xy^2 + a_{13}y^3 = 0 \\ &\ddot{y} + \omega_2^2 y + 2a_{22}yx^2 + 3a_{13}xy^2 + 4a_4y^3 = 0 \end{aligned} \tag{5.56}$$

Looking at the x equation when $y \equiv 0$ we find

$$\ddot{x} + x + x^2 = 0 \tag{5.57}$$

Using our previous work (5.14) as a model, we will assume a time transformation of the form

$$dt = \frac{d\tau}{\sqrt{g(\tau)}} \tag{5.14}$$

and solve for $g(\tau)$. Using eq 5.14 to transform Eq 5.57 results in

$$x''g + \frac{1}{2}x'g' + x + x^2 = 0 \tag{5.58}$$

For this example we want the time transformation to give us $x(\tau) = A_0 + A_1 \cos 2\tau$. We substitute $x(\tau) = A_0 + A_1 \cos 2\tau$ into eq 5.58 and solve for $g(\tau)$.

$$g' + \frac{4}{\tan 2\tau} g + \frac{-2}{A_1 \sin 2\tau} \Big((A_0 + A_1 \cos 2\tau) + (A_0 + A_1 \cos 2\tau)^2 \Big) = 0 \tag{5.59}$$

We will look for a solution to eq 5.59 of the form

$$g(\tau) = \frac{K(\tau)}{\sin^2 2\tau} \tag{5.60}$$

Plugging this into Eq. 5.59 and solving for $K'(\tau)$ yields

$$K'(\tau) = \frac{\sin 2\tau}{A_1} \Big((A_0 + A_1 \cos 2\tau) + (A_0 + A_1 \cos 2\tau)^2 \Big) \tag{5.61}$$

thus,

$$K(\tau) = \int \frac{\sin 2\tau}{A_1} \Big((A_0 + A_1 \cos 2\tau) + (A_0 + A_1 \cos 2\tau)^2 \Big) d\tau \tag{5.62}$$

We solve the integral using the substitution $u = \cos 2\tau$ and find

$$K(\tau) = \frac{1}{2A_1^2} \Big(C - \frac{1}{2}(A_0 + A_1 \cos 2\tau)^2 - \frac{1}{3}(A_0 + A_1 \cos 2\tau)^3 \Big) \tag{5.63}$$

This gives us a $g(\tau)$ of the form

$$g(\tau) = \frac{-1}{2A_1^2 \sin^2 2\tau}\left(C - \frac{1}{2}(A_0 + A_1 \cos 2\tau)^2 - \frac{1}{3}(A_0 + A_1 \cos 2\tau)^3\right) \tag{5.64}$$

We wish to chose C such that there are no singularities at $\tau = 0$ or $\pi/2$. To do this, we let $\cos^2\tau = 1 - \sin^2\tau$ and simplify.

$$g(\tau) = \frac{-1}{2A_1^2 \sin^2 2\tau}\left(q_0 + p_0 \cos 2\tau + (q_1 + p_1 \cos 2\tau)\sin^2 2\tau\right) \tag{5.65}$$

where

$$q_0 = -3A_0^2 - 6A_0A_1^2 - 3A_1^2 - 2A_0^3 + 6C$$

$$p_0 = -6A_0^2A_1 - 6A_0A_1 - 2A_1^3$$

$$q_1 = 3A_1^2 + 6A_0A_1^2$$

$$p_1 = 2A_1^3$$

To achieve a nonsingular $g(\tau)$, q_0 and p_0 must be zero when the denominator ($\sin^2(2\tau) = 0$). $q_0 = 0$ is easily achieved by setting $C = \frac{1}{2}A_0^2 - A_0A_1^2 + \frac{1}{2}A_1^2 - \frac{1}{3}A_0^3$. Setting $p_0 = 0$ results in

$$A_0 = -\frac{1}{2} \pm \frac{1}{6}\sqrt{9 - 12A_1^2} \qquad \text{or} \qquad \mathrm{A}_1 = 0, \pm\sqrt{-3\mathrm{A}_0^2 - 3\mathrm{A}_0} \tag{5.66}$$

Where $0 \leq A_0 \leq 1$ and $-\sqrt{3/4} \leq A_1 \leq \sqrt{3/4}$. We will work with

$$A_0 = -\frac{1}{2} + \frac{1}{6}\sqrt{9 - 12A_1^2} \tag{5.67}$$

We find $g(\tau)$ to be

$$g(\tau) = \frac{1}{12}\left(\sqrt{9 - 12A_1^2} + 2A_1\cos(2\tau)\right) \tag{5.68}$$

Which is a time transformation of the form

$$dt = \frac{d\tau}{\sqrt{\frac{1}{12}\left(\sqrt{9 - 12A_1^2} + 2A_1\cos(2\tau)\right)}} \tag{5.69}$$

We can compare this to the time transformation in eq 5.5 by using the identities we previously developed (5.52-5.54) and $A_1 = \frac{1}{2}B_1$. The appropriate substitutions result in

$$dt = \frac{d\tau}{\alpha\sqrt{(1 - k^2 \sin^2(\tau))}} \tag{5.70}$$

which is the same as eq 5.5.

We will continue the stability analysis in the interest of seeing if the final form of the stability equation remains in Ince's form. Using eq 5.69 to transform the y equation of eq 5.56 results in

$$\begin{aligned} \frac{1}{12}\left(\sqrt{9 - 12A_1^2} + 2A_1 \cos(2\tau)\right) y'' - \frac{1}{3}A_1 \sin 2\tau \, y' \\ +(\omega_2^2 + 2a_{22}x^2)\, y + 3a_{13}x\, y^2 + 4a_{04}\, y^3 \; = 0 \end{aligned} \tag{5.71}$$

We do a linear stability analysis and set

$$\begin{aligned} x &= A_0 + A_1 \cos 2\tau + u(\tau) \\ y &= v(\tau) \end{aligned} \tag{5.72}$$

where $u(\tau)$ and $v(\tau)$ are small. Substituting these into eq 5.71 and linearizing in $u(\tau)$ and $v(\tau)$ results in

$$\begin{aligned} \left[\frac{1}{12}\sqrt{9 - 12A_1^2} - \frac{1}{6}A_1 \cos(2\tau)\right] v'' - \left[\frac{1}{3}A_1 \sin(2\tau)\right] v' \\ + \Bigg[\omega_2^2 + a_{22}\left(1 + \frac{1}{3}\sqrt{9 - 12A_1^2} + \frac{1}{3}A_1^2\right) \\ + \left(2a_{22}A_1 + \frac{2}{3}a_{22}\sqrt{9 - 12A_1^2}A_1\right)\cos(2\tau) + a_{22}A_1^2\cos(4\tau)\Bigg] v \; = 0 \end{aligned} \tag{5.73}$$

which is in the generalized Ince's form

$$(a_0 + a_2 \cos(2\tau))v'' + b_2 \sin(2\tau)v' + (\delta + c_2 \cos(2\tau) + c_4 \cos(4\tau))v = 0 \tag{5.74}$$

where

$$
\begin{aligned}
a_0 &= \tfrac{1}{12}\sqrt{9-12A_1^2} \\
a_2 &= \tfrac{1}{6}A_1 \\
b_2 &= -\tfrac{1}{3}A_1 \\
c_2 &= 2a_{22}A_1 + \tfrac{2}{3}a_{22}\sqrt{9-12A_1^2}A_1 \\
c_4 &= a_{22}A_1^2 \\
\delta &= \omega_2^2 + a_{22}\left(1+\tfrac{1}{3}\sqrt{9-12A_1^2}+\tfrac{1}{3}A_1^2\right)
\end{aligned}
\tag{5.75}
$$

5.7 Trigonometrification Revisited

In this section we generalize the trigonometrification process to include a more general class of oscillator equations. We again start with the form

$$\ddot{x} + f(x) = 0 \tag{5.76}$$

but no longer assume that $f(x)$ is odd. We do however assume that the system (5.76) exhibits an oscillating solution. We seek to stretch the time in eq 5.76 so that $x(\tau) = Q(\tau)$ where $Q(\tau)$ is periodic, the specific form of $Q(\tau)$ is to be determined at a later point. We again assume that the time transformation takes the general form

$$dt = \frac{d\tau}{\sqrt{g(\tau)}} \tag{5.77}$$

The transformation (5.77) turns eq 5.76 into

$$x''g + \frac{1}{2}x'g' + f(x) = 0 \tag{5.78}$$

Substituting $x(\tau) = Q(\tau)$ into eq 5.78 yields

$$g' + 2\frac{Q''}{Q'}g + 2\frac{f(Q)}{Q'} = 0 \tag{5.79}$$

We assume the solution to eq 5.79 is of the form

$$g(\tau) = \frac{K(\tau)}{Q'^2} \tag{5.80}$$

Plugging this into eq 5.79 and solving for $K'(\tau)$ we find

$$K'(\tau) = -2f(Q)Q' \tag{5.81}$$

Integrating,

$$K(\tau) = -2F(Q) + C \tag{5.82}$$

where $F'(Q)$=$f(Q)$, i.e., $F(Q)$ is the antiderivative of $f(Q)$. Our equation for g, eq 5.80, then becomes

$$g(\tau) = \frac{-2F(Q) + C}{Q'^2} \tag{5.83}$$

We wish to choose C such that there are no singularities at τ^* (where τ^* is defined such that $Q'(\tau^*) = 0$). Thus, we choose $C = 2\ F(Q)|_{\tau=\tau^*}$.

$$g(\tau) = 2\ \frac{F(Q)|_{\tau=\tau^*} - F(Q)}{Q'^2} \tag{5.84}$$

Note that the more complicated $Q(\tau)$ becomes, the more τ^* exist and the harder it will be to remove the singularities for all τ^*. However, it can be shown (section 5.9) that the ansatz

$$Q(\tau) = A_0 + A_1 \cos 2\tau \tag{5.85}$$

is sufficient to treat systems for which $f(x)$ is an arbitrary polynomial. Assuming the form (5.85) for $Q(\tau)$, we find

$$g(\tau) = 2\ \frac{F(A_0 + A_1 \cos 2\tau^*) - F(A_0 + A_1 \cos 2\tau)}{4A_1^2 \sin^2 2\tau} \tag{5.86}$$

Singularities exist at $\tau^* = 0$ and at $\tau^* = \pi/2$. If we choose C=$2\ F(Q)|_{\tau=0}$=$F(A_0+A_1)$, we remove the singularity at $\tau^* = 0$. To remove the singularity at $\tau^* = \pi/2$, we must also determine an appropriate relationship between A_0 and A_1. To do this, we expand the numerator of $g(\tau)$ in eq 5.86 in a Fourier series and convert all even powers of $\cos 2\tau$ to even powers of $\sin 2\tau$ via the identity $\cos^2 2\tau{=}1-\sin^2 2\tau$. This results in the following expression for the numerator of $g(\tau)$:

$$\begin{aligned} \text{numerator}(g(\tau)) = q_0 + p_0 \cos 2\tau + \sin^2 2\tau\,(q_1 + p_1 \cos 2\tau) + \cdots \\ + \sin^{2n} 2\tau\,(q_n + p_n \cos 2\tau) \end{aligned} \tag{5.87}$$

where $q_n = q_n(A_0, A_1)$ and $p_n = p_n(A_0, A_1)$. The $\sin^{2n} 2\tau$ in front of $q_n + p_n \cos 2\tau$ $(n \geq 1)$ eliminates any possible singularities coming from these terms. Moreover our choice of $C = F(A_0 + A_1)$ removed the singularity at $\tau^* = 0$, which requires that $q_0 + p_0 = 0$ i.e., $q_0{=}{-}p_0$. It remains to remove the singularity at $\tau^* = \pi/2$, which requires that $q_0 - p_0{=}{-}2p_0{=}0$. Finally, p_0 is made to vanish by choosing an appropriate relationship between A_0 and A_1. It turns out that the resulting equation $p_0(A_0, A_1) = 0$ is an $(n+1)^{th}$ degree polynomial equation where n is the polynomial degree of $f(x)$. The procedure is illustrated by the following example.

5.8 Example 3

We take as an example the strongly nonlinear system [15]

$$\ddot{x} + x^3 + x^4 = 0 \tag{5.88}$$

Here $f(x) = x^3 + x^4$ which gives the antiderivative $F(x) = \frac{1}{4}x^4 + \frac{1}{5}x^5$. Assuming a trigonometrified solution , eq 5.85, and substituting into eq 5.86 results in a time transformation of the form $dt = d\tau/\sqrt{g(\tau)}$ where

$$g(\tau) = \frac{\frac{1}{4}\left(A_0 + A_1\right)^4 + \frac{1}{5}\left(A_0 + A_1\right)^5 - \frac{1}{4}\left(A_0 + A_1 \cos 2\tau\right)^4 - \frac{1}{5}\left(A_0 + A_1 \cos 2\tau\right)^5}{2A_1^2 \sin^2 2\tau} \tag{5.89}$$

which simplifies to the form

$$g(\tau) = \frac{1}{2A_1^2 \sin^2 2\tau} \left(q_0 + p_0 \cos 2\tau + (q_1 + p_1 \cos 2\tau) \sin^2 2\tau + (q_2 + p_2 \cos 2\tau) \sin^4 2\tau\right) \tag{5.90}$$

where

$$\begin{aligned}
q_0 &= A_0^4 A_1 + A_0^3 A_1 + 2A_0^2 A_1^3 + A_0 A_1^3 + \tfrac{1}{5} A_1^5 = -p_0 \\
p_0 &= -A_0^4 A_1 - A_0^3 A_1 - 2A_0^2 A_1^3 - A_0 A_1^3 - \tfrac{1}{5} A_1^5 \\
q_1 &= 2A_0^3 A_1^2 + \tfrac{3}{2} A_0^2 A_1^2 + 2A_0 A_1^4 + \tfrac{1}{2} A_1^4 \\
p_1 &= A_0 A_1^3 + 2A_0^2 A_1^3 + \tfrac{2}{5} A_1^5 \\
q_2 &= -A_0 A_1^4 - \tfrac{1}{4} A_1^4 \\
p_2 &= -\tfrac{1}{5} A_1^5
\end{aligned} \tag{5.91}$$

As stated in the previous section, the choice of $C = F(A_0 + A_1)$ is responsible for $q_0 = -p_0$, cf. eqs 5.91. Thus, setting $p_0 = 0$ will define a relationship between A_0 and A_1 that eliminates the singularities in $g(\tau)$. With this in mind we set $q_0 = -p_0 = 0$ to find

$$A_0^4 A_1 + A_0^3 A_1 + 2A_0^2 A_1^3 + A_0 A_1^3 + \frac{1}{5} A_1^5 = 0 \tag{5.92}$$

Note that this is a 5^{th} degree polynomial equation, which is one degree higher than that of $f(x) = x^3 + x^4$, as stated at the end of the previous section. Here the relationship between A_0 and A_1 produces real solutions for a certain set of A_0 and A_1 values. Assuming a real solution, we obtain the following final expression for $g(\tau)$:

$$g(\tau) = \frac{1}{2A_1^2} \left(q_1 + p_1 \cos 2\tau + (q_2 + p_2 \cos 2\tau) \sin^2 2\tau\right) \tag{5.93}$$

which becomes

$$
\begin{aligned}
g(\tau) = \quad & A_0^3 + \frac{3}{4}A_0^2 + A_0A_1^2 + \frac{1}{4}A_1^2 + \left(\frac{1}{2}A_0A_1 + A_0^2A_1 + \frac{1}{5}A_1^3\right)\cos 2\tau \\
& - \left(\frac{1}{2}A_0A_1^2 + \frac{1}{8}A_1^2\right)\sin^2 2\tau - \frac{1}{10}A_1^3 \cos 2\tau \sin^2 2\tau
\end{aligned} \tag{5.94}
$$

In order to visualize the process of trigonometrification, we show in figure 5.1 the periodic solution to eq 5.88 for the initial condition $x(0) = 0.6058$, $\dot{x}(0) = 0$. Then in figure 5.2 we show the trigonometrified solution, which is of the form $x(\tau) = A_0 + A_1 \cos 2\tau$, where $A_0 = -0.1948$ and $A_1 = 0.8006$. These values for A_0 and A_1 are obtained by simultaneously solving the initial condition $A_0 + A_1 = x(0)$ together with eq 5.92. These two figures also show the relative time compression involved in the trigonometrification process. Figure 5.3 shows the corresponding relationship between the original time t and transformed time τ defined by $dt = d\tau/\sqrt{g(\tau)}$ where $g(\tau)$ is given by eq 5.94.

Figures 5.4 and 5.5 compare a variety of solutions to eq 5.88 for different initial conditions with their trigonometrified counterparts. Note that the level curves of the original system are particularly distorted as they approach a separatrix with a saddle point at $x{=}-1$, $v{=}\dot{x}{=}0$. Since our method is limited to periodic solutions of the differential equation, we are limited to looking inside the separatrix. This turns out to yield a maximum permissible value for A_1, namely $A_1{=}\ 0.8029$, which corresponds to $A_0{=}\ -0.1971$. Thus, for this problem A_0 ranges from -0.1971 to 0.

As an application of this result, suppose we are interested in the stability of the NNM which lies in the $y = 0$ invariant manifold of the following system [15]:

$$
\ddot{x} + x^3 + x^4 + xy^2 = 0, \qquad \ddot{y} + \omega^2 y + x^2 y = 0 \tag{5.95}
$$

Note that the absence of a linear term in the equation for the $x-$mode, eq 5.88,

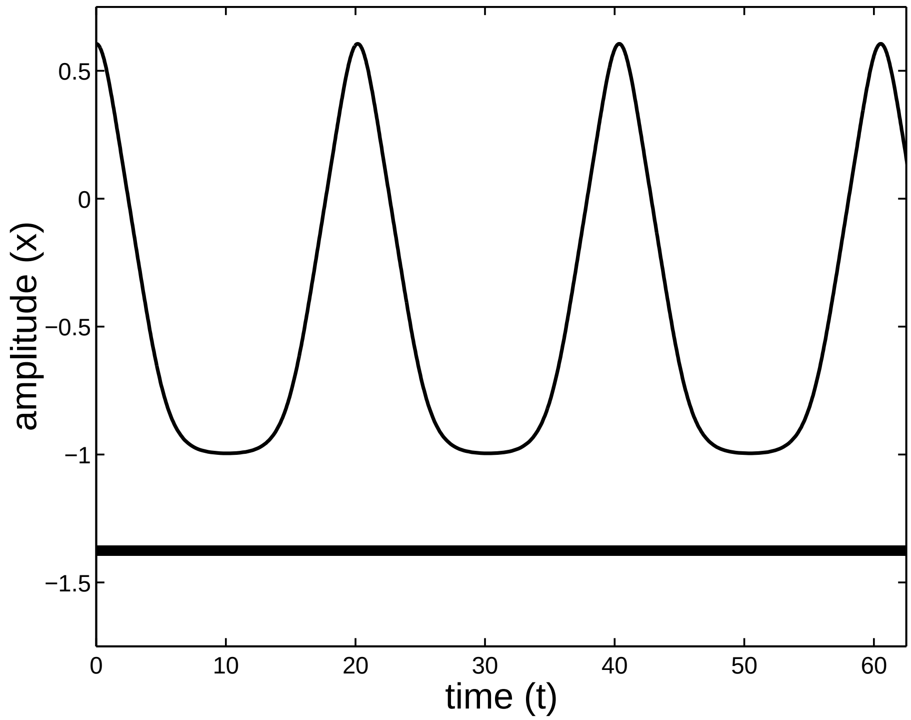

Figure 5.1: *Periodic solution $x(t)$ to eq 5.88 for initial condition $x(0) = 0.6058$, $\dot{x}(0) = 0$. Result obtained by numerical integration. Note that there is no relative time compression, corresponding to the original time.*

makes it difficult to obtain an expression for the NNM, and therefore makes the stability problem difficult without trigonometrification. Using eq 5.94 for $g(\tau)$ to define the time transformation results in stability of the $x-$mode in eq 5.95 being governed by

$$h_1(\tau)v'' + h_2(\tau)v' + \left(\omega^2 + h_3(\tau)\right) v = 0 \tag{5.96}$$

where

$$\begin{aligned}
h_1 &= -\tfrac{1}{8A_1^2}\left(p_2 \cos 6\tau + 2q_2 \cos 4\tau - p_2 \cos 2\tau - 4p_1 \cos 2\tau - 2q_2 - 4q_1\right) \\
h_2 &= \tfrac{1}{8A_1^2}\left(3p_2 \sin 6\tau + 4q_2 \sin 4\tau - p_2 \sin 2\tau - 4p_1 \sin 2\tau\right) \\
h_3 &= \tfrac{1}{2}A_1^2 \cos 4\tau + A_0 A_1 \cos 2\tau + \tfrac{1}{2}A_1^2 + A_0
\end{aligned} \tag{5.97}$$

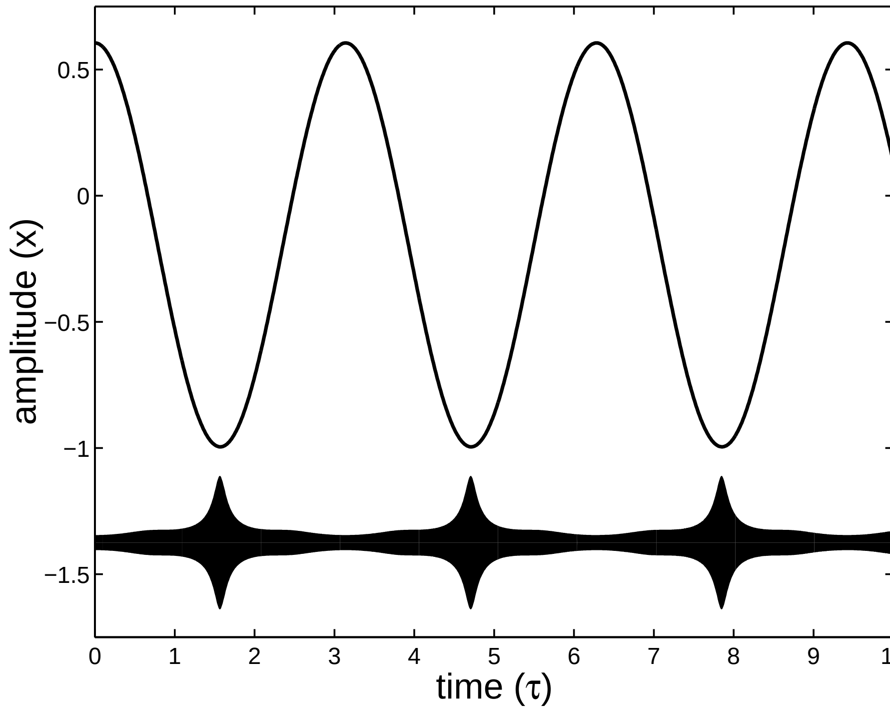

Figure 5.2: *Trigonometrified solution* $x(\tau) = -0.1948 + 0.8006\cos 2\tau$ *corresponding to original periodic solution of figure 5.1. Comparison with figure 5.1 shows that the relative time compression is greatest where the original periodic motion is stalled, that is, where the plot in figure 5.1 has nearly flat horizontal segments.*

Eq 5.96 is a generalized Ince's equation [14] and can be investigated by using harmonic balance.

5.9 Example 4

In this section we will look specifically at problems that are of polynomial form. We assume a system with an x NNM ($y = 0$) defined by

$$\ddot{x} + 2\,\Gamma_1 x + 3\,\Gamma_2 x^2 + 4\,\Gamma_3 x^3 + ... + (n+1)\,\Gamma_n x^n = 0 \tag{5.98}$$

or equivalently

$$\ddot{x} + \sum_{n=1}^{s} (n+1)\,\Gamma_n\, x^n = 0 \tag{5.99}$$

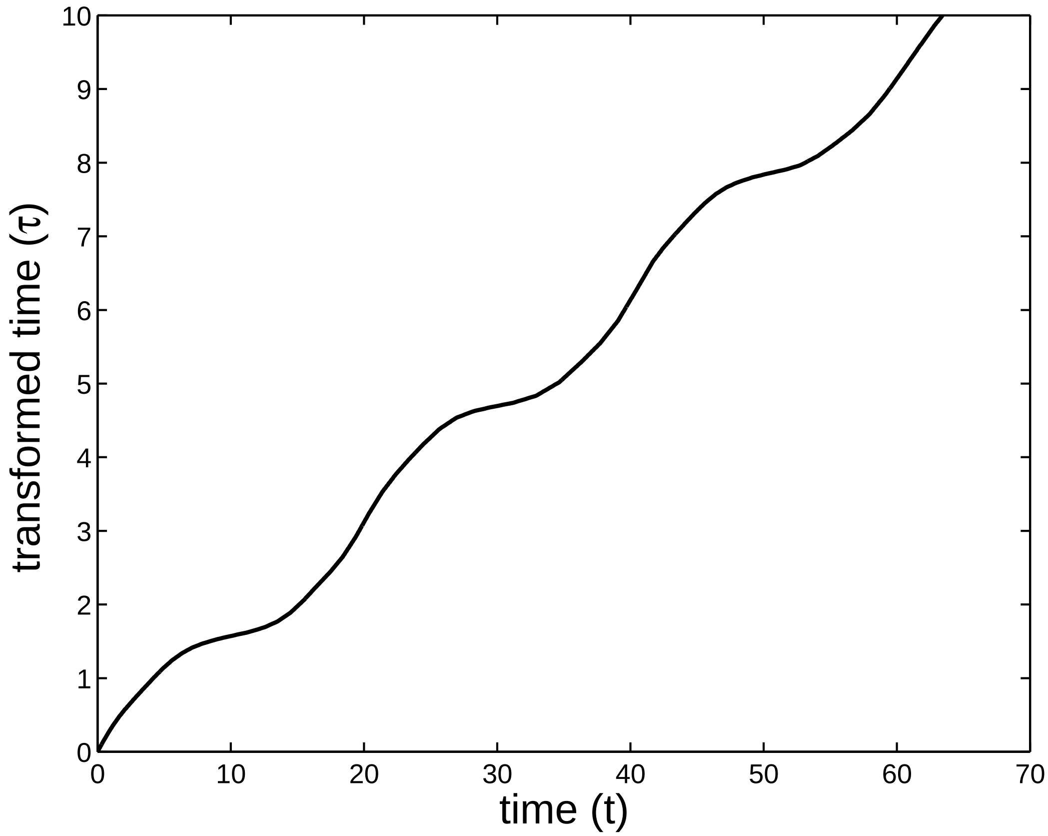

Figure 5.3: *Transformed time τ shown as a function of original time t. Result obtained by numerical integration of $dt = d\tau/\sqrt{g(\tau)}$ where $g(\tau)$ is given by eq 5.94, and where $A_0 = -0.1948$ and $A_1 = 0.8006$.*

Per our method,

$$f(x) = \sum_{n=1}^{s} (n+1)\,\Gamma_n\, x^n \tag{5.100}$$

which makes

$$F(x) = \sum_{n=1}^{s} \Gamma_n\, x^{n+1} \tag{5.101}$$

Substituting this into eq 5.86 results in a time transformation of the form

$$dt = \frac{d\tau}{\sqrt{g(\tau)}} \tag{5.102}$$

where

$$g(\tau) = 2\sum_{n=1}^{s} \Gamma_n (A_0 + A_1)^{n+1} - \sum_{n=1}^{s} \Gamma_n (A_0 + A_1 \cos 2\tau)^{n+1} 2A_1^2 \sin^2 2\tau \tag{5.103}$$

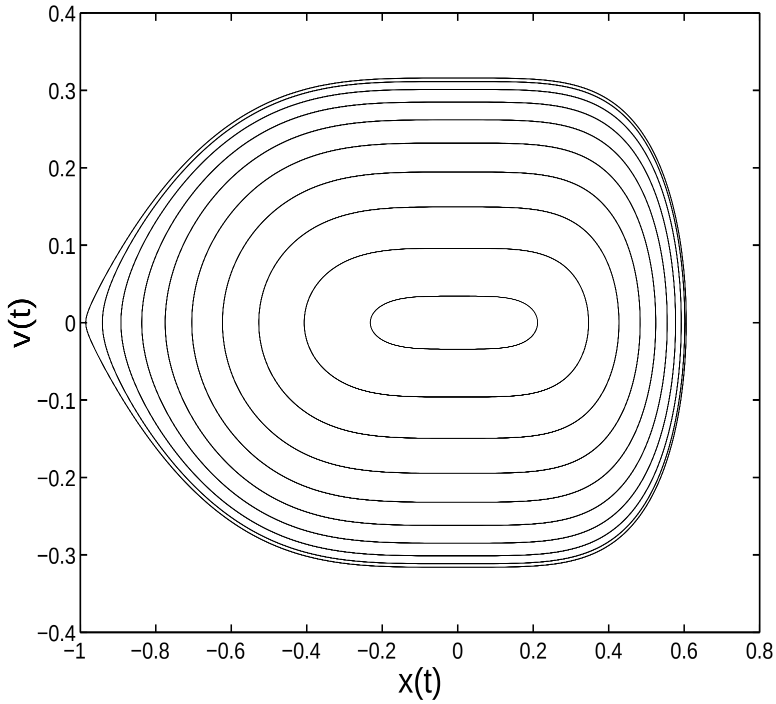

Figure 5.4: *Phase plane plots of solutions to eq 5.88. Comparison below*

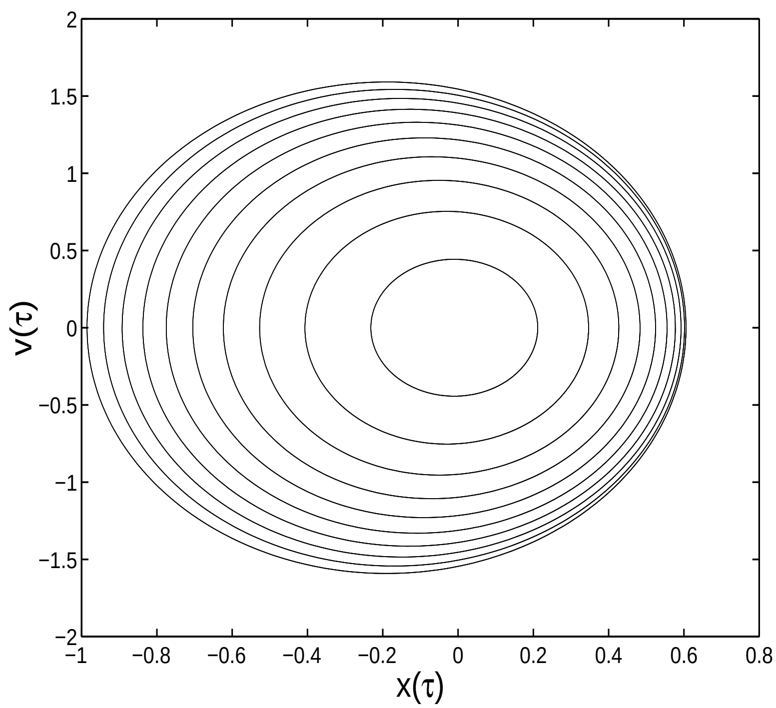

Figure 5.5: *Phase plane plots of trigonometrified solutions to eq 5.88.*

which simplifies to the form

$$
\begin{aligned}
g(\tau) \quad = \quad & \frac{1}{2A_1^2 \sin^2 2\tau} \Big(q_0 + p_0 \cos 2\tau + [q_1 + p_1 \cos 2\tau] \sin^2 2\tau \\
& + [q_2 + p_2 \cos 2\tau] \sin^4 2\tau + ... + [q_n + p_n \cos 2\tau] \sin^{2n} 2\tau \Big) \qquad (5.104)
\end{aligned}
$$

where

$$q_0 \quad = \quad -p_0 \qquad (5.105)$$

$$p_i \quad = \quad \sum_{k=1}^{s} \sum_{n=3}^{s} (-1)^{i+1} \binom{n}{2k} \binom{k-1}{i} \frac{2k}{n} A_1^{2k-1} A_0^{n-2k} \Gamma_{n-2} \qquad (5.106)$$

$$q_i \quad = \quad \sum_{k=1}^{s} \sum_{n=3}^{s} (-1)^{i+1} \binom{n}{2k-1} \binom{k-1}{i} \frac{2k-1}{n} A_1^{2(k-1)} A_0^{n-(2k-1)} \Gamma_{n-2} \qquad (5.107)$$

The relationship between A_0 and A_1 that eliminates the singularity of $g(\tau)$ is

$$q_0 = -p_0 = \sum_{k=1}^{s} \sum_{n=3}^{s} \binom{n}{2k} \frac{2k}{n} A_1^{2k-1} A_0^{n-2k} \Gamma_{n-2} = 0 \qquad (5.108)$$

Assuming that we know this relationship, we find that $g(\tau)$ now takes the form

$$
\begin{aligned}
g(\tau) \quad = \quad & \frac{1}{2A_1^2} \big([q_1 + p_1 \cos 2\tau] + [q_2 + p_2 \cos 2\tau] \sin^2 2\tau + ... \big) \\
& ... + [q_n + p_n \cos 2\tau] \sin^{2n-2} 2\tau \big) \qquad (5.109)
\end{aligned}
$$

or

$$g(\tau) = \frac{1}{2A_1^2} \left(\sum_{i=2}^{n} (q_i + p_i \cos 2\tau) \sin^{2i-2} 2\tau \right) \qquad (5.110)$$

Lets look at an example. Given a system defined by the following energies:

$$T = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 \qquad (5.111)$$

$$V = F(x) + \frac{1}{2} y^2 \left(\omega^2 + C(x) \right) + \sum_{n=3}^{k} y^n M_n(x) \qquad (5.112)$$

where $F(x)$, $C(x)$, and $M_n(x)$ are polynomials in x and $F(x)$ has the form $F(x) = \sum_{n=1}^{s} \Gamma_n x^{n+1}$ (see eq 5.101). For simplicity, we set

$$m(x, y) = \frac{1}{2} y^2 \left(\omega^2 + C(x) \right) + \sum_{n=3}^{k} y^n M_n(x) \qquad (5.113)$$

Lagrange's equations for this system are of the form:

$$\ddot{x} + f(x) + m_x(x, y) = 0 \tag{5.114}$$

$$\ddot{y} + m_y(x, y) = 0 \tag{5.115}$$

where we remind ourselves that $f(x)$ is of the form of eq 5.100 and $m_x(x, y)\big|_{y=0} = m_y(x, y)\big|_{y=0} = 0$. When $y = 0$ eq 5.114 is identical to eq 5.99. Using eq 5.110 we can transform the system into

$$x''g + \tfrac{1}{2}x'g' + f(x) + m_x(x, y) = 0 \tag{5.116}$$

$$y''g + \tfrac{1}{2}y'g' + m_y(x, y) = 0 \tag{5.117}$$

where (5.116) has a solution $x(\tau) = A_0 + A_1 \cos 2\tau$. To find the stability of the x-NNM, we substitute

$$x(\tau) = A_0 + A_1 \cos 2\tau + u(\tau) \tag{5.118}$$

$$y(\tau) = 0 + v(\tau) \tag{5.119}$$

into (5.116) and (5.117). The resulting stability equation is

$$v''g + \frac{1}{2}v'g' + \omega^2 v + C(A_0 + A_1 \cos 2\tau)v = 0 \tag{5.120}$$

where $g(\tau)$ is defined by eq 5.110. This equation in the form of a generalized Ince's equation.

5.10 Conclusions

We have presented a scheme for reparametrizing time such that the periodic motion of a general class of conservative nonlinear oscillators is able to be represented by a simple cosine function. Specifically, if the oscillator is of the form

$$\frac{d^2x}{dt^2} + f(x) = 0 \tag{5.121}$$

then, when expressed in the new time τ, the periodic motion may be written in the form

$$x(\tau) = A_0 + A_1 \cos 2\tau \qquad \text{for general } f(x) \text{, and} \tag{5.122}$$

$$x(\tau) = A \cos \tau \qquad \text{for } f(x) \text{ odd, i.e. } f(-x) = -f(x) \tag{5.123}$$

We have shown that this procedure has application to the stability of NNMs in two degree of freedom systems. Specifically, the stability problem is reduced to the study of a linear ODE with trigonometric coefficients. See e.g. eqs 5.12 and 5.96. Note that the reason this works is because the question of stability is invariant under reparametrization in time. Other applications, not covered in this paper, would include bifurcation of periodic orbits resulting from changes in stability. In the case of conservative two degree of freedom systems like that of eqs 5.6-5.9, or of eqs 5.95, this would involve trigonometrification of both nonlinear equations.

We note that the although the process of trigonometrification has the obvious advantage of replacing the original time dependence of the periodic motion in question with a trigonometrically simplified representation, it does so at the cost of a) including a first derivative term in an ODE that originally had none, and b) including time dependent terms in an ODE which was originally autonomous. As an example of this, see section 5.6 where the original ODE, eq 5.40, is replaced by the trigonometrified ODE, eq 5.78.

Finally we note that although trigonometrification totally simplifies a particular periodic solution of the original ODE (5.121), expressing it in one of the forms (5.122) or (5.123), it does not simplify the general solution of the original ODE.

Chapter 6

Future Work

To conclude this thesis we thought it would be useful to point out projects that stem from this work. These projects are presented mostly in the form of questions. The projects are questions raised during the research or opportunities the research presented.

6.1 Expansions on Coexistence Research

To begin, we look at our analysis of coexistent systems. We would like to ask four brief questions to instigate future work.

Symmetry

We saw from chapter 4 that the breaking the symmetry of the coexistent Ince equation with damping appears to destroy coexistence. One question we would like explore is how the breaking of other symmetries effects coexistence. For example in the particle in the plane the springs were symmetric. Does the loss of these symmetries preclude coexistence? What methods become important when studying non-symmetric systems?

Damping

A logical expansion on chapter 4 is a perturbation analysis starting at $\epsilon = \infty$ rather than $\epsilon = 0$. Such an asymptotic analysis was done for equations of the Mathieu type in [22]. This type of perturbation would provide a more rigorous proof for the damping analysis.

Minimizing Unstable Regions

In Chapter 3 we were able to completely remove regions of instability by tuning the parameters in such a was as to create a coexistent state. However, we also showed that many problems do not exhibit coexistence. For a designed system, the next best solution is a system where the unstable regions are as small as possible. Is it possible to minimize the unstable regions? What physical options do designers have to achieve the minimization?

Delay

Finally, we approach the subject of differential delay equations (DDE). A Mathieu equation with delay has tongues of instability in the stability diagram. Does a system with delay allow for coexistence [6]? Can we find DDEs that exhibit coexistent behavior?

6.2 Expansions on Trigonometrification

In chapter 5 we have presented a scheme for reparametrizing time such that the periodic motion of a general class of conservative nonlinear oscillators is able to be represented by a simple cosine function. We have shown that this procedure has application to the stability of NNMs in two degree of freedom systems. Specifically, the stability problem is reduced to the study of a linear ODE with trigonometric coefficients.

It was possible to apply this to the stability of NNMs because of the invariance of stability under a time transformation. We would like to extend this research to other system behaviors that are invariant under a time transformation. The first proposal is to study the bifurcation of periodic orbits resulting from changes in stability. In the case of conservative two degree of freedom systems, this would involve trigonometrification of both nonlinear equations.

A second proposal is to look at using the trigonometrification method to study the stability and bifurcation of limit cycle oscillators in multi-degree of freedom systems.

6.3 Final Remarks

In conclusion, we have developed tools for designers to avoid hidden regions of instability, or design a systems with the instabilities removed. We have also created a method that simplifies the analysis of differential equations with periodic NNMs. We see many future extensions for this work and hope the reader finds it useful in their research.

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