

"IS THERE A MARRIAGE FUNCTION YET?"

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Abstract

The transmission dynamics of HIV/AIDS, and of sexually transmitted diseases (STDs) in general, is highly dependent on the population social/sexual mixing structure, i.e., how many partners and who they are. STD epidemic models require mathematical descriptions of mixing which must satisfy various natural constraints. A few such solutions to the homosexual mixing problem have been found, and there has been some confusion as to which is the “correct” one to use. In this note we encapsulate our solution to the mixing problem for homosexual and heterosexual mixing populations and show how it reduces the problem of choosing a mixing function to one of choosing parameters.

The development of predictive models of the transmission dynamics of HIV, gonorrhea, syphilis, etc. has been severely hampered by the lack of a mathematical framework allowing a realistic description of social/sexual mixing. This is true both of one-sex (homosexual) and two-sex (heterosexual) epidemics. The problem has been discussed in detail in the literature (Barbour, 1978; Nold, 1980; Anderson et.al., 1987, 1989; Blythe and Castillo-Chavez, 1989; Hyman and Stanley, 1989; Castillo-Chavez and Blythe, 1989; Jacquez et.al., 1989; Sattenspiel, 1989; Busenberg and Castillo-Chavez, 1989, 1991, Castillo-Chavez and Busenberg 1991, Castillo-Chavez *et al.* 1991, Blythe 1991, and Blythe *et al.* 1991) and arises because descriptions of mixing (who everyone's partners are) are specified only by a set of constraints (eg. Castillo-Chavez and Blythe, 1989). In a one-sex population with N distinct groups, if $p_{ij}(t)$ is the mixing function (fraction of partners taken by people in group i among those in group j at time t), then the constraints amount to making the $\{p_{ij}(t)\}$ a set of probabilities, conserving the rate of partnership acquisition between each pair of groups, and ensuring that no one can take partners from an empty group. Following Busenberg and Castillo-Chavez (1989, 1991), we describe this mixing/pair-formation framework using two continuously distributed variables; chronological age and risk. Risk is a measure of the degree of sexual activity.

The mixing, contact, or pair-formation function describes the proportion of "dates" between individuals in distinct groups or sexual partnerships or sexual contacts between these individuals per unit time. It can be generalized to include the geographical distribution or movement of individuals by interpreting it as the proportion of partnerships formed between individuals from clearly defined groups (socially, demographically, etc.) at a particular geographical location and then linking the local geographical heterogeneities through the specification of migration or movement matrices as was done in Sattenspiel (1987a, b) and Sattenspiel and Simon (1988). Applications to vector-host interactions and population genetics mating systems are also possible (see Blythe *et al.* 1991.) In this note, however, we concentrate on the characterization of localized mixing functions in the context, implicit or explicit, of sexually-transmitted diseases.

Consider the interactions of a single, socially-homogeneous group of individuals who are structured according to age = a ; and sexual activity or risk level = r . $T(r,a,t)$ denotes the total population density per unit age and activity, at time t , the sexual mixing is defined through the mixing function

$\rho(r,a,r',a') \equiv$ the proportion of partners of an (r,a) individual (i.e., a person of activity level r at age a), with (r',a') individuals,

and the sexual activity is specified by the nonnegative function

$C(r,a, W(T(.,.,t))) \equiv$ the expected or average number of partners per unit time of an (r,a) individual given that the "effective" population size is $W(T(.,.,t))$ at time t .

The following natural conditions characterize the mixing function, that is, the probability that a

“typical” (a, r) -individual mixes with a “typical: (a', r') given that they have mixed:

- (i) $\rho \geq 0$,
- (ii) $\int_0^\infty \int_0^\infty \rho(r, a, r', a', t) dr' da' = 1$, if $C(r, a, W(T(\cdot, \cdot, t)))T(r, a, t) \neq 0$,
- (iii) $\rho(r, a, r', a', t)C(r, a, W(T(\cdot, \cdot, t)))T(r, a, t) = \rho(r', a', r, a, t)C(r', a', W(T(\cdot, \cdot, t)))T(r', a', t)$,
- (iv) $C(r, a, W(T(\cdot, \cdot, t)))T(r, a, t)C(r', a', W(T(\cdot, \cdot, t)))T(r', a', t) = 0 \Rightarrow \rho(r, a, r', a', t) = 0$.

Conditions (i) and (ii) imply that ρ is a conditional probability density, condition (iii) states that the rate at which (r, a) individuals mix with (r', a') individuals equals the rate at which (r', a') individuals mix with (r, a) individuals (all of this per unit of age), and condition (iv) expresses the fact that there is no mixing in the age and activity levels where there are no active individuals, that is, on the set

$$\mathfrak{S} = \{(r, a, r', a') : C(r, a, W(T(\cdot, \cdot, t)))T(r, a, t)C(r', a', W(T(\cdot, \cdot, t)))T(r', a', t) = 0\}.$$

To consider mixing functions ρ , which are Dirac delta functions (see Blythe and Castillo-Chavez 1989) or more general distributions, we modify properties (i) and (iv) by choosing appropriate spaces of test functions. whose generic elements we denote by f , replacing them with:

- (i') $\rho \geq 0$ in the sense of distributions; i.e.,

$$\int_0^\infty \int_0^\infty \rho(r, a, r', a') f(r', a') dr' da' \geq 0 \text{ for all } f \geq 0, \text{ and}$$

- (iv') $\rho = 0$ on a set \mathfrak{S} , which means

$$\int_{\mathfrak{S}} \int \rho(r, a, r', a') f(r', a', r, a) dr' da' dr da = 0 \text{ for all } f.$$

In writing the conditions characterizing ρ we have suppressed, for notational convenience, their dependence on t and T . In Busenberg and Castillo-Chavez (1989) the following result was established:

Theorem 1. The only separable mixing function ρ satisfying conditions (i)–(iv) is the total proportionate (that is, random mixing in age and risk) mixing function $\bar{\rho}$ given by (1).

$$\bar{\rho}(r, a, r', a', t) = \frac{C(r', a')T(r', a', t)}{\int_0^\infty \int_0^\infty C(u, v)T(u, v, t) du dv} . \quad (1)$$

Busenberg and Castillo-Chavez (1991) characterized all solutions to conditions (i) – (iv) through

the following representation theorem:

Theorem 2. Let $\phi: \mathbb{R}_+^4 \rightarrow \mathbb{R}$ be measurable and jointly symmetric: $\phi(r, a, r', a') = \phi(r', a', r, a)$, and suppose that

$$\int_0^\infty \int_0^\infty \bar{\rho}(r', a') \phi(r, a, r', a') dr' da' \leq 1,$$

and

$$\int_0^\infty \int_0^\infty \bar{\rho}(r, a) \left(\int_0^\infty \int_0^\infty \bar{\rho}(r', a') \phi(r, a, r', a') dr' da' \right) dr da < 1.$$

Let

$$\rho_1(r, a) = 1 - \int_0^\infty \int_0^\infty \bar{\rho}(r', a') \phi(r, a, r', a') dr' da', \quad (2)$$

then

$$\rho(r, a, r', a') = \bar{\rho}(r', a') \left[\frac{\rho_1(r, a) \rho_1(r', a')}{\int_0^\infty \int_0^\infty \bar{\rho}(r', a') \rho_1(r', a') dr' da'} + \phi(r, a, r', a') \right] \quad (3)$$

is a mixing function. Conversely, for every mixing function ρ there exists a ϕ that satisfies the hypotheses of the theorem such that ρ is given by (2) with ρ_1 defined by (3). Time-dependence on ϕ has been suppressed. In general ϕ will be given by an explicit or implicit function of time.

A variety of special mixing functions have been used in the past; examples in the context of STD's models are found in Blythe and Castillo-Chavez (1989), Castillo-Chavez and Blythe (1989), Hyman and Stanley (1989), Castillo-Chavez *et al.* (1991), and others. Particular mixing functions have been used extensively in age-structured epidemic models (see Dietz and Schenzle 1985, Anderson and May 1985, Castillo-Chavez *et al.* 1988, 1989, and references therein.) Theorem 2 provides the relationship between all these mixing solutions and the underlying mixing assumptions. A further subdivision between pair-formation models (see Dietz and Haderer 1988, Castillo-Chavez *et al.* 1991, and references therein) and contact models (see Hethcote and Yorke and references therein) has also been emphasized. Theorem 2 allows for the incorporation of contact and pair-formation mixing functions within a single modeling framework (see Castillo-Chavez *et al.* 1991); Equation (3) reduces the mixing problem to that of estimating ϕ . This estimation problem may be as technically difficult as the original mixing problem, however, it is conceptually simpler. In the remainder of this note, we illustrate the flexibility of Equation (3) by showing how some of the most popular, and a few new mixing functions, fit into this framework.

(I) Proportionate mixing in the age variable only.

Individuals in the (r, a) class, when choosing partners, do not show preference for any age group, that is, the proportion of contacts of an (r, a) individual (per capita of active population) with

(r', a') individuals is of the general form at time t ,

$$\tilde{\rho}(r, a, r') \frac{C(r', a')T(r', a', t)}{\int_0^\infty C(r', a')T(r', a', t)da'} ,$$

where $\tilde{\rho}(r, a, r')$ is a function of three variables that is only constrained by axioms (i) – (iv).

(II) Proportionate mixing in both the age and partner variables .

$$\rho(r, a, r', a') = \tilde{\rho}(r, a, a') \frac{C(r', a')T(r', a')}{\int_0^\infty \int_0^\infty C(r', a')T(r', a')da'dr'} .$$

(I) and (II) assume that the persons selecting partners have criteria of selection which depend on the class to which they belong. Situations where this is not the case are found in Busenberg and Castillo-Chavez (1991).

(III) Assortative or like-with-like mixing on only one of the variables (risk or age).

$$\rho(s, r) = (1-\alpha) \frac{C(r)T(r)}{\int_0^\infty C(u)T(u)du} + \alpha \delta(s-r).$$

Here δ denotes the Dirac delta function (see Castillo-Chavez and Blythe 1989). Assortative mixing is given as a the convex linear combination of the Dirac delta (a mixing function) and proportionate mixing.

A large class of mixing functions is given by the following corollary to Theorem 3.

Corollary: Let $\phi \geq 0$, $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}^+$ be jointly even: $\phi(r, a) = \phi(-r, -a)$, and suppose that for some $\alpha > 0$,

$$\alpha \int_0^\infty \int_0^\infty \bar{\rho}(r', a') \phi(r-r', a-a') dr' da' < 1, \quad \text{for } r, a \in [0, \infty) .$$

Then

$$\rho(r, a, r', a') = \bar{\rho}(r', a') \left[\frac{\rho_1(r, a) \rho_1(r', a')}{\int_0^\infty \int_0^\infty \bar{\rho}(r, a) \rho_1(r, a) dr da} + \alpha \phi(r-r', a-a') \right] \quad (4)$$

is a mixing function, where

$$\rho_1(r,a) = 1 - \alpha \int_0^\infty \int \bar{\rho}(r',a') \phi(r-r',a-a') dr' da'.$$

Equation (4) includes the mixing functions used by Blythe and Castillo-Chavez (1989), Castillo-Chavez and Blythe (1989), and Hyman and Stanley (1989). Equation (4) models like-with-like or assortative mixing by age and risk.

Several mixing matrices (discrete number of groups) have also been found. Four solutions $p_{ij}(t)$ to the one-sex discrete (multigroup) mixing problem are described in Table I. The earliest, best known and most widely used is ((A) in Table I) proportionate or random mixing (see Barbour 1977, Nold 1980; Hethcote and Yorke 1984; and references therein), where mixing among groups occurs in proportion to the total number of partnerships formed by all the people in each group (also see Equation 1.)

In "preferred mixing" ((B) in Table I, the $p_{ij}(t)$ are formed by having the members of each group reserve a constant fraction of their partners among themselves, the rest being spread randomly over all groups (see Nold, 1980; Hethcote and Yorke, 1984; Jacquez et.al., 1988). Koopman et.al. (1989) also present a mixing model ((C) in Table I) which includes Morris' (Skökloster Workshop 1990) solution (set $c_i = c$ for all i and f_{ij} is a set of appropriate constants.) The fourth solution, ((D) in Table I) is generalized mixing, that is, the one-variable discrete version of Equation (3). Naturally all other discrete solutions may be written in this form, i.e., as multiplicative perturbations of random mixing. The key to (D) is the set of parameters $\{\phi_{ij}\}$. If the $\{\phi_{ij}\}$ are constants, then since they determine the nature of the multiplicative perturbation, they prescribe, in some sense, the degree of affinity (ϕ budgets partner acquisition rates) between individuals of groups i and j (see Blythe *et al.* 1991.)

Explicit relationships between (D) and other solutions require finding a set of $\{\phi_{ij}\}$ which recover them. The third column in Table I lists appropriate $\{\phi_{ij}\}$ for the various solutions. The relationship is non-unique, there exist many distinct sets of $\{\phi_{ij}\}$, which will recover any particular solution. This is particularly clear for proportionate mixing. The equivalent $\{\phi_{ij}\}$ for "preferred mixing" (B) are frequency-dependent and hence time-dependent. This is not surprising as the reserved fractions do not depend on group population sizes. If the general solution to the mixing problem is assumed to have the form $p_{ij}(t) = \theta_{ij}(t)\bar{p}_j(t)$ (θ , unspecified), then (C) can be written as

$$\frac{f_{ij}}{\sum_{k=1}^N f_{ik}\bar{p}_k} = \theta_{ij}(t). \quad (5)$$

then (5) is in this very general sense a representation of all solutions. However, this representation makes the $\{f_{ij}\}$ time-dependent in an arbitrary manner (e.g. Koopman et.al., 1989).

The source of the problem may be made abundantly clear as follows. Suppose (as do Koopman

et.al., 1989, and Morris, Skökloster Workshop, 1990.) that sexual interaction is comprised of two separate processes. First, individuals mix socially, so that we have a "pre-cursor" social mixing matrix, $\{p_{ij}^{(1)}\}$ say. Then, individuals who have met decide whether or not to have sex, according to some set of rules which may be summed up in some matrix. Combining these two processes produces a sexual mixing matrix, say $\{p_{ij}^{(2)}\}$. This has two effects: first, the number of sexual contacts per unit time for each individual becomes, in general, a function of time (less than or equal to the number of social contacts in the "pre-cursor" mixing); this presents no problems. However, arbitrary choice of sexual acceptance parameters has a second effect, namely that because such a description does not take account of group-size changes with time (or does so at best in an ad hoc manner), the overall mixing process $\{p_{ij}^{(2)}\}$ in a sense decouples individual behavior from population dynamics. For example, the implication is that if a group i and a group j person meet, their overall probability of becoming sexual partners is unrelated to the abundance or scarcity of individuals in those, or any other, groups (or related in an ad hoc manner, c.f. Koopmen et.al., 1989).

We feel that selection of sexual partners from social contacts will depend on the relative abundance of perceived groups; for example, people becoming more selective if a desirable group is abundant, and vice versa. If we, for example, assume constant $\{\phi_{ij}\}$, then we establish an underlying, invariant structure of mutual absolute preferences or affinities amongst all the groups in the population and, consequently, constant ϕ -matrix gives a large class of useful and biologically transparent mixing functions because Equation (D) in Table I can be interpreted easily. Individuals in group i have a total rate of pair-formation $C_i T_i(t)$ or $C_i(t) T_i(t)$ which they budget among all groups. $C_i T_i(t) \phi_{ij}(t) p_{ij}(t)$ is budgeted for group- j mixing ($j=1,2,3,..N$) while the remainder $C_i T_i(t) [1 - R_i]$ is budgeted at random (proportionate mixing) among all groups (see Blythe *et al.* 1991) and, consequently, the matrix ϕ provides a measure of social/sexual affinity between groups. These considerations motivate the following definition.

A heterogeneous sexually-active mixing population has a fixed or invariant affinity mixing structure if ϕ is a matrix of constants.

We cannot generate all possible social/sexual mixing structures through Equation (D) with constant matrix ϕ . However, changes in activity (the $\{c_i(t)\}$) and/or group sizes (the $\{T_i(t)\}$) have no effect on this absolute, albeit not completely general, affinity structure.

The empirical approach of Gupta et al. (17), where the $\{p_{ij}(t)\}$ are held constant at their initial ($t=0$) values by judiciously altering the group activity levels ($\{c_i(t)\}$ in Table I), does maintain an invariant social structure (constant distance from proportionate mixing), but only at the cost of altering the proportionate mixing functions ($\{\bar{p}_i(t)\}$ in Table I) themselves through their assumption of adaptive sexual behavior change.

In the absence of social/medical factors which cause a change in absolute preference, individuals

in the population have some ranking of other individuals a priori as regards desirability for forming sexual partnerships (an apriori fixed budgeting of pair-formation rates among groups through the ϕ -matrix), then a constant $\{\phi_{ij}\}$ description should be (at least approximately) correct and, Equation (3) will provide a large class of useful mixing matrices. The alternative representation (using $\{f_{ij}\}$) is fine if the $\{f_{ij}(t)\}$ are known for all time but in that case no advantage is to be gained from not using (D) directly. In the apparently beguiling case where the $\{f_{ij}\}$ are constructed from behaviors in a succession of mixing steps (à la Koopman et.al., 1989; Morris Sköklöster Workshop 1990), the process of decoupling the steps (e.g. meet, talk, have sex) removes any invariance in the underlying structure of preference for sexual partners per se. The implications of this approach for parameter estimation are discussed in Blythe, Castillo-Chavez, and Casella (1991.)

Finally, we use the set of mixing probabilities $\{p_{ij}(t) \text{ and } q_{ji}(t): i = 1, \dots, L \text{ and } j = 1, \dots, N\}$ to describe the mixing/pair formation in a heterosexually active population. We let

$p_{ij}(t)$: fraction of partnerships of males in group i with females in group j at time t ,

$q_{ji}(t)$: fraction of partnerships of females in group j with males in group i at time t ,

$T_i^m(t)$: male population size of group i at time t ,

$T_j^f(t)$: female population size of group j at time t .

c_i : average (constant) number of female partners per unit time of males in group i , or the i^{th} -group rate of (male) pair-formation,

b_j : average (constant) number of male partners per unit time of females in group j , or the j^{th} -group rate of (female) pair-formation,

and define a two-sex mixing matrix as follows:

Def $(p_{ij}(t), q_{ji}(t))$ is called a mixing/pair-formation matrix iff it satisfies the following properties (at all times):

$$(A1) \quad 0 \leq p_{ij} \leq 1, \quad 0 \leq q_{ji} \leq 1,$$

$$(A2) \quad \sum_{j=1}^N p_{ij} = 1 = \sum_{i=1}^L q_{ji},$$

$$(A3) \quad c_i T_i^m p_{ij} = b_j T_j^f q_{ji}, \quad i = 1, \dots, L, \quad j = 1, \dots, N.$$

$$(A4) \quad \text{If for some } i, 0 \leq i \leq L \text{ and/or some } j, 0 \leq j \leq N \text{ we have that } c_i b_j T_i^m T_j^f = 0, \text{ then we define } p_{ij} \equiv q_{ji} \equiv 0.$$

All solutions to the two-sex mixing problem can be represented as ϕ -perturbations (see Castillo-Chavez and Busenberg 1991) of the Ross solutions (familiar from the study of vector-host interactions see Roos 1911 and Lotka 1925):

$$\bar{p}_j = \frac{b_j T_j^f}{\sum_{i=1}^L c_i T_i^m}, \quad \bar{q}_i = \frac{c_i T_i^m}{\sum_{j=1}^N b_j T_j^f}; \quad j = 1, \dots, N \quad \text{and} \quad i = 1, \dots, L.$$

Ross solutions (and their ϕ -multiplicative perturbations) make the relationship between pair-formation and contact models clear (see Castillo-Chavez *et al.* 1991.) The general representation solution for heterosexual populations allows for the systematic generation of mixing or pair-formation or mating solutions which we are currently studying. The main objective of this note is to point out these formulae and their relationship to the current theory hoping to increase interest on the study of the role of pair-formation in epidemiology, demography, sociology, and genetics. The mixing problem, eloquently expressed by Parlett in the title of his 1972 article: is there a marriage function?, and studied by many theoreticians, remains still open

Useful solutions to this question are of theoretical and practical value. The role of social structure in disease dynamics needs to be seriously studied in light of the devastating HIV/AIDS epidemics. Long-term forecasting of HIV prevalence cannot be provided seriously without clear understanding of social dynamics. At present, we would suggest that while there is a great deal of choice for solutions of the mixing axioms, for descriptive modeling or parameter estimation the general representation solution (D) is the only one that makes sense under all contingencies.

Table I
Relationship Between Mixing Functions

Mixing function	$p_{ij}(t)$	$\phi_{ij}(t)$
(A) Proportionate mixing ^a	$\bar{p}_j(t) \equiv \frac{c_j(t)T_j(t)}{\sum_{k=1}^n c_k(t)T_k(t)}$	$\alpha, \text{ all } i, j$ $0 < \alpha < 1$
(B) Preferred mixing ^b	$\delta_{ij}f_i + (1-f_i) \frac{(1-f_j)\bar{p}_j(t)}{\sum_{k=1}^n (1-f_k)\bar{p}_k(t)}$	$\delta_{ij}f_i/\bar{p}_i(t)$
(C) Koopman et.al.	$\frac{f_{ij}\bar{p}_j(t)}{\sum_{k=1}^n f_{ik}\bar{p}_k(t)}$	$f_{ij} \equiv F_{ij} \text{ in (C)}$
(D) General solution	$\bar{p}_j \left[\frac{R_i(t)R_j(t)}{\sum_{k=1}^n \bar{p}_k(t)R_k(t)} + \phi_{ij}(t) \right]$	Any $\phi_{ij}(t)$ such that $R_i(t) \geq 0$, all i and t , $\phi_{ij}(t) = \phi_{ji}(t)$, all i, j, t .
	$R_i(t) = 1 - \sum_{k=1}^N \bar{p}_k(t)\phi_{ih} \quad , \text{ all } i$	

Particular solutions (A-C) and the general solution (D) to the N-group one-sex mixing problem. The $\{p_{ij}(t)\}$ is the mixing matrix or function itself (explanation in text), and $\phi_{ij}(t)$ are parameters used in the General Solution to recover the particular solutions.

^a $c_i(t)$ and $T_i(t)$ are respectively the number of new partners taken by an individual per unit time and the total population of group i .

^b $\delta_{ij} = 1$ if $i = j$, zero elsewhere

References

Anderson, R. M. and May, R. M. (1984). Spatial, temporal and genetic heterogeneity in host populations and the design of immunization programmes. *IMA J. of Math. Applied in Med. and Biol.* 1: 233-266.

- Anderson, R.M. and R.M. May. (1987). Transmission dynamics of HIV infection. *Nature* 326, 137-142.
- Blythe, S.P. (1991) Heterogeneous sexual mixing in populations with arbitrarily connected multiple groups. *Math. Pop. Stud.* (to appear).
- Blythe, S.P. and Castillo-Chavez, C. (1989) Like-with-like preference and sexual mixing models. *Math. Biosci.* 96, 221-238 (1989).
- Busenberg, S. and Castillo-Chavez, C. (1989) Interaction, pair formation and force of infection terms in sexually transmitted diseases. In (C. Castillo-Chavez, ed.) *Mathematical and Statistical Approaches to AIDS Epidemiology*. Lecture Notes in Biomathematics 83, 289-300. Springer-Verlag.
- Castillo-Chavez, C. and Blythe, S.P. (1989) Mixing framework for social/sexual behavior. In (C. Castillo-Chavez, ed.) *Mathematical and Statistical Approaches to AIDS Epidemiology*. Lecture Notes in Biomathematics 83, 275-288. Springer-Verlag.
- Castillo-Chavez, C., Busenberg, S. and Gerow, K. (1991) Pair formation in structured populations. In: *Differential Equations with Applications in Biology, Physics and Engineering* (J. Goldstein, F. Kappel, W. Schappacher, eds.), Marcel Dekker, New York (in press).
5. Dietz, K. and Hader, K.P. Epidemiological models for sexually transmitted diseases. *J. Math. Biol.* 26, 1-25 (1988).
6. Anderson, R.M., S.P. Blythe, S. Gupta, and E. Konings. The transmission dynamics of the Human Immunodeficiency Virus Type 1 in the male homosexual community in the United Kingdom: The influence of changes in sexual behavior. *Phil Trans R Soc London B* 325, 45-89 (1989).
7. Busenberg, S. and Castillo-Chavez, C. On the role of preference in the solution of the mixing problem, and its application to risk- and age- structured epidemic models. (Submitted to *IMA J. of Math. Applic. to Med. and Biol.*, 1989).
8. Barbour, A.D. Macdonald's model and the transmission of bilharzia. *Trans. Roy. Soc. Trop. Med. Hyg.* 72, 6-15 (1978).
9. Dietz, K. and Schenzle, D. Proportionate mixing models for age-dependent infection transmission. *J. Math. Biol.* 22, 117-120 (1985).
10. Nold, A. Heterogeneity in disease-transmission modeling. *Math. Biosci.* 52, 227-240 (1980).
11. Hethcote, H.W. and Yorke, J.A. *Gonorrhea, Transmission Dynamics, and Control*. (Lecture Notes in Biomathematics 56. Springer-Verlag (1984).
12. Jacquez, J.A., Simon, C.P., Koopman, J., Sattenspiel, L., Perry, T. Modeling and analyzing HIV transmission: the effects of contact patterns. *Math. Biosci.* 92, 119-199 (1988).
13. Hyman, J.M. and Stanley, E.A. The effect of social mixing patterns on the spread of AIDS. In (C. Castillo-Chavez, S.A. Levin and C. Shoemaker, eds.) *Mathematical Approaches to Problems in Resource Management and Epidemiology*. Lecture Notes in Biomathematics 81, 119-190, Springer-Verlag (1989).

14. Blythe, S.P., Castillo-Chavez, C., and Casella, G. Empirical methods for the estimation of the mixing probabilities for socially structured populations from a single survey sample. Submitted to *J. of AIDS* (1990).
 17. Gupta, S., Anderson, R.M., May, R.M. Network of sexual contacts: implications for the pattern of spread of HIV. *AIDS* 3, 1-11 (1989).
- Morris, M. (1990) "An integrated framework for modelling the role of selective mixing in the spread of AIDS, " lecture at the Skökloster Workshop, Spread of Epidemics: Stochastic Modelling and Data Analysis.
18. Sattenspiel, L. Population structure and the spread of disease. *Human Biology* 59, 411-438 (1987).
 19. Sattenspiel, L. Epidemics in nonrandomly mixing populations: a simulation. *American Journal of Physical Anthropology* 73, 251-265 (1987).
 20. Sattenspiel, L. and Simon, C.P. The spread and persistence of infectious diseases in structured populations. *Math. Biosci.* 90, 341-366 (1988).
 21. Jacquez, J.A., Simon, C.P. and Koopman, J. Structured mixing: heterogeneous mixing by the definition of mixing groups. *Mathematical and Statistical Approaches to AIDS Epidemiology* (C. Castillo-Chavez, ed.) Lecture Notes in Biomathematics 83, 301-315. Springer-Verlag (1989).
 22. Sattenspiel, L. and Castillo-Chavez, C. Environmental context, social interactions, and the spread of HIV. *American J. of Human Biology*, Vol. 2, No. 4 (in press).
- Castillo-Chavez, C. and Busenberg, S. On the solution of the two-sex problem. In: *Proceedings of the Interantional Conference on Differential Equations and Applications to Biology and Population Dynamics* (S. Busenberg and M. Martelli, eds.), Lecture Notes in Biomathematics, Springer-Verlag (in press).
- Parlett, B. (1972) Can there be a marriage function? In (Greville, T. N. E., ed.), *Population Dynamics*. Academic Press, New York London: 107-135.
- Koopman, J.S., Simon, C.P., Jacquez, J.A. and Tae Sung Park (1989) Selective contact within structured mixing with an application to HIV transmission risk from oral and anal sex. In (C. Castillo-Chavez, ed.) *Mathematical and Statistical Approaches to AIDS Epidemiology*. Lecture Notes in Biomathematics 83, 316-348. Springer-Verlag
- Lotka, A. J. (1923). Contributions to the analysis of malaria epidemiology. *Amer. J. Hygiene* 3. Jan. Sppl.
- Ross R. (1911) . *The prevention of malaria. (@nd edition, with Addendum)*. John Murray, London.