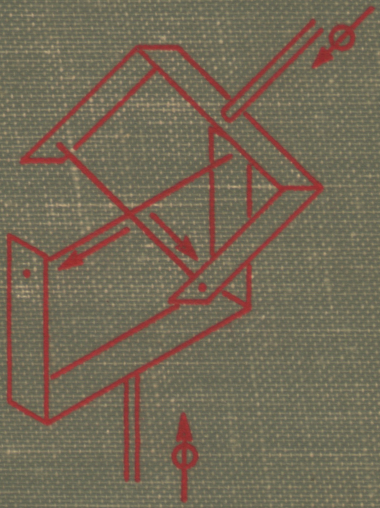

Analytical Elements of Mechanics
Volume 1

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PREFACE

This book is the first of two volumes intended for use in courses in classical mechanics. My objectives in writing it were to provide students and teachers with a text consistent in content and format with my ideas regarding the subject matter and teaching of mechanics, and to disseminate these ideas. In the paragraphs which follow, they are discussed with specific references to relevant portions of the book.

The ultimate purpose of courses in mechanics is to teach the student how to solve physically meaningful problems arising in a variety of fields. As the number of such problems, and even classes of problems, is so great that one cannot hope to cover them individually, the subject should be presented on a level of generality sufficiently high to encompass the entire range of phenomena to be considered, and in sufficient detail to permit relatively direct application to specific situations. This requires a suitable symbolic language, and vector analysis appears to be the best one available. After using it with sophomore students for nearly ten years, I am convinced that it can simplify both the presentation of theory and the solution of problems. (For example, the topic discussed in Section 3.3.6 becomes practically unmanageable in scalar form.) Accordingly, Chapter 1 contains a detailed exposition of vector algebra, and no prior knowledge of this subject is required. Vector methods are then employed throughout the remainder of the book, emphasis being placed on using them as guides to thought, rather than with slavish adherence to a formalism (see, for instance, Sections 3.2.6, 3.4.7, 3.6.2).

Proofs and derivations are given in considerable detail. The reason for this is that I favor spending only a very limited amount of classroom time on these, but believe they should be available to students. In class, it seems preferable to discuss the meaning and applications of a theorem and, in general, to concentrate on that which, apparently, cannot be taught very well in print, that is, anything which requires simultaneous use of visual and audio means of communication, such as, for example, the "setting-up" of problems. Also, the book contains very little discursive material—introductory remarks, allusions to topics of philosophical or historical interest, explanations of the physical significance of mathematically defined quantities, etc.—the underlying assumption being that it is best to deal with these matters verbally, spontaneously, and in a way suited to the background of each group of students. I have used the book with undergraduates, spending three hours per week for fifteen weeks, and with graduate students, covering the same material in ten to twelve lecture hours. In short, the book is not meant to be the entire course. It is concerned with the trees, leaving the teacher free, and indeed obligating him, to describe the forest.

The conviction that the difficult task of relating physics to mathematics is facilitated by initially keeping the two separated in a clear-cut way has strongly influenced me in the choice of contents for Chapters 2 and 3. In the former, the topic of mass centers is presented as a logical extension of concepts introduced in connection with centroids, no attempt being made at this juncture to relate mass centers to centers of gravity. (For the sake of completeness, methods of integral calculus are discussed, but it is not essential that the student have a mastery of this subject. On the contrary, believing that the solution of the majority of practical problems requires skill in locating mass centers without the use of integration, this aspect of the subject is stressed.) In Chapter 3, a theory of moments and couples is constructed without reference to forces, these being mentioned only in illustrative examples. This is done because it eventually becomes necessary to apply the theory to systems of vectors which are not forces, such as momenta and impulses. Particularly in connection with couples, I have broadened the more generally used definition (see Section 3.4) in order to establish a sound basis for later presentation of D'Alembert's Principle.

Equilibrium, per se, is not mentioned until Chapter 4, and even then the discussion is preceded by extended examination of the concept of force. The attempt to deal with forces in an intuitive way probably has been the source of most of the difficulties encountered in teaching mechanics. Hence, taking concepts of space, mass, and time for granted, and using the analytical tools previously developed for this purpose, concepts of force are presented in an essentially axiomatic fashion. Of course, this means that students do not get so much practice in solving equilibrium problems as they would if this work were begun earlier in the course. Consequently, their proficiency in dealing with certain classes of situations may not immediately become very great. But I am convinced that, in the long run, increased *understanding* more than compensates for lack of facility. For example, I have found that the very important ability to draw correct free-body diagrams is not developed most efficiently by drill, but by a theoretically sound exposition of the ideas underlying the construction of these diagrams. This is not to say that practice is unnecessary. On the contrary, problem solving is the most important activity on the part of the student. The attempt has, therefore, been made to provide problems which are properly correlated with matters treated in the text, and to give just enough of these to insure adequate coverage, without requiring the expenditure of excessive amounts of students' time. Experience indicates that thorough study of very nearly all of the problems is both necessary and sufficient for mastery of the subject.

One of the immediate goals of a first course in mechanics is to prepare students for subsequent study of other subjects. While I do not subscribe to the idea that extensive instruction in strength of materials and statics of fluids should be made an integral part of a mechanics course, it is certainly desirable to go so far that later work in mechanics of continua can be undertaken without any loss in continuity. Problem Set 16 and the related sections of the text are meant to accomplish this.

T. R. Kane

Philadelphia, Pennsylvania
May, 1959

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1 VECTOR ALGEBRA

PROBLEM SETS

1 VECTOR ALGEBRA

1.1 Terminology

The magnitude, orientation, and sense of a vector are called the *characteristics* of the vector.

1.1.1 The *magnitude* of a vector is specified by a positive number and a unit having appropriate dimensions. No unit is stated if the dimensions are those of a pure number.

Examples: Magnitude of a velocity vector: 30 mph. Magnitude of a force vector: 15 lb. Magnitude of a position vector: 6 ft.

1.1.2 The *orientation* of a vector is specified by the relationship between the vector and given reference lines and/or planes.

Example: Orientation of the velocity vector of a point P moving on a circular path: parallel to the tangent to the circle at P.

1.1.3 The *sense* of a vector is specified by the order of two points on a line parallel to the vector.

Example: Sense of the force exerted by a smooth sphere, center at point A, on a smooth sphere, center at point B, when the two spheres are pressed together: AB.

1.1.4 Orientation and sense together determine the *direction* of a vector.

1.1.5 The *dimensions* of a vector are those of its magnitude.

1.1.6 When a vector is associated with a particular point P in space, it is called a *bound* vector; otherwise, a *free* vector. The point P is known as the vector's *point of application*, and the line passing through P and parallel to the vector is called the *line of action* of the vector.

1.1.7 The operations of vector analysis deal only with the characteristics of vectors and apply, therefore, to both bound and free vectors. By the same token, they furnish no information regarding the point of application of any vector resulting from these operations.

1.2 Notation

Vectors are denoted by bold face letters, e.g., $\mathbf{a}$, $\mathbf{b}$, $\mathbf{A}$, $\mathbf{B}$.

The symbol $|\mathbf{v}|$ represents the magnitude of the vector $\mathbf{v}$; e.g., if the velocity $\mathbf{v}$ of a point has a magnitude of 30 miles per hour, this may be indicated by writing

$$|\mathbf{v}| = 30 \text{ mph}$$

It follows from 1.1.1 that the symbol $|\mathbf{v}|$ never represents a negative quantity.

1.2.1 Pictorially, vectors are represented either by straight or curved arrows. A vector represented by a straight arrow has the direction (see 1.1.4) indicated by that arrow. The direction of a vector represented by a curved arrow is the same as the direction in which a right-handed screw moves when the screw's axis is normal to the plane in which the arrow is drawn and the screw is

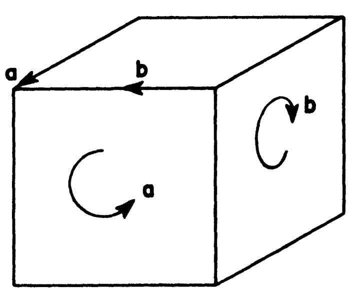


FIG. 1.2.1

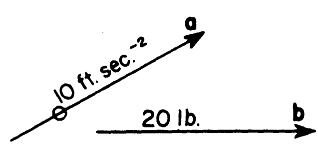


FIG. 1.2.2

rotated as indicated by the arrow. For example, Fig. 1.2.1* shows two representations of each of two vectors $\mathbf{a}$ and $\mathbf{b}$.

1.2.2 If a sketch contains arrows representing vectors whose dimensions (see 1.1.5) differ from each other, attention is drawn to this fact by modifying some of the arrows, e.g., by drawing circles through them. For example, in Fig. 1.2.2, $\mathbf{a}$ is an acceleration vector, $\mathbf{b}$ a force vector.

1.3 Equality

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be equal to each other when they have the same characteristics (see 1.1). One then writes

$$\mathbf{a} = \mathbf{b}$$

1.3.1 Equality does not imply physical equivalence of any sort. For instance, two forces represented by equal vectors do not necessarily cause identical motions of a body on which they act.

1.4 The product of a vector v and a scalar s: sv or vs

Definition: $s\mathbf{v}$ is a vector having the following characteristics:
Magnitude:

$$|s\mathbf{v}| \equiv |\mathbf{v}s| = |s||\mathbf{v}|$$

where $|s|$ denotes the absolute value of the scalar s.

Orientation: $s\mathbf{v}$ is parallel to $\mathbf{v}$. If $s = 0$, no definite orientation is attributed to $s\mathbf{v}$.

Sense: If $s > 0$, the sense of $s\mathbf{v}$ is the same as that of $\mathbf{v}$. If $s < 0$, the sense of $s\mathbf{v}$ is opposite to that of $\mathbf{v}$. If $s = 0$, no definite sense is attributed to $s\mathbf{v}$.

Problem (a): A particle has a mass m of 0.05 slugs and the acceleration $\mathbf{a}$ shown in Fig. 1.2.2. Determine the magnitudes of (1) the vector $m\mathbf{a}$ and (2) the vector $(-2)\mathbf{a}$.

* Each of the figures has the same number as the section in which it is first cited.

Solution (1):

$$|m\mathbf{a}| = |m||\mathbf{a}| = (0.05)(10) = 0.5 \text{ slug ft sec}^{-2}$$

Solution (2):

$$|(-2)\mathbf{a}| = |-2||\mathbf{a}| = (2)(10) = 20 \text{ ft sec}^{-2}$$

Problem (b): Two vectors, $\mathbf{p}$ and $\mathbf{q}$, are given by

$$\mathbf{p} = 2\mathbf{b} \qquad \mathbf{q} = -2\mathbf{b}$$

where $\mathbf{b}$ is the vector shown in Fig. 1.2.2. Draw a sketch showing $\mathbf{b}$, $\mathbf{p}$ and $\mathbf{q}$.

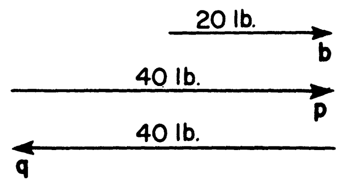


FIG. 1.4

Solution: See Fig. 1.4.

1.4.1 If s_1 and s_2 are any two scalars, then

$$s_1(s_2\mathbf{v}) = (s_1s_2)\mathbf{v} = s_2(s_1)\mathbf{v}$$

This follows from the definition of equality of two vectors (see 1.3), the definition of $s\mathbf{v}$, and the commutativity law for the multiplication of scalars. Hence parentheses are unnecessary, and one writes $s_1s_2\mathbf{v}$.

1.4.2 The vector $(-1)\mathbf{v}$ is called *the negative of* $\mathbf{v}$ and is denoted by the symbol $-\mathbf{v}$. Its magnitude is the same as that of $\mathbf{v}$, its direction opposite to that of $\mathbf{v}$. And

$$-(s\mathbf{v}) = (-s)\mathbf{v} = s(-\mathbf{v})$$

so that parentheses are unnecessary, and one writes $-s\mathbf{v}$.

1.5 Zero vectors

When a vector $\mathbf{v}$ is multiplied by the scalar zero, the result is a vector which does not have a definite direction (see 1.1.4 and 1.4) and whose magnitude is equal to zero. Any such vector is called a zero vector, and all zero vectors are regarded as being equal to each other. The symbol used to denote a zero vector is the same as that representing the scalar zero, i.e., 0.

1.6 The quotient of a vector v and a scalar s: $\mathbf{v}/s$ or $\frac{\mathbf{v}}{s}$

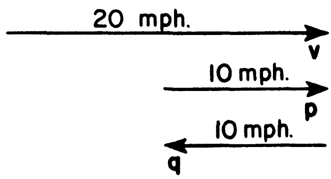


FIG. 1.6

Definition:

$$\mathbf{v}/s \equiv (1/s)\mathbf{v}$$

Example: In Fig. 1.6,

$$\mathbf{p} = \mathbf{v}/2$$

and

$$\mathbf{q} = \frac{\mathbf{v}}{-2}$$

1.7 Unit vectors

A vector whose magnitude is the pure number 1 is called a unit vector.

1.7.1 Given a vector $\mathbf{v}$, a unit vector $\mathbf{n}$ having the same direction as $\mathbf{v}$ is obtained by forming the quotient of $\mathbf{v}$ and $|\mathbf{v}|$:

$$\mathbf{n} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

1.7.2 The direction of a unit vector $\mathbf{n}$ is called the $\mathbf{n}$ *direction.* The opposite direction is called the $-\mathbf{n}$ *direction.*

1.7.3 A vector $\mathbf{v}$ can always be regarded as the product of a scalar v, called a *measure number*, and a unit vector $\mathbf{n}$ which has the same orientation as $\mathbf{v}$.

Proof: Let $\mathbf{n}$ be a unit vector having the same orientation as $\mathbf{v}$, and take

$$v = |\mathbf{v}|$$

or

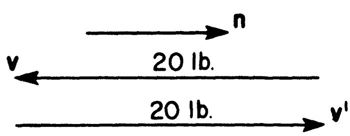
$$v = -|\mathbf{v}|$$

according as $\mathbf{n}$ has the same sense as $\mathbf{v}$, or the sense opposite to that of $\mathbf{v}$. Then (see 1.4) $v\mathbf{n}$ has the same characteristics as $\mathbf{v}$, and (see 1.3)

$$\mathbf{v} = v\mathbf{n}$$

Problem: Express each of the vectors $\mathbf{v}$ and $\mathbf{v}'$, shown in Fig. 1.7.3, as the product of a measure number and the unit vector $\mathbf{n}$.

FIG. 1.7.3



Solution:

$$\mathbf{v} = -20\,\mathbf{n}\ \text{lb}, \qquad \mathbf{v}' = 20\,\mathbf{n}\ \text{lb}$$

1.7.4 When a vector $\mathbf{v}$ is expressed as the product of a measure number v and a unit vector $\mathbf{n}$, the absolute value of the measure number is equal to the magnitude of the vector:

$$|v| = |\mathbf{v}|$$

Proof: If

$$\mathbf{v} = v\mathbf{n}$$

then*

$$|\mathbf{v}| \underset{(1.3)}{=} |v\mathbf{n}| \underset{(1.4)}{=} |v||\mathbf{n}| \underset{(1.7)}{=} |v|$$

* Numbers beneath equal signs refer to the corresponding sections of the text. When these numbers are preceded by P, E, or F, this device indicates a reference to specific problems, examples, and figures, respectively. For example, (P 1.8.2) is to be read "see the problem discussed in Section 1.8.2." When more than one example, figure, or problem is discussed in a particular section these will be numbered (a), (b), etc. and referred to as, for example, Fig. 1.7.5a, Problem 1.4(b).

1.7.5 Vectors are sometimes depicted as shown in Fig. 1.7.5a, i.e., by means of a straight or curved arrow together with a measure number. The corresponding vector is then regarded as having the direction indicated by the arrow if the measure number is

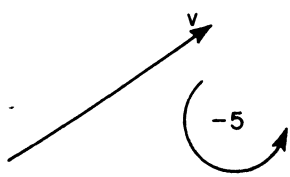


FIG. 1.7.5a

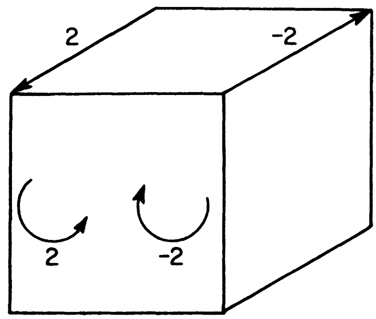


FIG. 1.7.5b

positive, and the opposite direction if it is negative. As will be seen later, this mode of representation is particularly convenient when the orientation of a vector is known while its magnitude and sense are unknown.

Example: Figure 1.7.5b shows four versions of the same vector.

VECTOR ADDITION

1.8 The sum of a vector $\mathbf{v}_1$ and a vector $\mathbf{v}_2$: $\mathbf{v}_1 + \mathbf{v}_2$ or $\mathbf{v}_2 + \mathbf{v}_1$

Definition: $\mathbf{v}_1 + \mathbf{v}_2$ is a vector whose characteristics are found either by graphical or analytical processes based on any one of Figs. 1.8a, b or c. In these figures, the length of each arrow is proportional to the magnitude of the vector represented by the arrow.

1.8.1 The vector $\mathbf{v}_1 + \mathbf{v}_2$ is called the *resultant* of $\mathbf{v}_1$ and $\mathbf{v}_2$.

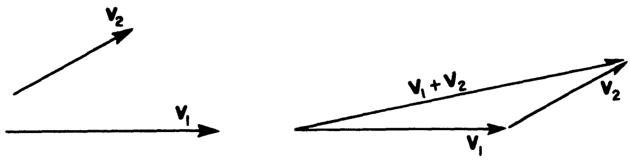


FIG. 1.8a

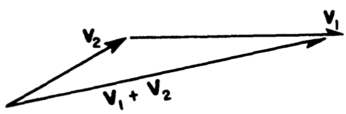


FIG. 1.8b

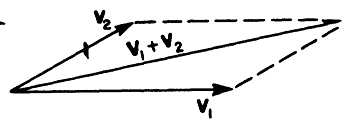


FIG. 1.8c

1.8.2 The definition of addition supplies no information regarding the point of application of the resultant (see 1.1.7).

Problem: In Fig. 1.8.2a, $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit vectors. Find the magnitude of the resultant of the two vectors $\mathbf{p}$ and $\mathbf{q}$ if $\mathbf{p} = 3\mathbf{n}_1$ and $\mathbf{q} = -4\mathbf{n}_2$.

Solution: Sketch the vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{p} + \mathbf{q}$, as shown in Fig. 1.8.2b. Use the law of cosines.

$$|\mathbf{p} + \mathbf{q}| = [3^2 + 4^2 - 2(3)(4) \cos 60°]^{\frac{1}{2}} = 3.61$$

1.8.3. The following is an immediate consequence of the definition of vector addition: If two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, are each rotated through an angle θ in the plane P determined by $\mathbf{v}_1$ and $\mathbf{v}_2$, thus forming two new vectors, $\mathbf{v}_1'$ and $\mathbf{v}_2'$, the resultant of $\mathbf{v}_1'$ and $\mathbf{v}_2'$ is equal to the vector obtained when the resultant of $\mathbf{v}_1$ and $\mathbf{v}_2$ is rotated through the angle θ in plane P.

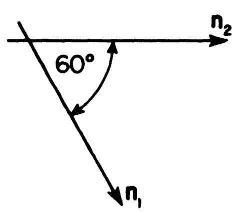


FIG. 1.8.2a

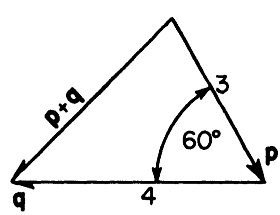


FIG. 1.8.2b

1.8.4 The sum $\mathbf{v}_1 + (-\mathbf{v}_2)$ is called the *difference of* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ and is denoted by $\mathbf{v}_1 - \mathbf{v}_2$ (see Fig. 1.8.4a).

FIG. 1.8.4a

Problem: Determine the magnitude of the vector $\mathbf{p} - \mathbf{q}$ for the vectors $\mathbf{p}$ and $\mathbf{q}$ described in Problem 1.8.2.

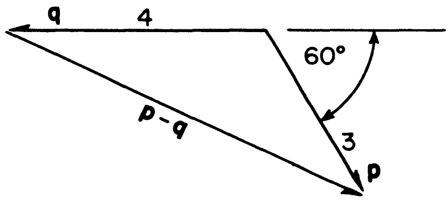


FIG. 1.8.4b

Solution: Sketch the vectors $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{p} - \mathbf{q}$, as shown in Fig. 1.8.4b, then use the law of cosines:

$$|\mathbf{p} - \mathbf{q}| = [3^2 + 4^2 + (2)(3)(4) \cos 60°]^{\frac{1}{2}} = 6.09$$

1.9 The sum of n vectors $\mathbf{v}_i$, $i = 1, 2 \ldots, n$: $\sum_{i=1}^{n} \mathbf{v}_i$ or $\mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_n$

Definition: $\sum_{i=1}^{n} \mathbf{v}_i$ is a vector whose characteristics are found either by graphical or analytical processes based on Fig. 1.9. In this figure, the length of each arrow is proportional to the magnitude of the vector represented by the arrow.

1.9.1 The vector $\sum_{i=1}^{n} \mathbf{v}_i$ is called the *resultant* of the vectors $\mathbf{v}_i$, $i = 1, 2, \ldots, n$.

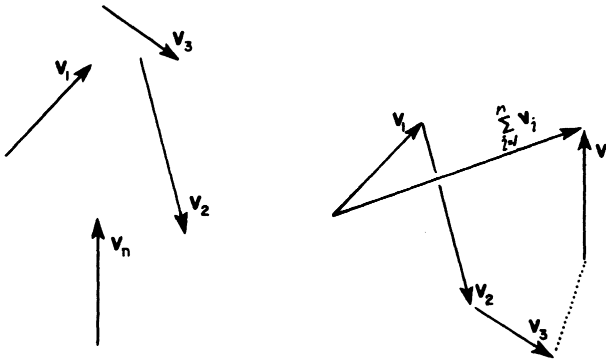


FIG. 1.9

1.9.2 Vector addition is *commutative* and *associative;* i.e., the characteristics of the resultant are independent of the order in which the vectors are added (commutativity) and are not affected by the manner in which the vectors are grouped when the sum is regarded as the resultant of a number of partial sums (associativity).

Example:

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{v}_3 + \mathbf{v}_1 + \mathbf{v}_2 = (\mathbf{v}_1 + \mathbf{v}_3) + \mathbf{v}_2$$

1.9.3 Vector addition obeys the following laws of *distributivity:*

$$\mathbf{v} \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} (\mathbf{v} s_i) \qquad \text{(a)}$$

$$s \sum_{i=1}^{n} \mathbf{v}_i = \sum_{i=1}^{n} (s \mathbf{v}_i) \qquad \text{(b)}$$

Proof (a): For $n = 2$,

$$\mathbf{v} \sum_{i=1}^{n} s_i = \mathbf{v}(s_1 + s_2)$$

while

$$\sum_{i=1}^{n} (\mathbf{v} s_i) = \mathbf{v} s_1 + \mathbf{v} s_2$$

Determine the magnitude and direction of $\mathbf{v}(s_1 + s_2)$ by using 1.4, that of $\mathbf{v}s_1 + \mathbf{v}s_2$ by using 1.4 and 1.8; then use 1.3. This concludes the proof for $n = 2$. The validity of Eq. (a) for all values of n follows from the fact that the sum of any number of vectors can be obtained by successive additions of two vectors.

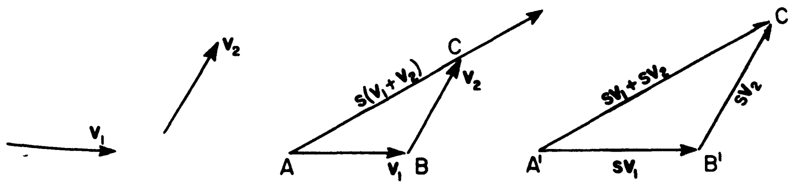


FIG. 1.9.3

Problem: Use the distributivity law to show that

$$3\mathbf{v} + 4\mathbf{v} - 5\mathbf{v} = 2\mathbf{v}$$

Solution:

$$3\mathbf{v} + 4\mathbf{v} - 5\mathbf{v} = (3 + 4 - 5)\mathbf{v} = 2\mathbf{v}$$

Proof (b): For $n = 2$,

$$s\sum_{i=1}^{n} \mathbf{v}_i = s(\mathbf{v}_1 + \mathbf{v}_2)$$

while

$$\sum_{i=1}^{n} (s\mathbf{v}_i) = s\mathbf{v}_1 + s\mathbf{v}_2$$

These two vectors are shown (for a positive value of s) in Fig. 1.9.3. Equality follows from the fact that triangles ABC and $A'B'C'$ are similar, by construction (see 1.8 and 1.4). As the sum of any number of vectors can be obtained by successive additions of two vectors, Eq. (b) is valid for all values of n.

Example:

$$\tfrac{3}{5}\mathbf{v}_1 + \tfrac{4}{5}\mathbf{v}_2 = \tfrac{1}{5}\,(3\mathbf{v}_1 + 4\mathbf{v}_2)$$

COMPONENTS OF A VECTOR

1.10 Resolution of a vector v into n components

Every vector can be regarded as the sum of n vectors ($n = 2, 3, \ldots$), of which all but one can be selected arbitrarily. Each of these n vectors is called a *component* of $\mathbf{v}$.

Problem (a): Resolve the vector $\mathbf{v}$ shown in Fig. 1.10a into four components, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$.

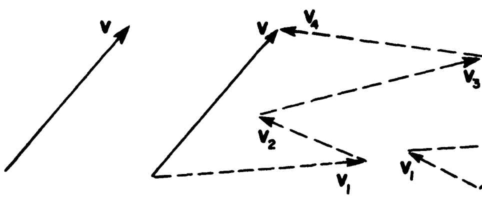


FIG. 1.10a

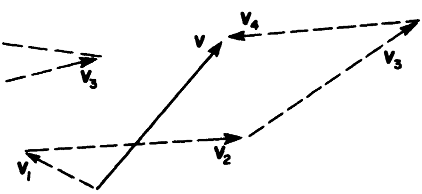


FIG. 1.10b

Solution: Fig. 1.10b shows two solutions.

Problem (b): The force $\mathbf{F}$ shown in Fig. 1.10c has a magnitude of 10 lb. Resolve it into two components, one parallel to line L_1,

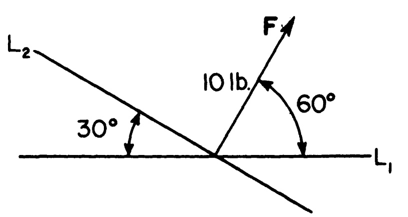


FIG. 1.10c

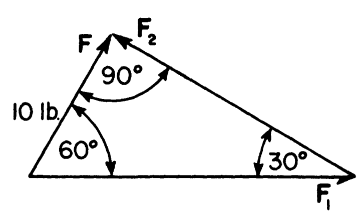


FIG. 1.10d

the other parallel to line L_2, and determine the magnitude of each component.

Solution: Call the components $\mathbf{F}_1$ and $\mathbf{F}_2$. Then

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

and the vectors $\mathbf{F}$, $\mathbf{F}_1$, $\mathbf{F}_2$ must form the triangle shown in Fig. 1.10d. Use the law of sines to determine the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$:

$$\frac{|\mathbf{F}_1|}{10} = \frac{\sin 90°}{\sin 30°} = \frac{1}{0.5}; \ |\mathbf{F}_1| = 20 \text{ lb}$$

$$\frac{|\mathbf{F}_2|}{10} = \frac{\sin 60°}{\sin 30°} = \frac{0.866}{0.5}; \ |\mathbf{F}_2| = 17.3 \text{ lb}$$

1.10.1 Problem 1.10(b) shows that it is possible to resolve a vector into components which are larger than the vector itself and that one of these components may be perpendicular to the vector.

1.10.2 If $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are any three unit vectors not parallel to the same plane, there exist three unique scalars v_1, v_2, v_3, such that a given vector $\mathbf{v}$ can be expressed as

$$\mathbf{v} = \sum_{i=1}^{3} v_i \mathbf{n}_i$$

The vector $v_i\mathbf{n}_i$ is called the $\mathbf{n}_i$ *component of* $\mathbf{v}$ and v_i is called the $\mathbf{n}_i$ *measure number of* $\mathbf{v}$.

Proof: Given $\mathbf{v}$ and the unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, there exists one and only one parallelepiped with $\mathbf{v}$ for its diagonal and with edges parallel to $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ (see Fig. 1.10.2). The edges can be used to construct arrows which represent the vectors $v_1\mathbf{n}_1$, $v_2\mathbf{n}_2$, $v_3\mathbf{n}_3$.

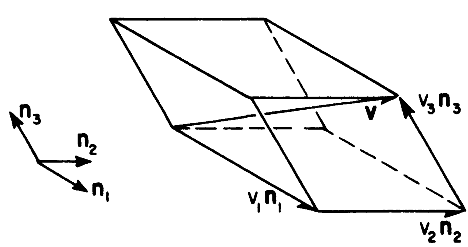


FIG. 1.10.2

1.10.3 When a zero vector $\mathbf{v}$ is expressed in the form

$$\mathbf{v} = \sum_{i=1}^{3} v_i \mathbf{n}_i$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are unit vectors not parallel to the same plane, then

$$v_i = 0, \quad i = 1, 2, 3$$

because the lengths of the edges of a parallelepiped are equal to zero whenever the length of the diagonal is equal to zero.

1.10.4 Every vector equation is equivalent to three scalar equations.

Proof: Every vector equation can be put into the form

$$\mathbf{v} = 0$$

or, letting $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ be unit vectors not parallel to the same plane, into the form (see 1.10.2)

$$v_1\mathbf{n}_1 + v_2\mathbf{n}_2 + v_3\mathbf{n}_3 = 0$$

It then follows from 1.10.3 that

$$v_1 = 0, \quad v_2 = 0, \quad v_3 = 0$$

and these are three scalar equations.

Problem: Given

$$\mathbf{a} = 6\mathbf{n}_1 - 4\mathbf{n}_2$$

and

$$\mathbf{b} = 3\mathbf{n}_1 - \mathbf{n}_2 - 2\mathbf{n}_3$$

determine c_1, c_2, c_3, the $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ measure numbers of a vector $\mathbf{c}$ defined by the vector equation

$$\mathbf{c} = \mathbf{a} - 2\mathbf{b}$$

Solution:

$$\mathbf{c} = c_1\mathbf{n}_1 + c_2\mathbf{n}_2 + c_3\mathbf{n}_3$$

Hence,

$$\begin{aligned} c_1\mathbf{n}_1 + c_2\mathbf{n}_2 + c_3\mathbf{n}_3 &= 6\mathbf{n}_1 - 4\mathbf{n}_2 - 2(3\mathbf{n}_1 - \mathbf{n}_2 - 2\mathbf{n}_3) \\ &= -2\mathbf{n}_2 + 4\mathbf{n}_3 \end{aligned}$$

and

$$c_1\mathbf{n}_1 + (c_2 + 2)\mathbf{n}_2 + (c_3 - 4\mathbf{n}_3) = 0$$

Thus,

$$c_1 = 0, \quad c_2 + 2 = 0, \quad c_3 - 4 = 0$$

and the measure numbers c_1, c_2, c_3 have the values

$$c_1 = 0, \quad c_2 = -2, \quad c_3 = 4$$

1.10.5 When a vector $\mathbf{v}$ is expressed in the form

$$\mathbf{v} = v_1\mathbf{n}_1 + v_2\mathbf{n}_2 + v_3\mathbf{n}_3$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors, the magnitude of $\mathbf{v}$ is given by

$$|\mathbf{v}| = (v_1^2 + v_2^2 + v_3^2)^{\frac{1}{2}}$$

Proof: Draw a sketch showing $\mathbf{v}$, the components $v_1\mathbf{n}_1$, $v_2\mathbf{n}_2$, $v_3\mathbf{n}_3$, and the vector $v_1\mathbf{n}_1 + v_2\mathbf{n}_2$. (See Fig. 1.10.5.) Triangle ACD

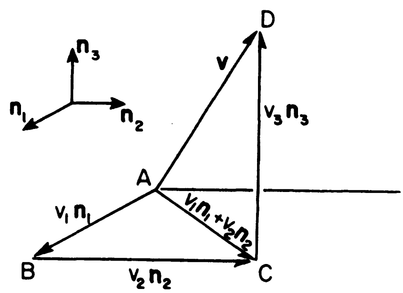


FIG. 1.10.5

is a right triangle whose sides have the lengths $|v_1\mathbf{n}_1 + v_2\mathbf{n}_2|$, $|v_3\mathbf{n}_3|$ and $|\mathbf{v}|$. Therefore,

$$|\mathbf{v}| = (|v_1\mathbf{n}_1 + v_2\mathbf{n}_2|^2 + |v_3\mathbf{n}_3|^2)^{\frac{1}{2}} \tag{1}$$

ABC is a right triangle with sides of lengths $|v_1\mathbf{n}_1|$, $|v_2\mathbf{n}_2|$, $|v_1\mathbf{n}_1 + v_2\mathbf{n}_2|$. Thus,

$$|v_1\mathbf{n}_1 + v_2\mathbf{n}_2|^2 = |v_1\mathbf{n}_1|^2 + |v_2\mathbf{n}_2|^2$$

Substitute into Eq. (1):

$$|\mathbf{v}| = (|v_1\mathbf{n}_1|^2 + |v_2\mathbf{n}_2|^2 + |v_3\mathbf{n}_3|^2)^{\frac{1}{2}} \tag{2}$$

Use 1.7.4:

$$|v_i\mathbf{n}_i| = |v_i|, \quad i = 1, 2, 3$$

Consequently,

$$|v_i\mathbf{n}_i|^2 = |v_i|^2, \quad i = 1, 2, 3$$

The square of the absolute value of a number is equal to the square of the number. Hence,

$$|v_i\mathbf{n}_i|^2 = v_i^2, \quad i = 1, 2, 3$$

Substitute into Eq. (2):

$$|\mathbf{v}| = (v_1^2 + v_2^2 + v_3^2)^{\frac{1}{2}}$$

Problem (a): A force $\mathbf{F}$ is given by

$$\mathbf{F} = 2\mathbf{n}_1 - 3\mathbf{n}_2 - 6\mathbf{n}_3 \text{ lb}$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors. Determine $|\mathbf{F}|$.

Solution:

$$|\mathbf{F}| = (4 + 9 + 36)^{\frac{1}{2}} = 7 \text{ lb}$$

Problem (b): Determine whether or not $\mathbf{n}$ is a unit vector if $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors and

$$\mathbf{n} = 3\mathbf{n}_1 - 2\mathbf{n}_2 + 4\mathbf{n}_3$$

Solution:

$$9 + 4 + 16 = 29$$

Hence,

$$|\mathbf{n}| = (29)^{\frac{1}{2}} \neq 1$$

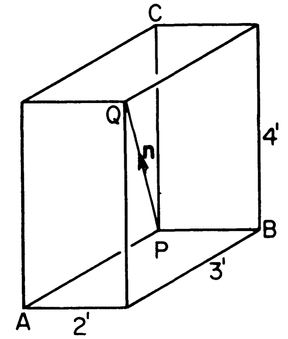
and $\mathbf{n}$ is not a unit vector.

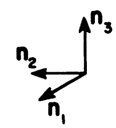
1.11 Methods for resolving a vector into three mutually perpendicular components

See the problems given below.

Problem (a): Resolve the unit vector $\mathbf{n}$ shown in Fig. 1.11a into components parallel to the edges of the rectangular parallelepiped.

Solution: Set up unit vectors parallel to the edges PA, PB, PC. There are many ways to do this; one is shown in Fig. 1.11b. ($\mathbf{n}_1$ points from the paper toward the reader.)





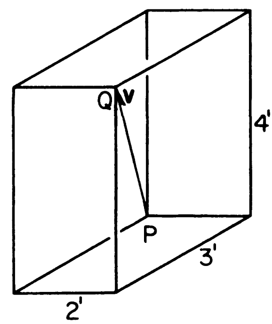


FIG. 1.11a

FIG. 1.11b

Let $\mathbf{v}$ be a vector joining P to Q, as shown in Fig. 1.11b. Then

$$\mathbf{v} = 3\mathbf{n}_1 - 2\mathbf{n}_2 + 4\mathbf{n}_3 \text{ ft}$$

(One may think of $\mathbf{v}$ in terms of "going" from P to Q by first going 3 ft in the $\mathbf{n}_1$ direction, then 2 ft in the $-\mathbf{n}_2$ direction, finally 4 ft in the $\mathbf{n}_3$ direction.)

Find $|\mathbf{v}|$:

$$|\mathbf{v}| = (9 + 4 + 16)^{\frac{1}{2}} = 5.39 \text{ ft}$$

Use 1.7.1:

$$\mathbf{n} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{5.39}(3\mathbf{n}_1 - 2\mathbf{n}_2 + 4\mathbf{n}_3)$$

$$\underset{(1.9.3)}{=} \frac{3}{5.39}\mathbf{n}_1 - \frac{2}{5.39}\mathbf{n}_2 + \frac{4}{5.39}\mathbf{n}_3$$

$$= 0.556\mathbf{n}_1 - 0.371\mathbf{n}_2 + 0.742\mathbf{n}_3$$

Check: $(0.556)^2 + (0.371)^2 + (0.742)^2 = 0.308 + 0.148 + 0.550 = 1.006 \approx 1.000$

Problem (b): Resolve the force $\mathbf{F}$ shown in Fig. 1.11c into components parallel to the edges of the rectangular parallelepiped, find the magnitudes of these components, and determine the sense of each component.

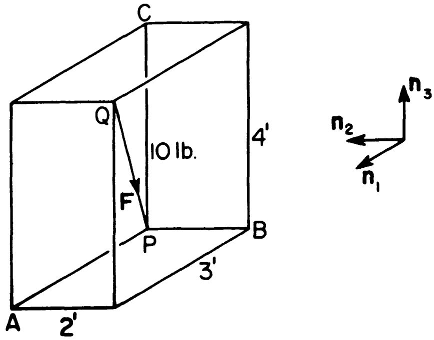


FIG. 1.11c

Solution: Set up unit vectors parallel to the edges (see Fig. 1.11c), and construct a unit vector $\mathbf{n}$ having the same orientation as $\mathbf{F}$. From Problem 1.11(a),

$$\mathbf{n} = 0.556\mathbf{n}_1 - 0.371\mathbf{n}_2 + 0.742\mathbf{n}_3$$

Noting that $\mathbf{n}$ and $\mathbf{F}$ have opposite senses, use 1.7.3:

$$\mathbf{F} = -10\mathbf{n} = -5.56\mathbf{n}_1 + 3.71\mathbf{n}_2 - 7.42\mathbf{n}_3 \text{ lb}$$

The magnitudes of the three components of $\mathbf{F}$ (see 1.7.4) are

$$5.56 \text{ lb}, \quad 3.71 \text{ lb}, \quad 7.42 \text{ lb}$$

and the signs in the expression for $\mathbf{F}$ show that the three components have the senses AP, BP, CP.

Problem (c): Four forces, $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$ are applied to the box shown in Fig. 1.11d. Force $\mathbf{P}$ has the magnitude and direction shown in the sketch. The forces $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$ have lines of action respectively parallel to OA, OB, OC. If the resultant of these four forces is equal to zero, what is the magnitude and sense of each of the forces $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$?

Solution: Draw a sketch showing the four forces, assigning the senses of $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ arbitrarily and using the notation described in 1.7.5. Also, set up unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, as shown in Fig. 1.11e.

Proceeding as in Problem 1.11(a), construct a unit vector parallel to the line of action of each of the four forces, then express each force as the product of a measure number and the appropriate unit vector:

$$\begin{aligned}
\mathbf{P} &= 10(-0.8\mathbf{n}_2 + 0.6\mathbf{n}_3) \\
\mathbf{Q} &= Q(-0.8\mathbf{n}_2 - 0.6\mathbf{n}_3) \\
\mathbf{R} &= \frac{R}{\sqrt{13}}(2\mathbf{n}_1 + 3\mathbf{n}_3) \\
\mathbf{S} &= \frac{S}{\sqrt{20}}(2\mathbf{n}_1 + 4\mathbf{n}_2)
\end{aligned}$$

Write the three scalar equations (see 1.10.3) corresponding to the vector equation

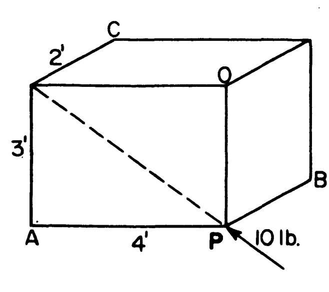
$$\mathbf{P} + \mathbf{Q} + \mathbf{R} + \mathbf{S} = 0$$

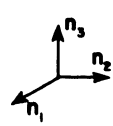
which states that the resultant of the four vectors is equal to zero. These are

$$\begin{aligned}
\frac{2}{\sqrt{13}}R + \frac{2}{\sqrt{20}}S &= 0 \\
-8 - 0.8Q + \frac{4}{\sqrt{20}}S &= 0 \\
6 - 0.6Q + \frac{3}{\sqrt{13}}R &= 0
\end{aligned}$$

Solve this set of simultaneous equations:

$$Q = 0, \quad R = -2\sqrt{13}, \quad S = 2\sqrt{20}$$





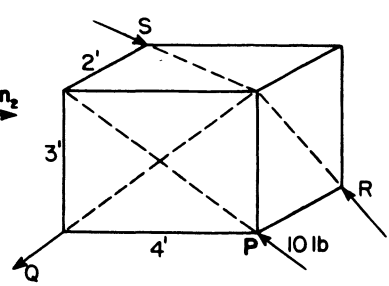


Fig. 1.11d

Fig. 1.11e

Refer to 1.7.3 and 1.7.4 to determine the magnitude and sense of each force (see Table 1.11).

TABLE 1.11

Force	Magnitude	Sense
$\mathbf{Q}$	0	—
$\mathbf{R}$	7.22 lb	OB
$\mathbf{S}$	8.96 lb	CO

1.11.1 The presentation of information in tabular form frequently saves time and space; e.g., all of the equations used in the solution of Problem 1.11(c) are contained, essentially, in Table 1.11.1.

TABLE 1.11.1

Force	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{n}_3$
$\mathbf{P}$	0	-8	6
$\mathbf{Q}$	0	-0.8	-0.6
$\mathbf{R}$	$2/\sqrt{13}$	0	$3/\sqrt{13}$
$\mathbf{S}$	$2/\sqrt{20}$	$4/\sqrt{20}$	0

1.12 Resolutes of a vector v

Definition: When a vector $\mathbf{v}$ is resolved into two components, $\mathbf{v}_L$ and $\mathbf{v}_P$, with $\mathbf{v}_L$ parallel to a line L (or to a unit vector $\mathbf{n}$) and $\mathbf{v}_P$ perpendicular to L, $\mathbf{v}_L$ is called the *L (or $\mathbf{n}$) resolute of* $\mathbf{v}$ and $\mathbf{v}_P$ the *resolute of* $\mathbf{v}$ *perpendicular to L.* In terms of $\mathbf{v}_L$ and $\mathbf{v}_P$, $\mathbf{v}$ is given by

$$\mathbf{v} = \mathbf{v}_L + \mathbf{v}_P$$

Problem: Determine the magnitudes of the L_1 and L_2 resolutes of the force $\mathbf{F}$ shown in Fig. 1.12a.

Solution: Let $\mathbf{F}_1$ be the L_1 resolute of $\mathbf{F}$, $\mathbf{F}_1'$ the resolute of $\mathbf{F}$ perpendicular to L_1 (see Fig. 1.12b). Then

$$|\mathbf{F}_1| = 10 \cos 60° = 5 \text{ lb}$$

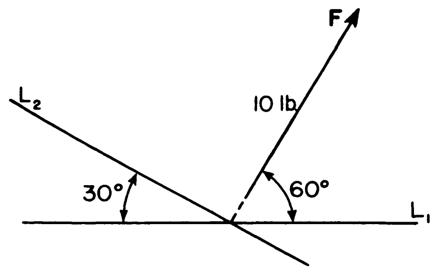


FIG. 1.12a

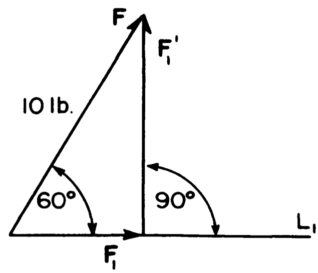


FIG. 1.12b

Next, note that L_2 is perpendicular to **F**. It follows that the magnitude of the L_2 resolute of **F** is equal to zero.

1.12.1 No resolute of a vector **v** ever has a magnitude which is greater than that of **v**; and if a line L is perpendicular to **v**, the L resolute of **v** is equal to zero. (Compare with 1.10.1.)

SCALAR MULTIPLICATION OF VECTORS

1.13 The angle between a vector a and a vector b: (a, b) or (b, a)

Definition: Given two vectors **a** and **b**, move either vector parallel to itself (leaving its sense unaltered) until their initial points coincide. The four situations which can arise are illustrated by Figs. 1.13a, b, c, d. The angle θ in Figs. 1.13a and b is called

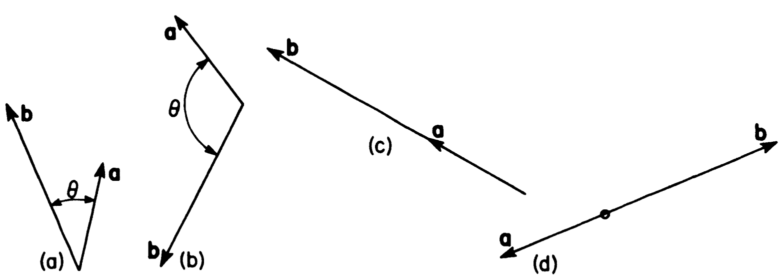


FIG. 1.13a FIG. 1.13b FIG. 1.13c FIG. 1.13d

the angle between $\mathbf{a}$ *and* $\mathbf{b}$ and is denoted by the symbols $(\mathbf{a}, \mathbf{b})$ or $(\mathbf{b}, \mathbf{a})$. Fig. 1.13c represents the case $(\mathbf{a}, \mathbf{b}) = 0$; Fig. 1.13d, the case $(\mathbf{a}, \mathbf{b}) = 180°$.

1.14 The scalar (dot) product of a vector a and a vector b: a • b or b • a

Definition:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\mathbf{a}, \mathbf{b})$$

Problem: Find the dot product of the two vectors shown in Fig. 1.14.

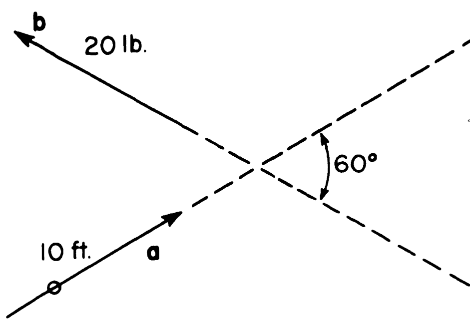


FIG. 1.14

Solution:

$$\mathbf{a} \cdot \mathbf{b} = 10(20) \cos 120°$$
$$= 10(20)(-0.5) = -100 \text{ ft lb}$$

1.14.1 If $\mathbf{a}$ is perpendicular to $\mathbf{b}$, $\mathbf{a} \cdot \mathbf{b} = 0$; but if $\mathbf{a} \cdot \mathbf{b} = 0$, $\mathbf{a}$ is not necessarily perpendicular to $\mathbf{b}$.

Proof: If $\mathbf{a}$ is perpendicular to $\mathbf{b}$, then $(\mathbf{a}, \mathbf{b}) = 90°$, $\cos(\mathbf{a}, \mathbf{b}) = 0$, and $\mathbf{a} \cdot \mathbf{b} = 0$. On the other hand, $\mathbf{a} \cdot \mathbf{b} = 0$ implies only that the product $|\mathbf{a}|\,|\mathbf{b}| \cos(\mathbf{a}, \mathbf{b})$ is equal to zero, and this is the case whenever $|\mathbf{a}| = 0$, or $|\mathbf{b}| = 0$, or $\cos(\mathbf{a}, \mathbf{b}) = 0$.

1.14.2 For any two vectors $\mathbf{a}$ and $\mathbf{b}$ and any scalar s,

$$(s\mathbf{a})\cdot\mathbf{b} = s(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(s\mathbf{b})$$

Hence parentheses are unnecessary, and one writes $s\mathbf{a}\cdot\mathbf{b}$.

Problem: Evaluate $-3\mathbf{a}\cdot\mathbf{b}$ by finding $-3(\mathbf{a}\cdot\mathbf{b})$, $\mathbf{a}\cdot(-3\mathbf{b})$, and $(-3\mathbf{a})\cdot\mathbf{b}$ for the vectors $\mathbf{a}$ and $\mathbf{b}$ shown in Fig. 1.14.

Solution:

$$-3(\mathbf{a}\cdot\mathbf{b}) \underset{\text{(P1.14)}}{=} -3(-100) = 300 \text{ ft lb}$$

$$\mathbf{a}\cdot(-3\mathbf{b}) = 10(60)\cos 60^\circ = 300 \text{ ft lb}$$

$$(-3\mathbf{a})\cdot\mathbf{b} = 30(20)\cos 60^\circ = 300 \text{ ft lb}$$

1.14.3 Scalar multiplication of vectors is distributive:

$$\mathbf{a}\cdot\sum_{i=1}^{n}\mathbf{b}_i = \sum_{i=1}^{n}(\mathbf{a}\cdot\mathbf{b}_i)$$

Proof: For $n = 2$,

$$\mathbf{a}\cdot\sum_{i=1}^{n}\mathbf{b}_i = \mathbf{a}\cdot(\mathbf{b}_1 + \mathbf{b}_2)$$

while

$$\sum_{i=1}^{n}(\mathbf{a}\cdot\mathbf{b}_i) = \mathbf{a}\cdot\mathbf{b}_1 + \mathbf{a}\cdot\mathbf{b}_2$$

Draw $\mathbf{a}$, $\mathbf{b}_1$ and $\mathbf{b}_2$ with a common initial point P, and construct $\mathbf{b}_1 + \mathbf{b}_2$. Drop perpendiculars from the terminal points of $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_1 + \mathbf{b}_2$, on the line passing through $\mathbf{a}$. This gives the points Q_1, Q_2 and Q shown in Fig. 1.14.3a.

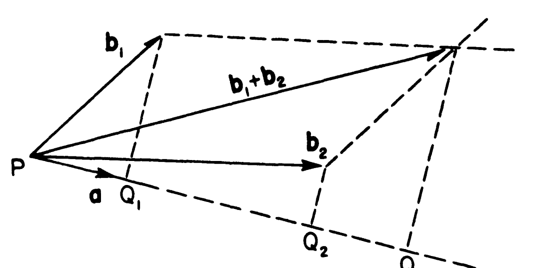


FIG. 1.14.3a

Note that $\overline{PQ_2} = \overline{Q_1Q}$, and evaluate $\mathbf{a}\cdot\mathbf{b}_1$, $\mathbf{a}\cdot\mathbf{b}_2$ and $\mathbf{a}\cdot(\mathbf{b}_1 + \mathbf{b}_2)$:

$$\mathbf{a}\cdot\mathbf{b}_1 = |\mathbf{a}|\,|\mathbf{b}_1| \cos(\mathbf{a}, \mathbf{b}_1) = |\mathbf{a}|\,\overline{PQ_1}$$
$$\mathbf{a}\cdot\mathbf{b}_2 = |\mathbf{a}|\,|\mathbf{b}_2| \cos(\mathbf{a}, \mathbf{b}_2) = |\mathbf{a}|\,\overline{PQ_2} = |\mathbf{a}|\,\overline{Q_1Q}$$
$$\mathbf{a}\cdot(\mathbf{b}_1 + \mathbf{b}_2) = |\mathbf{a}|\,|\mathbf{b}_1 + \mathbf{b}_2| \cos(\mathbf{a}, \mathbf{b}_1 + \mathbf{b}_2) = |\mathbf{a}|\,\overline{PQ}$$

Then

$$\mathbf{a}\cdot\mathbf{b}_1 + \mathbf{a}\cdot\mathbf{b}_2 = |\mathbf{a}|\,\overline{PQ_1} + |\mathbf{a}|\,\overline{Q_1Q} = |\mathbf{a}|\,(\overline{PQ_1} + \overline{Q_1Q})$$
$$= |\mathbf{a}|\,\overline{PQ} = \mathbf{a}\cdot(\mathbf{b}_1 + \mathbf{b}_2)$$

(F1.14.3a)

The validity of the theorem for all values of n follows from the fact that the vector $\sum_{i=1}^{n} \mathbf{b}_i$ can be regarded as the sum of two vectors, for example, $\mathbf{b}_1$ and $\sum_{i=2}^{n} \mathbf{b}_i$, and the scalar $\sum_{i=1}^{n} (\mathbf{a}\cdot\mathbf{b}_i)$ can be regarded as the sum of two scalars, e.g., $\mathbf{a}\cdot\mathbf{b}_1$ and $\sum_{i=2}^{n} (\mathbf{a}\cdot\mathbf{b}_i)$.

Problem: Fig. 1.14.3b shows three coplanar vectors, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Evaluate $\mathbf{c}\cdot(\mathbf{b} + 2\mathbf{a})$.

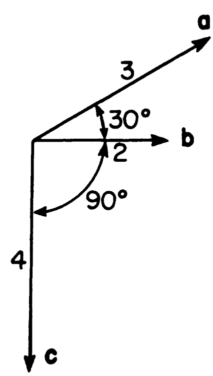


FIG. 1.14.3b

Solution:

$$\mathbf{c}\cdot(\mathbf{b} + 2\mathbf{a}) = \mathbf{c}\cdot\mathbf{b} + 2\mathbf{c}\cdot\mathbf{a}$$
$$= 0 + 2(4)(3)\cos 120°$$
$$= -12 \text{ in}^2$$

1.14.4 If

$$\mathbf{a} = a_1\mathbf{n}_1 + a_2\mathbf{n}_2 + a_3\mathbf{n}_3$$

and

$$\mathbf{b} = b_1\mathbf{n}_1 + b_2\mathbf{n}_2 + b_3\mathbf{n}_3$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors, then

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Proof: Use distributivity (see 1.14.3) and the relationships

$$\mathbf{n}_1\cdot\mathbf{n}_1 = \mathbf{n}_2\cdot\mathbf{n}_2 = \mathbf{n}_3\cdot\mathbf{n}_3 = 1$$

$$\mathbf{n}_1\cdot\mathbf{n}_2 = \mathbf{n}_2\cdot\mathbf{n}_3 = \mathbf{n}_3\cdot\mathbf{n}_1 = 0$$

Problem: Determine the scalar product of the force $\mathbf{F}$ and the unit vector $\mathbf{n}$ shown in Fig. 1.14.4.

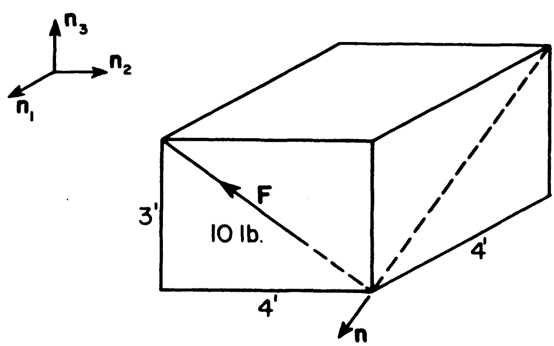


FIG. 1.14.4

Solution: Set up unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ as shown in Fig. 1.14.4, and express $\mathbf{F}$ and $\mathbf{n}$ in terms of these:

$$\mathbf{F} = -8\mathbf{n}_2 + 6\mathbf{n}_3 \text{ lb}$$

$$\mathbf{n} = 0.8\mathbf{n}_1 - 0.6\mathbf{n}_3$$

Then

$$\mathbf{F}\cdot\mathbf{n} = (0)(0.8) + (-8)(0) + (6)(-0.6) = -3.6 \text{ lb}$$

1.14.5 As the expression for $\mathbf{a}\cdot\mathbf{b}$ given in 1.14.4 does not involve the angle $(\mathbf{a}, \mathbf{b})$ between $\mathbf{a}$ and $\mathbf{b}$, it can be used in conjunction with the definition of $\mathbf{a}\cdot\mathbf{b}$ to evaluate this angle:

$$(\mathbf{a}, \mathbf{b}) \underset{(1.14,\ 1.14.4)}{=} \text{arc cos}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{|\mathbf{a}|\ |\mathbf{b}|}\right)$$

The arc cosine is a multi-valued function. However, the angle between two vectors (see 1.13) never exceeds 180 degrees. Only one of the values of this function is, therefore, appropriate in any given case.

Problem: Find the angle $(\mathbf{n}_a, \mathbf{n}_b)$ between the unit vectors $\mathbf{n}_a$ and $\mathbf{n}_b$ shown in Fig. 1.14.5.

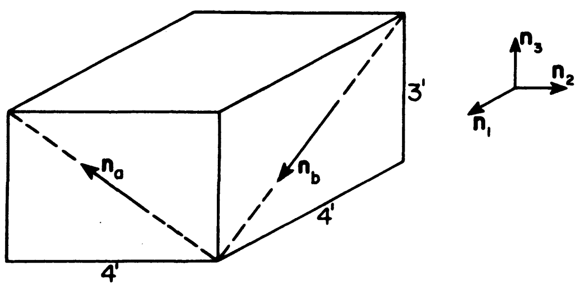


FIG. 1.14.5

Solution: Express $\mathbf{n}_a$ and $\mathbf{n}_b$ in terms of the unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ shown in Fig. 1.14.5:

$$\mathbf{n}_a = -0.8\mathbf{n}_2 + 0.6\mathbf{n}_3$$

$$\mathbf{n}_b = 0.8\mathbf{n}_1 - 0.6\mathbf{n}_3$$

Then

$$(\mathbf{n}_a, \mathbf{n}_b) = \text{arc cos}\,\frac{(0)(0.8) + (-0.8)(0) + (0.6)(-0.6)}{(1)(1)}$$

$$= \text{arc cos}\,(-0.36)$$

$$\text{arc cos}\,(-0.36) = 111.1°, 201.1°, 471.1°, \ldots$$

Hence,

$$(\mathbf{n}_a, \mathbf{n}_b) = 111.1°$$

1.14.6 The dot product is useful in evaluating resolutes of a vector (see 1.12):

$$\mathbf{v}_L = \mathbf{n}\cdot\mathbf{v}\mathbf{n}$$

Notation (see Fig. 1.14.6a):

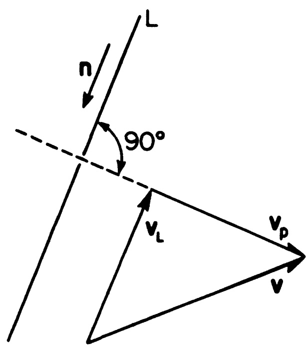


FIG. 1.14.6a

- $\mathbf{v}$ a vector
- L a line
- $\mathbf{v}_L$ the L resolute of $\mathbf{v}$
- $\mathbf{v}_P$ the resolute of $\mathbf{v}$ perpendicular to L
- $\mathbf{n}$ a unit vector parallel to L

Proof: By definition, $\mathbf{v}_L$ is parallel to $\mathbf{n}$. Hence $\mathbf{v}_L$ can be expressed as the product of $\mathbf{n}$ and a measure number v_L (see 1.7.3):

$$\mathbf{v}_L = v_L\mathbf{n} \tag{1}$$

Also,

$$\mathbf{v} \underset{(1.12)}{=} \mathbf{v}_L + \mathbf{v}_P \tag{2}$$

Substitute from Eq. (1) into Eq. (2):

$$\mathbf{v} = v_L\mathbf{n} + \mathbf{v}_P$$

Take the dot product of both sides of this equation with $\mathbf{n}$:

$$\mathbf{n}\cdot\mathbf{v} = v_L\mathbf{n}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{v}_P$$

But

$$\mathbf{n}\cdot\mathbf{n} = 1$$

and

$$\mathbf{n}\cdot\mathbf{v}_P \underset{(1.14.1)}{=} 0$$

Hence,

$$\mathbf{n}\cdot\mathbf{v} = v_L$$

Substitute into Eq. (1).

Problem: Determine the magnitude and sense of $\mathbf{F}_L$, the L resolute of the force $\mathbf{F}$ shown in Fig. 1.14.6b.

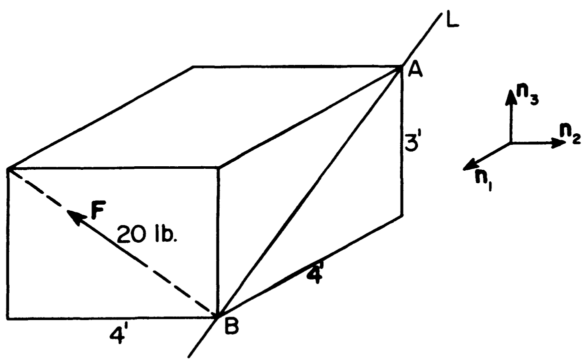


FIG. 1.14.6b

Solution: Let $\mathbf{n}$ be a unit vector parallel to line L and having the sense AB. Express $\mathbf{F}$ and $\mathbf{n}$ in terms of their $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ components:

$$\mathbf{F} = -16\mathbf{n}_2 + 12\mathbf{n}_3 \text{ lb}$$

$$\mathbf{n} = 0.8\mathbf{n}_1 - 0.6\mathbf{n}_3$$

Then

$$\mathbf{n}\cdot\mathbf{F} \underset{(1.14.4)}{=} -0.6(12)$$

$$= -7.2 \text{ lb}$$

and

$$\mathbf{F}_L = \mathbf{n}\cdot\mathbf{F}\mathbf{n} = -7.2\mathbf{n} \text{ lb}$$

Hence,

$$|\mathbf{F}_L| \underset{(1.7.4)}{=} 7.2 \text{ lb}$$

and the sense of $\mathbf{F}_L$ is opposite to that of $\mathbf{n}$; i.e., $\mathbf{F}_L$ has the sense BA (see Fig. 1.14.6b).

1.14.7 Given a line L and n vectors $\mathbf{v}_i$, $i = 1, 2, \ldots, n$, the resultant of the L resolutes of these vectors is equal to the L resolute of the vectors' resultant.

Proof: Let $\mathbf{n}$ be a unit vector parallel to L. Then the resultant of the L resolutes of the vectors $\mathbf{v}_i$, $i = 1, \ldots, n$, is the vector (see 1.14.6)

$$\sum_{i=1}^{n} (\mathbf{n}\cdot\mathbf{v}_i\,\mathbf{n})$$

while the L resolute of the resultant of the vectors $\mathbf{v}_i$, $i = 1, 2, \ldots, n$, is the vector

$$\mathbf{n}\cdot\left(\sum_{i=1}^{n}\mathbf{v}_i\right)\mathbf{n}$$

The equality of these two vectors follows from 1.9.3 and 1.14.3.

1.14.8 Given n vectors $\mathbf{v}_i$, $i = 1, \ldots, n$, the resultant of the resolutes of these vectors perpendicular to a line L is equal to the resolute of the vectors' resultant perpendicular to L.

Proof: With self-explanatory notation,

$$\sum_{i=1}^{n}(\mathbf{v}_i)_P \underset{(1.12)}{=} \sum_{i=1}^{n}[\mathbf{v}_i - (\mathbf{v}_i)_L] \underset{(1.9.2)}{=} \sum_{i=1}^{n}\mathbf{v}_i - \sum_{i=1}^{n}(\mathbf{v}_i)_L$$
$$\underset{(1.14.7)}{=} \sum_{i=1}^{n}\mathbf{v}_i - \left(\sum_{1=i}^{n}\mathbf{v}_i\right)_L \underset{(1.12)}{=} \left(\sum_{i=1}^{n}\mathbf{v}_i\right)_P$$

1.14.9 Every vector $\mathbf{v}$ can be expressed in the form

$$\mathbf{v} = \mathbf{n}_1\cdot\mathbf{v}\mathbf{n}_1 + \mathbf{n}_2\cdot\mathbf{v}\mathbf{n}_2 + \mathbf{n}_3\cdot\mathbf{v}\mathbf{n}_3$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors. (While this identity is of little value for purposes of computation, it is useful in certain proofs.)

Proof: $\mathbf{v}$ can always be expressed as

$$\mathbf{v} \underset{(1.10.2)}{=} v_1\mathbf{n}_1 + v_2\mathbf{n}_2 + v_3\mathbf{n}_3 \tag{1}$$

Dot multiply both sides of Eq. (1) with $\mathbf{n}_1$:

$$\mathbf{n}_1\cdot\mathbf{v} \underset{(1.14.2,\ 1.14.3)}{=} v_1\mathbf{n}_1\cdot\mathbf{n}_1 + v_2\mathbf{n}_1\cdot\mathbf{n}_2 + v_3\mathbf{n}_1\cdot\mathbf{n}_3$$

But,

$$\mathbf{n}_1\cdot\mathbf{n}_1 = 1$$

and

$$\mathbf{n}_1\cdot\mathbf{n}_2 = \mathbf{n}_1\cdot\mathbf{n}_3 = 0$$

Hence,

$$\mathbf{n}_1\cdot\mathbf{v} = v_1$$

Similarly,

$$\mathbf{n}_2\cdot\mathbf{v} = v_2 \quad \text{and} \quad \mathbf{n}_3\cdot\mathbf{v} = v_3$$

Substitute into Eq. (1).

1.14.10 While every resolute of a vector $\mathbf{v}$ is a component of $\mathbf{v}$, not every component is a resolute (see 1.10 and 1.12). However, when $\mathbf{v}$ is resolved into three components parallel to mutually perpendicular unit vectors $\mathbf{n}_i$, $i = 1, 2, 3$, the $\mathbf{n}_i$ component of $\mathbf{v}$ is equal to the $\mathbf{n}_i$ resolute of $\mathbf{v}$. For, $\mathbf{v}$ is then given by

$$\mathbf{v} \underset{(1.14.9)}{=} \mathbf{n}_1\cdot\mathbf{v}\mathbf{n}_1 + \mathbf{n}_2\cdot\mathbf{v}\mathbf{n}_2 + \mathbf{n}_3\cdot\mathbf{v}\mathbf{n}_3$$

so that, for example, $\mathbf{n}_1\cdot\mathbf{v}\mathbf{n}_1$ is the $\mathbf{n}_1$ component of $\mathbf{v}$; and, in accordance with 1.14.6, $\mathbf{n}_1\cdot\mathbf{v}\mathbf{n}_1$ is equal to the $\mathbf{n}_1$ resolute of $\mathbf{v}$.

Problem: Find $\mathbf{F}_1$, the $\mathbf{n}_1$ resolute of the force $\mathbf{F}$ shown in Fig. 1.11c.

Solution: From Problem 1.11(b),

$$\mathbf{F} = -5.56\mathbf{n}_1 + 3.71\mathbf{n}_2 - 7.42\mathbf{n}_3 \text{ lb}$$

Hence,

$$\mathbf{F}_1 = -5.56\mathbf{n}_1 \text{ lb}$$

1.14.11 The scalar obtained when a vector $\mathbf{v}$ is dot multiplied with itself is called *the square of* $\mathbf{v}$ and is denoted by the symbol $\mathbf{v}^2$, i.e.,

$$\mathbf{v}^2 \equiv \mathbf{v}\cdot\mathbf{v}$$

From the definition of the dot product it follows that

$$\mathbf{v}^2 = |\mathbf{v}|^2$$

VECTOR MULTIPLICATION OF VECTORS

1.15 The vector (cross) product of a vector $\mathbf{a}$ and a vector $\mathbf{b}$: $\mathbf{a} \times \mathbf{b}$

Definition:

$$\mathbf{a} \times \mathbf{b} \equiv |\mathbf{a}|\,|\mathbf{b}| \sin(\mathbf{a}, \mathbf{b})\mathbf{n}$$

where $\mathbf{n}$ is a unit vector whose direction is the same as the direction of advance of a right-handed screw rotated from $\mathbf{a}$ toward $\mathbf{b}$, through the angle $(\mathbf{a}, \mathbf{b})$, when the axis of the screw is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Examples: See Fig. 1.15.

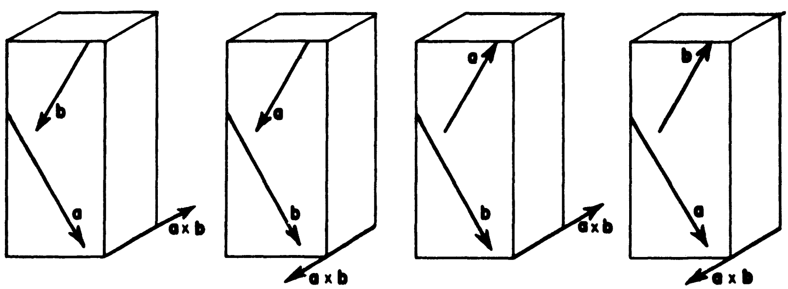


FIG. 1.15

1.15.1 The definition of $\mathbf{a} \times \mathbf{b}$ contains no information regarding the point of application of $\mathbf{a} \times \mathbf{b}$ (see 1.1.7).

1.15.2 The magnitude of $\mathbf{a} \times \mathbf{b}$ is given by (see 1.7.4)

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin(\mathbf{a}, \mathbf{b})$$

Problem: Determine the magnitude of the cross product of the vectors $\mathbf{a}$ and $\mathbf{b}$ shown in Fig. 1.15.2.

Solution:

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}| &= 10(20) \sin 120^\circ \\ &= 10(20)(0.866) = 173.2 \text{ ft lb} \end{aligned}$$

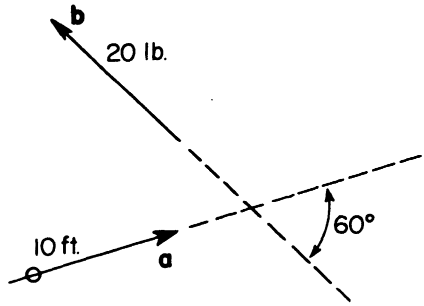


FIG. 1.15.2

1.15.3 If $\mathbf{a}$ is parallel to $\mathbf{b}$, then $\mathbf{a} \times \mathbf{b} = 0$; but if $\mathbf{a} \times \mathbf{b} = 0$, $\mathbf{a}$ is not necessarily parallel to $\mathbf{b}$.

Proof: If $\mathbf{a}$ is parallel to $\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ is either equal to zero or 180 degrees, $\sin(\mathbf{a}, \mathbf{b}) = 0$, and $\mathbf{a} \times \mathbf{b} = 0$. On the other hand, $\mathbf{a} \times \mathbf{b} = 0$ implies only that the product $|\mathbf{a}|\,|\mathbf{b}| \sin(\mathbf{a}, \mathbf{b})$ is equal to zero, and this is the case whenever $|\mathbf{a}| = 0$, or $|\mathbf{b}| = 0$, or $\sin(\mathbf{a}, \mathbf{b}) = 0$.

1.15.4 For any two vectors $\mathbf{a}$ and $\mathbf{b}$ and any scalar s,

$$(s\mathbf{a}) \times \mathbf{b} = s(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (s\mathbf{b})$$

Hence parentheses are unnecessary, and one writes $s\mathbf{a} \times \mathbf{b}$.

1.15.5 The sense of the unit vector $\mathbf{n}$ which appears in the definition of $\mathbf{a} \times \mathbf{b}$ depends on the order of the factors $\mathbf{a}$ and $\mathbf{b}$ in such a way that

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

1.15.6 Vector multiplication obeys the following law of distributivity:

$$\mathbf{a} \times \sum_{i=1}^{n} \mathbf{b}_i = \sum_{i=1}^{n} (\mathbf{a} \times \mathbf{b}_i)$$

Proof: For $n = 2$,

$$\mathbf{a} \times \sum_{i=1}^{n} \mathbf{b}_i = \mathbf{a} \times (\mathbf{b}_1 + \mathbf{b}_2)$$

while

$$\sum_{i=1}^{n} (\mathbf{a} \times \mathbf{b}_i) = \mathbf{a} \times \mathbf{b}_1 + \mathbf{a} \times \mathbf{b}_2$$

Introduce the following (see Fig. 1.15.6):

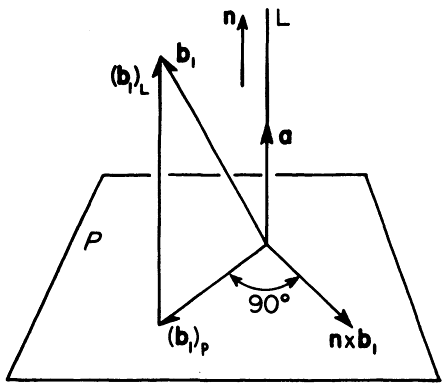


FIG. 1.15.6

L	a line, parallel to $\mathbf{a}$
P	a plane, perpendicular to $\mathbf{a}$
$\mathbf{n}$	a unit vector having the same direction as $\mathbf{a}$
$(\mathbf{b}_1)_L$, $(\mathbf{b}_2)_L$	the L resolutes of $\mathbf{b}_1$, $\mathbf{b}_2$
$(\mathbf{b}_1 + \mathbf{b}_2)_L$	the L resolute of $\mathbf{b}_1 + \mathbf{b}_2$
$(\mathbf{b}_1)_P$, $(\mathbf{b}_2)_P$	the resolutes of $\mathbf{b}_1$, $\mathbf{b}_2$ perpendicular to L
$(\mathbf{b}_1 + \mathbf{b}_2)_P$	the resolute of $\mathbf{b}_1 + \mathbf{b}_2$ perpendicular to L

Consider the cross product of $\mathbf{n}$ and $\mathbf{b}_1$. $\mathbf{n} \times \mathbf{b}_1$ is parallel to P and perpendicular to $(\mathbf{b}_1)_P$; and it has the same magnitude as $(\mathbf{b}_1)_P$. Thus $\mathbf{n} \times \mathbf{b}_1$ is equal to the vector obtained when $(\mathbf{b}_1)_P$ is rotated through 90 degrees in plane P. Similarly, $\mathbf{n} \times \mathbf{b}_2$ and $\mathbf{n} \times (\mathbf{b}_1 + \mathbf{b}_2)$ can be obtained by rotating $(\mathbf{b}_2)_P$ and $(\mathbf{b}_1 + \mathbf{b}_2)_P$ through 90 degrees in plane P. Now,

$$(\mathbf{b}_1 + \mathbf{b}_2)_P \underset{(1.14.8)}{=} (\mathbf{b}_1)_P + (\mathbf{b}_2)_P$$

Hence,

$$\mathbf{n} \times (\mathbf{b}_1 + \mathbf{b}_2) \underset{(1.8.3)}{=} \mathbf{n} \times \mathbf{b}_1 + \mathbf{n} \times \mathbf{b}_2$$

Make the substitution

$$\mathbf{n} \underset{(1.7.1)}{=} \frac{\mathbf{a}}{|\mathbf{a}|}$$

and multiply both sides of the resulting equation with $|\mathbf{a}|$. This concludes the proof for the case $n = 2$. The validity of the theorem for all values of n follows from the fact that $\sum_{i=1}^{n} \mathbf{b}_i$ and $\sum_{i=1}^{n} (\mathbf{a} \times \mathbf{b}_i)$ can each be regarded as a sum of only two vectors.

Problem: Show that

$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2\mathbf{a} \times \mathbf{b}$$

Solution:

$$\begin{aligned}(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) &= (\mathbf{a} - \mathbf{b}) \times \mathbf{a} + (\mathbf{a} - \mathbf{b}) \times \mathbf{b} \\ &= \mathbf{a} \times \mathbf{a} - \mathbf{b} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{b} \\ &= 0 + \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{b} - 0 = 2\mathbf{a} \times \mathbf{b}\end{aligned}$$

1.15.7 A set of mutually perpendicular unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ is called *right-handed* if $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{n}_3$. If $\mathbf{n}_1 \times \mathbf{n}_2 = -\mathbf{n}_3$, the set is called *left-handed.*

Example: Three right-handed sets of unit vectors are shown in Fig. 1.15.7.

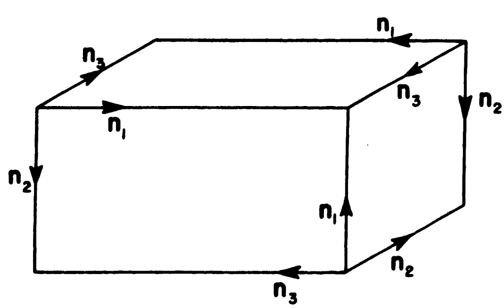


FIG. 1.15.7

1.15.8 If

$$\mathbf{a} = a_1\mathbf{n}_1 + a_2\mathbf{n}_2 + a_3\mathbf{n}_3$$

and

$$\mathbf{b} = b_1\mathbf{n}_1 + b_2\mathbf{n}_2 + b_3\mathbf{n}_3$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ is a right-handed set of mutually perpendicular unit vectors, then

$$\begin{aligned}\mathbf{a} \times \mathbf{b} = &(a_2b_3 - a_3b_2)\mathbf{n}_1\\ &+ (a_3b_1 - a_1b_3)\mathbf{n}_2\\ &+ (a_1b_2 - a_2b_1)\mathbf{n}_3\end{aligned}$$

Proof: Use distributivity (see 1.15.6) and the relationships

$$\mathbf{n}_1 \times \mathbf{n}_1 = \mathbf{n}_2 \times \mathbf{n}_2 = \mathbf{n}_3 \times \mathbf{n}_3 = 0$$

$$\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{n}_3, \quad \mathbf{n}_2 \times \mathbf{n}_3 = \mathbf{n}_1, \quad \mathbf{n}_3 \times \mathbf{n}_1 = \mathbf{n}_2$$

Problem: Referring to Fig. 1.14.4, resolve $\mathbf{F} \times \mathbf{n}$ into its $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ components.

Solution (see Problem 1.14.4):

$$\mathbf{F} = -8\mathbf{n}_2 + 6\mathbf{n}_3 \text{ lb}$$

$$\mathbf{n} = 0.8\mathbf{n}_1 - 0.6\mathbf{n}_3$$

Hence,

$$\begin{aligned}\mathbf{F} \times \mathbf{n} = &[(-8)(-0.6) - (6)(0)]\mathbf{n}_1\\ &+ [(6)(0.8) - (0)(-0.6)]\mathbf{n}_2\\ &+ [(0)(0) - (-8)(0.8)]\mathbf{n}_3\\ &= 4.8\mathbf{n}_1 + 4.8\mathbf{n}_2 + 6.4\mathbf{n}_3 \text{ lb}\end{aligned}$$

1.15.9 Using the same notation as in 1.15.8, $\mathbf{a} \times \mathbf{b}$ can be expressed in the following determinantal form:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Proof:

Expand the determinant by minors of the elements of the first row:

$$\begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{n}_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{n}_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{n}_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \mathbf{n}_1(a_2b_3 - a_3b_2) - \mathbf{n}_2(a_1b_3 - a_3b_1) + \mathbf{n}_3(a_1b_2 - a_2b_1)$$

$$= (a_2b_3 - a_3b_2)\mathbf{n}_1 + (a_3b_1 - a_1b_3)\mathbf{n}_2 + (a_1b_2 - a_2b_1)\mathbf{n}_3$$

Problem: Repeat Problem 1.15.8.

Solution:

$$\mathbf{F} \times \mathbf{n} = \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ 0 & -8 & 6 \\ 0.8 & 0 & -0.6 \end{vmatrix}$$

$$= (-8)(-0.6)\mathbf{n}_1 + (6)(0.8)\mathbf{n}_2 + (8)(0.8)\mathbf{n}_3$$

$$= 4.8\mathbf{n}_1 + 4.8\mathbf{n}_2 + 6.4\mathbf{n}_3 \text{ lb}$$

PRODUCTS OF THREE VECTORS

1.16 The scalar triple product of three vectors a, b, c: [a, b, c]

Definition:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a}\cdot(\mathbf{b} \times \mathbf{c})$$

1.16.1 The parentheses in the expression $\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c})$ are unnecessary because $(\mathbf{a}\cdot\mathbf{b}) \times \mathbf{c}$ is meaningless, $(\mathbf{a}\cdot\mathbf{b})$ being a scalar. Hence one writes $\mathbf{a}\cdot\mathbf{b} \times \mathbf{c}$.

1.16.2 The justification for using the symbol $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ to denote the scalar triple product of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is that it does not matter whether the dot is placed between $\mathbf{a}$ and $\mathbf{b}$, and the cross between $\mathbf{b}$ and $\mathbf{c}$, or vice versa; i.e.,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\cdot\mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b}\cdot\mathbf{c}$$

Proof: Resolve each of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ into $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ components ($\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ being a right-handed set of mutually perpendicular unit vectors), and carry out the indicated operations.

1.16.3 A change in the order of the factors appearing in a scalar triple product at most changes the sign of the product; i.e.,

$$[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{a}, \mathbf{b}, \mathbf{c}] \tag{a}$$

and

$$[\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}] \tag{b}$$

Proof (a):

$$[\mathbf{b}, \mathbf{a}, \mathbf{c}] \underset{(1.16.2)}{=} \mathbf{b} \times \mathbf{a} \cdot \mathbf{c}$$

$$\underset{(1.15.5)}{=} -\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} \underset{(1.16.2)}{=} -[\mathbf{a}, \mathbf{b}, \mathbf{c}]$$

Proof (b):

$$[\mathbf{b}, \mathbf{c}, \mathbf{a}] \underset{(1.16.2)}{=} \mathbf{b} \times \mathbf{c} \cdot \mathbf{a}$$

$$\underset{(1.14)}{=} \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \underset{(1.16.2)}{=} [\mathbf{a}, \mathbf{b}, \mathbf{c}]$$

1.16.4 If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are parallel to the same plane, or if any two of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are parallel to each other,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$$

Proof: In the first case, $\mathbf{b} \times \mathbf{c}$ is perpendicular to $\mathbf{a}$. Hence,

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \underset{(1.14.1)}{=} 0$$

In the second case, the two parallel vectors can be put in adjoining positions, and the cross can then be placed between them. Use 1.15.3.

1.16.5 Using the same notation as in 1.15.8, $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ can be expressed in the following determinantal form:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Proof: Resolve each of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ into three mutually perpendicular components, and carry out the operations indicated by $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Expand the determinant given above. Compare the results.

Problem: Referring to Fig. 1.14.6b, evaluate $[\mathbf{F}, \mathbf{n}_1, \mathbf{n}_2]$.

Solution:

$$[\mathbf{F}, \mathbf{n}_1, \mathbf{n}_2] = \begin{vmatrix} 0 & -16 & 12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 12 \text{ lb}$$

1.17 The vector triple product of three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

The expression $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ denotes the cross product of the vectors $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$. The parentheses are essential because $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is not, in general, equal to $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

Problem: Evaluate (1) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and (2) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ with $\mathbf{a} = \mathbf{n}_1$, $\mathbf{b} = \mathbf{n}_2$, $\mathbf{c} = \mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3$, and $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ a right-handed set of mutually perpendicular unit vectors.

Solution (1):

$$\mathbf{b} \times \mathbf{c} = \mathbf{n}_2 \times (\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3) = -\mathbf{n}_3 + \mathbf{n}_1$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{n}_1 \times (-\mathbf{n}_3 + \mathbf{n}_1) = \mathbf{n}_2$$

Solution (2):

$$\mathbf{a} \times \mathbf{b} = \mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{n}_3$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{n}_3 \times (\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3) = \mathbf{n}_2 - \mathbf{n}_1$$

1.17.1 For any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \mathbf{c}\mathbf{b} - \mathbf{a} \cdot \mathbf{b}\mathbf{c}$$

Proof: Resolve each of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ into three mutually perpendicular components, carry out the indicated operations for both sides of the equation, then compare.

Problem (a): Given a vector $\mathbf{v}$ and a unit vector $\mathbf{n}$, show that $\mathbf{v}_P$, the resolute of $\mathbf{v}$ perpendicular to $\mathbf{n}$, is given by

$$\mathbf{v}_P = \mathbf{n} \times (\mathbf{v} \times \mathbf{n})$$

Solution:

$$\mathbf{n} \times (\mathbf{v} \times \mathbf{n}) = \mathbf{n} \cdot \mathbf{n}\mathbf{v} - \mathbf{n} \cdot \mathbf{v}\mathbf{n} = \mathbf{v} - \mathbf{n} \cdot \mathbf{v}\mathbf{n}$$

$$= \mathbf{v} - \underset{(1.14.6)}{\mathbf{v}_L} \underset{(1.12)}{=} \mathbf{v}_P$$

Problem (b): Show that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{c} \cdot \mathbf{a}\mathbf{b} - \mathbf{c} \cdot \mathbf{b}\mathbf{a}$$

Solution:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &\underset{(1.15.5)}{=} -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \\ &= -\,\mathbf{c}\cdot\mathbf{b}\mathbf{a} + \mathbf{c}\cdot\mathbf{a}\mathbf{b} \\ &\underset{(1.9.2)}{=} \mathbf{c}\cdot\mathbf{a}\mathbf{b} - \mathbf{c}\cdot\mathbf{b}\mathbf{a}\end{aligned}$$

2 CENTROIDS AND MASS CENTERS

2.1 The position vector of one point relative to another

Definition: The position vector of a point P relative to a point O is a vector $\mathbf{p}$ having the following characteristics:

Magnitude: The length of line OP

Orientation: Parallel to line OP

Sense: OP

When a sketch of $\mathbf{p}$ is drawn, it is customary to show $\mathbf{p}$ as an arrow connecting O to P (see Fig. 2.1).

FIG. 2.1

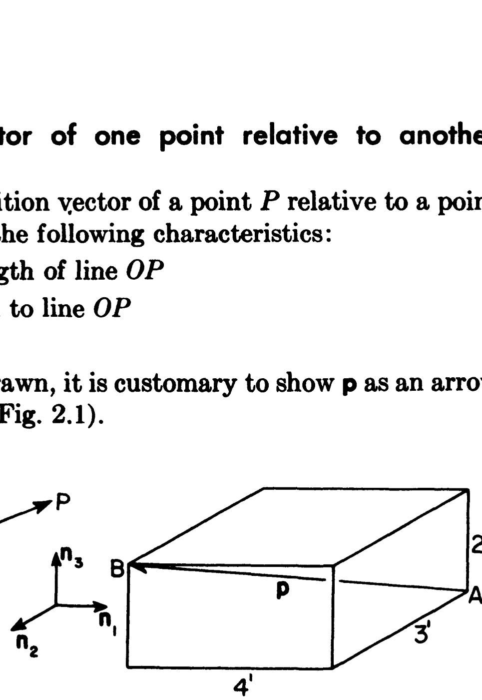


FIG. 2.1.1a

2.1.1 The position vector of a point P relative to P is a zero vector. Conversely, the only point whose position vector relative to a point P is a zero vector is P.

Problem (a): Referring to Fig. 2.1.1a, express (1) the position vector $\mathbf{p}$ of B relative to A and (2) the position vector $\mathbf{p}'$ of A relative to B in terms of their $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ components.

Solution (1):

$$\mathbf{p} = -4\mathbf{n}_1 + 3\mathbf{n}_2 + 2\mathbf{n}_3 \text{ ft}$$

Solution (2): $\mathbf{p}'$ differs from $\mathbf{p}$ only in sense, i.e.,

$$\mathbf{p}' = -\mathbf{p} = 4\mathbf{n}_1 - 3\mathbf{n}_2 - 2\mathbf{n}_3 \text{ ft}$$

Problem (b): Show that the distance d from a point P to a line L is given by

$$d = |\mathbf{n} \times \mathbf{p}|$$

where $\mathbf{n}$ is a unit vector parallel to L, and $\mathbf{p}$ is the position vector of P relative to any point O on L.

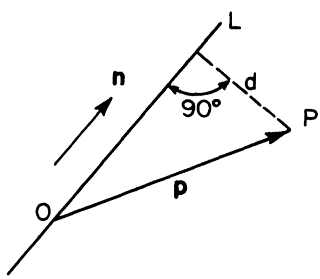


FIG. 2.1.1b

Solution (See Fig. 2.1.1b):

$$d = |\mathbf{p}| \sin(\mathbf{n}, \mathbf{p})$$
$$\underset{(1.7)}{=} |\mathbf{n}|\,|\mathbf{p}| \sin(\mathbf{n}, \mathbf{p})$$
$$\underset{(1.15.2)}{=} |\mathbf{n} \times \mathbf{p}|$$

2.2 The relationship between the position vector p and the coordinates x, y, z, of a point P

Notation (See Fig. 2.2a):

O, P	two points
X, Y, Z	oblique axes of a cartesian coordinate system whose origin is at O

x, y, z	the coordinates of P
$\mathbf{p}$	the position vector of P relative to O
$\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$	unit vectors, parallel to X, Y, Z, and having the senses of the positive X, Y, Z axes

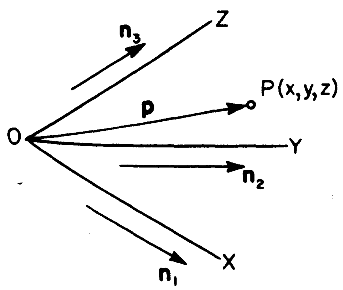


FIG. 2.2a

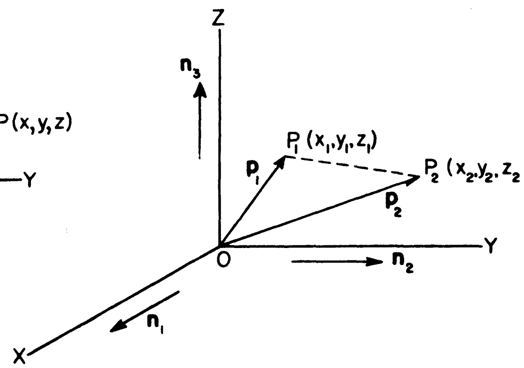


FIG. 2.2b

When $\mathbf{p}$ is resolved into three components respectively parallel to $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$, the $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ measure numbers of $\mathbf{p}$ (see 1.10.2) are respectively equal to x, y, z:

$$\mathbf{p} = x\mathbf{n}_1 + y\mathbf{n}_2 + z\mathbf{n}_3$$

This relationship is the link between vector analysis and scalar analytic geometry.

Problem: The rectangular cartesian coordinates of a point P_1 are x_1, y_1, z_1; those of a point P_2, x_2, y_2, z_2. Derive the distance formula of scalar analytic geometry; i.e., show that the distance d between P_1 and P_2 is given by

$$d = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{\frac{1}{2}}$$

Solution: Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the position vectors of P_1 and P_2 relative to the origin of the coordinate system, and let $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ be unit vectors such that (see Fig. 2.2b)

$$\mathbf{p}_1 = x_1\mathbf{n}_1 + y_1\mathbf{n}_2 + z_1\mathbf{n}_3$$

$$\mathbf{p}_2 = x_2\mathbf{n}_1 + y_2\mathbf{n}_2 + z_2\mathbf{n}_3$$

Then d is equal to the magnitude of the vector $\mathbf{p}_2 - \mathbf{p}_1$:

$$\begin{aligned} d &= |\mathbf{p}_2 - \mathbf{p}_1| \\ &= |(x_2 - x_1)\mathbf{n}_1 + (y_2 - y_1)\mathbf{n}_2 + (z_2 - z_1)\mathbf{n}_3| \\ &= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{\frac{1}{2}} \end{aligned}$$

2.3 The first moment of a point P with respect to a point O

If $\mathbf{p}$ is the position vector of a point P relative to a point O, and N is a scalar associated with P (e.g., the mass of a particle situated at P), the vector $N\mathbf{p}$ is called the first moment of P with respect to O. N is called the *strength* of P.

Problem: The point B shown in Fig. 2.1.1a has a strength of 10 slugs. Find the first moment of B with respect to A.

Solution: Let $\mathbf{p}$ be the position vector of B relative to A, N the strength of B. Then

$$\mathbf{p} = -4\mathbf{n}_1 + 3\mathbf{n}_2 - 2\mathbf{n}_3 \text{ ft}$$

and

$$N = 10 \text{ slug}$$

Hence,

$$N\mathbf{p} = -40\mathbf{n}_1 + 30\mathbf{n}_2 - 20\mathbf{n}_3 \text{ slug ft}$$

SETS OF POINTS

2.4 The centroid of a set of points

Notation (See Fig. 2.4):

S	a set of n points
$P_i,\ i = 1, 2, \ldots, n$	the points of S
$N_i,\ i = 1, 2, \ldots, n$	the strengths of the points of S; i.e., n scalars, all having the same dimensions, and each associated with one of the points of S

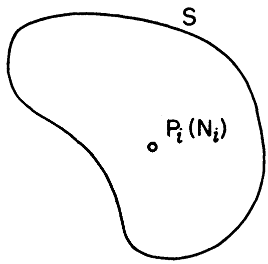


FIG. 2.4

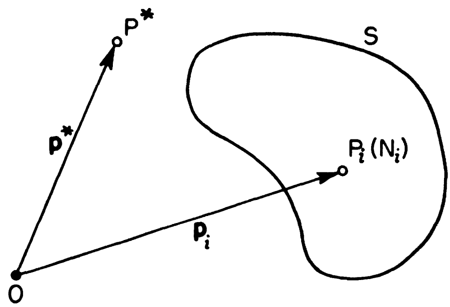


FIG. 2.4.1

Definition: The centroid of the set S is the point P^* with respect to which the sum of the first moments of the points of S is equal to zero.

2.4.1 The position vector $\mathbf{p}^*$ of P^*, relative to an arbitrarily selected reference point O, is given by

$$\mathbf{p}^* = \frac{\sum_{i=1}^{n} N_i \mathbf{p}_i}{\sum_{i=1}^{n} N_i}$$

where $\mathbf{p}_i$ is the position vector of P_i relative to O (see Fig. 2.4.1).

Proof: The position vector of P_i relative to P^* is $\mathbf{p}_i - \mathbf{p}^*$. The sum of the first moments of the points P_i with respect to P^* is $\sum_{i=1}^{n} N_i (\mathbf{p}_i - \mathbf{p}^*)$. If P^* is to be centroid of S, this sum must be equal to zero:

$$\sum_{i=1}^{n} N_i (\mathbf{p}_i - \mathbf{p}^*) = 0$$

Expand the left-hand member of this equation:

$$\sum_{i=1}^{n} N_i \mathbf{p}_i - \mathbf{p}^* \sum_{i=1}^{n} N_i = 0$$

Solve for $\mathbf{p}^*$:

$$\mathbf{p}^* = \frac{\sum_{i=1}^{n} N_i \mathbf{p}_i}{\sum_{i=1}^{n} N_i}$$

2.4.2 If

$$\sum_{i=1}^{n} N_i = 0$$

the centroid is not defined.

Problem (a): The points P_1 and P_2 shown in Fig. 2.4.2a have the strengths $N_1 = 2$ slugs and $N_2 = 6$ slugs. Locate their centroid P^*, and show it on a sketch.

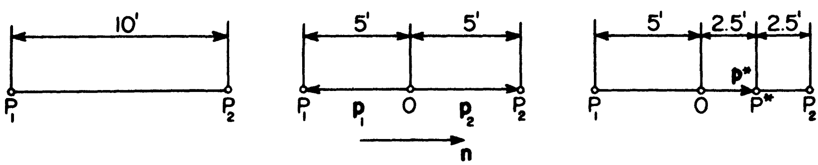


FIG. 2.4.2a FIG. 2.4.2b FIG. 2.4.2c

Solution: Introduce the following (see Fig. 2.4.2b):

- O the midpoint of line P_1P_2. (O is an arbitrarily selected reference point.)
- $\mathbf{p}_1$, $\mathbf{p}_2$ the position vectors of P_1 and P_2 relative to O
- P^* the centroid of the set P_1, P_2
- $\mathbf{p}^*$ the position vector of P^* relative to O
- $\mathbf{n}$ a unit vector parallel to line P_1P_2

Then

$$\mathbf{p}_1 = -5\mathbf{n} \text{ ft}, \quad \mathbf{p}_2 = 5\mathbf{n} \text{ ft}$$

and

$$\mathbf{p}^* = \frac{2(-5\mathbf{n}) + 6(5\mathbf{n})}{2+6}$$
$$= 2.5\mathbf{n} \text{ ft}$$

Result: See Fig. 2.4.2c.

Problem (b): The strengths of the points $P_1, \ldots, P_4$ shown in Fig. 2.4.2d are $N_1 = 2$, $N_2 = -3$, $N_3 = 5$, $N_4 = 3$. Locate the centroid of this set of points.

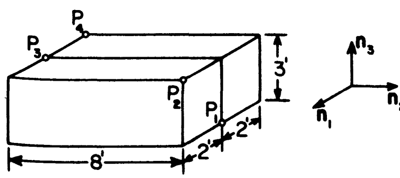


FIG. 2.4.2d

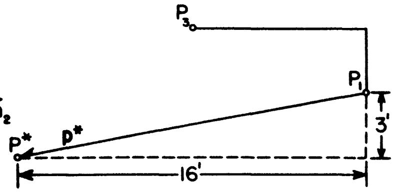


FIG. 2.4.2e

Solution: The strength of each point, the position vector of each point relative to the arbitrarily selected reference point P_1, and the first moment of each point with respect to P_1 are recorded in Table 2.4.2.

TABLE 2.4.3

Point	Strength	Position vector			First moment		
		n_1	n_2	n_3	n_1	n_2	n_3
P_1	2	0	0	0	0	0	0
P_2	−3	2	0	3	−6	0	−9
P_3	5	0	−8	3	0	−40	15
P_4	−3	−2	−8	3	6	24	−9
Totals	+1				0	−16	−3

$\mathbf{p}^*$, the position vector of the centroid P^* relative to P_1, is given by

$$\mathbf{p}^* = \frac{0\mathbf{n}_1 - 16\mathbf{n}_2 - 3\mathbf{n}_3}{1} = -16\mathbf{n}_2 - 3\mathbf{n}_3 \text{ ft}$$

Result: See Fig. 2.4.2e.

2.4.3 The centroid of a set of points of given strengths is a unique point: its location, as given in 2.4.1, is independent of the choice of reference point O.

Proof: Suppose that two distinct points, P^* and $\bar{P}^*$, were obtained when two different reference points, O and $\bar{O}$ (see Fig. 2.4.3), are used to locate the centroid. Then the vector $\mathbf{s}$, joining P^* to $\bar{P}^*$, would not be a zero vector (see 2.1.1). Conversely, if

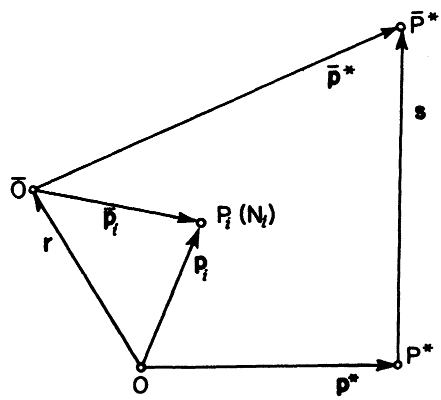


FIG. 2.4.3

$\mathbf{s} = 0$, then the points P^* and $\bar{P}^*$ coincide (see 2.1.1), and the centroid is a unique point.

The position vectors of P^* and $\bar{P}^*$ relative to O and $\bar{O}$ are, respectively, the vectors $\mathbf{p}^*$ and $\bar{\mathbf{p}}^*$ given by

$$\mathbf{p}^* = \frac{\sum_{i=1}^{n} N_i \mathbf{p}_i}{\sum_{i=1}^{n} N_i}, \qquad \bar{\mathbf{p}}^* = \frac{\sum_{i=1}^{n} N_i \bar{\mathbf{p}}_i}{\sum_{i=1}^{n} N_i}$$

From Fig. 2.4.3,

$$\mathbf{s} = \mathbf{r} + \bar{\mathbf{p}}^* - \mathbf{p}^*$$

Hence,

$$\mathbf{s} = \mathbf{r} + \frac{\left(\sum_{i=1}^{n} N_i \bar{\mathbf{p}}_i - \sum_{i=1}^{n} N_i \mathbf{p}_i\right)}{\sum_{i=1}^{n} N_i} = \mathbf{r} + \frac{\sum_{i=1}^{n} N_i (\bar{\mathbf{p}}_i - \mathbf{p}_i)}{\sum_{i=1}^{n} N_i}$$

But,

$$\bar{\mathbf{p}}_i - \mathbf{p}_i = -\mathbf{r}$$

Thus,

$$\mathbf{s} = \mathbf{r} + \frac{\sum_{i=1}^{n} N_i(-\mathbf{r})}{\sum_{i=1}^{n} N_i} = \mathbf{r} - \mathbf{r} = 0$$

2.4.4 If one assigns to the centroid of a set of points a strength equal to the sum of the strengths of the points of the set, then the first moment of the centroid with respect to any point is equal to the sum of the first moments of the points of the set with respect to this point. In this sense, the centroid can be regarded as representing the entire set of points.

Proof (see Fig. 2.4.4 for notation): The sum of the first moments of the n points P_i, $i = 1, \ldots, n$, with respect to O is equal

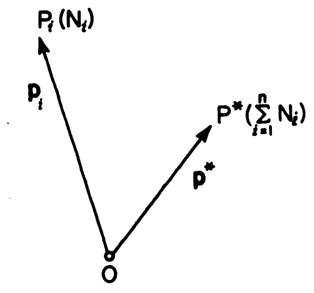


FIG. 2.4.4

to $\sum_{i=1}^{n} N_i \mathbf{p}_i$. The first moment of the point P^* (regarded as having the strength $\sum_{i=1}^{n} N_i$ with respect to O is given by

$$\left(\sum_{i=1}^{n} N_i\right)\mathbf{p}^* \underset{(2.4.1)}{=} \left(\sum_{i=1}^{n} N_i\right)\frac{\sum_{i=1}^{n} N_i \mathbf{p}_i}{\sum_{i=1}^{n} N_i} = \sum_{i=1}^{n} N_i \mathbf{p}_i$$

2.4.5 The cartesian coordinates of the centroid P^* of a set of points P_i, $i = 1, 2, \ldots, n$, of strengths N_i, $i = 1, \ldots, n$, are given by three expressions of the form

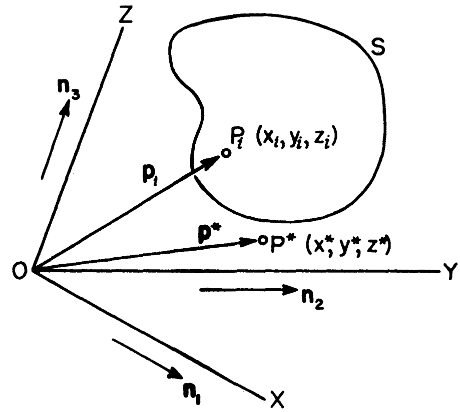


FIG. 2.4.5

$$x^* = \frac{\sum_{i=1}^{n} N_i x_i}{\sum_{i=1}^{n} N_i}$$

Proof (see Fig. 2.4.5 for notation):

$$\mathbf{p}_i \underset{(2.2)}{=} x_i\mathbf{n}_1 + y_i\mathbf{n}_2 + z_i\mathbf{n}_3, \quad \mathbf{p}^* \underset{(2.2)}{=} x^*\mathbf{n}_1 + y^*\mathbf{n}_2 + z^*\mathbf{n}_3$$

Substitute these into

$$\mathbf{p}^* = \frac{\sum_{i=1}^{n} N_i \mathbf{p}_i}{\sum_{i=1}^{n} N_i}$$

and write the three scalar equations corresponding to the vector equation thus obtained.

2.4.6 If the points of a set are arranged in such a way that corresponding to every point on one side of a certain plane there exists a point of equal strength on the other side, the two points being equidistant from the plane, but not necessarily lying on the same normal to it, then the centroid of the set lies in this plane. Such a plane is called a *plane of symmetry*.

Proof: Let the plane of symmetry be the X-Y plane of a rectangular cartesian coordinate system. Using the same notation as in 2.4.5, it must then be shown that $z^* = 0$, i.e., that $\sum_{i=1}^{n} N_i z_i = 0$. Now, this sum may contain terms arising from points which lie in the X-Y plane. The Z-coordinate of each such point is equal to zero; hence, these points contribute nothing to $\sum_{i=1}^{n} N_i z_i$. The remaining terms in $\sum_{i=1}^{n} N_i z_i$ can be grouped into sums of the type $Nz + N(-z)$, and each such sum is equal to zero.

Problem: Fig. 2.4.6 shows the strength and location of each of five points. Locate the centroid of this set of points by using symmetry considerations.

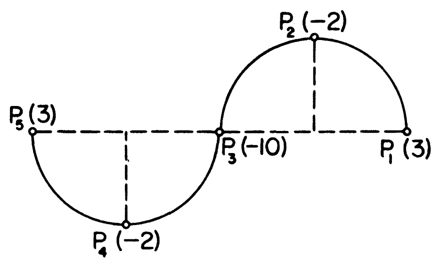


FIG. 2.4.6

Solution: The following planes are planes of symmetry: The plane passing through P_3 and normal to line P_1P_5; the plane in which the points lie, i.e., the plane of the paper; the plane passing through P_1 and P_5 and normal to the plane of the paper. The centroid must lie in each of these planes. Hence it coincides with their point of intersection, the point P_3.

2.4.7 A set S' of points is called a *subset* of a set S if every point of S' is a point of S. The centroid of a set S may be located as follows:

(a) Divide S into a number of subsets.

(b) Locate the centroid of each subset.

(c) Assign to each centroid a strength proportional to the sum of the strengths of the points of the corresponding subset.

(d) Locate the centroid of this set of centroids.

This method for locating the centroid is called the *method of decomposition.*

Proof: The vector $\mathbf{p}^*$ which locates P^* relative to an arbitrarily selected reference point O is obtained (see 2.4.1) by adding the first moments (see 2.3) of the points P_i and then dividing by the sum of the strengths of these points. Now, the sum of the first moments can be found by adding the first moments for all the points in each subset and then adding these sums. In accordance with 2.4.4, each of these sums is, however, equal to the first moment of the centroid of the corresponding subset. $\mathbf{p}^*$ is, therefore, equal to the sum of the first moments of the centroids of the subsets, divided by the sum of the strengths of the points of S. But the sum of the strengths of the points of S is equal to the sum of the strengths of the centroids of the subsets. Hence $\mathbf{p}^*$ is equal to the sum of the first moments of the centroids, divided by the

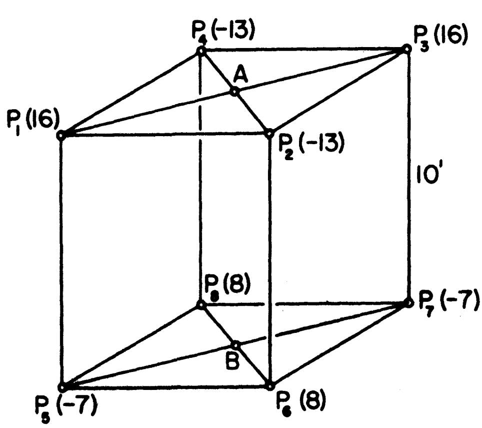


FIG. 2.4.7a

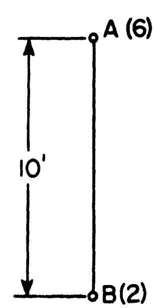


FIG. 2.4.7b

sum of the strengths of these centroids; but this is the vector which locates the centroid of the set of centroids.

Problem: The points $P_1, \ldots, P_8$ shown in Fig. 2.4.7a have the strengths indicated in parentheses. Locate the centroid of this set of points.

Solution: From symmetry considerations, the centroid of the subset $P_1, \ldots, P_4$ is known to be at point A; that of subset $P_5, \ldots, P_8$, at B. The points A and B, with strengths $16 - 13 + 16 - 13 = 6$ and $-7 + 8 - 7 + 8 = 2$, form a set of centroids (see Fig. 2.4.7b), and the centroid of this set lies on line AB, 2.5 ft from A, 7.5 ft from B.

CURVES, SURFACES, AND SOLIDS

2.5 The centroid of a curve, surface, or solid

Definition:

(a) Divide the curve, surface, or solid into n elements of arbitrary size and shape.

(b) Pick a point in each element.

(c) Assign to each point a strength proportional to the length, area, or volume of the corresponding element.

(d) Locate the centroid of the set of points.

(e) Find the point P^* which the centroid of the set of points approaches as n tends to infinity and each element shrinks to a point. P^* is the centroid of the curve, surface, or solid.

Problem: Locate the centroid of the semicircular curve shown in Fig. 2.5a.

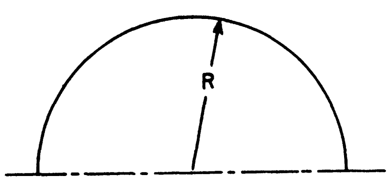


FIG. 2.5a

Solution:

(a) Division of the curve into n elements: Choose elements having equal arc lengths, $\pi R/n$. Note that each element subtends an angle of π/n radians at the center of the circle. Number the elements as shown in Fig. 2.5b.

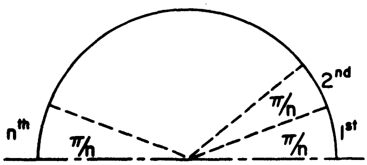


Fig. 2.5b

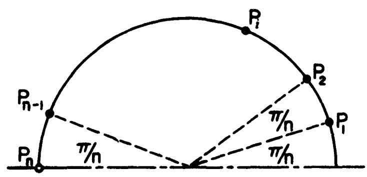


Fig. 2.5c

(b) Selection of a point in each element: use the left-most point of each element, calling these points $P_1, P_2, \ldots, P_n$, as shown in Fig. 2.5c. P_i is a typical point of this set of points.

(c) Strengths of the points P_i $(i = 1, \ldots, n)$: let N_i be the strength of P_i. The length of each element is $\pi R/n$, as noted in (a). Hence, all of the N_i $(i = 1, \ldots, n)$ must be taken equal to each other. Take

$$N_i = 1, i = 1, 2, \ldots, n \tag{1}$$

(d) Location of the centroid of the set of points P_i $(i = 1, \ldots, n)$: All of these points lie in a plane. This plane is, therefore, a plane of symmetry (see 2.4.6), and the centroid lies in this plane. Set up rectangular cartesian coordinate axes, X and Y, as shown in Fig. 2.5d, and let x_i and y_i be the coordinates of P_i. Note that angle $P_iOX = i\pi/n$ radians. Thus,

$$x_i = R\cos(i\pi/n), \quad y_i = R\sin(i\pi/n) \tag{2}$$

Let $\bar{x}$ and $\bar{y}$ be the coordinates of a point $\bar{P}$, the centroid of the set of points P_i $(i = 1, \ldots, n)$. Then

$$\bar{x} \underset{(2.4.5)}{=} \frac{\sum_{i=1}^{n} N_i x_i}{\sum_{i=1}^{n} N_i}, \quad \bar{y} \underset{(2.4.5)}{=} \frac{\sum_{i=1}^{n} N_i y_i}{\sum_{i=1}^{n} N_i} \tag{3}$$

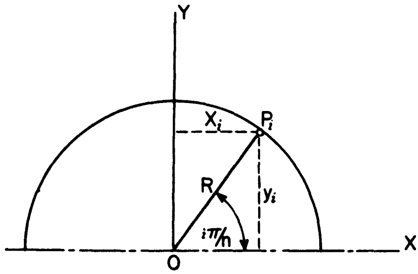


FIG. 2.5d

Use Eqs. (1) and (2):

$$\sum_{i=1}^{n} N_i x_i = \sum_{i=1}^{n} R \cos (i\pi/n) = R \sum_{i=1}^{n} \cos (i\pi/n)$$

$$\sum_{i=1}^{n} N_i y_i = \sum_{i=1}^{n} R \sin (i\pi/n) = R \sum_{i=1}^{n} \sin (i\pi/n)$$

$$\sum_{i=1}^{n} N_i = N_1 + N_2 + \dots + N_n = 1 + 1 + \dots + 1 = n$$

Substitute into Eqs. (3):

$$\bar{x} = \frac{R}{n} \sum_{i=1}^{n} \cos (i\pi/n), \quad \bar{y} = \frac{R}{n} \sum_{i=1}^{n} \sin (i\pi/n) \tag{4}$$

These results can be simplified as follows: For any angle θ not equal to 0, 2π, 4π, . . . radians, the sum $\sum_{i=1}^{n} \cos (i\theta)$ can be written (see W. E. Byerly, "Fourier Series and Spherical Harmonics," p. 32, Ginn & Co., Boston) as

$$\sum_{i=1}^{n} \cos (i\theta) = -\frac{1}{2} + \frac{1}{2} \frac{\sin [(2n + 1)\theta/2]}{\sin (\theta/2)}$$

and, similarly,

$$\sum_{i=1}^{n} \sin (i\theta) = \frac{\sin (n\theta/2) \sin [(n + 1)\theta/2]}{\sin (\theta/2)}$$

Hence, for $\theta = \pi/n$,

$$\sum_{i=1}^{n} \cos(i\theta) = \sum_{i=1}^{n} \cos(i\pi/n) = -\frac{1}{2} + \frac{1}{2}\frac{\sin(\pi + \pi/2n)}{\sin(\pi/2n)}$$

$$= -\frac{1}{2} - \frac{1}{2} = -1$$

$$\sum_{i=1}^{n} \sin(i\theta) = \sum_{i=1}^{n} \sin(i\pi/n) = \frac{\sin(\pi/2)\sin\left(\frac{\pi}{2} + \frac{\pi}{2n}\right)}{\sin(\pi/2n)}$$

$$= \operatorname{cotan}(\pi/2n)$$

Substitute into Eqs. (4):

$$\bar{x} = -\frac{R}{n}, \quad \bar{y} = \frac{R}{n}\operatorname{cotan}(\pi/2n) \tag{5}$$

The centroid $\bar{P}$ of the set of points P_i $(i = 1, \ldots, n)$ is shown in Fig. 2.5e.

(e) Limiting position of $\bar{P}$: as the elements were chosen in such a way that each element automatically shrinks to a point as

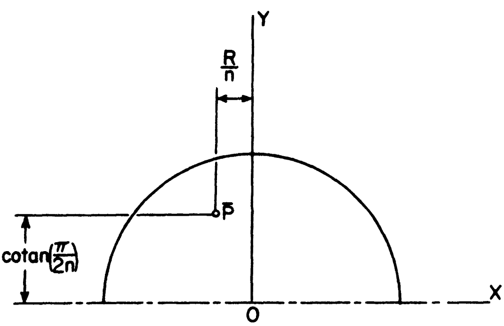


FIG. 2.5e

n tends to infinity, the coordinates x^* and y^* of the centroid of the curve are obtained by proceeding to the limit in Eq. (5), i.e.,

$$x^* = \lim_{n\to\infty} \bar{x} = \lim_{n\to\infty}\left(-\frac{R}{n}\right) = 0$$

$$y^* = \lim_{n\to\infty} \bar{y} = \lim_{n\to\infty}\left[\frac{R}{n}\operatorname{cotan}(\pi/2n)\right] = R\frac{\infty}{\infty}$$

To evaluate this indeterminate form, express cotan $(\pi/2n)$ as follows (see B. O. Peirce, "A Short Table of Integrals," p. 91, Ginn & Co., Boston): For any ϕ, such that $\phi^2 < \pi^2$,

$$\text{cotan } \phi = \frac{1}{\phi} - \frac{\phi}{3} - \frac{\phi^3}{45} - \ldots \text{ (odd powers of } \phi)$$

Hence, for $\phi = \pi/2n$,

$$\text{cotan } \phi = \text{cotan } (\pi/2n) = \frac{2n}{\pi} - \frac{1}{3}\frac{\pi}{2n} - \frac{1}{45}\left(\frac{\pi}{2n}\right)^3 - \ldots$$

Thus,

$$y^* = R \lim_{n\to\infty} \left[\frac{2}{\pi} - \frac{1}{3}\frac{1}{n}\frac{\pi}{2n} - \frac{1}{45}\frac{1}{n}\left(\frac{\pi}{2n}\right)^3 - \ldots\right] = R\left(\frac{2}{\pi}\right)$$

The centroid P^* of the semi-circular curve is shown in Fig. 2.5f.

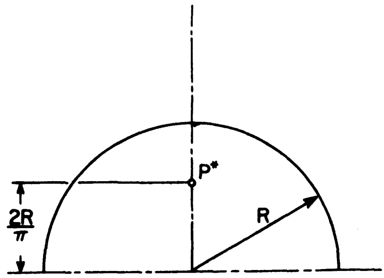


FIG. 2.5f

2.5.1 Problem 2.5 shows that it is possible to locate the centroid of a curve by performing the steps described in the definition. It also shows that, at least in this particular case, the process is arduous and requires the use of formulas which may not be readily available. The integral calculus furnishes a means for solving such problems in less cumbersome fashion.

Notation (See Fig. 2.5.1):

F	a curve, surface, or solid; any one of these is called a "figure"
τ	the total length, area, or volume of F
F_i, $i = 1, \ldots, n$	the n elements of F
τ_i	the length, area, or volume of F_i

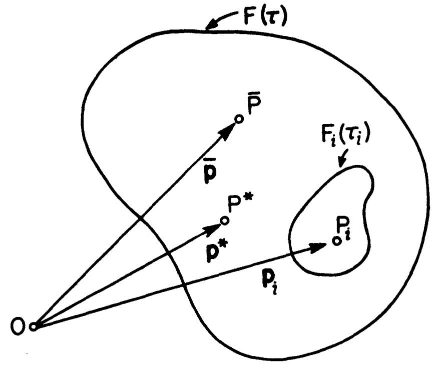


FIG. 2.5.1

P_i	a point of F_i
O	a point
$\mathbf{p}_i$	the position vector of P_i relative to 0
$\bar{P}$	the centroid of the set of points P_i, $i = 1, \ldots, n$, of strengths τ_i, $i = 1, \ldots, n$
$\bar{\mathbf{p}}$	the position vector of $\bar{P}$ relative to 0
P^*	the centroid of F
$\mathbf{p}^*$	the position vector of P^* relative to 0
$L[Q]$	a symbol denoting the limit approached by the quantity Q as n tends to infinity and each of the elements F_i, $i = 1, 2, \ldots, n$, shrinks to a point

In accordance with the definition in 2.5, $\mathbf{p}^*$ is given by

$$\mathbf{p}^* = L[\bar{\mathbf{p}}]$$

From 2.4.1,

$$\bar{\mathbf{p}} = \frac{\sum_{i=1}^{n} \tau_i \mathbf{p}_i}{\sum_{i=1}^{n} \tau_i}$$

Hence,

$$\mathbf{p}^* = L\left[\frac{\sum_{i=1}^{n} \tau_i \mathbf{p}_i}{\sum_{i=1}^{n} \tau_i}\right]$$

The theory of limits shows that limit of the quotient $\sum_{i=1}^{n} \tau_i \mathbf{p}_i / \sum_{i=1}^{n} \tau_i$ is equal to the quotient of the limits $L[\sum_{i=1}^{n} \tau_i \mathbf{p}_i]$ and $L[\sum_{i=1}^{n} \tau_i]$; that is,

$$\mathbf{p}^* = \frac{L\left[\sum_{i=1}^{n} \tau_i \mathbf{p}_i\right]}{L\left[\sum_{i=1}^{n} \tau_i\right]} \tag{1}$$

Each of the two limits in this expression is called an "integral over the figure F," and one writes

$$L\left[\sum_{i=1}^{n} \tau_i \mathbf{p}_i\right] \equiv \int_F \mathbf{p}\, d\tau \tag{2}$$

and

$$L\left[\sum_{i=1}^{n} \tau_i\right] \equiv \int_F d\tau \tag{3}$$

In these expressions, $\mathbf{p}$ denotes the position vector of a typical point of F, relative to 0, and $d\tau$ is the length, area, or volume of a differential element of F.

From Eqs. (1), (2) and (3),

$$\mathbf{p}^* = \frac{\int_F \mathbf{p}\, d\tau}{\int_F d\tau} \tag{4}$$

The integral $\int_F d\tau$ gives the total length, area, or volume of F, that is,

$$\int_F d\tau = \tau \tag{5}$$

(This follows from Eq. (3) and the fact that the total length, area, or volume of F is the sum of the lengths, areas, or volumes of the elements τ_i, $i = 1, \ldots, n$.)

Substitute from Eq. (5) into Eq. (4):

$$\mathbf{p}^* = \frac{1}{\tau} \int_F \mathbf{p}\, d\tau \tag{6}$$

Eq. (6) is of no use unless one knows how to evaluate the integral in the right-hand member. The properties of this integral can be studied by using 2.2 and Eq. (2) to reduce the integral to a sum involving integrals of scalar functions, the theory of integration of scalar functions being presumed known.

Introduce the following:

X, Y, Z	oblique axes of a cartesian coordinate system with origin at O
x_i, y_i, z_i	the coordinates of P_i
$\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$	unit vectors parallel to X, Y, Z and having the senses of the positive X, Y, Z axes

Then

$$\mathbf{p}_i \underset{(2.2)}{=} x_i\mathbf{n}_1 + y_i\mathbf{n}_2 + z_i\mathbf{n}_3$$

From Eq. (2),

$$\begin{aligned}\int_F \mathbf{p}\,d\tau &= L\left[\sum_{i=1}^{n} \tau_i(x_i\mathbf{n}_1 + y_i\mathbf{n}_2 + z_i\mathbf{n}_3)\right]\\ &= L\left[\left(\sum_{i=1}^{n} \tau_i x_i\right)\mathbf{n}_1 + \left(\sum_{i=1}^{n} \tau_i y_i\right)\mathbf{n}_2 + \left(\sum_{i=1}^{n} \tau_i z_i\right)\mathbf{n}_3\right]\\ &= L\left[\left(\sum_{i=1}^{n} \tau_i x_i\right)\mathbf{n}_1\right] + L\left[\left(\sum_{i=1}^{n} \tau_i y_i\right)\mathbf{n}_2\right] + L\left[\left(\sum_{i=1}^{n} \tau_i z_i\right)\mathbf{n}_3\right]\end{aligned}$$

$\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are not affected by the limiting process:

$$\int_F \mathbf{p}\,d\tau = \mathbf{n}_1 L\left[\sum_{i=1}^{n} \tau_i x_i\right] + \mathbf{n}_2 L\left[\sum_{i=1}^{n} \tau_i y_i\right] + \mathbf{n}_3 L\left[\sum_{i=1}^{n} \tau_i z_i\right]$$

Each of the three limits in this expression is the integral of a scalar function:

$$L\left[\sum_{i=1}^{n} \tau_i x_i\right] = \int_F x\,d\tau, \text{ etc.} \tag{7}$$

where x, y, z are the coordinates of a typical point of F. Thus

$$\int_F \mathbf{p}\,d\tau = \mathbf{n}_1 \int_F x\,d\tau + \mathbf{n}_2 \int_F y\,d\tau + \mathbf{n}_3 \int_F z\,d\tau \tag{8}$$

Let x^*, y^*, z^* be the coordinates of P^*. Then

$$\mathbf{p}^* = x^*\mathbf{n}_1 + y^*\mathbf{n}_2 + z^*\mathbf{n}_3 \tag{9}$$

Substitute from Eqs. (8) and (9) into Eq. (6), and write the three scalar equations corresponding to the resulting vector equation:

$$x^* = \frac{1}{\tau}\int_F x\,d\tau, \quad y^* = \frac{1}{\tau}\int_F y\,d\tau, \quad z^* = \frac{1}{\tau}\int_F z\,d\tau \tag{10}$$

2.5.2 If the coordinate axes X, Y, Z are mutually perpendicular, then $|x^*|$, $|y^*|$, $|z^*|$ are the distances from the centroid to the coordinate planes.

Problem (a): Locate the centroid of the semicircular curve shown in Fig. 2.5.2a.

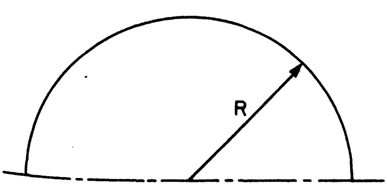


FIG. 2.5.2a

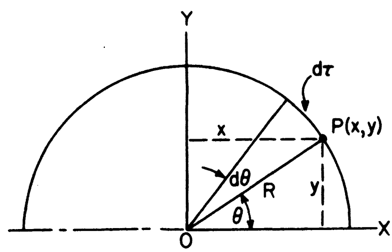


FIG. 2.5.2b

Solution: Introduce the following (see Fig. 2.5.2b):

X, Y	rectangular axes
P	a typical point of the curve
x, y	the coordinates of P
θ	the angle POX
x^*, y^*	the coordinates of the centroid
$d\tau$	the length of a differential element of the curve

Express x, y, $d\tau$ in terms of θ:

$$x = R\cos\theta, \quad y = R\sin\theta, \quad d\tau = R\,d\theta$$

Then

$$\tau = \int_F d\tau = \int_0^\pi R\,d\theta = \pi R$$

$$\int_F x\,d\tau = \int_0^\pi (R\cos\theta)\,R\,d\theta = 0$$

$$\int_F y\,d\tau = \int_0^\pi (R\sin\theta)\,R\,d\theta = 2R^2$$

$$x^* = \frac{1}{\tau}\int_F x\,d\tau = \frac{0}{\pi R} = 0$$

$$y^* = \frac{1}{\tau}\int_F y\,d\tau = \frac{2R^2}{\pi R} = \frac{2R}{\pi}$$

Result: The point P^* shown in Fig. 2.5f is the centroid of the semicircular curve.

Problem (b): Find the distance from the centroid of a hemispherical surface of radius R to the plane determined by the circular boundary of the surface.

Solution: Introduce the quantities shown in Fig. 2.5.2c. Let

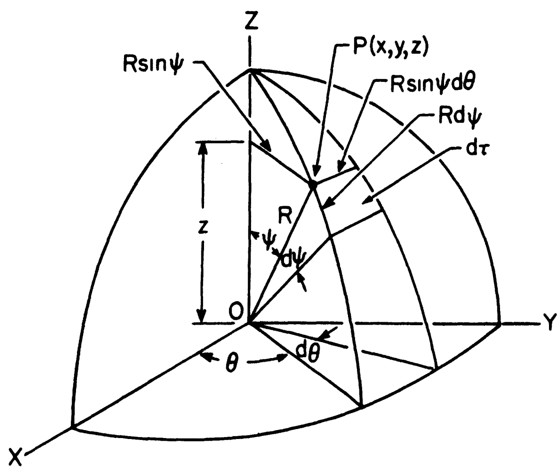


FIG. 2.5.2c

z^* be the distance from the centroid to the X-Y plane. This is the desired distance.

Express z and $d\tau$ in terms of θ and ψ:

$$z = R\cos\psi, \quad d\tau = (R\sin\psi\,d\theta)(R\,d\psi) = R^2\sin\psi\,d\theta\,d\psi$$

Then

$$
\begin{aligned}
\tau &= \int_F d\tau = \int_0^{2\pi} d\theta \int_0^{\pi/2} R^2 \sin\psi \, d\psi \\
&= \int_0^{2\pi} \Big[-R^2 \cos\psi\Big]_0^{\pi/2} d\theta \\
&= \int_0^{2\pi} [-R^2(0-1)]\, d\theta = R^2\theta \Big|_0^{2\pi} = 2\pi R^2
\end{aligned}
$$

$$
\begin{aligned}
\int_F z\, d\tau &= \int_0^{2\pi} d\theta \int_0^{\pi/2} R^3 \sin\psi \cos\psi \, d\psi \\
&= \int_0^{\pi/2} \left[-\frac{R^3}{4}\cos 2\psi\right]_0^{\pi/2} d\theta \\
&= \int_0^{2\pi} \left[-\frac{R^3}{4}(-1-1)\right] d\theta = \frac{R^3}{2}\theta \Big|_0^{2\pi} = \pi R^3
\end{aligned}
$$

$$
z^* = \frac{1}{\tau}\int_F z\, d\tau = \frac{\pi R^3}{2\pi R^2} = \frac{R}{2}
$$

Problem (c): Find the distance from the centroid of a hemispherical solid of radius R to its plane boundary.

Solution: Introduce the quantities shown in Fig. 2.5.2d. Let z^* be the distance from the centroid to the X-Y plane. This is the desired distance.

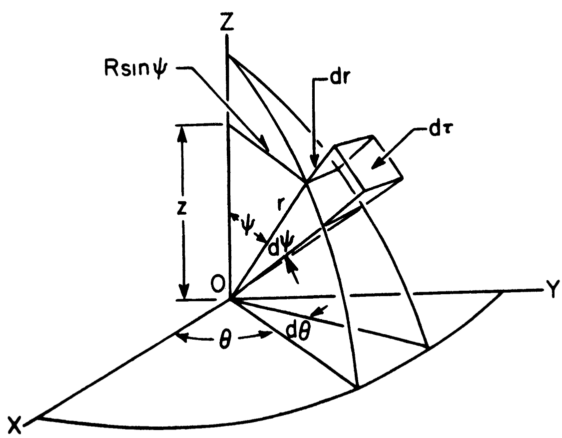


FIG. 2.5.2d

Express z and $d\tau$ in terms of r, θ and ψ:

$$z = r \cos \psi$$

$$d\tau = (r \sin \psi \, d\theta)(r \, d\psi) \, dr = r^2 \sin \psi \, d\theta \, d\psi \, dr$$

Then

$$\tau = \int_F d\tau = \int_0^R dr \int_0^{2\pi} d\theta \int_0^{\pi/2} r^2 \sin \psi \, d\psi = \frac{2\pi R^3}{3}$$

$$\int_F z \, d\tau = \int_0^R dr \int_0^{2\pi} d\theta \int_0^{\pi/2} r^3 \sin \psi \cos \psi \, d\psi = \frac{\pi R^4}{4}$$

$$z^* = \frac{1}{\tau} \int_F z \, d\tau = \frac{\pi R^4/4}{2\pi R^3/3} = \frac{3R}{8}$$

2.5.3 The centroids of a number of figures, located by performing calculations similar to those in the above examples, are shown in the Appendix. In subsequent sections it is shown how these results may be used to locate, without integration, the centroids of many more figures.

2.5.4 If, corresponding to every point of a figure on one side of a certain plane, there exists a point of the figure on the other side, the two points being equidistant from the plane, but not necessarily lying on the same normal to it, then the centroid of the figure lies in this plane. Such a plane is called a *plane of symmetry*.

Proof: Similar to the proof in 2.4.6.

Problem: Locate the centroid of the S-shaped curve shown in Fig. 2.5.4.

Solution: Each of the three coordinate planes is a plane of symmetry. Hence the centroid lies in each of these planes, i.e., it coincides with their point of intersection, the point O.

2.5.5 A *method of decomposition* (see 2.4.7) may be used to locate the centroid of a figure F:

(a) Divide F into a number of "contributing" figures.

(b) Locate the centroid of each figure.

(c) Assign to each centroid a strength proportional to the length, area, or volume of the corresponding figure.

(d) Locate the centroid of this set of centroids.

Proof: Similar to the proof in 2.4.7.

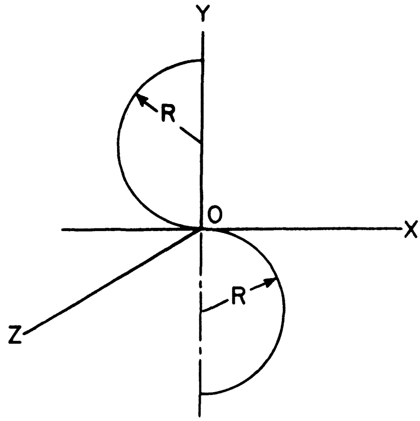


FIG. 2.5.4

2.5.6 The centroid of a figure which is composed of more than one type of figure—e.g., of curves and surfaces, or surfaces and solids—is not defined.

Problem: Determine x^*, the X coordinate of the centroid of the plane surface shown in Fig. 2.5.6a.

Solution: Call the rectangular portion of the figure F_1, the semicircular sector F_2. Use symmetry considerations and the Appendix to locate P_1^* and P_2^*, the centroids of F_1 and F_2 (see Fig. 2.5.6b). Determine the areas, A_1 and A_2, of F_1 and F_2:

$$A_1 = 4 \times 6 = 24 \text{ in.}^2$$

$$A_2 = \frac{\pi(3)^2}{2} = 14.1 \text{ in.}^2$$

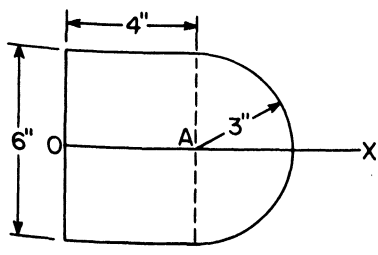


FIG. 2.5.6a

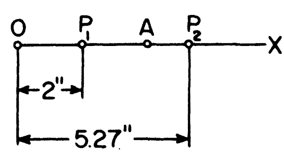


FIG. 2.5.6b

Assign to P_1^* the strength 24, to P_2^* the strength 14.1. Then

$$x^* = \frac{2 \times 24 + 5.27 \times 14.1}{24 + 14.1} = 3.21 \text{ in.}$$

2.5.7 Sometimes it is convenient to regard one or more of the contributing figures (see 2.5.5) as contributing negatively.

Problem: Determine x^*, the X coordinate of the shaded plane surface shown in Fig. 2.5.7a.

Solution: Regard the figure as being composed of the rectangle F_1 and the semicircular sector F_2, as shown in Fig. 2.5.7b. Use

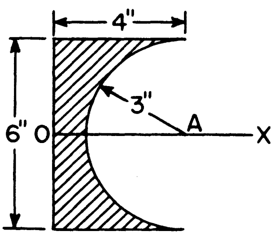


FIG. 2.5.7a

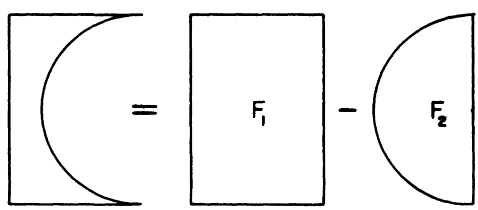


FIG. 2.5.7b

symmetry considerations and the Appendix to locate P_1^* and P_2^*, the centroids of F_1 and F_2 (see Fig. 2.5.7c). Find the areas, A_1 and A_2, of F_1 and F_2:

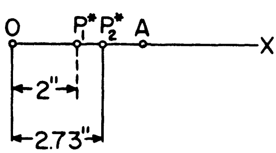
$$A_1 = 24 \text{ in.}^2$$

$$A_2 = 14.1 \text{ in.}^2$$

Assign to P_1^* the strength 24, to P_2^* the strength -14.1. Then

$$x^* = \frac{2 \times 24 + 2.73 \times (-14.1)}{24 \times (-14.1)} = 0.96 \text{ in.}$$

FIG. 2.5.7c



2.5.8 A surface S (see Fig. 2.5.8a) which can be generated by letting a plane curve C move in such a way that the points of C are displaced equal amounts h along lines perpendicular to the plane of C is called a *right-cylindrical surface.* Its centroid lies on a line which passes through the centroid of C and is perpendicular to the plane of C.

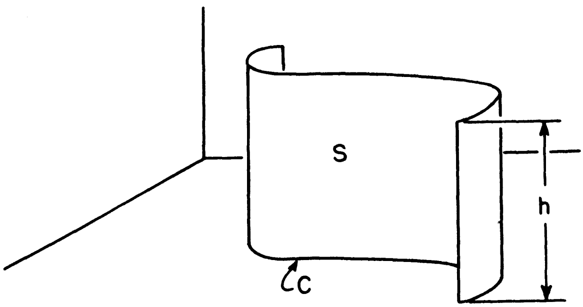


FIG. 2.5.8a

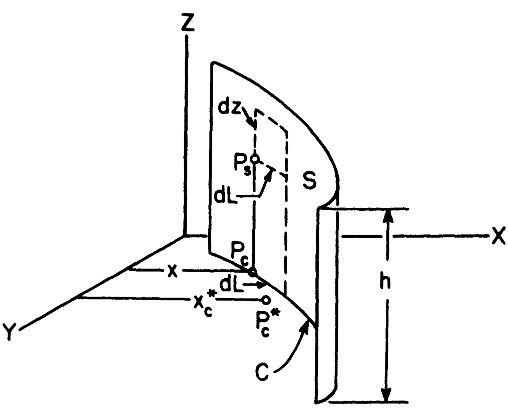


FIG. 2.5.8b

Proof: Introduce the following (see Fig. 2.5.8b):

X, Y, Z rectangular cartesian coordinate axes, the X-Y plane containing the curve C

$P_c{}^*$, $P_s{}^*$ the centroids of C and S

x_c^*, x_s^*	the X coordinates of P_c^* and P_s^*
P_c, P_s	typical points of C and S
x	the X coordinate of either P_c or P_s
z	the Z coordinate of P_s
L	the length of the curve C
A	the area of the surface S
dL	the length of a differential element of C
dA	the area of a differential element of S

It will be shown that $x_s^* = x_c^*$.
From 2.5.1, Eq. (5),

$$A = \int_s dA, \quad L = \int_c dL \tag{1}$$

From Fig. 2.5.8b,

$$dA = dz\, dL \tag{2}$$

Hence,

$$A \underset{(1)}{=} \int_s dA \underset{(2)}{=} \int_c \left[\int_0^h dz\right] dL = h \int_c dL \underset{(1)}{=} hL \tag{3}$$

From 2.5.1, Eq. (10),

$$x_s^* = \frac{1}{A}\int_s x\, dA, \quad x_c^* = \frac{1}{L}\int_c x\, dL \tag{4}$$

Evaluate $\int_s x\, dA$:

$$\int_s x\, dA \underset{(2)}{=} \int_c \left[\int_0^h x\, dz\right] dL = h \int_c x\, dL \tag{5}$$

Substitute from Eqs. (3) and (5) into Eq. (4):

$$x_s^* = \frac{1}{hL} h \int_c x\, dL \underset{(4)}{=} x_c^*$$

Similarly it can be shown that the Y coordinates of P_c^* and P_s^* are equal to each other and, furthermore, that the Z coordinate of P_s^* is given by

$$z_s^* = h/2$$

Problem: Determine the coordinates x^*, y^*, z^* of the centroid of the surface shown in Fig. 2.5.8c. (X, Y, Z are mutually perpendicular axes.)

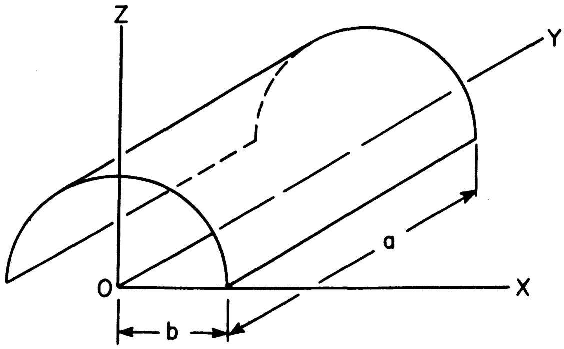


FIG. 2.5.8c

Solution: The surface is a right-cylindrical surface obtained by moving a semicircle of radius b parallel to the Z-X plane. From the Appendix, the centroid of the semicircle is the point $(0, 0, 2b/\pi)$. Hence, the X and Z coordinates of the centroid of the surface are

$$x^* = 0, \quad z^* = 2b/\pi$$

The Y coordinate of the centroid is

$$y^* = a/2$$

2.5.9 A *right-cylindrical solid* is a solid bounded by a right-cylindrical surface (see 2.5.8) and by two parallel plane surfaces. Each of the two plane surfaces is called a *base* of the solid (see Fig. 2.5.9a). The centroid of a cylindrical solid is the midpoint of the line which connects the centroids of the bases.

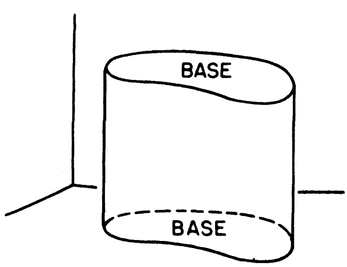


FIG. 2.5.9a

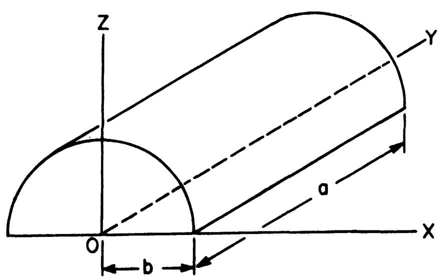


FIG. 2.5.9b

Proof: Similar to the proof in 2.5.8.

Problem: Determine the coordinates x^*, y^*, z^* of the centroid of the solid shown in Fig. 2.5.9b. (X, Y, Z are mutually perpendicular axes.)

Solution: The solid is a right-cylindrical solid whose bases are semicircular sectors of radius b. From the Appendix, the centroids of the bases are at the points $(0, 0, 4b/3\pi)$ and $(0, a, 4b/3\pi)$. The midpoint of the line joining these two points is the point $(0, a/2, 4b/3\pi)$. Hence,

$$x^* = 0, \quad y^* = a/2, \quad z^* = 4b/3\pi$$

SETS OF PARTICLES

2.6 The mass center of a set of particles

Definition: The mass center of a set of particles is the centroid of the set of points at which the particles are situated, the strength of each point being taken equal to the mass of the corresponding particle.

Problem: Particles of masses 2 slugs and 6 slugs are situated at the points P_1 and P_2 shown in Fig. 2.6. Locate their mass center.

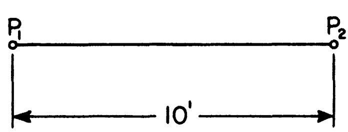


FIG. 2.6

Solution: From Problem 2.4.1(a) it follows that the mass center lies on line $P_1 P_2$, 7.5 ft from P_1, 2.5 ft from P_2.

CONTINUOUS BODIES

2.7 The mass center of a continuous body

Definition: The mass center of a continuous body is a point located by means of a limiting process similar to that described in 2.5:

(a) Divide the body into n elements of arbitrary size and shape.

(b) Pick a point in each element.

(c) Assign to each point a strength proportional to the mass of the corresponding element.

(d) Locate the centroid of the set of points.

(e) Find the point P^* which the centroid of the set of points approaches as n tends to infinity and each element shrinks to a point. P^* is the mass center of the body.

Proceeding as in 2.5.1, the following relationships, analogous to 2.5.1, Eqs. (5), (6) and (10), are obtained:

$$m = \int_F \rho \, d\tau \tag{1}$$

$$\mathbf{p}^* = \frac{1}{m} \int_F \mathbf{p}\rho \, d\tau \tag{2}$$

$$x^* = \frac{1}{m} \int_F x\rho \, d\tau, \quad y^* = \frac{1}{m} \int_F y\rho \, d\tau, \quad z^* = \frac{1}{m} \int_F z\rho \, d\tau \tag{3}$$

where the symbols are defined as follows (see Fig. 2.7a):

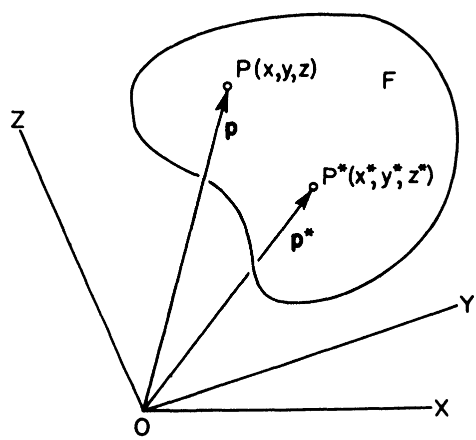


FIG. 2.7a

F	the figure (curve, surface, solid) occupied by the body
P	a typical point of the body
O	a point
$\mathbf{p}$	the position vector of P relative to O

ρ the mass density of the body at P. If F is a curve, ρ is the mass per unit of length; if F is a surface, the mass per unit of area; if F is a solid, the mass per unit of volume

$d\tau$ the length, area, or volume of a differential element of F

m the total mass of the body

P^* the mass center of the body

$\mathbf{p}^*$ the position vector of P^* relative to O

X, Y, Z oblique axes of a cartesian coordinate system with origin at O

x, y, z the coordinates of P

x^*, y^*, z^* the coordinates of P^*

Problem: ρ, the mass per unit of length of a thin semicircular wire, is given by

$$\rho = (1 + 0.5\theta) \times 10^{-3} \text{ slug ft}^{-1}$$

where θ is the angle shown in Fig. 2.7b. Locate the mass center of the wire.

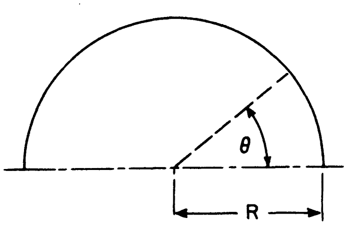


FIG. 2.7b

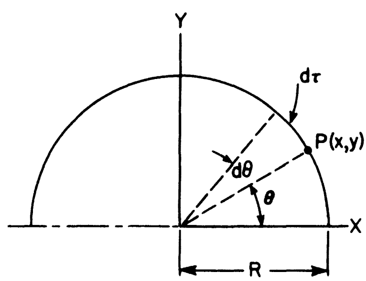


FIG. 2.7c

Solution: Introduce the following (see Fig. 2.7c):

X, Y rectangular axes

P a typical point on the semicircle

x, y the coordinates of P

x^*, y^* the coordinates of the mass center

$d\tau$ the length of a differential element of the semicircle

Express x, y, $d\tau$ in terms of θ:

$$x = R\cos\theta, \quad y = R\sin\theta, \quad d\tau = R\,d\theta$$

Then

$$m = \int_F \rho\, d\tau = \int_0^\pi (1 + 0.5\theta) \times 10^{-3} R\, d\theta$$

$$= 10^{-3} R(1 + 0.25\pi)\pi$$

Next,

$$\int_F x\rho\, d\tau = \int_0^\pi R \cos\theta\, (1 + 0.5\theta) \times 10^{-3} R\, d\theta = -10^{-3}R^2$$

and

$$\int_F y\rho\, d\tau = \int_0^\pi R \sin\theta\, (1 + 0.5\theta) \times 10^{-3} R\, d\theta = 10^{-3} R^2(2 + 0.5\pi)$$

Hence,

$$x^* = \frac{1}{m}\int_F x\rho\, d\tau = \frac{-10^{-3} R^2}{10^{-3} R(1 + 0.25\pi)\pi} = -0.178R$$

and

$$y^* = \frac{1}{m}\int_F y\rho\, d\tau = \frac{10^{-3} R^2(2 + 0.5\pi)}{10^{-3} R(1 + 0.25\pi)\pi} = \frac{2R}{\pi}$$

2.7.1 The mass center of a continuous body does not, in general, coincide with the centroid of the figure occupied by the body. (Compare the results obtained in Problem 2.7 with those of Problem 2.5.2(a).)

Problem: Problem 2.7 is rather academic, because it is not the sort of problem most frequently encountered in practical situations. Generally, the mass density of a body is not given as a function of the coordinates of a point of the body, and this function must, therefore, be constructed by the analyst. For example, Fig. 2.7.1 shows the cross-section of a steel shell whose inner

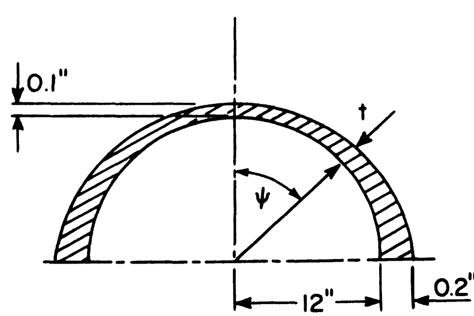


FIG. 2.7.1

surface is hemispherical, and whose wall-thickness t varies linearly with the angle ψ. Locate the mass center of the shell.

Solution: Regard the shell as matter distributed with variable density on a hemispherical surface of radius $R = 12$ in. This is an approximation which may be expected to give good results if the thickness of the shell is sufficiently small in comparison with the radius.

Let P be a typical point on the hemispherical surface, and introduce the quantities shown in Fig. 2.5.2c. Express the wall-thickness t in terms of ψ: As t varies linearly with ψ, t is given by

$$t = \alpha + \beta\psi$$

where the constants α and β can be evaluated by noting that $t = 0.1$ for $\psi = 0$, and $t = 0.2$ for $\psi = \pi/2$:

$$t_{\psi=0} = \alpha = 0.1 \text{ in.,} \quad t_{\psi=\pi/2} = \alpha + \beta\pi/2 = 0.2 \text{ in.}$$

$$\beta = \frac{2}{\pi}(0.2 - \alpha) = \frac{2}{\pi}(0.2 - 0.1) = \frac{0.2}{\pi} \text{ in.}$$

Hence,

$$t = 0.1 + \frac{0.2}{\pi}\psi$$

Assume that the steel of which the shell is made has the same properties at all points, and take ρ, the mass per unit of area of the inner surface at point P, proportional to the thickness at P:

$$\rho = kt = k\left(0.1 + \frac{0.2}{\pi}\psi\right)$$

Find the mass m of the shell:

$$m = \int_F \rho\, d\tau = \int_0^{2\pi} d\theta \int_0^{\pi/2} k\left(0.1 + \frac{0.2}{\pi}\psi\right) R^2 \sin\psi\, d\psi$$

$$= 2\pi k\, R^2(0.1 + 0.2/\pi)$$

Evaluate $\int_F z\rho\, d\tau$:

$$\int_F z\rho\, d\tau = \int_0^{2\pi} d\theta \int_0^{\pi/2} (R\cos\psi)\left[k\left(0.1 + \frac{0.2}{\pi}\psi\right)\right] R^2 \sin\psi\, d\psi$$

$$= 0.15\pi k\, R^3$$

The Z coordinate of the mass center of the shell is given by

$$z^* = \frac{1}{m}\int_F z\rho\, d\tau = \frac{R}{2}\left(\frac{1.5\pi}{2+\pi}\right)$$

For $R = 12$ in.,

$$z^* = 5.5 \text{ in.}$$

This is the distance from the mass center of the shell to the X-Y plane.

The approximate result here obtained can be compared with those corresponding to other types of approximation. For example, a rather "rough" approximation is that of neglecting the fact that the shell has a variable thickness and regarding all of the matter contained in the shell to be distributed uniformly on a hemispherical surface of radius 12 in. This leads to $z^* = 6$ in.

2.7.2 If the mass density of a body is the same at all points of the body, the density, as well as the body, are said to be ***uniform***. The mass center of a uniform body coincides with the centroid of the figure occupied by the body.

Proof: Introduce the following symbols (see Fig. 2.7.2):

B a uniform body
F the figure occupied by B
ρ the mass density of B at all points of F
τ the length, area, or volume of F
m the mass of B

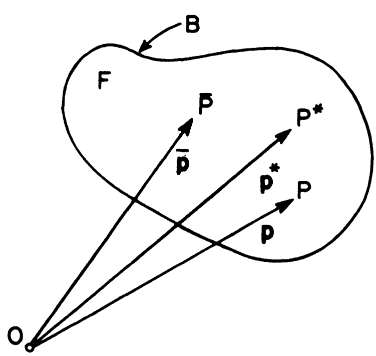


FIG. 2.7.2

P^* the mass center of B
$\bar{P}$ the centroid of F
P a typical point of F
O a point
$\mathbf{p}^*$ the position vector of P^* relative to O
$\bar{\mathbf{p}}$ the position vector of $\bar{P}$ relative to O
$\mathbf{p}$ the position vector of P relative to O

It must be shown that $\mathbf{p}^* = \bar{\mathbf{p}}$. Now,

$$\mathbf{p}^* = \frac{1}{m}\int_F \mathbf{p}\rho \, d\tau, \quad \bar{\mathbf{p}} = \frac{1}{\tau}\int_F \mathbf{p} \, d\tau$$

If ρ is independent of the position of P, then

$$\int_F \mathbf{p}\rho \, d\tau = \rho \int_F \mathbf{p} \, d\tau$$

while

$$m = \int_F \rho \, d\tau = \rho \int_F d\tau = \rho\tau$$

Hence,

$$\mathbf{p}^* = \frac{1}{\rho\tau}\rho \int_F \mathbf{p} \, d\tau = \frac{1}{\tau}\int_F \mathbf{p} \, d\tau = \bar{\mathbf{p}}$$

2.7.3 A *method of decomposition* may be used to locate the mass center of a body B:

(a) Divide B into a number of "contributing" bodies. (These bodies may be of various types, i.e., particles, bodies occupying curves, surfaces or solids. Compare this statement with 2.5.6.)

(b) Locate the mass center of each body.

(c) Assign to each mass center a strength proportional to the mass of the corresponding body (e.g., the weight of the body).

(d) Locate the centroid of this set of mass centers.

Proof: Similar to the proof in 2.4.7.

Problem: Parts A, B, C, D of the body shown in Fig. 2.7.3a are made of the following materials:

A steel (489 lb/ft^3)
B sheet metal (3.5 lb/ft^2)
C aluminum (169 lb/ft^3)
D brass (527 lb/ft^3)

Find x^*, the X coordinate of the mass center of the body.

Solution: Construct Table 2.7.3, by analyzing each of the seven bodies shown in Fig. 2.7.3b.

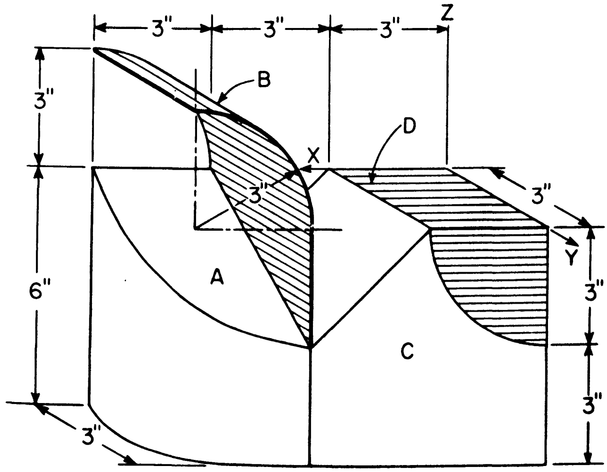


FIG. 2.7.3a

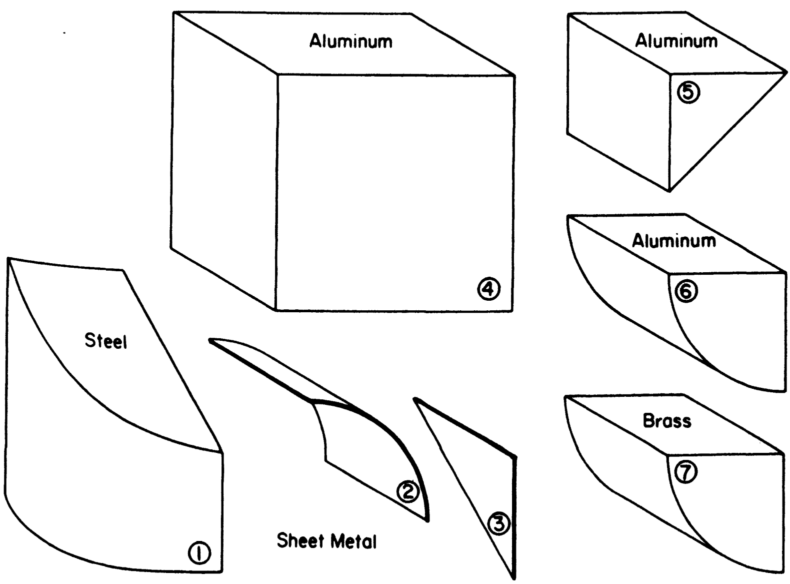


FIG. 2.7.3b

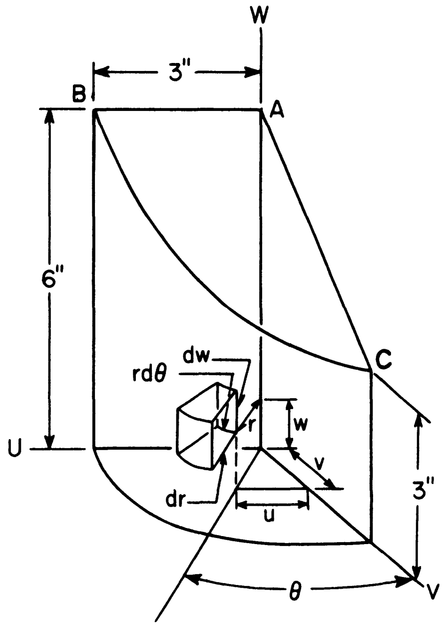


FIG. 2.7.3c

Body 1. (See 2.7.2, 2.5.1 and Fig. 2.7.3c.)

$$u^* = \frac{1}{\tau}\int_F u\,d\tau; \quad \tau = \int_F d\tau$$

$$u = r\sin\theta, \quad v = r\cos\theta$$

$$d\tau = r\,d\theta\,dr\,dw$$

The equation of the plane passing through the points A, B and C is

$$w = 6 - v = 6 - r\cos\theta$$

Hence,

$$\tau = \int_F d\tau = \int_0^{\pi/2} d\theta \int_0^3 dr \int_0^{6-r\cos\theta} r\,dw = 33.4 \text{ in.}^3$$

$$\int_F u\,d\tau = \int_0^{\pi/2} d\theta \int_0^3 dr \int_0^{6-r\cos\theta} r^2\sin\theta\,dw = 33.8 \text{ in.}^4$$

$$u^* = \frac{33.8}{33.4} = 1.01 \text{ in.}$$

X coordinate of mass center:

$$6 + 1.01 = 7.01 \text{ in.}$$

Specific weight:

$$\frac{489 \text{ lb ft}^{-3}}{1728 \text{ in.}^3 \text{ ft}^{-3}} = 0.283 \text{ lb in.}^{-3}$$

Strength:

$$(33.4 \text{ in.}^3)(0.283 \text{ lb in.}^{-3}) = 9.45 \text{ lb}$$

Body 2. (See 2.7.2 and 2.5.8.) Use the Appendix to locate the centroid P^* of the arc AB shown in Fig. 2.7.3d. The result is shown in Fig. 2.7.3e.

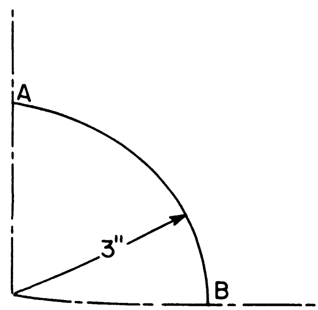


FIG. 2.7.3d

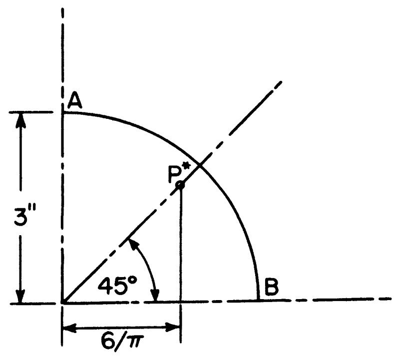


FIG. 2.7.3e

X coordinate of mass center:

$$9 - \frac{2 \times 3}{\pi} = 7.09 \text{ in.}$$

Area (see 2.5.8, Eq. (3)):

$$A = 3\left(\frac{3\pi}{2}\right) = 14.1 \text{ in.}^2$$

Specific weight:

$$\frac{3.5 \text{ lb ft}^{-2}}{144 \text{ in.}^2 \text{ ft}^{-2}} = 0.0243 \text{ lb in.}^{-2}$$

Strength:

$$(14.1 \text{ in.}^2)(0.0243 \text{ lb in.}^{-2}) = 0.343 \text{ lb}$$

Body 3. (See 2.5.4 and note that the plane of the figure is a plane of symmetry.)

X coordinate of the mass center: 6 in.
Specific weight: 0.0243 lb in.$^{-2}$
Area: 4.5 in.^{2}

Strength:

$$4.5 \times 0.0243 = 0.109 \text{ lb}$$

Body 4. Use 2.7.2 and 2.5.4.
Bodies 5, 6, 7. Use 2.7.2 and 2.5.9.

TABLE 2.7.3

Body	Area or volume	Specific weight	Strength (weight)	X coordinate of mass center	First moment
1	33.4	0.283	9.45	7.01	66.20
2	14.1	0.0243	0.343	7.09	2.43
3	4.5	0.0243	0.109	6.00	0.65
4	108.0	0.0978	10.6	3.00	31.80
5	13.5	0.0978	−1.32	5.00	−6.60
6	21.2	0.0978	−2.07	1.91	−3.96
7	21.2	0.305	6.46	1.91	12.30
			23.53 lb		102.82 in. lb

The last column in Table 2.7.3 contains the products of the numbers in the preceding two columns. In accordance with 2.4.5,

$$x^* = \frac{102.82}{23.53} = 4.37 \text{ in.}$$

3 MOMENTS AND COUPLES

3.1 The moment of a bound vector about a point

Notation (see Fig. 3.1a):

$\mathbf{v}$	a bound vector (see Sec. 1.2)
L	the line of action of $\mathbf{v}$
A	a point
B	any point on line L
$\mathbf{p}$	the position vector of B relative to A
$\mathbf{M}^{\mathbf{v}/A}$	the moment of $\mathbf{v}$ about A

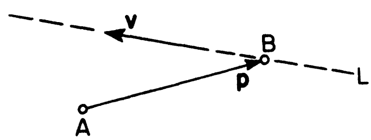


FIG. 3.1a

Definition:

$$\mathbf{M}^{\mathbf{v}/A} = \mathbf{p} \times \mathbf{v}$$

Problem: Referring to Fig. 3.1b, determine the moment of the force $\mathbf{F}$ about point S.

Solution: Let $\mathbf{p}$ be the position vector of R relative to S. Then

$$\mathbf{p} = 3\mathbf{n}_1 \text{ ft}$$

In accordance with 1.7.3,

$$\mathbf{F} = -5\mathbf{n}_3 \text{ lb}$$

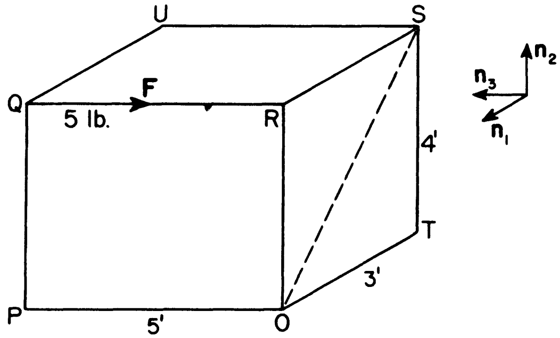


FIG. 3.1b

Hence,

$$\mathbf{M}^{\mathbf{F}/S} = \mathbf{p} \times \mathbf{F} = 3\mathbf{n}_1 \times (-5\mathbf{n}_3)$$
$$= -15\mathbf{n}_1 \times \mathbf{n}_3 = 15\mathbf{n}_2 \text{ ft lb}$$

3.1.1 $\mathbf{M}^{A/\mathbf{v}}$ is a free vector, i.e., a vector associated neither with a definite line nor with a definite point.

3.1.2 If the line of action of a vector $\mathbf{v}$ passes through a point P, or if $\mathbf{v}$ is a zero vector,

$$\mathbf{M}^{\mathbf{v}/P} = 0$$

Conversely, if

$$\mathbf{M}^{\mathbf{v}/P} = 0$$

then either the line of action of $\mathbf{v}$ passes through P, or $\mathbf{v}$ is a zero vector. This follows from 1.15.3 and the definition of $\mathbf{M}^{\mathbf{v}/P}$.

3.2 The moment of a bound vector about a line

Definition: The moment $\mathbf{M}^{\mathbf{v}/L}$ of a bound vector $\mathbf{v}$ about a line L is the L resolute (see 1.12) of the moment of $\mathbf{v}$ about any point on L.

Problem: Referring to Fig. 3.1b, determine the moment of the force $\mathbf{F}$ about line OS.

Solution: S is a point on line OS. $\mathbf{M}^{\mathbf{F}/S}$ was found in Problem 3.1. Fig. 3.2 shows $\mathbf{M}^{\mathbf{F}/S}$ and a unit vector $\mathbf{n}$ parallel to line OS. The $\mathbf{n}$ resolute of $\mathbf{M}^{\mathbf{F}/S}$ is given by

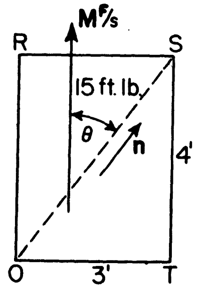


FIG. 3.2

$$\mathbf{M}^{\mathbf{F}/OS} = 15 \cos\theta\, \mathbf{n} = 15(\tfrac{4}{5})\mathbf{n} = 12\mathbf{n} \text{ ft lb}$$

3.2.1 The moment of a vector about a line is a free vector.

3.2.2 The magnitude of $\mathbf{M}^{\mathbf{v}/L}$ is given by

$$|\mathbf{M}^{\mathbf{v}/L}| = |[\mathbf{n}, \mathbf{p}, \mathbf{v}]|$$

where $\mathbf{n}$ is a unit vector parallel to L, and $\mathbf{p}$ is the position vector of a point on the line of action of $\mathbf{v}$ relative to a point on L.

Proof (see Fig. 3.2.2): $\mathbf{M}^{\mathbf{v}/L}$ is the L resolute of $\mathbf{M}^{\mathbf{v}/A}$. Hence,

$$\mathbf{M}^{\mathbf{v}/L} \underset{(1.14.6)}{=} \mathbf{n}\cdot\mathbf{M}^{\mathbf{v}/A}\mathbf{n}$$

$$\underset{(3.1)}{=} \mathbf{n}\cdot(\mathbf{p}\times\mathbf{v})\mathbf{n}$$

$$\underset{(1.16)}{=} [\mathbf{n}, \mathbf{p}, \mathbf{v}]\mathbf{n}$$

and

$$|\mathbf{M}^{\mathbf{v}/L}| \underset{(1.7.4)}{=} |[\mathbf{n}, \mathbf{p}, \mathbf{v}]|$$

Problem: Referring to Fig. 3.1b, determine the magnitude of the moment of the force $\mathbf{F}$ about line OS.

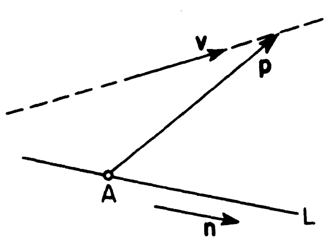


FIG. 3.2.2

Solution: R is a point on the line of action of $\mathbf{F}$, S a point on line OS. Let $\mathbf{p}$ be the position vector of R relative to S, and let $\mathbf{n}$ be a unit vector parallel to OS and having the sense SO. Then

$$\mathbf{p} = 3\mathbf{n}_1 \text{ ft}, \quad \mathbf{F} = -5\mathbf{n}_3 \text{ lb}$$

$$\mathbf{n} = \tfrac{1}{5}(3\mathbf{n}_1 - 4\mathbf{n}_2)$$

and

$$[\mathbf{n}, \mathbf{p}, \mathbf{F}] \underset{(1.15.9)}{=} \begin{vmatrix} 3 & 0 & 0 \\ 0 & 0 & -5 \\ \frac{3}{5} & -\frac{4}{5} & 0 \end{vmatrix} = -12 \text{ ft lb}$$

Hence

$$|\mathbf{M}^{\mathbf{F}/S}| = |[\mathbf{n}, \mathbf{p}, \mathbf{F}]| = |-12| = 12 \text{ ft lb}$$

3.2.3 If a line L intersects the line of action of $\mathbf{v}$, then

$$\mathbf{M}^{\mathbf{v}/L} = 0$$

Proof: Referring to 3.2.2, $\mathbf{p}$ can be chosen in such a way that

$$\mathbf{p} = 0$$

Then

$$[\mathbf{n}, \mathbf{p}, \mathbf{v}] = [\mathbf{n}, 0, \mathbf{v}] = 0$$

and

$$\mathbf{M}^{\mathbf{v}/L} \underset{(1.7.3)}{=} [\mathbf{n}, \mathbf{p}, \mathbf{v}]\mathbf{n} = 0$$

3.2.4 If a line L is parallel to the line of action of a vector $\mathbf{v}$, then

$$\mathbf{M}^{\mathbf{v}/L} = 0$$

Proof: Using the notation of 3.2.2,

$$\mathbf{M}^{\mathbf{v}/L} = [\mathbf{n}, \mathbf{p}, \mathbf{v}]\mathbf{n} \underset{(1.16.4)}{=} 0$$

3.2.5 If a line L is perpendicular to the line of action of a vector $\mathbf{v}$, and s is the shortest distance between these two lines, then

$$|\mathbf{M}^{\mathbf{v}/L}| = s|\mathbf{v}|$$

Proof (see Fig. 3.2.5, noting that $\mathbf{n}$ is perpendicular to both L and $\mathbf{v}$):

$$|\mathbf{M}^{\mathbf{v}/L}| \underset{(3.2.2)}{=} |[\mathbf{n}, \mathbf{p}, \mathbf{v}]| \underset{(1.16)}{=} |\mathbf{n}\cdot(\mathbf{p} \times \mathbf{v})|$$

$$\underset{(1.15)}{=} |\mathbf{n}\cdot(|\mathbf{p}|\,|\mathbf{v}| \sin(\mathbf{p}, \mathbf{v})\mathbf{n})| = |\mathbf{p}|\,|\mathbf{v}|$$

$$= s|\mathbf{v}|$$

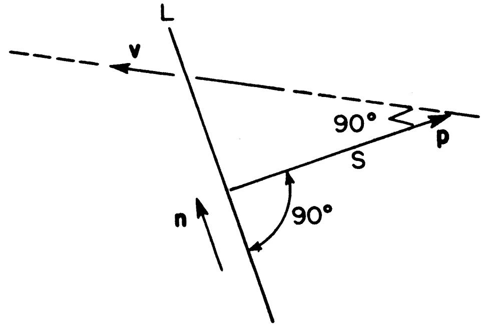


FIG. 3.2.5

Problem: Referring to Fig. 3.1b, determine the magnitude of the moment of the force **F** about line OT.

Solution: OR is the common perpendicular to the line of action of **F** and the line OT. Its length is 4 ft. Hence,

$$|\mathbf{M}^{\mathbf{F}/OT}| = (4)(5) = 20 \text{ ft lb}$$

3.2.6 When the line of action of a vector **v** is perpendicular to a line L, the direction of $\mathbf{M}^{\mathbf{v}/L}$ can readily be determined by inspection: $\mathbf{M}^{\mathbf{v}/L}$ is parallel to L and has the same sense as the vector $\mathbf{p} \times \mathbf{v}$ (see Fig. 3.2.5). Using 3.2.5, one can, therefore, evaluate $\mathbf{M}^{\mathbf{v}/L}$ without explicitly performing any dot or cross multiplications, whenever the line of action of **v** is perpendicular to L.

Problem: Referring to Fig. 3.1b, determine the moment of the force **F** about line ST.

Solution:

$$\mathbf{M}^{\mathbf{F}/ST} = (3)(5)\mathbf{n}_2 = 15\mathbf{n}_2 \text{ ft lb}$$

3.2.7 The moment of a bound vector about a point is equal to the sum of the moments of the vector about three mutually perpendicular lines which intersect at the point.

Proof: Let **v** be the vector, P the point, L_i, $i = 1, 2, 3$ the mutually perpendicular lines intersecting at P.

When $\mathbf{M}^{\mathbf{v}/P}$ is resolved into components respectively parallel to the three lines, the component parallel to L_i is equal to the L_i

resolute of $\mathbf{M}^{\mathbf{v}/P}$ (see 1.14.10); but the L_i resolute of $\mathbf{M}^{\mathbf{v}/P}$ is, by definition (see 3.2), the moment of $\mathbf{v}$ about L_i. Hence

$$\mathbf{M}^{\mathbf{v}/P} = \mathbf{M}^{\mathbf{v}/L_1} + \mathbf{M}^{\mathbf{v}/L_2} + \mathbf{M}^{\mathbf{v}/L_3}$$

Example: Problem 3.1 may be solved as follows:

$$\mathbf{M}^{\mathbf{F}/S} = \mathbf{M}^{\mathbf{F}/SR} + \mathbf{M}^{\mathbf{F}/ST} + \mathbf{M}^{\mathbf{F}/SU}$$

$$= \underset{(3.2.3)}{0} + \underset{(3.2.6)}{15\mathbf{n}_2} + \underset{(3.2.4)}{0}$$

Hence,

$$\mathbf{M}^{\mathbf{F}/S} = 15\mathbf{n}_2 \text{ ft lb}$$

3.2.8 When a bound vector $\mathbf{v}$ is resolved into n components $\mathbf{v}_i$, $i = 1, \ldots, n$, whose lines of action pass through one point on the line of action of $\mathbf{v}$, the moment of $\mathbf{v}$ about any point or line is equal to the sum of the moments of the vectors $\mathbf{v}_i$, $i = 1, \ldots, n$, about that point or line.

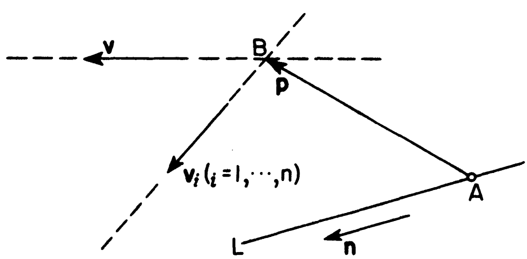


FIG. 3.2.8a

Proof: (see Fig. 3.2.8a): It must be shown that

$$\mathbf{M}^{\mathbf{v}/A} = \sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/A} \tag{1}$$

and that

$$\mathbf{M}^{\mathbf{v}/L} = \sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/L}$$

Now,

$$\mathbf{M}^{\mathbf{v}_i/A} \underset{(3.1)}{=} \mathbf{p} \times \mathbf{v}_i$$

Hence,

$$\sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/A} = \sum_{i=1}^{n} \mathbf{p} \times \mathbf{v}_i \underset{(1.15.6)}{=} \mathbf{p} \times \sum_{i=1}^{n} \mathbf{v}_i \underset{(1.10)}{=} \mathbf{p} \times \mathbf{v} \underset{(3.1)}{=} \mathbf{M}^{\mathbf{v}/A}$$

Next,

$$\mathbf{M}^{\mathbf{v}/L} \underset{(3.2,1.14.6)}{=} \mathbf{n} \cdot \mathbf{M}^{\mathbf{v}/A}\mathbf{n} \quad \text{and} \quad \mathbf{M}^{\mathbf{v}_i/L} = \mathbf{n} \cdot \mathbf{M}^{\mathbf{v}_i/A}\mathbf{n} \tag{2}$$

Dot multiply both sides of Eq. (1) with $\mathbf{n}$:

$$\mathbf{n} \cdot \mathbf{M}^{\mathbf{v}/A} = \mathbf{n} \cdot \sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/A} \underset{(1.14.3)}{=} \sum_{i=1}^{n} \mathbf{n} \cdot \mathbf{M}^{\mathbf{v}_i/A}$$

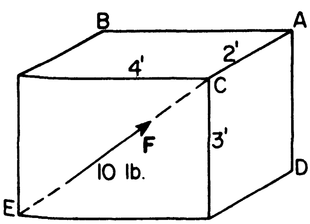
Multiply both sides of this equation with $\mathbf{n}$:

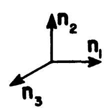
$$\mathbf{n} \cdot \mathbf{M}^{\mathbf{v}/A}\mathbf{n} = \left(\sum_{i=1}^{n} \mathbf{n} \cdot \mathbf{M}^{\mathbf{v}_i/A}\right)\mathbf{n} \underset{(1.9.3)}{=} \sum_{i=1}^{n} (\mathbf{n} \cdot \mathbf{M}^{\mathbf{v}_i/A}\mathbf{n})$$

Use Eqs. (2):

$$\mathbf{M}^{\mathbf{v}/L} = \sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/L}$$

Problem (a): Referring to Fig. 3.2.8b, find the moment of the force $\mathbf{F}$ about line AB.





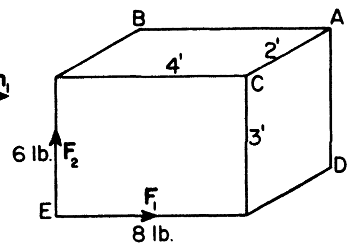


FIG. 3.2.8b

FIG. 3.2.8c

Solution: Resolve $\mathbf{F}$ into two components, $\mathbf{F}_1$ and $\mathbf{F}_2$, one parallel to line AB, the other parallel to line AD, and both having lines of action which pass through the same point E on the line of action of $\mathbf{F}$ (see Fig. 3.2.8c). Then

$$\mathbf{M}^{\mathbf{F}/AB} = \mathbf{M}^{\mathbf{F}_1/AB} + \mathbf{M}^{\mathbf{F}_2/AB}$$
$$= \underset{(3.2.4)}{0} + \underset{(3.2.6)}{(2)(6)}(-\mathbf{n}_1)$$
$$= -12\mathbf{n}_1 \text{ ft lb}$$

Problem (b): Fig. 3.2.8d shows five forces, drawn to a scale of 1 ft = 10 lb (for example, $|\mathbf{F}_4| = (10)(3) = 30$ lb). The force $\mathbf{F}$ is the resultant of the four forces $\mathbf{F}_1, \ldots, \mathbf{F}_4$ and has been drawn,

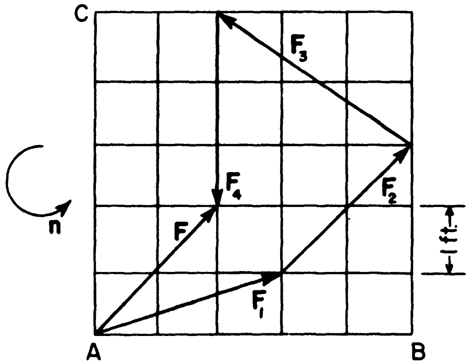


FIG. 3.2.8d

arbitrarily, with a line of action passing through A. Find (1) the moment of $\mathbf{F}$ about A and (2) the sum of the moments of the forces $\mathbf{F}_1, \ldots, \mathbf{F}_4$ about A. ($\mathbf{n}$ is a unit vector.)

Solution (1):

$$\mathbf{M}^{\mathbf{F}/A} \underset{(3.1.2)}{=} 0$$

Solution (2): Find the moment of each force about point A, then add these moments:

$$\mathbf{M}^{\mathbf{F}_1/A} \underset{(3.1.2)}{=} 0$$
$$\mathbf{M}^{\mathbf{F}_2/A} \underset{(\text{F3.2.8e})}{=} (1)(20)(-\mathbf{n}) + 3(20)\mathbf{n} = 40\mathbf{n}$$
$$\mathbf{M}^{\mathbf{F}_3/A} \underset{(\text{F3.2.8f})}{=} (3)(30)\mathbf{n} + 5(20)\mathbf{n} = 190\mathbf{n}$$
$$\mathbf{M}^{\mathbf{F}_4/A} \underset{(\text{F3.2.8d})}{=} (2)(30)(-\mathbf{n}) = -60\mathbf{n}$$
$$\mathbf{M}^{\mathbf{F}_1/A} + \ldots + \mathbf{M}^{\mathbf{F}_4/A} = (40 + 190 - 60)\mathbf{n} = 170\mathbf{n} \text{ ft lb}$$

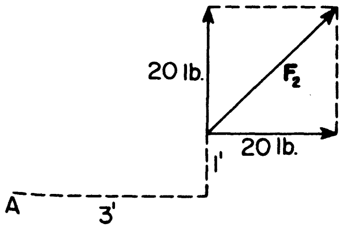


FIG. 3.2.8e

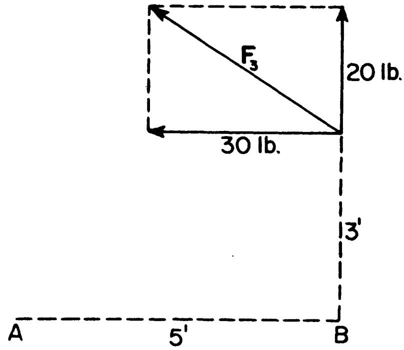


FIG. 3.2.8f

3.2.9 Problem 3.2.8(b) shows that the sum of the moments of the components of a vector $\mathbf{v}$ is not necessarily equal to the moment of $\mathbf{v}$, or, in other words, it demonstrates the importance of that part of 3.2.8 which states that the lines of action of the components of $\mathbf{v}$ must all pass through *one* point on the line of action of $\mathbf{v}$.

Problem: For the forces shown in Fig. 3.2.8d, locate the point P on line AB (or line AB extended) at which the line of action of the resultant of $\mathbf{F}_1, \ldots, \mathbf{F}_4$ must intersect line AB if the moment of this resultant about point A is to be equal to the sum of the moments of $\mathbf{F}_1, \ldots, \mathbf{F}_4$ about A.

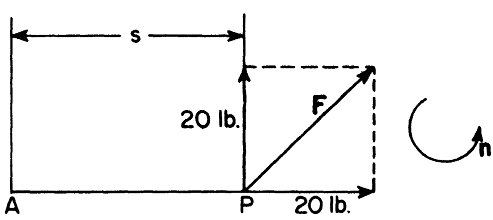


FIG. 3.2.9

Solution: Let s be the unknown distance AP, find the moment of $\mathbf{F}$ about A (see Fig. 3.2.9), and set this moment equal to $170\mathbf{n}$ ft lb:

$$\mathbf{M}^{F/A} = 20s\mathbf{n} = 170\mathbf{n}$$

The scalar equation corresponding to this vector equation is

$$20s = 170$$

Hence,

$$s = 8.5 \text{ ft}$$

and P is 8.5 ft to the right of point A.

3.2.10 Problem 3.2.9 shows that it may be possible to place the resultant of a system of vectors in such a way that the moment of the resultant about a certain point is equal to the sum of the moments of the vectors about that point. Problem 3.2.10 will demonstrate that it is not always possible to do this.

Problem: Fig. 3.2.10 shows three forces, drawn to a scale of 1 ft = 10 lb (for example, $|\mathbf{F}_2| = 50$ lb). Letting $\mathbf{F}$ be the resultant

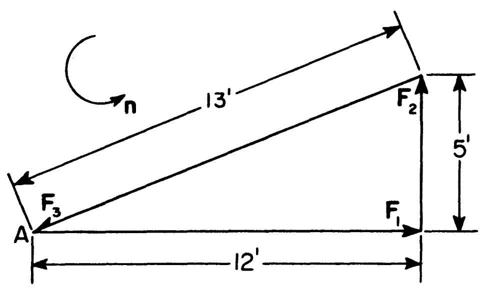


FIG. 3.2.10

of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, find (1) the moment of $\mathbf{F}$ about point A and (2) the sum of the moments of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ about A.

Solution (1):

$$\mathbf{F} \underset{(1.9)}{=} 0$$

Hence the moment of the resultant about point A (and about all other points) is equal to zero (see 3.1.2).

Solution (2):

$$\mathbf{M}^{\mathbf{F}_1/A} = 0, \quad \mathbf{M}^{\mathbf{F}_2/A} = (12)(50)\mathbf{n}, \quad \mathbf{M}^{\mathbf{F}_3/A} = 0$$

Hence,

$$\mathbf{M}^{\mathbf{F}_1/A} + \mathbf{M}^{\mathbf{F}_2/A} + \mathbf{M}^{\mathbf{F}_3/A} = 600\mathbf{n} \text{ ft lb}$$

and, for the purpose of taking moments about point A, it is impossible to replace the given force system with its resultant.

3.3 Moments of a system of bound vectors

Notation:

S	a system of bound vectors
$\mathbf{v}_i,\ i = 1, 2, \ldots, n$	the n vectors of S
P	a point
L	a line
$\mathbf{M}^{S/P}$	the moment of S about P
$\mathbf{M}^{S/L}$	the moment of S about L

Definitions:

$$\mathbf{M}^{S/P} = \sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/P}$$

$$\mathbf{M}^{S/L} = \sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/L}$$

Problem: Find the moment of the system S of three forces shown in Fig. 3.3 (1) about point A and (2) about line AB.

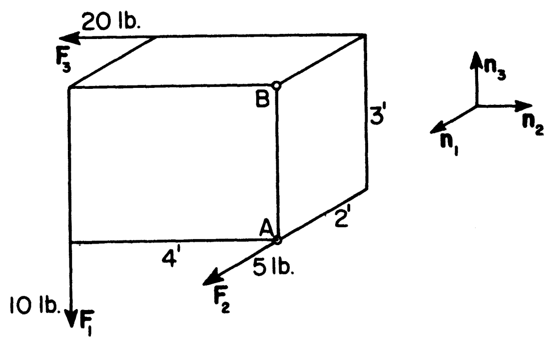


FIG. 3.3

Solution (1):

$$\begin{aligned} \mathbf{M}^{S/A} &= \mathbf{M}^{\mathbf{F}_1/A} + \mathbf{M}^{\mathbf{F}_2/A} + \mathbf{M}^{\mathbf{F}_3/A} \\ &= 40\mathbf{n}_1 + 0 + 60\mathbf{n}_1 + 40\mathbf{n}_3 \\ &= 100\mathbf{n}_1 + 40\mathbf{n}_3 \text{ ft lb} \end{aligned}$$

Solution (2):

$$\begin{aligned} \mathbf{M}^{S/AB} &= \mathbf{M}^{\mathbf{F}_1/AB} + \mathbf{M}^{\mathbf{F}_2/AB} + \mathbf{M}^{\mathbf{F}_3/AB} \\ &= 0 + 0 + 40\mathbf{n}_3 \\ &= 40\mathbf{n}_3 \text{ ft lb} \end{aligned}$$

3.3.1 The moments of a system of vectors about points and lines are free vectors.

3.3.2 The moment $\mathbf{M}^{S/L}$ of a system S of bound vectors $\mathbf{v}_i$, $i = 1, \ldots, n$, about a line L is equal to the L resolute of the moment $\mathbf{M}^{S/A}$ of S about any point A on line L.

Proof: Let $\mathbf{n}$ be a unit vector parallel to L. Then

$$\begin{aligned} \mathbf{M}^{S/L} &\underset{(3.3)}{=} \sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/L} \underset{(3.2,1.14.6)}{=} \sum_{i=1}^{n} (\mathbf{n}\cdot\mathbf{M}^{\mathbf{v}_i/A}\mathbf{n}) \\ &\underset{(1.9.3,1.14.3)}{=} \mathbf{n}\cdot\left(\sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/A}\right)\mathbf{n} \underset{(3.3)}{=} \mathbf{n}\cdot\mathbf{M}^{S/A}\mathbf{n} \end{aligned}$$

and this vector is equal to the L resolute of $\mathbf{M}^{S/A}$ (see 1.14.6).

Problem: Repeat part (2) of Problem 3.3.

Solution:

$$\begin{aligned} \mathbf{M}^{S/AB} &= \mathbf{n}_3\cdot\mathbf{M}^{S/A}\mathbf{n}_3 \\ &= \mathbf{n}_3\cdot(100\mathbf{n}_1 + 40\mathbf{n}_3)\mathbf{n}_3 \\ &= 40\mathbf{n}_3 \text{ ft lb} \end{aligned}$$

3.3.3 The moments $\mathbf{M}^{S/A}$ and $\mathbf{M}^{S/A'}$ of a system S of bound vectors, about two points A and A', are related to each other as follows:

$$\mathbf{M}^{S/A} = \mathbf{M}^{S/A'} + \mathbf{p} \times \mathbf{R}$$

where $\mathbf{p}$ is the position vector of A' relative to A, and $\mathbf{R}$ is the resultant of S.

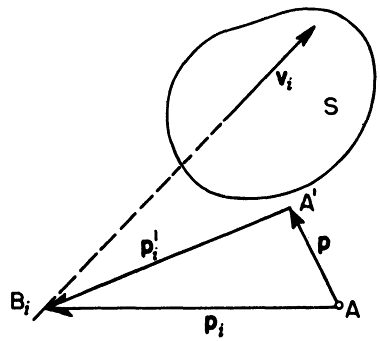


FIG. 3.3.3

Proof: Let $\mathbf{v}_i$, $i = 1, \ldots, n$, be the vectors comprising S, B_i a point on the line of action of $\mathbf{v}_i$, $\mathbf{p}_i$ and $\mathbf{p}_i'$ the position vectors of B_i relative to A and A' (see Fig. 3.3.3). Then

$$\begin{aligned}
\mathbf{M}^{S/A} &\underset{(3.3)}{=} \sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/A} \underset{(3.1)}{=} \sum_{i=1}^{n} \mathbf{p}_i \times \mathbf{v}_i \\
&\underset{(F3.3.3)}{=} \sum_{i=1}^{n} (\mathbf{p}_i' + \mathbf{p}) \times \mathbf{v}_i \underset{(1.15.6)}{=} \sum_{i=1}^{n} (\mathbf{p}_i' \times \mathbf{v}_i + \mathbf{p} \times \mathbf{v}_i) \\
&\underset{(1.9.2)}{=} \sum_{i=1}^{n} \mathbf{p}_i' \times \mathbf{v}_i + \sum_{i=1}^{n} \mathbf{p} \times \mathbf{v}_i \\
&\underset{(1.15.6)}{=} \sum_{i=1}^{n} \mathbf{p}_i' \times \mathbf{v}_i + \mathbf{p} \times \sum_{i=1}^{n} \mathbf{v}_i \\
&= \underbrace{\sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/A'}}_{(3.1)} + \mathbf{p} \times \underset{(1.9.1)}{\mathbf{R}} \\
&= \underset{(3.3)}{\mathbf{M}^{S/A'}} + \mathbf{p} \times \mathbf{R}
\end{aligned}$$

Problem: Referring to Problem 3.3, evaluate $\mathbf{M}^{S/B}$.

Solution: Let $\mathbf{p}$ be the position vector of A relative to B, $\mathbf{R}$ the resultant of S:

$$\mathbf{p} = -3\mathbf{n}_3 \text{ ft}$$

$$\mathbf{R} = 5\mathbf{n}_1 - 20\mathbf{n}_2 - 10\mathbf{n}_3 \text{ lb}$$

From Problem 3.3,

$$\mathbf{M}^{S/A} = 100\mathbf{n}_1 + 40\mathbf{n}_3 \text{ ft lb}$$

Hence,

$$\begin{aligned}\mathbf{M}^{S/B} &= \mathbf{M}^{S/A} + \mathbf{p} \times \mathbf{R} \\ &= 100\mathbf{n}_1 + 40\mathbf{n}_3 - 3(5\mathbf{n}_2 + 20\mathbf{n}_1) \\ &= 40\mathbf{n}_1 - 15\mathbf{n}_2 + 40\mathbf{n}_3 \text{ ft lb}\end{aligned}$$

3.3.4 The moments of a system S of bound vectors about all points of any line parallel to the resultant $\mathbf{R}$ of S are equal to each other.

Proof: Referring to 3.3.3, let A and A' lie on a line parallel to $\mathbf{R}$. Then

$$\mathbf{p} \times \mathbf{R} \underset{(1.15.3)}{=} 0$$

and

$$\mathbf{M}^{S/A} = \mathbf{M}^{S/A'}$$

3.3.5 The moments of a system S of bound vectors about all lines parallel to the resultant $\mathbf{R}$ of S are equal to each other.

Proof: Referring to 3.3.3, place A and A' respectively on two lines L and L' parallel to $\mathbf{R}$, and let $\mathbf{n}$ be a unit vector parallel to $\mathbf{R}$. Then

$$\begin{aligned}\mathbf{M}^{S/L} &\underset{(3.3.2)}{=} \mathbf{n}\cdot\mathbf{M}^{S/A}\mathbf{n} \\ &\underset{(3.3.3)}{=} \mathbf{n}\cdot\mathbf{M}^{S/A'}\mathbf{n} + \mathbf{n}\cdot\mathbf{p} \times \mathbf{R}\mathbf{n} \\ &= \underset{(3.3.2)}{\mathbf{M}^{S/L'}} + \underset{(1.16.4)}{0}\end{aligned}$$

3.3.6 If the resultant $\mathbf{R}$ of a system S of bound vectors is equal to zero, the moments of S about all points are equal to each other (see 3.3.3). If $\mathbf{R}$ is not equal to zero, the points about which S has a minimum moment ($\mathbf{M}^*$) lie on a line (L^*) which is parallel to $\mathbf{R}$ and passes through a point (P^*) whose position vector ($\mathbf{p}^*$) relative to an arbitrarily selected reference point (O) is given by

$$\mathbf{p}^* = \frac{\mathbf{R} \times \mathbf{M}^{S/O}}{\mathbf{R}^2}$$

L^* is called the *central axis* of S. $\mathbf{M}^*$ is given by

$$\mathbf{M}^* = \frac{\mathbf{R}\cdot\mathbf{M}^{S/O}}{\mathbf{R}^2}\mathbf{R}$$

Proof: For an arbitrarily selected reference point O, $\mathbf{M}^{S/O}$ is not, in general, parallel to $\mathbf{R}$, and the $\mathbf{R}$ resolute of $\mathbf{M}^{S/O}$ is, therefore, smaller than $\mathbf{M}^{S/O}$. Suppose it were possible to find a point A, such that $\mathbf{M}^{S/A}$ is parallel to $\mathbf{R}$. Then $\mathbf{M}^{S/A}$ would be smaller than $\mathbf{M}^{S/O}$, because (a) $\mathbf{M}^{S/A}$ would be equal to the $\mathbf{R}$ resolute of $\mathbf{M}^{S/A}$; (b) the $\mathbf{R}$ resolute of $\mathbf{M}^{S/A}$ is equal to the $\mathbf{R}$ resolute of $\mathbf{M}^{S/O}$ (see 3.3.2 and 3.3.5); (c) the $\mathbf{R}$ resolute of $\mathbf{M}^{S/O}$ is smaller than $\mathbf{M}^{S/O}$. Thus $\mathbf{M}^{S/A}$ would be the minimum moment $\mathbf{M}^*$ of S, and $\mathbf{M}^*$ would be given by

$$\mathbf{M}^* \underset{(1.14.6)}{=} \frac{\mathbf{R}}{|\mathbf{R}|} \cdot \mathbf{M}^{S/O} \frac{\mathbf{R}}{|\mathbf{R}|} \underset{(1.14.11)}{=} \frac{\mathbf{R} \cdot \mathbf{M}^{S/O}}{\mathbf{R}^2} \mathbf{R}$$

To show that points such as A exist, let $\mathbf{p}^*$ be the position vector of a point P^* relative to O, and impose on P^* the requirement

$$\mathbf{M}^{S/P^*} \equiv \mathbf{M}^* = \frac{\mathbf{R} \cdot \mathbf{M}^{S/O}}{\mathbf{R}^2} \mathbf{R}$$

In accordance with 3.3.3,

$$\mathbf{M}^{S/P^*} = \mathbf{M}^{S/O} - \mathbf{p}^* \times \mathbf{R}$$

Hence $\mathbf{p}^*$ must satisfy the equation

$$\mathbf{M}^{S/O} - \mathbf{p}^* \times \mathbf{R} = \frac{\mathbf{R} \cdot \mathbf{M}^{S/O}}{\mathbf{R}^2} \mathbf{R}$$

or

$$\mathbf{p}^* \times \mathbf{R} = \mathbf{M}^{S/O} - \frac{\mathbf{R} \cdot \mathbf{M}^{S/O}}{\mathbf{R}^2} \mathbf{R} \underset{(1.17.1)}{=} \frac{\mathbf{R} \times \mathbf{M}^{S/O}}{\mathbf{R}^2} \times \mathbf{R}$$

This equation is satisfied identically by

$$\mathbf{p}^* = \frac{\mathbf{R} \times \mathbf{M}^{S/O}}{\mathbf{R}^2}$$

Thus there exists at least one point (P^*) about which the moment of S is equal to $\mathbf{M}^*$. From 3.3.4 it follows that the moments of S about all points of a line L^* which is parallel to $\mathbf{R}$ and passes through P^* are also equal to $\mathbf{M}^*$.

It remains to be shown that the points of L^* are the only points about which the moment of S is equal to $\mathbf{M}^*$.

The moment of S about a point P not lying on L^* is given by

$$\mathbf{M}^{S/P} \underset{(3.3.3)}{=} \mathbf{M}^* + \mathbf{p} \times \mathbf{R}$$

where the position vector $\mathbf{p}$ of P relative to P^* is not parallel to $\mathbf{R}$. Hence $\mathbf{p} \times \mathbf{R}$ is not equal to zero, and $\mathbf{M}^{S/P}$ is not equal to $\mathbf{M}^*$.

Problem: Referring to Problem 3.3, determine the minimum moment $\mathbf{M}^*$ of S, and find the shortest distance d^* from point A to the central axis of S.

Solution: The resultant $\mathbf{R}$ of S is given by

$$\mathbf{R} = 5\mathbf{n}_1 - 20\mathbf{n}_2 - 10\mathbf{n}_3 \text{ lb}$$

while

$$\mathbf{M}^{S/A} \underset{(P3.3)}{=} 100\mathbf{n}_1 + 40\mathbf{n}_3 \text{ ft lb}$$

Hence,

$$\mathbf{R} \cdot \mathbf{M}^{S/A} = 100 \text{ ft lb}^2$$

and

$$\mathbf{M}^* = \frac{\mathbf{R} \cdot \mathbf{M}^{S/A}}{\mathbf{R}^2} \mathbf{R}$$

$$= \frac{100}{525} (5\mathbf{n}_1 - 20\mathbf{n}_2 - 10\mathbf{n}_3)$$

$$= 0.952\mathbf{n}_1 - 3.808\mathbf{n}_2 - 1.904\mathbf{n}_3 \text{ ft lb}$$

Next, the position vector $\mathbf{p}^*$ of a point on the central axis, relative to point A, is given by

$$\mathbf{p}^* = \frac{\mathbf{R} \times \mathbf{M}^{S/A}}{\mathbf{R}^2}$$

$$= \frac{1}{525} (-800\mathbf{n}_1 + 800\mathbf{n}_2 + 2000\mathbf{n}_3)$$

Now, $\mathbf{p}^*$ is perpendicular to the central axis (as can be seen by recalling that the central axis is parallel to $\mathbf{R}$ and noting that $\mathbf{R} \cdot \mathbf{p}^* = 0$). Hence

$$d^* = |\mathbf{p}^*|$$

$$= \frac{100}{525} (64 + 64 + 400)^{\frac{1}{2}}$$

$$= 4.38 \text{ ft}$$

COUPLES

3.4 Definition

A couple is a system of bound vectors whose resultant is equal to zero and whose moment about some point is not equal to zero.

Example: The system of three forces described in Problem 3.2.10 is a couple.

3.4.1 Couples are not vectors; for, a system of vectors is not a vector, any more than a system of points is a point.

3.4.2 A couple consisting of only two vectors is called a *simple couple.* The vectors comprising a simple couple have equal magnitudes, parallel lines of action, and opposite senses.

Example: The two forces shown in Fig. 3.4.2 constitute a simple couple.

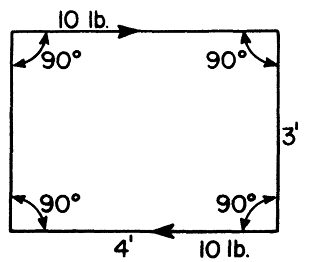


FIG. 3.4.2

3.4.3 Most writers use the word "couple" to denote what has here been called a simple couple.

3.4.4 The moment of a couple about a point is called the *torque* of the couple. It is unnecessary to refer to a specific point, because the moment of a couple about one point is equal to the moment of the couple about any other point. This follows from 3.3.3 and the fact that the resultant of a couple is equal to zero.

3.4.5 Although couples are not vectors, and torques are vectors, the words "couple" and "torque" are often used interchangeably. One encounters, for instance, such phrases as "the magnitude of a couple," "the direction of a couple," "a torque applied to a body," etc.

Problem (a): The system of four forces shown in Fig. 3.4.5a is a couple. Find the torque $\mathbf{T}$ of this couple.

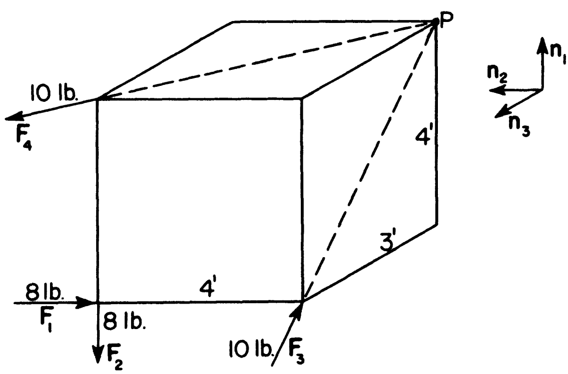


FIG. 3.4.5a

Solution: Select a convenient point. Find the moment of each force about this point. Add these moments:

$$\mathbf{T} = \mathbf{M}^{\mathbf{F}_1/P} + \mathbf{M}^{\mathbf{F}_2/P} + \mathbf{M}^{\mathbf{F}_3/P} + \mathbf{M}^{\mathbf{F}_4/P}$$

$$= 64\mathbf{n}_1 - 24\mathbf{n}_2 + 24\mathbf{n}_3 \text{ ft lb}$$

Problem (b): $\mathbf{F}_1$ and $\mathbf{F}_2$ are two forces of a system S of 96 forces, the remaining 94 comprising a couple C whose torque $\mathbf{T}$ is shown in Fig. 3.4.5b. (The circle drawn through the arrow representing $\mathbf{T}$ is meant to call attention to the fact that the dimensions of $\mathbf{T}$ are different from those of $\mathbf{F}_1$ and $\mathbf{F}_2$.) Find (1) the resultant $\mathbf{F}$ of S, (2) the moment $\mathbf{M}^{S/A}$ of S about point A, and (3) the moment $\mathbf{M}^{S/B}$ of S about point B.

Solution (1): The couple C contributes nothing to the resultant of S. Hence

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = 5\mathbf{n}_1 + 10\mathbf{n}_2 \text{ lb}$$

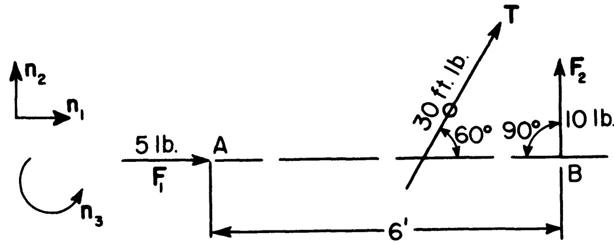


FIG. 3.4.5b

Solution (2):

$$\begin{aligned}
\mathbf{M}^{S/A} &= \mathbf{M}^{\mathbf{F}_1/A} + \mathbf{M}^{\mathbf{F}_2/A} + \mathbf{M}^{C/A} \\
&= 0 + (6)(10)\mathbf{n}_3 + \mathbf{T} \\
&= 60\mathbf{n}_3 + 30 \cos 60^\circ\, \mathbf{n}_1 + 30 \sin 60^\circ\, \mathbf{n}_2 \\
&= 15\mathbf{n}_1 + 25.98\mathbf{n}_2 + 60\mathbf{n}_3 \text{ ft lb}
\end{aligned}$$

Solution (3):

$$\begin{aligned}
\mathbf{M}^{S/B} &= \mathbf{M}^{\mathbf{F}_1/B} + \mathbf{M}^{\mathbf{F}_2/B} + \mathbf{M}^{C/B} \\
&= 0 + 0 + \mathbf{T} \\
&= 15\mathbf{n}_1 + 25.98\mathbf{n}_2 \text{ ft lb}
\end{aligned}$$

3.4.6 The magnitude of the torque $\mathbf{T}$ of a simple couple is given by

$$|\mathbf{T}| = s|\mathbf{v}|$$

where s is the distance between the lines of action of the two vectors comprising the couple, and $\mathbf{v}$ is one of these vectors.

Proof (see Fig. 3.4.6): $\mathbf{T}$ is the sum of the moments of $\mathbf{v}$ and $-\mathbf{v}$ about any point. Take moments about point A:

$$\begin{aligned}
\mathbf{T} &= \mathbf{M}^{\mathbf{v}/A} + \mathbf{M}^{-\mathbf{v}/A} \\
&= \underset{(3.1)}{\mathbf{p} \times \mathbf{v}} + \underset{(3.1.2)}{0}
\end{aligned}$$

Hence,

$$|\mathbf{T}| = |\mathbf{p} \times \mathbf{v}| \underset{(1.15.2)}{=} |\mathbf{p}|\,|\mathbf{v}| \sin(\mathbf{p}, \mathbf{v}) = s|\mathbf{v}|$$

Problem: Determine the magnitude of the torque $\mathbf{T}$ of the couple shown in Fig. 3.4.2.

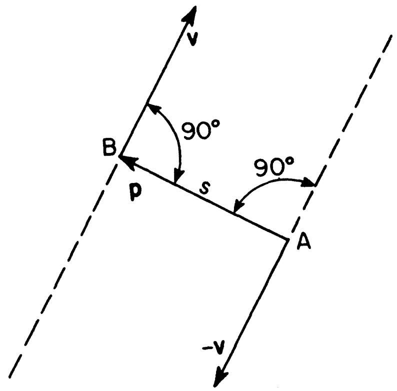


Fig. 3.4.6

Solution:

$$|\mathbf{T}| = 3(10) = 30 \text{ ft lb}$$

3.4.7 The direction of the torque of a simple couple can readily be determined by inspection: $\mathbf{T}$ is perpendicular to the plane determined by the lines of action of the two vectors comprising the couple, and the sense of $\mathbf{T}$ is the same as that of $\mathbf{p} \times \mathbf{v}$ (see Fig. 3.4.6). Using 3.4.6, one can, therefore, find the torque of a simple couple without explicitly performing any cross multiplications.

Problem (a): A body is subjected to the action of a system S of forces, S consisting of two simple couples, C and C', as shown in Fig. 3.4.7a. Evaluate the moment of S about point A.

Solution:

$$\begin{aligned} \mathbf{M}^{S/A} &= \mathbf{T}^{C} + \mathbf{T}^{C'} \\ &= 3(10)\mathbf{n}_2 + 3(5)(-\mathbf{n}_1) \\ &= 30\mathbf{n}_2 - 15\mathbf{n}_1 \text{ ft lb} \end{aligned}$$

Problem (b): Draw three simple couples whose torques are respectively parallel to the unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ shown in Fig. 3.4.7b and whose resultant moment about any point in space is equal to that of the force system described in Problem 3.4.5(a).

Solution: Let C_1, C_2, C_3 be the three simple couples, $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ their torques. Then

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 = 64\mathbf{n}_1 - 24\mathbf{n}_2 + 24\mathbf{n}_3 \text{ ft lb}$$

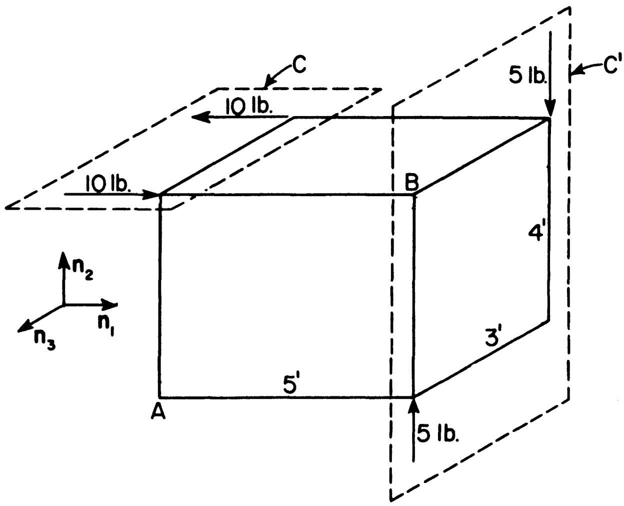


FIG. 3.4.7a

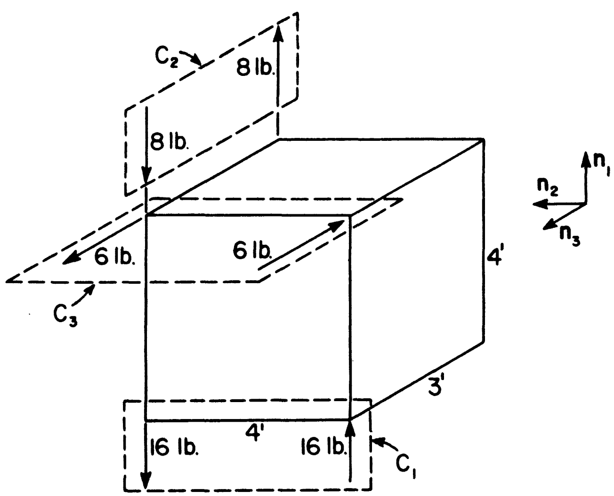


FIG. 3.4.7b

This equation may be satisfied by taking

$$\mathbf{T}_1 = 64\mathbf{n}_1 \text{ ft lb}, \quad \mathbf{T}_2 = -24\mathbf{n}_2 \text{ ft lb}, \quad \mathbf{T}_3 = 24\mathbf{n}_3 \text{ ft lb}$$

One system of simple couples having these torques is shown in Fig. 3.4.7b.

3.4.8 The moment of a couple about a line L is equal to the L resolute of the torque of the couple.

Proof: Use 3.4, 3.3.2 and 3.4.4.

Problem (a): Referring to Problem 3.4.5(b) determine the moment of S about line AB.

Solution: $\mathbf{F}_1$ and $\mathbf{F}_2$ contribute nothing to $\mathbf{M}^{S/AB}$. The moment of C about line AB is equal to the AB resolute of $\mathbf{T}$. Hence,

$$\mathbf{M}^{S/AB} = 30 \cos 60° \, \mathbf{n}_1 = 15\mathbf{n}_1 \text{ ft lb}$$

Problem (b): Evaluate the moments of the couple C shown in Fig. 3.4.8, about (1) line L_1, (2) line L_2, (3) line L_3, (4) line L_4.

Solution (1): The torque $\mathbf{T}$ of the couple C is given by

$$\mathbf{T} \underset{(3.4.7)}{=} 30\mathbf{n}_3 \text{ ft lb}$$

The resolute of $\mathbf{T}$ parallel to L_1 is $\mathbf{T}$ itself. Hence,

$$\mathbf{M}^{C/L_1} = 30\mathbf{n}_3 \text{ ft lb}$$

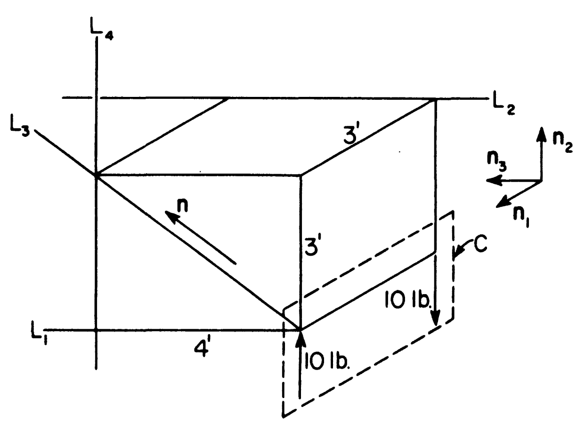


FIG. 3.4.8

Solution (2): The resolute of $\mathbf{T}$ parallel to L_2 is equal to the resolute of $\mathbf{T}$ parallel to L_1. Hence,

$$\mathbf{M}^{C/L_2} = \mathbf{M}^{C/L_1} = 30\mathbf{n}_3 \text{ ft lb}$$

Solution (3): Let $\mathbf{n}$ be a unit vector parallel to L_3 (see Fig. 3.4.8). Then

$$\mathbf{M}^{C/L_3} \underset{(1.14.6)}{=} \mathbf{n}\cdot\mathbf{T}\mathbf{n} = 24\mathbf{n} \text{ ft lb}$$

Solution (4): $\mathbf{T}$ is perpendicular to L_4. Hence the L_4 resolute of $\mathbf{T}$ is equal to zero, and

$$\mathbf{M}^{C/L_4} = 0$$

3.4.9 The moments of a couple about two parallel lines are equal to each other.

Proof: Use 3.4.8

3.4.10 When the torque $\mathbf{T}$ of a couple C is resolved into three mutually perpendicular components, each component is equal to the moment of the couple about any line parallel to that component.

Proof: When $\mathbf{T}$ is resolved into components respectively parallel to mutually perpendicular unit vectors $\mathbf{n}_i$, $i = 1, 2, 3$, the $\mathbf{n}_i$ component of $\mathbf{T}$ is equal to the $\mathbf{n}_i$ resolute of $\mathbf{T}$ (see 1.14.10); and the $\mathbf{n}_i$ resolute of $\mathbf{T}$ is equal to the moment of C about any line parallel to $\mathbf{n}_i$ (see 3.4.8 and 3.4.9).

Problem: The box shown in Fig. 3.4.10a is subjected to the action of a system S of forces, S consisting of a force $\mathbf{F}$ and a couple

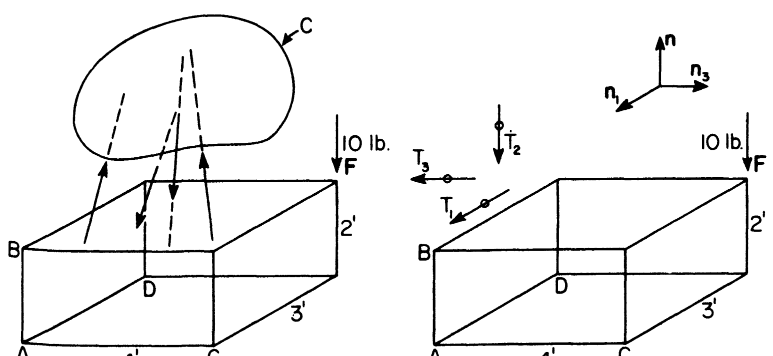


FIG. 3.4.10a FIG. 3.4.10b

C. The moments of S about lines AB, AC and AD are equal to zero. Determine the magnitude of the torque $\mathbf{T}$ of the couple C.

Solution: Let T_1, T_2, T_3 be the $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ measure numbers of $\mathbf{T}$ (see Fig. 3.4.10b), and take moments about lines AB, AC, AD:

$$\begin{aligned}
\mathbf{M}^{S/AB} &= \mathbf{M}^{F/AB} + \mathbf{M}^{C/AB} \\
&= 0 + T_2(-\mathbf{n}_2) = 0 \\
\therefore \quad T_2 &= 0 \\
\mathbf{M}^{S/AC} &= \mathbf{M}^{F/AC} + \mathbf{M}^{C/AC} \\
&= 3(10)(-\mathbf{n}_3) + T_3(-\mathbf{n}_3) = 0 \\
\therefore \quad T_3 &= -30 \text{ ft lb} \\
\mathbf{M}^{S/AD} &= \mathbf{M}^{F/AD} + \mathbf{M}^{C/AD} \\
&= 4(10)(-\mathbf{n}_1) + T_1\mathbf{n}_1 = 0 \\
\therefore \quad T_1 &= 40 \text{ ft lb}
\end{aligned}$$

The magnitude of $\mathbf{T}$ is given by

$$|\mathbf{T}| \underset{(1.10.5)}{=} (T_1^2 + T_2^2 + T_3^2)^{\frac{1}{2}} = (1600 + 0 + 900)^{\frac{1}{2}} = 50 \text{ ft lb}$$

The fact that T_3 is negative means that the $\mathbf{n}_3$ component of $\mathbf{T}$ has a sense opposite to that shown in Fig. 3.4.10b.

EQUIVALENCE, REPLACEMENT, AND REDUCTION

3.5 Definition

Two systems S and S' of bound vectors are said to be *equivalent* when *both* of the following conditions are fulfilled: (a) The resultant of S is equal to the resultant of S'. (b) There exists at least one point about which S and S' have equal moments.

If S and S' are equivalent, then either is called a *replacement* of the other, and if S' contains fewer vectors than S, S' is called a *reduction* of S. The process by means of which a replacement (reduction) is obtained, is also called replacement (reduction).

Example: Fig. 3.5 shows a force system S consisting of three forces ($\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$) and a couple (of torque $\mathbf{T}$); further, a system S, consisting of a force ($\mathbf{F}$) and two couples (of torques $\mathbf{T}_1'$, $\mathbf{T}_2'$). S is equivalent to S', as may be seen by evaluating their resultants, $\mathbf{R}$ and $\mathbf{R}'$, and the moments, $\mathbf{M}^{S/A}$ and $\mathbf{M}^{S'/A}$, of S and S' about A:

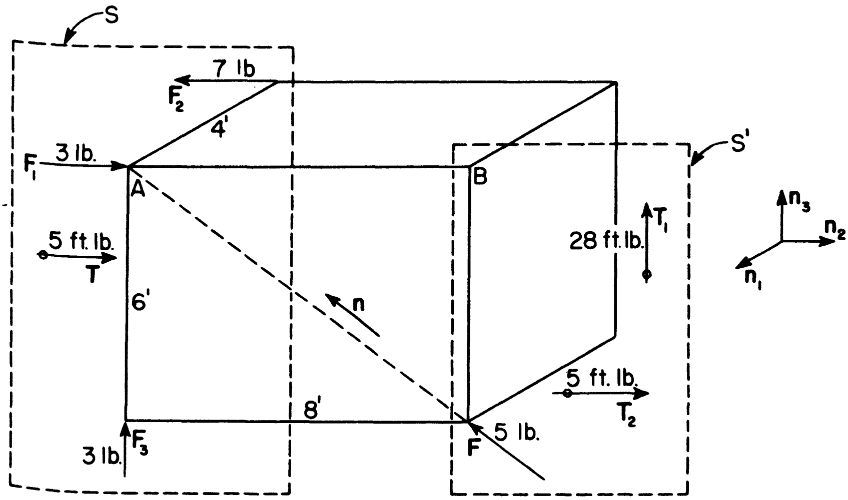


FIG. 3.5

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 3\mathbf{n}_2 - 7\mathbf{n}_2 + 3\mathbf{n}_3 = -4\mathbf{n}_2 + 3\mathbf{n}_3 \text{ lb}$$

$$\mathbf{R}' = \mathbf{F} = 5\mathbf{n} = 5(-0.8\mathbf{n}_2 + 0.6\mathbf{n}_3) = -4\mathbf{n}_2 + 3\mathbf{n}_3 \text{ lb}$$

$$\begin{aligned} \mathbf{M}^{S/A} &= \mathbf{M}^{\mathbf{F}_1/A} + \mathbf{M}^{\mathbf{F}_2/A} + \mathbf{M}^{\mathbf{F}_3/A} + \mathbf{T} \\ &= 0 + 4(7)\mathbf{n}_3 + 0 + 5\mathbf{n}_2 = 5\mathbf{n}_2 + 28\mathbf{n}_3 \text{ ft lb} \\ \mathbf{M}^{S'/A} &= \mathbf{M}^{\mathbf{F}/A} + \mathbf{T}_1' + \mathbf{T}_2' \\ &= 0 + 28\mathbf{n}_3 + 5\mathbf{n}_2 \text{ ft lb} \end{aligned}$$

3.5.1 The word "equivalence" is not to be regarded as implying physical equivalence of any sort: Figs. 3.5.1a and 3.5.1b each

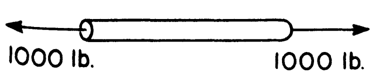


FIG. 3.5.1a

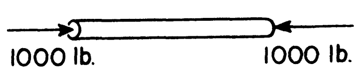


FIG. 3.5.1b

show a rod subjected to the action of a pair of forces. The two pairs of forces are equivalent; but their effects on the rod are quite different from each other.

3.5.2 Given a line L and two equivalent systems S and S' of bound vectors, the sum of the L resolutes of the vectors in S is equal to the sum of the L resolutes of the vectors in S'.

Proof: The sums of the L resolutes of the vectors in S and S' are respectively equal to the L resolutes of the resultants of S and S' (see 1.14.7). These resultants are equal to each other (see 3.5); hence they have equal L resolutes.

Example: Referring to Example 3.5, it may be verified that the sums of the $\mathbf{n}_2$ resolutes of the forces in S and S' are each equal to $-4\mathbf{n}_2$ lb.

3.5.3 The moments of two equivalent systems of bound vectors, about *any* point, are equal to each other.

Proof (see Fig. 3.5.3): S and S' have equal resultants, $\mathbf{R}$ and

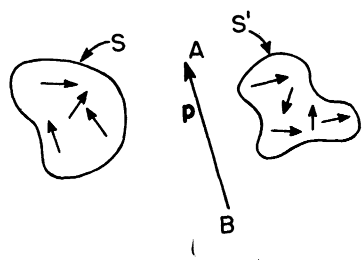


FIG. 3.5.3

$\mathbf{R}'$, and there exists at least one point, say A, about which S and S' have equal moments. Hence

$$\mathbf{R} = \mathbf{R}'$$

and

$$\mathbf{M}^{S/A} = \mathbf{M}^{S'/A}$$

It must be shown that S and S' have equal moments about a point such as B, i.e., that

$$\mathbf{M}^{S/B} = \mathbf{M}^{S'/B}$$

Note that

$$\mathbf{M}^{S/B} \underset{(3.3.3)}{=} \mathbf{M}^{S/A} + \mathbf{p} \times \mathbf{R}$$

Hence,

$$\mathbf{M}^{S/B} = \mathbf{M}^{S'/A} + \mathbf{p} \times \mathbf{R}'$$

But

$$\mathbf{M}^{S'/A} + \mathbf{p} \times \mathbf{R}' \underset{(3.3.3)}{=} \mathbf{M}^{S'/B}$$

Thus,

$$\mathbf{M}^{S/B} = \mathbf{M}^{S'/B}$$

3.5.4 The moments of two equivalent systems of bound vectors, about *any* line L, are equal to each other.

Proof: The moments of the two systems about L are respectively equal to the L resolutes of the moments of the systems about a point A on L (see 3.3.2). These moments about A are equal to each other (see 3.5.3); hence they have equal L resolutes.

Problem: A force system S consists of two forces ($\mathbf{F}_1$, $\mathbf{F}_2$) and two couples (of torques $\mathbf{T}_1$, $\mathbf{T}_2$), as shown in Fig. 3.5.4a. Reduce S

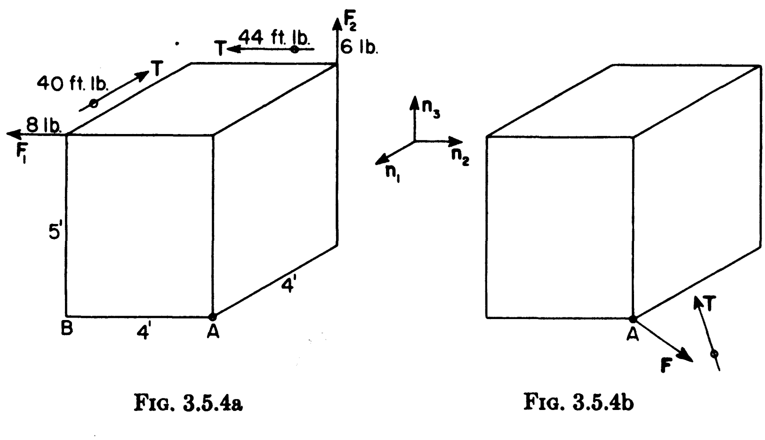


FIG. 3.5.4a FIG. 3.5.4b

to a force system S' consisting of a simple couple C and a force $\mathbf{F}$ whose line of action passes through point A.

Solution: Let $\mathbf{T}$ be the torque of C. Show $\mathbf{F}$ and $\mathbf{T}$ on a sketch (see Fig. 3.5.4b). Evaluate the resultants, $\mathbf{R}$ and $\mathbf{R}'$, of S and S':

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = -8\mathbf{n}_2 + 6\mathbf{n}_3 \text{ lb}$$
$$\mathbf{R}' = \mathbf{F}$$

Then

$$\mathbf{F} = -8\mathbf{n}_2 + 6\mathbf{n}_3 \text{ lb}$$

Next,

$$\mathbf{M}^{S/A} \underset{(F3.5.4a)}{=} \mathbf{M}^{\mathbf{F}_1/A} + \mathbf{M}^{\mathbf{F}_2/A} + \mathbf{T}_1 + \mathbf{T}_2$$

64 $$= 40\mathbf{n}_1 + 24\mathbf{n}_2 - 40\mathbf{n}_1 - 44\mathbf{n}_2$$

-8 $$= -20\mathbf{n}_2 \text{ ft lb}$$

and

$$\mathbf{M}^{S'/A} \underset{(F3.5.4b)}{=} \mathbf{M}^{\mathbf{F}/A} + \mathbf{T} = 0 + \mathbf{T}$$

Hence (see 3.5.3),

$$\mathbf{T} = -20\mathbf{n}_2 \text{ ft lb}$$

A simple couple C, whose torque is equal to $-20\mathbf{n}_2$ ft lb, and the

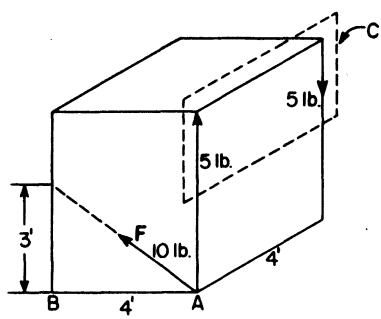


Fig. 3.5.4c

force $\mathbf{F}$ are shown in Fig. 3.5.4c.

3.5.5 If S is equivalent to S', and S' is equivalent to S'', then S is equivalent to S''. This property of the equivalence relation is called ***transitivity***. It is an immediate consequence of 3.5.

3.5.6 Every system S of bound vectors can be replaced with a system S' consisting of a couple C and a single bound vector $\mathbf{v}$ whose line of action passes through an arbitrarily selected ***base point*** O. The torque $\mathbf{T}$ of C depends on the choice of base point:

$$\mathbf{T} = \mathbf{M}^{S/O}$$

The vector $\mathbf{v}$ is independent of the choice of base point:

$$\mathbf{v} = \mathbf{R}$$

where $\mathbf{R}$ is the resultant of S.

Proof (see Fig. 3.5.6): If

$$\mathbf{v} = \mathbf{R}$$

and

$$\mathbf{T} = \mathbf{M}^{S/O}$$

then S and S' have equal resultants and equal moments about point O. See 3.5.

Problem: Referring to Problem 3.5.4, replace S with a force system consisting of a couple of torque $\mathbf{T}$ and a force $\mathbf{F}$ whose line

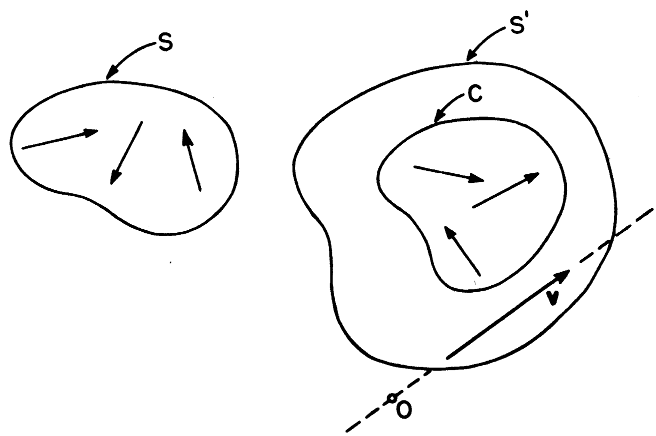


Fig. 3.5.6

of action passes (1) through point A and (2) through point B. Determine $\mathbf{F}$ and $\mathbf{T}$ for each case.

Solution (1):

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = -8\mathbf{n}_2 + 6\mathbf{n}_3 \text{ lb}$$

$$\mathbf{T} = \mathbf{M}^{S/A} \underset{(\text{P3.5.4})}{=} -20\mathbf{n}_2 \text{ ft lb}$$

Solution (2):

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = -8\mathbf{n}_2 + 6\mathbf{n}_3 \text{ lb}$$

$$\mathbf{T} = \mathbf{M}^{S/B} \underset{(3.3.3)}{=} \mathbf{M}^{S/A} + 4\mathbf{n}_2 \times \mathbf{F}$$

$$= -20\mathbf{n}_2 + 24\mathbf{n}_1 \text{ ft lb}$$

3.5.7 A couple C can be replaced with any system of couples the sum of whose torques is equal to the torque of C. This follows from 3.5.6 and 3.4.4.

3.5.8 When a system of bound vectors consists of a couple of torque $\mathbf{T}$ and a single vector parallel to $\mathbf{T}$, it is called a *wrench*.

If the resultant of a system S of bound vectors is not equal to zero, S can be replaced with a wrench W, consisting of the resultant $\mathbf{R}$ of S, placed on the central axis L^* of S (see 3.3.6), and a couple, whose torque $\mathbf{T}^*$ is equal to the minimum moment $\mathbf{M}^*$ of S.

Proof: When W is constructed as described above, S and W have equal resultants and equal moments about any point of L^*. Hence they are equivalent.

Problem: A force system S consists of 4 forces and a couple of torque $\mathbf{T}$, as shown in Fig. 3.5.8a. Replace S with a wrench.

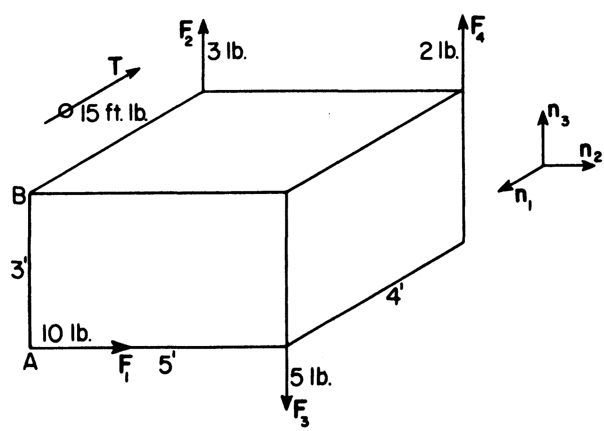


FIG. 3.5.8a

Solution: Let $\mathbf{F}^*$ be the force and $\mathbf{T}^*$ the torque of the couple associated with the wrench. Then

$$\mathbf{F}^* = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = 10\mathbf{n}_2 \text{ lb}$$

and the line of action of $\mathbf{F}^*$ passes through the point P^* whose position vector relative to point A is given by

$$\mathbf{p}^* \underset{(3.3.6)}{=} \frac{\mathbf{F}^* \times \mathbf{M}^{S/A}}{\mathbf{F}^{*2}}$$

$$= \frac{10\mathbf{n}_2 \times (-30\mathbf{n}_1 + 20\mathbf{n}_2)}{100} = 3\mathbf{n}_3 \text{ ft}$$

That is, P^* coincides with point B. $\mathbf{T}^*$ is given by

$$\mathbf{T}^* \underset{(3.3.6)}{=} \frac{\mathbf{F}^* \cdot \mathbf{M}^{S/A}}{\mathbf{F}^{*2}} \mathbf{F}^* = 20\mathbf{n}_2 \text{ ft lb}$$

Fig. 3.5.8b represents the wrench equivalent to S.

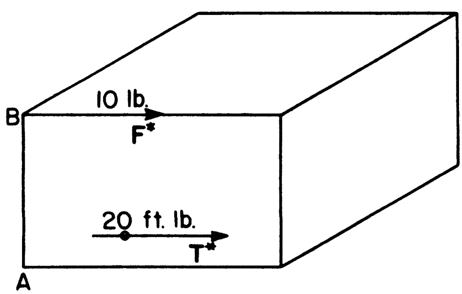


FIG. 3.5.8b

3.5.9 The solution of practical problems frequently involves reductions of systems of bound vectors whose lines of action intersect at a point, pass through a line, lie in a plane, or are parallel to each other. In the sections which follow, reductions of such systems are examined in detail.

3.5.10 A system of vectors $\mathbf{v}_i$, $i = 1, \ldots, n$, whose lines of action intersect at a point O, can be replaced with a single vector whose line of action passes through O.

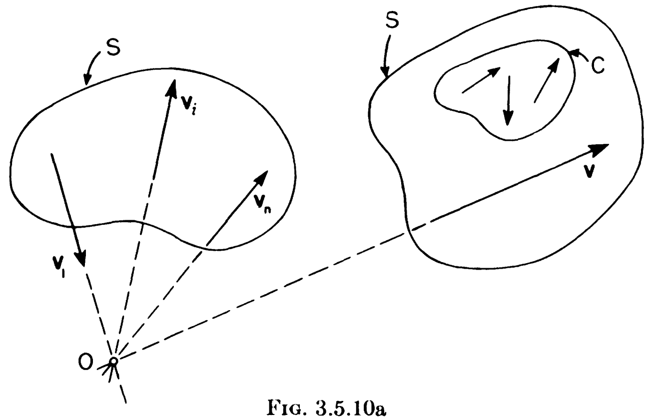


FIG. 3.5.10a

Proof: Replace S in accordance with 3.5.6, using point O as base point (see Fig. 3.5.10a). Then the torque $\mathbf{T}$ of the couple C is given by

$$\mathbf{T} \underset{(3.5.6)}{=} \mathbf{M}^{S/O} \underset{(3.3)}{=} \sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/O} \underset{(3.1.2)}{=} 0$$

and S' consists entirely of $\mathbf{v}$.

Problem: The system S of three forces shown in Fig. 3.5.10b is to be replaced with (1) a single force $\mathbf{F}$ and (2) a simple couple and a force $\mathbf{F}$ whose line of action passes through point B. Show each of these replacements in a sketch.

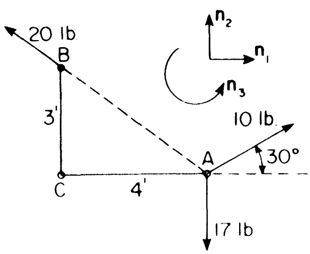


FIG. 3.5.10b

FIG. 3.5.10c

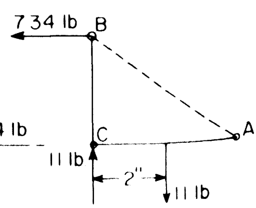


FIG. 3.5.10d

Solution (1): The single force $\mathbf{F}$ is equal to the resultant of S:

$$\mathbf{F} = -7.34\mathbf{n}_1 \text{ lb}$$

The line of action of $\mathbf{F}$ must pass through point A. See Fig. 3.5.10c.

Solution (2): The force $\mathbf{F}$ is again equal to the resultant of S. The torque $\mathbf{T}$ of the simple couple is equal to the moment of S about point B:

$$\mathbf{T} = -22\mathbf{n}_3 \text{ ft lb}$$

The desired replacement is shown in Fig. 3.5.10d.

3.5.11 A system S of vectors $\mathbf{v}_i$, $i = 1, \ldots, n$, whose lines of action intersect a line L can be replaced with a couple whose torque is perpendicular to L, together with a vector whose line of action intersects L.

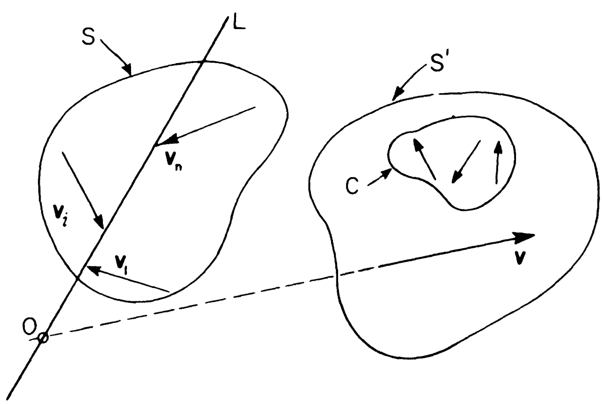


FIG. 3.5.11

Proof: Replace S in accordance with 3.5.6, using a point O on line L as base point (see Fig. 3.5.11). Then the torque $\mathbf{T}$ of the couple C is given by

$$\mathbf{T} \underset{(3.5.6)}{=} \mathbf{M}^{S/O} \underset{(3.3)}{=} \sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/O}$$

Now,

$$\mathbf{M}^{\mathbf{v}_i/O} \underset{(3.2.7)}{=} \mathbf{M}^{\mathbf{v}_i/L} + \mathbf{M}^{\mathbf{v}_i/L'} + \mathbf{M}^{\mathbf{v}_i/L''}$$

where L' and L'' are lines intersecting at O and perpendicular to L. Furthermore,

$$\mathbf{M}^{\mathbf{v}_i/L} \underset{(3.2.3)}{=} 0$$

and $\mathbf{M}^{\mathbf{v}_i/L'}$ is parallel to L', $\mathbf{M}^{\mathbf{v}_i/L''}$ to L'', so that both are perpendicular to L. Hence $\mathbf{M}^{\mathbf{v}_i/O}$ is perpendicular to L. It follows that $\mathbf{T}$ is perpendicular to L.

3.5.12 When the lines of action of the vectors $\mathbf{v}_i$, $i = 1, \ldots, n$, of a system S lie in a plane P, S is called a system of *coplanar* vectors, and S can be replaced with a couple whose torque is perpendicular to P, together with a vector whose line of action lies in P.

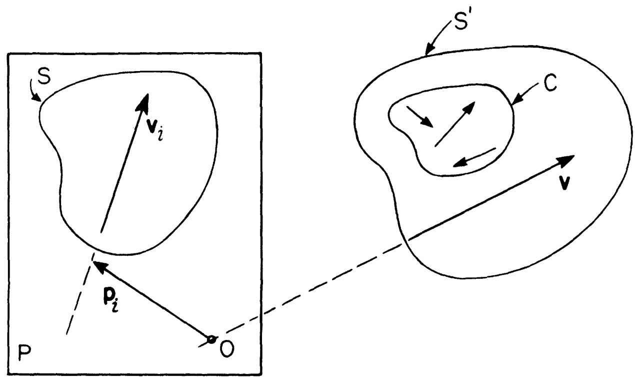


FIG. 3.5.12

Proof: Replace S in accordance with 3.5.6, using a point O of plane P as base point (see Fig. 3.5.12). Then the torque $\mathbf{T}$ of the couple C is given by

$$\mathbf{T} \underset{(3.5.6)}{=} \mathbf{M}^{S/O} \underset{(3.3)}{=} \sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/O} \underset{(F3.5.12)}{=} \sum_{i=1}^{n} \mathbf{p}_i \times \mathbf{v}_i$$

Now, $\mathbf{p}_i \times \mathbf{v}_i$ is perpendicular to both $\mathbf{p}_i$ and $\mathbf{v}_i$, hence to P. It follows that $\mathbf{T}$ is perpendicular to P.

The line of action of $\mathbf{v}$ lies in plane P because it passes through O and is parallel to the resultant of S.

3.5.13 If the resultant of a system S of coplanar vectors is not equal to zero, S can be replaced with a single vector. This vector is the resultant $\mathbf{R}$ of S, placed on the central axis of S.

Proof: Reduce S to a wrench W (see 3.5.8). The minimum moment of S is given by

$$\mathbf{M}^* \underset{(3.3.6)}{=} \frac{\mathbf{R}\cdot\mathbf{M}^{S/O}}{\mathbf{R}^2}\mathbf{R}$$

and $\mathbf{M}^{S/O}$ is perpendicular to $\mathbf{R}$. Hence

$$\mathbf{M}^* = 0$$

and W consists of $\mathbf{R}$, placed on the central axis of S.

3.5.14 A system S of vectors $\mathbf{v}_i$, $i = 1, \ldots, n$, whose lines of action are parallel to a line L can be replaced with a couple whose

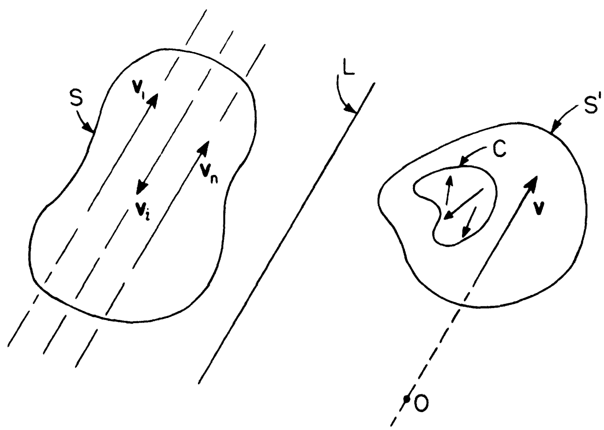


FIG. 3.5.14

torque is perpendicular to L, together with a single vector whose line of action is parallel to L.

Proof: Replace S in accordance with 3.5.6, using any point O as base point (see Fig. 3.5.14). Then the line of action of $\mathbf{v}$ is parallel to L, and the torque $\mathbf{T}$ of C is given by

$$\mathbf{T} \underset{(3.5.6)}{=} \mathbf{M}^{S/O} \underset{(3.3)}{=} \sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/O}$$

Now,

$$\mathbf{M}^{\mathbf{v}_i/O} \underset{(3.2.7)}{=} \mathbf{M}^{\mathbf{v}_i/L_1} + \mathbf{M}^{\mathbf{v}_i/L_2} + \mathbf{M}^{\mathbf{v}_i/L_3}$$

where L_1, L_2, L_3 are mutually perpendicular lines intersecting at O. If L_3 is taken parallel to L, then

$$\mathbf{M}^{\mathbf{v}_i/L_3} \underset{(3.2.4)}{=} 0$$

and, as $\mathbf{M}^{\mathbf{v}_i/L_1}$ and $\mathbf{M}^{\mathbf{v}_i/L_2}$ are respectively parallel to L_1 and L_2, $\mathbf{M}^{\mathbf{v}_i/O}$ is perpendicular to L. Hence, $\mathbf{T}$ is perpendicular to L.

3.5.15 If the resultant of a system S of parallel vectors $\mathbf{v}_i$, $i = 1, 2, \ldots, n$, is not equal to zero, S can be replaced with a single vector. This vector is the resultant $\mathbf{R}$ of S, placed on the central axis of S. Furthermore, the central axis passes through the centroid P^* of a set of points P_i, $i = 1, 2, \ldots, n$, of strengths v_i, $i = 1, 2, \ldots, n$, obtained as follows: P_i is any point on the line of action of $\mathbf{v}_i$, and v_i is a measure number satisfying the equation

$$\mathbf{v}_i = v_i\mathbf{n}$$

where $\mathbf{n}$ is a unit vector parallel to the vectors of S.

Proof: The minimum moment $\mathbf{M}^*$ of S is given by

$$\mathbf{M}^* \underset{(3.3.6)}{=} \frac{\mathbf{R}\cdot\mathbf{M}^{S/O}}{\mathbf{R}^2}\mathbf{R}$$

where $\mathbf{R}$ is the resultant of S and $\mathbf{M}^{S/O}$ the moment of S about an arbitrarily selected reference point. It was shown in 3.5.14 that $\mathbf{M}^{S/O}$ is perpendicular to $\mathbf{R}$. Hence

$$\mathbf{M}^* = 0$$

and when S is reduced to a wrench (see 3.5.8), the wrench consists entirely of $\mathbf{R}$, placed on the central axis of S.

The central axis passes through the point P^* whose position vector $\mathbf{p}^*$ relative to O is given by

$$\mathbf{p}^* \underset{(3.3.6)}{=} \frac{\mathbf{R}\times\mathbf{M}^{S/O}}{\mathbf{R}^2}$$

Now, if

$$\mathbf{v}_i = v_i\mathbf{n}$$

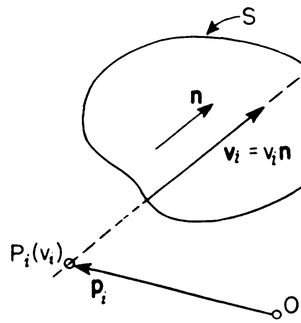


FIG. 3.5.15a

and $\mathbf{p}_i$ is the position vector of P_i (see Fig. 3.5.15a) relative to O, then

$$\mathbf{R} = \sum_{i=1}^{n} \mathbf{v}_i = \sum_{i=1}^{n} v_i\mathbf{n} \underset{(1.9.3)}{=} \left(\sum_{i=1}^{n} v_i\right)\mathbf{n}$$

and

$$\mathbf{M}^{S/O} \underset{(3.3)}{=} \sum_{i=1}^{n} \mathbf{M}^{\mathbf{v}_i/O} \underset{(3.1)}{=} \sum_{i=1}^{n} \mathbf{p}_i \times (v_i\mathbf{n})$$

$$\underset{(1.15.4,1.15.6)}{=} \left(\sum_{i=1}^{n} \mathbf{p}_i v_i\right) \times \mathbf{n}$$

Hence,

$$\mathbf{p}^* = \frac{\left[\left(\sum_{i=1}^{n} v_i\right)\mathbf{n}\right] \times \left[\left(\sum_{i=1}^{n} \mathbf{p}_i v_i\right) \times \mathbf{n}\right]}{\left(\sum_{i=1}^{n} v_i\right)^2} \underset{(1.17.1.)}{=} \frac{\sum_{i=1}^{n} \mathbf{p}_i v_i}{\sum_{i=1}^{n} v_i} + \lambda\mathbf{n}$$

where λ is an appropriately selected scalar. The first term in this expression is the position vector, relative to O, of the centroid of the set of points P_i, $i = 1, \ldots, n$, of strengths v_i, $i = 1, \ldots, n$. (See 2.4.1.)

Problem: The system of five forces shown in Fig. 3.5.15b is to be replaced with a single force $\mathbf{F}$. Show $\mathbf{F}$ in a sketch.

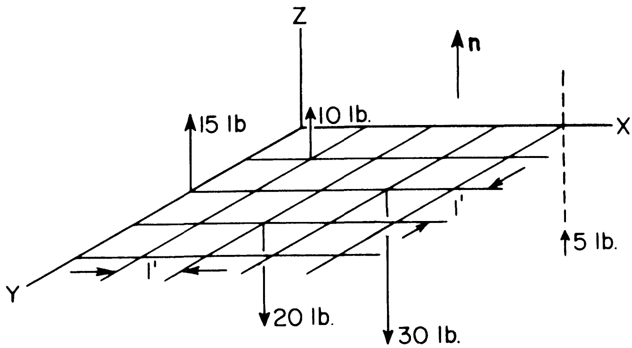


FIG. 3.5.15b

Solution: Let $\mathbf{n}$ be a unit vector parallel to the lines of action of the five forces (see Fig. 3.5.15b). $\mathbf{F}$ is the resultant of these forces:

$$\mathbf{F} = (15 + 10 + 5 - 20 - 30)\mathbf{n} = -20\mathbf{n} \text{ lb}$$

Let x^* and y^* be the X and Y coordinates of the centroid of the set of points at which the lines of action of the forces intersect the X-Y plane, and take the strengths of these points equal to the measure numbers of the corresponding forces. Then (see 2.4.5)

$$x^* = \frac{15(0) + 10(1) + 5(4) + (-20)(2) + (-30)(3)}{15 + 10 + 5 + (-20) + (-30)} = 5 \text{ ft}$$

$$y^* = \frac{15(2) + 10(1) + 5(0) + (-20)(3) + (-30)(2)}{15 + 10 + 5 + (-20) + (-30)} = 4 \text{ ft}$$

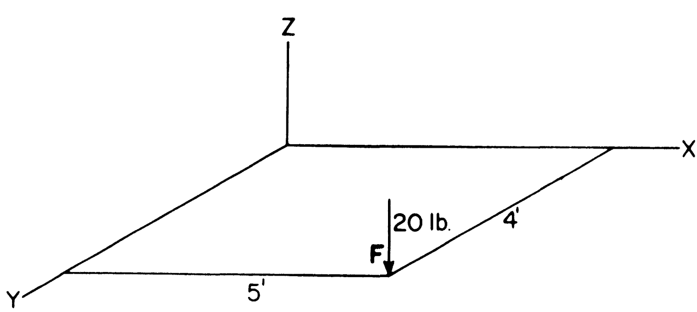


FIG. 3.5.15c

$\mathbf{F}$ is shown in Fig. 3.5.15c.

ZERO SYSTEMS

3.6 Definition

A system S of bound vectors is called a *zero system* if *both* of the following conditions are fulfilled: (a) The resultant of S is equal to zero. (b) There exists at least one point about which the moment of S is equal to zero.

3.6.1 Given any line L, the sum of the L resolutes of the vectors of a zero system S is equal to zero.

Proof: The sum of the L resolutes of the vectors in S is equal to the L resolute of the resultant of S (see 1.14.7). The resultant of S is equal to zero. Hence its L resolute is equal to zero.

Problem: If the system of forces shown in Fig. 3.6.1a is a zero system, what is the value of F?

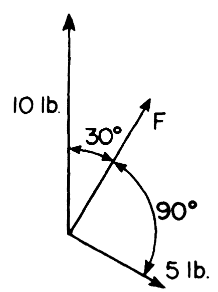


FIG. 3.6.1a

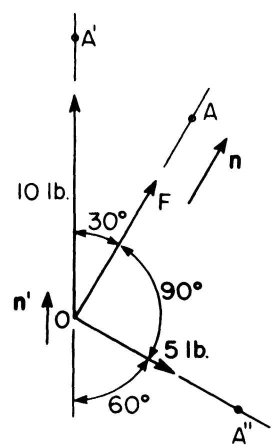


FIG. 3.6.1b

Solution: Let $\mathbf{n}$ be a unit vector parallel to OA (see Fig. 3.6.1b), and set the sum of the $\mathbf{n}$ resolutes of the three forces equal to zero:

$$10 \cos 30° \, \mathbf{n} + F\mathbf{n} = 0$$

or

$$(0.866 + F)\mathbf{n} = 0$$

Hence,

$$8.66 + F = 0$$

and

$$F = -8.66 \text{ lb}$$

3.6.2 The resolute of a vector parallel to a line can always be expressed as the product of a unit vector $\mathbf{n}$ and a measure number. When this is done for the resolute of each vector of a zero system, and these resolutes are then added, there results a scalar equation governing the measure numbers (in Problem 3.6.1, the equation $8.66 + F = 0$). This equation can often be written by inspection, once the unit vector $\mathbf{n}$ has been chosen in order to establish a "sign convention" for the measure numbers.

Problem: Referring to Problem 3.6.1 and Fig. 3.6.1b, determine F by setting the sum of the $\mathbf{n}'$ resolutes of the three forces equal to zero.

Solution:

$$10 + F \cos 30° - 5 \cos 60° = 0$$

$$F = \frac{-10 + 5(\frac{1}{2})}{0.866} = -8.66 \text{ lb}$$

3.6.3 The form of the scalar equation obtained by setting a sum of resolutes parallel to a line equal to zero depends on the line. Frequently, one line is more convenient than another; e.g., in Problem 3.6.1, line OA is more convenient than line OA', and line OA'' cannot be used at all, for the purpose of evaluating F.

3.6.4 When a zero system is composed either partly or entirely of couples, the vectors comprising the couples contribute nothing to equations obtained by setting sums of resolutes equal to zero.

Problem: A zero force system consists of four forces, a simple couple C_1, and a couple of torque $\mathbf{T}_2$, as shown in Fig. 3.6.4. Determine F.

Solution: Set the sum of the AB resolutes of all forces equal to zero. This gives

$$-20 + \tfrac{4}{5}F = 0; \quad F = 25 \text{ lb}$$

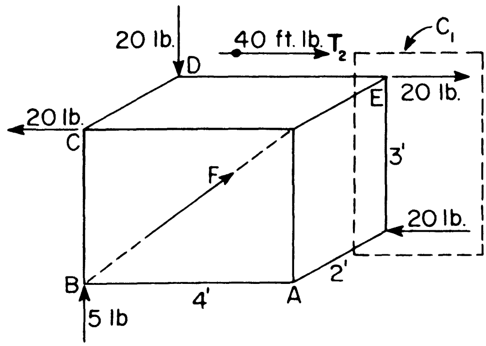


FIG. 3.6.4

3.6.5 Given any point A, the moment of a zero system S about A is equal to zero.

Proof: There exists at least one point, say A', about which S has zero moment. Keeping in mind that the resultant of S is equal to zero, it follows from 3.3.3 that

$$\mathbf{M}^{S/A} = \mathbf{M}^{S/A'} = 0$$

Problem: A zero system S consists of only two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$. Show that the lines of action of $\mathbf{v}_1$ and $\mathbf{v}_2$ coincide.

Solution: Let P be a point on the line of action of $\mathbf{v}_1$. Then

$$\mathbf{M}^{S/P} \underset{(3.3)}{=} \mathbf{M}^{\mathbf{v}_1/P} + \mathbf{M}^{\mathbf{v}_2/P} \underset{(3.1.2)}{=} \mathbf{M}^{\mathbf{v}_2/P}$$

As S is a zero system,

$$\mathbf{M}^{S/P} \underset{(3.6.5)}{=} 0$$

Hence,

$$\mathbf{M}^{\mathbf{v}_2/P} = 0$$

It follows from 3.1.2 that the line of action of $\mathbf{v}_2$ passes through point P. As P is any point on the line of action of $\mathbf{v}_1$, the line of action of $\mathbf{v}_2$ thus passes through every point on the line of action of $\mathbf{v}_1$; i.e., the lines of action of $\mathbf{v}_1$ and $\mathbf{v}_2$ coincide.

3.6.6 Given any line L, the moment of a zero system S about L is equal to zero.

Proof: Let A be a point on L. $\mathbf{M}^{S/L}$ is equal to the L resolute of $\mathbf{M}^{S/A}$ (see 3.3.2). Now,

$$\mathbf{M}^{S/A} \underset{(3.6.5)}{=} 0$$

Hence the L resolute of $\mathbf{M}^{S/A}$ is equal to zero.

Problem (a): A zero system S consists of three vectors, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Show (1) that the lines of action of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are coplanar, and (2) that these lines of action are either concurrent or parallel.

Solution (1): Let P_i, $i = 1, 2, 3$, be points on the lines of action of $\mathbf{v}_i$, $i = 1, 2, 3$, as shown in Fig. 3.6.6; and let L_i, $i = 1, 2, 3$,

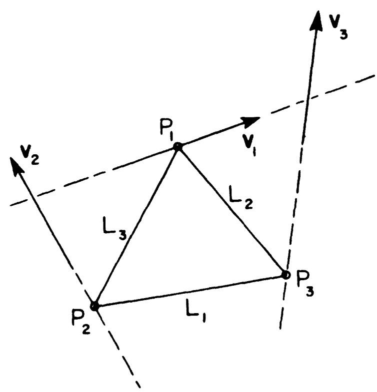


Fig. 3.6.6

be lines joining these points. Find the moment of S about each of these lines:

$$\mathbf{M}^{S/L_i} \underset{(3.3)}{=} \mathbf{M}^{\mathbf{v}_1/L_i} + \mathbf{M}^{\mathbf{v}_2/L_i} + \mathbf{M}^{\mathbf{v}_3/L_i}$$

$$\underset{(3.2.3)}{=} \mathbf{M}^{\mathbf{v}_i/L_i}, \quad i = 1, 2, 3$$

As S is a zero system,

$$\mathbf{M}^{S/L_i} \underset{(3.6.6)}{=} 0, \qquad i = 1, 2, 3$$

Hence,

$$\mathbf{M}^{\mathbf{v}_i/L_i} = 0, \qquad i = 1, 2, 3$$

and the line of action of $\mathbf{v}_i$ either is parallel to L_i, or intersects L_i. In either case the line of action of $\mathbf{v}_i$ lies in the plane determined by the three points P_i, $i = 1, 2, 3$.

Solution (2): First, suppose that no two of the lines of action are parallel. Let P be the point of intersection of any two. The third line of action must pass through P, in order that the moment of S about P be equal to zero. Next, if two of the lines of action are parallel to each other while the third intersects these two at points A and B, the moment of S about neither A nor B is equal to zero. Hence A and B cannot exist; i.e., the third line of action must be parallel to the other two.

Problem (b): Solve Problem 3.6.4 by setting the moment of the given force system about line CD equal to zero.

Solution: Let $\mathbf{n}$ be a unit vector parallel to CD and having the sense CD. Then

$$-3(\tfrac{4}{5}F)\mathbf{n} + 3(20)\mathbf{n} = 0$$

or

$$(-\tfrac{4}{5}F + 20)\mathbf{n} = 0$$

Hence,

$$-\tfrac{4}{5}F + 20 = 0$$

and

$$F = 25 \text{ lb}$$

3.6.7 The moment of a vector (or couple) about a line can always be expressed as the product of a unit vector $\mathbf{n}$ and a measure number. When this is done for the moment of each vector of a zero system, and these moments are then added, there results a scalar equation governing the measure numbers (in Problem 3.6.6(b), the equation $-\tfrac{4}{5}F + 20 = 0$). This equation can often be written by inspection, once the unit vector $\mathbf{n}$ has been chosen in order to establish a "sign convention" for the measure numbers.

Problem: Solve Problem 3.6.4 by setting the moment of the given force system about line DE equal to zero.

Solution:

$$-2(5) - 2(\tfrac{3}{5}F) + 40 = 0$$

$$F = 25 \text{ lb}$$

3.6.8 The form of the scalar equation obtained by setting a sum of moments about a line equal to zero depends on the line. Frequently, one line is more convenient than another; e.g., comparing the solutions of Problems 3.6.6(b) and 3.6.7, line CD is seen to be more convenient than line DE, and line AB (see Fig. 3.6.4) cannot be used at all, for the purpose of evaluating F.

3.6.9 The *orthogonal projection* of a bound vector $\mathbf{v}$, on a plane N (see Fig. 3.6.9) which is perpendicular to a unit vector $\mathbf{n}$, is a

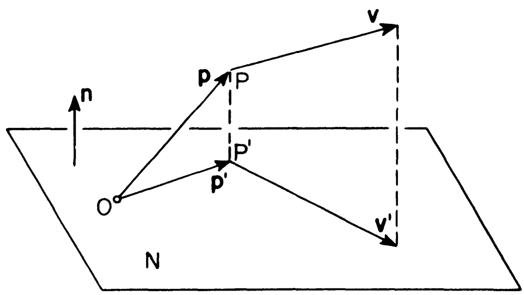


FIG. 3.6.9

bound vector $\mathbf{v}'$ equal to the resolute of $\mathbf{v}$ perpendicular to $\mathbf{n}$. The point of application of $\mathbf{v}'$ is the point P' at which a line parallel to $\mathbf{n}$ and passing through the point of application P of $\mathbf{v}$ intersects N.

If S is a system of bound vectors, and S' consists of the orthogonal projections of the vectors in S, on plane N, then (a) if S is a zero system, S' is a zero system, and (b) if S is a couple of torque $\mathbf{T}$, S' is either a couple whose torque $\mathbf{T}'$ is equal to the $\mathbf{n}$ resolute of $\mathbf{T}$, or S' is a zero system.

Proof: Let $\mathbf{v}$ be a typical vector of S, $\mathbf{v}'$ the corresponding vector of S'. Then

$$\mathbf{v}' \underset{(\text{P1.17.1(a)})}{=} \mathbf{n} \times (\mathbf{v} \times \mathbf{n})$$

If $\mathbf{R}$ is the resultant of S, the resolute of $\mathbf{R}$ perpendicular to $\mathbf{n}$ is equal to $\mathbf{n} \times (\mathbf{R} \times \mathbf{n})$. As shown in 1.14.8, this resolute of $\mathbf{R}$ is equal to the resultant $\mathbf{R}'$ of S'. Thus

$$\mathbf{R}' = \mathbf{n} \times (\mathbf{R} \times \mathbf{n}) \tag{1}$$

Let O be a point of N, $\mathbf{p}$ and $\mathbf{p}'$ the position vectors of P and P' relative to O. Then the moments of S and S' about O are given by

$$\mathbf{M}^{S/O} = \sum \mathbf{p} \times \mathbf{v}, \quad \mathbf{M}^{S'/O} = \sum \mathbf{p}' \times \mathbf{v}'$$

But $\mathbf{p}'$ is the resolute of $\mathbf{p}$ perpendicular to $\mathbf{n}$; that is,

$$\mathbf{p}' = \mathbf{n} \times (\mathbf{p} \times \mathbf{n})$$

Hence,

$$\sum \mathbf{p}' \times \mathbf{v}' = \sum [\mathbf{n} \times (\mathbf{p} \times \mathbf{n})] \times [\mathbf{n} \times (\mathbf{v} \times \mathbf{n})]$$
$$= \mathbf{n} \cdot \left(\sum \mathbf{p} \times \mathbf{v}\right) \mathbf{n}$$

and

$$\mathbf{M}^{S'/O} = \mathbf{n} \cdot \mathbf{M}^{S/O}\mathbf{n} \tag{2}$$

Part (*a*): If S is a zero system,

$$\mathbf{R} = 0$$

and

$$\mathbf{M}^{S/O} = 0$$

Substitution into Eqs. (1) and (2) shows that S' is a zero system.

Part (*b*): If S is a couple of torque $\mathbf{T}$,

$$\mathbf{R} = 0$$

and

$$\mathbf{M}^{S/O} = \mathbf{T}$$

Substitution into Eqs. (1) and (2) shows that

$$\mathbf{R}' = 0$$

and

$$\mathbf{M}^{S'/O} = \mathbf{n} \cdot \mathbf{T}\mathbf{n}$$

Thus, S' has a zero resultant, and the moment of S' about O is equal to the $\mathbf{n}$ resolute of $\mathbf{T}$. If $\mathbf{T}$ is not perpendicular to $\mathbf{n}$, S' is, therefore, a couple whose torque $\mathbf{T}'$ is equal to the $\mathbf{n}$ resolute of $\mathbf{T}$; and if $\mathbf{T}$ is perpendicular to $\mathbf{n}$, S' is a zero system.

3.6.10 When a system S of bound vectors is a zero system, infinitely many scalar equations governing the vectors of S can be written by setting the moments of S about various lines L, or the sums of the L resolutes of the vectors of S, equal to zero. At most six such equations are both independent of each other and non-trivial. This follows from the fact that each of the two conditions imposed on the vectors of S by the definition of a zero system can be expressed as one vector equation.

One way to obtain the largest number of independent, non-trivial, scalar equations available in a given case, is to (a) set the sums of the L_1, L_2, L_3 resolutes of the vectors of S equal to zero, and (b) set the moments of S about L_1, L_2, L_3 equal to zero, L_1, L_2, L_3 being mutually perpendicular lines. If this procedure yields fewer than six equations, no other will yield a greater number.

Problem: The lines of action of the vectors of a zero system S lie in a plane P. Show that at least three independent, trivial, scalar equations governing the vectors of S can be written. (Consequently, at most three independent, non-trivial, scalar equations are available in this case.)

Solution: Let L_1, L_2, L_3 be mutually perpendicular, concurrent lines, L_1 and L_2 lying in plane P. Then the sum of the L_3 resolutes of the vectors of S; the moment of S about L_1; and the moment of S about L_2 are each equal to zero, whether or not S is a zero system, and no information about the vectors of S is obtained by setting these quantities equal to zero. The corresponding equations are, therefore, trivial.

4 STATIC EQUILIBRIUM

4.1 Force

In the study of certain physical phenomena it is convenient to employ a class of bound vectors called forces. The sections which follow contain descriptions of forces associated with specific situations.

GRAVITATIONAL FORCES

4.2 Mutual gravitational attraction of particles

With every pair of distinct particles in the universe there are associated two forces, $\mathbf{F}$ and $\mathbf{F}'$, given by

$$\mathbf{F} = -Gmm'\mathbf{p}(\mathbf{p}^2)^{-\frac{3}{2}} \tag{1}$$

$$\mathbf{F}' = -\mathbf{F} \tag{2}$$

where G is a positive constant, the same for all particles; m and m' are the masses of the particles; $\mathbf{p}$ is the position vector of P relative

Fig. 4.2

to P' (see Fig. 4.2), P and P' being the points at which the particles are situated.

The points of application of $\mathbf{F}$ and $\mathbf{F}'$ are P and P', respectively. (The lines of action of both forces thus coincide with line PP'.)

$\mathbf{F}$ and $\mathbf{F}'$ are called, respectively, the *gravitational force* exerted on the particle at P by the particle at P' and the gravitational force exerted on the particle at P' by the particle at P.

4.2.1 The word "distinct" in 4.2 rules out the possibility $\mathbf{p} = 0$: Two distinct particles cannot be situated at the same point.

4.2.2 Equation (2), sometimes called a *law of action and reaction,* is not independent of Eq. (1), because the procedure by means of which $\mathbf{F}$ is constructed must apply equally well to the construction of $\mathbf{F}'$. Eq. (2) is, therefore, an immediate consequence of Eq. (1) and of the fact that $\mathbf{p}'$, the position vector of P' relative to P, is equal to $-\mathbf{p}$.

4.2.3 The magnitudes of the gravitational forces $\mathbf{F}$ and $\mathbf{F}'$ are equal to each other and are inversely proportional to the square of the distance between P and P':

$$|\mathbf{F}| = |\mathbf{F}'| = Gmm'/\mathbf{p}^2$$

Proof:

$$|\mathbf{F}| \underset{(4.2)}{=} |-Gmm'\mathbf{p}(\mathbf{p}^2)^{-\frac{3}{2}}|$$

$$\underset{(1.4)}{=} Gmm'(\mathbf{p}^2)^{-\frac{3}{2}}|\mathbf{p}|$$

$$\underset{(1.14.11)}{=} Gmm'(\mathbf{p}^2)^{-\frac{3}{2}}(\mathbf{p}^2)^{\frac{1}{2}} = Gmm'/\mathbf{p}^2$$

4.2.4 The *dimensions* of force (F), length (L), mass (M), and time (T) are related to each other as follows:

$$(F)(L)^{-1}(M)^{-1}(T)^2 = 1$$

Problem: A particle of mass 2 grams exerts a force of magnitude 6.66×10^{-5} dynes on a particle of mass 5 grams when the two are separated by a distance of 0.1 centimeters. Determine the constant G, expressing the result in units of (1) force, length and mass, and (2) length, mass and time.

Solution (1): Use 4.2.3, with

$$|\mathbf{F}| = 6.66 \times 10^{-5} \text{ dyne}$$

$$m = 2 \text{ gm}, \quad m' = 5 \text{ gm}, \quad \mathbf{p}^2 = (0.1)^2 = 0.01 \text{ cm}^2$$

This gives

$$G = |\mathbf{F}^5|\mathbf{p}^2/mm' = 6.66 \times 10^{-8} \times 0.01/2 \times 5$$
$$= 6.66 \times 10^{-8} \text{ dyne cm}^2 \text{ gm}^{-2}$$

Solution (2) Use 4.2.4 to express the unit of force (the dyne) in terms of units of mass, length and time:

$$(1 \text{ dyne})(1 \text{ cm})^{-1}(1 \text{ gm})^{-1}(1 \text{ sec})^2 = 1$$
$$1 \text{ dyne} = 1 \text{ cm gm sec}^{-2}$$

Substitute into the result obtained in (1):

$$G = 6.66 \times 10^{-8} \text{ cm gm sec}^{-2} \text{ cm}^2 \text{ gm}^{-2}$$
$$= 6.66 \times 10^{-8} \text{ cm}^3 \text{ gm}^{-1} \text{ sec}^{-2}$$

4.2.5 In general, three relationships are required to establish the connection between two systems of units. In the case of the *metric absolute* (dyne, centimeter, gram, second) and *British gravitational* (pound, foot, slug, second) systems, only two relationships are required, because the unit of time is the same in the two systems.

Problem: Use the results of Problem 4.2.4 to express the constant G in units of force, length and time of the British gravitational system of units, taking

$$1 \text{ cm} = 0.0328 \text{ ft}, \quad 1 \text{ gm} = 6.86 \times 10^{-5} \text{ slug}$$

Solution: Use 4.2.4 to re-write the result obtained in Problem 4.2.4 as

$$G = 6.66 \times 10^{-8} \text{ dyne}^{-1} \text{ cm}^4 \text{ sec}^{-4}$$

Find the relationship between the dyne and the pound by using 4.2.4 together with the given relationships between units of length and units of mass:

$$1 \text{ dyne} = 1 \text{ cm gm sec}^{-2} = 0.0328 \times 6.86 \times 10^{-5} \text{ ft slug sec}^{-2}$$
$$= 2.25 \times 10^{-6} \text{ ft slug sec}^{-2} = 2.25 \times 10^{-6} \text{ lb}$$

Substitute:

$$G = \frac{6.66 \times 10^{-8} \times (0.0328)^4}{2.25 \times 10^{-6}} = 3.42 \times 10^{-8} \text{ lb}^{-1} \text{ ft}^4 \text{ sec}^{-4}$$

4.2.6 The system of gravitational forces exerted on a particle P by the particles of a set S is equivalent to a single force $\mathbf{F}$ whose line of action passes through P. $\mathbf{F}$ is called *the gravitational force exerted on P by S.*

In general, the line of action of $\mathbf{F}$ does not pass through the mass center P^* of S. However, if the distances from the particles of S to the point P^* are sufficiently small in comparison with the distance between P^* and P, $\mathbf{F}$ is nearly equal to the gravitational force $\mathbf{F}^*$ exerted on P by a particle of mass m^* situated at P^*, m^* being the total mass of the particles in S; and the line of action of $\mathbf{F}$ then nearly passes through P^*.

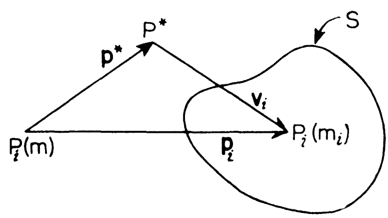


FIG. 4.2.6a

Proof: Introduce the following (see Fig. 4.2.6a):

P	a particle of mass m
P_i, $i = 1, 2, \ldots, n$	the particle of a set S
m_i	the mass of P_i
$\mathbf{F}_i$	the gravitational force exerted on P by P_i
$\mathbf{F}$	the resultant of the forces $\mathbf{F}_i$, $i = 1, \ldots, n$
P^*	the mass center of S
$\mathbf{p}^*$	the position vector of P^* relative to P
$\mathbf{p}_i$	the position vector of P_i relative to P
$\mathbf{r}_i$	the position vector of P_i relative to P^*
m^*	the total mass of S
$\mathbf{F}^*$	the gravitational force exerted on P by a particle of mass m^* at P^*

It must be shown that (a) the system of forces $\mathbf{F}_i$, $i = 1, 2, \ldots, n$, is equivalent to a single force whose line of action passes through P, and that (b) this force is nearly equal to $\mathbf{F}^*$ and has a line of action which nearly passes through P^*, when $|\mathbf{r}_i|$ ($i = 1, 2, \ldots, n$) is sufficiently small in comparison with $|\mathbf{p}^*|$.

The following relationships will be used:

$$\mathbf{F}_i \underset{(4.2)}{=} Gmm_i\mathbf{p}_i(\mathbf{p}_i^2)^{-\frac{3}{2}} \tag{1}$$

$$\mathbf{F} \underset{(1.9.1)}{=} \sum_{i=1}^{n} \mathbf{F}_i \tag{2}$$

$$\mathbf{F}^* \underset{(4.2)}{=} Gmm^*\mathbf{p}^*(\mathbf{p}^{*2})^{-\frac{3}{2}} \tag{3}$$

$$m^* = \sum_{i=1}^{n} m_i \tag{4}$$

$$\mathbf{p}_i \underset{(F4.2.6a)}{=} \mathbf{p}^* + \mathbf{r}_i \tag{5}$$

$$\sum_{i=1}^{n} m_i\mathbf{r}_i \underset{(2.6,\ 2.4)}{=} 0 \tag{6}$$

Part (*a*): In accordance with 4.2, the lines of action of the forces $\mathbf{F}_i$, $i = 1, 2, \ldots, n$, intersect at P. It follows from 3.5.10 that the system of forces $\mathbf{F}_i$, $i = 1, 2, \ldots, n$, is equivalent to the single force $\mathbf{F}$ placed in such a way that its line of action passes through P. This is indicated schematically in Fig. 4.2.6b.

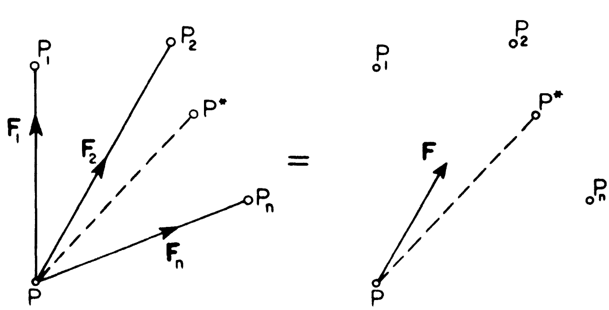


FIG. 4.2.6b

Part (*b*): Express $\mathbf{F}_i$ in a form suitable for taking into account the possibility

$$|\mathbf{r}_i| < |\mathbf{p}^*|$$

From Eqs. (1) and (5),

$$\begin{aligned}\mathbf{F}_i &= Gmm_i(\mathbf{p}^* + \mathbf{r}_i)[(\mathbf{p}^* + \mathbf{r}_i)^2]^{-\frac{3}{2}} \\ &\underset{(1.14.11)}{=} Gmm_i(\mathbf{p}^* + \mathbf{r}_i)[\mathbf{p}^{*2} + 2\mathbf{p}^*\cdot\mathbf{r}_i + \mathbf{r}_i^2]^{-\frac{3}{2}} \\ &= Gmm_i(\mathbf{p}^{*2})^{-\frac{3}{2}}(\mathbf{p}^* + \mathbf{r}_i)\left[1 + \frac{2\mathbf{p}^*\cdot\mathbf{r}_i}{\mathbf{p}^{*2}} + \frac{\mathbf{r}_i}{\mathbf{p}^{*2}}\right]^{-\frac{3}{2}}\end{aligned}$$

Use the Binomial Theorem, dropping all terms whose magnitudes depend on the second or higher powers of the ratio $|\mathbf{r}_i|/|\mathbf{p}^*|$:

$$\mathbf{F}_i \approx Gmm_i(\mathbf{p}^{*2})^{-\frac{3}{2}}\left(\mathbf{p}^* + \mathbf{r}_i - \frac{3\mathbf{p}^*}{\mathbf{p}^{*2}}\,\mathbf{p}^*\cdot\mathbf{r}_i\right)$$

Substitute into Eq. (2):

$$\mathbf{F} \approx Gm(\mathbf{p}^{*2})^{-\frac{3}{2}}\left[\mathbf{p}^*\sum_{i=1}^{n} m_i + \sum_{i=1}^{n} m_i\mathbf{r}_i - \frac{3\mathbf{p}^*}{\mathbf{p}^{*2}}\,\mathbf{p}^*\cdot\sum_{i=1}^{n} m_i\mathbf{r}_i\right]$$

Use Eqs. (4), (6), and (3):

$$\mathbf{F} \approx Gm(\mathbf{p}^{*2})^{-\frac{3}{2}}\mathbf{p}^*m^* = \mathbf{F}^*$$

The line of action of $\mathbf{F}^*$ passes through P^* (see 4.2). Hence the line of action of $\mathbf{F}$ nearly passes through P^*.

4.2.7 The statements made in 4.2.6 remain correct when the words "on" and "by" are replaced with, respectively, "by" and "on."

4.2.8 Referring to 4.2.6, the particle P must be distinct from each of the particles P_i, $i = 1, 2, \ldots, n$. If this is not the case, no meaning is attributed to the phrase "the force exerted on P by S." For example, if $n = 2$, particle P_1 cannot be said to exert a force on the set of particles P_1 and P_2.

Problem: 100 particles, each having a mass of 0.01 gm, are placed at equal intervals along a circle of radius 1 cm. Determine, approximately, the gravitational force $\mathbf{F}$ exerted by this set of particles on a particle of mass 0.5 gm placed at a point P whose distance from the center C of the circle is 10 cm. Show $\mathbf{F}$ in a sketch.

Solution: $\mathbf{F}$ is approximately equal to the force exerted on the particle at P by a particle of mass 100×0.01 gm at C:

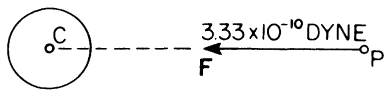


FIG. 4.2.8

$$|\mathbf{F}| \underset{(4.2.3)}{\approx} G\frac{(100 \times 0.01)(0.5)}{10^2}$$

$$\underset{(P4.2.4)}{=} 6.66 \times 10^{-8} \times 0.005 = 3.33 \times 10^{-10} \text{ dyne}$$

See Fig. 4.2.8.

4.3 Mutual gravitational attraction of a particle and a continuous body

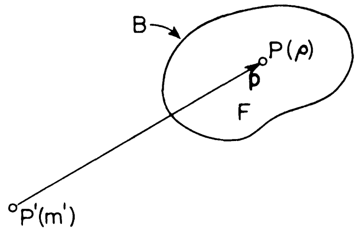
Divide the figure occupied by the body B into elements of arbitrary size and shape. Pick a point in each element, and determine the gravitational force (see 4.2.6) exerted by a particle P' of mass m' on a set of particles situated at these points, each particle of the set having the mass of the associated element. The limit approached by this force, when the number of elements tends to infinity and each element shrinks to a point, is a force $\mathbf{F}$ whose line of action passes through P'. $\mathbf{F}$, called *the gravitational force exerted on B by P'*, is given by

$$\mathbf{F} = -Gm' \int_F \mathbf{p}(\mathbf{p}^2)^{-\frac{3}{2}}\rho \, d\tau$$

Proof: Introduce the following (see Fig. 4.3):

- F the figure (curve, surface, solid) occupied by the body B
- P a typical point of F
- ρ the mass density of B at P
- $d\tau$ the length, area, or volume of a differential element of F
- $\mathbf{p}$ the position vector of P relative to P'
- $\mathbf{F}$ the gravitational force exerted on B by P'

FIG. 4.3



F is, by definition, the limit of a force of the kind considered in 4.2.6, i.e., a force whose line of action passes through P' and which is given by an expression of the form

$$-Gm' \sum_{i=1}^{n} m_i \mathbf{p}_i (\mathbf{p}_i{}^2)^{-\frac{3}{2}}$$

As n tends to infinity and the elements of F shrink to points, the sum $\sum_{i=1}^{n} m_i \mathbf{p}_i (\mathbf{p}_i{}^2)^{-\frac{3}{2}}$ approaches the integral $\int_F \mathbf{p}(\mathbf{p}^2)^{-\frac{3}{2}} \rho \, d\tau$ as a limit.

4.3.1 The particle P' and the body B must be distinct from each other in order for the phrase "force exerted on B by P'" to be meaningful.

Problem: To explain the statement "a sphere attracts as if concentrated at its center," (1) determine the gravitational force exerted on a particle P' of mass m' by a sphere of mass m, the density ρ at a point P of the sphere depending only on the distance between P and the center C of the sphere. (The distance "a" between C and the particle must be taken greater than the radius R of the sphere.) (2) Determine the gravitational force $\mathbf{F}'$ exerted on P' by a particle of mass m situated at C.

Solution (1) (see Fig. 4.3.1a):
Express $\mathbf{p}$ and $d\tau$ in terms of r, θ, ψ:

$$\mathbf{p} = r \sin\psi \cos\theta \, \mathbf{n}_1 + r \sin\psi \sin\theta \, \mathbf{n}_2 + (a + r\cos\psi)\, \mathbf{n}_3$$

$$d\tau \underset{(\text{P2.5.2(c)})}{=} r^2 \sin\psi \, d\theta \, d\psi \, dr$$

m, the mass of the sphere, is given by

$$m = \int_F \rho \, d\tau = \int_0^R dr \int_0^{2\pi} d\theta \int_0^{\pi} r^2 \sin\psi \, \rho \, d\psi$$

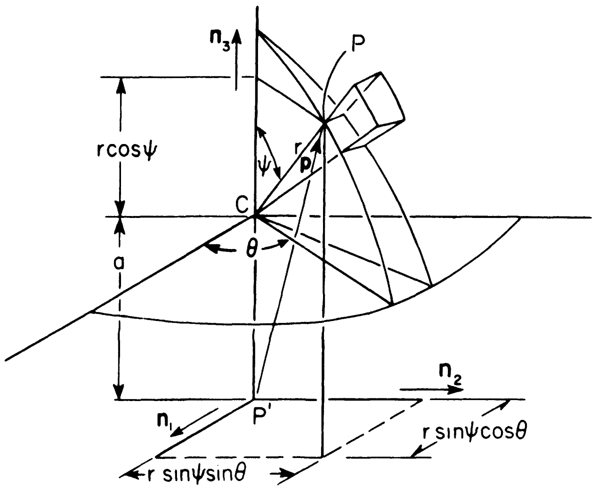


FIG. 4.3.1a

Integrate with respect to θ and ψ. (Integration with respect to r cannot be performed explicitly, because ρ depends on r in an unspecified way.)

$$m = 4\pi \int_0^R r^2 \rho\, dr \tag{1}$$

$\mathbf{F}$ is the negative of the gravitational force exerted on the sphere by the particle:

$$\mathbf{F} \underset{(4.3)}{=} Gm' \int_F \mathbf{p}(\mathbf{p}^2)^{-\frac{3}{2}} \rho\, d\tau$$

Using the law of cosines,

$$\mathbf{p}^2 = a^2 + r^2 + 2ar \cos\psi$$

Hence,

$$\mathbf{F} = Gm'(I_1\mathbf{n}_1 + I_2\mathbf{n}_2 + I_3\mathbf{n}_3) \tag{2}$$

where

$$I_1 = \int_0^R dr \int_0^\pi d\psi \int_0^{2\pi} \frac{\sin^2\psi \cos\theta\, r^3}{(a^2 + r^2 + 2ar\cos\psi)^{\frac{3}{2}}} \rho\, d\theta$$

$$I_2 = \int_0^R dr \int_0^\pi d\psi \int_0^{2\pi} \frac{\sin^2\psi \sin\theta\, r^3}{(a^2 + r^2 + 2ar\cos\psi)^{\frac{3}{2}}} \rho\, d\theta$$

$$I_3 = \int_0^R dr \int_0^\pi d\psi \int_0^{2\pi} \frac{\sin\psi(a + r\cos\psi) r^2}{(a^2 + r^2 + 2ar\cos\psi)^{\frac{3}{2}}} \rho\, d\theta$$

Integrate with respect to θ:

$$I_1 = 0, \quad I_2 = 0 \tag{3}$$

$$I_3 = 2\pi \int_0^R dr \int_0^\pi \frac{\sin\psi(a + r\cos\psi)r^2}{(a^2 + r^2 + 2ar\cos\psi)^{\frac{3}{2}}} \rho \, d\psi \tag{4}$$

Introduce a new variable v, by letting

$$a^2 + r^2 + 2ar\cos\psi = v^2 \tag{5}$$

Then

$$r\cos\psi = (v^2 - a^2 - r^2)/2a$$

and

$$a + r\cos\psi = (v^2 + a^2 - r^2)/2a \tag{6}$$

Differentiate Eq. (5) with respect to v:

$$-2ar\sin\psi \frac{d\psi}{dv} = 2v$$

or

$$\sin\psi \, d\psi = -\frac{v}{ar} dv \tag{7}$$

Use Eq. (5) to determine the values of v corresponding to $\psi = 0$ and $\psi = \pi$: For $\psi = 0$,

$$v = (a^2 + r^2 - 2ar)^{\frac{1}{2}} = a + r$$

For $\psi = \pi$,

$$v = (a^2 + r^2 - 2ar)^{\frac{1}{2}} = a - r \quad (a > r)$$

Substitute from Eqs. (5), (6) and (7) into Eq. (4):

$$\begin{aligned} I_3 &= -\frac{\pi}{a^2} \int_0^R r\rho\, dr \int_{a+r}^{a-r} \frac{v^2 + a^2 - r^2}{v^2} dv \\ &= -\frac{\pi}{a^2} \int_0^R \left[\int_{a+r}^{a-r} dv + (a^2 - r^2) \int_{a+r}^{a-r} v^{-2}\, dv \right] r\rho\, dr \\ &= -\frac{\pi}{a^2} \int_0^R \left[a - r - (a + r) - (a^2 - r^2)\left(\frac{1}{a - r} - \frac{1}{a + r} \right) \right] r\rho\, dr \\ &= \frac{4\pi}{a^2} \int_0^R r^2 \rho\, dr \end{aligned}$$

Use Eq. (1):

$$I_3 = \frac{m}{a^2} \tag{8}$$

Substitute from Eqs. (3) and (8) into Eq. (2):

$$\mathbf{F} = \frac{Gmm'}{a^2}\mathbf{n}_3$$

The gravitational force $\mathbf{F}$ exerted on P' by the sphere is shown in Fig. 4.3.1b.

Solution (2): The gravitational force $\mathbf{F}'$ exerted on P' by a particle of mass m situated at C is identical with the force $\mathbf{F}$ shown in Fig. 4.3.1b. This follows from 4.2.

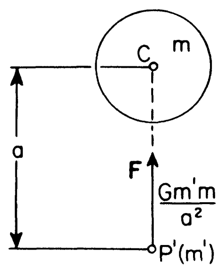


FIG. 4.3.1b

4.3.2 Problem 4.3.1 shows that the gravitational force exerted by a particle on a continuous body *may* be identical with the force exerted by the particle on a single particle placed at the mass center of the continuous body. In general, this is *not* the case; but if the distances from all points of a continuous body B to its mass center P^* are small in comparison with the distance from P^* to a particle P, the force exerted by P on B is nearly equal to the force exerted by P on a single particle placed at P^* and having the same mass as B. This is an immediate consequence of 4.2.6 and 2.7.

Problem: Determine F_1 and F_2, the $\mathbf{n}_1$ and $\mathbf{n}_2$ measure numbers of the force $\mathbf{F}$ exerted on a particle P' of mass m' by a thin, uniform wire of mass m and length l, when the particle and the wire occupy the positions shown in Fig. 4.3.2a. Show that the line of action of $\mathbf{F}$ does not pass through the mass center P^* of the wire.

Solution: As the wire is uniform, its mass density, ρ, is given by

$$\rho = \frac{m}{l}$$

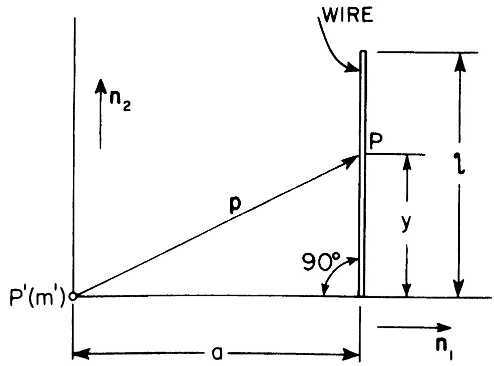


FIG. 4.3.2a

$\mathbf{p}$, the position vector of a typical point P of the wire, relative to P', is given by

$$\mathbf{p} = a\mathbf{n}_1 + y\mathbf{n}_2$$

and $d\tau$ by

$$d\tau = dy$$

Hence,

$$\begin{aligned}\mathbf{F} &= Gm' \int_F \mathbf{p}(\mathbf{p}^2)^{-\frac{3}{2}}\rho \, d\tau \\ &= \frac{Gmm'}{l}\left[\mathbf{n}_1 \int_0^l a(a^2 + y^2)^{-\frac{3}{2}}\, dy + \mathbf{n}_2 \int_0^l y(a^2 + y^2)^{-\frac{3}{2}}\, dy\right]\end{aligned}$$

and the $\mathbf{n}_1$ and $\mathbf{n}_2$ measure numbers of $\mathbf{F}$ are

$$F_1 = \frac{Gmm'}{l}\int_0^l a(a^2 + y^2)^{-\frac{3}{2}}\, dy = \frac{Gmm'}{a(l^2 + a^2)^{\frac{1}{2}}}$$

$$F_2 = \frac{Gmm'}{l}\int_0^l y(a^2 + y^2)^{-\frac{3}{2}}\, dy = \frac{Gmm'}{a(l^2 + a^2)^{\frac{1}{2}}}\left[\left(1 + \frac{a^2}{l^2}\right)^{\frac{1}{2}} - \frac{a}{l}\right]$$

Fig. 4.3.2b shows the point Q at which the line of action of $\mathbf{F}$ intersects the wire. The distance d from Q to one end of the wire is given by

$$\begin{aligned}d &= a \tan \theta = aF_2/F_1 \\ &= a\left[\left(1 + \frac{a^2}{l^2}\right)^{\frac{1}{2}} - \frac{a}{l}\right]\end{aligned}$$

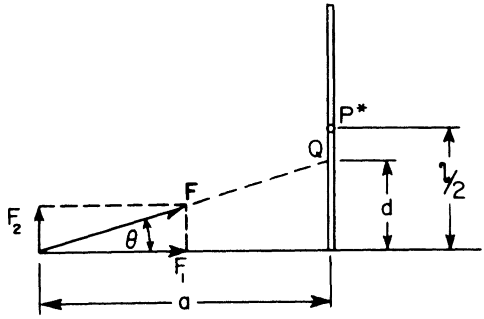


FIG. 4.3.2b

The line of action of **F** passes through P^* if $d = l/2$, that is, if

$$\frac{2a}{l}\left[\left(1 + \frac{a^2}{l^2}\right)^{\frac{1}{2}} - \frac{a}{l}\right] = 1$$

There exists no finite value of the ratio a/l which satisfies this equation. Consequently, the line of action of **F** does not pass through P^*. But (in agreement with 4.3.2) the equation can be satisfied to any desired degree of accuracy, by assigning a sufficiently large value to a/l.

For future reference, note that F_1 and F_2 can be expressed in the form

$$F_1 = \frac{Gmm'}{ab}, \quad F_2 = \frac{Gmm'}{ab}\frac{b - a}{l}$$

where b is the distance between P' and that end of the wire which is farthest from P'.

4.4 Mutual gravitational attraction of two continuous bodies

Divide the figure occupied by one of the bodies (B) into elements of arbitrary size and shape. Pick a point in each element, and regard the mass of the element as concentrated at this point in the form of a particle. Determine the forces exerted on these particles by the other body (B'), and replace them with a couple and a force whose line of action passes through an arbitrarily selected base point. The limits approached by this force and by the torque of the couple, when the number of elements tends to infinity and each element shrinks to a point, are, respectively, a force **F** whose line of action passes through the base point and a couple of torque **T**. **F** is called *the gravitational force*, and **T** *the torque of the gravitational couple, exerted on B by B'*. (The bodies B and B' must be distinct from each other; i.e., neither may contain any matter contained in the other.)

Problem: Fig. 4.4 shows a thin, semicircular wire B lying in a plane which passes through the end point of, and is normal to, a thin, straight wire B'. Both wires are uniform. Their masses are

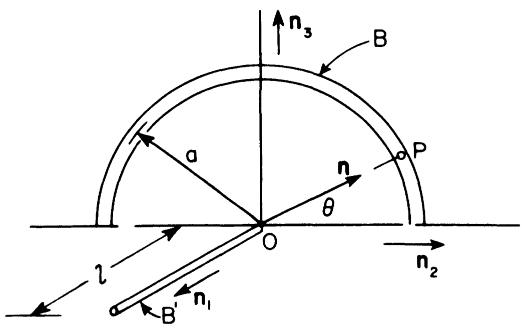


FIG. 4.4

M and M', respectively. Reduce the system of gravitational forces exerted on B by B', to a couple of torque **T** and a force **F** whose line of action passes through point O.

Solution: The mass density ρ of B is given by

$$\rho = \frac{M}{a\pi}$$

Hence the mass of an element of B situated at P (see Fig. 4.4) and having a length $a\,d\theta$ is

$$\rho a\,d\theta = \frac{M}{\pi}\,d\theta$$

and, letting this mass play the part of m', and M' that of m, in the results obtained in Problem 4.3.2, the gravitational force exerted by B' on this element of B is seen to be

$$\frac{GMM'}{\pi ab}\left(-\mathbf{n} + \frac{b-a}{l}\mathbf{n}_1\right)d\theta$$

where $\mathbf{n}$ and $\mathbf{n}_1$ are the unit vectors shown in Fig. 4.4, and b is the distance between P and that end of B' which is farthest from P.

The line of action of this force passes through P. Hence the moment of the force about point O is given by

$$a\mathbf{n} \times \left[\frac{GMM'}{\pi ab}\left(-\mathbf{n} + \frac{b-a}{l}\mathbf{n}_1\right)d\theta\right] = \frac{GMM'}{\pi bl}(b-a)\mathbf{n} \times \mathbf{n}_1\, d\theta$$

Thus, when the system of all such forces is replaced with a couple of torque $\mathbf{T}$ and a force $\mathbf{F}$ whose line of action passes through O,

$$\mathbf{T} = \frac{GMM'}{\pi bl}(b-a)\int_0^\pi \mathbf{n} \times \mathbf{n}_1\, d\theta$$

and

$$\mathbf{F} = \frac{GMM'}{\pi ab}\int_0^\pi \left(-\mathbf{n} + \frac{b-a}{l}\mathbf{n}_1\right)d\theta$$

Now,

$$\mathbf{n} = \cos\theta\, \mathbf{n}_2 + \sin\theta\, \mathbf{n}_3$$

Hence,

$$\mathbf{T} = \frac{GMM'}{\pi bl}(b-a)(I_3\mathbf{n}_2 - I_2\mathbf{n}_3)$$

and

$$\mathbf{F} = \frac{GMM'}{\pi ab}\left(\frac{b-a}{l}I_1\mathbf{n}_1 - I_2\mathbf{n}_2 - I_3\mathbf{n}_3\right)$$

where

$$I_1 = \int_0^\pi d\theta = \pi$$

$$I_2 = \int_0^\pi \cos\theta\, d\theta = 0$$

$$I_3 = \int_0^\pi \sin\theta\, d\theta = 2$$

Thus,

$$\mathbf{T} = \frac{2GMM'(b-a)}{\pi bl}\mathbf{n}_2$$

and

$$\mathbf{F} = \frac{GMM'}{ab}\left(\frac{b-a}{l}\mathbf{n}_1 - \frac{2}{\pi}\mathbf{n}_3\right)$$

4.4.1 There exist bodies for which the choice of base point can always be made in such a way that **T** vanishes; i.e., the system of forces exerted on such a body by *any* other body can be replaced with a single force. (One example, the sphere, has already been encountered.) These bodies are said to be *centrobaric;* and, while centrobaric bodies are the exception rather than the rule, every body B is approximately centrobaric as regards its interaction with bodies B' whose points are separated from the mass center P^* of B by distances large in comparison with the distances from P^* to the points of B. For, under these circumstances, the system of gravitational forces exerted on B by B' can be replaced, approximately, with a single force whose line of action passes through P^*. This follows from 4.3.2 and 4.4.

4.4.2 When a non-centrobaric body B (mass m) whose largest dimension is small in comparison with the radius of a sphere S (mass m') is located near the surface of S (see Fig. 4.4.2), the

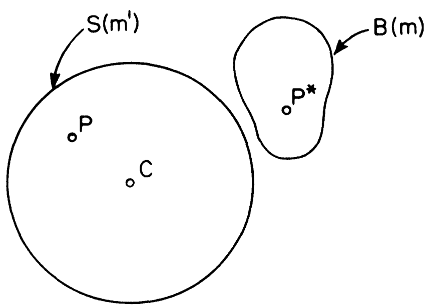


FIG. 4.4.2

conditions under which B is approximately centrobaric are not satisfied, because not all points of S are separated from the mass center P^* of B by distances large in comparison with the distances from P^* to the points of B. Nevertheless, the system of gravitational forces exerted on B by S can be replaced with a single force, provided only that the density ρ at a point P of S depend solely on the distance between P and the center C of S (see Problem 4.3.1); and (see 4.3.2) this force is nearly the same as that exerted on a particle of mass m at P^* by a particle of mass m' at C.

Problem: Regarding the earth as a sphere having a radius of 3960 miles, a mass of 4.11×10^{23} slugs, and a mass density which, at any point P, depends only on the distance between P and the center of the sphere, show that the magnitude of the gravitational force $\mathbf{F}$ exerted by the earth on a small (in comparison with the earth) body B located near the earth's surface is proportional to the mass m of B. Let g be the constant of proportionality and show that g is equal to 32.2 lb/slug when expressed in units of the British gravitational system.

Solution: $\mathbf{F}$ is approximately equal to the gravitational force exerted on a particle of mass m by a particle of mass m', the particles being separated by a distance of 3960 miles. Hence,

$$|\mathbf{F}| \underset{(4.2.3)}{=} \frac{Gmm'}{\mathbf{p}^2} = gm$$

where

$$G \underset{(P4.2.5)}{=} 3.42 \times 10^{-8} \text{ lb ft}^2 \text{ slug}^{-2}$$

$$m' = 4.11 \times 10^{23} \text{ slug}$$

$$\mathbf{p}^2 = (3960 \times 5280)^2 \text{ ft}^2$$

and

$$g = \frac{Gm'}{\mathbf{p}^2} = \frac{3.42 \times 4.11 \times 10^{15}}{(3960 \times 5280)^2} = 32.2 \text{ lb slug}^{-1}$$

4.4.3 The gravitational force exerted on a body B by the earth is so large in comparison with gravitational forces exerted on B by other bodies that the latter forces may frequently be neglected.

Problem: Taking the mass density of lead equal to 22.5 slug/ft^3, evaluate (1) the magnitude of the force exerted by the earth on a lead sphere of radius 1 ft and (2) the magnitude of the force exerted on this sphere by another, identical one, when their centers are three feet apart.

Solution (1): The mass m of the lead sphere is

$$m = \frac{4\pi}{3} \times 22.5 = 30\pi \text{ slug}$$

Hence the force exerted on this sphere by the earth has a magnitude of (see Problem 4.4.2)

$$32.2m = 32.2 \times 30\pi = 3040 \text{ lb}$$

Solution (2): The two lead spheres attract each other as if each were concentrated at its center. Hence the force each exerts on the other has a magnitude of

$$\frac{Gm^2}{\mathsf{p}^2} = \frac{3.42 \times 10^{-8} \times (30\pi)^2}{3^2} = 0.0000338 \text{ lb}$$

The force exerted on one of the spheres by the earth is seen to be nearly ninety million times as large as the force exerted on that sphere by the other one.

4.4.4 A certain force, which depends, in part, on the earth's motion, and whose line of action is the plumb line passing through the mass center of a body B, has a magnitude called the *weight* of B. This force is *not* equal to the gravitational force exerted on B by the earth; but the two are so nearly equal that they are frequently used interchangeably.

4.4.5 In view of 4.4.2–4.4.4, one frequently may regard the system of all gravitational forces exerted on a body B (mass m) by other bodies in the universe as approximately equivalent to a single force directed from the mass center P^* of B toward the earth, along the plumb line passing through P^*, and having a magnitude equal to the product mg ($g = 32.2$ lb/slug) or to the weight of B.

4.4.6 Gravitational forces belong to a class of forces called *forces exerted at a distance.* Other forces belonging to the same class are those associated with the phenomena of magnetism and electricity.

CONTACT FORCES

4.5 Force systems associated with contact surfaces

With every contact surface σ between two distinct bodies B and B' there are associated two systems of forces whose points of applications are points of σ. The forces in one system are called *contact forces exerted on B by B' across σ*; those in the other, *contact forces exerted on B' by B across σ.*

4.5.1 The word "distinct" in 4.5 rules out the possibility of contact forces being exerted on a body by any of its parts: Two bodies B and B' are distinct only if neither contains any matter contained in the other, which is not the case if B' is a part of B.

4.5.2 Every contact between two bodies involves two *physical surfaces* and one *mathematical surface.* In some situations it is necessary to distinguish these from each other—for instance, when one wishes to discuss contact of a rough body with a smooth body—but generally the term "surface" is taken to mean any one of

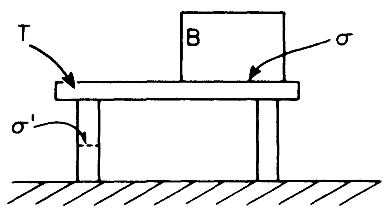


FIG. 4.5.2

the three. Furthermore, a surface may be one of *actual contact,* for instance, the surface σ of contact of the block B and the table T shown in Fig. 4.5.2, or it may be one of *imagined contact,* such as the surface σ' which separates the upper and lower portions of the leg of the table.

4.6 Equilibrium equations

Equations governing the gravitational and contact forces exerted on a body B by other bodies are obtained by using the properties of a zero system, together with the following proposition:

In the absence of magnetic and electric effects, the force system consisting of (a) *all gravitational forces exerted on a body B by other bodies in the universe and* (b) *all contact forces exerted on B by other bodies is a zero system whenever B is at rest on the earth.*

In this proposition, motions of the earth are left out of account. As knowledge of kinematics is prerequisite to understanding a quantitative description of the errors thus introduced, further consideration of this question is deferred to Vol. II.

Equations based on 3.6(a) or 3.6.1 are called *force equations;* those based on 3.6.5 or 3.6.6, *moment equations.* The term *equilibrium equation* applies to all of them.

Problem: A steel sphere (specific weight 489 lb/ft^3) is attached to an inclined board by means of a screw, as shown in Fig. 4.6a.

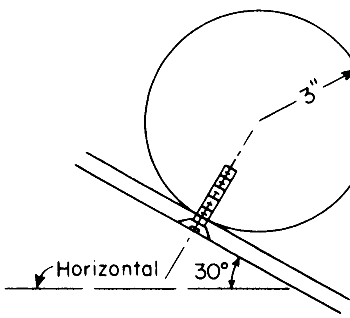


FIG. 4.6a

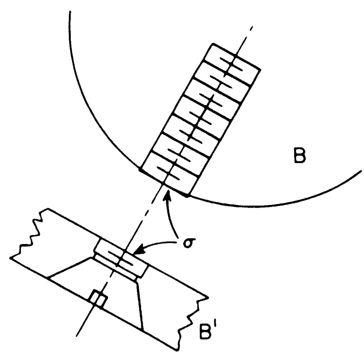


FIG. 4.6b

Regarding the sphere and that portion of the screw which is outside of the board as a body B in contact over a surface σ (see Fig. 4.6b) with a body B' consisting of the remainder of the screw and a portion of the board, describe the system of contact forces exerted on B by B' across σ.

Solution: The system of all gravitational forces exerted on B by other bodies in the universe is approximately equivalent to a single force (see 4.4.5) of magnitude

$$489 \times \tfrac{4}{3}\pi\left(\tfrac{3}{12}\right)^3 = 32 \text{ lb}$$

and having the line of action shown in Fig. 4.6c. (The mass center and the weight of B are taken to be those of the steel sphere, it being assumed that the screw's volume is so small, in comparison with the sphere's, that errors introduced by neglecting any possible difference in the mass densities of the screw and sphere are negligible.)

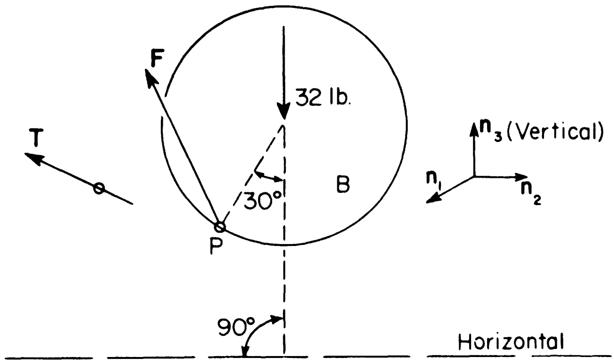


Fig. 4.6c

The only body with which B is in contact is B'. In accordance with 3.5.6, the system of all contact forces exerted on B by B' across σ can be replaced with a couple of torque $\mathbf{T}$ and a single force $\mathbf{F}$ whose line of action passes through the point P shown in Fig. 4.6c.

Use the force equation

$$\mathbf{F} - 32\mathbf{n}_3 \underset{(3.6)}{=} 0$$

and the moment equation (moments about point P)

$$\mathbf{T} - 32(3 \sin 30°)\mathbf{n}_1 \underset{(3.6.5)}{=} 0$$

to determine $\mathbf{F}$ and $\mathbf{T}$:

$$\mathbf{F} = 32\mathbf{n}_3 \text{ lb}, \quad \mathbf{T} = 48\mathbf{n}_1 \text{ in lb}$$

Result: The system of forces exerted on B by B' across σ is equivalent to the system of forces shown in Fig. 4.6d.

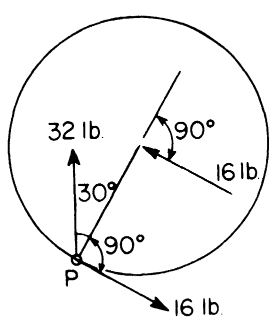
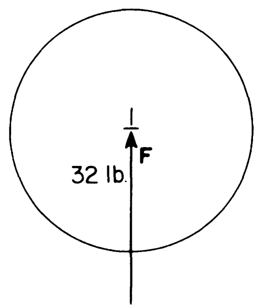


Fig. 4.6d

4.6.1 The solution of equilibrium equations does not furnish detailed descriptions of individual contact forces exerted on one body by another. It leads only to descriptions of force systems equivalent to systems of such forces.

Any force system equivalent to the system of contact forces exerted on a body B by a body B' across a surface σ is called the

FIG. 4.6.1



reaction of B' on B across σ. Note that a given reaction can be represented in infinitely many ways, because every force system can be replaced by infinitely many others.

Problem: Referring to Problem 4.6, show that the reaction of B' on B across σ can be represented by the single vertical force shown in Fig. 4.6.1.

Solution: The force $\mathbf{F}$ shown in Fig. 4.6.1 is equivalent to the force system shown in Fig. 4.6d. The latter is equivalent to the system S of contact forces exerted on B by B' across σ. Hence $\mathbf{F}$ is equivalent to S (see 3.5.5) and, therefore, represents the reaction of B' on B across σ.

4.6.2 Before writing equilibrium equations for a body B, it is always advisable to make a sketch representing B and some system of forces equivalent to the system of *all* gravitational and contact forces exerted on B by other bodies. (The presence of couples is indicated by arrows representing the torques of the couples.) Such a sketch is called a *free-body diagram* of B.

Example: Fig. 4.6c is a free-body diagram of the body B described in Problem 4.6. A second free-body diagram of B is shown in Fig. 4.6.2. Figs. 4.6d and 4.6.1 are not free-body diagrams, because the gravitational forces exerted on B by other bodies are not represented in these sketches.

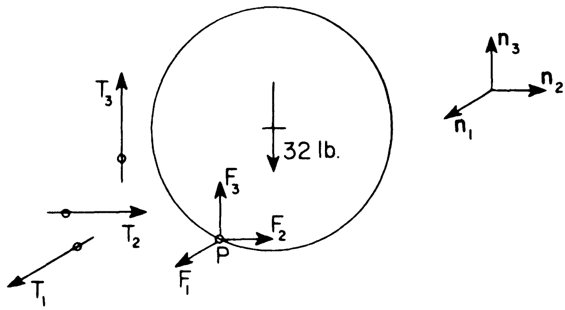


FIG. 4.6.2

4.6.3 A sketch representing the orthogonal projections of a body B and of all gravitational and contact forces exerted on B by other bodies, on a plane N, is called a *plane free-body diagram* of B. (See 3.6.9(b): In a plane free-body diagram, the presence of couples is indicated by arrows representing only the $\mathbf{n}$ resolutes of the torques of these couples, $\mathbf{n}$ being a unit vector *normal* to N). The system of forces represented on such a sketch is a zero system whenever B is at rest on the earth (see 4.6 and 3.6.9(a)). Hence, although a plane free-body diagram is *not* a free-body diagram, it can be used as a basis for writing equations which furnish information about the forces acting on B.

Problem: Draw plane free-body diagrams of the body B described in Problem 4.6, for planes N_1, N_2, N_3, respectively normal to the unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ shown in Fig. 4.6c. Write three force and three moment equations governing the unknown forces and torques appearing in each diagram; and solve these equations, thus determining the reaction of B' on B.

Solution: Figs. 4.6.3 a, b and c are plane free-body diagrams corresponding to the free-body diagram shown in Fig. 4.6.2.

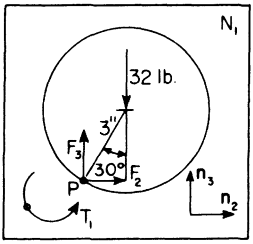


FIG. 4.6.3a

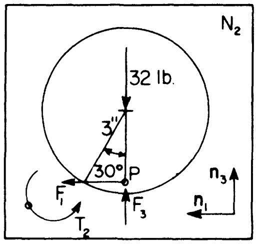


FIG. 4.6.3b

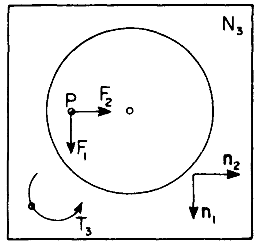


FIG. 4.6.3c

Force equations:

Fig. 4.6.3a:	$F_2 = 0,\ F_3 - 32 = 0$
Fig. 4.6.3b:	$F_3 - 32 = 0,\ F_1 = 0$
Fig. 4.6.3c:	$F_1 = 0,\ F_2 = 0$

Moment equations (moments about P):

Fig. 4.6.3a:	$T_1 - 32(3 \sin 30°) = 0$
Fig. 4.6.3b:	$T_2 = 0$
Fig. 4.6.3c:	$T_3 = 0$

Result:

$$F_1 = 0, \quad F_2 = 0, \quad F_3 = 32 \text{ lb}$$

$$T_1 = 48 \text{ in lb}, \quad T_2 = 0, \quad T_3 = 0$$

4.6.4 As will be seen later, there exist situations in which some of the quantities appearing in a free-body diagram cannot be determined, while all of the remaining ones appear in certain plane free-body diagrams. These are the situations in which it is particularly advantageous to use plane free-body diagrams.

4.7 The relationship between reactions across a given surface

Taken together, the two systems of forces associated with a contact surface σ between two bodies B and B' (see 4.5) which are at rest on the earth constitute a zero system. Consequently, if the reaction of B' on B is a couple of torque $\mathbf{T}$ and a force $\mathbf{F}$ whose line of action passes through a point P, and the reaction of B on B' is a couple of torque $\mathbf{T}'$ and a force $\mathbf{F}'$ whose line of action passes through the *same* point P, then

$$\mathbf{F}' = -\mathbf{F}$$

and

$$\mathbf{T}' = -\mathbf{T}$$

Proof: Let S be the reaction of B' on B across σ; S' the reaction of B on B' across σ; $\bar{S}$ the system of all gravitational and contact forces exerted on B, with the exception of the contact forces exerted on B by B' across σ; $\bar{S}'$ the system of all gravitational and contact forces exerted on B', with the exceptions of the contact forces exerted on B' by B across σ. These force systems are shown

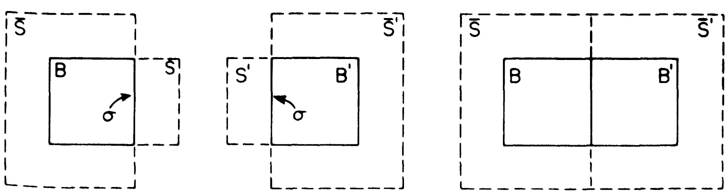


FIG. 4.7a FIG. 4.7b FIG. 4.7c

in free-body diagrams of B, B', and the body consisting of both B and B', in Figs. 4.7a, b, and c. (S and S' do not appear in Fig. 4.7c, because neither contains forces exerted on the body consisting of B and B'.)

It must be shown that

$$S + S' = 0$$

(This is a convenient symbolic form of the statement "the system of forces consisting of all forces in S and all forces in S' is a zero system.")

As B, B', and $B + B'$ are each at rest on the earth, the system of all forces exerted on each of these bodies is a zero system. Hence,

$$S + \bar{S} = 0$$

$$S' + \bar{S}' = 0$$

$$\bar{S}' + \bar{S} = 0$$

Add the first two of these "equations," then use the third. This gives

$$S + S' = 0$$

Now let S be equivalent to a couple of torque $\mathbf{T}$, together with a force $\mathbf{F}$ whose line of action passes through a point P, and S' equivalent to a couple of torque $\mathbf{T}'$, together with a force $\mathbf{F}'$ whose line of action passes through the same point P. Then the resultant of $S + S'$ is equal to $\mathbf{F} + \mathbf{F}'$, and the moment of $S + S'$ about P is equal to $\mathbf{T} + \mathbf{T}'$, so that, if $S + S'$ is a zero system,

$$\mathbf{F} + \mathbf{F}' = 0$$

and

$$\mathbf{T} + \mathbf{T}' = 0$$

Hence,

$$\mathbf{F}' = -\mathbf{F}$$

and

$$\mathbf{T}' = -\mathbf{T}$$

Problem: Referring to Problem 4.6, determine the reaction of B on B' across σ.

Solution: In Problem 4.6.1, it was shown that the reaction of B' on B across σ is the single vertical force $\mathbf{F}$ shown in Fig. 4.6.1. Hence the reaction of B on B' is the single force $\mathbf{F}'$ given by

$$\mathbf{F}' = -\mathbf{F} = -32\mathbf{n}_3 \text{ lb}$$

and having a line of action passing through the center of the sphere.

4.8 Contact forces and physical properties

By using only equilibrium equations, the system of *all* contact forces exerted on a body B by other bodies can be reduced to a known force $\mathbf{F}$ and a couple of known torque $\mathbf{T}$ whenever all gravitational forces exerted on B by other bodies are known: $\mathbf{F}$ is equal to the negative of the resultant of the gravitational forces and has a line of action passing through an arbitrarily selected point P; $\mathbf{T}$ is equal to the negative of the moment of the gravitational forces about point P.

On the other hand, equilibrium equations, alone, never furnish sufficient information for the determination of the reactions on a body B of individual bodies with which B is in contact: Each such reaction involves two unknown vector quantities, so that $2n$ vector equations are needed for the determination of n reactions; but only 2 independent vector equations can be written for a given body. Thus, additional information is required. Such information is obtained from experiments which disclose relationships between physical properties of bodies and forces exerted on, or by, these bodies.

The discussion of the results of experiments inevitably involves words associated with undefinable qualities (e.g., flexibility, roughness.) The ability to understand these words has its roots in sensory perceptions and is the product of experience. It must, therefore, be presupposed. Furthermore, it is clearly impossible to compile a complete list of all such results. The paragraphs which follow may be regarded as supplying a partial one.

4.8.1 The system of all contact forces exerted on a portion B of a thin, *flexible cable* by a contiguous portion B' (see Fig. 4.8.1a) is equivalent to a single force $\mathbf{F}$ whose line of action coincides with the tangent to the cable at the point P where B and B' meet. $\mathbf{F}$ has the sense OP, O being a point on that half of the tangent which is associated with B. The magnitude of $\mathbf{F}$ is called the *cable tension* at P.

Problem: Figure 4.8.1b illustrates a body B of weight W, B being supported by three light, non-coplanar cables, attached at a

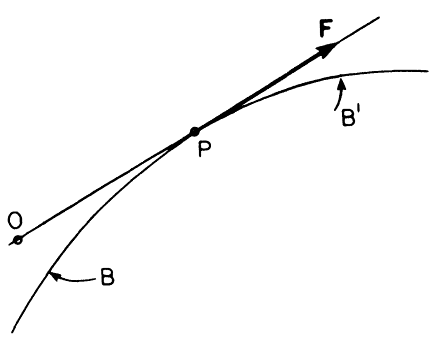


Fig. 4.8.1a

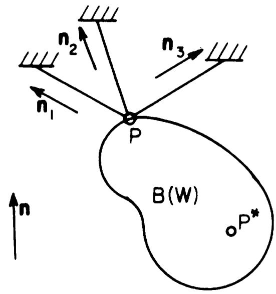


FIG. 4.8.1b

point P. P^* is B's mass center, $\mathbf{n}$ is a unit vector directed vertically upward, and $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are unit vectors respectively parallel to the cables. Show that P is vertically above P^* whenever B is at rest; and determine the cable tensions T_1, T_2, T_3 at arbitrarily selected points of the cables.

Solution: Let P_1, P_2, P_3 be points on the three cables, and draw a free-body diagram of the body consisting of B and the portions of the cables between P_1 and P; P_2 and P; and P_3 and P (see Fig. 4.8.1c).

As the cables are "light," their weights can be neglected in comparison with that of B. The system of all gravitational forces exerted on the body under consideration is then approximately equivalent to a single force in the $-\mathbf{n}$ direction and of magnitude W, as shown in Fig. 4.8.1c. The system of all contact forces is

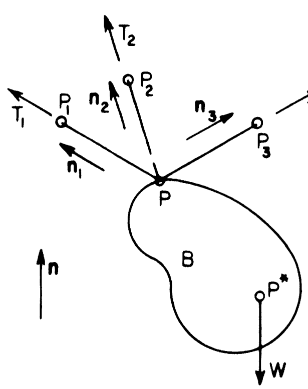


FIG. 4.8.1c

equivalent to three forces whose lines of action coincide with the cables and whose magnitudes are T_1, T_2, T_3 respectively.

To show that P is vertically above P^* whenever B is at rest, note that the system of three forces $T_1\mathbf{n}_1$, $T_2\mathbf{n}_2$, $T_3\mathbf{n}_3$ can be replaced with a single force $\mathbf{F}$ whose line of action passes through P (see 3.5.10). $\mathbf{F}$ and $-W\mathbf{n}$ then constitute a zero system whenever B is at rest. The lines of action of these two forces must coincide (see Problem 3.6.5), and this is possible only if P lies vertically above P^*.

The most convenient way to find the cable tensions is to write force equations for directions perpendicular to two of the cables. For example, let $\mathbf{n}_1'$ be a unit vector perpendicular to both $\mathbf{n}_2$ and $\mathbf{n}_3$. Then the forces $T_2\mathbf{n}_2$ and $T_3\mathbf{n}_3$ contribute nothing to the sum of the $\mathbf{n}'_1$ resolutes of the forces shown in Fig. 4.8.1c, and

$$\mathbf{n}'_1\cdot(T_1\mathbf{n}_1) + \mathbf{n}'_1\cdot(-W\mathbf{n}) = 0$$

Hence,

$$T_1 = W\frac{\mathbf{n}\cdot\mathbf{n}_1'}{\mathbf{n}_1\cdot\mathbf{n}_1'}$$

In terms of $\mathbf{n}_2$ and $\mathbf{n}_3$, $\mathbf{n}_1'$ is given by

$$\mathbf{n}_1' = \frac{\mathbf{n}_2\times\mathbf{n}_3}{|\mathbf{n}_2\times\mathbf{n}_3|}$$

Thus,

$$T_1 = W\frac{[\mathbf{n}, \mathbf{n}_2, \mathbf{n}_3]}{[\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3]}$$

Similarly,

$$T_2 = W\frac{[\mathbf{n}, \mathbf{n}_3, \mathbf{n}_1]}{[\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3]}$$

and

$$T_3 = W\frac{[\mathbf{n}, \mathbf{n}_1, \mathbf{n}_2]}{[\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3]}$$

(It is instructive to study the limits approached by these expressions as $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ tend toward parallelism with (a) a vertical plane and (b) a horizontal plane.)

4.8.2 The lines of action of contact forces exerted across a *smooth* surface σ are normal to σ at their points of intersection with σ.

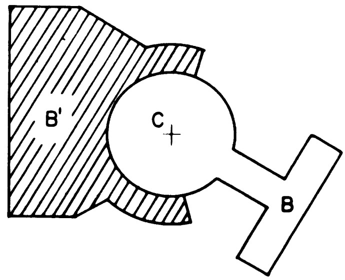


FIG. 4.8.2a

Problem (a): Figure 4.8.2a illustrates two rigid bodies, B and B', attached to each other by means of a ball-and-socket connection. Show that the system S of contact forces exerted on B by B' can be reduced to a single force whose line of action passes through the center C of the ball, if the surfaces of the ball and socket are smooth.

Solution: S consists of forces whose lines of action are normal to the (spherical) surface of the ball and thus pass through C. It follows from 3.5.10 that S can be replaced with a force whose line of action passes through C.

Problem (b): Figure 4.8.2b shows a uniform block A which

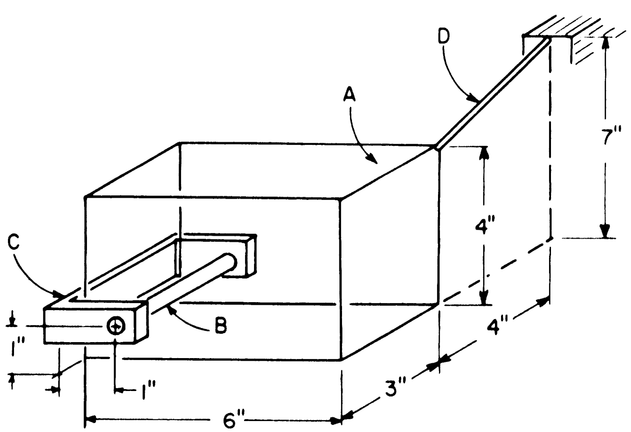


FIG. 4.8.2b

weighs 9 lb and is supported by a thin, cylindrical pin B, a bracket C, and a light, flexible cable D. Assuming that the surfaces over which the block is in contact with the pin and bracket as smooth, determine the cable tension T at an arbitrarily selected point of D, and find the reaction of the block on the pin and bracket.

Solution: Draw a free-body diagram of the body consisting of A and a portion D' of D (see Fig. 4.8.2c), based on following considerations:

If the cable is light, and pin B is thin (so that the volume of the cylindrical hole containing the pin is small in comparison with the

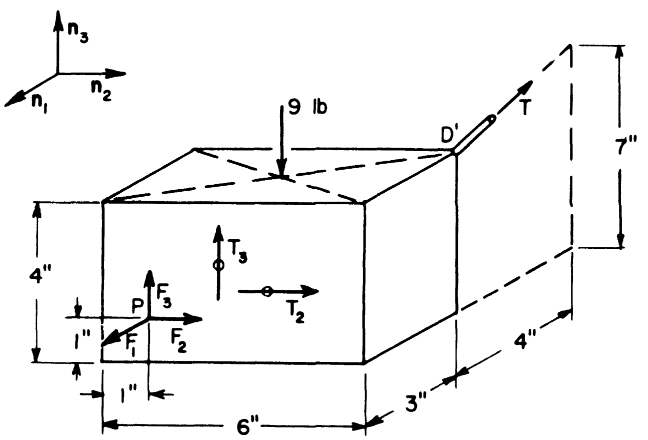


FIG. 4.8.2c

volume of A), the system of all gravitational forces exerted on $A + D'$ is approximately equivalent to the 9 lb force shown in Fig. 4.8.2c.

Contact forces are exerted on $A + D'$ (a) by a portion of D contiguous to D'; (b) by B; and (c) by C. These can be replaced with (a) (see 4.8.1) a force of magnitude T, line of action coincident with the cable; (b) (see 3.5.11, keeping in mind that forces exerted by B have lines of action normal to the cylindrical surface of B) a couple, whose torque is perpendicular to the pin axis, together with a force, whose line of action is perpendicular to the pin axis and intersects it; and (c) (see 3.5.14) a couple, whose torque is perpendicular to the pin axis, together with a force, whose line of action is parallel to the pin axis.

Together, the forces and couples in (b) and (c) can be reduced to a single force and couple, the force having the three components, and the torque of the couple the two components, shown in Fig. 4.8.2c.

T and the measure numbers F_1, F_2, F_3, T_2, T_3 can now be found by solving equilibrium equations.

Force equations:

$$\mathbf{n}_1: \quad F_1 - \tfrac{4}{5}T = 0$$

$$\mathbf{n}_2: \quad F_2 = 0$$

$$\mathbf{n}_3: \quad F_3 - 9 + \tfrac{3}{5}T = 0$$

Moment equations (lines through P):

$$\mathbf{n}_1: \quad -2(9) + 5(\tfrac{3}{5}T) = 0$$

$$\mathbf{n}_2: \quad -1.5(9) + 3(\tfrac{3}{5}T) - 3(\tfrac{4}{5}T) + T_2 = 0$$

$$\mathbf{n}_3: \quad 5(\tfrac{4}{5}T) + T_3 = 0$$

These give

$$T = 6 \text{ lb}$$

$$F_1 = 4.8 \text{ lb}, \quad F_2 = 0, \quad F_3 = 5.4 \text{ lb}$$

$$T_2 = 17.1 \text{ in lb}, \quad T_3 = -24 \text{ in lb}$$

The reaction of the block on the pin and bracket is a couple of torque $\mathbf{T}$ and a force $\mathbf{F}$ whose line of action passes through P, with (see 4.7)

$$\mathbf{T} = -(T_1\mathbf{n}_1 + T_3\mathbf{n}_3) = -17.1\mathbf{n}_2 + 24\mathbf{n}_3 \text{ in. lb}$$

and

$$\mathbf{F} = -(F_1\mathbf{n}_1 + F_2\mathbf{n}_2 + F_3\mathbf{n}_3) = -4.8\mathbf{n}_1 - 5.4\mathbf{n}_3 \text{ lb}$$

4.8.3 If a contact surface σ is sufficiently small, the moment of a system S of contact forces associated with σ, about any point P of σ, is so small, that S may be replaced with a single force whose line of action passes through P.

Problem: A uniform, solid hemisphere H of weight W rests on a horizontal support S and is attached to a wall by means of a

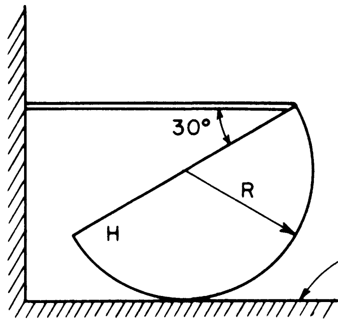


FIG. 4.8.3a

horizontal string (see Fig. 4.8.3a). Draw a sketch showing the reaction of S on H.

Solution: Draw a plane free-body diagram of the body consisting of the hemisphere and a portion of the string (see Fig. 4.8.3b), representing the reaction of S on H with three forces whose lines of action pass through the contact point P.

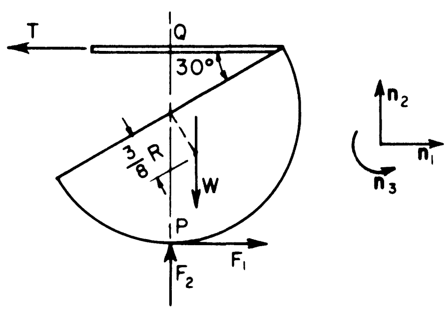


FIG. 4.8.3b

Take moments about a line passing through Q and parallel to $\mathbf{n}_3$:

$$(R + R \sin 30°)F_1 - (\tfrac{3}{8}R \sin 30°)W = 0$$

Use two force equations:

$$\mathbf{n}_2: \qquad F_2 - W = 0$$

$$\mathbf{n}_3: \qquad F_3 = 0$$

The reaction of S on H is shown in Fig. 4.8.3c.

4.8.4 If the points of a contact surface σ lie sufficiently near a curve γ, the lines of action of contact forces associated with σ may be regarded as intersecting γ.

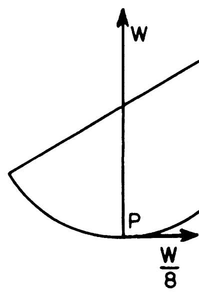


FIG. 4.8.3c

Problem: A uniform, solid half-cylinder C of weight W rests on a horizontal support S and is attached to a wall by means of a horizontal string (see Fig. 4.8.4a). Determine the reaction of S on C.

Solution: In the three plane free-body diagrams of the body consisting of the cylinder and a portion of the string, the reaction of S on C can be represented (see 3.5.11) by a couple, whose torque

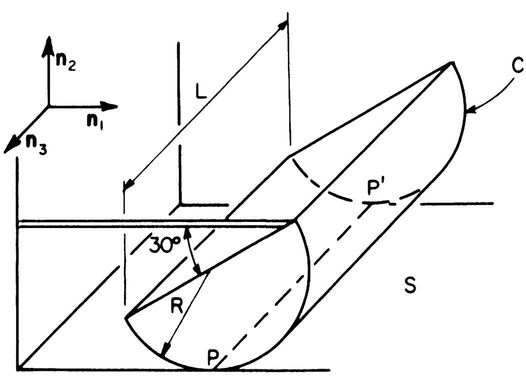


FIG. 4.8.4a

(T_1, T_2) is perpendicular to the line of contact PP' between C and S, and a force (F_1, F_2, F_3), whose line of action intersects this line at point P (see Fig. 4.8.4b, c, d).

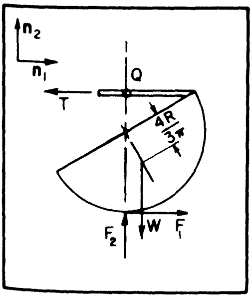


Fig. 4.8.4b

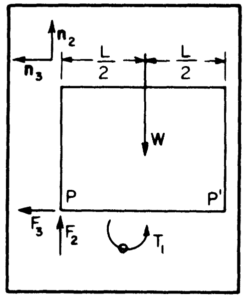


Fig. 4.8.4c

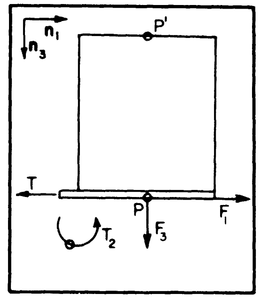


Fig. 4.8.4d

Referring to Fig. 4.8.4b, find F_1 by taking moments about a line passing through Q and parallel to $\mathbf{n}_3$, and use a force equation to determine F_2:

$$F_1 = \frac{4W}{9\pi}$$

$$F_2 = W$$

From Fig. 4.8.4c,

$$F_3 = 0$$

$$T_1 = \frac{WL}{2}$$

and from Fig. 4.8.4d,

$$T_2 = 0$$

Thus the reaction of S on C is a couple whose torque $\mathbf{T}$ is given by

$$\mathbf{T} = \frac{WL}{2}\mathbf{n}_1$$

and a force $\mathbf{F}$ whose line of action passes through P, $\mathbf{F}$ being given by

$$\mathbf{F} = \frac{4W}{9\pi}\mathbf{n}_1 + W\mathbf{n}_2$$

As the couple can be replaced (see 3.5.7) with two forces, $-W\mathbf{n}_2$ and $W\mathbf{n}_2$, whose lines of action pass respectively through P and through the midpoint of line PP', the reaction of S on C can be represented by the two forces shown in Fig. 4.8.4e.

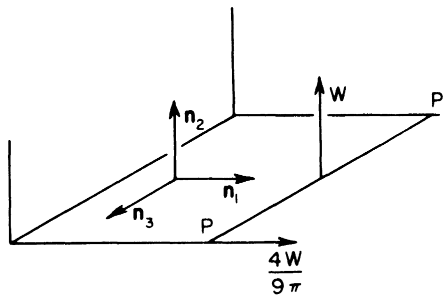


FIG. 4.8.4e

4.8.5 When the ends A and A' of a light, helical spring are attached to two bodies B and B', the systems of contact forces exerted on the spring by B and B' are respectively equivalent to forces $\mathbf{F}$ and $\mathbf{F}'$, described as follows: If L is the *natural length* of the spring, i.e., the distance between A and A' when the spring is not attached to other bodies (see Fig. 4.8.5a), and x is the *deformation* of the spring, i.e., the amount by which the distance between A and A' exceeds L, or is smaller than L, when the spring is attached to B and B' (see Figs. 4.8.5b and c), then

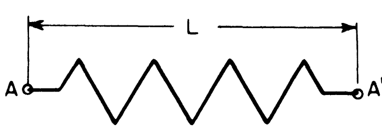


FIG. 4.8.5a

$$|\mathbf{F}| = |\mathbf{F}'| = f(x)$$

where $f(x)$ is a function of x. The points of application and directions of $\mathbf{F}$ and $\mathbf{F}'$ are those shown in Figs. 4.8.5b and c.

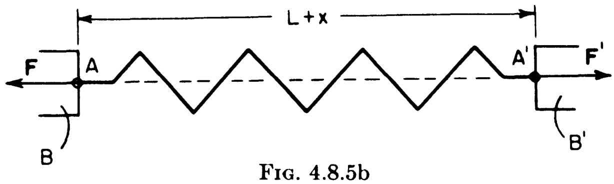


FIG. 4.8.5b

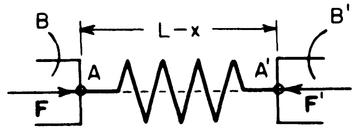


FIG. 4.8.5c

If there exists a range of values of x in which

$$f(x) = kx$$

where k is a constant, the spring is said to be *linear* in this range, and k is called its *spring constant* or *modulus*.

Problem: A uniform sphere, weighing 15 lb, is free to slide on a smooth vertical shaft and is attached to two light, helical springs, each having a natural length of 5 inches (see Fig. 4.8.5d). The upper spring has a spring constant of 2 lb/inch; the lower, a spring constant of 3 lb/inch. Determine h.

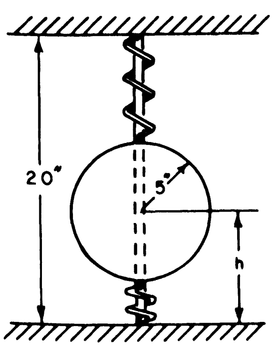


FIG. 4.8.5d

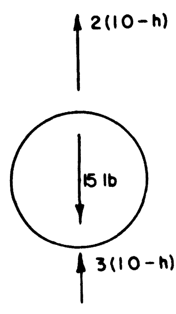


FIG. 4.8.5e

Solution: Fig. 4.8.5e is a free-body diagram of the sphere. (The forces exerted on the sphere by the springs are found, in part, by using 4.7.)

The force equation

$$2(10 - h) - 15 + 3(10 - h) = 0$$

leads to

$$h = 7 \text{ in.}$$

4.9 Several-body problems

The equilibrium equations used in the solution of all problems considered so far were, in each case, equilibrium equations for a single body. The examples which follow, show that the determination of certain quantities requires that equilibrium equations be written for several bodies associated with a given system.

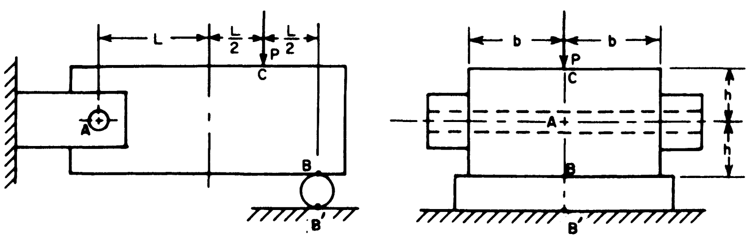


FIG. 4.9a

Problem (a): Figure 4.9a shows two views of a uniform beam of weight w per unit of length, the beam being supported at one end by a smooth pin and bracket; at the other, by a cylindrical roller. A load P is applied at point C; i.e., a body (not shown) exerts contact forces on the beam, across a surface in the neighborhood of C, and this system of contact forces is equivalent to a single force of magnitude P, as shown in Fig. 4.9a. Determine the reaction of the pin and bracket on the beam.

Solution: Draw free-body diagrams (see Figs. 4.9b and c) of (1) the beam and (2) the roller, based on the following considerations:

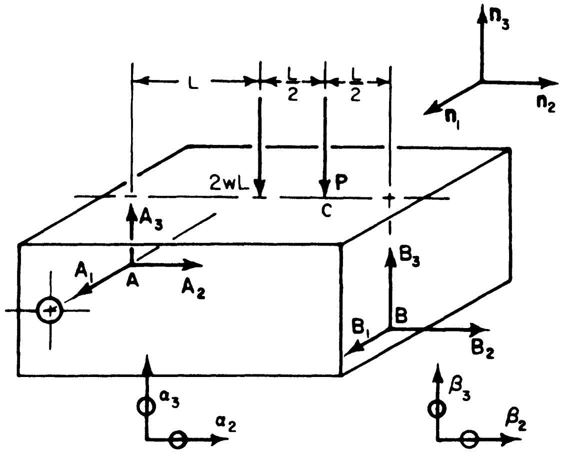


Fig. 4.9b

(1) The system of all gravitational forces exerted on the beam is replaced with a single force of magnitude $2wL$ (see 4.4.5).

The reaction of the pin and bracket on the beam is a force (A_1, A_2, A_3) whose line of action passes through the point A on the pin-axis, together with a couple whose torque (α_2, α_3) is perpendicular to the pin-axis (see Problem 4.8.2(b)).

In accordance with 4.8.4 and 3.5.11, the system of contact forces exerted on the beam by the roller is replaced with a force (B_1, B_2, B_3) whose line of action passes through point B, together with a couple whose torque (β_2, β_3) is perpendicular to the line of contact of the beam and roller.

(2) A single force of magnitude W represents the gravitational forces exerted on the roller.

A replacement of the system of contact forces exerted on the roller by the beam is obtained by using 4.7.

The system of contact forces exerted on the roller by the support on which the roller rests is replaced in accordance with 4.8.4 and 3.5.11.

None of the five quantities which are to be determined $(A_1, A_2, A_3, \alpha_2, \alpha_3)$ can be found by solving equilibrium equations written for the beam, as can be seen by examining the following force and moment equations:

Force equations:

$$\mathbf{n}_1: \quad A_1 + B_1 = 0 \tag{1}$$

$$\mathbf{n}_2: \quad A_2 + B_2 = 0 \tag{2}$$

$$\mathbf{n}_3: \quad A_3 + B_3 - 2wL - P = 0 \tag{3}$$

Moment equations (lines through B):

$$\mathbf{n}_1: \quad -2LA_3 - hA_2 + 2wL^2 + PL/2 = 0 \tag{4}$$

$$\mathbf{n}_2: \quad hA_1 + \alpha_2 + \beta_2 = 0 \tag{5}$$

$$\mathbf{n}_3: \quad LA_1 + \alpha_3 + \beta_3 = 0 \tag{6}$$

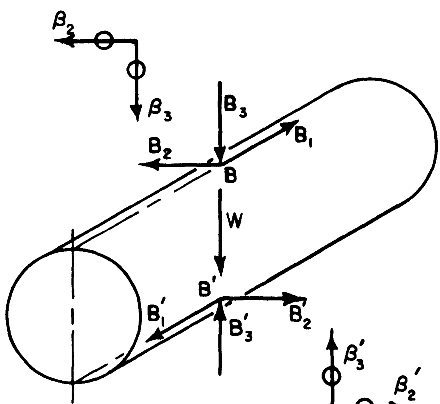


FIG. 4.9c

Equations (2) and (4) can be solved for A_2 and A_3, once B_2 is known. B_2 is found by taking moments about the line of contact of the roller and the support on which the roller rests (see Fig. 4.9c):

$$B_2 = 0$$

Thus, from Eqs. (2) and (4),

$$A_2 = 0$$

$$A_3 = wL + \frac{P}{4}$$

A_1, α_2, and α_3 cannot be evaluated, because it is impossible to write further equilibrium equations for the roller without involving at least one of the unknown quantities B'_1, B'_2, B'_3, β'_2, β'_3.

Note that the quantities which *can* be evaluated by solving equilibrium equations for the beam and the roller are precisely those appearing in plane free-body diagrams (for a plane normal to $\mathbf{n}_1$) of these two bodies. These diagrams are shown in Figs. 4.9d

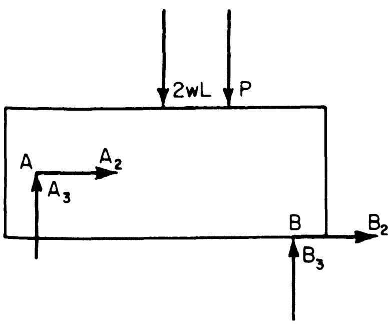


FIG. 4.9d

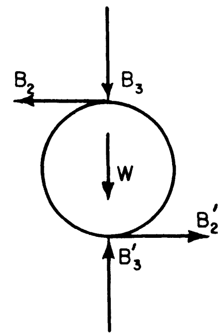


FIG. 4.9e

and e. Hence nothing is "lost" by using these diagrams in place of the free-body diagrams shown in Figs. 4.9b and c. On the contrary, it is advantageous to do so (see 4.6.4).

Problem (b): Figure 4.9f represents schematically a device known as "Hooke's joint," described as follows:

Two shafts, S and S', are mounted in fixed bearings, B and B', the axes of the shafts being respectively parallel to unit vectors $\mathbf{n}$ and $\mathbf{n}'$ and intersecting at a point A. Each shaft terminates in a "yoke," and these yokes, Y and Y', are connected to each other by a rigid cross, one of whose arms is supported by bearings D and E, the other by bearings D' and E'. The arms of the cross have equal lengths, form a right angle with each other, and are respectively perpendicular to $\mathbf{n}$ and $\mathbf{n}'$.

Supposing that this system is at rest when subjected to the action of two couples exerted on the shafts by contiguous portions (not shown), these couples having torques $T\mathbf{n}$ and $T'\mathbf{n}'$, respectively, express the ratio of T to T' in terms of $\mathbf{n}$, $\mathbf{n}'$, $\boldsymbol{\nu}$, and $\boldsymbol{\nu}'$, where

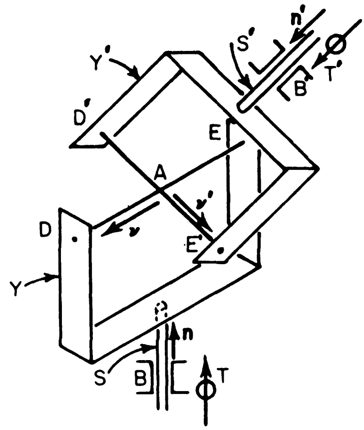


FIG. 4.9f

$\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$ are unit vectors parallel to the arms of the cross, as shown in Fig. 4.9f. (Neglect all gravitational forces.)

Solution: Draw free-body diagrams (see Figs. 4.9g, h, i) of (1) the lower shaft and yoke, (2) the upper shaft and yoke, and (3) the cross, based on the following consideration: The lines of action of contact forces exerted on a shaft across the surfaces of a well constructed bearing either very nearly intersect or are very nearly parallel to the axis of the shaft.

(1) The system of forces exerted on S by B is replaced with a force $\mathbf{B}$ whose line of action passes through the axis of S, together

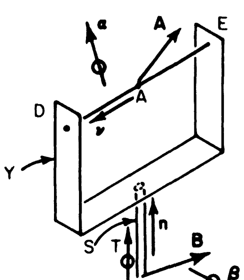


FIG. 4.9g

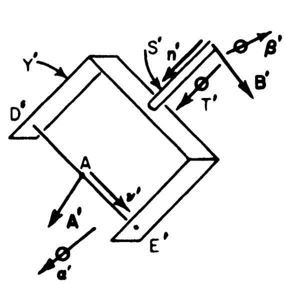


FIG. 4.9h

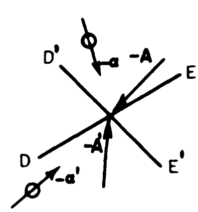


FIG. 4.9i

with a couple whose torque $\boldsymbol{\beta}$ is perpendicular to $\mathbf{n}$ (see 3.5.11 and 3.5.14), so that

$$\mathbf{n}\cdot\boldsymbol{\beta} = 0 \tag{1}$$

Similarly, the system of forces exerted on Y across the bearing surfaces at D and E is replaced with a force $\mathbf{A}$ whose line of action passes through A, together with a couple whose torque $\boldsymbol{\alpha}$ is perpendicular to $\boldsymbol{\nu}$:

$$\boldsymbol{\nu}\cdot\boldsymbol{\alpha} = 0 \tag{2}$$

(2) With slight changes in notation, the statements made in (1) apply equally well to the present case. Hence

$$\mathbf{n}'\cdot\boldsymbol{\beta}' = 0 \tag{3}$$

and

$$\boldsymbol{\nu}'\cdot\boldsymbol{\alpha}' = 0 \tag{4}$$

(3) In accordance with 4.7, the systems of forces exerted on the cross by Y and Y' are equivalent to two forces, $-\mathbf{A}$ and $-\mathbf{A}'$, whose lines of action pass through A, together with two couples, whose torques are $-\boldsymbol{\alpha}$ and $-\boldsymbol{\alpha}'$, respectively.

Equilibrium equations:

Fig. 4.9g, moments about the axis of S:

$$T + \mathbf{n}\cdot\boldsymbol{\beta} + \mathbf{n}\cdot\boldsymbol{\alpha} = 0$$

or, using Eq. (1),

$$T + \mathbf{n}\cdot\boldsymbol{\alpha} = 0 \tag{5}$$

Fig. 4.9h, moments about the axis of S':

$$T' + \mathbf{n}'\cdot\boldsymbol{\beta}' + \mathbf{n}'\cdot\boldsymbol{\alpha}' = 0$$

or, using Eq. (3),

$$T' + \mathbf{n}'\cdot\boldsymbol{\alpha}' = 0 \tag{6}$$

Fig. 4.9i, moments about point A:

$$-\boldsymbol{\alpha} - \boldsymbol{\alpha}' = 0 \tag{7}$$

Equations (2), (4), (5), (6) and (7) must now be solved for the ratio T/T'. Use (7) to eliminate $\boldsymbol{\alpha}'$ from (4) and (6):

$$\boldsymbol{\nu}'\cdot\boldsymbol{\alpha} = 0 \tag{8}$$

$$T' - \mathbf{n}'\cdot\boldsymbol{\alpha} = 0 \tag{9}$$

Solve (5) for T, (9) for T', and divide the results:

$$\frac{T}{T'} = -\frac{\mathbf{n} \cdot \boldsymbol{\alpha}}{\mathbf{n}' \cdot \boldsymbol{\alpha}} \tag{10}$$

From (2) and (8) it follows that $\boldsymbol{\alpha}$ is perpendicular to both $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$. Hence $\boldsymbol{\alpha}$ can be expressed as

$$\boldsymbol{\alpha} = \lambda \boldsymbol{\nu} \times \boldsymbol{\nu}'$$

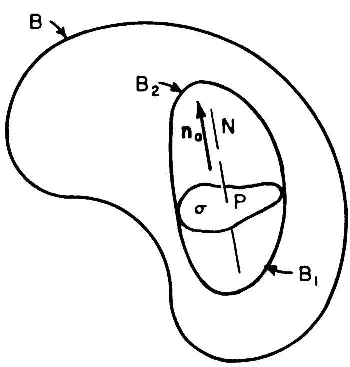
where λ is an appropriately selected scalar. Substitute in Eq. (10):

$$\frac{T}{T'} = -\frac{[\mathbf{n}, \boldsymbol{\nu}, \boldsymbol{\nu}']}{[\mathbf{n}', \boldsymbol{\nu}, \boldsymbol{\nu}']}$$

4.10 Traction

The analysis of certain phenomena requires that "local" properties of contact force systems be taken into account. This is done by using an experimentally verifiable fact (see 4.8.3) as the basis for the definition of a quantity called traction, in terms of which one can then deal with contact over a surface of any size, by regarding it as a limiting case of many contacts over small surfaces.

FIG. 4.10a



Notation (see Fig. 4.10a):

B	a continuous body
P	a point of B
$\mathbf{n}_a$	a unit vector

N	a line passing through P, parallel to $\mathbf{n}_a$
σ	a surface which contains P and whose normal at P is N
A	the area of σ
B_1, B_2	two distinct parts of B, in contact over the surface σ, numbered in such a way that a point moving on N in the direction $\mathbf{n}_a$ leaves B_1 and enters B_2 when crossing σ; $\mathbf{n}_a$ is called the *outward normal* to B_1 at P.

Definition: It is assumed that, as σ shrinks to the point P (while N remains unchanged), the system of contact forces exerted on B_1 by B_2 across σ tends toward equivalence with a single force $\mathbf{F}$ whose line of action passes through P and whose magnitude is proportional to A. In other words, the smaller σ is, the more nearly is the system of forces exerted on B_1 by B_2 across σ equivalent to a single force $\mathbf{F}$ given by

$$\mathbf{F} = A\tau_a{}^P$$

where $\tau_a{}^P$ is a vector whose characteristics depend only on $\mathbf{n}_a$. $\tau_a{}^P$ is called *the traction at* P, *for the direction* $\mathbf{n}_a$. (In general, $\tau_a{}^P$ is *not* parallel to $\mathbf{n}_a$.)

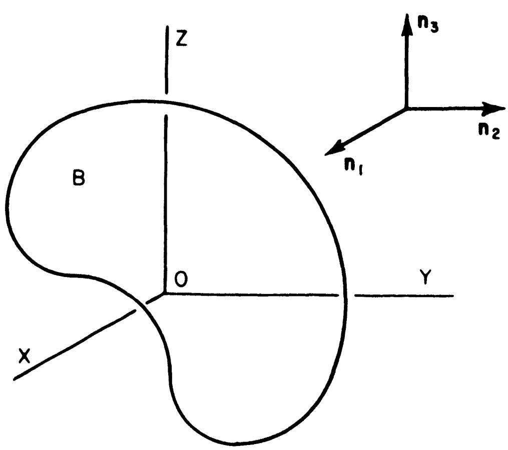


FIG. 4.10b

Problem: In Fig. 4.10b, O is a point of a continuous body B. The tractions at O, for the directions $\mathbf{n}_1$ and $\mathbf{n}_2$, are given by

$$\boldsymbol{\tau}_1{}^O = -12\mathbf{n}_3 \text{ lb in}^{-2}$$

$$\boldsymbol{\tau}_2{}^O = 2\mathbf{n}_1 - 4\mathbf{n}_2 \text{ lb in}^{-2}$$

The bodies B_1, B_2, B_3, shown in Fig. 4.10c, are parts of B. Assuming that ϵ is very small, give descriptions of approximate reductions of the systems of forces exerted (1) on B_1 by B_2, (2) on B_2 by B_1, (3) on B_1 by the body consisting of B_2 and B_3.

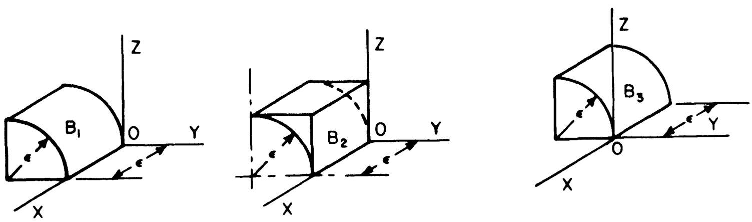


FIG. 4.10c

Solution (1): The area of the surface of contact of B_1 and B_2 is equal to $\pi\epsilon^2/2$. O plays the part of the point P; the Y axis, that of line N; $\mathbf{n}_2$, that of $\mathbf{n}_a$. Hence the system of contact forces exerted on B_1 by B_2 is approximately equivalent to a force $\mathbf{F}$ whose line of action passes through O, $\mathbf{F}$ being given by

$$\mathbf{F} = (\pi\epsilon^2/2)\boldsymbol{\tau}_2{}^O = (\mathbf{n}_1 - 2\mathbf{n}_2)\pi\epsilon^2$$

$\mathbf{F}$ is shown in Fig. 4.10d.

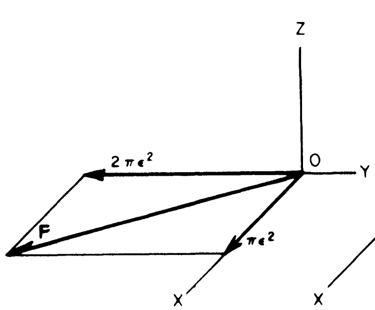


FIG. 4.10d

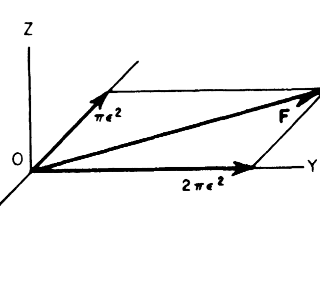


FIG. 4.10e

Solution (2): Use 4.7: The system of contact forces exerted by B_1 on B_2 is approximately equivalent to the force $\mathbf{F}'$ shown in Fig. 4.10e.

Solution (3): The system of contact forces exerted on B_1 by the body consisting of B_2 and B_3 is composed of the two systems of contact forces exerted on B_1 by B_2 and B_3. The latter is approxi-

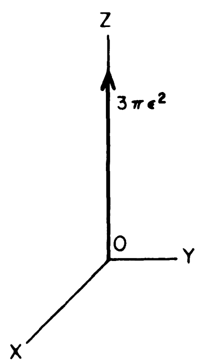


FIG. 4.10f

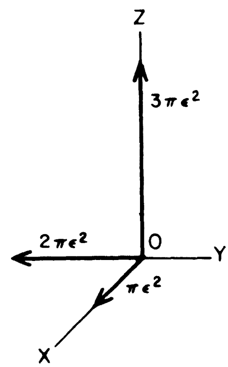


FIG. 4.10g

mately equivalent to the single force shown in Fig. 4.10f. Hence Fig. 4.10g represents an approximate reduction of the force system in question.

4.10.1 If $\boldsymbol{\tau}_a{}^P$ and $\boldsymbol{\tau}_b{}^P$ are the tractions at a point P, for the directions $\mathbf{n}_a$ and $\mathbf{n}_b$, and

$$\mathbf{n}_b = -\mathbf{n}_a$$

then

$$\boldsymbol{\tau}_b{}^P = -\boldsymbol{\tau}_a{}^P$$

This is an immediate consequence of 4.7 and the definition of traction.

Problem: Referring to Problem 4.10, determine the traction at O, for the direction $-\mathbf{n}_1$.

Solution: Let

$$\mathbf{n}_a = -\mathbf{n}_1$$

and call the desired traction $\boldsymbol{\tau}_a{}^O$. Then

$$\boldsymbol{\tau}_a{}^O = -\boldsymbol{\tau}_1{}^O = 12\mathbf{n}_3 \text{ lb in}^{-2}$$

4.10.2 The point P appearing in the definition of traction may be a point of an actual surface of a body B. If B is not in contact with any other body at P, and $\mathbf{n}_a$ is a unit vector perpendicular to the surface of B at P, then

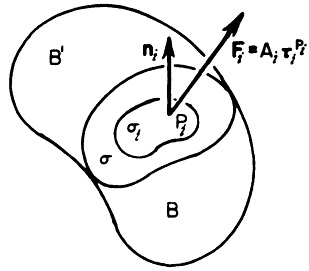
$$\boldsymbol{\tau}_a{}^P = 0$$

because either B_1 or B_2 vanishes during the limiting process used for the determination of $\boldsymbol{\tau}_a{}^P$. Under these circumstances, the surface of B is said to be *traction-free* at P.

4.10.3 The system of contact forces exerted on a body B by a body B' across a surface σ is related to the tractions at points of σ as follows (see Fig. 4.10.3a):

Divide σ into elements σ_i, $i = 1, 2, \ldots, n$. Let A_i be the area of σ_i; P_i, a point of σ_i; $\mathbf{n}_i$, the outward normal to B at P_i; $\boldsymbol{\tau}_i{}^{P_i}$,

FIG. 4.10.3a



the traction at P_i, for the direction $\mathbf{n}_i$; $\mathbf{F}_i$, a force whose line of action passes through P_i and which is given by

$$\mathbf{F}_i = A_i\boldsymbol{\tau}_i{}^{P_i}$$

As n tends toward infinity and the elements σ_i, $i = 1, \ldots, n$, shrink to points, the system of forces $\mathbf{F}_i$, $i = 1, 2, \ldots, n$, approaches equivalence with the system of forces exerted on B by B' across σ.

Problem: A uniform block of weight W rests on a horizontal support (see Fig. 4.10.3b). Assuming that the traction at all points of the surface of contact between the block and support, for the direction $\mathbf{n}_a$, is the same, say $\boldsymbol{\tau}_a$, determine this traction.

Solution: $\mathbf{n}_a$ is the outward normal to the block at *every* point of the surface of contact between the block and support. Draw a free-body diagram of the block, replacing the system of contact forces exerted on the block by the support, across elements σ_i, $i = 1, \ldots, n$, of the contact surface, with forces $A_i\boldsymbol{\tau}_a^{P_i}$, $i = 1, \ldots, n$, as shown in Fig. 4.10.3c. Let $L[Q]$ denote the limit approached

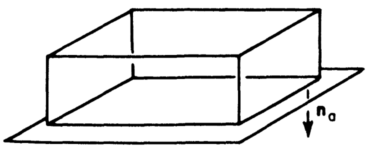


FIG. 4.10.3b

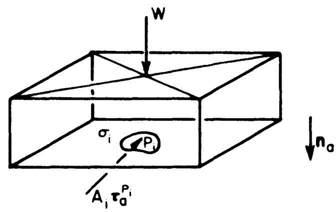


FIG. 4.10.3c

by the quantity Q as n tends to infinity and the elements σ_i, $i = 1, \ldots, n$, shrink to points. Then the following force equation is justified:

$$W\mathbf{n}_a + \mathrm{L}\left[\sum_{i=1}^{n} A_i\boldsymbol{\tau}_a^{P_i}\right] = 0 \tag{1}$$

Assuming that

$$\boldsymbol{\tau}_a^{P_i} = \boldsymbol{\tau}_a, \quad i = 1, 2, \ldots, n$$

the limit appearing in (1) may be evaluated:

$$\mathrm{L}\left[\sum_{i=1}^{n} A_i\boldsymbol{\tau}_a^{P_i}\right] = \mathrm{L}\left[\sum_{i=1}^{n} A_i\boldsymbol{\tau}_a\right]$$

$$= \boldsymbol{\tau}_a\mathrm{L}\left[\sum_{i=1}^{n} A_i\right] = \boldsymbol{\tau}_a A$$

where A is the area of the contact surface. Substitute into Eq. (1) and solve for $\boldsymbol{\tau}_a$:

$$\boldsymbol{\tau}_a = -\frac{W}{A}\mathbf{n}_a$$

4.10.4 Limits of the kind encountered in Problem 4.10.3 are similar to those considered in 2.5.1 and can frequently be expressed as integrals; e.g.,

$$\mathrm{L}\left[\sum_{i=1}^{n} A_i \tau_a^{P_i}\right] = \int_{\sigma} \tau_a^{P}\, dA$$

Correspondingly, a force such as $A_i\tau_a^{P_i}$ in Fig. 4.10.3c is then replaced with a *differential force* $\tau_a^P\, dA$, dA being the area of a differential element $d\sigma$ of the contact surface, as shown in Fig. 4.10.4.

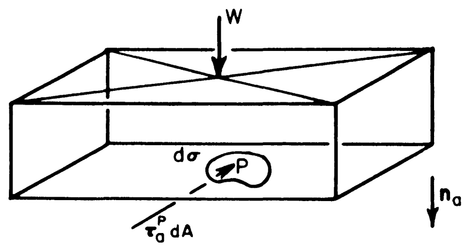
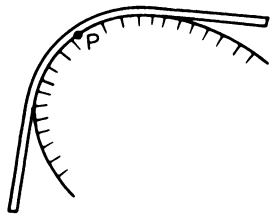


Fig. 4.10.4

4.10.5 From 4.10 and 4.8.2 it follows that the traction at a point P, for the direction $\mathbf{n}_a$, is parallel to $\mathbf{n}_a$ whenever P is a point of a smooth surface and $\mathbf{n}_a$ is perpendicular to this surface, at P.

Problem: A taut, flexible cable is supported by a smooth cylindrical surface, the cable lying in a plane normal to the generators of the surface (see Fig. 4.10.5a). Neglecting the weight of the cable, show that (1) the cable tension does not vary from point to point along the cable, and (2) the magnitude of the traction at a point P, for the direction normal to the support, is directly propor-

Fig. 4.10.5a



tional to the tension at P, inversely proportional to the radius of curvature of the cable at P, and inversely proportional to the width w of the surface of contact between the cable and the support.

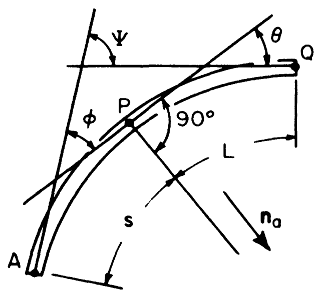


FIG. 4.10.5b

Solution: Introduce the following (see Fig. 4.10.5b):

A a point of the cable
ϕ the angle between the tangents to the cable at A and P
s the length of the arc AP
Q a point of the cable, so chosen that the arc AQ has a length $s + L$
ψ the angle between the tangents to the cable at A and Q
θ the angle between the tangents to the cable at P and Q
$\mathbf{n}_a$ a unit vector normal to the supporting surface at P, pointing from the cable toward the surface
T the cable tension at P
τ the magnitude of the traction at P, for the direction $\mathbf{n}_a$
ρ the radius of curvature of the cable at P

For later use, it is convenient to express θ in terms of L, ρ and s. This is done as follows: By construction,

$$\theta = \psi - \phi \tag{1}$$

Regard ϕ as a function of s; that is, let

$$\phi = \phi(s)$$

Then

$$\psi = \phi(s + L)$$

or, using Taylor's Theorem,

$$\psi = \phi(s) + \frac{d\phi}{ds} L + \frac{d^2\phi}{ds^2}\frac{L^2}{2!} + \cdots$$

Hence,

$$\theta \underset{(1)}{=} \frac{d\phi}{ds} L + \frac{d^2\phi}{ds^2}\frac{L^2}{2!} + \cdots$$

For any plane curve,

$$\frac{d\phi}{ds} = \frac{1}{\rho}$$

Substitute:

$$\theta = \frac{L}{\rho} + \frac{d^2\phi}{ds^2}\frac{L^2}{2} + \cdots \tag{2}$$

Regarding T as a function of s, it is to be shown that

$$\frac{dT}{ds} = 0$$

and that

$$\tau = \frac{T}{w\rho}$$

Draw a free-body diagram of the portion of the cable between points P and Q. The contact forces acting on this body are (a) those exerted by contiguous portions of the cable at P and Q, and (b) those exerted by the support.

The system of forces exerted on PQ by AP is equivalent to the single force of magnitude T shown in Fig. 4.10.5c. As T is a function of s, the system of forces exerted by PQ on a portion of the cable to the right of point Q is equivalent to the force shown in Fig. 4.10.5d. Using 4.7, the system of forces exerted on PQ by a portion of the cable to the right of Q is thus equivalent to the force

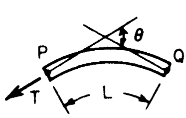


FIG. 4.10.5c

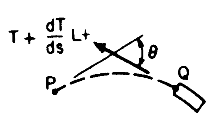


FIG. 4.10.5d

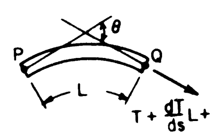


FIG. 4.10.5e

shown in Fig. 4.10.5e.

The system of forces exerted on PQ by the supporting surface can be replaced, approximately, with a single force whose line of action passes through P and is normal to the (smooth) supporting surface, and whose magnitude is equal to $wL\tau$, where w is the width of the contact surface at P. The complete free-body diagram is

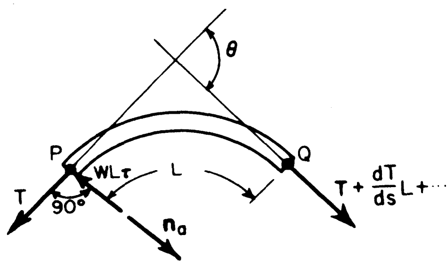


Fig. 4.10.5f

shown in Fig. 4.10.5f.

Assuming that w is so small that the surface of contact between the body PQ and the support may be regarded as shrinking to a point when L approaches zero, the force equations

$$-T + \left(T + \frac{dT}{ds}L + \ldots\right)\cos\theta = 0 \tag{3}$$

and

$$wL\tau - \left(T + \frac{dT}{ds}L + \ldots\right)\sin\theta = 0 \tag{4}$$

apply when L approaches zero.

Replace $\sin\theta$ and $\cos\theta$ with, respectively,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \ldots, \quad \cos\theta = 1 - \frac{\theta^2}{2!} + \ldots$$

Use Eq. (2) to eliminate θ, then arrange the terms in each equation in order of ascending powers of L:

$$\frac{dT}{ds} - T\frac{L}{2\rho^2} + \ldots = 0 \tag{5}$$

$$w\tau - \frac{T}{\rho} - \frac{dT}{ds}\frac{L}{\rho} + \ldots = 0 \tag{6}$$

Let L approach zero:

$$\frac{dT}{ds} \underset{(5)}{=} 0$$

$$\tau \underset{(6)}{=} \frac{T}{w\rho}$$

4.11 Friction

Notation (See Fig. 4.11a):

B, B'	continuous bodies
σ	a surface of actual contact of B and B'
P	a point of σ
$\mathbf{n}_a$	the outward normal to B at P
$\boldsymbol{\tau}_a{}^P$	the traction at P, for the direction $\mathbf{n}_a$
$\boldsymbol{\tau}_n$	the $\mathbf{n}_a$ resolute of $\boldsymbol{\tau}_a{}^P$; $\boldsymbol{\tau}_n$ is called the *normal component* of $\boldsymbol{\tau}_a{}^P$
$\boldsymbol{\tau}_f$	the resolute of $\boldsymbol{\tau}_a{}^P$ perpendicular to $\mathbf{n}_a$; $\boldsymbol{\tau}_f$ is called the *friction component* of $\boldsymbol{\tau}_a{}^P$

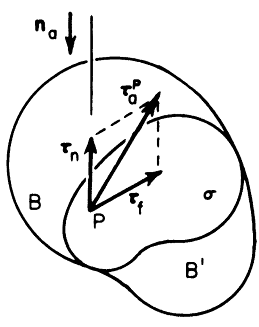


FIG. 4.11a

The following relationships between $\boldsymbol{\tau}_n$ and $\boldsymbol{\tau}_f$ are valid for bodies having dry, clean surfaces, and are called *Laws of Dry Friction:*

I When B is at rest relative to B' at P,

$$|\boldsymbol{\tau}_f| \leqslant \mu |\boldsymbol{\tau}_n|$$

where μ is a number which depends only on the materials of which B and B' are composed at P. μ is called the *coefficient of static friction* for B and B'. Typical values are 0.2 for metal on metal, 0.6 for metal on wood.

II When B is in a state of impending, tangential motion relative to B' at P,

$$|\tau_f| = \mu|\tau_n|$$

and τ_f is directed in such a way as to oppose relative motion.

III When B is moving tangentially relative to B' at P,

$$|\tau_f| = \mu'|\tau_n|$$

where μ' is a number called the *coefficient of kinetic friction* for B and B'. The value of μ', for a given pair of materials, is usually slightly smaller than the value of μ for the same materials. τ_f is again directed so as to oppose relative motion.

Problem (a): A uniform block of weight W rests on a horizontal surface and is attached to a horizontal cable, as shown in Fig. 4.11b. The tension in the cable is increased gradually, reaching

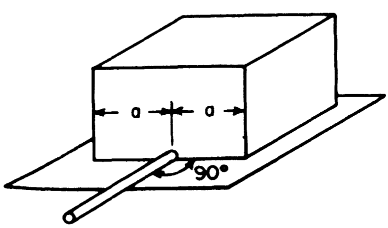


FIG. 4.11b

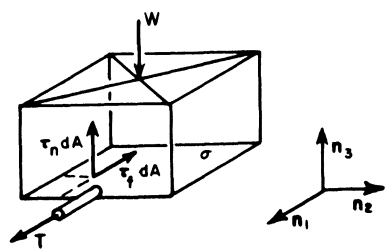


FIG. 4.11c

a value T when the block begins to slip. The coefficient of friction for the block and surface has the value 0.2. Determine T.

Solution: In the free-body diagram, Fig. 4.11c, the forces exerted on the block by the support are represented by the typical, differential forces of magnitude $\tau_n\, dA$ and $\tau_f\, dA$, the direction of the latter force being so chosen that the force opposes impending motion. (The direction of impending motion is determined by considerations of symmetry; if the cable were attached at one of the corners of the block, or did not make an angle of ninety degrees with the edge, the direction of impending motion at a typical point of the contact surface would not be known.)

τ_n, being unknown, must be presumed to vary from point to point of the contact surface σ. From *II*, τ_f depends on τ_n:

$$\tau_f = \mu\tau_n \tag{1}$$

Force equations:

$$\mathbf{n}_3: \qquad -W + \int_\sigma \tau_n \, dA = 0 \tag{2}$$

$$\mathbf{n}_1: \qquad T - \int_\sigma \tau_f \, dA = 0 \tag{3}$$

Substitute from Eq. (1) into Eq. (3), and solve for T:

$$T = \mu \int_\sigma \tau_n \, dA \tag{4}$$

The integration in the right-hand member of Eq. (4) cannot be performed explicitly, because it is not known how τ_n varies from point to point over the surface of contact; but it is unnecessary to perform this integration, because Eq. (2) furnishes the value of the integral in question:

$$\int_\sigma \tau_n \, dA = W$$

Substitute into Eq. (4):

$$T = \mu W$$

With $\mu = 0.2$,

$$T = 0.2W$$

Problem (b): A uniform cylinder of weight W rests in a horizontal trough and is attached to a horizontal cable, as shown in Fig. 4.11d. The tension in the cable is increased gradually, reaching a value T when the cylinder begins to slip. The coefficient of friction for the cylinder and trough has the value 0.2. Determine T.

Solution: Draw the free-body diagram shown in Fig. 4.11e. Note that τ_n, being unknown, must be regarded as an unknown function of θ and x (see Fig. 4.11e). From *II*,

$$\tau_f = \mu\tau_n \tag{1}$$

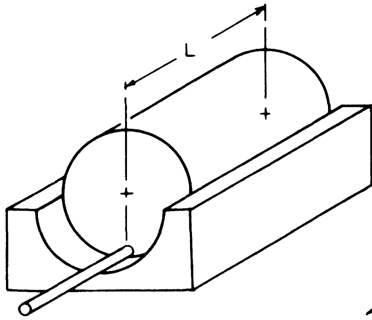


FIG. 4.11d

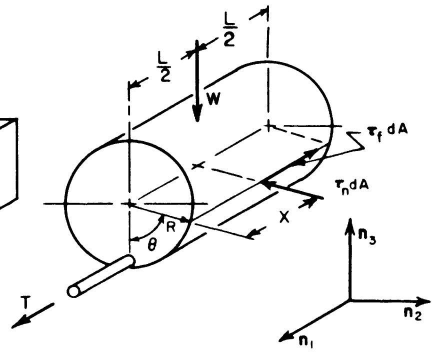


FIG. 4.11e

Force equations:

$$\mathbf{n}_3: \qquad -W + \int_\sigma \tau_n \cos\theta \, dA = 0 \tag{2}$$

$$\mathbf{n}_1: \qquad T - \int_\sigma \tau_f \, dA = 0 \tag{3}$$

Substitute from Eq. (1) into Eq. (3), and solve for T:

$$T = \mu \int_\sigma \tau_n \, dA \tag{4}$$

The integration in the right-hand member of Eq. (4) cannot be performed explicitly, because it is not known how τ_n varies from point to point over the surface of contact; and in the present problem, as contrasted with Problem 4.11(a), it is necessary to perform this integration, because the integrals appearing in Eqs. (2) and (4) are not identical with each other, as they were previously. It is therefore necessary to make an assumption regarding the dependence of τ_n on θ and x.

Assume that τ_n is an even function of θ, which attains its maximum value when $\theta = 0$ and vanishes for $\theta = \pm\pi/2$. One of the simplest functions fulfilling these requirements is

$$\tau_n(\theta, x) = p(x) \cos\theta$$

where $p(x)$ is an unspecified function of x.

Substitute into Eqs. (2) and (4) (after replacing dA with $R\,d\theta\,dx$):

$$-W + R\int_0^L dx \int_{-\pi/2}^{\pi/2} p\cos^2\theta\,d\theta = 0 \tag{5}$$

$$T = \mu R\int_0^L dx \int_{-\pi/2}^{\pi/2} p\cos\theta\,d\theta = 0 \tag{6}$$

Integrate with respect to θ in Eqs. (5) and (6):

$$-W + \frac{R\pi}{2}\int_0^L p\,dx = 0 \tag{7}$$

$$T = 2\mu R\int_0^L p\,dx = 0 \tag{8}$$

Solve Eq. (7) for $\int_0^L p\,dx$, and substitute into Eq. (8):

$$T = \frac{4\mu W}{\pi}$$

With $\mu = 0.2$,

$$T = 0.254W$$

4.11.1 The two forces $\mathbf{N}$ and $\mathbf{F}$ defined as (see 4.11 for notation)

$$\mathbf{N} = \int_\sigma \boldsymbol{\tau}_n\,dA, \quad \mathbf{F} = \int_\sigma \boldsymbol{\tau}_f\,dA$$

are called, respectively, the *normal force* and the *friction force* exerted on B by B' across σ. Their magnitudes, $|\mathbf{N}|$ and $|\mathbf{F}|$, are not, in general, related to each other in the same way as are those of $\boldsymbol{\tau}_n$ and $\boldsymbol{\tau}_f$; i.e., the Laws of Dry Friction do not, in general, apply to $\mathbf{N}$ and $\mathbf{F}$.

Problem: Determine the ratio of the magnitude of the friction force to that of the normal force for the situations described (1) in Problem 4.11(a) and (2) in Problem 4.11(b).

Solution (1) (See Fig. 4.11c):

$$\begin{aligned}\mathbf{F} &= \int_\sigma \boldsymbol{\tau}_f\,dA = \int_\sigma (-\tau_f\mathbf{n}_1)\,dA \\ &= \int_\sigma (-\mu\tau_n\mathbf{n}_1)\,dA = -\mu\mathbf{n}_1\int_\sigma \tau_n\,dA\end{aligned}$$

$$\begin{aligned}\mathbf{N} &= \int_\sigma \boldsymbol{\tau}_n\,dA \\ &= \int_\sigma (\tau_n\mathbf{n}_3)\,dA = \mathbf{n}_3\int_\sigma \tau_n\,dA\end{aligned}$$

Hence,

$$\frac{|\mathbf{F}|}{|\mathbf{N}|} = \mu$$

and

$$\frac{|\tau_f|}{|\tau_n|} = \frac{|\mathbf{F}|}{|\mathbf{N}|}$$

Solution (2) (See Fig. 4.11e):

$$\mathbf{F} = \int_\sigma \tau_f \, dA = \int_\sigma (-\tau_f \mathbf{n}_1) \, dA$$

$$= \int_\sigma (-\mu \tau_n \mathbf{n}_1) \, dA = -\mu \mathbf{n}_1 \int_\sigma \tau_n \, dA$$

$$\mathbf{N} = \int_\sigma \tau_n \, dA$$

$$= \int_\sigma \tau_n(-\sin\theta \mathbf{n}_2 + \cos\theta \mathbf{n}_3) \, dA$$

$$= -\mathbf{n}_2 \int_\sigma \tau_n \sin\theta \, dA + \mathbf{n}_3 \int_\sigma \tau_n \cos\theta \, dA$$

As in Problem 4.11(b), let

$$\tau_n(\theta, x) = p(x)\cos\theta$$

and evaluate the integrals appearing above:

$$\int_\sigma \tau_n \, dA = R\int_0^L dx \int_{-\pi/2}^{\pi/2} p\cos\theta \, d\theta = 2R\int_0^L p \, dx$$

$$\int_\sigma \tau_n \sin\theta \, dA = R\int_0^L dx \int_{-\pi/2}^{\pi/2} p\cos\theta\sin\theta \, d\theta = 0$$

$$\int_\sigma \tau_n \cos\theta \, dA = R\int_0^L dx \int_{-\pi/2}^{\pi/2} p\cos^2\theta \, d\theta = \frac{R\pi}{2}\int_0^L p \, dx$$

Substitute.

$$\mathbf{F} = -2\mu R \mathbf{n}_1 \int_0^L p \, dx$$

$$\mathbf{N} = \frac{R\pi}{2}\mathbf{n}_3 \int_0^L p \, dx$$

Hence,

$$\frac{|\mathbf{F}|}{|\mathbf{N}|} = \frac{4\mu}{\pi}$$

and

$$\frac{|\tau_f|}{|\tau_n|} \neq \frac{|\mathbf{F}|}{|\mathbf{N}|}$$

4.11.2 When the surface of contact between two bodies is so small that the lines of action of the forces exerted by one body on the other may be regarded as passing through a single point (see 4.8.3), the Laws of Dry Friction *do* apply to the associated normal force ($\mathbf{N}$) and friction force ($\mathbf{F}$). For, as the surface σ shrinks to a point, the integrals $\int_\sigma \tau_n \, dA$ and $\int_\sigma \tau_f \, dA$ approach, respectively, $\tau_n \int_\sigma dA$ and $\tau_f \int_\sigma dA$; that is,

$$\mathbf{N} \to \tau_n A, \quad \mathbf{F} \to \tau_f A$$

where A is the (very small) area of the contact surface σ. Hence

$$|\tau_n| \to \frac{|\mathbf{N}|}{A}, \quad |\tau_f| \to \frac{|\mathbf{F}|}{A}$$

and τ_n and τ_f may be replaced with, respectively, $\mathbf{N}$ and $\mathbf{F}$ throughout *I*, *II* and *III* in 4.11.

Problem (a): Referring to Problem 4.8.3, determine the minimum value of the coefficient of friction, μ, for the hemisphere and support.

Solution: The normal force $\mathbf{N}$ at the point of contact P is given by

$$\mathbf{N} = F_2 \mathbf{n}_2 = W \mathbf{n}_2$$

while the friction force $\mathbf{F}$ is

$$\mathbf{F} = F_1 \mathbf{n}_1 + F_3 \mathbf{n}_3 = (W/8) \mathbf{n}_1$$

As the hemisphere is at rest relative to the support,

$$|\mathbf{F}| \leqslant \mu |\mathbf{N}|$$

or

$$\mu \geqslant \frac{|\mathbf{F}|}{|\mathbf{N}|} = \frac{W/8}{W} = \frac{1}{8}$$

Thus, $\frac{1}{8}$ is the minimum value of μ.

Problem (b): In Problem 4.10.5, it was shown that the tension in a light, taut, flexible cable supported by a smooth cylindrical surface does not vary from point to point along the cable. If the supporting surface is not smooth, the tension *does* vary. Referring

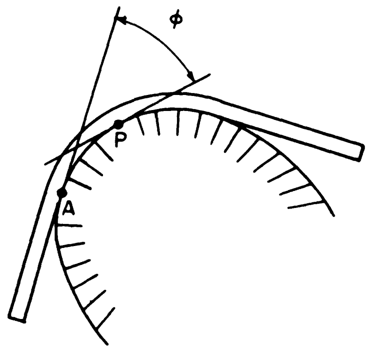


FIG. 4.11.2a

to Fig. 4.11.2a, show that, when the cable is on the verge of slipping from A toward P, the cable tensions, T and T_A, at P and A are related to each other as follows:

$$T = T_A e^{\mu\phi}$$

where μ is the coefficient of friction for the cable and support.

Solution: The free-body diagram shown in Fig. 4.10.5f must be modified by the addition of a force of magnitude $\mu w\tau L$, directed as shown in Fig. 4.11.2b. Accordingly, Eq. (3) of Problem 4.10.5 is replaced with

$$-T - \mu w\tau L + \left(T + \frac{dT}{ds} L + \ldots\right) \cos\theta = 0$$

while Eq. (4) remains unaltered:

$$wL\tau - \left(T + \frac{dT}{ds} L + \ldots\right) \sin\theta = 0$$

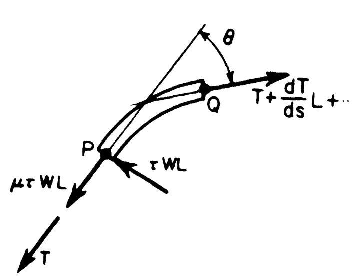


FIG. 4.11.2b

Proceeding as in Problem 4.10.5, reduce these equations to

$$-\mu w\tau + \frac{dT}{ds} = 0$$

and

$$\tau = \frac{T}{w\rho}$$

Eliminate τ, and replace $1/\rho$ with $d\phi/ds$:

$$-\mu T \frac{d\phi}{ds} + \frac{dT}{ds} = 0$$

or

$$T \frac{d}{ds}(-\mu\phi + \log T) = 0$$

This equation shows that, when $T \neq 0$,

$$-\mu\phi + \log T = C$$

where C is a constant.
Evaluate C by noting that $T = T_A$ when $\phi = 0$; that is,

$$0 + \log T_A = C$$

Then

$$-\mu\phi + \log T = \log T_A$$

and

$$T = T_A e^{\mu\phi}$$

Problem (c): A thirty-four pound weight is supported by a cable which is wound on a fixed horizontal drum and is attached to a wall, as shown in Fig. 4.11.2c. The coefficient of friction for

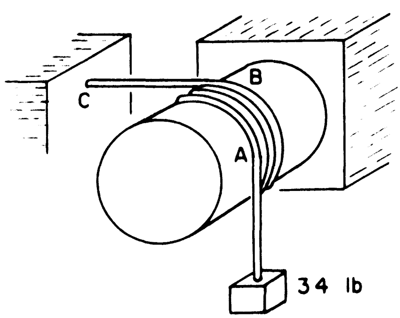


FIG. 4.11.2c

the drum and cable has the value 0.2. Determine the reaction of the cable on the wall.

Solution: Draw plane free-body diagrams of (1) the portion of the cable between points C and B, (2) the portion of the cable

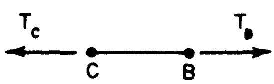


FIG. 4.11.2d

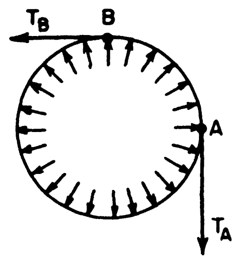


FIG. 4.11.2e

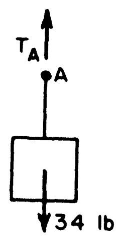


FIG. 4.11.2f

in contact with the drum, (3) the body consisting of the weight and the portion of the cable below point A (see Figs. 4.112d, e, f).

Force equations:

Fig. 4.11.2d: $$T_C - T_B = 0; T_C = T_B \tag{1}$$

Fig. 4.11.2f: $$T_A - 34 = 0; T_A = 34 \tag{2}$$

If the weight is on the point of moving upward,

$$T_B = T_A e^{\mu\phi} \tag{3}$$

If it is on the point of moving downward,

$$T_A = T_B e^{\mu\phi} \tag{4}$$

With $\phi = 4.5\pi$, $\mu = 0.2$, $T_A = 34$ lb, Eq. (3) gives

$$T_B = 34e^{2.83} = 34 \times 17 = 578 \text{ lb} \tag{5}$$

while Eq. (4), solved for T_B, leads to

$$T_B = 34e^{-2.83} = \tfrac{34}{17} = 2 \text{ lb} \tag{6}$$

Hence, for impending motion upward,

$$T_C \underset{(1,5)}{=} 578 \text{ lb}$$

while, for impending motion downward,

$$\underset{(1,6)}{T_C = 2\ \text{lb}}$$

The force exerted on the wall by the cable therefore has a minimum magnitude of 2 lb and a maximum magnitude of 578 lb. The actual value cannot be determined solely on the basis of information contained in the statement of the problem.

PROBLEM SETS

PROBLEM SET 1

(See Sections 1.1–1.7 of the text)

(a) A force **F** has a magnitude of 20 pounds. Two scalars, s_1 and s_2, have the values $s_1 = -\frac{1}{2}$, $s_2 = 3$ ft. Using scales of 1 in. = 10 lb and 1 in. = 20 ft lb, draw **F** and the vectors $s_1\mathbf{F}$, $s_2\mathbf{F}$; $s_1(s_2\mathbf{F})$, $s_2(s_1\mathbf{F})$, $(s_1s_2)\mathbf{F}$; $(-s_1)\mathbf{F}$, $s_1(-\mathbf{F})$, $-(s_1\mathbf{F})$; $\mathbf{F}/s_1$.

(b) Letting **n** be a unit vector having the same direction as the force **F** in Problem 1(a),* each of the vectors drawn in Problem 1(a) can be expressed in the form $s\mathbf{n}$. Determine s for each case.

Results: 20 lb, −10 lb, 60 ft lb; −30 ft lb, −30 ft lb, −30 ft lb; 10 lb, 10 lb, 10 lb; −40 lb.

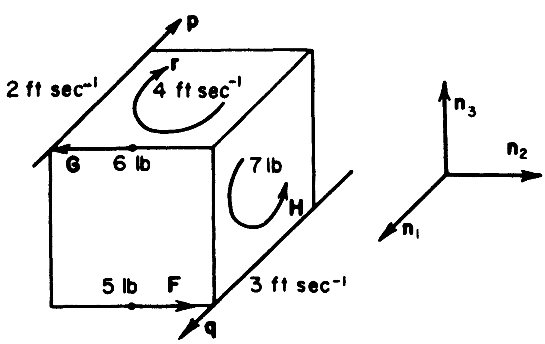


Fig. 1c

(c) In Fig. 1(c),† $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are unit vectors parallel to the edges of a parallelepiped. **F**, **G** and **H** represent forces; **p**, **q** and **r**, velocities. Express each force and velocity as the product of a scalar and one of the unit vectors.

* In the sixteen Problem Sets particular problems are referred to as follows: The arabic numeral indicates the number of the Problem Set, and the letter enclosed in parentheses indicates the paragraph in which the problem is presented.

† In the Problem Sets each figure has the same number as the problem to which it pertains.

Results: $5\mathbf{n}_2$ lb, $-6\mathbf{n}_2$ lb, $7\mathbf{n}_2$ lb; $-2\mathbf{n}_1$ ft sec^{-1}, $3\mathbf{n}_1$ ft sec^{-1}, $-4\mathbf{n}_3$ ft sec^{-1}.

(d) A force $\mathbf{F}'$ is given by

$$\mathbf{F}' = -15\mathbf{n} \text{ lb}$$

where $\mathbf{n}$ is a unit vector having the same direction as the force $\mathbf{F}$ in Problem 1(a). Express $\mathbf{F}'$ as the quotient of $\mathbf{F}$ and a scalar.

Result: $\mathbf{F}' = \mathbf{F}/(-\frac{4}{3})$.

(e) Referring to Fig. 1(c), determine the scalar s for which

$$\mathbf{n}_3 = s\mathbf{r}$$

Result: $-\frac{1}{4}$ ft^{-1} sec.

(f) A force $\mathbf{F}$ has a magnitude of 10 lb and the direction of the vector $\mathbf{p}$ shown in Fig. 1(c). Express $\mathbf{F}$ as the product of a scalar and the vector $\mathbf{q}$.

Result: $\mathbf{F} = -\frac{10}{3}\mathbf{q}$ lb.

(g) A vector $\mathbf{v}$ is given by

$$\mathbf{v} = -\frac{\mathbf{G}}{|\mathbf{P}|}$$

where $\mathbf{G}$ and $\mathbf{p}$ are two of the vectors shown in Fig. 1(c). Determine the magnitude of $\mathbf{v}$.

Result: 3 lb sec ft^{-1}.

PROBLEM SET 2

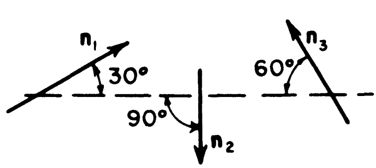
(See Sections 1.8–1.9 of the text)

All problems in this set deal with three forces, $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$, defined as follows:

$$\mathbf{F}_1 = 10\mathbf{n}_1 \text{ lb}, \quad \mathbf{F}_2 = -15\mathbf{n}_2 \text{ lb}, \quad \mathbf{F}_3 = 20\mathbf{n}_3 \text{ lb}$$

where $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are the unit vectors shown in Fig. 2(a).

FIG. 2a



(a) Determine $|\mathbf{F}_1 + \mathbf{F}_2|$ analytically, then check the result graphically and compare it with $|\mathbf{F}_1| + |\mathbf{F}_2|$. Is the magnitude of the sum of two vectors ever equal to the sum of the magnitudes of the vectors? Draw a sketch illustrating a situation in which the magnitude of the sum of two vectors is equal to the difference of their magnitudes.

(b) Determine $|\mathbf{F}_1 - \mathbf{F}_2|$ analytically, check graphically, and compare with $|\mathbf{F}_1| - |\mathbf{F}_2|$.

(c) Find two scalars, s_1 and s_2, for which $s_1\mathbf{F}_1 + s_2\mathbf{F}_2 = 0$. Is the solution of this problem unique?

(d) Find two scalars, x_1 and x_2, for which $x_1\mathbf{F}_1 + x_2\mathbf{F}_2 = \mathbf{F}_3$.

Result: $-\frac{2}{3}\sqrt{3}$, $\frac{8}{9}\sqrt{3}$.

(e) Determine the magnitude of the resultant of $\mathbf{F}_1$, $\mathbf{F}_2$ and $-\mathbf{F}_3/2$.

Result: 17.8 lb.

(f) Determine whether or not $|5\mathbf{F}_1 - 5\mathbf{F}_2|$ is equal to $5|\mathbf{F}_1 - \mathbf{F}_2|$, then either prove that $\left|\sum_{i=1}^{n} s\mathbf{v}_i\right|$ is equal to $s\left|\sum_{i=1}^{n} \mathbf{v}_i\right|$, or give a counter-example.

(g) $\mathbf{n}$ is a unit vector having the same direction as $\mathbf{F}_1 + \mathbf{F}_2$ and satisfying the equation

$$\mathbf{n} + A\mathbf{n}_1 + B\mathbf{n}_2 = 0$$

Evaluate A and B.

Result: $-2/\sqrt{19}$, $3/\sqrt{19}$.

PROBLEM SET 3

(See Section 1.10 of the text)

(a) The force $\mathbf{F}$ shown in Fig. 3(a) is to be resolved into four coplanar components, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$, three of which are shown. Determine the magnitude of $\mathbf{F}_4$.

Result: 4.03 lb.

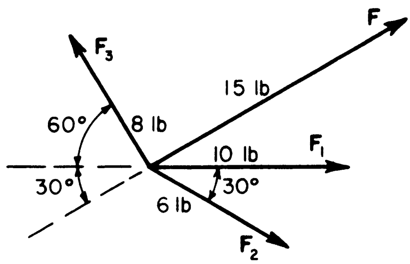


FIG. 3a

(b) A vector is resolved into two components, each of which has a magnitude n times as large as that of the vector. Assuming that n is large in comparison with unity, determine the (smallest) angle between the lines of action of the components.

Result: $1/n$ rad.

(c) Determine the values of x, y and z for which the two vectors

$$(2x - 3y)\mathbf{n}_1 - z\mathbf{n}_2 + (x - z)\mathbf{n}_3$$

and

$$(z - 1)\mathbf{n}_1 + (2y - x)\mathbf{n}_2 - z\mathbf{n}_3$$

are equal to each other, $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ being unit vectors not parallel to the same plane.

Result: 0, 1, -2.

Would this result be altered if $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ were parallel to the same plane?

(d) Letting $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ be the unit vectors shown in Fig. 2(a), determine the values of x and y for which the two vectors

$$x\mathbf{n}_1 - 2\mathbf{n}_2 + 3\mathbf{n}_3$$

and

$$7\mathbf{n}_1 + 2\mathbf{n}_2 + (3 + 2y)\mathbf{n}_3$$

are equal to each other.

Result: 5, $\sqrt{3}$.

Could these two vectors be equal to each other if $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ were not parallel to the same plane?

(e) Two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, expressed in terms of unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ which are not parallel to the same plane, are given by

$$\mathbf{F}_1 = 9\mathbf{n}_1 - 3\mathbf{n}_2 + \mathbf{n}_3 \text{ lb}$$

$$\mathbf{F}_2 = 5\mathbf{n}_2 - 4\mathbf{n}_3 \text{ lb}$$

A third force, $\mathbf{F}$, is defined as

$$\mathbf{F} = 4(\mathbf{F}_1 - 2\mathbf{F}_2)$$

Determine the $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ measure numbers of $\mathbf{F}$.

Result: 36 lb, −52 lb, 36 lb.

(f) Determine the magnitude of the force $\mathbf{F}$ of Problem 3(e), taking $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ to be the unit vectors shown in Fig. 3(f).

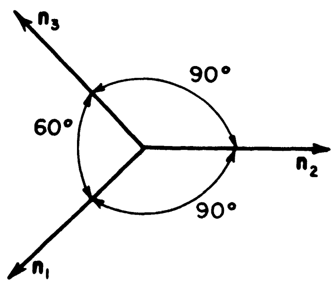


FIG. 3f

Result: 81.2 lb.

(g) Determine the magnitude of the force $\mathbf{F}$ of Problem 3(e), letting $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ be mutually perpendicular.

Result: 72.8 lb.

PROBLEM SET 4

(See Sections 1.11–1.12 of the text)

(a) Determine the magnitude of the resultant of the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ shown in Fig. 4(a).

Result: 31 lb.

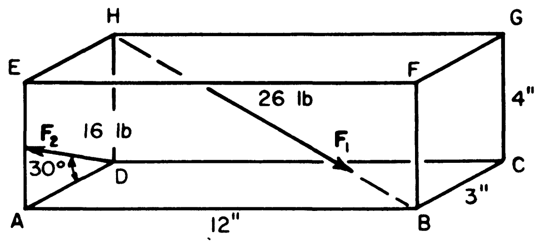


FIG. 4a

(b) Determine the $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ measure numbers of the unit vector $\mathbf{n}$ shown in Fig. 4(b), assuming that $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors.

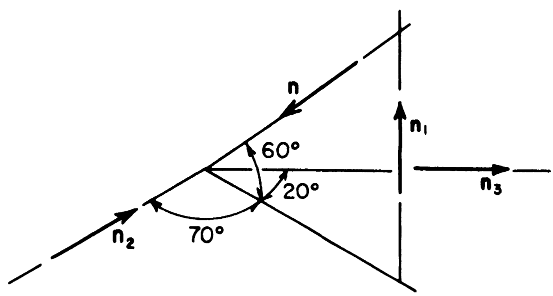


FIG. 4b

Result: -0.866, 0.171, -0.470.

(c) Two forces, $\mathbf{F}'_1$ and $\mathbf{F}'_2$, each of magnitude F, are to be added to the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ shown in Fig. 4(a). If the line of action of $\mathbf{F}'_1$ passes through points B and G, and the resultant of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}'_1$, $\mathbf{F}'_2$ is equal to zero, what is the value of F?

Answer: 40.3 lb.

(d) For the vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ shown in Fig. 2(a), determine α_2, α_3; β_3, β_1; γ_1, γ_2, such that

$$\mathbf{n}_1 = \alpha_2\mathbf{n}_2 + \alpha_3\mathbf{n}_3$$

$$\mathbf{n}_2 = \beta_3\mathbf{n}_3 + \beta_1\mathbf{n}_1$$

$$\mathbf{n}_3 = \gamma_1\mathbf{n}_1 + \gamma_2\mathbf{n}_2$$

Compare $\alpha_2^2 + \alpha_3^2$, $\beta_3^2 + \beta_1^2$, and $\gamma_1^2 + \gamma_2^2$ with, respectively, $|\mathbf{n}_1|$, $|\mathbf{n}_2|$, $|\mathbf{n}_3|$, and state, in general terms, a conclusion based on these comparisons.

Results: -2, $-\sqrt{3}$; $-\sqrt{3}/2$, $-\frac{1}{2}$; $-1/\sqrt{3}$, $-2/\sqrt{3}$.

(e) The vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_1'$ and $\mathbf{n}_2'$ shown in Fig. 4(e) are unit vectors parallel to the plane of the paper. Express $\mathbf{n}_1$ and $\mathbf{n}_2$ in terms of θ, $\mathbf{n}_1'$ and $\mathbf{n}_2'$; $\mathbf{n}_1'$ and $\mathbf{n}_2'$ in terms of θ, $\mathbf{n}_1$ and $\mathbf{n}_2$.

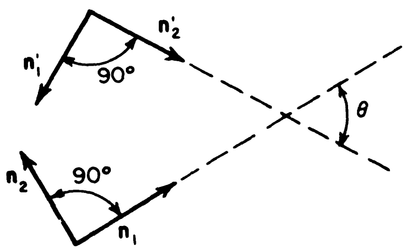


FIG. 4e

Results:

$$\mathbf{n}_1 = -\sin\theta\mathbf{n}_1' + \cos\theta\mathbf{n}_2', \quad \mathbf{n}_2 = -\cos\theta\mathbf{n}_1' - \sin\theta\mathbf{n}_2'$$

$$\mathbf{n}_1' = -\sin\theta\mathbf{n}_1 - \cos\theta\mathbf{n}_2, \quad \mathbf{n}_2' = \cos\theta\mathbf{n}_1 - \sin\theta\mathbf{n}_2$$

(f) Four rectangular parallelepipeds, R_i, $i = 1, 2, 3, 4$, are arranged as shown in Fig. 4(f). The unit vectors $\mathbf{n}_{ij}$, $j = 1, 2, 3$, are respectively parallel to the edges of R_i. The configuration shown is one in which the angles ϕ, θ, ψ (called "Eulerian" angles) are regarded as positive.

The vectors $\mathbf{n}_{1i}$, $i = 1, 2, 3$, can each be expressed in the form

$$\mathbf{n}_{1i} = \sum_{j=1}^{3} A_{ij}\mathbf{n}_{4j}$$

where the nine quantities A_{ij} $(i, j = 1, 2, 3)$ are functions of the Eulerian angles. Determine these functions.

Suggestion: Noting that $\mathbf{n}_{13} = \mathbf{n}_{23}$, $\mathbf{n}_{21} = \mathbf{n}_{31}$, and $\mathbf{n}_{33} = \mathbf{n}_{43}$, proceed as in Problem 4(e), to express $\mathbf{n}_{11}$, $\mathbf{n}_{12}$ and $\mathbf{n}_{13}$ in terms of $\mathbf{n}_{21}$, $\mathbf{n}_{22}$

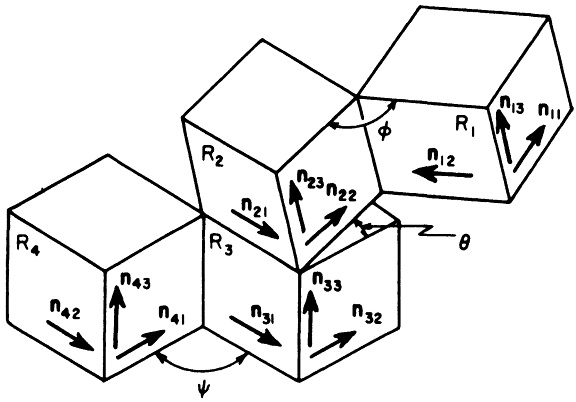


FIG. 4f

and $\mathbf{n}_{23}$; next, $\mathbf{n}_{21}$, $\mathbf{n}_{22}$, $\mathbf{n}_{23}$ in terms of $\mathbf{n}_{31}$, $\mathbf{n}_{32}$, $\mathbf{n}_{33}$; finally, $\mathbf{n}_{31}$, $\mathbf{n}_{32}$, $\mathbf{n}_{33}$ in terms of $\mathbf{n}_{41}$, $\mathbf{n}_{42}$, $\mathbf{n}_{43}$. Then substitute.

Result:

i	$j = 1$	$j = 2$	$j = 3$
1	$\cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi$	$\cos\phi\sin\psi + \cos\theta\sin\phi\cos\psi$	$\sin\theta\sin\phi$
2	$-\sin\phi\cos\psi - \cos\theta\cos\phi\sin\psi$	$-\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi$	$\sin\theta\cos\phi$
3	$\sin\theta\sin\psi$	$-\sin\theta\cos\psi$	$\cos\theta$

(g) A force $\mathbf{F}$ is given by

$$\mathbf{F} = 2\mathbf{n}_{11} - 3\mathbf{n}_{12} + 4\mathbf{n}_{13} \text{ lb}$$

where $\mathbf{n}_{1i}$, $i = 1, 2, 3$, are three of the unit vectors shown in Fig. 4(f). Resolving $\mathbf{F}$ into three components respectively parallel to $\mathbf{n}_{41}$, $\mathbf{n}_{42}$, $\mathbf{n}_{43}$, determine the $\mathbf{n}_{42}$ measure number of $\mathbf{F}$, for $\phi = -30°$, $\theta = 60°$, $\psi = 180°$.

Result: $(2 + 11\sqrt{3})/4$ lb.

(h) The L_1 resolute of the force $\mathbf{F}$ shown in Fig. 4(h) has a magnitude of 5 lb. When $\mathbf{F}$ is resolved into two components, one

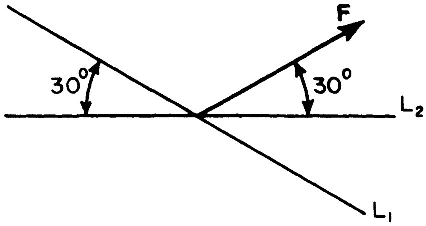


FIG. 4h

parallel to L_1, the other parallel to L_2, what is the magnitude of the component parallel to L_1?

Answer: 10 lb.

(i) A force **F** is parallel to line L_1 of Fig. 4(h) and has a magnitude of 20 lb. Determine the magnitude of the L_1 resolute of the resolute of **F** perpendicular to L_2.

Result: 5 lb.

PROBLEM SET 5

(See Sections 1.13–1.14 of the text)

(a) For the vectors **a** and **b** shown in Fig. 5(a), determine the angles (**a**, **b**), (−**a**, **b**), (−**a**, −**b**), (**a**, −**b**).

Results: 50°, 130°, 50°, 130°.

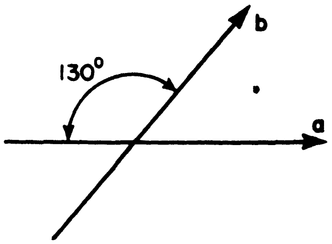


FIG. 5a

(b) Show that, for any two vectors **a** and **b**,

$$\sin(-\mathbf{a}, \mathbf{b}) = \sin(\mathbf{a}, \mathbf{b})$$

and

$$\cos(-\mathbf{a}, \mathbf{b}) = -\cos(\mathbf{a}, \mathbf{b})$$

(c) Referring to Problem Set 2, evaluate $\mathbf{F}_2 \cdot \mathbf{n}_1$ and $\mathbf{F}_2 \cdot \mathbf{n}_2$.
Results: 7.5 lb, -15 lb.

(d) Letting $s_2 = 2$ ft lb^{-1} and $s_3 = -3$, and referring to Fig. 3(a), evaluate

$$(s_2\mathbf{F}_2) \cdot (s_3\mathbf{F}_3)$$

Result: $144\sqrt{3}$ ft lb.

(e) Referring to Fig. 3(a), determine the angle between $\mathbf{F}_3$ and $\mathbf{F}_1 + \mathbf{F}_2$, evaluate $\cos\,[\mathbf{F}_3, (\mathbf{F}_1 + \mathbf{F}_2)]$, then use the definition of the dot product to find $\mathbf{F}_3 \cdot (\mathbf{F}_1 + \mathbf{F}_2)$. Check the result by evaluating $\mathbf{F}_3 \cdot \mathbf{F}_1$ and $\mathbf{F}_3 \cdot \mathbf{F}_2$ and adding these. Is one of these two ways of evaluating $\mathbf{F}_3 \cdot (\mathbf{F}_1 + \mathbf{F}_2)$ "better" than the other?

(f) Referring to Problem 3(e), evaluate $\mathbf{F}_1 \cdot \mathbf{F}_2$, first taking $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ to be the unit vectors shown in Fig. 3(f), then letting $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ be mutually perpendicular unit vectors.
Results: -37 lb^2, -19 lb^2.

(g) Referring to Problem 4(f), show that

$$\mathbf{n}_{1i} \cdot \mathbf{n}_{4j} = A_{ij}$$

and use this fact to show that when the vectors $\mathbf{n}_{4j}$, $j = 1, 2, 3$, are expressed as

$$\mathbf{n}_{4j} = \sum_{k=1}^{3} B_{jk}\mathbf{n}_{1k}$$

the quantities B_{jk} are given by

$$B_{jk} = A_{kj}$$

(h) Referring to Fig. 4(a), let $\mathbf{p}$ be a vector joining B to G and having the sense BG. Without first finding the angle between $\mathbf{p}$ and $\mathbf{F}_1$, evaluate $\mathbf{p} \cdot \mathbf{F}_1$.
Result: -50 in. lb.

(i) A force $\mathbf{F}_1$ has a magnitude of 10 lb; a force $\mathbf{F}_2$, a magnitude of 8 lb. Evaluate

$$(\mathbf{F}_1 + \mathbf{F}_2) \cdot (\mathbf{F}_1 - \mathbf{F}_2)$$

Result: 36 lb^2.

(j) Repeat Problem 3(f), by evaluating $(\mathbf{F}^2)^{\frac{1}{2}}$.

(k) Referring to Problem 4(f), show that

$$\sum_{j=1}^{3} (A_{ij})^2 = 1$$

(l) Referring to Problem 4(f), determine the angle between $\mathbf{n}_{12}$ and $\mathbf{n}_{42}$, for $\phi = 60°$, $\theta = 30°$, $\psi = 60°$.

Result: 123°.

(m) Referring to Problem 5(h), determine the angle between $\mathbf{p}$ and $\mathbf{F}_1$.

Result: 112.6°.

(n) A unit vector $\mathbf{n}$, expressed in terms of mutually perpendicular unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, is given by

$$\mathbf{n} = \lambda \mathbf{n}_1 + \mu \mathbf{n}_2 + \nu \mathbf{n}_3$$

Determine the "direction cosines" of $\mathbf{n}$, i.e., the cosines of the angles between $\mathbf{n}$ and $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$.

(o) $\mathbf{n}$ and $\mathbf{n}'$ are unit vectors making angles α, β, γ and α', β', γ' with mutually perpendicular unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$. Show that

$$\cos(\mathbf{n}, \mathbf{n}') = \cos\alpha \cos\alpha' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma'$$

(p) A force $\mathbf{F}$ is parallel to the force $\mathbf{F}_1$ shown in Fig. 4(a). The CF resolute of $\mathbf{F}$ has a magnitude of 70 lb and the sense CF. Determine the magnitude and sense of $\mathbf{F}$.

Results: 650 lb, BH.

(q) The edges PP_1, PP_2, PP_3 of the tetrahedron shown in Fig. 5(q) are respectively parallel to the mutually perpendicular unit

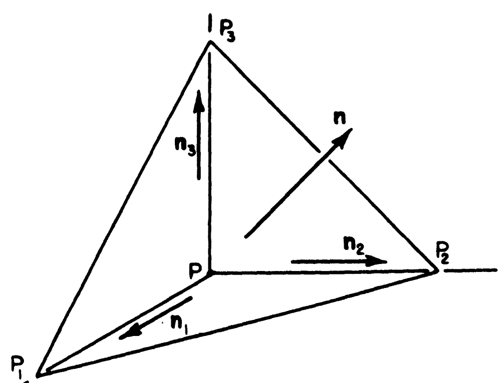


FIG. 5q

vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$. $\mathbf{n}$ is a unit vector, perpendicular to the face P_1, P_2, P_3 of the tetrahedron. Letting A be the area of this face, A_1 the area of the face which is perpendicular to $\mathbf{n}_1$, etc., show that

$$\frac{A_i}{A} = \mathbf{n}_i \cdot \mathbf{n}, \quad i = 1, 2, 3$$

PROBLEM SET 6

(See Sections 1.15–1.17 of the text)

(a) The vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$, shown in Fig. 6(a), have magnitudes of 4, 5, 3, and 5 feet, respectively. Draw sketches showing the 9 vectors

$$\mathbf{a} \times \mathbf{b},\ \mathbf{b} \times \mathbf{a},\ (-\mathbf{a}) \times \mathbf{b},\ -(\mathbf{a} \times \mathbf{b}),\ (2\mathbf{a}) \times (3\mathbf{b}),$$
$$6(\mathbf{a} \times \mathbf{b}),\ \mathbf{a} \times (\mathbf{b} + \mathbf{c}),\ \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{b},\ \mathbf{c} \times \mathbf{d}$$

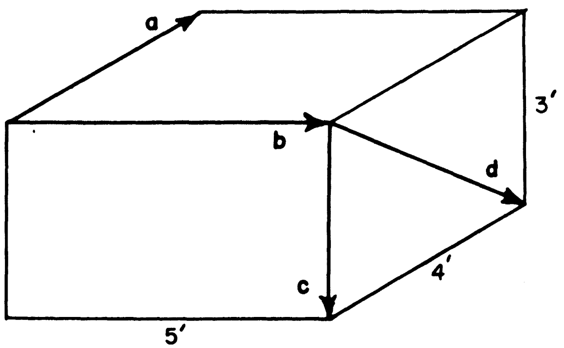


FIG. 6a

and determine the minimum number of vectors which must be drawn for this purpose.

Result: 5.

(b) The angle between the two forces described in Problem 5(i) is 30 degrees. Determine the magnitude of

$$(\mathbf{F}_1 + \mathbf{F}_2) \times (\mathbf{F}_1 - \mathbf{F}_2)$$

Result: 80 lb^2.

(c) Letting $\mathbf{F}'$ be the BG resolute of the force $\mathbf{F}_1$ shown in Fig. 4(a), explain why it is difficult to evaluate $\mathbf{F}_1 \times \mathbf{F}'$ "directly," i.e., without first resolving $\mathbf{F}_1$ and $\mathbf{F}'$ into mutually perpendicular components. Then find $\mathbf{F}_1 \times \mathbf{F}'$ and determine the magnitude and sense of the BC resolute of this vector.

Result: 192 lb^2, BC.

(d) Two unit vectors, $\mathbf{n}_1$ and $\mathbf{n}_2$, are respectively parallel to two lines, L_1 and L_2. Letting $\mathbf{n}$ be a unit vector perpendicular to both L_1 and L_2, show that

$$\mathbf{n} = \pm\frac{\mathbf{n}_1 \times \mathbf{n}_2}{|\mathbf{n}_1 \times \mathbf{n}_2|}$$

(e) Referring to Fig. 4(a), determine the (smallest) angle between line AB and a line which is perpendicular to both HB and ED.

Result: 68.2°.

(f) $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are mutually perpendicular unit vectors. Letting

$$\mathbf{a} = 2\mathbf{n}_1 - 3\mathbf{n}_2, \quad \mathbf{b} = \mathbf{n}_1 - 2\mathbf{n}_3, \quad \mathbf{c} = -\mathbf{n}_1 + 3\mathbf{n}_2 - 2\mathbf{n}_3$$

show that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$. Does this prove that $\mathbf{b}$ and $\mathbf{c}$ are equal to each other?

(g) Evaluate $[\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3]$ for (1) the unit vectors shown in Fig. 3(f), (2) a right-handed system of mutually perpendicular unit vectors, (3) a left-handed system of mutually perpendicular unit vectors.

Results: $\sqrt{3}/2$, 1, -1.

(h) Referring to Problem 3(e), use the simplest possible method to evaluate $[\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}]$ and $[\mathbf{F}_1, 3\mathbf{F}_1, 5\mathbf{F}_2]$. Check the results by evaluating the appropriate determinants, assuming that the unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular.

(i) A vector $\mathbf{v}$ is related to two vectors $\mathbf{a}$ and $\mathbf{b}$ as follows:

$$\mathbf{v} = \mathbf{a} \times [\mathbf{b} \times (\mathbf{a} \times \mathbf{b})]$$

Express the vector $\mathbf{b} \times [\mathbf{a} \times (\mathbf{b} \times \mathbf{a})]$ entirely in terms of $\mathbf{v}$.

(j) A vector $\mathbf{a}$ and the crossproduct of $\mathbf{a}$ with a vector $\mathbf{b}$ are given by

$$\mathbf{a} = \mathbf{n}_1 + \mathbf{n}_2, \quad \mathbf{a} \times \mathbf{b} = -2\mathbf{n}_3$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors. Express the vector

$$\mathbf{b} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}^2}\,\mathbf{a}$$

in terms of $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$.

Result: $\pm(\mathbf{n}_1 - \mathbf{n}_2)$.

(k) Letting $\mathbf{a}$ and $\mathbf{b}$ be any two vectors, determine the crossproduct of the resolutes of $\mathbf{a}$ and $\mathbf{b}$ perpendicular to a unit vector $\mathbf{n}$.

Result: $[\mathbf{n}, \mathbf{a}, \mathbf{b}]\mathbf{n}$.

PROBLEM SET 7

(See Sections 2.1–2.4 of the text)

(a) P, Q and R are the vertexes of a triangle; $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ the position vectors of these points relative to a point O (not necessarily lying in the plane of the triangle). Express the position vectors of P relative to Q, Q relative to R, and R relative to P in terms of $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$.

Result: $\mathbf{p} - \mathbf{q}$, $\mathbf{q} - \mathbf{r}$, $\mathbf{r} - \mathbf{p}$.

(b) Letting $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of two vertexes of a triangle, relative to the third vertex, show that the area of the triangle is equal to $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$, and use this fact to determine the area of a triangle whose vertexes have the rectangular cartesian coordinates $(-2, 3, -1)$, $(0, 0, -1)$, $(4, 0, 1)$, the coordinates being measured in feet.

Result: 7 ft^2.

(c) Two wires are attached to vertical posts, as shown in Fig. 7(c). Assuming that the wires are straight, determine the value of h for which the shortest distance between the wires is one foot.

Suggestion: Show first that the shortest distance between any two lines is equal to $|\mathbf{p}\cdot\mathbf{n}|$, where $\mathbf{p}$ is the position vector of any point on one of the lines relative to any point on the other line, and $\mathbf{n}$ is a unit vector perpendicular to both lines.

Result: 5.8 ft.

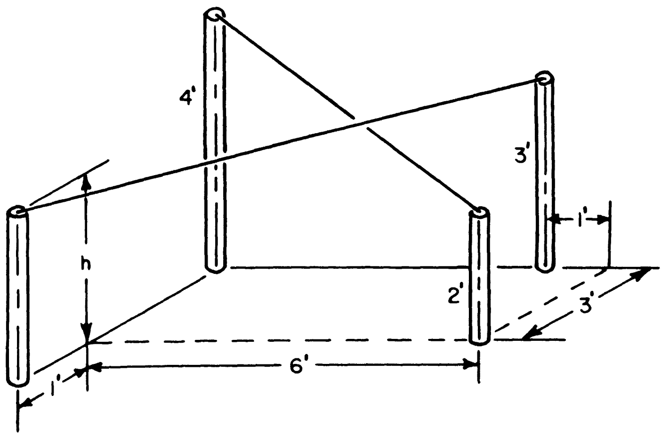


FIG. 7c

(d) In Fig. 7(d), $\mathbf{n}$ and $\mathbf{n}'$ are unit vectors respectively parallel to two lines L and L', $\mathbf{p}$ is the position vector of a point A' on L' relative to a point A on L, and $\boldsymbol{\nu}$ is a unit vector parallel to line BB', the shortest line joining L and L'. Express the distance between A and B in terms of $\mathbf{p}$, $\mathbf{n}$, $\mathbf{n}'$, and $\boldsymbol{\nu}$.

Suggestion: Write an equation which expresses the fact that the vector $\overrightarrow{AB} + \overrightarrow{BB'}$ is equal to the vector $\overrightarrow{AA'} + \overrightarrow{A'B'}$, then dot-multiply both sides with $\mathbf{n}' \times \boldsymbol{\nu}$.

Result: $[\mathbf{p}, \mathbf{n}', \boldsymbol{\nu}]/[\mathbf{n}, \mathbf{n}', \boldsymbol{\nu}]$.

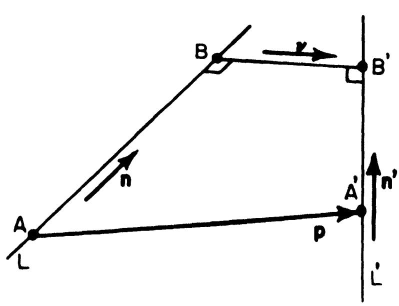


FIG. 7d

(e) If the points F, B and H in Fig. 4(a) have strengths of 2, 4, and -7 pounds, what is the distance from their centroid to point C?

Answer: 88.2 in.

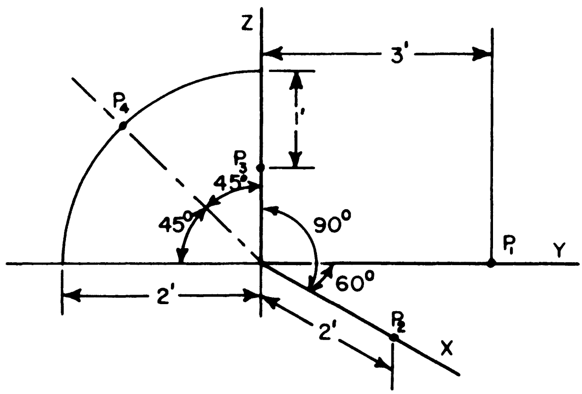


FIG. 7f

(f) The points P_1, P_2, P_3, and P_4, shown in Fig. 7(f), have strengths of 1, 2, 3, and 4 units. Noting that the X axis is not perpendicular to the Y axis, determine the coordinates x^*, y^* and z^* of the centroid of this set of points.

Result: $x^* = 0.4$ ft, $y^* = -0.266$ ft, $z^* = 0.866$ ft.

(g) The points $P_1, \ldots, P_8$, shown in Fig. 7(g), have the fol-

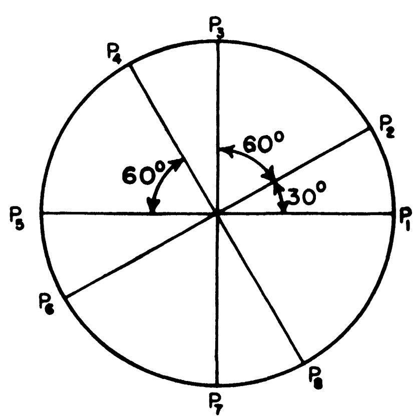


FIG. 7g

lowing strengths: 1, −2, 3, −4, 1, −2, 3, −4. Determine the first moment of this set of points with respect to point P_5, by (1) adding the first moments of the points of the set and (2) using symmetry considerations to locate the centroid and regarding the centroid as a "representative" point. Is one of these two methods "better" than the other?

PROBLEM SET 8

(See Section 2.5 of the text)

(a) Determine the distance between the centroid and one end of a straight line, by carrying out the limiting process described in the text. Check the result, by integration.

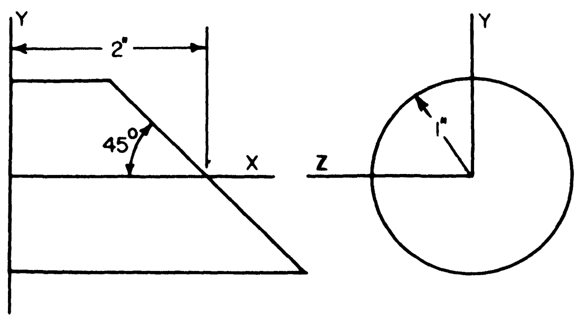


FIG. 8b

(b) Determine, by integration, the coordinates of the centroid of the figure shown (in two views) in Fig. 8(b), regarding the figure as (i) a surface possessing no plane portions, and (ii) a solid.

Results:

	x^*	y^*	z^*
(i)	$\frac{9}{8}$ in.	$-\frac{1}{4}$ in.	0
(ii)	$\frac{17}{16}$ in.	$-\frac{1}{8}$ in.	0

(c) Determine, without integration, the coordinates of the centroid of the figure shown in Fig. 8(c), regarding the figure as

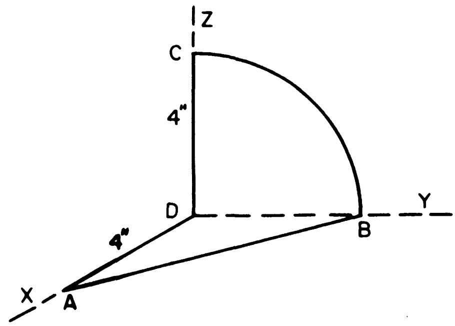


FIG. 8c

(i) the curve $ABCDA$, and (ii) a surface consisting of a triangle and a quarter of a circle.

Results:

	x^*	y^*	z^*
(i)	0.970 in.	1.37 in.	1.20 in.
(ii)	0.517 in.	1.56 in.	1.04 in.

(d) Determine, without integration, the X-coordinate of the centroid of the figure shown (in two views) in Fig. 8(d), regarding the figure as (i) a surface possessing no plane portions, and (ii) a solid.

Results:

$$\frac{l}{3}\,\frac{R+2r}{R+r},\quad \frac{l}{4}\,\frac{R^2+2rR+3r^2}{R^2+rR+r^2}$$

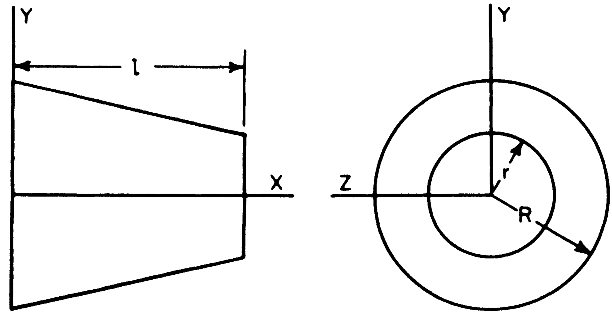


FIG. 8d

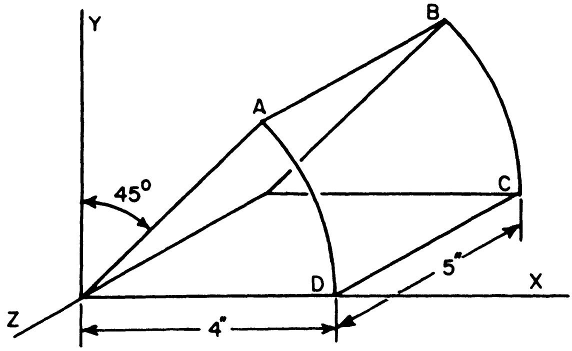


FIG. 8e

(e) Locate, without integration, the centroid of the figure shown in Fig. 8(e), regarding the figure as (i) a surface consisting of two rectangles and the surface $ABCD$, and (ii) a solid.

Results:

	x^*	y^*	z^*
(i)	2.24 in.	0.927 in.	−2.5 in.
(ii)	2.39 in.	0.988 in.	−2.5 in.

(f) A plane curve C of length L is revolved about a line lying in the plane of C and not intersecting C. Show that the area of the surface of revolution thus generated is equal to the product of L and the circumference of the circle described by the centroid of C, and use this theorem, together with the fact that the surface area of a sphere of radius R is equal to $4\pi R^2$, to locate the centroid of a semicircular curve.

(g) A plane region R of area A is revolved about a line lying in the plane of R and not intersecting R. Show that the volume of the solid of revolution thus generated is equal to the product of A and the circumference of the circle described by the centroid of R, and use this theorem, together with the fact that the volume of a sphere of radius R is equal to $4\pi R^3/3$, to locate the centroid of a semicircular sector.

(The two theorems stated in Problem 8(f) and Problem 8(g) are sometimes called "Theorems of Pappus" or "Guldin's Rules.")

PROBLEM SET 9

(See Sections 2.6–2.7 of the text)

(a) The point at which the centroid of a set of points is located when all of the points of the set are assigned equal strengths is called the "center of mean position" of the set. Referring to Fig. 4(a), determine the distance between the center of mean position of the points F, B, H and the mass center of three particles of masses 10 gm, 2 gm, 3 gm placed at these points.

Result: 1.82 ft.

(b) Particle weighing 0.001, 0.002, 0.003, and 0.004 pounds are placed at the points P_1, P_2, P_3, and P_4 shown in Fig. 7(f). Determine the coordinates of the mass center of this set of particles, and compare the results with those of Problem 7(f).

(c) The mass per unit of length at a point P of a straight wire is a linear function of the distance from P to the end of the wire. The distance from one end of the wire to the mass center is equal

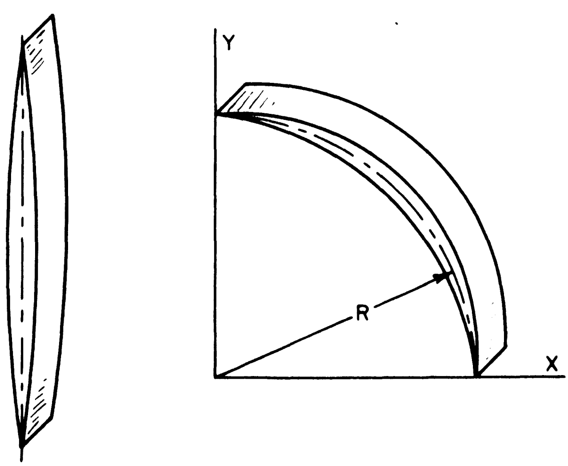


Fig. 9d

to $\frac{5}{12}$ of the length of the wire. Determine the ratio of the mass densities at the two ends of the wire.

Result: 3:1.

(d) A thin metal strip of non-uniform thickness is bent into the form of a quarter circle, as shown in Fig. 9(d). Determine the X-coordinate of the mass center approximately, (1) assuming that the thickness varies sinusoidally, and (2) regarding the thickness as uniform.

Results: $2R/3$, $2R/\pi$.

(e) A wire, ABC, is attached to a triangular piece of sheet metal, ACD, as shown in Fig. 9(e). The total weight of the wire is one tenth that of the sheet metal. Determine the coordinates of the mass center of this assembly.

Results: 1.21 in., 1.31 in., 0.212 in.

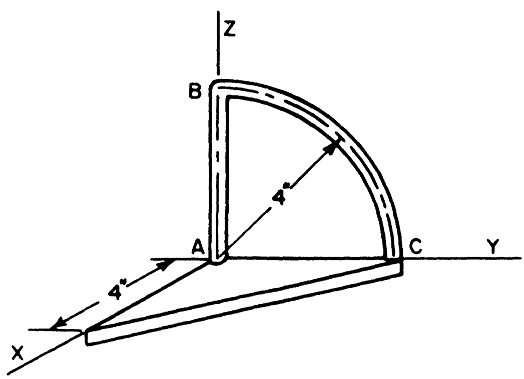


FIG. 9e

(f) A steel (489 lb ft^{-3}) wedge has a core in the form of a right-circular cylinder, as shown in Fig. 9(f). Determine the distance between the mass centers of this body when the core is made of brass (527 lb ft^{-3}), on the one hand, and aluminum (169 lb ft^{-3}), on the other.

Result: 0.052 in.

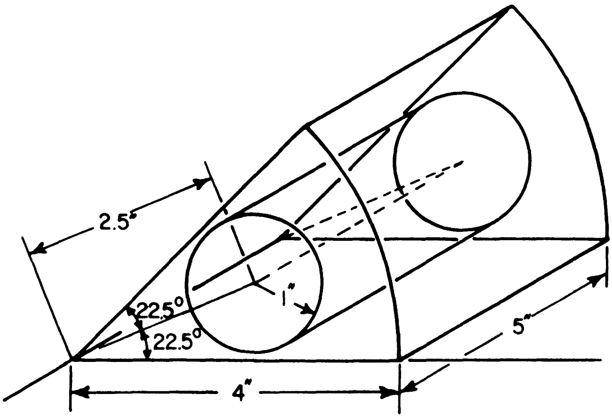


Fig. 9f

(g) Figure 9(g) represents a thin-walled, open, cylindrical vessel, filled to a height h with a fluid having a uniform weight density of 60 lb ft^{-3}. The walls and base of the vessel are made of a uniform material weighing 5 lb per square foot. Determine the value of h for which the mass center of the entire system (vessel plus fluid) is at the lowest possible point.

Result: 6 in.

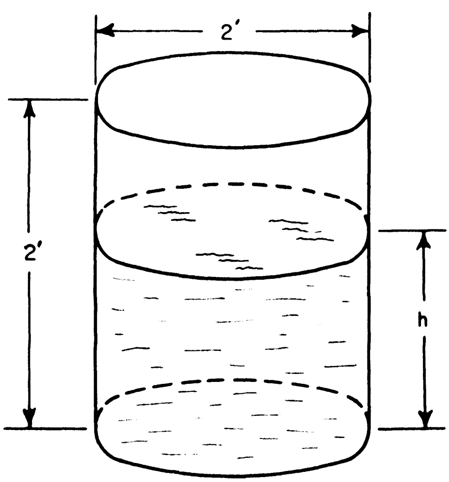


Fig. 9g

PROBLEM SET 10

(See Sections 3.1–3.2 of the text)

Problems 10 (a)–(g) deal with moments of the two forces $\mathbf{F}$ and $\mathbf{G}$ shown in Fig. 10(a). The lines L_1, L_2 and L_3 are mutually perpendicular, and the unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are parallel to lines L_1, L_2 and L_3, respectively.

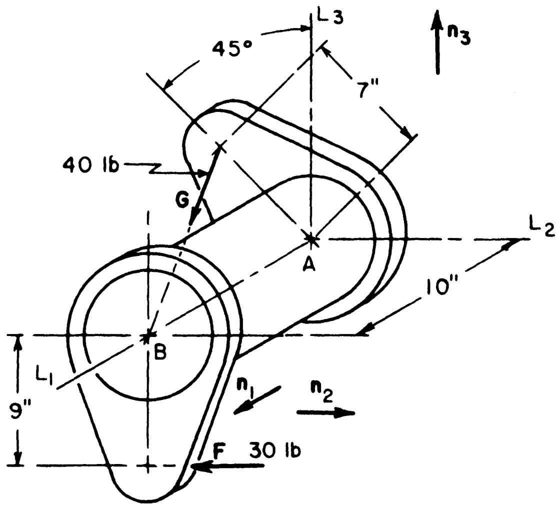


FIG. 10a

(a) Determine the moments of $\mathbf{F}$ and $\mathbf{G}$ about point A and line L_3, by using the definitions of the moment of a vector about a point and about a line.

Results:

$$\mathbf{M}^{F/A} = -270\mathbf{n}_1 - 300\mathbf{n}_3 \text{ in. lb}$$

$$\mathbf{M}^{G/A} = 162(\mathbf{n}_2 + \mathbf{n}_3) \text{ in. lb}$$

$$\mathbf{M}^{F/L_3} = -300\mathbf{n}_3 \text{ in. lb}$$

$$\mathbf{M}^{G/L_3} = 162\mathbf{n}_3 \text{ in. lb}$$

(b) Use the results of Problem 10(a) to determine the magnitudes of the moments of $\mathbf{F}$ and $\mathbf{G}$ about line L_3.

Results: 300 in. lb, 162 in. lb.

(c) Determine the magnitude of the moment of $\mathbf{F}$ about L_3, by evaluating the product of $|\mathbf{F}|$ and the distance s between L_3 and the line of action of $\mathbf{F}$. Compare this method for finding $|\mathbf{M}^{F/L_3}|$ with that used in Problem 10(*b*), and explain why this method is not suitable for the evaluation of $|\mathbf{M}^{G/L_3}|$.

(d) Determine the moments of $\mathbf{F}$ about line L_2 and of $\mathbf{G}$ about line L_1, by inspection.

(e) Find $\mathbf{M}^{F/A}$, by regarding it as the sum of the moments of $\mathbf{F}$ about L_1, L_2 and L_3 and evaluating these moments by inspection. Compare this method for finding $\mathbf{M}^{F/A}$ with that used in Problem 10(a), and explain why it is not a useful method for the evaluation of $\mathbf{M}^{G/A}$.

(f) Resolving $\mathbf{G}$ into 3 mutually perpendicular components whose lines of action intersect at point B, use these components to evaluate the moments of $\mathbf{G}$ about point A and line L_3, and compare this method with that used in Problem 10(a).

(g) Show that there exists no force whose line of action passes through point B and whose moment about point A is equal to the sum of the moments of $\mathbf{F}$ and $\mathbf{G}$ about A.

(h) Letting $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ be a left-handed set of unit vectors respectively parallel to the axes of a rectangular cartesian coordinate system, determine the $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ measure numbers of the moment of a force $\mathbf{F}$ about the origin, the $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ measure numbers of $\mathbf{F}$ being F_1, F_2, F_3, and $\mathbf{F}$ being applied at the point whose coordinates are x_1, x_2, x_3.

Result: $x_3F_2 - x_2F_3,\ x_1F_3 - x_3F_1,\ x_2F_1 - x_1F_2$

(i) A force has a magnitude of 21.3 lb. The moment of the force about a point P has a magnitude of 270 in. lb. Determine the shortest distance from P to the line of action of the force.

Result: 12.7 in.

PROBLEM SET 11

(See Sections 3.3–3.4 of the text)

Problems 11(a)–(h) deal with moments of the force system S consisting of the two forces $\mathbf{F}$ and $\mathbf{G}$ shown in Fig. 10(a).

(a) Using the definitions of the moment of a system of vectors about a point and about a line, determine the moments of S about point A and line L_3.

Results: $-270\mathbf{n}_1 + 162\mathbf{n}_2 - 138\mathbf{n}_3$ in. lb; $-138\mathbf{n}_3$ in. lb.

(b) Determine the resultant of S, and use it, together with $\mathbf{M}^{S/A}$ as found in Problem 11(a), to evaluate the moment of S about point B. Check the result, by evaluating the sum of $\mathbf{M}^{\mathbf{F}/B}$ and $\mathbf{M}^{\mathbf{G}/B}$.

(c) If L_A and L_B are lines passing through A and B, respectively, and the moments of S about L_A and L_B are equal to each other, are L_A and L_B necessarily parallel to (1) the resultant of S and (2) each other?

Answers: (1) No, (2) Yes.

(d) Determine the magnitude of the minimum moment of S.

Result: 226 in. lb.

(e) Determine the distance from B to the central axis of S.

Result: 3.77 in.

(f) Explain why S is not a couple, and, letting $\mathbf{H}$ be a force such that the system of three forces $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$ is a couple and the line of action of $\mathbf{H}$ passes through point B, determine the magnitude of the torque $\mathbf{T}$ of this couple, and find the angle between $\mathbf{T}$ and $\mathbf{H}$.

Results: 270 in. lb, 33°.

(g) The line of action of one of the forces of a simple couple coincides with line L_3 in Fig. 10(a); that of the other passes through point B. The torque of the couple has a magnitude of 50 in. lb and the same direction as $\mathbf{n}_2$. Determine the magnitude and direction of the force whose line of action passes through B.

Results: 5 lb, $-\mathbf{n}_3$

(h) A force system S' consists of S and a couple whose moments about lines L_1 and L_3 are equal to zero. The moment of S' about line AC is equal to zero. Determine the moment of S' about line L_2.

Result: $-138\mathbf{n}_2$ in. lb.

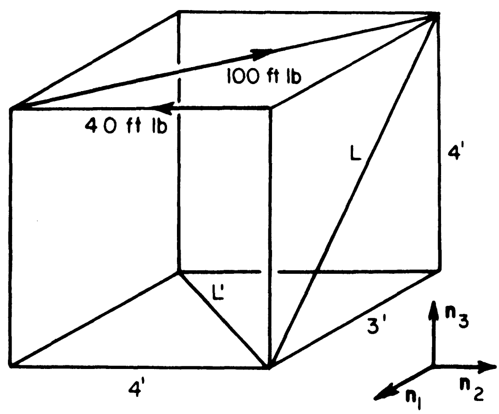


FIG. 11i

(i) A force system S consists of two couples whose torques are shown in Fig. 11(i). Determine the magnitudes of the moments of S about lines L and L'.

Result: 36 ft lb, 4 ft lb.

PROBLEM SET 12

(See Section 3.5 of the text)

(a) Replacing the system of two forces shown in Fig. 10(a) with (1) a couple of torque $\mathbf{T}$ and a force whose line of action passes through point A, and (2) a couple of torque $\mathbf{T}'$ and a force whose line of action passes through point B, determine the angle between $\mathbf{T}$ and $\mathbf{T}'$.

Result: 39.2°.

(b) The line of action of every force of a system S of forces applied to the surface $ABCD$ of the body shown in Fig. 8(e) is normal to that surface. The sum of the resolutes of the forces of S parallel to the X-axis has a magnitude of 80 lb and is directed to the left; the sum of the resolutes parallel to the Y-axis has a magnitude of 60 lb and is directed upward. The resolutes of the moment of S about the origin, parallel to the X- and Y-axes, have magnitudes of 60 and 80 in lb and are directed to the right and downward, respectively. If S is replaced with a single force **F** whose line of action passes through D, and a couple of torque **T**, what are the magnitudes of **F** and **T**?

Answers: 100 lb, 260 in. lb.

(c) Replacing the system of two forces shown in Fig. 10(a), with a single force and a couple, the line of action of the force being chosen in such a way as to make the torque of the couple as small as possible, determine (1) the (shortest) distance from point A to the line of action of the force, and (2) the magnitude of the torque of the couple.

Results: 6.57 in., 226 in. lb.

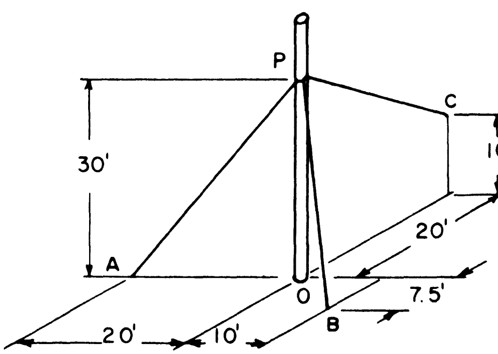


FIG. 12d

(d) The system S of forces exerted on a vertical mast by guy wires PA, PB and PC (see Fig. 12(d)) is equivalent to a force whose line of action passes through P. This force has a magnitude Q and is directed vertically downward. S is also equivalent to a system of three forces whose lines of action pass through P and are parallel to PA, PB and PC. These forces have magnitudes of R, S and 1000 pounds, respectively. Determine Q, R and S.

Results: 4,950 lb; 1,700 lb; 3,070 lb.

(e) Let $\mathbf{F}'$ and $\mathbf{G}'$ be a system S' of two forces, the line of action of $\mathbf{F}'$ being the line L_1 shown in Fig. 10(a). If S' is equivalent to the system of two forces $\mathbf{F}$ and $\mathbf{G}$, what are the magnitudes of $\mathbf{F}'$ and $\mathbf{G}'$, and what is the distance from B to the line of action of $\mathbf{G}'$?

Suggestion: Verify that when a system S of vectors can be replaced with two vectors, $\mathbf{v}$ and $\mathbf{v}'$, of which the line of action, L, of one, say $\mathbf{v}$, is chosen arbitrarily, then

$$\mathbf{v} = \frac{\mathbf{R}\cdot\mathbf{M}}{\mathbf{n}\cdot\mathbf{M}}\,\mathbf{n}, \quad \mathbf{v}' = \mathbf{R} - \mathbf{v}$$

and

$$\mathbf{p} = \frac{\mathbf{v}' \times \mathbf{M}}{(\mathbf{v}')^2}$$

where $\mathbf{R}$ is the resultant of S, $\mathbf{M}$ is the moment of S about any point O on L, $\mathbf{n}$ is a unit vector parallel to L, and $\mathbf{p}$ is the position vector of a point on the line of action of $\mathbf{v}'$, relative to O.

(If $\mathbf{M}$ is perpendicular to $\mathbf{n}$, but not to $\mathbf{R}$, S cannot be replaced with two vectors, one of which has L for its line of action.)

Results: 32.8 lb, 21.3 lb, 12.7 in.

(f) The system S of forces exerted on a pulley by two belts is equivalent to the four coplanar forces shown in Fig. 12(f).

(1) With

$$P_1 = 100 \text{ lb}, \quad P_2 = 90 \text{ lb}, \quad Q_1 = 80 \text{ lb}, \quad Q_2 = 70 \text{ lb}$$

determine the shortest distance from A to the line of action of a single force which is equivalent to S.

(2) With

$$P_1 = 100 \text{ lb}, \quad P_2 = 90 \text{ lb}, \quad Q_1 = 80 \text{ lb}$$

determine the value of Q_2 for which S can be replaced by a single force whose line of action passes through A.

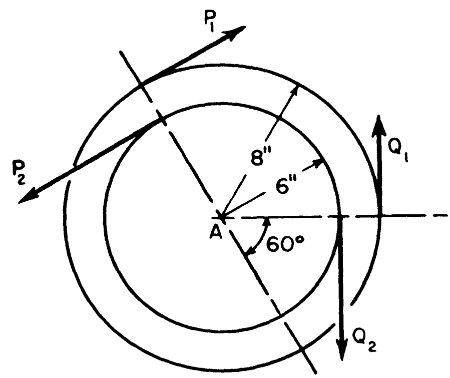


FIG. 12f

(3) With

$$P_1 = 100 \text{ lb}, \quad Q_1 = 80 \text{ lb}$$

determine the values of P_2 and Q_2 for which S can be replaced with a couple, and find the magnitude of the torque of this couple.

Results: (1) 2.31 in., (2) 63.3 lb; (3) 100 lb, 80 lb, 40 in. lb.

(g) Figure 12(g) represents a system of forces equivalent to some of the vertical forces acting on a portion of a horizontal beam. Draw a sketch showing the single force with which this system of five forces can be replaced.

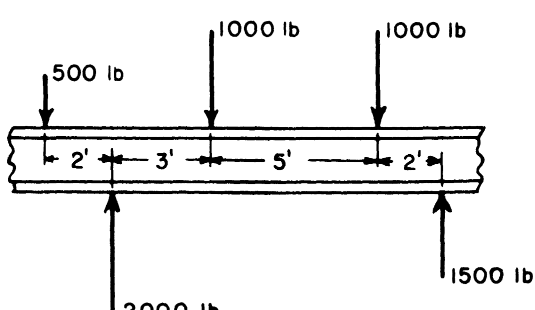


FIG. 12g

PROBLEM SET 13

(See Section 3.6 of the text)

(a) The system S of four forces shown in Fig. 13(a) is equivalent to the system of forces exerted on a ring by four cables. S is

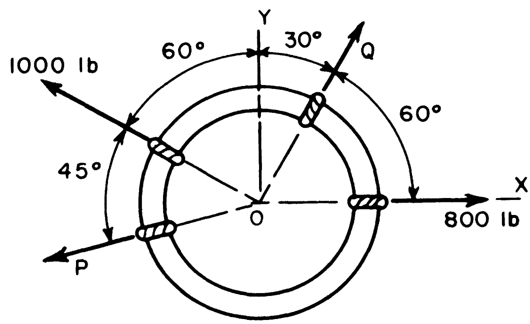


FIG. 13a

a zero system. Determine the values of P and Q, (1) by setting the sums of the resolutes of all forces parallel to the X- and Y-axes equal to zero; (2) by setting the sums of the resolutes of all forces parallel to the line of action of the 1000 pound force, and parallel to the Y-axis, equal to zero; (3) by setting the sum of the resolutes of all forces parallel to the line of action of the 1000 pound force, and the moment of S about some point (other than O) on the line of action of P, equal to zero; (4) graphically, by constructing the resultant of S, to scale.

Explain the "physical" significance of the signs of P and Q as found above.

Why is method (1) the "worst" of the four?

(b) Seven feet above a horizontal floor, a microphone is suspended from three wires, as shown in Fig. 13(b). Assuming that the system S of all forces acting on the microphone consists of the four forces shown in Fig. 13(b), that the wires are straight, and

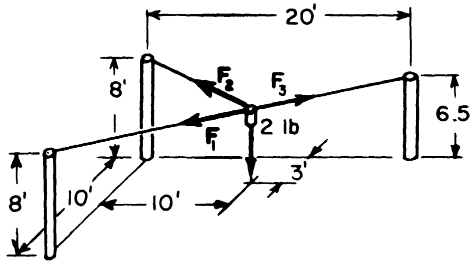


FIG. 13b

that S is a zero system, determine the magnitude of $\mathbf{F}_3$, by writing (and solving) a *single* (scalar) equation.

Result: 41.9 lb.

(c) A flexible shaft S is supported by a rigid housing H, as shown in Fig. 13(c). ($\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors.) This assembly is subjected to the action of a force system S consisting of the following: A force $\mathbf{A}$, whose line of action passes through point A and is perpendicular to $\mathbf{n}_1$; a force $\mathbf{B}$, whose line of action passes through point B and is perpendicular to $\mathbf{n}_3$; a couple, whose torque $\mathbf{T}$ is perpendicular to $\mathbf{n}_1$; two couples,

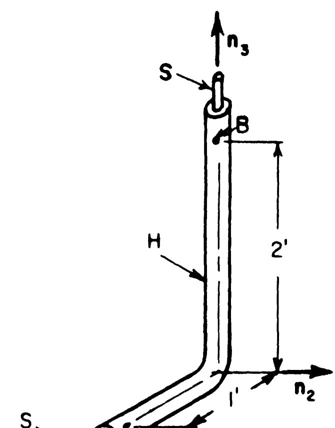


FIG. 13c

whose torques are equal to $-100\mathbf{n}_1$ ft lb and $-100\mathbf{n}_3$ ft lb, respectively. S is a zero system. Determine the magnitudes of $\mathbf{A}$ and $\mathbf{T}$.

Results: 50 lb, 50 ft lb.

(d) A system S of forces acting on a portion of a truss is equivalent to the six forces shown in Fig. 13(d). (The letter "K"

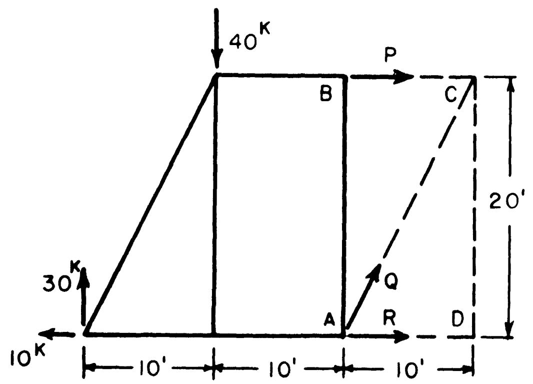


FIG. 13d

denotes a unit of force, called the "kip" or, more fully, the kilopound; i.e., 1K = 1000 lb.) Assuming that S is a zero system, determine P, Q, and R. Do this (1) by setting the sum of the moments of all forces about point A, the sum of the AB resolutes of all forces, and the sum of the moments of all forces about point C equal to zero, and (2) by setting the sum of the AD resolutes of all forces, the sum of the moments of all forces about point B, and the sum of the CD resolutes of all forces equal to zero.

Which of the two methods is "better"? Can the problem be solved without summing resolutes parallel to any line, i.e., by using only moment equations?

(e) A system S of forces acting on a cantilever beam consists of the three forces shown in Fig. 13(e), a force $\mathbf{F}$ whose line of action passes through point P, and a couple of torque $\mathbf{T}$. If S is a zero system, what are the magnitudes of $\mathbf{F}$ and $\mathbf{T}$, and what is the angle between $\mathbf{F}$ and $\mathbf{T}$?

Results: 4.58 K, 20 ft K, 90°.

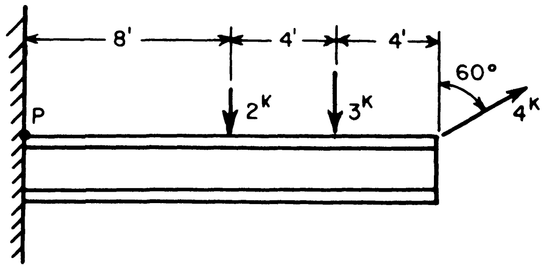


FIG. 13e

(f) A force system S consist of two couples whose torques are shown in Fig. 11(i). Letting S_1 be the orthogonal projection of S on a plane which is perpendicular to $\mathbf{n}_1$, and S_2 the orthogonal projection of S on a plane which is perpendicular to $\mathbf{n}_2$, determine the magnitudes of the moments of S_1 and S_2 about (1) line L and (2) line L', and compare the results with those of Problem 11(i).

Results: 36 ft lb, 0; 36 ft lb, 32 ft lb.

PROBLEM SET 14

(See Sections 4.1–4.4 of the text)

(a) The masses of particles situated at the points C, F, B, H shown in Fig. 4(a) are 10^{-3}, 2×10^{-3}, 3×10^{-3}, 4×10^{-3} slug. Determine the magnitude of the resultant of the gravitational forces exerted on the particle at C by the particle at F, and on the particle at B by the particle at H.

Result: 5.54×10^{-13} lb.

(b) Two particles, P_1 and P_2, are separated by a distance of 3 ft. A third particle, P, lies on the line joining P_1 and P_2, 1 ft from P_1, 2 ft from P_2. The masses of P_1 and P_2 are m and $2m$, respectively. Letting $\mathbf{F}$ be the gravitational force exerted on P by

P_1 and P_2, and $\mathbf{F}^*$ the gravitational force exerted on P by a particle of mass $3m$ placed at the mass center of P_1 and P_2, determine (1) the angle between $\mathbf{F}^*$ and $\mathbf{F}$, and (2) the ratio of the magnitudes of $\mathbf{F}^*$ and $\mathbf{F}$.

Results: 180°, 6.

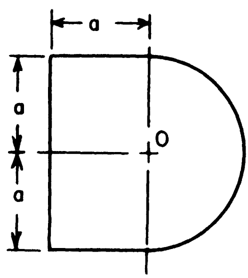
In view of these results, is $\mathbf{F}^*$ a satisfactory approximation to $\mathbf{F}$?

(c) Four particles of masses 2, 4, 6 and 8 grams are placed at the points P_1, P_2, P_3, and P_4 shown in Fig. 7(f). Determine, approximately, the magnitude of the gravitational force exerted on this set of particles by a particle of mass 1 gram, placed on the Z-axis, 5 ft above the X-Y plane.

Result: 83×10^{-12} dyne.

(d) A uniform wire of mass m is bent into the shape shown in Fig. 14(d), and a particle P is placed at point O. Letting $\mathbf{F}$ be the

FIG. 14d



gravitational force exerted on P by the wire, and $\mathbf{F}^*$ the gravitational force exerted on P by a particle of mass m placed at the mass center of the wire, determine the ratio of the magnitudes of $\mathbf{F}$ and $\mathbf{F}^*$.

(e) Particles of masses M and M' are placed on the axis of a thin, uniform, hemispherical shell, as shown in Fig. 14(e). Determine the value of M/M' for which the gravitational forces exerted on the shell by the particles are equal in magnitude.

Result: $\frac{1}{4}$.

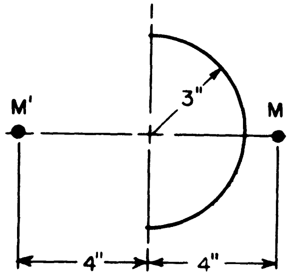


FIG. 14e

(f) Show that the gravitational force exerted by a thick, uniform, spherical shell on a particle placed inside the shell is equal to zero.

(g) In Fig. 14(g), lines AB and $A'B'$ represent identical, uniform wires; $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors; and the distance between A and A' is equal to the length of either wire.

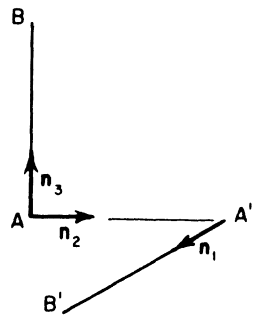


FIG. 14g

Replacing the system of gravitational forces exerted on $A'B'$ by AB, with a couple of torque $\mathbf{T}$ and a force $\mathbf{F}$ whose line of action passes through A, determine the ratio of the $\mathbf{n}_2$ and $\mathbf{n}_1$ measure numbers of (1) $\mathbf{F}$, and (2) $\mathbf{T}$.

Results: $-\frac{\pi}{3} \Big/ \log\left[\frac{(1+\sqrt{3})(1-\sqrt{2})}{(1-\sqrt{3})(1+\sqrt{2})}\right]$,

$$(2\sqrt{2} - 1 - \sqrt{3}) \Big/ \log\left(\frac{1+\sqrt{3}}{2+\sqrt{2}}\right).$$

(h) The earth satellite launched in Russia on October 4, 1957 was reported to have a diameter of twenty-two inches and a weight of 184 pounds. Determine the magnitude of the force exerted on the earth by the satellite at an altitude of 560 miles.

Result: 141 lb.

PROBLEM SET 15

(See Sections 4.5–4.9 of the text)

(a) A brass (527 lb ft^{-3}) sphere, radius 4 inches, rests on a horizontal table. Imagining the sphere as divided into two parts by a plane which passes through the sphere's center C and is inclined at an angle θ to the horizontal, reduce the system of forces exerted by the upper part on the lower part, to a couple of torque $\mathbf{T}$ and a force whose line of action passes through C. Determine the magnitude of $\mathbf{T}$ for $\theta = 0$, 45 and 89.99 degrees.

Results: 0, 43.3 in. lb, 61.2 in. lb.

(b) Figure 12(d) represents a vertical mast which is supported by three guy wires and by a ball-and-socket connection at O. Show that the reaction at O can be reduced to a vertical force, and determine the maximum magnitude of this force, if none of the cable tensions exceeds 5000 lb.

Result: 8,060 lb.

(c) A flexible shaft S (see Fig. 15(c)) is supported by a rigid housing H which is held in place by a fixed sleeve A and a fixed ring B. Each end of the shaft is subjected to the action of a couple exerted on that end by a contiguous portion of the shaft (not shown), these couples having torques of $-100\mathbf{n}_1$ and $-100\mathbf{n}_3$ ft lb. Assuming that the surface of the housing is smooth, and neglecting gravitational forces, determine the reactions of A and B on H.

Results: A force of $50\mathbf{n}_2$ lb, line of action passing through the point A shown in Fig. 13(c), and a couple having a torque of $50\mathbf{n}_3$ ft lb; a force of $-50\mathbf{n}_2$ lb, line of action passing through the point B shown in Fig. 13(c).

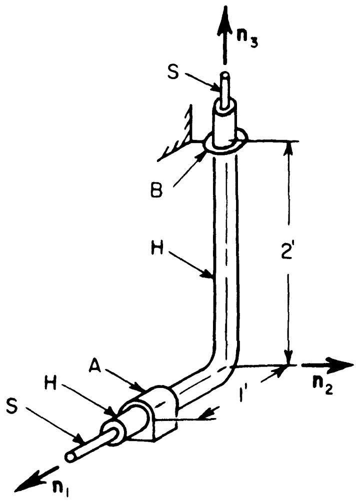


FIG. 15c

(d) Three flexible cables are attached to a cantilever beam, as shown in Fig. 15(d). The tensions in cables A, B, and C are equal to 2, 3 and 4 kips, respectively. Neglecting gravitational forces,

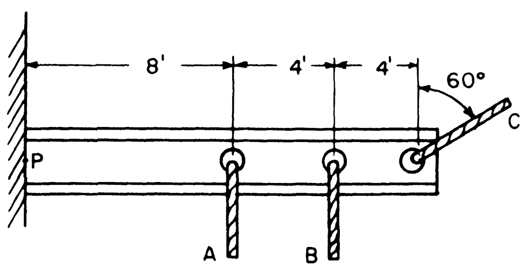


FIG. 15d

reduce the reaction of the beam on the wall to a couple of torque **T** and a force **F** whose line of action passes through point P, and determine the magnitudes of **F** and **T**.

Results: 4.8 kip, 20 ft kip.

(e) Three flexible cables are attached to a cantilever beam, as shown in Fig. 15(d). The tensions in cables A and B are equal to 2 and 3 kips, respectively. Cable C is attached to a fixed support. Explain why this information is insufficient for the determination of the reaction of the beam on the wall, then reduce this reaction to a couple of torque $\mathbf{T}$ and a force $\mathbf{F}$ whose line of action passes through P, and determine the magnitudes of $\mathbf{F}$ and $\mathbf{T}$, assuming that gravitational forces can be neglected and that the magnitude of $\mathbf{F}$ is equal to the tension in cable C.

Results: 5 kip, 12 ft kip.

(f) A steel (489 lb ft^{-3}) plate, $5' \times 5' \times \frac{1}{2}''$, is suspended from two vertical cables, each five feet long, as shown in Fig. 15(f′). Next, two cables are attached at A and B, and these are used to

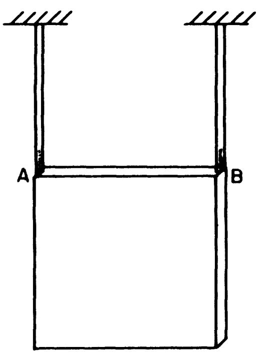


FIG. 15f′

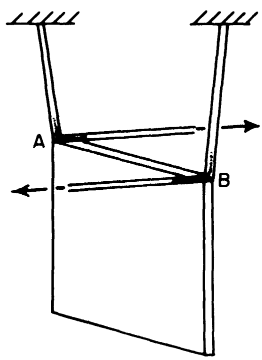


FIG. 15f″

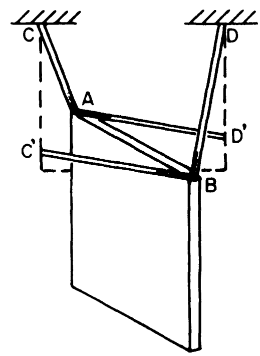


FIG. 15f‴

exert forces which cause the plate to rise through a distance of one foot while rotating about a vertical line passing through the plate's center (see Fig. 15(f″). These cables are then attached at C' and D', four feet below C and D, respectively (see Fig. 15(f‴)). Determine the tensions in cables AC and BC'.

Results: 319 lb, 255 lb.

(g) One end of a uniform, twenty-one foot long boom is supported in a spherical socket. When the boom is not in use, its upper end is attached to a five foot cable and rests on a vertical

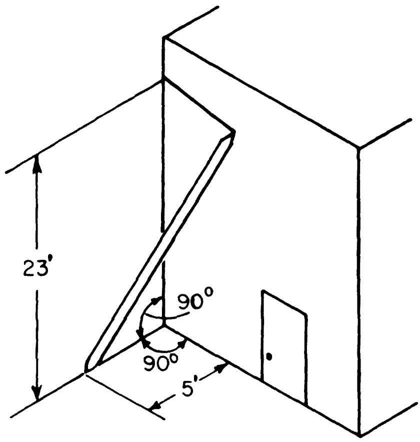


FIG. 15g

wall, as shown in Fig. 15(g). The boom weighs 920 pounds. Determine the tension in the cable, assuming that the wall is smooth.

Result: 100 lb.

(h) Fig. 15(h) illustrates an element of a seismic device consisting of a uniform, 15 pound sphere which is free to slide on a vertical shaft mounted in a box, and is attached to the sides of the

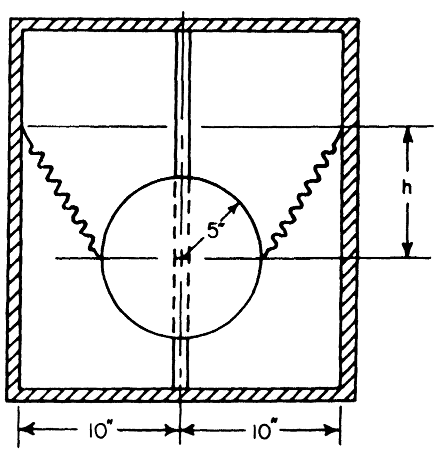


FIG. 15h

box by two springs, each of which has a natural length of 5 inches. The springs have moduli of two and three pounds per inch, respectively. Determine h.

Result: $h = 7$ in.

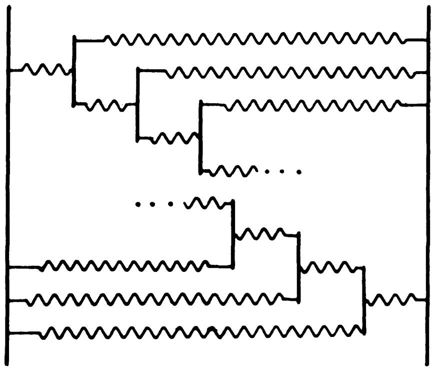


FIG. 15i

(i) Two bodies are connected by a very large number of springs, all of which have the same modulus, k. If the springs are arranged as shown in Fig. 15(i), what is the modulus of the single spring equivalent to this system of springs?

Answer: $(\sqrt{5} - 1)k$.

(j) One corner of a smooth, uniform plate rests on a post. The plate weighs 120 lb and is held in a horizontal position by a rod built into a wall, as shown in Fig. 15(j). Show that the reaction of

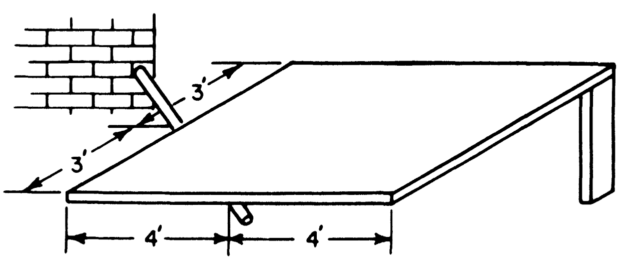


FIG. 15j

the rod on the wall can be reduced to a single vertical force, and, neglecting the weight of the rod, determine (1) the magnitude of this force and (2) the distance from the line of action of the force to that corner of the plate which rests on the post.

Results: 80 lb, 7.5 ft.

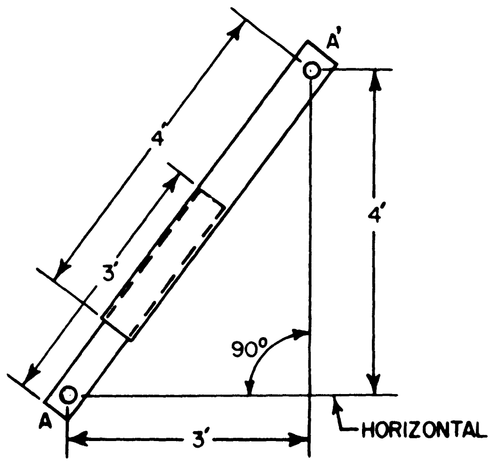


Fig. 15k

(k) Figure 15(k) represents two smooth, overlapping tubes, pinned to fixed supports at A and A'. The tubes have equal weights per unit of length. Reducing the reaction of the shorter tube on the support at A to a force $\mathbf{F}$ and a couple, and that of the longer tube on the support at A' to a force $\mathbf{F}'$ and couple, determine the ratio of the magnitudes of the vertical resolutes of $\mathbf{F}$ and $\mathbf{F}'$.

Result: 813/937.

(l) In Fig. 15(l), line AB represents the axis of a smooth hinge connecting two uniform, square plates, each of which weighs W pounds. The corners C and D are joined by a string, and this assembly rests on a smooth, horizontal plane. Determine the tension in the string.

Result: $(3/8)^{\frac{1}{2}}W$.

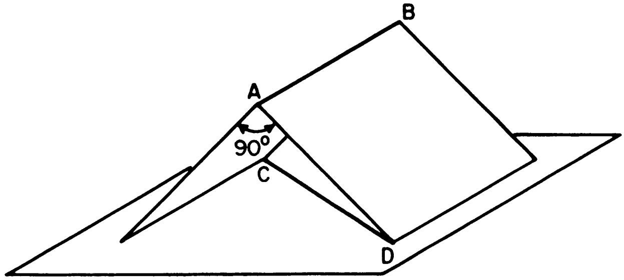


FIG. 15l

(m) In Fig. 15(m), S and S' represent shafts mounted in bearings B and B' and connected to each other with a shaft S'' which is attached to both S and S' by means of Hooke's joints. $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors. Lines XX and $X'X'$ are

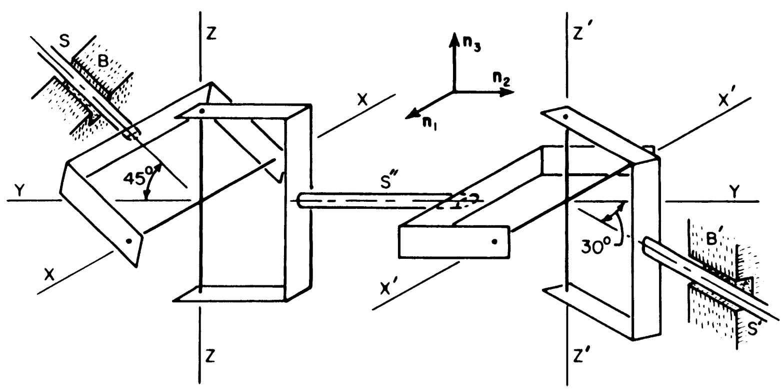


FIG. 15m

parallel to $\mathbf{n}_1$, YY is parallel to $\mathbf{n}_2$, ZZ and $Z'Z'$ are parallel to $\mathbf{n}_3$. S lies in the plane determined by YY and ZZ, S' in that determined by $X'X'$ and YY.

Two couples, whose torques are respectively parallel to S and S' and have magnitudes T and T', are applied to S and S'. Determine the ratio of T to T'.

Result: $\sqrt{2/3}$.

(n) Figure 15(n) shows two shafts, S and S' mounted in bearings, B and B', and connected to each other with an Oldham coupling, constructed as follows: Circular discs, D and D', are rigidly attached to S and S', respectively. Each disc has a groove in which one of two tongues on a third disc, C; can slide freely. These tongues are at right angles to each other.

A couple, whose torque is parallel to the axes of the shafts, is applied to each shaft. Show that the ratio of the magnitudes of these torques is equal to unity.

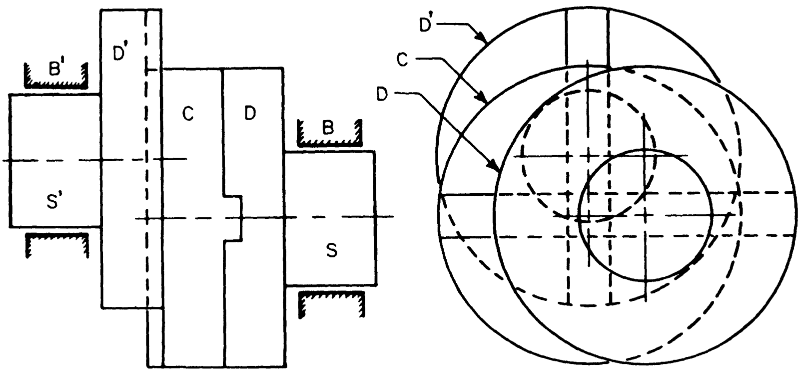


FIG. 15n

(o) Referring to Problem 15(g), replace the system of forces exerted on the upper half of the boom by the lower half, with a force and a couple. Determine the magnitude of the force, and show that the line of action of the force is parallel to the boom.

Result: 420 lb.

(p) A manually operated punch has the dimensions shown in Fig. 15(p). Assuming that the system of forces exerted by the hand on members A and B is equivalent to the two forces shown in the drawing, use symmetry considerations to reduce the system of forces exerted by member D on the piece C to a force whose

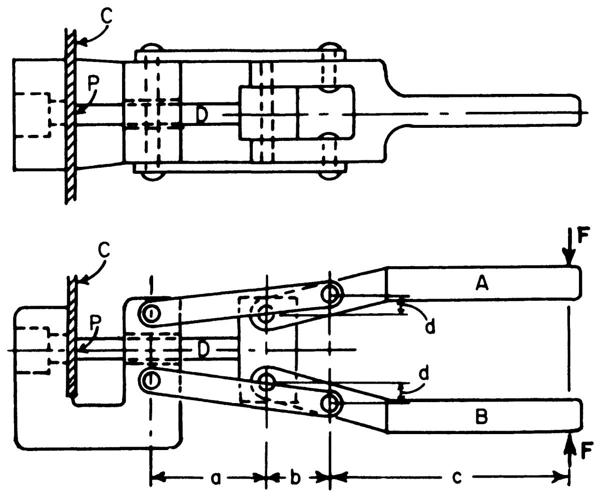


FIG. 15p

line of action passes through point P. Determine the magnitude of this force.

Result: $2F(a + b)(b + c)/ad$.

PROBLEM SET 16

(See Sections 4.10–4.11 of the text)

(a) When certain contact forces are applied to a rectangular beam having the dimensions shown in Fig. 16(a), the tractions at

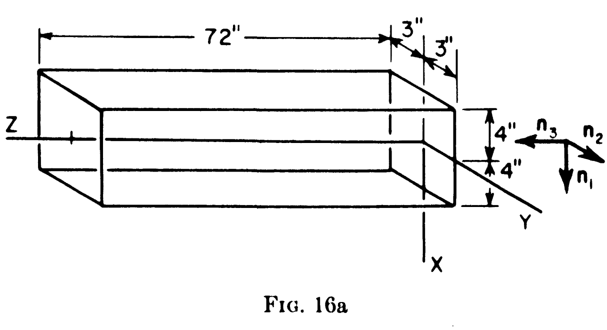


FIG. 16a

the point (x, y, z), for the directions $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, are respectively given by

$$\boldsymbol{\tau}_1 = 10(16 - x^2)\mathbf{n}_3 \text{ lb in.}^{-2}$$

$$\boldsymbol{\tau}_2 = 0$$

$$\boldsymbol{\tau}_3 = 10(16 - x^2)\mathbf{n}_1 - 20(72 - z)\, x\, \mathbf{n}_3 \text{ lb in.}^{-2}$$

where x, y and z are measured in inches.

(1) Letting P be the point at which the Z-axis intersects the face $z = 72$ of the beam, and σ a portion of this surface, σ containing P and having an area of 0.01 in.^{2}, reduce the system of contact forces exerted on the beam across the surface σ, to a couple of torque $\mathbf{T}$ and a force $\mathbf{F}$ whose line of action passes through P, and determine (approximately) the magnitudes of $\mathbf{F}$ and $\mathbf{T}$.

Results: 1.6 lb, 0.

(2) The surface $z = 36$ divides the beam into two parts. Letting B be the part in which z is greater than 36, and B' that in which z is less than 36, reduce the system of forces exerted by B' on B, to a couple of torque $\mathbf{T}$ and a force $\mathbf{F}$ whose line of action passes through the point (0, 0, 36), and determine the magnitudes of $\mathbf{F}$ and $\mathbf{T}$.

Results: 5120 lb, 15,360 ft lb.

(b) Referring to Problem 5(q), suppose that the tetrahedron is a portion of a continuous body at rest. Let $\boldsymbol{\tau}$ be the traction at a point of face $P_1P_2P_3$, for the direction $\mathbf{n}$; $\boldsymbol{\tau}_1^P$, $\boldsymbol{\tau}_2^P$, $\boldsymbol{\tau}_3^P$, the tractions at P, for the directions $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, respectively.

Assuming that the tetrahedron is small and that gravitational forces can be neglected, express $\boldsymbol{\tau}$ in terms of $\boldsymbol{\tau}_i^P$, $\mathbf{n}_i$ $(i = 1, 2, 3)$, and $\mathbf{n}$.

Result: $\boldsymbol{\tau} \approx \mathbf{n}\cdot\mathbf{n}_1\boldsymbol{\tau}_1^P + \mathbf{n}\cdot\mathbf{n}_2\boldsymbol{\tau}_2^P + \mathbf{n}\cdot\mathbf{n}_3\boldsymbol{\tau}_3^P$.

Note: When the face $P_1P_2P_3$ approaches point P in such a way that the orientation of $\mathbf{n}$ remains unaltered, the relationship found above becomes exact and thus furnishes a means for determining the traction at any point of a body, for any direction, whenever the tractions at that point, for three mutually perpendicular directions, are known.

(c) Referring to Problem 16(a), let A, B and C be the points $(4, -3, 36)$, $(-4, 3, 36)$, and $(0, 0, 36)$, respectively. Determine (1) the magnitude of the traction at C for the direction AB, and (2) the angle between this traction and line AB.

Results: 128 lb in.$^{-2}$, 90°.

(d) A "perfect fluid" is a body which, when at rest, exhibits the following property: The traction ($\boldsymbol{\tau}_a{}^P$) at any point P, for the direction $\mathbf{n}_a$, is parallel to $\mathbf{n}_a$. Hence $\boldsymbol{\tau}_a{}^P$ can be expressed as

$$\boldsymbol{\tau}_a{}^P = \tau_a{}^P \mathbf{n}_a$$

and, similarly, if $\mathbf{n}_b$ is any other direction,

$$\boldsymbol{\tau}_b{}^P = \tau_b{}^P \mathbf{n}_b$$

Use the result of Problem 16(b), together with the expressions

$$\mathbf{n}_a = \mathbf{n}_1 \cdot \mathbf{n}_a \mathbf{n}_1 + \mathbf{n}_2 \cdot \mathbf{n}_a \mathbf{n}_2 + \mathbf{n}_3 \cdot \mathbf{n}_a \mathbf{n}_3$$

$$\mathbf{n}_b = \mathbf{n}_1 \cdot \mathbf{n}_b \mathbf{n}_1 + \mathbf{n}_2 \cdot \mathbf{n}_b \mathbf{n}_2 + \mathbf{n}_3 \cdot \mathbf{n}_b \mathbf{n}_3$$

to show that

$$\tau_a{}^P = \tau_b{}^P$$

Note: The negative of the quantity $\tau_a{}^P$ is called "the pressure at point P."

(e) In Fig. 16(e), S represents a part of a shaft which is supported by a conical bearing surface and is subjected to the action of a system of forces equivalent to a single axial force of magnitude F. (This system of forces includes the gravitational forces exerted on the shaft by the earth.) Letting $\boldsymbol{\tau}$ be the traction at a point P of the bearing surface, for the direction of the normal to this surface

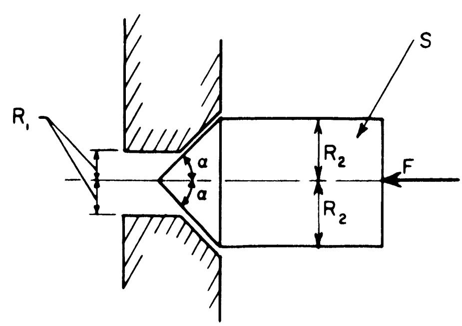


FIG. 16e

at P, determine the magnitude $|\boldsymbol{\tau}|$ of $\boldsymbol{\tau}$, assuming that the surface is smooth and that $|\boldsymbol{\tau}|$ is (1) independent of the position of P, and (2) inversely proportional to the distance r from P to the axis of the shaft.

Results:

$$\frac{F}{\pi(R_2^2 - R_1^2)}, \quad \frac{F}{2\pi r(R_2 - R_1)}$$

(f) Referring to Problem 16(e), suppose that the bearing surface is not smooth and that the coefficient of friction has the value f. Let $\mathbf{T}$ be the torque of a couple which is applied to the shaft, in addition to the system of forces described in Problem 16(e), in order to set the shaft in motion (rotation). If $\mathbf{T}$ is parallel to the axis of the shaft, and the same assumptions are made about $|\boldsymbol{\tau}|$ as were made in (1) and (2) of Problem 16(e), what is the magnitude of $\mathbf{T}$?

Answers:

$$\frac{2fF(R_2^3 - R_1^3)}{3 \sin \alpha(R_2^2 - R_1^2)}, \quad \frac{fF(R_1 + R_2)}{2 \sin \alpha}$$

(g) Referring to Problem 15(g), determine the smallest value of the coefficient of friction for the wall and the boom, such that the boom can remain in the position shown in 15(g), when the cable is removed.

Result: $(4/25)\sqrt{26}$.

(h) A thin-walled tube, resting on a horizontal support, is subjected to the action of a system of forces equivalent to a single

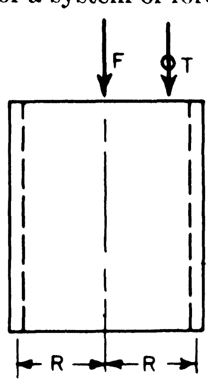


FIG. 16h

axial force of magnitude F, together with a couple whose torque is parallel to the axis of the tube and has a magnitude T, as shown in Fig. 16(h). Determine the value of T for which the tube is in a state of impending rotation about the tube's axis, and check your results by regarding the present situation as a limiting case of those considered in Problem 16(f).

(i) A cylindrical drum D is rigidly attached to a shaft whose axis coincides with that of the drum and which is mounted in smooth bearings. A couple, whose torque is parallel to the shaft axis and has a magnitude C, is applied to the shaft. In order to measure C, a band brake is used in the two ways shown in Figs. 16(i′) and 16(ii″). The brake consists of a brake band B, one of

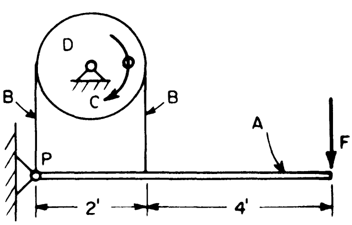


Fig. 16i′

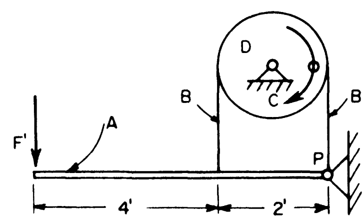


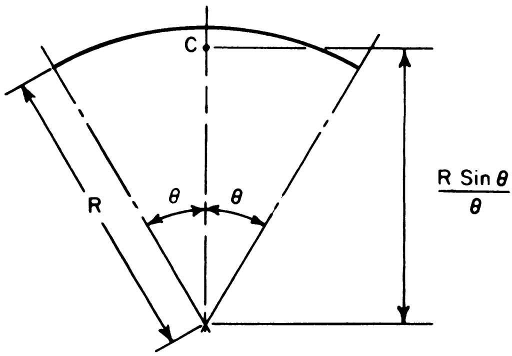
Fig. 16i″

whose ends is attached to a fixed point P, the other to the brake arm A. The brake arm is free to rotate about a fixed axis which passes through P and is parallel to the axis of the drum, and the brake is actuated by the application of a force perpendicular to the arm, this force having a magnitude F in one case, F' in the other. If F and F' are equal to 60 and 130 pounds, respectively, when the drum is just prevented from rotating, what is the value of C?

Answer: 210 ft lb.

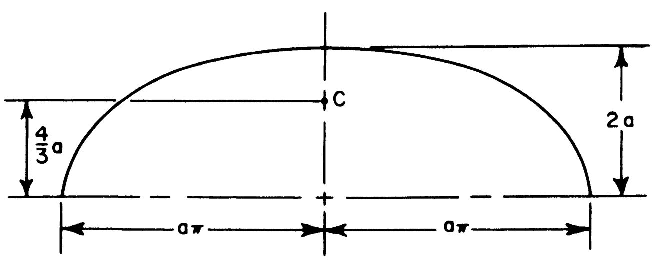
APPENDIX

Centroids of Curves, Surfaces, and Solids



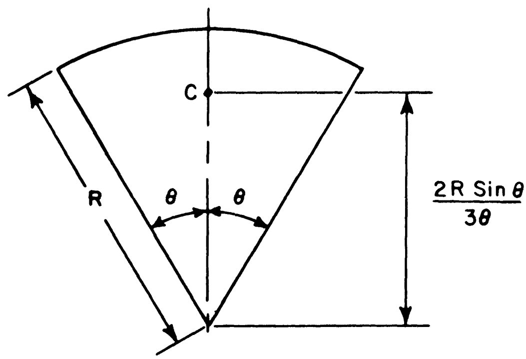
Length: 2Rθ

Fig. 1a. Circular Curve.



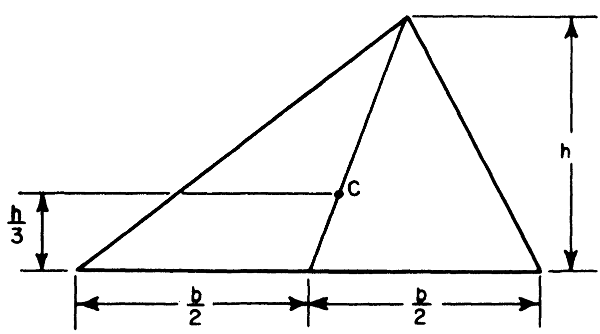
Length: 8a

Fig. 2a. Cycloid.



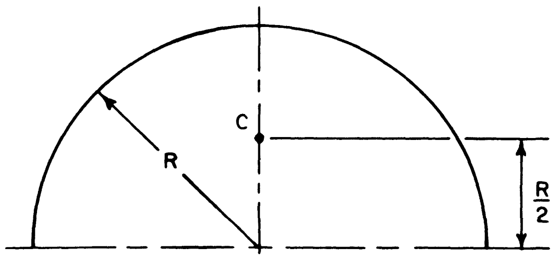
Area: ΘR^2

Fig. 3A. Circular Sector.



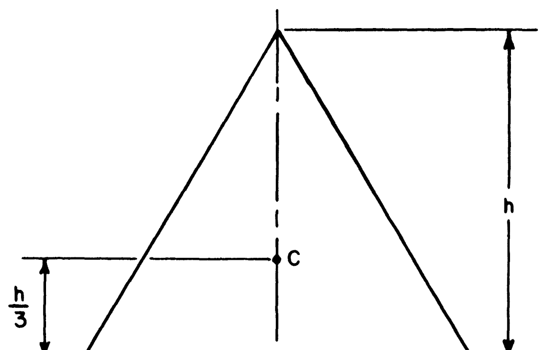
Area: bh/2

Fig. 4A. Triangle.



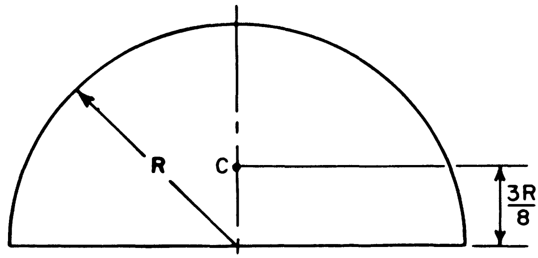
Area: $2\pi R^2$

FIG. 5A. HEMISPHERICAL SURFACE.



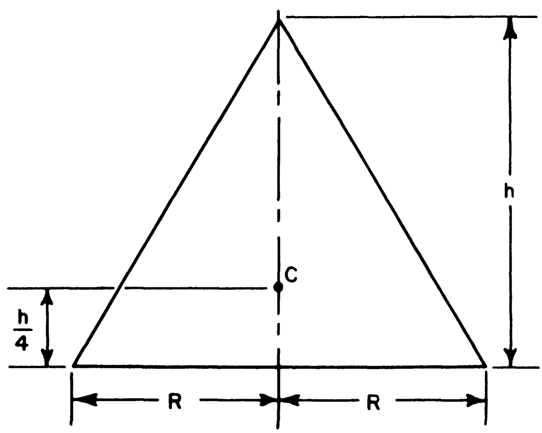
Area: $\pi R(h^2 + R^2)^{\frac{1}{2}}$

FIG. 6A. CONICAL SURFACE.



Volume: $2\pi R^3/3$

FIG. 7A. HEMISPHERICAL SOLID.



Volume: $\pi R^2 h/3$

FIG. 8A. CONICAL SOLID.

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