CHAPTER VII.

KINEMATIC NOTATION.

§ 52.

Necessity for a Kinematic Notation.

THE investigations concluded in the last article have conducted us again through the lower and higher pairs of elements to the kinematic chain, the form which, as we have already seen (§ 3), represents the general solution of the machine problem. What we found before to follow directly from simple fundamental propositions, we have now been led to a second time, indirectly, in the course of natural development, and we have seen further the employment of the kinematic chain extending rapidly in all directions. The glance which we have taken at the history of machine development has shown us the course of the mental processes which have produced the chain, and by a continuation of which—we may suppose—it may be still further improved and applied. We must now turn to the direct consideration of the thing itself.

Such an immense variety of cases—existing and possible—here present themselves, that it becomes increasingly difficult to comprehend them all. This difficulty presents itself specially in the indication by names of their separate characteristics, and in distinguishing between cases which ought to be separated; and it appears likely to become still greater in the future with the increase in the variety of chain-forms employed. It has become, too, equally necessary to be able to survey the inner relationships of mechanisms as well as their differences. We are here led involuntarily to look for some means of facilitating the expression of both.

In similar circumstances Mathematics, and afterwards Chemistry, have taken to their aid special symbolic notations, which have now become so essential to both sciences that neither could proceed without them. Both adopted them so soon as the real nature of their fundamental operations had been determined. The ideas connected with our subject are now so distinctly and individually before us, their mutual relations can be so definitely determined, that their concise expression by means of simple signs becomes not only justifiable but practicable. We shall therefore use these important aids to the furthest extent possible in our work.

It is very easy to see what an immense advantage there is in the possibility of so expressing a complex idea that when it is employed along with another of the same kind they may both be expressed by a single sign. The continual returning upon already defined conditions becomes unnecessary, while the conciseness of the expression allows conclusions to be arrived at as to the mutual relations of the parts combined, which with the common method of expression can only be formed with great difficulty, and can scarcely be communicated at all. The reader need not fear that any continual alteration of his accustomed ideas will be demanded from him in making himself familiar with the system of contractions which we are about to describe. For a scientific symbolic notation is in essence nothing else than a systematised method of contraction, it is not a hieroglyphic system, mysterious to the uninitiated. Our examination of it here will not be simply parenthetical, but will give us opportunity for examining more closely the real nature of several important kinematic chains.

§ 53.

Former Attempts.

Attempts have not been wanting to express machine combinations in some concise form. Clockmakers, among others, and writers upon subjects connected with horology, have employed a kind of notation for showing the sequence of the wheels and arbors in clockwork. Willis has entered somewhat closely into this method of symbolization. The following, for example, shows the arrangement of the wheel-work of a common clock in a form which he himself adopted :—

```
48
 6—————45
         6—————————30
```

Here the numerals stand for numbers of teeth, the lines indicate the connection of two wheels by an arbor, the placing of one figure over another shows that the corresponding wheels gear together. Putting the names of the wheels beside the numbers of their teeth we should have :—

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Great Wheel 48
      Pinion 6—————45 Second wheel
                  Pinion 6——————————30 Balance-wheel.
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Other writers have used methods somewhat differing from this.* It is evident, however, that the object here is the representation of a portion only of an isolated case,—and even that portion is not intended to include the general kinematic nature of the spur-gearing, but to cover merely the indication of its velocity-ratios, a very important matter, of course, in itself.

The method proposed by Mr. Babbage "for expressing by signs the action of machinery" was more important than these, and was indeed intended to be quite general in its application. Babbage, to whom no doubt the subject was suggested during the extremely difficult construction of his calculating machine, described his system in a small book, not much known,† in which he has illustrated it by two large examples,—a clock with working and striking trains and a hydraulic ram. His method is as follows :—The names of the whole of the moving parts are first put down in order, and then signs are placed in tabular form beside the name of

* Willis instances the following for the case supposed :—

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Oughtred (1677)      Derham (1696)           Allexandre (1735).
 30                  48)6—45)6—30             48
  6)45                                        45—6
    6)48                                      30—6
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† *A Method of Expressing by Signs the Action of Machinery*, by C. Babbage. London, 1826.

each part, to indicate its motion. The symbols employed are arrows of various kinds, full lines, dotted lines, brackets, crosses, hooks, and so on. It is certainly possible to interpret the action of the machine by the aid of the signs when the meanings of these have once been completely mastered. Notwithstanding this the method has never been used. No notice was taken of it by those practically interested in machinery, and by this want of attention they added unconsciously to the great irritation which displayed itself in the work which Babbage published shortly before his death. In this he struck about him most vehemently, like Timon of Athens with his spade, accusing his contemporaries of their want of comprehension and appreciation of his work. Without in the least depreciating, however, his most important labours in other directions, it must be said that the cause of the non-acceptance of his system of notation was due to its own defects, and not to those of the public.

What the symbolic memoranda of Babbage express, and were intended to express, is not the essential constitution of the machine, its different parts scientifically defined and recognizably indicated by the stenographic symbols, but merely the general nature of the motion of those pieces which were themselves described at length or by their names in the usual manner. We learn whether such and such a piece turns backwards or forwards, moves continuously or discontinuously, uniformly or with varying velocity, and in cases where there is turning about axes we have the velocity-ratio and so on, given. It is at once evident, however, that under this system mechanisms of completely different constructions might be represented by one and the same set of symbols. These extend merely to the external conditions accompanying certain characteristics of the single organs, not to their full meaning; they form simply a concise description of the action of the machine, not in any way showing its dependence upon general fundamental principles. If the symbols proposed by Mr. Babbage were placed upon the necessary drawing of the machine itself, they would express their meaning much more clearly than when used in the more abstract form of a table.*

* In a small pamphlet of half-a-dozen pages published in 1857 Mr. Babbage again proposed a very complicated "Mechanical Notation," no doubt the offspring of his own requirement in connection with his machines; but here he appears to have

For our purpose—the representation of kinematic chains by symbols,—Babbage's method is of no service; I therefore pass over Willis' attempts to make it more useful by certain alterations.*

§ 54.

Nature of the Symbols required.

The object of the kinematic notation which we wish to form is, like that of mathematical symbols, to express certain operations performed with, or supposed to be performed with, the bodies indicated by signs or otherwise; partly also its province is similar to that of chemical notation, for it must afford information, and indeed somewhat full information, as to the quality of the thing named. The symbols for kinematic bodies must not, therefore, be in themselves meaningless like those of Mathematics, where different letters indicate only the variations in magnitude of known,—and so far as their measurability is concerned similar,—things; but each letter must stand, as in Chemistry, for a particular class of bodies, the differences between the classes being here their geometrical properties. The letter must therefore stand for the name of the body—that is, of the kinematic element, the definite characteristics of which are sufficiently indicated by that name. The letter used in this way we shall call the class- or name-symbol of the element.

The sign for the general name of a kinematic element, as *e.g.* the sign for "screw,ô" "revolute,"† "prism," and so on, is seldom sufficient by itself. Most frequently some further indication is required as to the form of the body, as for example whether the screw be external or internal, that is whether the screw-spindle or the nut be meant. The geometrical basis figure is the same in both cases, but there is a great difference between the forms in which it is used. Signs serving to indicate this more exactly, which will be used in connection with the name-symbols, we shall call form-symbols.

In addition to these two classes of symbols, a third kind is

intended that the letters and symbols should be put on the drawings themselves, as Prof. Reuleaux suggests.

* Willis: *Principles of Mechanism*, p. 343. (2nd. Ed. 292.)

† See page 91.

required, its object being to show the mutual relation between two or more elements of a mechanism; whether two neighbouring elements be paired or linked together: if the latter, what their relative geometric position is; whether a link be fixed or movable —its relation, that is, to surrounding space—and so on. Signs for these purposes we shall call symbols of relation.

The more exactly and explicitly the signs explain the kinematic elements and their chaining the better will they serve the purpose for which they are intended. We shall, however, be content with a certain degree of completeness in order to avoid diffuseness; in all cases, however, the signs will express the real and general nature of the thing symbolized.

§ 55.

Class or Name-Symbols.

In choosing symbols to indicate the class to which, considered kinematically, a particular body belongs, we shall follow the example set us in chemical notation, and use Roman capital letters, making these where possible the initial letter of the name of the class. The following twelve signs will be used to stand for the bodies whose names are placed after theme—

S Screw,
R Solid of Revolution (Revolute),
P Prism,
C Cylinder,
K Cone (Konus),
H Hyperboloid,
G Sphere (Globe),
A Sector or sweep (Arc),
Z Tooth or projection,
V Vessel or chamber,
T Tension-organ,
Q Pressure-organ.

It may appear remarkable that the number of classes of elements is so small. In fact, however, forms which can be called kinematic all lie within such a limited circle that a greater number of signs is not required, and it is advisable in all cases to be content with as few signs as possible. The letters have been chosen with care so as—so far as possible—to suggest the form for which they stand, and also to be available in other European languages than our own. I can also say from experience that the recollection of the signs is no great tax upon the memory.

§ 56.

Form-Symbols.

In choosing signs suitable for kinematic form-symbols we are struck by the existence of a certain insufficiency, for our purposes that is, in some very usual geometrical ideas or methods. In geometry the name given to a body of a certain form is that of a portion of space limited by the same figure. In general it is the portion of space inclosed within this figure which is considered to be the form of the body having the same name. Evidently there is here a certain indefiniteness; because the two portions of space, the one outside and the other inside the figure, cannot both be meant at the same time.

For our purposes, however, it is necessary to distinguish between these two, the portion of space inclosed by the figure, and the portion inclosing it. If between two parallel planes, for instance, there be a circular cylinder, its axis at right angles to them, then the space inclosed between the planes and inside the cylinder must be distinguished from that which is between the planes but outside the cylinder;—in other words we must know upon which side of the cylindrical surface the material forming our element is placed. We call the body inclosed by the cylinder a full cylinder, and that in which it is inclosed an open cylinder. We shall use for the form-symbols of full and open* bodies respectively the ordinary signs for plus and minus.

The plane limitation of a solid of revolution requires also a sign. It lies equally between the limits + and −, and therefore may suitably be indicated by zero.

For curved profiles, that is profiles which are neither rectilinear nor circular, we may use the circumflex;—we therefore have

$^{+}$for full bodies	^{0} for plane bodies
$^{-}$for open bodies	$\hat{}$ for bodies having curved profiles.

These form-symbols will be placed above and to the right of the class-symbols (excepting the circumflex, which will be placed over the letter to which it refers), and in a smaller type. Thus

* I use these words as being at the same time shorter and more expressive than the commoner ones solid and hollow.

for instance we may use the following symbols for the forms named:—

C^+ Full cylinder, C^- Open cylinder,
S^+ Screw spindle, S^- Nut,
K^+ Full cone, K^- open cone,
K^0 Plane cone (Cone having a vertex angle of 180°.)
$\tilde{C}$ Cylinder upon a general curvilinear base,
$\tilde{C}^+$ the same cylinder full,
$\tilde{C}^-$ the same open,
$\tilde{P}$ Prism upon a general curvilinear base.

We have chosen the symbol V for a vessel of any kind. By a suitable form-symbol we can make its meaning more definite in certain special cases,—we can use V^+ for the reciprocal of the vessel,—that is, a body touching it all round upon its inner surface, so that V^- will stand for the vessel itself. V^+, for instance, might represent a piston, V^- the cylinder in which it works.

Small letters similar to those of the class-symbols will also be used as form-symbols. They will be placed below and to the right of the former, and permit a distinct separation to be made between the forms of various elements, so as to shut out all meanings but the one intended. The following may serve as a few examples of this:—

C_z Cylindrical spur-wheel, and from it
C_z^+ Spur-wheel with external teeth,
C_z^- Do. with internal teeth, or annular wheel,
K_z^+ Bevel wheel with external teeth, K_z^0 face wheel,*
H_z^+ Hyperboloidal toothed wheel.
H_z^0 Hyperboloidal face wheel,*
$\tilde{C}_z^+$ Non-circular spur wheel with external teeth.
P_z Rack.
C_z^+ Cylindrical screw wheel.
T_p Prismatic tension-organ, such as a flat belt.
T_p^+, T_p^- the same moving respectively towards or from its pulley.
T_s Rope, T_c wire, T_z common chain, T_r jointed chain.

* See *Der Constructeur*, 3rd. ed. p. 435 and 451.

For pressure-organs we require to distinguish between gases and liquids. We may use the Greek letters λ and γ for this purpose, and thus have

Q_λ liquid pressure-organ, water, etc.

Q_γ gaseous pressure-organ, gas, air, steam, etc.

In certain cases the pressure-organ consists of more or less round grains, which may with sufficient accuracy be taken as spherical, and therefore we have

Q_g or more exactly Q_g for a pressure-organ consisting of more or less globular portions.

We shall form and use further compound symbols as we have occasion.

§ 57.

Symbols of Relation.

Of the relations which one element in a chain can have to another the most important are those of pairing and linkage. The first we may indicate by a comma. C, C will thus stand for two cylinders rolling together, C^+,C^+ would be used if both were full, C^-,C^+ if one were full and one open. We shall always presuppose that the comma indicates both the possibility and the existence of correct pairing. Thus we shall not require any sign beyond C^+, C^+ to show that the axes of the cylinders are parallel, —while C^-,C^- is incorrect, for it is impossible to form a kinematic pair from two open cylinders.

Linkage will be denoted by a dot or dotted line. $C^+ \ldots\ldots C$ for instance is a link having two full cylinders for the elements which it connects, $C^- \ldots\ldots C^-$ a link connecting two open cylinders or eyes.

The fixing of a link may be indicated by underlining the dotted lines. $\underline{P^+ \ldots\ldots C^+}$ for instance stands for a fixed link connecting a full prism and a full cylinder.

It may occasionally be necessary to indicate that a link is elastic,—namely that it is a spring,—in which case a wavy line may be placed over the dotted one $\widetilde{\ldots\ldots\ldots}$

A number of other signs are partly the same as the common arithmetical signs, and partly based upon them. They are as follow :—

$=$ equal, $>$ greater than, $<$ less than, ∞ infinite;
$|$ conaxial, $\|$ parallel, $\angle$ oblique, $\perp$ normal;
$\not\vdash$ crossed obliquely; $+$ crossed at right angles;
$\ddagger$ equal and con-axial, $\#$ equal and parallel;
$\cong$ coincident;
$\square$ conplane,— lying in the same plane,
Z anti-parallel (in a quadrilateral),
$\leq$ isosceles, or having adjacent arms equal (in a quadrilateral).

The relations expressed by these signs may exist either between the elements of a pair or the links of a chain. In the former case the comma between the two elements is omitted, and the symbol itself is printed in a smaller type than in the latter.

If a hollow cylinder be paired with a geometrically equal solid cylinder they are equal and con-axial, so that the comma would have to be replaced by the sign $\ddagger$. If however such bodies, being equal, are to be paired, they must also by necessity be con-axial, so special indication of that relation may be omitted without any loss of distinctness, and we may write the pair $C^+_= C^-$. If this be a closed turning-pair, the conditions as to the prevention of cross-motions by a proper sectional profile (§ 15) must be fulfilled. We shall here always presuppose that two elements, the symbols for which are connected by the sign for pairing, form a closed pair, unless the contrary be expressly stated. We shall see further on that in cases where they are not closed the notation of the chain itself always makes it possible to do this. The three lower pairs, then, twisting pair, turning pair, and sliding pair, have for their symbols :—

$$S^+_= S^- \qquad R^+_= R^- \qquad P^+_= P^-$$

The curved discs in the triangular, quadrilateral, etc., hollow prisms, (Chap. III), can be indicated generally by the formula $\bar{C}^+, \bar{P}^-$; they fall therefore in one and the same class of pairs. With respect to the simple turning pair, $R^+_= R^-$, in which the most various profile forms may be used so long as the pair-closure remains, it will be noticed that as far as the relative motions of its elements are concerned it does not differ from the closed cylinder pair $C^+_= C^-$. In most cases it is therefore allowable to write $C^+_= C^-$, instead of $R^+_= R^-$. The idea is somewhat simpler, the cylinder instead of the revolute, and in machines actually the special case C of the body R is almost always used. It is only in very special cases that we shall find it necessary to adhere to the strictly general notation.

Symbols of relation between the elements of a link will be placed in the dotted line. Thus $C^+ \ldots \| \ldots C^+$ stands for the linkage of two parallel full cylinders; $C^- \ldots \| \ldots C^-$ for a linkage of two parallel open cylinders, that is, a connecting rod; $C^+_z \ldots | \ldots C^+$ a spur wheel attached to its con-axial shaft; $C^+_z \ldots | \ldots C^-$ a spur wheel with a con-axial open (bored-out) boss.

A special indication is sometimes required for incomplete pairs. The first necessity is here a symbol for incompleteness, and for this we use the ordinary sign of division, so as to allow the method of closing to be indicated by a divisor.

To indicate merely the incompleteness of a pair we may use the divisor 2, considering the piece or element as halved. If it be completed by force-closure, the divisor f (force) may be employed. For closure by a kinematic chain we choose the divisor k; if the chain-closure occur by means of a spring we substitute for this l; and, lastly, if closure be effected by a pair (§ 47), we shall use the divisor p. We therefore have the following:—

$\frac{C^-}{2}$	a portion of an open cylinder,	
$\frac{C^+}{f}$	a full cylinder paired by force-closure,	
$\frac{C^+}{k}$	do.	paired by chain-closure,
$\frac{C^+}{l}$	do.	closed by a spring,
$\frac{C^+}{p}$	do.	closed by a pair of elements.

If any link used for chain-closure have a special indicating letter, as a, b, c, etc. (as we shall see to be sometimes the case), this also can be placed in the divisor so as to indicate distinctly the method of closure. We shall find further on frequent applications of these methods of symbolization.

§ 58.

Formulæ for simple Kinematic Chains and Mechanisms.

In describing a complete kinematic chain by symbols written, as they must be, in lines, we cannot represent the returning back upon itself, or closure, of the chain, and must be content with merely indicating this. This disadvantage, if it be one, our notation shares with that of Chemistry, but the matter if fairly examined is seen

Fig. 179.

to be quite unimportant. In writing down a chain we begin with a link, and thus a single element of some pair must stand first in the formula, at the end of which, consequently, must be its partner element, and the sign of pairing annexed to the symbol of this latter sufficiently indicates the closure of the chain.

This will perhaps be made clearer by an illustration. Let it be required to write down the familiar chain shown in Fig. 179. The cylindric elements are indicated in the figure by the letters $b\,c$, $d\,e$, etc., and for distinctness' sake we may in the first case add these to the symbols. Beginning then with the link $b\,c$ we have to write

$$\underbrace{C^{+} \ldots \| \ldots C^{\pm}_{=}}_{b\,c}\ \underbrace{C^{-} \ldots \| \ldots C^{-}_{=}}_{d\,e}\ \underbrace{C^{+} \ldots \| \ldots C^{\pm}_{=}}_{f\,g}\ \underbrace{C^{-} \ldots \| \ldots C^{-}_{=}}_{h\,a}$$

The links, so far as the above form-symbols are concerned, appear identical. If the chain be fixed in the way shown in Fig. 180, the formula becomes (omitting the letterings of the special links):—

$$C^{+} \ldots \| \ldots C^{\pm}_{=}\, C^{-} \ldots \| \ldots C^{-}_{=}\, C^{+} \ldots \| \ldots \underline{C^{\pm}_{=}\, C^{-} \ldots \| \ldots C^{-}}$$

At first sight it might appear to be strange, and to show a defect in our notation, that it does not show any difference in form between the links. This I deny at once, for we know from § 16 that the form-symbols – and + are in the lower pairs absolutely interchange-able without alteration of the pairs, so that in the formula before us all the links might be indicated by the same symbols. Closer examination changes the defect into an advantage. The links appear alike in the formula because in their actual nature they are really alike. A clear indication of this is necessary

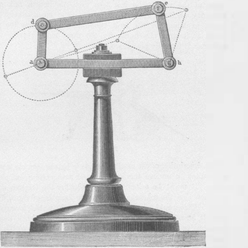

Fig. 180.

for the realization of the abstract form of the mechanism, for the perception of its essential nature under its material disguise. In the mechanism of Fig. 180 the ordinary mechanic sees a "beam" driving a "crank,"—fg is the arm of the beam, de the connecting rod, bc the crank, and ah the frame of the machine, formed in the most various ways of columns or castings or timber, and supported on built foundations. In the actual machine this framework, the fixed link of the chain, is of so varied a figure, so mixed up with portions of buildings, and complex in so many ways, as to afford no indication of its really simple nature; indeed, to very many people the mechanism presents itself as one of three links only,—beam, connecting-rod and crank. This has been carried so far that even in text-books, when a purely schematic or abstract representation of the mechanism has been used, the fourth link *ah* is entirely omitted from the figure. The inconsistency is no doubt explained by the tacit assumption made that the paper forms as it were the frame for the three moving parts whose existence is formally recognised.

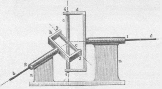

Fig. 181.

The description of the mechanism given by our formula is specially suited to remove this prejudice, or rather this confusion of ideas. It might be asked, however, whether the formula should not show also the relations between the lengths of the different links. This would certainly be possible; it would however be difficult, for of course the four lengths may have between them an enormous variety of numerical relations, and it would at the same time be of little use, for even if the lengths were added to the formula it would require a special study of the mechanism in each case to make any practical use of them. We shall, however, find it possible further on to employ a method of indicating these length-relations which is simple and very easily used.

We may take for a second illustration a universal joint, or Hooke's joint,* of which Fig. 181 gives a schematic representation.

* Prof. Reuleaux points out in his *Constructeur* that if this joint is to be named after its inventor it should be called Cardano's joint; for he was the first to point out its possibility.

The chain which constitutes this joint has four links, which are marked in the figure with the letters $a, b, c,$ and d. The link a is paired with b by the turning-pair 2. Normal to this turning-pair is another, 3, which has its open cylinder in the fork of b, and its full cylinder in the sloping arm of the piece c; the link b must therefore be written $C^+ \ldots \perp \ldots C^-$. It must be noted that the lower and upper arms of the fork form together one piece only, and must be reckoned as such; the same is true of the two ends of the arm of c, which kinematically form a single element only. The piece c consists of two solid cylinders, 3 and 4, having their axes crossing at right angles, and it must therefore be written $C^+ \ldots \perp \ldots C^+$. The third link, the fork and spindle d, is similar to b, and will be written in the same way. The fourth link a, lastly, consists of two open cylinders, 1 and 2, oblique to each other, and so must be written $C^- \ldots \angle \ldots C^-$; it is a fixed link, as its form in the figure shows. The complete formula, therefore (to which we have added the letters and numbers used above to distinguish the links and pairs), runs thus:—

$$\underbrace{\underset{2}{C^+} \ldots \perp \ldots \underset{3}{C^{\underline{-}}}}_{b}\ \underbrace{C^+ \ldots \perp \ldots \underset{4}{C^{\underline{\pm}}}}_{c}\ \underbrace{C^- \ldots \perp \ldots \underset{1}{C^{\underline{\pm}}}}_{d}\ \underbrace{\underline{C^- \ldots \angle \ldots C^-}\underset{2}{}}_{a}$$

There is one geometrical property of the chain which is not shown by our formula, namely that the axes of the pairs 1, 2, 3, and 4 have a common point of intersection. But unless the chain possessed this property it would not be possible, on our supposition that all its pairs are closed. No special indication of this property is therefore commonly necessary. Our formula shows, however, that the three links b, c, and d are again identical. This circumstance is very notable, and we shall later on have to deal with it in another form; the common construction of the joint so entirely conceals it as to make it almost unrecognizable.

The belt train, the kinematic nature of which we have already examined, will be written as follows:—

$$T^{\pm}_{p} \ldots \cong \angle \ldots T^{\mp}_{p},\ R^+ \ldots | \ldots \underline{C^{\underline{\pm}}\, C^- \ldots \| \ldots C^{\underline{\mp}}} C^+ \ldots | \ldots R^+,$$

The tension organ used here is a flat band, and is therefore marked with the suffix p (prismatic); it rolls both onoto and off each pulley, andt thereforet receives thet signs ± and∓, these beingt re-versedt because the portion of the beltrunning on totonepulley

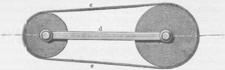

Fig. 182.

ist thet samet ast thatt runniugt offt thet other. Thet bandt fort the pulley *a* is identical—coincident—with that for *b*, the corre-

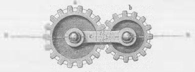

Fig. 183.

spondingsign ≌ musttthereforе bet placed in thetdotted linet; the belttis—lastly—open,tanditstwo sides aretinclined toteachtother

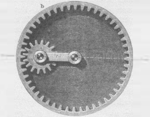

Fig. 184.

ontaccount of the inequality ofthe pulleys,tso that we must add the symbol ∠, oblique, to the sign ≌. If the belt were crossed we should have had to substitute for $\angle$ the symbol $\not-$. The rest of the formula is clear from what has gone before. Put into words, the whole stands thus: "Kinematic chain consisting of two unequal revolutes connected by an endless band touching them externally, each being provided with a con-axial cylindric spindle, and the two spindles working in parallel bearings in a stationary supporting piece."

The simple spur-gearing of Fig. 183 is written,

$$C^{+} \ldots | \ldots C_{z}^{+}, C_{z}^{+} \ldots | \ldots \underline{C^{+}_{=} C^{-} \ldots \| \ldots C^{-}}$$

The links coming first in the formula are the two spur-wheels with their shafts, the last is the (fixed) bar carrying the bearings. The annular gear of Fig. 184 is written:—

$$C^{+} \ldots | \ldots C_{z}^{+}, C_{z}^{-} \ldots | \ldots \underline{C^{+}_{-} C^{-} \ldots \| \ldots C^{-}_{=}}$$

The first link is here the wheel *a* having external teeth, the second is the annular wheel *b*, and the third is again the bar *c*, the latter being supposed fixed.

§ 59.

Contracted Formulæ.

If once the separation of a chain into links and the special examination of the latter has been completed—so that they may be assumed to be already known—the formula or symbolic description of the chain may be in many cases greatly shortened. There are several possible forms of contraction, which we shall examine in order.

Firstly, in the case of the lower pairs, and of some others in which the partner elements have the same name-symbol, one letter may frequently be made to suffice for a pair of elements if it be used along with some distinguishing mark. For this purpose a parenthesis can be used, so that we may employ as contractions:

(S)	for the twisting pair	$S^{+}_{=}S^{-}$
(C)	" " turning "	$C^{+}_{=}C^{-}$
(P)	" " sliding "	$P^{+}_{=}P^{-}$
(C_z)	for a pair of spur wheels	C_z, C_z
(K_z)	" " " bevel "	K_z, K_z

and so on. This allows us, for example, to write the chain represented in Fig. 179 as:

$$C^{+} \ldots \parallel \ldots (C) \ldots \parallel \ldots (C') \ldots \parallel \ldots (C) \ldots \parallel \ldots C^{-}_{=}$$

Here only the elements of the first (and last) link require to be written separately with their form signs—the one letter in the parenthesis standing for a pair of elements. The same method can be carried further; it allows us to write certain simple kinematic chains in a still shorter form, for where we have the parenthesis the relations of the linked elements are sufficiently defined without the use of the dotted line, which may therefore be omitted. This is certainly the case in the present instance, where the same relation—parallelism—exists between all the elements forming links. Indeed we may in these circumstances extend the parenthesis so as to include several, or the whole of the links. Thus for certain cases we may compress the above formula, without making its meaning uncertain, into the symbol (C''_4), in words "C parallel four," or "C four parallel," and meaning "a chain formed upon four turning pairs, consisting, that is, of four links, each connecting two parallel cylindric elements." Such a contraction presupposes in all cases a familiarity with the way in which the chain can be formed out of its elements; its form, however, is so concise as to leave nothing to be desired in this direction. The chain forming the universal joint, Fig. 181,—to take another example,—allows itself to be written $(C^{\perp}_{3}\, C^{\llcorner})$, in words, "$C$ normal three C oblique"; the spur-gearing of Fig. 183 may be written $(C^{+}_{z}C''_{2})$, in words, "C plus z C parallel two," and so on.

These concentrated symbolic forms seem at first suitable only for the kinematic chain, not for the mechanism formed by fixing one of its links; further on, however, we shall find means for making use of them in these cases also, within certain limits.

§ 60.

Formulæ for Compound Chains.

In the simple kinematic chains the choice of the link with which to begin the formula was to a certain extent arbitrary. This strikes us still more in compound chains, and makes it appear at first sight to some extent difficult to attain the required distinctness. A little experience, however, enables this to be obtained, as a few examples will show.

We have before examined (§ 3) the chain represented in Fig. 185, containing seven cylindric pairs. It is obtained from the familiar chain (C''_4) by the addition of two more links of the form $C \ldots \| \ldots C$, and possesses a certain symmetry of arrangement in having two opposite three-cylindered links twice connected by a pair of two-cylindered links,* altogether, that is, by four such links. This is made more distinct by the schematic representation in Fig. 186, in which also the dimensions are so chosen as to make the chain symmetrical. The turning-pairs are here numbered

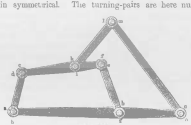

FIG. 186.

from 1 to 7. We may look at the whole chain as consisting of two five-linked cylinder-chains 1, 2, 3, 4, 5 and 1, 2, 6, 7, 5—in which the links 1, 2 and 1, 5 are common, the cylinders 2, 3, 6 united into one link containing three elements, and the cylinders 5, 4, 7 into another. To distinguish between links containing two, and links containing three elements, we may call them **binary** and **ternary links** respectively.

We may now proceed by first writing down these two five-linked cylinder-chains—neither of which is by itself constrainedly closed—singly, and then as it were adding them together—that is putting a single sign only where pieces are common to both chains, and bracketing the elements brought together in the ternary links.

* Firstly in the original connection by *a d* and *e h*, and secondly in the additional connection by *k l* and *m n*.

We obtain the following result (using the contracted symbols for all the inner pairs) :—

$$\begin{array}{cccccccccccc}
1 & & 2 & & 3 & & 4 & & 5 & & 1 \\
C^+ & \ldots \| \ldots & (C) & \ldots \| \ldots & (C) & \ldots \| \ldots & (C) & \ldots \| \ldots & (C) & \ldots \| \ldots & C^-_= \\
C^+ & \ldots \| \ldots & (C) & \ldots \| \ldots & (C) & \ldots \| \ldots & (C) & \ldots \| \ldots & (C) & \ldots \| \ldots & C^-_= \\
1 & & 2 & & 6 & & 7 & & 5 & & 1 \\
\hline
\end{array}$$

$$\begin{array}{ccccccccc}
1 & & 2 & & \begin{array}{ccc} 3 & & 4 \end{array} & & 5 & & 1 \\
C^+ & \ldots \| \ldots & (C) & \ldots \| & \left\{ \begin{array}{ccccc} \ldots & (C) & \ldots \| \ldots & (C) & \ldots \\ \ldots & (C) & \ldots \| \ldots & (C) & \ldots \end{array} \right\} & \| \ldots & (C) & \ldots \| \ldots & C^-_= \\
 & & & & \begin{array}{ccc} 6 & & 7 \end{array} & & & &
\end{array}$$

The compound formula resulting from the addition—and to which, for explanation's sake, we have added the numbers of the turning-pairs—may be considered as one which really allows the nature of the chain to be seen, for it distinctly reproduces its symmetrical arrangement. It is possible, however, to bring the formula into a still clearer shape, which may be useful in some cases. Noticing, namely, that the four turning-pairs within the brackets form by themselves a simple closed cylinder-chain (C''_4), and at the same time that this whole chain has taken the place before occupied by a pair of elements, we see that the whole formula may be written,—

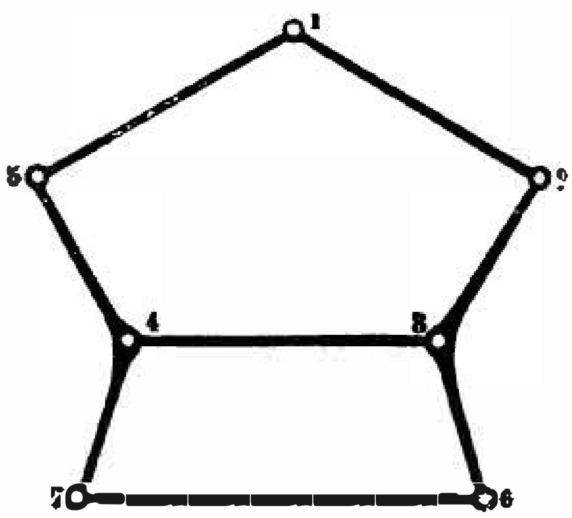

FIG. 186.

$$\begin{array}{ccccccccc}
1 & & 2 & & 3,4,7,6 & & 5 & & 1 \\
C^+ & \ldots \| \ldots & (C) & \ldots \| \ldots & (C''_4) & \ldots \| \ldots & (C) & \ldots \| \ldots & C^-_=
\end{array}$$

so as to take up much less space than before. The formula could be used in this shape for the mechanism, also, if the fixed link were 1.2, 2.3, 1.5 or 5.7—but if the fixed link be one of the inner group, 3, 4, 7, 6, the more extended formula must be employed.

We may choose a train of spur wheels as another illustration. Fig. 187 represents a compound mechanism of this kind with two pairs of wheels a, b and c d. The wheel c is fixed to b, the three spindles con-axial with the wheels have their bearings in the ternary link 1. 2. 3, which is here the fixed link. The formula may be arranged in several different ways.

Starting from the turning pair 2 we have, on each side, simply a pair oftspur-wheelst withttheirt connectingtlink. Wetwritetlhem singlytandtaddtthemttogether as followst:—

$$
\begin{array}{ccccccc}
2 & & b,\ a & & 1 & & 2 \\
C^{+}\ \ldots & | & \ldots\ C^{+}_{z}, C^{+}_{z}\ \ldots & | & \ldots\ (C)\ \ldots & \| & \ldots\ C^{-}_{=} \\
C^{+}\ \ldots & | & \ldots\ C^{+}_{z}, C^{+}_{z}\ \ldots & | & \ldots\ (C)\ \ldots & \| & \ldots\ C^{-}_{=} \\
2 & & c,\ d, & & 3 & & 2 \\
\hline
\end{array}
$$

$$
\begin{array}{c}
2 \qquad\qquad b,\ a, \qquad\qquad 1 \qquad\qquad 2 \\
C^{+}\ \ldots\ |\ \ldots \left\{\begin{array}{l} C^{+}_{z}, C^{+}_{z}\ \ldots\ |\ \ldots\ \underline{(C)\ \ldots} \\ C^{-}_{z}, C^{+}_{z}\ \ldots\ |\ \ldots\ \underline{(C)\ \ldots} \end{array}\right\} \underline{\ldots\ \|\ \ldots\ C^{-}_{=}} \\
c,\ d \qquad\qquad 3
\end{array}
$$

This shows two binary links a, 1 and d, 3, and twotternary links, 2, b, c and 2, 1, 3. Thetlattertare the coupledwheels b and c with

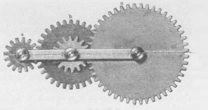

Fig. 187.

theirtcommont spindle,tandtthetbar withtthetbearings,twhich—as thetunderliningshows,—istfixed.

Wemaytobtain thetformula in anothertshape as follows. Beginning with the wheel a and its spindle, we then write the wheel b anditstconnections atonce as a ternary linkt; continue with d, and writet itslinkage with 3; and then unite the second cylinders of 2 and of 3 with 1. Fordistinctness' sakewcuse theuncontracted form

$$
\begin{array}{c}
1 \qquad\qquad\qquad\qquad 2 \qquad\qquad\qquad\qquad 1 \\
C^{+}\ \ldots\ |\ \ldots\ C^{+}_{z}, C^{+}_{z}\ \ldots\ | \left\{\begin{array}{l} \ldots\ldots\ldots\ C^{\pm}_{=}\ \underline{C^{-}}\ \ldots\ldots\ldots\ldots \\ \ldots\ C^{+}_{z}, C^{+}_{c}\ \ldots\ |\ \ldots\ C^{\pm}_{=}\ \underline{C^{-}\ \ldots} \end{array}\right\} \underline{\|\ \ldots\ C^{-}_{=}} \\
\qquad\qquad\qquad\qquad\qquad 3
\end{array}
$$

The meaning of this formula is exactly the same as that of the last. The part of it within the brackets, however, shows itself at once to be a simple kinematic chain, consisting of the wheels c and d and their connecting link 2, 3. If we use for this the contracted symbols of § 59, and contract also the symbol for the pair a, b, we have:

$$\begin{array}{ccccccc} 1 & & a,b & & c,d,3,2 & & 1 \\ C^{+} \ldots & \mid & \ldots (C^{+}_{z}) \ldots & \mid & \ldots (C^{+}_{z}\, \underline{C''_{2}) \ldots & \parallel & \ldots C^{-}_{=}} \end{array}$$

In some cases it will be quite sufficient to write this in a still shorter form—extending the use of our former method of contraction,—as $(C^{+}_{z2} C''_{3})$; or generally for an n-fold train of spur-gearing $(C^{+}_{zn} C''_{n+1})$, where for n pairs of wheels there are in general $n + 1$ axes, or rather turning-pairs (C), required.

§ 61.

Formulæ for Chains containing Pressure-organs.

In order to find the formula for a chain which contains a pressure-organ it is frequently advisable to imagine the substitution of a rigid element for the latter—the pairing being obviously somewhat altered in consequence—and then to transform the formula thus obtained by the re-insertion of the pressure-organ.

In order, for example, to express by a formula the water-wheel (Fig. 188) which we have already looked at, we may first replace the water by a rack with a prismatic guide (Fig. 189) so arranged as to drive the spur-wheel a by its own weight, its action being thus similar to that of the fluid for which it is substituted. The formula will run:

$$C^{+} \ldots \mid \ldots C^{+}_{z}, P_{z} \ldots \parallel \ldots P^{+}_{=} P^{-} \ldots + \ldots C^{-}_{=}$$

If we now change the link $P_{z} \ldots \parallel \ldots P^{+}$ into $Q_{\lambda} \ldots\ldots e\, Q_{\lambda}$,—replacing the water for its temporary substitute,—we must put V^{-}, the symbol for a vessel of any kind, for P^{-}, and so obtain as a formula for the water-wheel,

$$C^{+} \ldots \mid \ldots C^{+}_{z}, Q_{\lambda} \ldots\ldots Q_{\lambda}, V^{-} \ldots + \ldots C^{-}_{=}.$$

If it require further to be indicated that the channel is uncovered, the water being paired with it, that is, by force-closure, we must substitute $\frac{V^-}{f}$ for V^-. The constitution of this mechanism is the same as that of the lift- or flash-wheel. If we suppose the link $Q_\lambda \ldots Q_\lambda$ fixed instead of the link $V^- \ldots + \ldots C^-$, we obtain a very different but very familiar mechanism. The link $V^- \ldots + \ldots C^-$ moves in the (relatively to it) stationary water; the mechanism is that of the paddle-steamer. It will be seen that in this case we must presuppose that the link c possesses the requisite buoyancy.

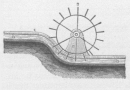

Fig. 111.

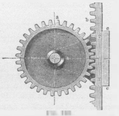

Fig. 112.

We may here also use the abridged notation. In the first formula the pair $C_z P_z$ presents a certain difficulty, for it is our object if possible not to use two capital letters in the contraction for one pair of elements, in order that there may never be any doubt as to whether each letter stands for a pair or not. So far as it is possible we wish that the number of capital letters in any contracted formula—with the addition of course of the repetitions, if any, indicated by the suffixes—shall show at once the number of pairs in the chain for which the formula stands. We must for this purpose have recourse to a convention. Without leading to any misunderstanding we may denote the pair $C_z P_z$ by the symbol (C_{zp}), and by doing so we obtain as an expression for the whole chain: $(C'' C_{zp} P^+)$. We add the sign $+$ ("crossed") to the symbol (P) of the sliding pair, so as to make the position of that pair quite determinate.

Similar difficulties as to double letters occur twice in the second formula, but treating them similarly to the one just discussed, we may write

$$(C_{z\lambda}) \text{ for } C_z, Q_\lambda$$
$$(V_\lambda) \text{ for } V^-, Q_\lambda .$$

The index λ stands as contraction for q_λ, and is sufficient to make the pairing of C_z and V respectively with a liquid quite distinct. For the kinematic chain of Fig. 188 we therefore obtain the concentrated formula $(C' \; C_{z\lambda} \; V_\lambda)$, which, as we have seen, serves as well for the water-wheel as for the lift-wheel and the paddle-steamer.

§ 62.

Contracted Formulæ for Single Mechanisms.

The abridged notation which we have described for kinematic chains cannot be applied to the same chains in the form of mechanisms without some additions,—for it shows only pairs and not links, and therefore does not in itself furnish any means for indicating the fixing of a link. It is, however, most important that we should have the means of extending these concise,—and yet for so many cases quite sufficient,—symbols to mechanisms.

Although this cannot be done by such logical generalizations as those by which the contractions were arrived at in the first instance, still in the chains which are most important to us the end can be obtained by various special means. These means are the giving of definite name-symbols, settled by agreement in each particular case, to the separate links of the chain. If this be done, and the name-symbol of the fixed link,—as the one about which something special requires to be indicated,—be assigned some particular and conspicuous position in the formula, we have obtained an abridged notation for the mechanism.

We choose the letters of the small Roman alphabet for the link symbols, beginning with *a* in each case, and going on as far as may be necessary; the letters indicate in themselves therefore no quality or form. To prevent any confusion arising between these letters and the form-signs, we give the former a specially distinctive position in the formula, namely, that of an exponent outside the brackets which inclose the symbols of the pairs. Only one letter, as a rule, will occupy this position, there being only one fixed link to be indicated. An illustration will make the method quite distinct.

Let it be required to write contracted formulæ for the mechanisms in the form of which the four-linked chain, Fig. 190, can be used. We first give to the four links the signs a, b, c, d in the way schematically indicated in Fig. 191; these signs are arbitrarily

Fig. 190.

chosen in the first instance, but once chosen they must of course be adhered to. The lengths of the links are so proportioned that if d be fixed the link a (the crank) can revolve while c swings in circular arcs. So long as the chain is unfixed its contracted formula we have already found to be (C''_4). If now d be fixed—as its form in Fig. 191 indicates—the formula will become $(C''_4)^d$—in words "C parallel four on d." The particle on indicates that the

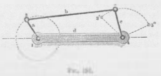

Fig. 191.

chain is, as it were, placed on the link d, that this link becomes its base. If d were released and a fixed instead, the new mechanism would be $(C''_4)^a$; in the same way the water-wheel of § 61 would be $(C'' C_{s\lambda}\ V_\lambda)^c$, the paddle-steamer $(C'' C_{s\lambda}\ V_\lambda)^b$, and so on. It will be seen that this very short method of symbolization enables us easily to distinguish by distinct symbols the different mechanisms which can be made from one and the same chain. It possesses always the limitation, however, that the letters a, b, c, etc., are here without any general qualitative meaning—there is no special connection between the symbols and the links for which they stand. The only direction in which they have a partly general character is in their alphabetic sequence—their order, that is, may be not without significance. Where possible we may begin with *a* at some specially distinctive link,—such as the crank in the case supposed,—and so greatly facilitate the recollection of the meaning agreed on for the symbols. The applications of this contracted notation, as we shall find in the sequel, prove it to be of the greatest value.

Our method of symbolization, lastly, allows a further and most useful piece of information to be brought into the formula. It is frequently important to indicate that link of the chain to which the driving effort is applied, or through which the mechanism is moved. For it is evident that there is an immense difference between two mechanisms—otherwise the same—if one be driven by an effort applied to the link *a* and the other by an effort applied to *b*. We had a striking example of this in the mechanism of the water-wheel and the lift-wheel. Both would be indicated by the symbol $(C'C_{\pi\lambda}V_{\lambda})^{c}$, while the transmission of motion in them would be essentially different.

It becomes evident on looking into this matter that this formula is of the nature of a general or indeterminate formula for both mechanisms,—which it must be our object to turn into a special or determinate formula for each of them. We may do this, and supply the information that is wanted, by putting the symbol for the driving link as a denominator in the exponent. The latter will then show the fixed link only in the general formula, but in the special formula it will be fractional, its numerator indicating the fixed, and its denominator the driving link. The choice of the fractional form is justified by the analogy with the symbols for force- and chain-closures which were fixed in § 57.

Thus for example the mechanism $(C''_4)^{d}$, if the crank be the driving link, will be written $(C''_4)^{\frac{d}{a}}$,—in words "*C* parallel four on *d* by *a*," the latter part being a contraction for "placed on *d*, driven by *a*." The same mechanism, if driven by the lever, has for its special formula $(C''_4)^{\frac{d}{c}}$; the general formula $(C''_4)^{d}$ being of course common to both mechanisms. The water-wheel will now be written $(C' C_{z\lambda} V_\lambda)^{\frac{c}{b}}$, the lift-wheel $(C' C_{z\lambda} V_\lambda)^{\frac{a}{a}}$; the steamer $(C' C_{z\lambda} V_\lambda)^{\frac{b}{a}}$. These last examples already show in the most distinct way the usefulness of our formulæ. For the mere transcription of them is sufficient on. the one hand to show the intimate connection between machines which constructively seem to stand so far apart, and on the other hand, to indicate definitely and simply the true differences between them.

The special formulæ of mechanisms are chiefly useful in the analysis of complete machines,—that is, in reference to the applications of mechanisms,—while the general formulæ commonly suffice for their abstract representation. Here, too, however, the special formulæ are often very valuable, as showing which of the link motions is to be considered as the independent variable. We may now proceed to the systematic application of the kinematic notation; in the following chapters we shall have to make extended use of both kinds of formulæ.