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Statistical Analyses for Multistage Experiment Designs

Walter T. Federer

Abstract

Suppose that t experiments are conducted simultaneously on the same set of experimental units. For example, suppose that t mutually orthogonal latin square experiment designs are used for the t experiments on n^2 experimental units. Statistical literature is voluminous on construction of such designs, but contains relatively little and incomplete results on statistical analyses for such designs. Six statistical analyses are presented for a pair of orthoginal latin square experiment designs. Then, the methods are generalized for t mutually orthogonal experiment designs. The results are also extended to a set of t mutually balanced Youden experiment designs.

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1. Introduction

Multistage experiment designs come in many forms and situations. Statistical procedures for summarizing the information from one multistage design will be varied and different from a second multistage design and from single stage designs. By far the vast majority of statistical procedures developed have been for single stage experiment designs. The same is true for multistage treatment designs (selection of treatments for inclusion in an experiment design, the arrangement of treatments in an experiment). We shall, however, restrict our attention to multistage experiment design, and shall be even more restrictive and consider only one type of multistage design, *viz.* a set of t mutually orthogonal latin square experiment designs, a MOL(n,t) set, or a set of t mutually balanced Youden experiment designs (see Preece, 1966b; Federer, 1972; Hedayat, Seiden and Federer, 1972), a MBY $(n=v=b,k,\lambda,t)$ set.

Some types of multistage experiments are listed below.

Type 1: Consider any completely randomized design, randomized block design, row by column design, etc. for which each sampling unit is measured over time. This has been called a repeated measures design. Several different concepts and statistical analyses have been derived for these designs (see, e.g., Kershner and Federer, 1981). For some repeated measures designs, a single treatment is applied to a given sampling unit either at the beginning of the experiment or continuously throughout the experiment. Other types of repeated measures designs involve changing the treatments at specified periods and are called cross-over or change-over designs (see Kershner and Federer, 1981). Additional concepts, definitions, and statistical analyses are required to summarize and apply the information available from these experiment designs.

The earliest uses of repeated measures experiment designs appear to have been in agriculture for continuous cropping studies, for crop rotation experiments, and for pasture experiments. Other early uses were in nutrition and feeding experiments on animals (see Brandt, 1938 and Cochran, Autrey and Cannon, 1941).

Type 2: A specified experiment design and a specified set of v treatments conducted at ℓ locations, sites, laboratories, etc., using a different set of rv experimental units for each experiment. Two classic papers on statistical analyses for this type of multistage design are Cochran (1936) and Yates and Cochran (1938). When the experiment design varies from site to site, one may use the statistical analyses described in Cochran and Cox (1957), chapter 14.

Type 3: There are v_i, $i=1,2,\cdots,\ell$, treatments in the experiment at site i; the number of replications may vary on each or all of the v_i treatments; the v_i treatments may differ from site to site; the experiment design may vary at each site; one wishes to summarize the information over the ℓ experiments for such purposes as to determine an optimum sampling fraction (see Yates and Zacopanay, 1935), to determine optimal allocation of resources for maximizing genetic advance (see Sprague and Federer, 1951), or to determine crop response to fertilizer applications (see Meisinger, 1976). The first two references above made use of unbiased estimates of ratios of variance components to summarize the information from an experiment. The unbiased estimates were then combined. Since one is dealing with a population of population parameters, unweighted combinations were utilized to combine the information from the ℓ experiments.

The last reference considers that the response is a function of a number of measured independent variables at each of the sites.

Type 4: The t separate sets of n=v treatments are carried out simultaneously or successively on the same set of rc experimental units. In some situations it is assumed that the treatments in one set do not affect those in the second set, and hence are not considered in the designing; in other situations the treatments in one set are in an orthogonal, a balanced, or a partially balanced arrangement with the treatments in another set. For example, a set of t mutually orthogonal latin squares may be used on the n^2 experimental units or a set of t mutually balanced Youden designs may be used on the nk experimental units. Other experiment designs may also be used.

Type 5: The v_i treatments are in a mixture such as found in intercropping, application of drugs, application of teaching or recreational programs, etc., where the different items may enter simultaneously, sequentially, or at different times. Individual responses may be measured on each item in a mixture, or only one response may be available for the k items in a mixture. Statistical designs and analyses have to be devised for the various types of responses, as well as for attaining the goals of an experiment.

2. Some Statistical Analyses for a Pair of Orthogonal Latin Square Designs Conducted Simultaneously

Consider a marketing research situation wherein an experimenter has n grocery stores available for n time periods. He wishes to use n merchandising treatments on one commodity, say apples, and n merchandising treatments on a second commodity, say carrots. He decides to use a pair of orthogonal latin square experiment designs for the experiment. After he obtains his n^2 responses,

Y_{hij}, on apples and n^2 responses W_{hij}, on carrots, the question now arises as to how he should analyze the $2n^2$ responses. Some possibilities with their assumptions and difficulties are listed below.

(i) The results from the two experiments are considered to be independent. It is further believed that the standard textbook ANOVA would be appropriate for these experiments. The ANOVA would be:

Apples		Carrots	
Source of variation	d.f.	Source of variation	d.f.
Total	n^2	Total	n^2
Correction for mean	1	Correction for mean	1
Stores (columns)	n-1	Stores (columns)	n-1
Periods (rows)	n-1	Periods (rows)	n-1
Treatments	n-1	Treatments	n-1
Remainder	(n-1)(n-2)	Remainder	(n-1)(n-2)

Such ANOVA's are related to the response model equations:

$$Y_{hij} = \mu_a + \rho_{ah} + \gamma_{ai} + \tau_{aj} + \epsilon_{ahij} \tag{2.1}$$

and

$$W_{hij} = \mu_c + \rho_{ch} + \gamma_{ci} + \tau_{cj} + \epsilon_{chij} \quad , \tag{2.2}$$

where μ_x, ρ_{xh}, γ_{xi}, and τ_{xj} are overall mean, row, column, and treatment effects for commodity x; x=a,c; h,i,j=1,2,$\cdots$,n; and ϵ_{xhij} are NIID$(0,\sigma^2_{\epsilon x})$. The experimenter may then use F-tests, some multiple comparisons procedure, or other statistical procedures to summarize the information and make the desired inferences.

The assumption that the two experiments are independent would be untenable for the above situation, since sales of apples and sales of carrots would most likely be related, that is, the purchase of apples affects the purchase of carrots, and *vice versa*. One should also note that the above procedure is the one followed in most textbooks and by most statisticians and investigators. The textbook ANOVA and response models (2.1) and (2.2) may be inappropriate. Instead, the differential gradient model of Cox (1958), a model including terms for non-additivity, or some other response model, may be more appropriate than equations (2.1) and (2.2).

The residuals using response model equations (2.1) and (2.2) are computed as:

$$\begin{aligned} e_{ahij} &= Y_{hij} - \bar{y}_{h..} - \bar{y}_{.i.} - \bar{y}_{..j} + 2\bar{y}_{...} \\ &= Y_{hij} - \bar{y}_{...} - \hat{\rho}_{ah} - \hat{\gamma}_{ai} - \hat{\tau}_{aj} \end{aligned} \qquad (2.3)$$

and

$$\begin{aligned} e_{chij} &= W_{hij} - \bar{w}_{h..} - \bar{w}_{.i.} - \bar{w}_{..j} + 2\bar{w}_{...} \\ &= W_{hij} - \bar{w}_{...} - \hat{\rho}_{ch} - \hat{\gamma}_{ci} - \hat{\tau}_{ch} \quad . \end{aligned} \qquad (2.4)$$

The "Remainder" sum of squares may be computed by squaring the n^2 residuals and obtaining their sum for each analysis.

Example 2i: Suppose that an experimenter conducted two experiments simultaneously in four grocery stores and for four time periods. Suppose that there were four treatments involving four methods of packaging apples and four treatments involving size of packages of carrots. Furthermore, suppose that the two latin square experiment designs used were orthogonal ones. The responses are considered to be pounds of a product (apples or carrots) sold to ten customers, and it is considered that the non-treatment period between treatments is sufficient to remove any carry-over effect of a treatment. The data are given in Table 2.1. For the analysis we use response model equations (2.1) and (2.2). The data were constructed to give integers for solutions of effects.

For the Y_{hij} the following values were used:

$\hat{\mu}_a = 10$		
$\hat{\rho}_{a1} = -1$	$\hat{\gamma}_{a1} = -3$	$\hat{\tau}_{aA} = -3$
$\hat{\rho}_{a2} = -1$	$\hat{\gamma}_{a2} = 3$	$\hat{\tau}_{aB} = -4$
$\hat{\rho}_{a3} = 2$	$\hat{\gamma}_{a3} = 0$	$\hat{\tau}_{aC} = 7$
$\hat{\rho}_{a4} = 0$	$\hat{\gamma}_{a4} = 0$	$\hat{\tau}_{aD} = 0$

The e_{ahij} values are given in Table 2.3.

For the W_{hij} the following values were used:

$\hat{\mu}_c = 10$		
$\hat{\rho}_{c1} = -1$	$\hat{\gamma}_{c1} = -3$	$\hat{\tau}_{ca} = 0$
$\hat{\rho}_{c2} = -1$	$\hat{\gamma}_{c2} = 3$	$\hat{\tau}_{cb} = -3$
$\hat{\rho}_{c3} = 0$	$\hat{\gamma}_{c3} = -3$	$\hat{\tau}_{cc} = -1$
$\hat{\rho}_{c4} = 2$	$\hat{\gamma}_{c4} = 3$	$\hat{\tau}_{cd} = 4$

The e_{chij} values are given in Table 2.3.

Period	Store (apple sales, pounds) 1	2	3	4	$Y_{\cdot h \cdot\cdot}$	$\bar{y}_{\cdot h \cdot\cdot}$	$\hat{\rho}_{ah}$
1	A 4	B 7	C 16	D 9	36	9	-1
2	B 1	A 10	D 9	C 16	36	9	-1
3	C 16	D 15	A 8	B 9	48	12	2
4	D 7	C 20	B 7	A 6	40	10	0
$Y_{\cdot\cdot i \cdot}$	28	52	40	40	160	-	0
$\bar{y}_{\cdot\cdot i \cdot}$	7	13	10	10	-	10	-
$\hat{\gamma}_{ai}$	-3	3	0	0	0	-	-

$Y_{\cdot\cdot\cdot A} = 28$	$\hat{\tau}_{aA} = -3$
$Y_{\cdot\cdot B \cdot} = 24$	$\hat{\tau}_{aB} = -4$
$Y_{\cdot\cdot C \cdot} = 68$	$\hat{\tau}_{aC} = 7$
$Y_{\cdot\cdot D \cdot} = 40$	$\hat{\tau}_{aD} = 0$
$Y_{a\cdot\cdot\cdot} = 42$	$\hat{\delta}_{aa} = 0.5$
$Y_{b\cdot\cdot\cdot} = 38$	$\hat{\delta}_{ab} = -0.5$
$Y_{c\cdot\cdot\cdot} = 42$	$\hat{\delta}_{ac} = 0.5$
$Y_{d\cdot\cdot\cdot} = 38$	$\hat{\delta}_{ad} = -0.5$

Period	Store (carrot sales, pounds) 1	2	3	4	$W_{\cdot h \cdot\cdot}$	$\bar{w}_{\cdot h \cdot\cdot}$	$\hat{\rho}_{ch}$
1	a 8	b 7	c 3	d 18	36	9	-1
2	d 8	c 13	a 5	b 10	36	9	-1
3	b 4	a 13	d 11	c 12	40	10	0
4	c 8	d 19	a 9	b 12	48	12	2
$W_{\cdot\cdot i \cdot}$	28	52	28	52	160	-	0
$\bar{w}_{\cdot\cdot i \cdot}$	7	13	7	13	-	10	-
$\hat{\gamma}_{ci}$	-3	3	-3	3	0	-	-

$W_{\cdot\cdot\cdot a} = 40$	$\hat{\tau}_{ca} = 0$
$W_{\cdot\cdot\cdot b} = 28$	$\hat{\tau}_{cb} = -3$
$W_{\cdot\cdot\cdot c} = 36$	$\hat{\tau}_{cc} = -1$
$W_{\cdot\cdot\cdot d} = 56$	$\hat{\tau}_{cd} = 4$
$W_{A\cdot\cdot\cdot} = 44$	$\hat{\delta}_{cA} = 1$
$W_{B\cdot\cdot\cdot} = 36$	$\hat{\delta}_{cB} = -1$
$W_{C\cdot\cdot\cdot} = 36$	$\hat{\delta}_{cC} = -1$
$W_{D\cdot\cdot\cdot} = 44$	$\hat{\delta}_{cD} = 1$

Table 2.1. Data from a pair of orthogonal latin square designs conducted simultaneously on the same 16 experimental units.

Apple Sales (Y_{hij})

Source of variation	d.f.	Sum of squares	Mean square
Total	16	2000	-
Correction for mean	1	1600	-
Stores	3	72	24
Periods	3	24	8
Apple treatments	3	296	296/3
Remainder	6	8	4/3

Carrot Sales (W_{hij})

Source of variation	d.f.	Sum of squares	Mean square
Total	16	1904	-
Correction for mean	1	1600	-
Stores	3	144	48
Periods	3	24	8
Carrot treatments	3	104	104/3
Remainder	6	32	16/3

Table 2.2. Analyses of variance for apple sales Y_{hij} and carrot sales W_{hij} using response model equations (2.1) and (2.2).

Residuals e_{ahij} for Y_{hij} values

Period	Store 1	2	3	4	Sum
1	A 1	B -1	C 0	D 0	0
2	B -1	A 1	D 0	C 0	0
3	C 0	D 0	A -1	B 1	0
4	D 0	C 0	B 1	A -1	0
Sum	0	0	0	0	0

Residuals e_{chij} for W_{hij} values

Period	Store 1	2	3	4	Sum
1	a 2	b -2	c -2	d 2	0
2	d -2	c 2	b 2	a -2	0
3	b 0	a 0	d 0	c 0	0
4	c 0	d 0	a 0	b 0	0
Sum	0	0	0	0	0

Table 2.3. Residuals for Y_{hij} and W_{hij} using equations (2.3) and (2.4).

(ii) The treatments in each of the two orthogonal latin square designs are blocked in relation to the treatments in the other experiment. Response model equations (2.1) and (2.2) are extended to include an additional term for this; thus,

$$Y_{ghij} = \mu_a + \delta_{ag} + \rho_{ah} + \gamma_{ai} + \tau_{aj} + \epsilon_{aghij} \qquad (2.5)$$

and

$$W_{ghij} = \mu_c + \delta_{cg} + \rho_{ch} + \gamma_{ci} + \tau_{cj} + \epsilon_{cghij} \quad , \qquad (2.6)$$

where δ_{ag} is the stratification (blocking) effect of the carrot treatments on the apple treatments, δ_{cg} is the blocking effect of the apple treatments on the carrot treatments, and the other terms are as described previously. Several authors have used response model equations of this form, such as, e.g., Anderson (1972), Anderson and Federer (1976), Bose and Srivastava (1964), Bradu (1965), Cheng (1978), Clarke (1963), Federer (1981), Freeman (1958), Freeman and Jeffers (1962), Potthoff (1962a,1962b), Preece (1966), Rees (1966a), Singh, et al. (1981), and Srivastava and Anderson (1970,1971).

The residuals are computed as:

$$e_{aghij} = Y_{ghij} - \bar{y}_{g\cdots} - \bar{y}_{\cdot h\cdot\cdot} - \bar{y}_{\cdot\cdot i\cdot} - \bar{y}_{\cdots j} + 3\bar{y}_{\cdots\cdot} \qquad (2.7)$$

and

$$e_{cghij} = W_{ghij} - \bar{w}_{g\cdots} - \bar{w}_{\cdot h\cdot\cdot} - \bar{w}_{\cdot\cdot i\cdot} - \bar{w}_{\cdots j} + 3\bar{w}_{\cdots\cdot} \quad . \qquad (2.8)$$

Example 2ii: Using the data from example 2i and response models (2.5) and (2.6) results in the analyses of variance given in Table 2.5. The values for $\hat{\mu}_a$, $\hat{\mu}_c$, $\hat{\rho}_{ah}$, $\hat{\rho}_{ch}$, $\hat{\gamma}_{ai}$, $\hat{\gamma}_{ci}$, $\hat{\tau}_{aj}$, and $\hat{\tau}_{cj}$ are the same as in Example 2i. The solutions for the $\hat{\delta}_{ag}$ and $\hat{\delta}_{cg}$ are:

$$\hat{\delta}_{aa} = 0.5 \qquad \hat{\delta}_{cA} = 1$$

$$\hat{\delta}_{ab} = -0.5 \qquad \hat{\delta}_{cB} = -1$$

$$\hat{\delta}_{ac} = 0.5 \qquad \hat{\delta}_{cC} = -1$$

$$\hat{\delta}_{ad} = -0.5 \qquad \hat{\delta}_{cD} = 1$$

The residuals for the Y_{ghij} and the W_{ghij} values are given in Table 2.4.

(iii) One might wish to combine the results from the two experiments. Note that sums and differences are orthogonal to each other. Therefore, one could obtain analyses of variance of the following forms using response model equations of the form of (2.5) and (2.6):

Sum of apple and carrot sales ($Y_{ghij} + W_{ghij} = S_{ghij}$)		Apple sales minus carrot sales ($Y_{ghij} - W_{ghij} = D_{ghij}$)	
Source of variation	d.f.	Source of variation	d.f.
Total	n^2	Total	n^2
Correction for mean	1	Correction for mean = Product (P)	1
Stores (columns)	n-1	Stores (columns) x P	n-1
Periods (rows)	n-1	Periods (rows) x P	n-1
Grouping 1 (apples)	n-1	Grouping 1 (apples) x P	n-1
Grouping 2 (carrots)	n-1	Grouping 2 (carrots) x P	n-1
Remainder	(n-1)(n-3)	Remainder	(n-1)(n-3)

The null hypothesis tested using Grouping 1 (apples) and remainder mean squares is that apples treatments in the presence of all carrot treatments do not differ. A similar hypothesis is tested using Grouping 2 (carrots) and remainder mean squares. With differences one is testing the null hypothesis of no difference

Residuals e_{aghij} for Y_{ghij} values

Period	Store 1	2	3	4	Sum
1	Aa 0.5	Bb -0.5	Cc -0.5	Dd 0.5	0
2	Bd -0.5	Ac 0.5	Db 0.5	Ca -0.5	0
3	Cb 0.5	Da -0.5	Ad -0.5	Bc 0.5	0
4	Dc -0.5	Cd 0.5	Ba 0.5	Ab -0.5	0
Sum	0	0	0	0	0

Residuals e_{cghij} for W_{ghij} values

Period	Store 1	2	3	4	Sum
1	Aa 1	Bb -1	Cc -1	Dd 1	0
2	Bd -1	Ac 1	Db 1	Ca -1	0
3	Cb 1	Da -1	Ac -1	Bd 1	0
4	Dc -1	Cd 1	Ba 1	Ab -1	0
Sum	0	0	0	0	0

Table 2.4. Residuals e_{aghij} and e_{cghij} for Y_{ghij} and W_{ghij} values.

Apple sales (Y_{ghij})

Source of variation	d.f.	Sum of squares	Mean square
Total	16	2000	-
Correction for mean	1	1600	-
Stores	3	72	24
Periods	3	24	8
Grouping by carrot treatments	3	4	4/3
Apple treatments	3	296	296/3
Remainder	3	4	4/3

Carrot sales (W_{ghij})

Source of variation	d.f.	Sum of squares	Mean square
Total	16	1904	-
Correction for mean	1	1600	-
Stores	3	144	48
Periods	3	24	8
Grouping by apple treatments	3	16	16/3
Carrot treatments	3	104	104/3
Remainder	3	16	16/3

Table 2.5. Analyses of variance for apple sales Y_{ghij} and carrot sales W_{ghij} using response model equations (2.5) and (2.6).

in the sales of the two products using the correction for the mean, a throw-away in ordinary textbook analyses, and the remainder mean squares. The remaining four mean squares are interactions with product.

Example 2iii: Sums and differences for the n^2 categories of Table 2.1 are presented in Table 2.6. Analyses of variance for the $S_{ghij} = Y_{ghij} + W_{ghij}$ values and the $D_{ghij} = Y_{ghij} - W_{ghij}$ values are presented in Tables 2.7. In addition, F-values, using the remainder mean square as the demoninator of F, are computed to demonstrate that the correction for mean, or the product mean square, is used in testing.

(iv) One might wish to combine total sales as in the previous section, or one might wish to take some economic value of responses. Suppose the cost per pound of apples is c_a and the cost per pound of product two, carrots, is c_c. Then one could combine the responses from each experiment as $P_{ghij} = c_a Y_{ghij} + c_c W_{ghij}$. One could then obtain an analysis of variance table as described for S_{ghij} values above, or one might wish to compute an ANOVA table of the following form on the P_{ghij} values:

Source of variation	d.f.	
Total	n^2	
Correction for mean	1	
Product one (apples)	n-1	
Product two (carrots)	n-1	
Interaction of products	$(n-1)^2$	
Stores (columns) plus a component of interaction		n-1
Periods (rows) plus a component of interaction		n-1
Remainder of interaction		(n-1)(n-3)

$$Y_{ghij} + W_{ghij} = S_{ghij}$$

Period	Store 1	2	3	4	$S_{\cdot h\cdot\cdot}$	$\bar{s}_{\cdot h\cdot\cdot}$	$\bar{s}_{\cdot h\cdot\cdot}-\bar{s}_{\cdot\cdot\cdot\cdot}$
1	Aa 12	Bb 14	Cc 19	Dd 27	72	18	-2
2	Bd 9	Ac 23	Db 14	Ca 26	72	18	-2
3	Cb 20	Da 28	Ad 19	Bc 21	88	22	2
4	Dc 15	Cd 39	Ba 16	Ab 18	88	22	2
$S_{\cdot\cdot i\cdot}$	56	104	68	92	320	-	0
$\bar{s}_{\cdot\cdot i\cdot}$	14	26	17	23	-	$20 = \bar{s}_{\cdot\cdot\cdot\cdot}$	
$\bar{s}_{\cdot\cdot i\cdot}-\bar{s}_{\cdot\cdot\cdot\cdot}$	-6	6	-3	3	0		

$S_{\cdots A} = 72$	$\bar{s}_{\cdots A} = 18$	$\hat{\tau}_{aA} = -2$	$S_{a\cdots} = 82$	$\bar{s}_{a\cdots} = 20.5$	$\hat{\tau}_{ca} = 0.5$
$S_{\cdots B} = 60$	$\bar{s}_{\cdots B} = 15$	$\hat{\tau}_{aB} = -5$	$S_{b\cdots} = 66$	$\bar{s}_{b\cdots} = 16.5$	$\hat{\tau}_{cb} = -3.5$
$S_{\cdots C} = 104$	$\bar{s}_{\cdots C} = 26$	$\hat{\tau}_{aC} = 6$	$S_{c\cdots} = 78$	$\bar{s}_{c\cdots} = 19.5$	$\hat{\tau}_{cc} = -0.5$
$S_{\cdots D} = 84$	$\bar{s}_{\cdots D} = 21$	$\hat{\tau}_{aD} = 1$	$S_{d\cdots} = 94$	$\bar{s}_{d\cdots} = 23.5$	$\hat{\tau}_{cd} = 3.5$

$$Y_{ghij} - W_{ghij} = D_{ghij}$$

Period	Store 1	2	3	4	$D_{\cdot h\cdot\cdot}$	$\bar{d}_{\cdot h\cdot\cdot}$	$\bar{d}_{\cdot h\cdot\cdot}-\bar{d}_{\cdot\cdot\cdot\cdot}$
1	Aa -4	Bb 0	Cc 13	Dd -9	0	0	0
2	Bd -7	Ac -3	Db 4	Ca 6	0	0	0
3	Cb 12	Da 2	AD -3	Bc -3	8	2	2
4	Dc -1	Cd 1	Ba -2	AB -6	-8	-2	-2
$D_{\cdot\cdot i\cdot}$	0	0	12	-12	0	-	0
$\bar{d}_{\cdot\cdot i\cdot}$	0	0	3	- 3	-	$0 = \bar{d}_{\cdot\cdot\cdot\cdot}$	
$\bar{d}_{\cdot\cdot i\cdot}-\bar{d}_{\cdot\cdot\cdot\cdot}$	0	0	3	-3	0		

$D_{\cdots A} = -16$	$\bar{d}_{\cdots A} = -4$	$\hat{\delta}_A = -4$	$D_{a\cdots} = 2$	$\bar{d}_{a\cdots} = 0.5$	$\hat{\delta}_a = 0.5$
$D_{\cdots B} = -12$	$\bar{d}_{\cdots B} = -3$	$\hat{\delta}_B = -3$	$D_{b\cdots} = 10$	$\bar{d}_{b\cdots} = 2.5$	$\hat{\delta}_b = 2.5$
$D_{\cdots C} = 32$	$\bar{d}_{\cdots C} = 8$	$\hat{\delta}_C = 8$	$D_{c\cdots} = 6$	$\bar{d}_{c\cdots} = 1.5$	$\hat{\delta}_c = 1.5$
$D_{\cdots D} = -4$	$\bar{d}_{\cdots D} = -1$	$\hat{\delta}_D = -1$	$D_{d\cdots} = -18$	$\bar{d}_{d\cdots} = -4.5$	$\hat{\delta}_d = -4.5$

Table 2.6. S_{ghij} and D_{ghij} values for data of Example 2i.

Apple sales plus carrot sales = S_{ghij}

Source of variation	d.f.	Sum of squares	Mean square	F-value
Total	16	7224	-	-
Correction for mean	1	6400	-	-
Stores (columns)	3	360	120	10
Periods (rows)	3	64	64/3	21/9
Apple treatments	3	264	88	22/3
Carrot treatments	3	100	100/3	25/9
Remainder	3	36	12	

Apple sales minus carrot sales = D_{ghij}

Source of variation	d.f.	Sum of squares	Mean square	F-value
Total	16	584	-	-
Product = P (correction for mean)	1	0	0	0
P x Stores	3	72	24	18
P x Period	3	32	32/3	8
P x Apple treatments	3	360	120	90
P x Carrot treatments	3	116	116/3	29
Remainder	3	4	4/3	

Table 2.7. Analyses of variance and F-values for S_{ghij} and D_{ghij} values.

Unless one were making inferences to a population of levels of each product and unless one has a random sample of levels, one could not utilize the "Remainder of interaction" mean square as an error term. Instead, one would need to obtain an error mean square from theory, from previous experiments, or elsewhere. A mean square with interaction components would be inappropriate for hypothesis testing of fixed main effects.

Example 2iv: One could select various c_a and c_c prices to obtain P_{ghij} values. For analysis purposes one need only consider the ratio of prices, say c_a/c_c, and obtain $P_{ghij} = (c_a/c_c)Y_{ghij} + W_{ghij}$ for various ratios to depict the range of prices incurred in practice. Since the computations would be straightforward for the data in Table 2.1, no analyses were performed on the data.

Other combinations of data such as total calories, total protein, etc. could also be used to combine results from two commodities such as beans and maize, cowpeas and soybeans, etc.

(v) The experimenter wishes to combine the results from the two experiments, and the levels of the two products are comparable, such that level one for product one is the same as level one for product two, etc. Then, for standard response model equations of the form of (2.1) and (2.2), an ANOVA would be

Source of variation	d.f.	
Total	$2n^2$	
Correction for mean	1	
Products (squares)	1	
Stores within products	2(n-1)	
Stores		n-1
Stores X Products		n-1
Periods within Products	2(n-1)	
Periods		n-1
Periods X Products		n-1
Treatments within Products	2(n-1)	
Levels		n-1
Levels X Products		n-1
Remainder within Products	2(n-1)(n-2)	

The levels would need to be more than nominal levels of product; they would need to have practical meaning and associated for all levels.

Example 2v: Suppose that for the data in Example 2i, treatments A and a are comparable, B and b are comparable, C and c are comparable, and D and d are comparable. For example, treatment C could be standard price of apples and treatment c could be standard price for carrots. Treatments D and d could be a 20% price increase over standard. Treatments b and B could be standard pricing, but a free gift is available for those purchasing a product. Treatments A and a could be a 20% price reduction. Using response model equations (2.1) and (2.2), an ANOVA and F-statistics for the data from Example 2i are given in Table 2.8. A two-way table of product by level totals is included to indicate how to compute the Treatment and Treatment X Product sums of squares.

Source of variation	d.f.	Sum of squares	Mean square	F-values
Total	32	3904	-	-
Correction for mean	1	3200	-	-
Product (P)	1	0	0	0
Stores within Products	6	216	-	-
Stores	3	180	60	18
Stores x P	3	36	12	3.6
Periods within Products	6	48	-	-
Periods	3	32	32/3	3.2
Periods x P	3	16	16/3	1.6
Treatments within Products	6	400	-	
*Treatments	3	220	220/3	22
*Treatments x P	3	180	60	18
Remainder within Products	12	40	10/3	

*Computed from the table

Product	A=a	B=b	C=c	D=d	Sum
Apples	28	24	68	40	1600
Carrots	40	28	36	56	1600
Sum	68	52	104	96	320
Difference	-12	- 4	32	-16	0

$$[(-12)^2 + (-4)^2 + 32^2 + (-16)^2]/8 - 0 = 180$$

Table 2.8. Analysis of variance and F-statistics when treatment levels are identical for the two products.

(vi) Another method for combining the results from the two experiments is to use a bivariate analysis of variance and appropriate multivariate procedure. For response model equations (2.5) and (2.6), a bivariate analysis of variance would be of the following form:

Source of variation	d.f.	Sums of products
Total	n^2	$\begin{pmatrix} T_{yy} & T_{yw} \\ & T_{ww} \end{pmatrix}$
Correction for mean	1	$\begin{pmatrix} Y^2_{\cdots}/n^2 & Y_{\cdots}W_{\cdots}/n^2 \\ & W^2_{\cdots}/n^2 \end{pmatrix}$
Stores	n-1	$\begin{pmatrix} S_{yy} & S_{yw} \\ & S_{ww} \end{pmatrix}$
Periods	n-1	$\begin{pmatrix} P_{yy} & P_{yw} \\ & P_{ww} \end{pmatrix}$
Treatments (Product 1)	n-1	$\begin{pmatrix} V_{yy} & V_{yw} \\ & V_{ww} \end{pmatrix}$
Treatments (Product 2)	n-1	$\begin{pmatrix} U_{yy} & U_{yw} \\ & U_{ww} \end{pmatrix}$
Remainder	(n-1)(n-3)	$\begin{pmatrix} E_{yy} & E_{yw} \\ & E_{ww} \end{pmatrix}$

The procedure for computing the sums of squares and products in the above MANOVA table is straight-forward except for perhaps the last three. To compute V_{yw} use totals $Y_{\cdots j}$ and $W_{g\cdots}$, where $j=g=1,2,\cdots,n$ equals number of product one treatments. U_{yw} uses totals $Y_{g\cdots j}$ and $W_{\cdots j}$ where $j=g=1,2,\cdots,n$ equals number of product two treatments. These cross products compare a treatment response for its own product with responses from the same experimental units and with all treatments of the other product on these n experimental units. E_{yw} is obtained as the sum of the products of the n^2 pairs of residuals given by formulas (2.7) and (2.8).

After one has the sums of squares and cross products in a MANOVA table, one then proceeds in the usual manner for multivariate analysis procedures.

Example 2vi: We shall now use the data from Example 2i to illustrate the application of multivariate techniques for data from two simultaneous experiments on the same $n^2 = 16$ experimental units. A MANOVA table is given in Table 2.9. To compute the sums of squares and cross-products for Treatments (apples) use the following data from Table 2.1:

Apple sales	$Y_{\cdots A} = 28$	$Y_{\cdots B} = 24$	$Y_{\cdots C} = 68$	$Y_{\cdots D} = 40$
Carrot sales	$W_{A\cdots} = 44$	$W_{B\cdots} = 36$	$W_{C\cdots} = 36$	$W_{D\cdots} = 44$

$$\tfrac{1}{4}[28(44) + 24(36) + 68(36) + 40(44)] - \tfrac{1}{16}(160)(160) = 1576 - 1600 = -24 .$$

Likewise, the treatments (carrots) sum of products is computed as:

$$\tfrac{1}{4}[42(40) + 38(28) + 42(36) + 38(56)] - \tfrac{1}{16}(160)(160) = 1596 - 1600 = -4 .$$

The sum of products of the $n^2 = 16$ residuals in Table 2.4 is computed as:

$$[0.5(1) - 0.5(-1) - 0.5(-1) + 0.5(1) + \cdots + 0.5(1) + 0.5(1) - 0.5(-1)] = 8 .$$

Since the residuals for W_{ghij} are twice those for Y_{ghij}, the correlation of residuals is one, i.e., $8/\sqrt{4(16)} = 1$.

Source of variation	d.f.	Sums of products	Mean squares and covariance
Total	16	$\begin{bmatrix} 2000 & 1660 \\ & 1904 \end{bmatrix}$	$\begin{bmatrix} - & - \\ & - \end{bmatrix}$
Correction for mean	1	$\begin{bmatrix} 1600 & 1600 \\ & 1600 \end{bmatrix}$	$\begin{bmatrix} - & - \\ & - \end{bmatrix}$
Stores (columns)	3	$\begin{bmatrix} 72 & 72 \\ & 144 \end{bmatrix}$	$\begin{bmatrix} 24 & 24 \\ & 48 \end{bmatrix}$
Periods (rows)	3	$\begin{bmatrix} 24 & 8 \\ & 24 \end{bmatrix}$	$\begin{bmatrix} 8 & 8/3 \\ & 8 \end{bmatrix}$
Treatments (apples)	3	$\begin{bmatrix} 296 & -24 \\ & 16 \end{bmatrix}$	$\begin{bmatrix} 296/3 & -8 \\ & 16/3 \end{bmatrix}$
Treatments (carrots)	3	$\begin{bmatrix} 4 & -4 \\ & 104 \end{bmatrix}$	$\begin{bmatrix} 4/3 & -4/3 \\ & 104/3 \end{bmatrix}$
Remainder	3	$\begin{bmatrix} 4 & 8 \\ & 16 \end{bmatrix}$	$\begin{bmatrix} 4/3 & 8/3 \\ & 16/3 \end{bmatrix}$

Table 2.9. MANOVA for the data in Example 2i.

This means that $\begin{vmatrix} E_{yy} & E_{yw} \\ E_{yw} & E_{ww} \end{vmatrix} = \begin{vmatrix} 4 & 8 \\ 8 & 16 \end{vmatrix} = 0$ and that statistics like

$\begin{vmatrix} E_{yy} & E_{yw} \\ E_{yw} & E_{ww} \end{vmatrix} \div \begin{vmatrix} E_{yy}+U_{yy} & E_{yw}+U_{yw} \\ E_{hw}+U_{yw} & E_{ww}+U_{ww} \end{vmatrix}$ are not practically useful. All the information on the remainder variances is given by either variable Y or W. One can easily construct examples where this is not the case.

3. Extension to More Than Two Designs

Suppose that one sets up t experiments in t mutually orthogonal latin square designs, or alternatively in t mutually balanced Youden designs. Straightforward extensions of the six analyses in section 2 are possible. For method (i), one simply computes the analyses for each of the t experiments separately without any reference to the remaining t-1 experiments.

For method (ii), one groups or stratifies one set of treatment for (t-1) other sets which results in a straightforward extension of equations (2.5) and (2.6) for possible response model equations of the form:

$$Y_{1fghij} = \mu_1 + \sum_{f=2}^{t} \delta_{1fg} + \rho_{1h} + \gamma_{1i} + \tau_{1j} + \epsilon_{1fghij} \qquad (3.1)$$

$$Y_{2fghij} = \mu_2 + \sum_{f=1\neq 2}^{t} \delta_{2fg} + \rho_{2h} + \gamma_{2i} + \tau_{2j} + \epsilon_{2fghij} \qquad (3.2)$$

$$\vdots$$

$$Y_{tfghij} - \mu_t + \sum_{f=1}^{t-1} \delta_{tfg} + \rho_{th} + \gamma_{ti} + \tau_{ti} + \epsilon_{tfghij} \qquad (3.3)$$

with corresponding residual equations:

$$e_{1fghij} = Y_{1fghij} - \sum_{f=2}^{t} \bar{y}_{1fg\cdots} - \bar{y}_{1\cdot\cdot h\cdot\cdot} - \bar{y}_{1\cdots i\cdot} - \bar{y}_{1\cdots\cdot j} + (t+1)\bar{y}_{1\cdots\cdot\cdot} \qquad (3.4)$$

$$e_{2fghij} = Y_{2fghij} - \sum_{f=1, f\neq 2}^{t} \bar{y}_{1fg\cdots} - \bar{y}_{2\cdot\cdot h\cdot\cdot} - \bar{y}_{2\cdots i\cdot} - \bar{y}_{2\cdots\cdot j} + (t+1)\bar{y}_{2\cdots\cdot\cdot} \qquad (3.5)$$

$$\vdots$$

$$\hat{e}_{tfghij} = Y_{tfghij} - \sum_{f=1}^{t-1} \bar{y}_{tfg\cdots} - \bar{y}_{t\cdot\cdot h\cdot\cdot} - \bar{y}_{t\cdots i\cdot} - \bar{y}_{t\cdots\cdot j} + (t+1)\bar{y}_{t\cdots\cdot\cdot} \qquad (3.6)$$

The parameters are as defined previously with obvious extensions. An ANOVA partitioning of degrees of freedom for, say, the first response equation for t mutually orthogonal latin squares, is:

Source of variation	d.f.
Total	n^2
Correction for mean	1
Rows	n-1
Columns	n-1
Stratification by t-1 sets of treatments	(t-1)(n-1)
First set of treatments	n-1
Remainder	n^2-1-(t+2)(n-1)

For t mutually balanced Youden designs, the partitioning of degrees of freedom for response model equations (3.1) to (3.3) for the response in (3.1) is:

Source of variation	d.f.
Total	nk
Correction for mean	1
Rows	k-1
Columns	n-1
Stratification by t-1 sets of treatments (eliminating column effects)	(t-1)(n-1)
First set of treatments (eliminating columns and (t-1) groupings by other sets of treatments)	n-1
Remainder	nk-k-(t+1)(n-1)

For method (iii), sums and differences, one may set up (t-1) orthogonal contrasts among the t sets of treatments and have t-1 sums and t-1 differences in the same type of analyses as for two sets of treatments, or one may set up a partitioning of degrees of freedom in an ANOVA as follows for t mutually orthogonal latin square designs:

Source of variation	d.f.
Total	tn^2
Correction for mean	1
Rows	n-1
Columns	n-1
Stratification	t(n-1)
Remainder	n^2-1-(t+2)(n-1)
t sets of treatments = T	t-1
T x rows	(t-1)(n-1)
T x columns	(t-1)(n-1)
T x t sets of treatments	t(t-1)(n-1)
T x Remainder	(t-1)[n^2-1-(t+2)(n-1)]

For method (iv), one may take any linear combination of the t responses, say $\Sigma_{e=1}^{t} c_{efghij} Y_{efghij}$, where c_{efghij} are costs, relative amount of protein, usefulness, etc., and conduct an ANOVA using these n^2 responses. The ANOVA would be similar to the first one in this section for an MOL(n,t)-set or for the second ANOVA for a set of t mutually balanced Youden designs.

For method (v) involving n comparable levels for the treatments in each of the t sets and for a MOL(n,t)-set of latin squares, one could partition the degrees of freedom as follows:

Source of variation	d.f.	
Total	tn^2	
Correction for mean	1	
Sets	t-1	
Rows within sets	t(n-1)	
Rows		n-1
Rows by sets		(t-1)(n-1)
Columns within sets	t(n-1)	
Columns		n-1
Columns x sets		(t-1)(n-1)
Treatments within sets	t(n-1)	
Levels		n-1
Levels x sets		(t-1)(n-1)
Stratification by other treatments within sets	t(t-1)(n-1)	
Remainder within sets	$t[n^2-1-(t+2)(n-1)]$	

For method (vi), the extension from the bivariate multivariate analysis to the t-variate multivariate analysis is straightforward. The sums of squares and products matrix becomes t x t instead of 2 x 2, and the MANOVA partitioning of degrees of freedom is that for the first ANOVA of this section.

4. Connectedness of Designs and Analysis

In using a MOL(n,t)-set of latin squares and whenever t = n-1, there are zero degrees of freedom for the remainder sum of squares for methods (ii) - (vi). Over-stratification can result in disconnected designs. For example, consider the following array where the first two rows define the rows and columns of the array, and the first four rows form an orthogonal array (a MOL(3,2)-set):

0 0 0	1 1 1	2 2 2
0 1 2	0 1 2	0 1 2
0 1 2	1 2 0	2 0 1
0 1 2	2 0 1	1 2 0
2 0 1	0 1 2	1 2 0
2 0 1	1 2 0	0 1 2
1 2 0	0 1 2	2 0 1
1 2 0	2 0 1	0 1 2

Use of more than two 3 X 3 latin squares results in disconnected designs.

Likewise, if one has n-1 MBY(n,n-1,n-2) Youden designs, there are only n(n-1) observations for one response variable for the n^2 degrees of freedom. In an ANOVA partitioning of degrees of freedom there would be:

Source of variation	d.f.
Total	n(n-1)
Correction for mean	1
Rows	n-2
Columns	n-1
(n-1) sets of treatments	$(n-1)^2$
Remainder	-(n-1)

Thus, the design is overparameterized by (n-1) degrees of freedom. To illustrate, consider the MOL(5,4)-set with the last rows of the squares omitted, i.e.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
omitted				

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
omitted				

1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
omitted				

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
omitted				

An ANOVA is:

Source of variation	d.f.
Total	20
Correction for mean	1
Rows	3
Columns	4
First set of treatments	4
Second set of treatments	4
Third set of treatments	4
Fourth set of treatments	4
Remainder	-4

-(n-1) = -4 degrees of freedom are associated with the Remainder line in the ANOVA, indicating the degree of overparameterization.

Whenever the number of rows k becomes smaller, the overparameterization becomes greater when using the full set of mutually balanced Youden designs. This fact has not been considered by researchers who construct sets of the various designs. Using more than n-1 latin squares could be useful in coding theory, as this is one method of widening a code by having more than n+1 rows in an array. The code is lengthened merely by repeating the array as many times as desired. For three symbols we obtained an 8-row by 9-column array above.

For four symbols one can obtain 23 rows by 16 columns as follows:

0 0 0 0	1 1 1 1	2 2 2 2	3 3 3 3
0 1 2 3	0 1 2 3	0 1 2 3	0 1 2 3
0 1 2 3	1 0 2 3	2 3 0 1	3 2 1 0
0 1 2 3	3 2 1 0	1 0 2 3	2 3 0 1
0 1 2 3	2 3 0 1	3 2 1 0	1 0 2 3
1 0 2 3	0 1 2 3	2 3 0 1	3 2 1 0
1 0 2 3	3 2 1 0	0 1 2 3	2 3 0 1
1 0 2 3	2 3 0 1	3 2 1 0	0 1 2 3
1 0 2 3	2 3 0 1	0 1 2 3	3 2 1 0
1 0 2 3	3 2 1 0	2 3 0 1	0 1 2 3
1 0 2 3	0 1 2 3	3 2 1 0	2 3 0 1
2 3 0 1	0 1 2 3	1 0 2 3	3 2 1 0
2 3 0 1	3 2 1 0	0 1 2 3	1 0 2 3
2 3 0 1	1 0 2 3	3 2 1 0	0 1 2 3
2 3 0 1	1 0 2 3	0 1 2 3	3 2 1 0
2 3 0 1	3 2 1 0	1 0 2 3	0 1 2 3
2 3 0 1	1 0 2 3	3 2 1 0	1 0 2 3
3 2 1 0	0 1 2 3	1 0 2 3	2 3 0 1
3 2 1 0	2 3 0 1	0 1 2 3	1 0 2 3
3 2 1 0	1 0 2 3	2 3 0 1	0 1 2 3
3 2 1 0	1 0 2 3	0 1 2 3	2 3 0 1
3 2 1 0	2 3 0 1	1 0 2 3	0 1 2 3
3 2 1 0	0 1 2 3	2 3 0 1	1 0 2 3

The first two rows define the rows and columns of the latin square, and the first five rows define an orthogonal array of five rows and 16 columns (a MOL(4,3)-set).

5. Discussion

An early use of a complete set of orthogonal 5 x 5 latin squares is described in Tippett (1936) and Fisher (1937), section 35.1. The latter points out that interactions between sets of treatments must be negligible or absent in order to make valid statements about the treatments in each set. This set is being used as a 5^{-4} fractional replicate of a 5^6 factorial. McNemar (1951) emphasizes this fact and again points out that a latin square can be used as a fractional replicate provided there is no interaction between factors. The last assumption he considered to be mostly untenable in psychological research. Grant (1948) essentially presented the response model equation and ANOVA for a pair of orthogonal latin squares as described in method (ii).

Although we have confined our discussion to sets of orthogonal latin squares and balanced Youdens, we could have used other types of row-column designs. For example, dropping a row from or adding one to a Youden results in partially balanced row-column designs. These could be used if desired. The computations become more difficult, due to the lack of orthogonality or balance. We also could have considered orthogonal F-squares, orthogonal latin and F-cubes and hyper-cubes. The concepts would be the same.

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