An analysis of the Longstaff-Schwartz algorithm for American option pricing

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Abstract

Recently, F.A. Longstaff and E.S. Schwartz proposed a Monte-Carlo method for the computation of American option prices, based on least squares regression. Under fairly general conditions, we prove the almost sure convergence of the algorithm. We also determine the rate of convergence and further prove that the normalized error is asymptotically Gaussian.

KEY WORDS: American options, optimal stopping, Monte-Carlo methods, least squares regression.
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1 Introduction

The computation of American option prices is a challenging problem, especially when several underlying assets are involved. The mathematical problem to solve is an optimal stopping problem. In classical diffusion models, this problem is associated with a variational inequality, for which, in higher dimensions, classical PDE methods are ineffective.

Recently, various authors introduced numerical methods based on Monte-Carlo techniques (see, among others, [1, 2, 3, 5]). The starting point of these methods is to replace the

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time interval of exercise dates by a finite subset. This amounts to approximating the American option by a so called *Bermuda* option. The solution of the corresponding discrete optimal stopping problem reduces to an effective implementation of the dynamic programming principle. The conditional expectations involved in the iterations of dynamic programming cause the main difficulty for the development of Monte-carlo techniques. One way of treating this problem is to use least squares regression on a finite set of functions as a proxy for conditional expectation. This is the method used by Longstaff and Schwartz [3]. Another type of least squares regression is proposed by Tsitsiklis and Van Roy [5].

The purpose of this paper is to analyze the Longstaff-Schwartz algorithm, which seems to have become popular among practitioners. More precisely, we will prove the convergence of the algorithm and establish a type of central limit theorem for the rate of convergence, thus providing the asymptotic normalized error. We note that partial convergence results are stated in [3], together with excellent empirical results, but with no study of the rate of convergence. On the other hand, convergence (but not the rate nor the error distribution) is provided in [5] for a somewhat different algorithm.

The paper is organized as follows. In Section 2, a precise description of the Longstaff-Schwartz algorithm and the notation is established. In Section 3, we prove the convergence of the algorithm. In Section 4, we study the rate of convergence.

2 The Longstaff-Schwartz algorithm and notations

2.1 Description of the algorithm

As mentioned in the introduction, the first step in all probabilistic approximation methods is to replace the original optimal stopping problem in continuous time by an optimal stopping problem in discrete time. Therefore, we will present the Longstaff-Schwartz algorithm in the context of discrete optimal stopping.

We will consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, equipped with a discrete filtration $(\mathcal{F}_j)_{j=0,\ldots,L}$. Here, the positive integer L denotes the (discrete) time horizon. Given an adapted payoff process $(Z_j)_{j=0,\ldots,L}$, where Z_0, Z_1,..., Z_L are square integrable random variables, we are interested in computing

$$\sup_{\tau \in \mathcal{T}_{0,L}} \mathbb{E} Z_\tau,$$

where $\mathcal{T}_{j,L}$ denotes the set of all stopping times with values in $\{j, \ldots, L\}$.

Following classical optimal stopping theory (for which we refer to [4], chapter 6), we introduce the Snell envelope $(U_j)_{j=0,\dots,L}$ of the payoff process $(Z_j)_{j=0,\dots,L}$, defined by

$$U_j = \text{ess-}\sup_{\tau \in \mathcal{T}_{j,L}} \mathbb{E}\left(Z_\tau \mid \mathcal{F}_j\right), \quad j = 0, \dots, L.$$

The dynamic programming principle can be written as follows:

$$\begin{cases} U_L = Z_L \\ U_j = \max\left(Z_j, \mathbb{E}\left(U_{j+1} \mid \mathcal{F}_j\right)\right), \quad 0 \le j \le L-1. \end{cases}$$

We also have $U_j = \mathbb{E}\left(Z_{\tau_j} \mid \mathcal{F}_j\right)$, with

$$\tau_j = \min\{k \ge j \mid U_k = Z_k\}.$$

In particular $\mathbb{E}U_0 = \sup_{\tau \in \mathcal{T}_{0,L}} \mathbb{E}Z_\tau = \mathbb{E}Z_{\tau_0}$.

The dynamic programming principle can be rewritten in terms of the optimal stopping times τ_j, as follows:

$$\begin{cases} \tau_L = L \\ \tau_j = j\mathbf{1}_{\{Z_j \ge \mathbb{E}(Z_{\tau_{j+1}} \mid \mathcal{F}_j)\}} + \tau_{j+1}\mathbf{1}_{\{Z_j < \mathbb{E}(Z_{\tau_{j+1}} \mid \mathcal{F}_j)\}}, \; j \le L-1, \end{cases}$$

This formulation in terms of stopping rules (rather than in terms of value functions) plays an essential role in the Longstaff-Schwartz method.

The method also requires that the underlying model be a Markov chain. Therefore, we will assume that there is an $(\mathcal{F}_j)$-Markov chain $(X_j)_{j=0,\dots,L}$ with state space $(E, \mathcal{E})$ such that, for $j = 0, \dots, L$,

$$Z_j = f(j, X_j),$$

for some Borel function $f(j, \cdot)$. We then have $U_j = V(j, X_j)$ for some function $V(j, \cdot)$ and $\mathbb{E}\left(Z_{\tau_{j+1}} \mid \mathcal{F}_j\right) = \mathbb{E}\left(Z_{\tau_{j+1}} \mid X_j\right)$. We will also assume that the initial state $X_0 = x$ is deterministic, so that U_0 is also deterministic.

The first step of the Longstaff-Schwartz algorithm is to approximate the conditional expectation with respect to X_j by the orthogonal projection on the space generated by a finite number of functions of X_j. Let us consider a sequence $(e_k(x))_{k \ge 1}$ of measurable real valued functions defined on E and satisfying the following conditions:

$\mathbf{A_1}$: For $j = 1$ to $L-1$, the sequence $(e_k(X_j))_{k \ge 1}$ is total in $L^2(\sigma(X_j))$.

$\mathbf{A_2}$: For $j = 0$ to $L-1$ and $m \ge 1$, if $\sum_{k=1}^{m} \lambda_k e_k(X_j) = 0$ a.s. then $\lambda_k = 0$ for $k = 1$ to m.

For $j = 1$ to $L-1$, we denote by P_j^m the orthogonal projection from $L^2(\Omega)$ onto the vector space generated by $\{e_1(X_j), \dots, e_m(X_j)\}$ and we introduce the stopping times $\tau_j^{[m]}$:

$$\begin{cases} \tau_L^{[m]} = L \\ \tau_j^{[m]} = j\mathbf{1}_{\left\{Z_j \geq P_j^m(Z_{\tau_{j+1}^{[m]}})\right\}} + \tau_{j+1}^{[m]}\mathbf{1}_{\left\{Z_j < P_j^m(Z_{\tau_{j+1}^{[m]}})\right\}}, \ j \leq L-1, \end{cases}$$

From these stopping times, we obtain an approximation of the value function:

$$U_0^m = \max\left(Z_0, \mathbb{E}Z_{\tau_1^{[m]}}\right). \tag{2.1}$$

Recall that $Z_0 = f(0,x)$ is deterministic. The second step of the algorithm is then to evaluate numerically $\mathbb{E}Z_{\tau_1^{[m]}}$ by a Monte-Carlo procedure. We assume that we can simulate N independent paths $(X_j^{(1)}), \dots, (X_j^{(n)}), \dots (X_j^{(N)})$ of the Markov chain (X_j) and we denote by $Z_j^{(n)}$ the associated payoff for $j = 0$ to L and $n = 1$ to N $(Z_j^{(n)} = f(j, X_j^{(n)}))$. For each path n, we then estimate recursively the stopping times $(\tau_j^{[m]})$ by:

$$\begin{cases} \tau_L^{n,m,N} = L \\ \tau_j^{n,m,N} = j\mathbf{1}_{\left\{Z_j^{(n)} \geq \alpha_j^{(m,N)} \cdot e^m(X_j^{(n)})\right\}} + \tau_{j+1}^{n,m,N}\mathbf{1}_{\left\{Z_j^{(n)} < \alpha_j^{(m,N)} \cdot e^m(X_j^{(n)})\right\}}, \ j \leq L-1, \end{cases}$$

Here, $x \cdot y$ denotes the usual inner product in $\mathbb{R}^m$, e^m is the vector valued function $(e_1, \dots, e_m)$ and $\alpha_j^{(m,N)}$ is the least square estimator:

$$\alpha_j^{(m,N)} = \arg\min_{a \in \mathbb{R}^m} \sum_{n=1}^{N} \left(Z^{(n)}_{\tau_{j+1}^{n,m,N}} - a \cdot e^m(X_j^{(n)})\right)^2,$$

Remark that for $j = 1$ to $L-1$, $\alpha_j^{(m,N)} \in \mathbb{R}^m$. Finally, from the variables $\tau_j^{n,m,N}$, we derive the following approximation for U_0^m:

$$U_0^{m,N} = \max\left(Z_0, \frac{1}{N}\sum_{n=1}^{N} Z^{(n)}_{\tau_1^{n,m,N}}\right). \tag{2.2}$$

In the next section, we prove that, for any fixed m, $U_0^{m,N}$ converges almost surely to U_0^m as N goes to infinity, and that U_0^m converges to U_0 as m goes to infinity. Before stating these results, we devote a short section to notation.

2.2 Notation

For $m \geq 1$ we denote by $e^m(x)$ the vector $(e_1(x), \dots, e_m(x))$ and for $j = 1$ to $L-1$ we note:

$$P_j^m(Z_{\tau_{j+1}^{[m]}}) = \alpha_j^m \cdot e^m(X_j) \tag{2.3}$$

We remark that the m dimensional parameter α_j^m has the explicit expression:

$$\alpha_j^m = (A_j^m)^{-1} I\!E(Z_{\tau_{j+1}^{[m]}} e^m(X_j)), \tag{2.4}$$

for $j = 1$ to $L-1$, where A_j^m is an $m \times m$ matrix, with coefficients given by

$$(A_j^m)_{1 \le k,l \le m} = I\!E(e_k(X_j)e_l(X_j)). \tag{2.5}$$

Similarly, the estimators $\alpha_j^{(m,N)}$ are equal to

$$\alpha_j^{(m,N)} = (A_j^{(m,N)})^{-1} \frac{1}{N} \sum_{n=1}^{N} Z^{(n)}_{\tau_{j+1}^{n,m,N}} e^m(X_j^{(n)}), \tag{2.6}$$

for $j = 1$ to $L-1$, where $A_j^{(m,N)}$ is an $m \times m$ matrix, with coefficients given by

$$(A_j^{(m,N)})_{1 \le k,l \le m} = \frac{1}{N} \sum_{n=1}^{N} e_k(X_j^{(n)}) e_l(X_j^{(n)}). \tag{2.7}$$

We note $\alpha^m = (\alpha_1^m, \dots, \alpha_{L-1}^m)$ and $\alpha^{(m,N)} = (\alpha_1^{(m,N)}, \dots, \alpha_{L-1}^{(m,N)})$.

Given a parameter $a^m = (a_1^m, \dots, a_{L-1}^m)$ in $I\!R^m \times \dots \times I\!R^m$ and deterministic vectors $z = (z_1, \dots, z_L) \in I\!R^L$ and $x = (x_1, \dots, x_L) \in E^L$, we define a vector field $F = (F_1, \dots, F_L)$ by:

$$\begin{aligned} F_L(a^m, z, x) &= z_L \\ F_j(a^m, z, x) &= z_j \mathbf{1}_{\{z_j \ge a_j^m \cdot e^m(x_j)\}} + F_{j+1}(a^m, z, x) \mathbf{1}_{\{z_j < a_j^m \cdot e^m(x_j)\}}, \text{ for } j = 1, \dots, L-1. \end{aligned}$$

We have

$$F_j(a^m, z, x) = z_j \mathbf{1}_{B_j^c} + \sum_{i=j+1}^{L-1} z_i \mathbf{1}_{B_j \dots B_{i-1} B_i^c} + z_L \mathbf{1}_{B_j \dots B_{L-1}},$$

with

$$B_j = \{z_j < a_j^m \cdot e^m(x_j)\}.$$

We remark that $F_j(a^m, Z, X)$ does not depend on $(a_1^m, \dots, a_{j-1}^m)$ and that we have

$$\begin{aligned} F_j(\alpha^m, Z, X) &= Z_{\tau_j^{[m]}} \\ F_j(\alpha^{(m,N)}, Z^{(n)}, X^{(n)}) &= Z^{(n)}_{\tau_j^{n,m,N}}. \end{aligned}$$

For $j = 2$ to L, we denote by G_j the vector valued function

$$G_j(a^m, z, x) = F_j(a^m, z, x) e^m(x_{j-1}),$$

and we define the functions ϕ_j and ψ_j by

$$\phi_j(a^m) = \mathbb{E} F_j(a^m, Z, X) \tag{2.8}$$

$$\psi_j(a^m) = \mathbb{E} G_j(a^m, Z, X). \tag{2.9}$$

Observe that with this notation, we have

$$\alpha_j^m = (A_j^m)^{-1}\psi_{j+1}(\alpha^m), \tag{2.10}$$

and similarly, for $j = 1$ to $L-1$,

$$\alpha_j^{(m,N)} = (A_j^{(m,N)})^{-1}\frac{1}{N}\sum_{n=1}^{N} F_{j+1}(\alpha^{(m,N)}, Z^{(n)}, X^{(n)})e^m(X_j^{(n)}). \tag{2.11}$$

3 Convergence

The convergence of U_0^m to U_0 is a direct consequence of the following result.

Theorem 3.1 *Assume that A_1 is satisfied, then for $j = 0$ to L we have*

$$\lim_{m\to+\infty} \mathbb{E}(Z_{\tau_j^{[m]}}|\mathcal{F}_j) = \mathbb{E}(Z_{\tau_j}|\mathcal{F}_j),$$

in L^2.

Proof: We proceed by induction on j. The result is true for $j = L$. Let us prove that if it holds for $j+1$, it is true for j $(j \le L-1)$. Since

$$Z_{\tau_j^{[m]}} = Z_j \mathbf{1}_{\{Z_j \ge \alpha_j^m \cdot e^m(X_j)\}} + Z_{\tau_{j+1}^{[m]}} \mathbf{1}_{\{Z_j < \alpha_j^m \cdot e^m(X_j)\}},$$

for $j \le L-1$, we have

$$\begin{aligned}
\mathbb{E}(Z_{\tau_j^{[m]}} - Z_{\tau_j}|\mathcal{F}_j) &= (Z_j - \mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j))\left(\mathbf{1}_{\{Z_j \ge \alpha_j^m \cdot e^m(X_j)\}} - \mathbf{1}_{\{Z_j \ge \mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j)\}}\right) \\
&\quad + \mathbb{E}(Z_{\tau_{j+1}^{[m]}} - Z_{\tau_{j+1}}|\mathcal{F}_j)\mathbf{1}_{\{Z_j < \alpha_j^m \cdot e^m(X_j)\}}.
\end{aligned}$$

By assumption, the second term of the right side of the equality converges to 0 and we just have to prove that B_j^m defined by

$$B_j^m = (Z_j - \mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j))\left(\mathbf{1}_{\{Z_j \ge \alpha_j^m \cdot e^m(X_j)\}} - \mathbf{1}_{\{Z_j \ge \mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j)\}}\right),$$

converges to 0 in L^2. Observe that

$$\begin{aligned}
|B_j^m| &= |Z_j - \mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j)||\mathbf{1}_{\{\mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j) > Z_j \ge \alpha_j^m \cdot e^m(X_j)\}} - \mathbf{1}_{\{\alpha_j^m \cdot e^m(X_j) > Z_j \ge \mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j)\}}| \\
&\le |Z_j - \mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j)|\mathbf{1}_{\{|Z_j - \mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j)| \le |\alpha_j^m \cdot e^m(X_j) - \mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j)|\}} \\
&\le |\alpha_j^m \cdot e^m(X_j) - \mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j)| \\
&\le |\alpha_j^m \cdot e^m(X_j) - P_j^m(\mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j))| + |P_j^m(\mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j)) - \mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j)|.
\end{aligned}$$

But

$$\alpha_j^m \cdot e^m(X_j) = P_j^m(Z_{\tau_{j+1}^{[m]}}) = P_j^m(\mathbb{E}(Z_{\tau_{j+1}^{[m]}}|\mathcal{F}_j)),$$

and consequently

$$||B_j^m||_2 \quad \leq \quad ||\mathbb{E}(Z_{\tau_{j+1}^{[m]}}|\mathcal{F}_j) - \mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j)||_2 + ||P_j^m(\mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j)) - \mathbb{E}(Z_{\tau_{j+1}}|\mathcal{F}_j)||_2.$$

The first term of the right side of this inequality tends to 0 by the induction hypothesis and the second one by A_1.

In what follows, we fix the value m and we study the properties of $U_0^{m,N}$ as N the number of Monte-Carlo simulations, goes to infinity. For notational simplicity, we drop the superscript m throughout the rest of the paper.

Theorem 3.2 *We assume A_1, A_2 and that for $j = 1$ to $L-1$, $\mathbb{P}(\alpha_j \cdot e(X_j) = Z_j) = 0$. Then $U_0^{m,N}$ converges almost surely to U_0^m as N goes to infinity.*

Note that with the notation of the preceeding section, we have to prove that

$$\lim_N \frac{1}{N} \sum_{n=1}^{N} F_1(\alpha^{(N)}, Z^{(n)}, X^{(n)}) = \phi_1(\alpha). \tag{3.1}$$

The proof is based on the following lemmas.

Lemma 3.1 *For $j = 1$ to L, we have :*

$$|F_j(a, Z, X) - F_j(b, Z, X)| \leq \sum_{i=j}^{L} |Z_i| \sum_{i=j}^{L-1} \mathbf{1}_{\{|Z_i - b_i \cdot e(X_i)| \leq |a_i - b_i||e(X_i)|\}}.$$

Proof : Let $B_j = \{Z_j \geq a_j \cdot e(X_j)\}$ and $\tilde{B}_j = \{Z_j \geq b_j \cdot e(X_j)\}$. We have :

$$\begin{aligned} F_j(a, Z, X) - F_j(b, Z, X) \quad &= \quad Z_j(\mathbf{1}_{B_j} - \mathbf{1}_{\tilde{B}_j}) \\ &\quad + \sum_{i=j+1}^{L-1} Z_i(\mathbf{1}_{B_j \ldots B_{i-1} B_i^c} - \mathbf{1}_{\tilde{B}_j \ldots \tilde{B}_{i-1} \tilde{B}_i^c}) \\ &\quad + Z_L(\mathbf{1}_{B_j^c \ldots B_{L-1}^c} - \mathbf{1}_{\tilde{B}_j^c \ldots \tilde{B}_{L-1}^c}). \end{aligned}$$

But

$$\begin{aligned} |\mathbf{1}_{B_j} - \mathbf{1}_{\tilde{B}_j}| \quad &= \quad \mathbf{1}_{\{a_j \cdot e(X_j) \leq Z_j < b_j \cdot e(X_j)\}} - \mathbf{1}_{\{b_j \cdot e(X_j) \leq Z_j < a_j \cdot e(X_j)\}} \\ &= \quad |\mathbf{1}_{\{b_j \cdot e(X_j) < Z_j \leq a_j \cdot e(X_j)\}} - \mathbf{1}_{\{a_j \cdot e(X_j) < Z_j \leq b_j \cdot e(X_j)\}}| \\ &= \quad \mathbf{1}_{\{b_j \cdot e(X_j) < Z_j \leq a_j \cdot e(X_j)\}} + \mathbf{1}_{\{a_j \cdot e(X_j) < Z_j \leq b_j \cdot e(X_j)\}} \\ &\leq \quad \mathbf{1}_{\{|Z_j - b_j \cdot e(X_j)| \leq |a_j - b_j||e(X_j)|\}} \end{aligned}$$

Moreover

$$
\begin{aligned}
|\mathbf{1}_{B_j\ldots B_{i-1}B_i^c} - \mathbf{1}_{\tilde{B}_j\ldots\tilde{B}_{i-1}\tilde{B}_i^c}| &\leq \sum_{k=j}^{i-1} |\mathbf{1}_{B_k} - \mathbf{1}_{\tilde{B}_k}| + |\mathbf{1}_{B_i^c} - \mathbf{1}_{\tilde{B}_i^c}| \\
&= \sum_{k=j}^{i} |\mathbf{1}_{B_k} - \mathbf{1}_{\tilde{B}_k}|,
\end{aligned}
$$

this gives

$$
|F_j(a,Z,X) - F_j(b,Z,X)| \leq \sum_{i=j}^{L} |Z_i| \sum_{i=j}^{L-1} |\mathbf{1}_{B_i} - \mathbf{1}_{\tilde{B}_i}|.
$$

Combining these inequalities, we obtain the result of Lemma 3.1.

Lemma 3.2 *Assume that for $j = 1$ to $L-1$, $I\!P(\alpha_j \cdot e(X_j) = Z_j) = 0$ then $\alpha_j^{(N)}$ converges almost surely to α_j.*

Proof: we proceed by induction on j. For $j = L-1$, the result is a direct consequence of the law of large numbers. Now, assume that the result is true for $i = j$ to $L-1$. We want to prove that it is true for $j-1$. We have

$$
\alpha_{j-1}^{(N)} = (A_{j-1}^{(N)})^{-1} \frac{1}{N} \sum_{n=1}^{N} G_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}).
$$

By the law of large numbers, we know that $A_{j-1}^{(N)}$ converges almost surely to A_{j-1} and it remains to prove that $\frac{1}{N}\sum_{n=1}^{N} G_j(\alpha^{(N)}, Z^{(n)}, X^{(n)})$ converges to $\psi_j(\alpha)$. From the law of large numbers, we have the convergence of $\frac{1}{N}\sum_{n=1}^{N} G_j(\alpha, Z^{(n)}, X^{(n)})$ to $\psi_j(\alpha)$ and it suffices to prove that :

$$
\lim_{N\to+\infty} \frac{1}{N} \sum_{n=1}^{N} \Big(G_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - G_j(\alpha, Z^{(n)}, X^{(n)}) \Big) = 0.
$$

We note $G_N = \frac{1}{N}\sum_{n=1}^{N} \Big(G_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - G_j(\alpha, Z^{(n)}, X^{(n)}) \Big)$. We have :

$$
\begin{aligned}
|G_N| &\leq \tfrac{1}{N} \sum_{n=1}^{N} |e(X_{j-1}^{(n)})| |F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - F_j(\alpha, Z^{(n)}, X^{(n)})| \\
&\leq \tfrac{1}{N} \sum_{n=1}^{N} |e(X_{j-1}^{(n)})| \sum_{i=j}^{L} |Z_i^{(n)}| \sum_{i=j}^{L-1} \mathbf{1}_{\{|Z_i^{(n)} - \alpha_i \cdot e(X_i^{(n)})| \leq |\alpha_i^{(N)} - \alpha_i| |e(X_i^{(n)})|\}}.
\end{aligned}
$$

Since, for $i = j$ to $L-1$, $\alpha_i^{(N)}$ converges almost surely to $\alpha_i^{(N)}$, we have for each $\epsilon > 0$:

$$
\begin{aligned}
\limsup |G_N| &\leq \limsup \tfrac{1}{N} \sum_{n=1}^{N} |e(X_{j-1}^{(n)})| \sum_{i=j}^{L} |Z_i^{(n)}| \sum_{i=j}^{L-1} \mathbf{1}_{\{|Z_i^{(n)} - \alpha_i \cdot e(X_i^{(n)})| \leq \epsilon |e(X_i^{(n)})|\}} \\
&= I\!E |e(X_{j-1})| \sum_{i=j}^{L} |Z_i| \sum_{i=j}^{L-1} \mathbf{1}_{\{|Z_i - \alpha_i \cdot e(X_i)| \leq \epsilon |e(X_i)|\}},
\end{aligned}
$$

where the last equality follows from the law of large numbers. Letting ϵ go to 0, we obtain the convergence to 0, since for $j=1$ to $L-1$, $I\!P(\alpha_j \cdot e(X_j) = Z_j) = 0$.

The proof of Theorem 3.2 is similar to the proof of Lemma 3.2.

4 Rate of convergence of the Longstaff-Schwartz algorithm

4.1 Tightness

In this section we are interested in the rate of convergence of $\frac{1}{N}\sum_{n=1}^{N} Z^{(n)}_{\tau_j^{n,N}}$, for $j=1$ to L. Recall that m is fixed.

We assume that :

H₁: The random variable Z and the functions $e_1, \ldots, e_m$ are bounded.

H₂: $\forall j = 1, \ldots, L-1$, $\limsup_{\epsilon \to 0} \frac{I\!P(|Z_j - \alpha_j \cdot e(X_j)| \leq \epsilon)}{\epsilon} = 0$.

Note that H_2 implies that $I\!P(Z_j = \alpha_j \cdot e(X_j)) = 0$ and, consequently, under H_2 we know from Section 3 that $\frac{1}{N}\sum_{n=1}^{N} F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)})$ converges almost surely to $\phi_j(\alpha)$. Remark too that H_2 is satisfied if the random variable $(Z_j - \alpha_j \cdot e(X_j))$ has a bounded density near zero.

Theorem 4.1 *Under H_1 and H_2, the sequences $\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N}(F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - \phi_j(\alpha))\right)_{N\geq 1}$ $(j=1,\ldots,L)$ and $(\sqrt{N}(\alpha_j^{(N)} - \alpha_j))_{N\geq 1}$ $(j = 1, \ldots, L-1)$ are tight.*

The proof of Theorem 4.1 is based on the following Lemma.

Lemma 4.1 *Let $(U^{(n)})$ be a sequence of identically distributed random variables such that*

$$\limsup_{\epsilon \to 0} \frac{I\!P(|U^{(1)}| \leq \epsilon)}{\epsilon} < +\infty,$$

and (θ_N) a sequence of positive random variables such that $(\sqrt{N}\theta_N)$ is tight, then the sequence $\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\mathbf{1}_{\{|U^{(n)}|\leq \theta_N\}}\right)_{N\geq 1}$ is tight.

Proof: We note $\sigma_N(\theta) = \frac{1}{\sqrt{N}}\sum_{n=1}^{N}\mathbf{1}_{\{|U^{(n)}|\leq \theta_N\}}$. Observe that σ_N is a non decreasing function of θ. Let $A > 0$, we have

$$\begin{aligned}
\mathbb{P}(\sigma_N(\theta_N) \geq A) &\leq \mathbb{P}(\sigma_N(\theta_N) \geq A, \sqrt{N}\theta_N \leq B) + \mathbb{P}(\sqrt{N}\theta_N > B) \\
&\leq \mathbb{P}(\sigma_N(\tfrac{B}{\sqrt{N}}) \geq A) + \mathbb{P}(\sqrt{N}\theta_N > B) \\
&\leq \tfrac{1}{A}\mathbb{E}\sigma_N(\tfrac{B}{\sqrt{N}}) + \mathbb{P}(\sqrt{N}\theta_N > B) \\
&= \tfrac{\sqrt{N}}{A}\mathbb{P}(|U^{(1)}| \leq \tfrac{B}{\sqrt{N}}) + \mathbb{P}(\sqrt{N}\theta_N > B).
\end{aligned}$$

From the assumption on $(U^{(n)})$ and the tightness of $(\sqrt{N}\theta_N)$, we deduce easily the tightness of $\sigma_N(\theta_N)$.

Proof of Theorem 4.1: We know from the classical Central Limit Theorem that the sequence $(1/\sqrt{N})\sum_{n=1}^{N}(F_j(\alpha, Z^{(n)}, X^{(n)}) - \phi_j(\alpha))$ is tight and it remains to prove the tightness of $\frac{1}{\sqrt{N}}\sum_{n=1}^{N}(F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - F_j(\alpha, Z^{(n)}, X^{(n)}))$, for $j = 1$ to L. Similarly, to prove the tightness of $(\sqrt{N}(\alpha_j^{(N)} - \alpha_j))_{N\geq 1}$, for $j = 1$ to $L-1$, we just have to prove the tightness of $\frac{1}{\sqrt{N}}\sum_{n=1}^{N}(G_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - G_j(\alpha, Z^{(n)}, X^{(n)}))$ (see Section 2 for the notation). We proceed by induction on j. The tightness of $\frac{1}{\sqrt{N}}\sum_{n=1}^{N}(F_L(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - F_L(\alpha, Z^{(n)}, X^{(n)}))$ and $(\sqrt{N}(\alpha_{L-1}^{(N)} - \alpha_{L-1}))$ is straightforward.

Assume that $\frac{1}{\sqrt{N}}\sum_{n=1}^{N}(F_i(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - F_i(\alpha, Z^{(n)}, X^{(n)}))$ and $(\sqrt{N}(\alpha_{i-1}^{(N)} - \alpha_{i-1}))$ are tight for $i = j$ to L. We set

$$F_N = \frac{1}{\sqrt{N}}\sum_{n=1}^{N}(F_{j-1}(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - F_{j-1}(\alpha, Z^{(n)}, X^{(n)})).$$

Now from Lemma 3.1, we have :

$$|F_N| \leq C\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\sum_{i=j-1}^{L-1}\mathbf{1}_{\{|Z_i^{(n)} - \alpha_i \cdot e(X_i^{(n)})| \leq |\alpha_i - \alpha_i^{(N)}|||e||_\infty\}}$$

from Lemma 4.1 and by the induction hypothesis, we deduce that F_N is tight. In the same way, we prove that $(\sqrt{N}(\alpha_{j-2}^{(N)} - \alpha_{j-2}))$ is tight.

4.2 A central limit theorem

We prove in this section that under some stronger assumptions than in section 4.1, the vector $\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N}(Z^{(n)}_{\tau_j^{n,N}} - \mathbb{E}Z_{\tau_j^{[m]}})\right)_{j=1,\ldots,L}$ converges weakly to a Gaussian vector. With the preceding notation, we have

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{N}(Z^{(n)}_{\tau_j^{n,N}} - I\!\!E Z_{\tau_j^{[m]}}) = \frac{1}{\sqrt{N}}\sum_{n=1}^{N}(F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - \phi_j(\alpha)). \tag{4.1}$$

In the following, we will denote by Y the couple (Z, X) and by $Y^{(n)}$ the couple $(Z^{(n)}, X^{(n)})$.

We make some more hypothesis:

$\mathbf{H_2^*}$: For $j = 1$ to $L-1$, there exists a neighborhood V_j of α_j such that for $a_j \in V_j$, $Z_j - a_j \cdot e(X_j)$ admits a density f_{a_j} near 0, such that, for some $\eta > 0$, $\sup_{a_j \in V_j, |z| \leq \eta} f_{a_j}(z) < +\infty$.

$\mathbf{H_3}$: For $j = 1$ to $L-1$, ϕ_j and ψ_j are $\mathcal{C}^1$ in a neighborhood of α.

Observe that H_2^* is stronger than H_2.

Theorem 4.2 *Under H_1, H_2^*, H_3, the vector*

$$\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N}(Z^{(n)}_{\tau_j^{n,N}} - I\!\!E Z_{\tau_j^{[m]}}), \sqrt{N}(\alpha_j^{(N)} - \alpha_j)\right)_{j=1,\ldots,L-1}$$

converges in law to a Gaussian vector as N goes to infinity.

To prove Theorem 4.2, we use the following decomposition :

$$\begin{aligned}\frac{1}{\sqrt{N}}\sum_{n=1}^{N}(F_j(\alpha^{(N)}, Y^{(n)}) - \phi_j(\alpha)) &= \frac{1}{\sqrt{N}}\sum_{n=1}^{N}\Big(F_j(\alpha^{(N)}, Y^{(n)}) - F_j(\alpha, Y^{(n)}) - (\phi_j(\alpha^{(N)}) - \phi_j(\alpha))\Big)\\ &+\frac{1}{\sqrt{N}}\sum_{n=1}^{N}(F_j(\alpha, Y^{(n)}) - \phi_j(\alpha)) + \sqrt{N}(\phi_j(\alpha^{(N)}) - \phi_j(\alpha)).\end{aligned}$$

From the classical Central Limit Theorem, we know that $\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N}(F_j(\alpha, Y^{(n)}) - \phi_j(\alpha))\right)_{j=1,\ldots,L}$ converges in law to a Gaussian vector. Moreover, we have :

$$\begin{aligned}\sqrt{N}(\alpha_j^{(N)} - \alpha_j) = (A_j^{(N)})^{-1}&\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\Big(G_{j+1}(\alpha^{(N)}, Y^{(n)}) - \psi_{j+1}(\alpha)\Big)\\ &-(A_j)^{-1}\sqrt{N}(A_j^{(N)} - A_j)(A_j^{(N)})^{-1}\psi_{j+1}(\alpha),\end{aligned}$$

where $A_j^{(N)}$ converges almost surely to A_j and $\sqrt{N}(A_j^{(N)} - A_j)$ converges in law. From these decompositions, it is straightforward to check that Theorem 4.2 is a consequence of the differentiability of the functions ϕ and ψ and of the following theorem.

Theorem 4.3 *Under H_1, H_2^*, H_3, the variables*

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\Big(F_j(\alpha^{(N)}, Y^{(n)}) - F_j(\alpha, Y^{(n)}) - (\phi_j(\alpha^{(N)}) - \phi_j(\alpha))\Big)$$

and

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\left(G_{j+1}(\alpha^{(N)},Y^{(n)})-G_{j+1}(\alpha,Y^{(n)})-(\psi_{j+1}(\alpha^{(N)})-\psi_{j+1}(\alpha))\right)$$

converge to 0 *in* L^2, *for* $j=1$ *to* $L-1$.

The proof of Theorem 4.3 requires some Lemmas. In the following, we denote by $I(Y_i,a_i,\epsilon)$ the function

$$I(Y_i,a_i,\epsilon)=\mathbf{1}_{\{|Z_i-a_i\cdot e(X_i)|\le||e||_\infty\epsilon\}}. \tag{4.2}$$

Note that $I(Y_i,a_i,\epsilon)\le I(Y_i,b_i,\epsilon+|b_i-a_i|)$.

Lemma 4.2 *For* $j=1$ *to* $L-1$, *and* a,b *in* $(I\!R^m)^{L-1}$, *we have under* H_1

$$\begin{aligned}|F_j(a,Y)-F_j(b,Y)| &\le ||Z||_\infty\sum_{i=j}^{L-1}I(Y_i,a_i,|a_i-b_i|)\\ |G_j(a,Y)-G_j(b,Y)| &\le ||Z||_\infty||e||_\infty\sum_{i=j}^{L-1}I(Y_i,a_i,|a_i-b_i|)\end{aligned}$$

This result follows easily from Lemma 3.1.

Lemma 4.3 *Assume* H_1 *and* H_2^*, *then for* $j=1$ *to* $L-1$, *we have, for all* $\delta>0$,

$$\lim_{N\to+\infty}N I\!P(|\alpha_j^{(N)}-\alpha_j|\ge\delta)=0.$$

Proof: Let us recall that if $(U_n)_n$ is a sequence of i.i.d. bounded variables, we have

$$\forall\delta>0,\quad \lim_{N\to+\infty}N I\!P\left(|\frac{1}{N}\sum_{i=1}^{N}U_i-I\!EU_1|\ge\delta\right)=0 \tag{4.3}$$

Observe that

$$\begin{aligned}\alpha_j^{(N)}-\alpha_j=(A_j^{(N)})^{-1}\frac{1}{N}\sum_{n=1}^{N}&(G_{j+1}(\alpha^{(N)},Y^{(n)})-\psi_{j+1}(\alpha))\\ &-(A_j)^{-1}(A_j^{(N)}-A_j)(A_j^{(N)})^{-1}\psi_{j+1}(\alpha),\end{aligned}$$

We note $\Omega_j^\epsilon=\{||A_j^{(N)}-A_j||\le\epsilon\}$ and we choose ϵ such that $||(A_j^{(N)})^{-1}||\le 2||(A_j)^{-1}||$ on Ω_j^ϵ. From (4.3), we know that $N I\!P((\Omega_j^\epsilon)^c)$ tends to 0, for $j=1$ to $L-1$ and that, on Ω_j^ϵ,

$$|\alpha_j^{(N)}-\alpha_j|\le K'\left|\frac{1}{N}\sum_{n=1}^{N}(G_{j+1}(\alpha^{(N)},Y^{(n)})-\psi_{j+1}(\alpha))\right|+K\epsilon.$$

Now, since $G_L(\alpha^{(N)}, Y^{(n)}) = Z_L^{(n)} e(X_{L-1}^{(n)})$, we deduce that

$$\lim_{N\to+\infty} N\mathbb{P}(|\alpha_{L-1}^{(N)} - \alpha_{L-1}| \geq \delta) = 0.$$

Assume now that the result of Lemma 4.3 is true for $j+1, \ldots, L-1$. We will prove that $\lim_{N\to+\infty} N\mathbb{P}(|\alpha_j^{(N)} - \alpha_j| \geq \delta) = 0$.

We have

$$\frac{1}{N}\sum_{n=1}^{N}(G_{j+1}(\alpha^{(N)}, Y^{(n)}) - \psi_{j+1}(\alpha)) = \frac{1}{N}\sum_{n=1}^{N}(G_{j+1}(\alpha^{(N)}, Y^{(n)}) - G_{j+1}(\alpha, Y^{(n)})) + \frac{1}{N}\sum_{n=1}^{N}(G_{j+1}(\alpha, Y^{(n)}) - \psi_{j+1}(\alpha)).$$

From Lemma 4.2, we obtain on Ω_j^ϵ,

$$|\alpha_j - \alpha_j^{(N)}| \leq K\epsilon + \frac{K'}{N}\sum_{n=1}^{N}\sum_{i=j+1}^{L-1} I(Y_i^{(n)}, \alpha_i, |\alpha_i - \alpha_i^{(N)}|) + \frac{K''}{N}\left|\sum_{n=1}^{N}(G_{j+1}(\alpha, Y^{(n)}) - \psi_{j+1}(\alpha))\right|.$$

The last term can be treated using (4.3). Therefore, it suffices to prove that

$$\forall\delta > 0 \lim_{N\to+\infty} N\mathbb{P}(S_N \geq \delta) = 0,$$

where $S_N = \frac{1}{N}\sum_{n=1}^{N}\sum_{i=j+1}^{L-1} I(Y_i^{(n)}, \alpha_i, |\alpha_i - \alpha_i^{(N)}|)$. But

$$\mathbb{P}(S_N \geq \delta) \leq \mathbb{P}\left(\frac{1}{N}\sum_{n=1}^{N}\sum_{i=j+1}^{L-1} I(Y_i^{(n)}, \alpha_i, \epsilon) \geq \delta\right) + \sum_{i=j+1}^{L-1}\mathbb{P}\left(|\alpha_i^{(N)} - \alpha_i| \geq \epsilon\right).$$

By assumption, for $i = j+1$ to $L-1$, we have $\lim_{N\to+\infty} N\mathbb{P}(|\alpha_i^{(N)} - \alpha_i| \geq \epsilon) = 0$. Moreover we know from H_2^* that $\sum_{i=j+1}^{L-1} \mathbb{E}I(Y_i^{(n)}, \alpha_i, \epsilon)$ goes to 0 as ϵ tends to 0, so for ϵ small enough we have $\delta - \sum_{i=j+1}^{L-1} \mathbb{E}I(Y_i^{(n)}, \alpha_i, \epsilon) > 0$ and from (4.3), we see that $N\mathbb{P}\left(\frac{1}{N}\sum_{n=1}^{N}\sum_{i=j+1}^{L-1} I(Y_i^{(n)}, \alpha_i, \epsilon) \geq \delta\right)$ tends to 0 as N goes to infinity. This completes the proof of Lemma 4.3.

We now state a technical Lemma which requires some more notation. Given C and C' two constant vectors in $\mathbb{R}_+^{L-1}$ and $a \in (\mathbb{R}^m)^{L-1}$, we define a sequence of random vectors

$\mathcal{U}^{(N-2-k)}$, for $k=0$ to $L-2$ by

$$\begin{aligned}
\mathcal{U}_{L-1}^{(N-2-k)}(C,C',a) &= \tfrac{C_{L-1}}{N}\\
\mathcal{U}_{j}^{(N-2-k)}(C,C',a) &= \tfrac{C_j}{N}+\tfrac{C'_j}{N}\sum_{n=1}^{N-2-k}\sum_{i=j+1}^{L-1} I\left(Y_i^{(n)},a_i,\mathcal{U}_i^{(N-2-k)}(C,C',a)\right),
\end{aligned}$$

for $j=1,\dots,L-2$. Note that $\mathcal{U}^{(N-2-k)}$ is $\sigma(Y^{(1)},\dots,Y^{(N-2-k)})$ measurable.

Lemma 4.4 *Assume H_1 and H_2^*, then for all $C,C'\in \mathbb{R}_+^{L-1}$ and $j=1$ to $L-2$, there exist some constant K such that*

$$\mathbb{E}\mathcal{U}_j^{(N-2)}(C,C',\alpha^{(N-2)})\le \frac{K}{N}.$$

Proof : For $j=1$ to $L-2$, we will prove by induction on k, $1\le k\le L-1-j$ that there exist $C_j(k)$ and $C'_j(k)$ in $\mathbb{R}_+^{L-1}$ such that

$$\mathbb{E}\mathcal{U}_j^{(N-2)}(C,C',\alpha^{(N-2)})\le \frac{K}{N}+K'\sum_{i=j+k}^{L-1}\mathbb{E}\mathcal{U}_i^{(N-2-k)}(C_j(k),C'_j(k),\alpha^{(N-2-k)}).$$

We have

$$\begin{aligned}
\mathbb{E}\mathcal{U}_j^{(N-2)}(C,C',\alpha^{(N-2)}) &= \frac{C_j}{N}+\frac{C'_j}{N}\sum_{n=1}^{N-2}\sum_{i=j+1}^{L-1}\mathbb{E}I(Y_i^{(n)},\alpha_i^{(N-2)},\mathcal{U}_i^{(N-2)})\\
&= \frac{C_j}{N}+\frac{C'_j}{N}(N-2)\sum_{i=j+1}^{L-1}\mathbb{E}I(Y_i^{(N-2)},\alpha_i^{(N-2)},\mathcal{U}_i^{(N-2)}).
\end{aligned}$$

But $I(Y_i^{(N-2)},\alpha_i^{(N-2)},\mathcal{U}_i^{(N-2)})\le I(Y_i^{(N-2)},\alpha_i^{(N-3)},\mathcal{U}_i^{(N-2)}+|\alpha_i^{(N-2)}-\alpha_i^{(N-3)}|)$. As in the proof of Lemma 4.3, we can choose Ω_N such that $NP(\Omega_N^c)$ goes to 0 as N goes to infinity and such that on Ω_N

$$\begin{aligned}
|\alpha_{L-1}^{(N-2)}-\alpha_{L-1}^{(N-3)}| &\le \tfrac{\tilde{C}_{L-1}}{N}\\
|\alpha_{j}^{(N-2)}-\alpha_{j}^{(N-3)}| &\le \tfrac{\tilde{C}_j}{N}+\tfrac{\tilde{C}'_j}{N}\sum_{n=1}^{N-3}\sum_{i=j+1}^{L-1} I\left(Y_i^{(n)},\alpha_i^{(N-3)},|\alpha_i^{(N-2)}-\alpha_i^{(N-3)}|\right).
\end{aligned}$$

So, on Ω_N, there exist $C(1)$ and $C'(1)$ in $\mathbb{R}_+^{L-1}$ such that, for $i=j+1$ to $L-1$,

$$\mathcal{U}_i^{(N-2)}(C,C',\alpha^{(N-2)})+|\alpha_i^{(N-2)}-\alpha_i^{(N-3)}|\le \mathcal{U}_i^{(N-3)}(C(1),C'(1),\alpha^{(N-3)}).$$

Consequently, we obtain

$$\mathbb{E}\mathcal{U}_j^{(N-2)}\le \frac{C_j}{N}+\frac{C'_j}{N}(N-2)\sum_{i=j+1}^{L-1}\mathbb{E}I\left(Y_i^{(N-2)},\alpha_i^{(N-3)},\mathcal{U}_i^{(N-3)}(C(1),C'(1),\alpha^{(N-3)})\right).$$

Conditioning on $\sigma(Y^{(1)},\dots,Y^{(N-3)})$ and using H_2^*, we finally get

$$\mathbb{E}\mathcal{U}_j^{(N-2)}(C,C',\alpha^{(N-2)}) \le \frac{K}{N} + K' \sum_{i=j+1}^{L-1} \mathbb{E}\mathcal{U}_i^{(N-3)}(C(1),C'(1),\alpha^{(N-3)}).$$

This proves the step $k=1$ of the induction. We prove the implementation from k to $k+1$ in a similar way. Applying this result to $k=L-1-j$ gives Lemma 4.4.

Lemma 4.5 *For $i,j=1$ to $L-1$, we have, under H_1, H_2^*,*

$$\lim_N N\mathbb{E}\left[I\left(Y_i^{(N-1)},\alpha_i^{(N-2)},|\alpha_i^{(N)}-\alpha_i^{(N-2)}|\right) I\left(Y_j^{(N)},\alpha_j,|\alpha_j^{(N)}-\alpha_j|\right)\right]=0$$
$$\lim_N N\mathbb{E}\left[I\left(Y_i^{(N)},\alpha_i^{(N-2)},|\alpha_i^{(N)}-\alpha_i^{(N-2)}|\right) I\left(Y_j^{(N-1)},\alpha_j,|\alpha_j^{(N)}-\alpha_j|\right)\right]=0$$

Proof: We choose $\tilde\Omega_N$ such that $NP(\tilde\Omega_N^c)$ tends to 0 as N goes to infinity and such that, on $\tilde\Omega_N$,

$$\begin{aligned}|\alpha_{L-1}^{(N)}-\alpha_{L-1}^{(N-2)}| &\le \tfrac{C_{L-1}}{N}\\ |\alpha_j^{(N)}-\alpha_j^{(N-2)}| &\le \tfrac{C_j}{N}+\tfrac{C_j'}{N}\sum_{n=1}^{N-2}\sum_{i=j+1}^{L-1} I(Y_i^{(n)},\alpha_i^{(N-2)},|\alpha_i^{(N)}-\alpha_i^{(N-2)}|),\end{aligned}$$

for $j=1$ to $L-2$. Now, with the notation $U^{(N-2)}=\mathcal{U}^{(N-2)}(C,C',\alpha^{(N-2)})$, we remark that for $j=1$ to $L-1$, $|\alpha_j^{(N)}-\alpha_j^{(N-2)}|\le U_j^{(N-2)}$ on $\tilde\Omega_N$. Define

$$\Omega_N=\tilde\Omega_N\bigcap\{|\alpha^{(N)}-\alpha|\le\delta\}\bigcap\{|\alpha^{(N-2)}-\alpha|\le\eta\}.$$

From Lemma 4.3, $N\mathbb{P}(\Omega_N^c)$ tends to 0. Now we have

$$\begin{aligned}&N\mathbb{E}\left[I\left(Y_i^{(N-1)},\alpha_i^{(N-2)},|\alpha_i^{(N)}-\alpha_i^{(N-2)}|\right) I\left(Y_j^{(N)},\alpha_j,|\alpha_j^{(N)}-\alpha_j|\right)\mathbf{1}_{\Omega_N}\right]\\ &\le\ N\mathbb{E}\left[I\left(Y_i^{(N-1)},\alpha_i^{(N-2)},U_i^{(N-2)}\right) I\left(Y_j^{(N)},\alpha_j,\delta\right)\mathbf{1}_{\{|\alpha^{(N-2)}-\alpha|\le\eta\}}\right].\end{aligned}$$

But

$$\mathbb{E}\left[I\left(Y_j^{(N)},\alpha_j,\delta\right)|Y^{(1)},\dots,Y^{(N-1)}\right]\le C\delta,$$

and, from H_2^*,

$$\mathbb{E}\left[I\left(Y_i^{(N-1)},\alpha_i^{(N-2)},U_i^{(N-2)}\right)\mathbf{1}_{\{|\alpha^{(N-2)}-\alpha|\le\eta\}}|Y^{(1)},\dots,Y^{(N-2)}\right]\le CU_i^{(N-2)},$$

this gives

$$N\mathbb{E}\left[I\left(Y_i^{(N-1)},\alpha_i^{(N-2)},|\alpha_i^{(N)}-\alpha_i^{(N-2)}|\right)I\left(Y_j^{(N)},\alpha_j,|\alpha_j^{(N)}-\alpha_j|\right)\mathbf{1}_{\Omega_N}\right]\le CN\mathbb{E}U_i^{(N-2)}\delta.$$

Applying Lemma 4.4, we know that $\mathbb{E}U_i^{(N-2)}\le K/N$ and Lemma 4.5 is proved.

Proof of Theorem 4.3*:* We prove that

$$\lim_N \frac{1}{\sqrt{N}}\sum_{n=1}^N\left(F_j(\alpha^{(N)},Y^{(n)})-F_j(\alpha,Y^{(n)})-(\phi_j(\alpha^{(N)})-\phi_j(\alpha))\right)=0$$

in L^2. The proof is similar for the second term of the Theorem. We introduce the notation $\Delta_j(a,b,Y)=F_j(a,Y)-F_j(b,Y)-(\phi_j(a)-\phi_j(b))$. We have to prove that

$$\lim_N \frac{1}{N}\mathbb{E}\left(\sum_{n=1}^N\Delta_j(\alpha^{(N)},\alpha,Y^{(n)})\right)^2=0.$$

Remark that for $n=1$ to N, the couples $(\alpha^{(N)},Y^{(n)})$ and $(\alpha^{(N)},Y^{(1)})$ have the same law, and for $n\neq m$, $(\alpha^{(N)},Y^{(n)},Y^{(m)})$ and $(\alpha^{(N)},Y^{(N)},Y^{(N-1)})$ have the same distribution. So we obtain

$$\begin{aligned}\tfrac{1}{N}\mathbb{E}(\sum_{n=1}^N\Delta_j(\alpha^{(N)},\alpha,Y^{(n)}))^2 &= \mathbb{E}\Delta_j^2(\alpha^{(N)},\alpha,Y^{(1)})\\ &\quad +(N-1)\mathbb{E}\Delta_j(\alpha^{(N)},\alpha,Y^{(N-1)})\Delta_j(\alpha^{(N)},\alpha,Y^{(N)}).\end{aligned}$$

But $|\Delta_j(\alpha^{(N)},\alpha,Y^{(1)})|\le 4||Z||_\infty$. Since the sequence $(\alpha^{(N)})$ goes to α almost surely and $\mathbb{P}(Z_j=\alpha_j\cdot e(X_j))=0$ for $j=1$ to L by assumption, we deduce that $\Delta_j(\alpha^{(N)},\alpha,Y^{(1)})$ goes to 0 almost surely. Consequently, we obtain that $\mathbb{E}\Delta_j^2(\alpha^{(N)},\alpha,Y^{(1)})$ tends to 0. It remains to prove that

$$\lim_N N\mathbb{E}\Delta_j(\alpha^{(N)},\alpha,Y^{(N-1)})\Delta_j(\alpha^{(N)},\alpha,Y^{(N)})=0. \tag{4.4}$$

We observe that

$$\mathbb{E}\left(\Delta_j(\alpha^{(N-2)},\alpha,Y^{(N)})|Y^{(1)},\dots,Y^{(N-1)}\right)=0,$$

since $\mathbb{E}\left(F_j(\alpha^{(N-2)},Y^{(N)})|Y^{(1)},\dots,Y^{(N-1)}\right)=\phi_j(\alpha^{(N-2)})$ almost surely. This gives

$$\mathbb{E}\Delta_j(\alpha^{(N-2)},\alpha,Y^{(N-1)})\Delta_j(\alpha^{(N-2)},\alpha,Y^{(N)})=0,$$

and we just have to prove that

$$\lim_N N\mathbb{E}\left(\Delta_j(\alpha^{(N)},\alpha,Y^{(N-1)})\Delta_j(\alpha^{(N)},\alpha,Y^{(N)})-\Delta_j(\alpha^{(N-2)},\alpha,Y^{(N-1)})\Delta_j(\alpha^{(N-2)},\alpha,Y^{(N)})\right)=0.$$

We have the equality

$$\Delta_j(\alpha^{(N)},\alpha,Y^{(N-1)})\Delta_j(\alpha^{(N)},\alpha,Y^{(N)})-\Delta_j(\alpha^{(N-2)},\alpha,Y^{(N-1)})\Delta_j(\alpha^{(N-2)},\alpha,Y^{(N)})=$$
$$\Delta_j(\alpha^{(N)},\alpha^{(N-2)},Y^{(N-1)})\Delta_j(\alpha^{(N)},\alpha,Y^{(N)})+\Delta_j(\alpha^{(N-2)},\alpha,Y^{(N-1)})\Delta_j(\alpha^{(N)},\alpha^{(N-2)},Y^{(N)}),$$

We want to prove that $\lim_{N\to\infty} N\mathbb{E}\Delta_j(\alpha^{(N)},\alpha^{(N-2)},Y^{(N-1)})\Delta_j(\alpha^{(N)},\alpha,Y^{(N)}) = 0$ and $\lim_{N\to\infty} N\mathbb{E}\Delta_j(\alpha^{(N-2)},\alpha,Y^{(N-1)})\Delta_j(\alpha^{(N)},\alpha^{(N-2)},Y^{(N)})=0$. Both equalities can be proved in a similar manner. We give some details for the first one. Using Lemma 4.2 and Lemma 4.4, we see that there exists a subset Ω_N such that $\lim_{N\to\infty} N\mathbb{P}(\Omega_N^c)=0$, on which

$$\left|F_j(\alpha^{(N)},Y^{(N-1)})-F_j(\alpha^{(N-2)},Y^{(N-1)})\right|\leq C\sum_{i=j}^{L-1} I\left(Y_i^{(N-1)},\alpha_i^{(N-2)},U_i^{(N-2)}\right),$$

with $U_i^{(N-2)}=\mathcal{U}_i^{(N-2)}(C_i,C_i',\alpha^{(N-2)})$ for some constant vectors C_i, C_i'. We may also assume that, for a given $\delta>0$, with a proper choice of Ω_N, we have, on Ω_N,

$$\left|\Delta_j(\alpha^{(N)},\alpha,Y^{(N)})\right|\leq C\sum_{i=j}^{L-1} I\left(Y_i^{(N)},\alpha_i,\delta\right)+\sup_{|\beta-\alpha|\leq\delta}|\phi_j(\beta)-\phi_j(\alpha)|.$$

We now condition with respect to $\sigma(Y^{(1)},\dots,Y^{(N-2)})$ to obtain

$$\mathbb{E}\left|F_j(\alpha^{(N)},Y^{(N-1)})-F_j(\alpha^{(N-2)},Y^{(N-1)})\right|\left|\Delta_j(\alpha^{(N)},\alpha,Y^{(N)})\right|\leq\rho(\delta)\mathbb{E}U_i^{(N-2)}+o(1/N),$$

where $\lim_{\delta\to 0}\rho(\delta)=0$. It follows that

$$\lim_{N\to\infty} N\mathbb{E}\left|F_j(\alpha^{(N)},Y^{(N-1)})-F_j(\alpha^{(N-2)},Y^{(N-1)})\right|\left|\Delta_j(\alpha^{(N)},\alpha,Y^{(N)})\right|=0.$$

It remains to show that

$$\lim_{N\to\infty} N\mathbb{E}\left|\phi_j(\alpha^{(N)})-\phi_j(\alpha^{(N-2)})\right|\left|\Delta_j(\alpha^{(N)},\alpha,Y^{(N)})\right|=0.$$

We can find Ω_N such that $\lim_{N\to\infty} N\mathbb{P}(\Omega_N^c)=O$, on which

$$\left|\phi_j(\alpha^{(N)})-\phi_j(\alpha^{(N-2)})\right|\leq C\sum_{k=j}^{L-1}\left|\alpha_k^{(N)}-\alpha_k^{(N-2)}\right|.$$

Here, we use the fact that ϕ_j is $\mathcal{C}^1$ near α. If $j=L-1$, the right handside can be estimated by C/N (on a suitable Ω_N). For $j<L-1$, it can be controlled by

$$\frac{C}{N}\sum_{n=1}^{N-2}\sum_{k=j+1}^{L-1} I\left(Y_k^{(n)},\alpha_k^{(N-2)},\alpha_k^{(N)}-\alpha_k^{(N-2)}\right),$$

and the expectation can be estimated in the same way as above.

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