CHAPTER VIII.

KINEMATIC ANALYSIS.

§ 63.

The Problems of Kinematic Analysis.

THE analysis of a kinematic arrangement as such consists in separating it into those parts which may be regarded kinematically as elements, and in determining the manner in which these are combined into pairs and kinematic chains. All constructive details are left out of the question. The notation which we have formed gives us the means of representing the results of the analysis in a form which can be easily surveyed, and which distinctly expresses the law of their connection. We shall now undertake a series of such investigations; partly in order to show how the method of analysis is applied, but principally in order to determine clearly the nature of certain important subdivisions of Machine-science. Our work will show us that hitherto there has been an entire want of definiteness about many fundamental ideas, with which nevertheless it has been thought easy to operate. We shall have to rectify many common notions; indeed we shall find necessary the destruction, or at least total transformation, of some propositions apparently universal. As compensation for this, however, we shall be able to place on a really scientific basis other conceptions of even greater meaning and weight.

§ 64.

The "Mechanical Powers" or "Simple Machines."

The mechanical arrangements which go by the name of "mechanical powers" or "simple machines" are familiar to all. Since the time of Galileo, or before it, they have been described in the majority of text-books as those arrangements to which, to a greater or less extent, all machines can be traced back,—of which, in other words, they may all be regarded as compounded. As to the h o w and the w h e t h e r, however, there has not been complete agreement; and it is specially noticeable, and at first sight astonishing, that the h i g h e r Mechanics has more and more separated itself from any connection with these arrangements. For if they have really the meaning put upon them,—and the contrary, in spite of the sceptics, is nowhere shown,—they should here only acquire a higher value. The highest science could not then venture to overlook them,—however homely or trifling they might appear to be,—while in point of fact the notion seems to be gaining ground that while the "simple machines" are good enough for elementary mechanics, they are worthless for the higher part of the science.

If we look more closely into the question, and compare one text-book with another, we discover everywhere a doubtfulness as to the real significance of the ideas of which they yet retain the outward form.[46] Even as to the number of "mechanical powers" there is no unanimity. Some speak of six—Lever, Inclined Plane, Wedge, Pulley, Wheel and Axle, Screw;—while others would include unconditionally the "funicular machine"* as a seventh. The definition of the "simple machine" fares even worse—no two books can agree upon one. The most various places also are given to them in the treatment of the subject. Sometimes they stand at the beginning, sometimes in the middle, sometimes at the end, somctimes taken in different chapters; sometimes they are treated of without being called by their traditional names, as if with the suspicion that if they were acknowledged nothing

* A cord suspended from both ends, and having weights attached to it at different points. I have not noticed this among the mechanical powers in English works, but here generally the "toothed wheel" takes its place— not to mention the "compound wheel and axle," &c., occasionally met with.

could be done with them. In short, such a comparison shows that there is no common idea really underlying the matter, for the differences are more than superficial; it rather leads to doubt as to whether the "simple machines" have any right whatever to their name.

And yet there is something specially characteristic in these arrangements,—at least in some of them, as, *e.g.*, the lever and the inclined plane,—which have so entirely passed from a special department into common language and ideas. There is something homely and familiar about them, they excite, I might almost say, a sentimental interest. Does this merely result from recollections of youthful mechanical study, or is it a breath from the childhood of science itself playing upon us? Or has this sympathy, to which even the most abstract theorist would probably have to acknowledge in his quiet moments, really no deeper

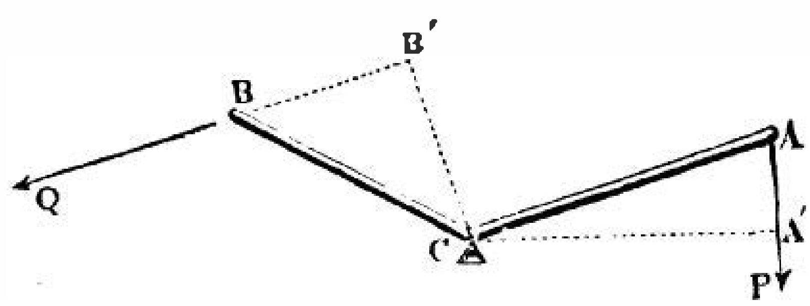


FIG. 192.

ground? Kinematic analysis must give us a distinct answer to these questions; it must show us whether we have really to give up these old heirlooms of mechanics, and if so it must enable us to remove them altogether, or whether there is not something really indestructible in them. Let us proceed with the examination.

The Lever.—A straight bar or knee-shaped body supported upon a fixed angular bearing, about which it can turn, (Fig. 192); two forces act on the bar on the one or the other side of the support; their equilibrium is to be studied. The problem has been stated thus since the time of Archimedes. In most cases the description is not exact. It is assumed, but not distinctly stated, that the support is so arranged that only plane motions can occur; it remains unsaid that in cases where the direction of the forces is such as to move the lever from the support, this does not occur, in other words that it is prevented by suitable restraint. We have here certainly an incomplc teness in the statemen ttof conditions whichtis very extraordinary in the case of ant important fundamental proposition. If we supply these defects we have the bodies, lever, and support so arranged that their relative motions are constrained, and that each is free only to rotate relatively to the other This, however, is nothing else than the arrangement of the turning pair $R_{=}^{+}R^{-}$, or (see § 57) $C_{=}^{+}C^{-}$, and will be called, according as one or the other elements be fixed

$$R_{=}^{-}R^{+} \text{ or } R_{=}^{+}R^{-}$$

or otherwise

$$C_{=}^{-}C^{+} \text{ or } C_{=}^{+}C^{-}$$

and the "principle of the lever" is simply the conditions of equilibrium of the forces in a turning pair. The pair is usually represented, however, as incomplete and force-closed, in principle as in Fig 193, for which the formula stands: $\frac{C_{=}^{-}\;C^{+}}{f}$.

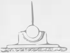


FIG. 193.

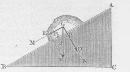


FIG. 194.

The Inclined Plane.—A surface oblique to theplane of the horizon, having a body resting upon it, touching it throughout a plane section, and tending by its weight to slide downwards (Fig. 194); the magnitude of the force necessary to prevent this sliding is studied. Here again the descriptiont leaves much to be wished. Itis, as a rule, left unexpressed that the body canonly slide parallel to the greatest slope of the plane,—that is, the necessary bodily restraint in other directions is imagined,—and means are also imagined to exist by which it is prevented from leaving the plane. In other words, it is tacitly assumed that the sliding body with the one below it are paired for rectilinear motion, and the pair under the supposed conditions is simply a sliding-pair, written, according as one or the other element be fixed.

$$\underline{P}^{\pm}_{=} P^{-} \text{ or } \underline{P}^{-}_{=} P^{+}.$$

The complete "principle of the inclined plane" gives the conditions of equilibrium of the forces in a sliding pair. The common representations show, as in the last case, an incomplete, force-closed pair, which would be written $\underline{P}_{=}\frac{P^{+}}{f}$,

The Wedge.—This arrangement is commonly represented in a very primitive form, and one almost entirely wanting in the strictness of machinal motion, namely, as a means for splitting a piece of wood, Fig. 195. In this very rural-looking apparatus the ratio of the driving effort to the lateral resistances against the sides of the wedge is investigated. If we complete the description—which

FIG. 195.

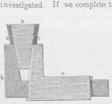


FIG. 196.

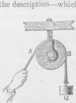


FIG. 197.

as a rule is so entirely wanting in definiteness—sufficiently to make it applicable to a machinal system, we may say shortly that the two sides of the wedge are imagined to be prismatically paired with surfaces against which they work,—and further, that the latter (the halves of the tree stem), having, as they are separated, relatively a rectilinear motion, are also imagined to be paired in the same way. The whole represents a mechanism formed of a three-linked prism chain (Fig. 196), of which the formula, taking the links in the order a, b, c, would be:

$$P^{+}_{\smile} \ldots \angle \ldots P^{\pm}_{=} P^{-} \ldots \angle \ldots \underline{P^{+}P \quad \ldots \angle \ldots P^{-}_{=}}.*$$

* More strictly, as the chain is force-closed throughout, this should be,

$$\frac{P^{+}}{f} \cdot \angle \cdot \frac{P^{+}_{=}}{f}\frac{P^{-}}{f} \quad \angle \quad \frac{P^{+}_{=}}{f}\underline{P^{-} \ldots \angle \ldots P^{-}_{\equiv}} \text{ or } \left(\frac{P^{\angle}_{3}}{f}\right)^{c}_{a}$$

The "principle of the wedge," if it be expressed in a sufficiently general form, gives the conditions of equilibrium of the forces in this chain. The traditional representation stands for a combination of bodies, force-closed throughout, which only roughly approximates to the combination really intended.

The Pulley.—A disc turning about a fixed pin, and having a grooved periphery over which rests a rope stretched at both ends (Fig. 197); the equilibrium of the forces acting at the ends of the rope and upon the pin is studied. The pulley takes a remarkable position among the simple machines. In the first place, we have here not two but three bodies used in combination. As a rule no mention is made of the assumption that the bearings of the pulley are supposed to be such as to prevent cross motions. Then again it is remarkable that while here a force-closed element, the rope, is employed, there is very insufficient recognition of its characteristic property of one-sided resistance. If the bracket for the pulley-spindle be considered as fixed, the kinematic formula for the chain is as follows :—

$$\underline{C^{\pm}}\, C^{-} \ldots \mid \ldots R^{+}, T \ldots \begin{cases} \ldots \dfrac{T^{-}}{f} \\ \ldots \dfrac{T^{+}}{f} \end{cases}$$

a mechanism of three links covering very indefinite motions, which approximate to machinal strictness only in consequence of force-closure.

The mechanism is commonly known as the fixed pulley, but under the head "pulley" another arrangement, the loose pulley (Fig. 198), is usually treated. Here the pulley frame is movable and loaded, and one end of the rope fixed, as in the formula.

$$\frac{C^{\pm}}{f}\, C^{-} \ldots \mid \ldots R^{+}, T \ldots \begin{cases} \ldots \dfrac{T^{-}}{f} \\ \ldots \underline{T^{+}} \end{cases}$$

This expression differs from the former only in the link which is fixed. The old mechanicians have busied themselves with the inversion of a kinematic chain! In the loose pulley also force-closure is applied to the fullest extent.

The Wheel and Axle.—Two drums of different diameters fixed together and having a common shaft, each having one end of a rope which is loaded at the other fixed to it; the shaft works in fixed bearings, or at least is imagined to do so, for the bearings are often enough omitted in the drawing (Fig. 199); the equilibrium of the forces is studied. Again the problem is wanting in clearness, and is only solved by the employment of a number of

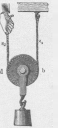


Fig. 198.

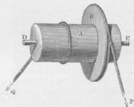


Fig. 199.

abstract assumptions, for the most part not expressed. Supposing the axle bearings fixed the chain runs

$$\underline{C_{=}}\, C^{+} \ldots \left\{ \begin{array}{l} \ldots R^{+}, \dfrac{T^{-}}{f} \\ \ldots R^{-}, \dfrac{T^{+}}{f} \end{array} \right.$$

All the indefiniteness which we saw existing in the assumptions in the former cases exists also here. Indeed they are increased by the helical winding off and on of the cords, which occurs, too, so that their axes must describe higher screw-lines if artificial means of preventing it are not supposed to exist, or if the difficulty be not got over by the supposition of infinitely thin cords. This last is very common. The sense of the necessity for eliminating these complicated motions of the cord has led many to omit it altogether, replacing it merely by tangential forces acting upon the peripheries of the drums. This, however, makes the problem simply a repetition of that of the lever, which was not its original meaning.

The Screw.—A screw placed vertically working in a fixed nut and loaded by a weight (Fig. 200); the force which has to be applied, normal to a radius and at some point not in the axis of the screw, in order to balance the load, is determined. We recognise at once the twisting-pair, written either

[illegible]

The "principle of the screw" is a very limited, indeed incomplete, case of the equilibrium of forces in a twisting-pair.

The Funicular Machine.—This, lastly, is a problem which,—apart from its value in pure Mechanics when put into an abstract form, is so far removed from the machinal idea by its extended force-closure and the indefiniteness of its motions, that it obviously has no right to a place among "simple machines," and we need not therefore consider it here.*

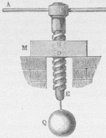


FIG. 200.

As a whole, the result at which we have arrived is very remarkable. We find in the simple machines, which of all others ought to appear harmoniously related, a crude mixture of kinematic problems—closed and unclosed pairs, and chains mistaken for pairs, arrangements mostly force-closed—among them the tension-organ with all its difficulties of treatment,—and in addition an experiment in the inversion of a mechanism. We have been compelled to recognise, too, that in their usual treatment there is an extraordinary inexactness in stating the problems, which can hardly tend to give the beginner clear ideas. The explanation of

* The "toothed wheel" in the form in which it appears among the "mechanical powers" is really the mechanism $(C_z^+ C_2'')^c$ which is shown in Fig. 183. Precisely the same chain placed upon another link, viz. $(C_z^+ C_2'')^a$, forms an epicyclic train, which is treated not as a simple machine but as a more or less difficult case of "aggregate motion."

all this may be found in the general mode of development of machinal ideas which we have already studied, and under which we have seen the early machines to have grown up gradually from force-closed combinations of fixed and moving bodies. In the history of machine-development the simple machines formed the first experiment at a scientific arrangement of existing material; the same train of ideas which governed its phenomena as a whole repeated itself upon a smaller scale in the early attempts at the scientific explanation of what had been empirically determined.

Beyond this, we may ask further whether, when the necessary strictness of conception and definition has been obtained, the "mechanical powers" do really constitute the elementary parts of all machines? The answer must be most distinctly negative.

Three of the simple machines indeed, stripped of their conventional disguise, are no other than the three lower pairs (R), (P) and (S),—and another the higher pair $\mathbf{R},\mathbf{T}$; but all the other higher pairs are wanting, while there is no representative of the pressure organs, not to speak of the springs. With steam-engines and pumps—the triumphs of pressure organs—before us, how is it possible to assert that the traditional simple machines have formed the foundation for all others? It seems scarcely conceivable that this should ever have been said. It has been so far modified as to be replaced by the statement that all the static problems of machinery were contained in the simple machines, and that it was this that gave them their importance and formed the real connection between them. This also, however, is incorrect. The "principle of the lever" does not teach the relations among forces in the higher cylinder-pairs—for that purpose we have to go back to the infinitely small instantaneous motions—nor in the hyperboloidic pair. There are many dynamic problems in machinery of which the simple machines teach us nothing. In themselves they teach nothing of couples, and they leave entirely without notice the application of fluid-organs as elements in machinery, although they recognise their contra-positives the tension-organs. In short, the assertion that all machines can be traced back to those which have received the name of "simple" is justified from no point of view whatever.

We can now well understand the increasing fear of recognising the simple machines, in spite of their historical position, which appears in modern text-books, and we see also the reason of the neglectful treatment they have received from the higher mechanics, but our investigations have shown us something which helps to explain the attachment to these old and well preserved problems. This no doubt rests chiefly upon the fact that three of them, the lever, inclined-plane and screw, represent pairs of elements,—perhaps also upon the existence in another, the pulley, of a timid step towards a free and exhaustive treatment of a kinematic chain.

It was therefore in the first place an indistinct feeling that the motions of a machine were founded upon those of pairs of bodies, which led to the "simple machines." In point of fact they have, as it were, felt the way in this direction. It is this that has allowed the lever, inclined-plane and screw—to which we arrived by a priori reasoning as the three lower pairs (§15)—to take such deep root. The faint trace of the law of the kinematic chain which appears in the two forms of pulley is both interesting and striking—only to this extent do the venerable problems seem justified. I think, however, that our examination of them has shown that this whole department of elementary Mechanics, whether treated by itself or as a part of Physics,—in text-books or orally,—absolutely requires a very searching revision.

§ 65.

The Quadric (Cylindric) Crank Chain (C''_4).

The kinematic chain which consists of four links connected by parallel cylinder pairs, and which has already repeatedly engaged our attention, is one of the most important chains occurring in practical machine-construction, and we shall now proceed to its analysis. Its complete treatment belongs to applied and not to theoretic Kinematics; our purpose here is not its exhaustive treatment, but simply the examination of the various forms in which it is applied as a mechanism. We shall find that they have very great variety.

We may look first at the train already described in § 62 and shown in Fig. 201,—where the four links are so proportioned that,

d beingfixed, a can revolve while c swings about its axis. For this wemust alwaysthavethetconditio ns

$$a + b + c > d \qquad a + d + c \geq b$$
$$a + b - c \leq b \qquad a + b - c \leq b$$

and a the smallest of the four links ; the lettershere standing for thelengths of thelinksbetweenthetcentres of the pins.

The parallelism of the cylinder pairs makes all the centroids plarrefigures, and allthe axoids cylinders. In its applied forms the link a is always kno wn as a crank, and from this we may call the chain a cylindric crank-quadrilateral, or, more concisely, a

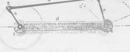


Fig. 201.

quadric (cylindric) crank-chain. The mechanisms obtained by fixing one or other of the links will then be called quadric (cylindric) crank mechanisms orttrains. Thedesignation cylindric requires to be retained, as we shall presently beco met acquainted with crank mechanisms of another kind. The mechanisms occurring are four in number, their contracted formulæ being $(C''_4)^a$, $(C''_4)^b$, $(C''_4)^a$, $(C''_4)^c$. We may takethemtbriefly in order.

The mechanism $(C''_4)^d$. We have met with this mechanism often eno ugh to be no w tolerably familiar with it. Its links possess such totally distinct functions that we may venture to use forthem distinct names, this will enable us make our descriptions shorter and more exact. We shall call

a the crank $\qquad$ c the lever
b the coupler $\qquad$ d the frame.*

The characteristic of themechanism (Fig. 201) is that it has both a crank and a lever among its moving links, the one turning while

* I propose to distinguish between links that can turn completely round their centres andthose that can only swing to andfro by callingthem cranks and levers respectively. I donot think thiswill lead to anyconfusion, and it often greatly simplifies the nomenclature of thetrains, as will be seen further on. For brevity's sake I have used coupler instead of the more common, but much longer name connecting rod.

themother swings; from this peculiarity wemay call it a lever-and-crank train, or simply a lever-crank

The mechanism $(C''_4)^b$. If wenow placethe chain on b, that is, release thenframe d and fixinstead of it the coupler b (Fig. 202) we obtain a mechanism inwhich a and c again turn and swing respec-

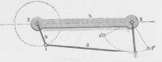


Fig. 202.

tively, butnow about the centres 2 and 3 instead of 1 and 4. The frame d has become the coupler, and thecoupler b the frame. The whole is still a lever-crank, andndiffers from the former only in the relativelengths of the coupler and frame. There is thereforenno difference in kind between the two mechanisms, and we have $(C'')^d n = (C''_4)^b$

The mechanism $(C''_4)^a$. If the link a be made the frame, Fig. 203, we obtain the entirely different mechanism, one which we have previously examined in § 9. The links b and d rotate abouttthe axes 2 and 1,—that is, they become cranks,—c, on the other hand, becomes the coupler. The mechanism is known in practice as a drag-link coupling, weshallcall itthedouble-crank. Thecranks move with varying angular velocity ratio in a way which we were able to represent conveniently by the aid of reduced centroids in Fig. 25.

The mechanism $(C''_4)^c$. In this last arrangement the links b and d swing to right and left about their axes 3 and 4; c has become the frame, and a the coupler. In the position 4 1' 2 3 shown in dotted lines, b has completed its swing to the right; as it returns, however, d can movensomewhat further to the right and thenwill swing in thensame direction until at 1" it reaches the leftnlimitof itstravel. As it returns b in its turn moves further tothe leftand thenreturns as d didbefore:—4 1''' 2''' 3 shows an intermediate position with the links crossed. We may call this mechanism,—which is frequentlymsed in the parallel motions of machinery, but then not to the limits of its motion,—the double lever. This will indicate its relation to the mechanism $(C''_4)^a$, in which the arms b and d turn instead of swinging.*

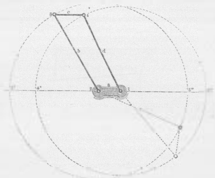


Fig. 203.

Hereowe have exhausted the methods of placing the chain (C''_4), and have found that three out of the four mechanisms

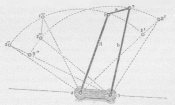


Fig. 204.

* Prof. Reuleaux uses " Revolving double crank " and "Oscillating double crank" for $(C''_4)^a$ and $(C''_4)^c$ respectively. By using the words crank and lever, as I have proposed, we can thus greatly shorten the names without, I think, making them indefinite.

belong to different classes. The three different kinds of motion obtained are, as we know, simply those relative motions in the chain which we have made absolute, or more strictly speaking absolute "for us," by fixing one or other of the links (see § 3.) The most frequently used of the four mechanisms is $(C_4)^d = (C''_4)^b$; or putting the two formulæ together, $(C''_4)^{d=b}$.

§ 66.

Parallel Cranks.

It is obvious that by altering the relative lengths of the links in the chain (C''_4) we alter the mechanisms to be obtained from it, and therefore the resulting motions,—for by extending the angle of oscillation we can convert relative swinging into rotation and *vice versâ*. We shall consider the most important special cases which arise here. In the original mechanism we had $a < c$, if the difference between them be reduced until $a = c$, and if at the same time b be made $= d$ the crank chain becomes a parallelogram, as Fig. 205. The lever c becomes a crank equal to a, and (d being fixed) it moves always through the same angle.

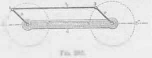


Fig. 205.

The contracted symbol for the chain, the opposite links being always parallel, is $(C''_2 \| C''_2)$. It is unnecessary to use the sign $\#$, for the $\|$ is by itself sufficient to exclude the crossing which, as far as the construction of the chain itself is concerned, is possible (§ 47). The sign of equality, on the other hand, would not be sufficient by itself, for the equality of pairs might be $a = b$ and $c = d$, which would allow $a <$ or $> c$, and would therefore be inconsistent with our conditions. The sign $\#$ may be reserved for the case where the parallelogram is a rhombus.

If the mechanism be placed on d, as in Fig. 205, its formula runs $(C''_2 \| C''_2)^d$. It falls into the same class whether it be placed on b or c or a; so that all the four mechanisms with which the chain furnishes us are similar. We shall call them Parallel Cranks.

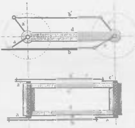


Fig. 206.

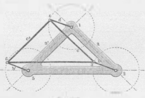


Fig. 207.

We have already seen that in the dead positions 2' 1 3' 4 and 1 2'' 4 3'' the chain is not constrainedly closed. If the it is to be used so that the points 2' and 2'' can be passed some special closure must be arranged. We have found (§ 46) that this could be done by the addition of another similar chain in the two ways, among others, shown in Figs. 206 and 207. We have now to find means for indicating these in our kinematic notation.

We have here chain-closure. It may therefore be indicated, as mentioned in § 57, by placing the sign k as a divisor below the original formula, so that both chains could be written $\frac{(C''_2 \parallel C''_2)}{k}$. But the addition to the k of the sign of equality, and the inclosure of both in brackets will allow us to make distinct that the closing chain is equal to the one closed. The formula would then be $\frac{(C''_2 \parallel C''_2)}{(k =)s}$, or in words: a pair of parallel cranks closed by another pair of parallel cranks. We may, however, choose a still more convenient way of indicating the combination of two chains which are both equal and reciprocally closing, namely, by adding the factor 2 to the formula for the single chain: $2\ (C''_2 \parallel C''_2)$.

There is, lastly, another doubtful point to make clear,—the difference between the two arrangements of Figs. 206 and 207. In the first case the cranks of the closing chain are rigidly connected into links with those of the other;—in the second, one of the cranks of the closing chain appears to be identical with one of those of the primary chain, the other being separately constructed but connected by a coupler also with the second primary crank. If however we compare the two chains more carefully, and in their most abstract forms,—so as to see distinctly what is actually before us,—we find that the two chains (not the mechanisms) are identical. The ternary links $a\,a'$ and $c\,c'$ of the chain Fig. 206 correspond to the ternary links $a\,a'$ and $c\,c'$ of Fig. 207,—and the binary links d, b and b' of the first to those similarly lettered in the second. If then, as the figures indicate, the chains be made into mechanisms by placing them upon $d\,d'$ and $a\,a'$ respectively, the second is nothing more than an inversion of the first, so that the difference between the two will be indicated in the general formulæ, $2\ (C''_2 \parallel C''_2)^d$ for Fig. 206, and $2\ (C''_2 \parallel C''_2)^a$ for Fig. 207. They are both formed from the same five-linked chain, and they are examples of the only two classes of mechanisms into which this chain can be formed.

§67

Anti-parallel Cranks.

By means of pair-closure we can, as we have already seen in § 47, convert the crank parallelogram into an anti-parallelogram, and this can be so constrained as to retain its special property in every position. Figs. 208 and 209 represent the two forms of this chain, pair-closed, which we have already considered. We may call the mechanisms to be formed from it anti-parallel

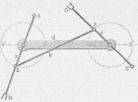
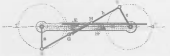
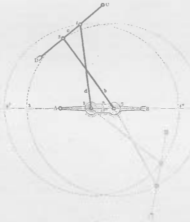


Fig. 208

crank trains. Two different results can be obtained by the different modes of placing the chain, one if it be placed on d or b, the other if a or c be the fixed link. If d be fixed, as is supposed in both the figures, the two cranks turn in opposite directions, or reversely, for which reason I have already given the mechanism the name of reverse cranks (§ 47, Fig. 155). If, however, the chain be placed on a (Fig. 210), so that c becomes the coupler and the former coupler and frame both become cranks, then b and d both revolve continuously, but in the same direction, or we may say conversely. In the first case the anti-parallelogram gives us reverse anti-parallel cranks, in the second converse anti-parallel cranks. It should be noticed that the nature of the relative rotations is the same in both cases. This arises from the equality, — through the anti-parallelism, of the angles 123 and 143. We may therefore use a pair of congruent ellipses as reduced centroids (§ 9) for the hyperbolæ (§ 47), the fo rm of which necessarily makes it somewhat difficult to realise the motio ns they re present.



The contracted formulæ for these mechanisms must, in the first place, make their characteristic property of anti-parallelism clear, —we therefore put its symbol between those of the cylinder



pairs. The chain, unfixed, will then be written $(C''_2 \angle C''_2)$. The reverse anti-parallel cranks will be $(C''_2 \quad C''_2)^d$ or $(C''_2 \angle C''_2)^b$, of which formulæ we need use only one,—let it be the former,—

unless we wish to combine the two expressions in $(C_2'' \mathbin{Z} C_2'')^{d\text{—}b}$. The converse anti-parallel cranks will be $(C_2'' \mathbin{Z} C_2'')^{a}$, if in the same way we omit the exponent c as superfluous, or $(C_2'' \mathbin{Z} C_2'')^{a\text{—}c}$ if we wish to express the fact that the chain placed either on a or on c gives the same mechanism.

The pair-closure has still to be indicated. This will only be necessary if the action of the mechanism extends over the dead points. If the closure exists, and if it be arranged as in Fig. 208, the formula will run $\dfrac{(C_2'' \mathbin{Z} C_2'')^{d}}{(p)\ a.c}$; if as in Fig. 209, $\dfrac{(C_2'' \mathbin{Z} C_2'')^{d}}{(p)\ b.d}$; where the existence of the pair-closure is denoted by p (see end of § 57), while the brackets and the addition of the symbols for the paired links sufficiently indicate the rest. It will frequently, however, be unnecessary specially to indicate the pair-closure, for the maintenance of the anti-parallelism,—the assumption, that is, of the continued validity of the sign $\mathbin{Z}$,—presupposes it. The anti-parallel cranks have here and there been used, but without being recognised; Dübs's locomotive coupling is an instance, and here the ellipses actually serve as profiles for the buffers.*

§ 68.

The Isosceles Crank-train.

We obtain a special case of the chain (C_4'') which has very great theoretical interest if we make $a = d$, $b = c$, and, as before, $a < c$. We have already described (§ 47) the pair-closure in a mechanism formed from this chain. Figs. 211 and 212 represent the mechanism first without, and then with, the pair-closure. A diagonal joining the points 2 and 4 of the quadrilateral divides it always into two isosceles triangles, for which reason we shall call the train Isosceles. The writing of the chain is easy after the foregoing; the formula must be,—using the symbol for isosceles given in § 47 $(C_2'' \leq C_2'')$. If the higher pairing of Fig. 212 have to be expressed this becomes $\dfrac{(C_2'' \leq C_2'')}{(p)\ a.c}$. As with the anti-parallel cranks, the higher pairing may here be arranged between

* Dübs and Copestake's patent coupling was illustrated and described in *Engineering*, vol. xi. p. 318.

d and b instead of between a and c;—or the pair-closure may be partly between a and c and partly between d and b, but this gives us no new results.

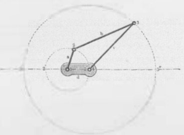


Fig. 211.

The chain gives us two kinds of mechanisms, one by placing it on d or a, the other by placing it on c or b.

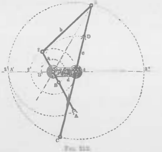


Fig. 212.

The mechanism of Fig. 211, which is placed on d, has the formula $(C''_2 \lessgtr C''_2)^d$. The motion of c is remarkable, for it now not merely swings but completely revolves,—and it has a mean angular velocity equal to half that of a, as we have already seen in § 47. By fixing c (or b) we obtain the mechanism shown in Fig. 213, for which the formula (including the expression for the pair-closure) is $\frac{(C''_2 \leq C''_4)^c}{(\overline{p})\ a.c}$. Its motion is no less characteristic than that of the first mechanism. It is in some respects similar to that of the lever-crank $(C''_4)^d$. The link d has become the crank, and a the coupler; b however swings about its axis 3

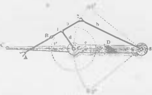


Fig. 213

symmetrically to c through such an angle that the greatest distance of 2 from 4, when the former is in either of the positions 2′ or 2″, is equal to $2a$. The points 2′ and 2″ therefore are nearly four crank lengths apart, while in the mechanism $(C''_4)^d$ the end 3 of the lever oscillates through a distance which approximates only to two crank lengths. We shall further on have occasion to return to this interesting case.

§ 69.

The Cylindric Slider-crank Chain $(C''_3 P^\perp)$.

Continuing our examination of the chain (C''_4) let us now somewhat alter its form. We can substitute for the lever c a small sector of an annular cylinder, and inclose this in a circular slot, (Fig. 214) rigidly connected with the eye at 1. If then centre of the slot and of the sector c be placed at a distance from 1 equal to the distance 1.4 in the former case, the sector has exactly the same relative motions as it would have had had it been connected to the lever c. We may therefore allow it to take the place of the latter:—we shall find further on that kinematically the two are identical. The new arrangement may be written, beginning with a,

$$C^+ \ldots \| \ldots C^{\pm}_{=} C^- \ldots \| \ldots C_= \; C^+ \ldots \| \ldots A^{\pm}_{=} \; A^- \ldots \| \ldots C_=$$

for we have already chosen (§ 57) the symbol A for a circular sector. The contracted formula is $(C''_3 A'')$. This shows even more distinctly than in the former case that the links must be so proportioned that c slides backwards and forwards in its curved path; for otherwise the pair $A^{\pm}_{=} A^-$ would be insufficient.

We can now, without introducing any constructive difficulties,

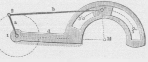


Fig. [illegible]

make the radius of A of any required magnitude; the only alteration will be that the slot and the slider become flatter than before. Let us therefore make this radius infinite. With this the distance of the centre 4 from the point 1, that is the length of the link d, must also become infinite. In other words the links c and d, or the distances 3.4 and 1.4, are made infinite simultaneously; so that

$$c = d = \infty.$$

Our last formula will then require alteration, for the arc A becomes a prism P, and the pair $A^{\pm}_{=} A^-$ is replaced by the prism-pair $P^{\pm}_{=} P^-$. It follows from the equality of c and d that the line in which 3 moves relatively to d passes through the point 1, and is perpendicular to both the axes 3 and 1. The new chain, therefore, which is shown in the following figure, and which is already known to us, must be written

$$C^+ \ldots \| \ldots C^{\pm}_{=} C^- \ldots \| \ldots C_= \; C^+ \ldots \perp \ldots P^{\pm}_{=} P^- \ldots \perp \ldots C_=$$

or moretshortly,

$$C^{+} \ldots \parallel \ldots (C) \ldots \parallel \ldots (C) \ldots \perp \ldots (P) \ldots \perp \ldots C^{-}_{=}$$

or in its contracted form $(C''_3P^{\perp})$, In this most importanttchain thelink c slides alongta straight linet insteadt of swinging in an arc as in (C''_4). We may call it shortly the cylindric slider-crank chain,—or simply the slider chain. We have now to examine the four mechanisms corresponding to its four positions.

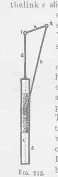


Fig. 215.

The mechanism $(C''_3P^{\perp})^d$. If we place the chain on d, as in Fig. 216 the link c slides backwards and forwards as the crank rotates, and we have before us one of the most familiar of mechanisms, one which appears constantly in direct acting steam-engines, in pumps, and in slotting and so many other machines. The link c we shall call the block, and the link d the slide, when we haveoccasionto name them. The whole mechanism we may call a turuing slide r-crank, on account of the characteristic rotation of a. In its applications to the steam-engine the block c becomes the driving link,so that the general formul a $(C''_3P^{\perp})^d$ gives us the special formula$(C''_3P^{\perp})^{\frac{d}{c}}$. In thetothertapplicationsof it which we mentioned thetcrank a is the driver, —theirspecial formula is therefore $(C''_3P^{\perp})^{\frac{d}{a}}$. The complex mo tion of the coupler b can be exactly determined byits centroids,

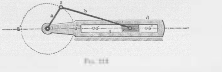


Fig. 216.

but these we must here leave unexamined, merely noticing that theyaretsymmetrical about theaxis 3·1.

Thetmechanism $(C''_3P^{\perp})^b$. Followingthe order formerly adopted let us now place the chain on b, Fig. 217.* The crankt$a$ now

* In Prof. Reuleaux's models for the $(C''_3P^{\perp})$ mechanisms he uses the standwith screwadjustment which is shown in Figs. 11 and 180.

turns about 2, which was formerly the crank-pin, the block c oscillates about the point 3, and causes the slide to turn about the same point in addition to following the motion of the crank. We shall call the mechanism a swinging-block (slider-crank). It will be remembered that we can reverse any of the lower pairs (§ 16) without altering their relative motions; by choosing the arrangement of slide and block shown in Fig. 218, therefore, we do not alter the mechanism. In this form it is exceedingly well known, although not so constantly employed as $(C''_s P^{\perp})^a$. A

Fig. 217.

familiar illustration is the oscillating engine, of which Fig. 218 at once reminds us. Here the slide d, in the shape of the piston, is the driving link, the special formula being therefore $(C''_s P^{\perp})^{b}_{d}$. There have been various attempts to elucidate the connection between the mechanisms of the oscillating and the common direct-acting steam-engine, the explanations being generally founded on some process of altering the relative dimensions of their parts.

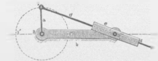


Fig. 218.

It has been said, for instance, that the former is simply the direct-acting engine with the length of its connecting rod reduced to zero, and at the same time, in order that motion may be possible, with its cylinder made so that it can oscillate about an axis. There is here obviously something more than a mere alteration of dimensions, and the whole process remains indistinct. We see now how entirely different and at the same time how completely clear the connection between the two is ;—that the whole matter

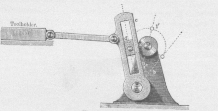


FIG. 219.

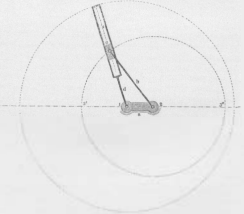


FIG. 220.

lies simply in the inversion of the kinematic chain which forms theo basisoof'bothoequally.o Onotheodifferenceobetweenothe two steam-engines themselves we shall have more to say further on (§ 80).

The mechanism $(C_3'' P{\perp})^{b}$ in the form $(C_3'' P{\perp})^{\frac{b}{a}}$ has found another application in shaping and slotting machines. In this the peculiar motion of the link c,—swinging un-uniformly while the crank rotates uniformly,—is made use of. The crank a is the driver,—during the semi-revolution through 1′ it imparts to the block a much smaller mean angular velocity than during the semi-revolution through 1″. By connecting the link c with the holder of a cutting tool, as Fig. 219, we can therefore obtain a slow (mean) forward motion of the tool while cutting, and a quick (mean) return. Mechanisms of this kind are known as "quick return" motions. The mechanism $(C_3'' P{\perp})^{b}$ is known and familiar to the engineer in this and other ways.

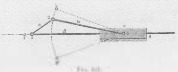


Fig. 219.

The mechanism $(C_3'' P{\perp})^{a}$. By placing the chain on a we obtain a third mechanism, Fig. 220. The link b which was the coupler now revolves about the axis 2,—it has, that is, become a crank; the crank a becomes the frame. The slide d, driven by the block c, turns about the axis 1. Its rotation, if the crank b turn uniformly, is un-uniform, the latter imparting to it at 3‴ a minimum and at 3′ a maximum velocity. On account of this property Whitworth and others have used it in the form $(C_3'' P{\perp})^{\frac{a}{b}}$ as a quick return. According to Goodeve* the mechanism is an old one, and was long ago used to represent the motion of the moon relatively to the earth. We shall call it the turning-block (slider-crank).

The mechanism $(C_3'' P{\perp})^{c}$. The fourth and last mechanism is obtained by fixing the block c instead of the link a, Fig. 221. The coupler b now swings about the fixed axis 3,—the slide d moves rectilinearly to and fro in the block c, now become the frame,—

* Goodeve, *Elements of Mechanism*, p. 68.

and the crank a becomes a coupler and makes complex oscillations. We shall call the mechanism, on account of the swinging of the link b, a swinging slider-crank. This mechanism is little known, but does here and there find applications. Among others there is the apparatus sketched in Fig. 222, which is used in disc-polishing machines in order to give the polishing wheel a to-and-fro motion axially along with its rotation. The train is set in motion by the link a (by means of the worm), and its special formula is therefore $(C_3''P^{\perp})_{a}^{c}$. I have elsewhere described* another application of this mechanism in the same form, and I shall come further on to another very notable one.

Our analysis has shown that four mechanisms can be obtained from the chain $(C_3''P^{\perp})$, of which the first is extremely familiar, the last very little known;—the connection between them, however,

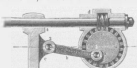


Fig. 222.

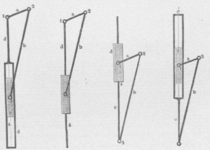
has remained until now completely unseen. At the same time we see that we have exhausted the chain which is before us;—we know that no more mechanisms than these four can be formed out of it. If we now put these together and consider them once more, we shall be able to recognise, by the help of their formulæ, still closer relationships between them. With this object let us write the formulæ at length, one above the other, so that they may be easily compared. We then have

$$\begin{aligned}
(C_3''P^{\perp})^d &= \overbrace{C''^{+} \ldots \parallel \ldots (C)}^{a} \overbrace{\ldots \parallel \ldots (C)}^{b} \overbrace{\ldots \perp \ldots (P)}^{c} \overbrace{\ldots \perp \ldots C^{=}}^{d} \\
(C_3''P^{\perp})^b &= C^{+} \ldots \parallel \ldots \underline{(C) \ldots \parallel \ldots (C)} \ldots \perp \ldots (P) \ldots \perp \ldots C^{=} \\
\underline{(C_3''P^{\perp})^a} &\underline{= C^{+} \ldots \parallel \ldots (C)} \ldots \parallel \ldots (C) \ldots \perp \ldots (P) \ldots \perp \ldots C^{=} \\
(C_3''P^{\perp})^c &= C^{+} \ldots \parallel \ldots (C) \ldots \parallel \ldots \underline{(C) \ldots \perp \ldots (P)} \ldots \perp \ldots C^{=}
\end{aligned}$$

If we recollect that these formulæ are expressions which return upon themselves, that they can also be read, or written, from

* *Civil-ingénieur*, vol. iv. 1858, p. 4.

either end, we see that the second and third mechanisms have absolutely the same formula. Both are placed upon a link of the form $C\ldots\|\ldots C$ having for its adjacent links one like itself and one of the form $C\ldots\perp\ldots P$, these two last being again connected by a link $C\ldots\perp\ldots P$. The difference between the mechanisms, which, as we have seen, is very great, lies solely in the ratio between the lengths of the links a and b. The names we have chosen for the mechanisms,—swinging and turning block



slider-cranks respectively,—give expression both to the relationship and the difference between them.

An exactly analogous connection exists between the first and the fourth mechanisms. The fixed link is in each case $C\ldots\perp\ldots P$, having on the one side a link $C\ldots\|\ldots C$ and on the other a link $C\ldots\perp\ldots P$, these in their turn being connected by one of the form $C\ldots\|\ldots C$. Here also the difference between the mechanisms depends upon the relative lengths of a and b, and we have again employed names which indicate that relationship in calling them turning and swinging slider-cranks.

We notice lastly, what is very striking, that in all four mechanisms the two adjacent links of the form $C\ldots\perp\ldots P$, the block and the slide, are represented by exactly the same symbols,—that between them, therefore, there is absolutely no kinematic difference.

However extraordinary this may seem at first, it is perfectly true, and requires moreover to be well remembered by anyone who wishes readily to understand existing mechanisms;—it is sufficient to cite Fig. 219 as an illustration of this. The chains which are represented in the four figures 223 to 226 are kinematically absolutely identical throughout. The external differences which appear in each case are merely due to that reversal of lower pairs which we emphasised so strongly,—it can now be seen with how good reason,—in an earlier chapter (§ 16).

§ 70.

The Isosceles Slider-crank Chain.

We have seen that the difference between the two mechanisms $(C_3'' P^{\perp})^b$ and $(C_3'' P^{\perp})^a$ is simply dne to our having taken $b > a$; the difference between $(C_3'' P^{\perp})^d$ and $(C_3'' P^{\perp})^c$ is due entirely to the same cause. We must therefore obtain an intermediate form for each pair of cases if we make $a = b$;—the chain thus obtained is the one already described in § 47 and shown in Fig. 227. The links a and b are here made equal; the links c and d are also equal, for they are the two infinite links which always form part of the chain $(C_3'' P^{\perp})$. The equal links are adjacent in each case, so that the general conditions of the chain are the same as in the isosceles crank train of Fig. 211; the chain before us is simply a special case of the former, and we shall therefore give it a similar name, calling it an isosceles slider-crank chain.

We have already considered its centroids in § 47. They are two pairs of Cardanic circles, the smaller being the centroids for the links a and b, the larger those for c and d. The peripheral ratio which appears here is a general property of the isosceles quadric crank chain,—we found it before where the centroids had unlike and complex forms, and we find it here also in the limiting case, which is one, as we see, of peculiar simplicity.

The four mechanisms of the slider-crank here become two only, of which the first is shown in Fig. 227. Omitting the symbols for the higher pairing, its formula will be $(C_2'' \lessdot C'' P^{\perp})^{d=c}$. The same mechanism is obtained whether the chain be placed on c or d, which is indicated by their equality in the exponent. We shall call it,—carrying out the system of nomenclature already adopted,—an isosceles turning slider-crank.

If the chain be placed on a or b wet obtain the second of its possible mechanisms, for which the formula runs $(C_2'' \leqq C'' P^{\perp})^{a=b}$. It is represented in Fig. 228. The crank a has become the frame, the coupler b the crank. The block c transmits thetrotation of the latter to the slide d, or *vice versa*. We shall call thetmechanism an isosceles turning block.*

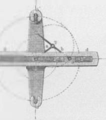


Fig. 228.

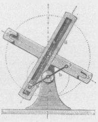


Fig. 229.

The links b and d both rotate, they revolve in the same direction, and have the constant angulartvelocity ratio 2 : 1 ; the motion is exactly what it would be if b and d were two spur wheels having internal contact and having the ratio 1 : 2 between the numbers of their teeth. In faet the toothed gearing shown in Fig. 229,—in which the smaller wheel a has two teeth with cylindrical profiles (pin-teeth),—is very similar to thetmechanism beforeus, although it has one link less. The four-toothed wheel b corresponds to the turning slide d. The similarity becomes less apparent if we make the numbers of tee tht 3 and 6, as in Fig. 230, and disappearstalmost entirely if other forms of teeth be usedt The real relation between the mechanisms is however very obvious; they have identical centroids. The whole matter gives us an interesting illustration

* Here, as in former cases, the words " slider-crank " can be added to the designation given, should it be necessary to do so. I think that it will very seldom be required.

of the solution of one and the same kinematic problem by quite different mechanisms.

The motion of the block c (Fig. 228) is also remarkable. Its centroid relatively to a is a great Cardanic circle, described about the centre 3, and the smaller centroid with which this rolls must be

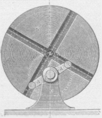


Fig. 229.

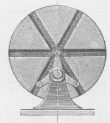


Fig. 230.

imagined to be fixed to a, and to have 2 for its centre. It therefore coincides with the circular centroid of the link b. The motion of c can thus be realised by remembering that its centroid rolls about the fixed smaller centroid of a. The point-paths of the block are therefore all peri-trochoids

§ 71.

Expansion of Elements in the Slider-crank Chain.

We have not hitherto concerned ourselves at all with the diameter of the cylinder pairs in the crank mechanisms. We know that alterations in the dimensions of the elements do not affect their motions, so long as the centroids remain unaltered. It will be well, however, to give them some special consideration here, for these external alterations sometimes so conceal the real nature of the mechanism as to cause much indistinctness in its ordinary kinematic treatment. In considering this subject we shall confine ourselves in the first place to the changes of the relative dimensions of the three cylinder pairs in the chain $(C_3''P^{\perp})$ Theextensiontof our results to other cases will then be quite easy

Each of the four links of the slider-crank chain $(C_3''P^{\perp})$ Fig. 231 is more or less closelyconnected with itst threecylinder pairs 1, 2, and 3, and their forms are therefore dependent upon the

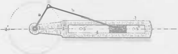


Fig. 231.

relativetsizes of the latter, although, as wehavetsaid, the natureof theirt motiont is nottaffectedt by thet samet cause Evidently, for instance, wetdo nottaltertthechain kinematicallyif we give to the fulltcylinder,tor pin 1, on whichtthe cranka revolves, atdiametertso large thatthe profileof thetpin 2 fallswithinit. Suchantenlargement, we shall call it an expansion,* of thetpin is showntin Fig.

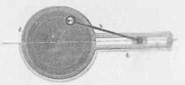


Fig. 232.

232.t The open cylinder of d must now obviously be enlarged to exactly the same extentf so that the pair may still be closed. This arrangement, which may bet shortly described as "2 within 1," occurs in practice in somet slotting and shearingt machines, and in other cases when a short crank forms one piece with its own shaft.†

* Compare the idea of expansion with that of equidistant profiles in § 35.

† It is a very common arrangement too for working a pump,—on board ship or elsewhere, from the end of the crank-shaft.

If we expand the pin 2 instead of 1, and make it large enough to include 1 within its profile, we obtain the form of chain shown in Fig. 233. If this be placed on d and driven by a, we have the mechanism $(C_3'' P^{\perp})^{d}_{a}$ in the form which is so familiar to us as an

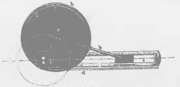


Fig. 233.

eccentric and rod. It can be seen at once that it differs only in its constructive form from the common slider-crank. This expansion is also used in practice placed on a, so as to give us the turning block slider-crank, $(C_3''' P^{\perp})^{a}$. Fig. 234 shows a form in which

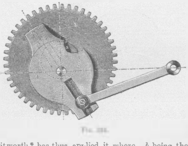


Fig. 234.

Mr. Whitworth* has thus applied it, where—b being the driving link—its special formula becomes $(C_3'' P^{\perp})^{a}_{b}$; it is here used as a quick-return motion, and has already been described and named by Redtenbacher.† The driving rod and parts connected with it do

* See Prof. Shelley's *Workshop Appliances*, p. 253.

† *Die Bewegungs-Mechanismen* (Basserman).

not concern us here, nor does the spur-wheel. In the body of the latter, however, we can recognise the coupler, the two elements of which are represented by the cylinder 2 and the pin 3. The latter fits into and carries the block c, which in its turn moves in the open prism of the slide d.

If the pin 2 be further expanded until it includes also 3, we obtain the arrangement shown in Fig. 235. In this case we have made, as we are always at liberty to do, the element of the cylinder pair 2 which belongs to the crank a as an open figure. The

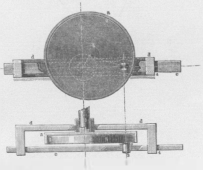


Fig. 235.

coupler b becomes an eccentric disc which swings about the full cylinder 3 of the block c, while it remains always in contact with the open disc 2 of the crank a.

Instead of placing 3 within 2 we may allow 2 to fall within 3, as in Fig. 236. The coupler b is again an eccentric disc; but it now oscillates in a ring forming part of the block c, while the crank pin drives it by internal contact. The reader whose eye is not yet accustomed to detect the abstract form of such mechanisms as these behind their constructive outline, and to whom therefore they may be somewhat difficult to understand, will find the dotted centre lines which we have shown to be of considerable assistance.

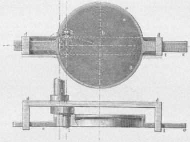


FIG. 236.

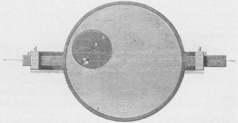


FIG. [illegible]

We have thus considered four methods of pin-expansion in the slider-crank chain, obtained by placing

2 within 1	(Fig. 232)	3 within 2	(Fig. [illegible])
1 within 2	(Fig. 232)	2 within 3	(Fig. [illegible])

We have therefore exhausted the praticable combinations of the pins 1, 2, and 3 in pairs. We may, however, go further, and make one of the pins include two others. 1 can be placed in 2, for instance, at the same time that 2 is lying in 3, so that we can place

1 within 2 within 3

and also 3 within 2 within 1.

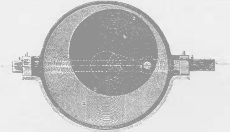


Fig. 237.

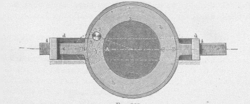


Fig. 238.

These two arrangements are shown in Figs. 237 and 238; both being placed on the frame d; both are turning slider-cranks, $(C_3''P^{\perp})^4$.

The reader may perhaps think that this idea of pin-expansion, carried so far beyond practical limits, can have but little importance in Applied Kinematics. This, however, is not the case, as we shall now proceed to show.

Turning again to the fourth method of expansion, 2 within 3, a closer examination of it shows us that the link c may be made with a concentric cylindrical projection, which can be fitted into a corresponding opening or eye in the link b, Fig. 239. Suppose the mechanism placed on d as before. The coupler b has now become a ring of rectangular cross-section which makes oscillatory motions in the annular groove of the block c. We have in no way altered the mechanism by this, for so long as we keep the pair 3 as a closed turning pair we can alter its profile at will. This condition allows us to go still further. Let us suppose the crank a to be the driver, the mechanism having for its special formula $(C''_3 P^{\perp})^{\frac{d}{a}}$, we then have simply the rectilinear reciprocation of the block c to consider. The coupler b, as it is moved to the right, drives c both at A and at D, and as it is moved to the left both at B and at C, we may certainly replace this double contact in each direction by a single one; and this can be done in several ways. We shall attain the object very conveniently if we substitute a sector of the ring b for the whole of it, choosing the sector so as to include the pin 2, as is shown by the dotted lines. This can then drive the block to the right through A, and to the left through B, the motion of the coupler itself being always an oscillation in the annular ring of c. Of the latter we require to use no more than a piece large enough to afford room on each side of the centre line for the swing of the sector b.

Fig. 240 represents the arrangement altered in this way. It must not be forgotten that b is still the coupler as it was before, and that its motion as a link in the chain remains quite unaltered and completely constrained. Kinematically it consists of just the same parts as before, as does also the link c. The form of the link b is still, $C \ldots \| \ldots C$, one of the cylinders being the eye enclosing the crank pin 2, while the other works, with sufficient restraint, in the portion of an open cylinder belonging to c, relatively to which it has exactly its former motion. If we wish to write the links b and c in a manner corresponding to their constructive form we must use the symbol for sector, A, instead of that for the complete cylinders in the pair 2, and we thus have:

$$\overbrace{C^- \ldots \| \ldots A^{\underline{\pm}}}^{b}\,\overbrace{A^- \ldots \perp \ldots P^+}^{c}.$$

This shows us that a pair of the form $C^{\pm}C^{-}$ or $C^{\mp}C^{+}$ moving only in oscillations of small angle, may be replaced by a pair of the form $A^{\pm}A^{-}$ or $A^{\mp}A^{+}$, so that in such cases

$$(C) = (A)$$

We have already (§ 69) had occasion to employ this substitution of the pair sector and curved slot, (A), for a cylinder pair. It can now be seen that we were fully justified in doing so, that the change did not in any way alter the nature of the chain.

This circumstance is very frequently taken advantage of in practice; pin expansion, that is to say, occurs there very frequently. The mechanism of Fig. 240 is both known and used, although it has not hitherto been considered identical with the turning slider-crank. It has been shown * that the block when driven by the crank moves

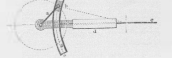


FIG. 244

exactly in the same way that it would move were it connected with the latter by means of a connecting rod having a length equal to the radius of the slot. It has not been noticed, however, that the little sector b really is itself this connecting rod or coupler. The small space which it occupies makes this form of mechanism very convenient in some cases.† We are very familiar with its employment in reversing gear, both in links of the common form, and in Gooch's link, and others. These mechanisms are compound and not simple crank chains, so that we have not to consider them here, but the replacement of (C) by (A) occurs in them in various forms, forming an essential part of each (compare § 16).

* See for instance Giulio, *Cinematica*, p. 109.

† I have added in a note, p. 320, a somewhat interesting example of the use of this form of expansion in such a way as to render it impossible to alter a mechanism from $(C_3'' P^{\perp})$ to $(C_3'' P^{+})$ a thing which it is sometimes very convenient to have the means of doing.

The mechanism shown in the following figure, which occurs sometimes in slotting machines, furnishes us with another illustration of pin-expansion. The whole forms a turning slider crank having the formula $(C_3''P^{\perp})_a^d$. The link b, the coupler, is formed essentially as it was in the last case, but here the profiles against which it works are concave on both sides of the pin 2, the upper profile being of large, and the lower of very small radius, but both forming part of the block c. The block c is in this case so closed by the binder d that the profiles representing the pin 3 lie entirely within the prism-pair 4. We have here, therefore, the elements of the slider crank chain so proportioned that 2 lies within 3 and 3 within 4, an illustration of the way in which the method of expansion can be applied also to the fourth pair. To show how, conversely, we may place 4 within 3 we may take the chain shown in Fig. 242. Here the pin 3 of the link c is made so large that the open prism of the pair 4 can be formed within it. Upon 3 there now oscillates the open cylinder of the coupler b, which, however, must be cutaway at the sides so as to allow it to clear the frame d. The extent to which this

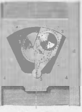


Fig. 241.

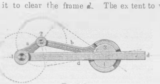


Fig. 242.

cutting away must take place can be found, as the figure shows, by drawing d in the two positions in which it encloses the greatest angles with b. These occur when a and d are at right angles. We shall have to consider some applications of this form of expansion further on.

In general the expansion of elements occasions, as we have seen, extraordinary alterations in the form of a mechanism, alterations which on the one hand tend very much to conceal its original and real nature, and on the other hand frequently offer great constructive advantages. This is true also for other mechanisms besides those we have been considering. Many familiar arrangements appear in a new and unexpected light if we replace slots and sectors by the complete cylindric forms, $C \ldots \| \ldots C$, which they represent; others again, by the reversed process, can be put into a form which allows of their use in practice where otherwise this would be impossible.*

§ 72.

The Normal Double Slider-Crank Chain. $(C''_2 P^{\perp}_2)$.

We have already, in § 68, considered the limiting case of the substitution of the pair (A) for a swinging (C) pair, in taking the lever c of infinite length. If we apply the method there used to the coupler b of the slider-crank chain, which appears already in Fig. 240 as a sector working in a slot, we can make it also infinite. The slot of the block c will then be straight and at right angles to the line 1, 3, the coupler becomes a prismatic slide with a cylinder normal to it, as Fig. 243 shows. If we write the new chain in full, beginning with the crank a, we have:

$$\overbrace{C^{+} \ldots \| \ldots C^{\pm}_{=}}^{a}\;\overbrace{C^{-} \ldots \perp \ldots P^{\pm}_{=}}^{b}\;\overbrace{P^{-} \ldots \perp \ldots P^{\pm}_{=}}^{c}\;\overbrace{P^{-} \ldots \perp \ldots C^{-}_{=}}^{d}$$

The block c has become a pair of prisms at right angles to each other, one of them (as in Fig. 243,) or both (as in Fig. 248) being open, or in the form of slots. We shall call it a cross-block, or in particular a normal cross-block, and the whole chain (as it now contains two sliding-pairs) a normal double slider-chain. The crank a remains as before, the coupler, however, has assumed the form $C \ldots \perp \ldots P$.

* In the *Constructeur* I have for a long time made use of the method of expansion of elements, but I have not there been able to analyse it causally, for this, as we have seen, is a matter which requires a somewhat lengthy investigation. I do not wonder therefore that it has remained greatly misunderstood, and has been sometimes pronounced unimportant, and even superfluous.—*R.*

To put the formula into the contracted shape we have to notice that the chain consists of two parallel cylinder-pairs and two pairs of prisms normal to each other; we must therefore write it $(C_2''P_2^{\perp})$.

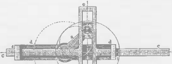


Fig. 243.

The mechanism $(C_2''P_2^{\perp})^{d=b}$. In considering the mechanisms which can be formed from the chain before us, we may begin, as before, by placing it upon d. We notice at once that the coupler b is kinematically exactly equal to the frame d, and lies exactly similarly in the chain, having namely a link $C \ldots \parallel \ldots C$ on one side of it,

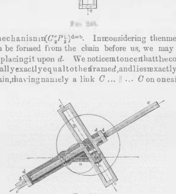


Fig. 244.

and a link $P \ldots \perp \ldots P$ on the other; the mechanisms $(C_2''P_2^{\perp})^d$ and $(C_2''P_2^{\perp})^b$ are therefore identical. In Fig. 243, d is made the fixed link. Following the analogy of our former nomenclature we may call the train a turning double slider-crank, or shortly turning double slider, for we obtain it by the addition of a second sliding pair to the turning slider-crank. The train $(C_2''P_2^{\perp})^b$, which is formed from the swinging block, we may call a swinging cross-block. The motion produced is very simple. The centroids of a and c are Cardanic circles, the smaller (for a) having a diameter $= 1.2$, and the larger described from the centre of the cross-block with a radius $= a$, as in Fig. 243. The primary centroids for b and d are infinite, and must be replaced by secondaries, which are here omitted; they would show that every point in b describes circles relating to d, the whole piece moving always parallel to itself.

The turning double slider-crank is not unfrequently used, its most common application being in the driving gear of steam pumps in the form $(C''_2 P^{\perp}_{2})^{\frac{d}{a}}$. It is often of value also from the fact that if the crank revolve uniformly, it imparts to the cross-block c a simple harmonic motion.

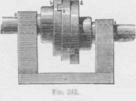


FIG. 245.

The mechanism $(C''_2 P^{\perp}_{2})^{a}$. If the chain be placed on a, the links b and d move about fixed axes 2 and 1. The cross-block revolves, its centroid rolling always upon that of a, as is shown in Fig. 244. We shall call it a turning cross-block.* The links b and d are kinematically identical, although constructively different; their angular motions are always the same.

There have been many practical applications of this train. The well known Oldham's coupling (Fig 245) gives us one interesting illustration. The object of this mechanism is the communication of a uniform rotation between two parallel shafts, its special formula is therefore $(C''_2 P^{\perp}_{2})^{\frac{a}{c}}$ or $(C''_2 P_2^{\perp})^{\frac{a}{c}}$. The bed-plate of the two shafts is the frame a (the crank originally), the middle disc is the link c, the cross-block $P \ldots \perp \ldots P$; the two shafts and their connected discs are the links b and d. To make the construction of the coupling clearer, the three last-named links are shown separately in Fig. 246. The special property of the train $(C''_2 P^{\perp}_{2})^{a}$ which is utilized in Oldham's coupling is the uniformity of the rotation

* Compare "turning block" for $(C''_3 P^{\perp})$ p. 299.

of the links b and d. It was applied in an original manner in Mr. Winan's "Cigar-boat." *

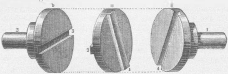


FIG. 246.

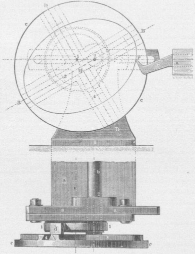


FIG. 247.

The "elliptic chuck" shown in Fig. 247—which so far as we know was invented by Leonardo da Vinci, and was certainly

* *Practical Mechanic's Journal*, vol. xix., (1866-7), p. 271

investigated by him—is a very remarkable application of the mechanism before us. Use is here made of the fact that all points connected with the smaller centroid, that is in this case all points connected with the fixed link a, describe ellipses* relatively to the piece to which the larger Cardanic circle belongs. In the apparatus itself the cross is formed upon the back of the disc c. In one of its two slots it encloses the full prism 3, which is attached to the lathe-spindle. The headstock a forms the cylinder pair 2 along with the spindle b. The cylinder of the pair 1 which belongs to a is attached to the headstock by screws; it is made annular, so that the spindle b passes through it, in other words, it is expanded sufficiently to allow 2 to lie within 1. The piece d is made as a ring:—its inner surface forms the hollow cylinder paired with a, while it carries outside the full prism of the pair 4 (divided into two) which works in the second slot of the cross c. The describing point or tool P forms a part of the fixed link or frame a. The ellipses which are described relatively to the disc by the point of P, have—if P lie beyond a—a difference between their semi-axes equal to the length a; if P lie between 1 and 2, a is equal to the sum of the semi-axes. The enlargement of the pin 1 allows the magnitude of a (that is the distance 1, 2) to be varied within certain limits, and this, together with alterations in the position of P, allows very great variety in the ellipses produced by this apparatus. The link b being the driving link, the complete formula is $(C_2'' P\frac{\perp}{2})^{\frac{a}{b}}$. The mechanism might also be so arranged that d, which is kinematically equal to b, became the driving link. (Compare § 76). It must be remembered that the point-paths of the disc c are those determined by the larger Cardanic circle, and are therefore peri-trochoids, including the particular case of cardioids. The path of the centre M of the disc c is the smaller centroid, through which it passes twice for each revolution of b or d.

The mechanism $(C_2'' P\frac{\perp}{2})^{c}$. We have now left only the train obtained by placing the chain on c. This may be called the swinging double slider-crank, or shortly swinging double slider; it is the mechanism familiar to us as the "trammel" used by draughts-

* Laboulaye (*Cinématique*, 1861, p. 863) attempts to show that the curves described in this apparatus are not ellipses, but he is mistaken. I shall afterwards (§ 76) come to the form of the mechanism given by him, which differs somewhat from the one represented above.—*R.*

men, or in thetclumsier form of Fig.248 employed for drawing on plaster;* and so on. Its special formula is $(C''_2 P^{\|}_{22})^c_a$. The connection between this mechanism and the last—which we call the inversion of a chain—was discovered by Chasles, as I have already mentioned in a noteto § 3; he missed, however, the principletreally underlying it.

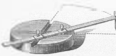


Fig. 248.

The chain $(C''_2 P^{\perp}_2)$ is, as we havet seen, frequentlytused in machinery. Its real nature, however, is even moret hidden than that of most chains by its constructive form; and on that account its real connection with other chains has hitherto remained unrecognised.

§ 73.

The CrossedtSlider-crank Chain, $(C''_3 P^{+})$

Thevery considerable number of forms in which we have now seen thequadrict crankchain by no means exhausts it,for in the slider-crank chain $(C''_3 P^{\perp})$ and those derived from it it is always possible to make a difference in length between the infinitely long links. By using different points in our mechanismtas starting points, as it were, fort thet infinitetlengths, we cantmaket between these a finite difference of any desired magnitude. We shall very briefly considerthealterations which canthus bemade in the chain.

If in the crankchain (C''_4), Fig. 249, we make the link c infinitely long, and therefore at the sametime maket d infinite, but arrangethem so that c be longer than d,† we obtainthe chain shown in Figs. 250and 251. Thet direction of motiont(relatively to d) of thepin 3 no longer passes through 1, but at adistance fromit equal to the finite difference between c and d. We shall call these

* Called in Germany *Stuckateur zirkel*,tand in France also *Compas de menuisier*.

† Perhaps it would be better to sayrather that the links are so arrangedthat if their point of intersection were at any imaginable finite distance, c would be longer than d.

mechanisms crossed slider-cranks, and their uncontracted formula will be

$$C^{+} \ldots \| \ldots C^{\pm}_{=} C^{-} \ldots \| \ldots C^{-}_{=} C^{+} \ldots \perp \ldots P^{\pm}_{=} P^{-} \ldots + \ldots C^{-}_{=} \quad \text{(Fig. 250)}$$

or

$$C^{+} \ldots \| \ldots C^{\pm}_{=} C^{-} \ldots \text{n} \ldots C^{-}_{=} C^{+} \ldots + \ldots P^{\pm}_{=} P^{-} \ldots \perp \ldots C^{-}_{=} \quad \text{(Fig. 251)}$$

In place of oneof theformer symbols for normal we havehere, either inthe link c or in d, the symbol for normally crossed, or crossedat right angles.

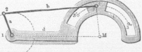


Fig. 250.

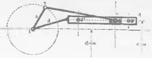


Fig. 251.

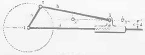


Fig. 252.

Innitsncontracted shape thenformulanwill run $(C''_{3}P^{+})$. The chain,nlikethe more simple one $(C''_{3}P^{\perp})$, gives us fourmechanisms, corresponding toits fourpositions on d, b, a, and c. We mayuse for them the same names as before, prefixing the word crossed in eachncase. We haventhusnthe crossed turning slider-crank, $(C''_{3}P^{+})^{d}$ the crossed swinging block $(C''_{3}P^{+})^{b}$, and so on. The motions occurring innthese chainsnare morencomplexnthanninnthe former cases, the links being no longer symmetrical; necessarily, however, they are very closely related to them. Their applications are far less common. Schwartzkopf's adjusting spanner, Fig. 252, is a very original example of one of them. It is a self-acting universal spanner. To produce the required pressure on the movable cheek c, the mechanism of the crossed turning slider-crank is used in the form $(C_3'' P^{+})^d$. The link a, a portion of which serves as a handle, is the crank, turning about the pin 1 carried by the frame d. At 2 it is connected by a pin with the coupler b, which joins the block c at the cylinder-pair 3, and moves it in the direction CD. 4 is the prism-pair between the links c and d. If the handle be pressed in the direction of the arrow, the mechanism grips the nut between the cheeks of c and d, and holds it the more

Fig. 252.

firmly the harder a be pressed. The nut and the spanner thus become virtually one piece, and the action of the handle a is simply that of a lever attached to the nut, by which it can be tightened, or by reversing the spanner, loosened, at will. Each time the spanner is placed upon a nut, we have the mechanism $(C_3'' P^{+})_a^d$ in action for a short period.*

* Another illustration of the mechanism $(C_3'' P^{+})^d$ is shown in the accompanying figure, which represents an arrangement proposed by Deprez, (*Engineering*, June 18, 1875), and before him by Mr Henry Davey, and very possibly by others also, as a reversing gear requiring one eccentric only. Here the link c is made with a curved slot having a radius equal to the length of the coupler, and in this there is placed a sector b' which carries the pin or full cylinder of the pair 3. From what has been already said (p. 310) it will be recognised that this sector is kinematically identical with the coupler b. When the mechanism is working it retains a fixed position relatively to the slot in c. If it be fixed in the position shown by the full lines the mechanism is simply $(C_3'' P^{\perp})^d$, but by giving it any other position, as for instance, 3', the train becomes $(C_3'' P^{+})^d$ and the motion received by c from a (and therefore, in the machine itself, by the valve from the eccentric) undergoes

The crossed-chains formed from the isosceles slider-crank chains are of less importance than the foregoing. They are formed by a method analogous to that described in § 69, and will be called isosceles crossed turning slider-crank $(C''_2 \leqq C''P^+)^{d=c}$ and isosceles crossed swinging block $(C''_2 \leqq C''P^+)^{b=a}$.

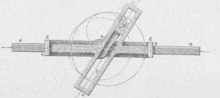
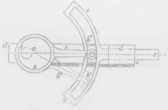


FIG. [illegible]

If we make b also $= \infty$ however, we obtain some remarkable special cases. The normal cross-block then becomes oblique, or to

just that alteration of phase which is required to change the cut-off or the direction of the engine's motion. The excessive friction in the pair 4, when the amount of crossing is large, has prevented any great use being made of $(C''_3 P^+)$ for this purpose.



The mechanism $(C''_2 P\frac{L}{2})$, Fig. 253, might be used in the same way if it were constructed so that the angle of skew of the cross could be altered. Such an arrangement would be in some respects better than the one just described, but it presents some constructive difficulties in cases where an eccentric takes the place of the crank.

use a shorter word well known to engineers, skew, as in Fig. 253. The chain will be written:

$$\overbrace{C^{+} \ldots \parallel \ldots C^{\pm}_{=}}^{a}\ \overbrace{C^{-} \ldots \perp \ldots P^{\pm}_{=}}^{b}\ \overbrace{P^{-} \ldots \angle \ldots P^{\pm}_{=}}^{c}\ \overbrace{P^{-} \ldots \perp \ldots C^{-}_{=}}^{d}$$

The symbol for crossed disappears, and makes room for that of oblique. If, as in former cases, we make prominent in the contracted formula the characteristic symbol of relation of the links, it will in this case be $(C''_2 P^{\angle}_2)$. The links b and d are again equal and similarly placed, so that the chain gives us, like $(C''_2 P^{\perp}_2)$, three mechanisms, namely—

The turning skew (double) slider or swinging skew cross-block	$(C''_2 P^{\angle}_2)^{d=b}$
The turning skew cross-block	$(C''_2 P^{\angle}_2)^{a}$
The swinging skew (double) slider	$(C''_2 P^{\angle}_2)^{c}$

Besides these special cases, the crossed-slider chain has, lastly, two more special forms, which we can only mention here. These are the forms obtained if, instead of three only, we take all four links of infinite length.

If we make $c = d = \infty$ as before, and then make b and a also infinite but having a finite difference, we get the chain shown in Fig. 254, of which the following is the formula:—

$$\overbrace{C^{+} \ldots \perp \ldots P^{\pm}_{=}}^{a}\ \overbrace{P^{-} \ldots + \ldots C^{\pm}_{=}}^{b}\ \overbrace{C^{-} \ldots \perp \ldots P^{\pm}_{=}}^{c}\ \overbrace{P^{-} \ldots \perp \ldots C^{-}_{=}}^{d}$$

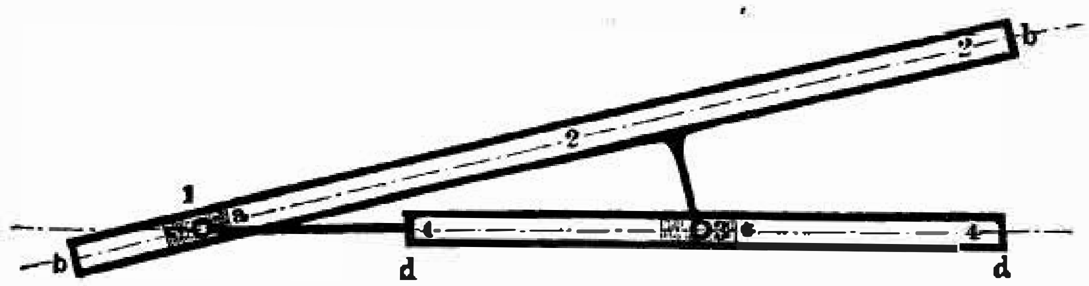


FIG. 254.

We may call this a single crossed-slide chain, and write it shortly as $(CP^{+}\, CP^{\perp})$. All its links are dissimilarly placed, it therefore gives us four mechanisms. If the lengths of c and d have a difference as well as those of b and a, but the two differences are unequal, we obtain the chain of Fig. 255, which we may call a double crossed slide chain $(C P^{+})_2$. The links a and c are here similarly placed in the chain, as are also *b* and *d*, it gives us therefore only two mechanisms. In all these mechanisms the centroids

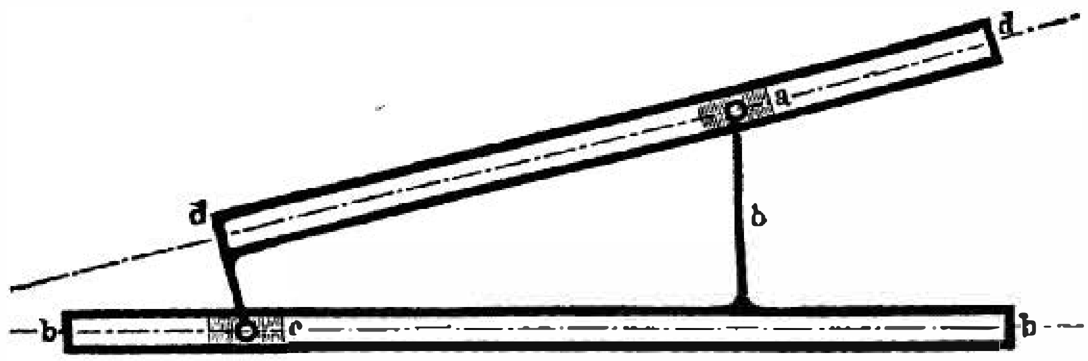


Fig. 255.

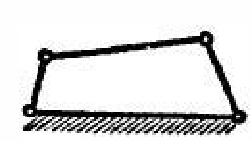
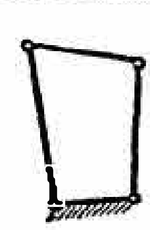
have only infinitely distant points. The single crossed slide has sometimes been used in machinery.

§ 74.

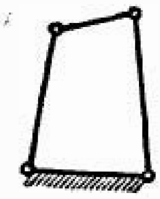
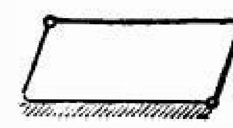
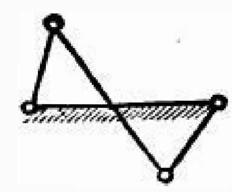
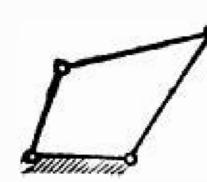
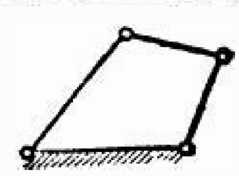
Recapitulation of the Cylindric Crank Trains.

The number of important mechanisms which we have formed or derived from the chain (C''_4) has been so large that in order that their mutual relations may be more clearly surveyed it will be well to place them together in a tabular form. This has been done in the following pages, with the addition of a small schematic outline of each mechanism, the fixed link being in every case shaded. The higher pairing, where it occurs, is omitted.

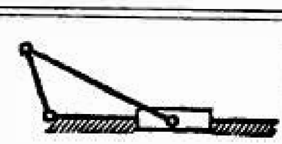
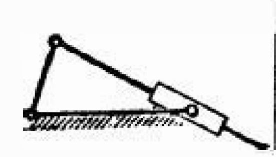
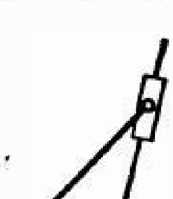
A. Quadric Crank Chain (C''_4).*

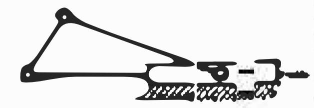
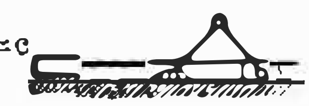
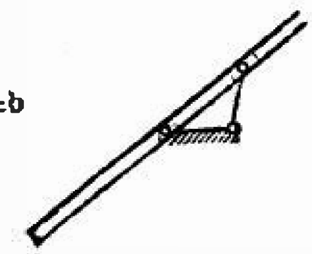
1. Lever-crank	$(C''_4)^{d=b}$	
2. Double-crank	$(C''_4)^{a}$	

* In this table I have put in brackets words which, although they form an essential part of the name of the mechanism, might yet very often be omitted without indistinctness in referring to it.

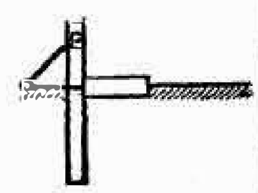
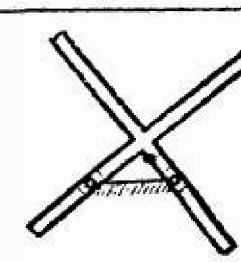
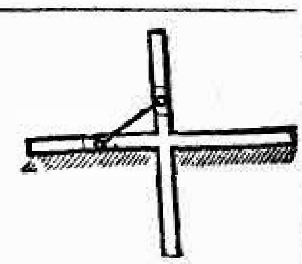
3. Double-lever	$(C_4'')^c$	
4. Parallel-cranks	$(C_2'' \parallel C_2'')^{d=b=a=c}$	
5. Reverse anti-parallel cranks .	$(C_2'' \nparallel C_2'')^{d=b}$	
6. Converse anti-parallel cranks .	$(C_2'' \nparallel C_2'')^{a=c}$	
7. Isosceles double-crank . . .	$(C_2'' \leq C_2'')^{d=a}$	
8. Isosceles double-lever . . .	$C_2'' \leq C_2'')^{b=c}$	

B. Slider-Crank Chain $(C_3''P^{\perp})$.

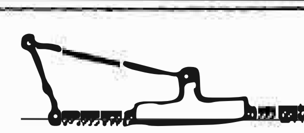
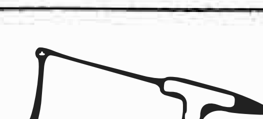
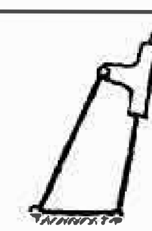
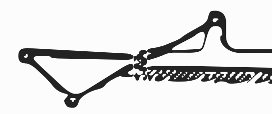
9. (Turning) slider-crank . . .	$(C_3''P^{\perp})^d$	
10. Swinging block (slider-crank) .	$(C_3''P^{\perp})^b$	
11. Turning block (slider-crank) .	$(C_3''P^{\perp})^a$	

12. Swinging slider-crank . .	$(C_3'' P^{\perp})^c$	
13. Isosceles (turning) slider-crank	$(C_3'' \lessdot C'' P^{\perp})^{d=c}$	
14. Isosceles turning block (slider-crank)	$(C_2'' \lessdot C'' P^{\perp})^{a=b}$	

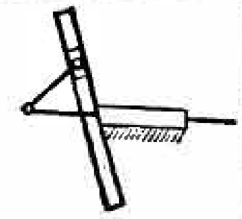
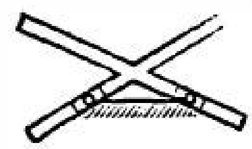
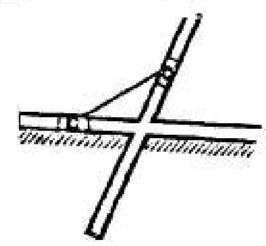
C. Normal double slider-crank chain $(C_2'' P_2^{\perp})$.

15. Turningdoubleslider(-crank) or swinging cross-block .	$(C_2'' P_2^{\perp})^{d=b}$	
16. Turning cross-block . . .	$(C_2'' P_2^{\perp})^a$	
17. Swinging doubleslider(-crank)	$(C_2'' P_2^{\perp})^c$	

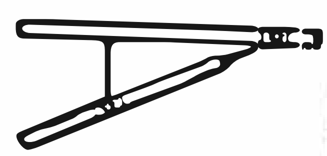
D. Crossed slider-crank chain $(C_3'' P^{+})$.

18. Crossed (turning) slider-crank.	$(C_3'' P^{+})^d$	
19. Crossed swinging block . .	$(C_3'' P^{+})^b$	
20. Crossed turning block . .	$(C_3'' P^{+})^a$	
21. Crossed swinging slider-crank	$(C'' P^{+})^c$	

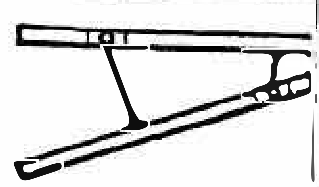
E. Skew double slider-crank chain $(C''_2 P^{\angle}_2)$.

22. (Turning) skew double slider, or swinging skew cross-block	$(C''_2 P^{\angle}_2)^{d=b}$	
23. (Turning) skew cross-block	$(C''_2 P^{\angle}_2)^{a}$	
24. Swinging skew double slider	$(C''_2 P^{\angle}_2)^{c}$	

F. Single crossed-slide chain $(CP^{+}\, CP^{\perp})$

25. to 28. Four mechanisms.	

G. Double crossed-slide chain $(CP^{+})_2$.

29 & 30. Two mechanisms.	

This recapitulation furnishes the best possible proof of the necessity of our previous kinematic analysis to acquaint ourselves even with chains apparently so simple as (C''_4) and those derived from it. We also see how absolutely necessary it was to choose definite names for those of the mechanisms found by our analysis which occur most frequently. These names have been chosen with care and systematically, and they can be easily remembered, especially in connection with their formulæ. The removal of unessentials, which they greatly promote, is an enormous help to the recognition of the real kinematic nature of the constructively complex forms which occur in actual machinery. We shall also see immediately that we have in no way exhausted the list of mechanisms which can be formed from the four cylinder-pairs, notwithstanding its necessary limitations; indeed that we have yet to examine another great family of them, quite different from those we have been considering.

§ 75.

The Conic Quadric Crank Chain ($C_4^{\angle}$)

If the axes of the four cylinder pairs of the chain (C''_4) be not parallel, but have a common point of intersection at a finite distance, the chain remains movable, and (the former conditions being again fulfilled) also closed. The axoids will no longer be cylinders but cones,—as all the instantaneous axes have the point of intersection in common,—and the motion of the links will be determined by their conic rolling, the general nature of which we examined in § 10. If the lengths of the links measured as arcs of great circles upon a sphere drawn about the point of intersection (M) of the axes—fulfil the conditions laid down for those of (C''_4) in § 65, we have a chain of such a form as is shown in Fig. 256. We may call it a conic quadric crank chain, or

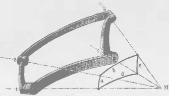


Fig. 256.

four-linked conic crank chain. It stands in a very close relation to the cylindric crank chain, which indeed may be considered as the special case of it when the point of axial intersection, is at an infinite distance.[48] The formula for the chain is

$$\overbrace{C^{+} \ldots \angle \ldots C^{\pm}_{=}}^{a}\ \overbrace{C^{-} \ldots \angle \ldots C^{-}_{=}}^{b}\ \overbrace{C^{+} \ldots \angle \ldots C^{\pm}_{=}}^{c}\ \overbrace{C^{-} \ldots \angle \ldots C^{-}_{=}}^{d}.$$

It can be contracted into the very simple form ($C_4^{\angle}$), in using which we understand not only that the pairs are oblique to each other, but also that their axes have a common point of intersection, as is shown above in Fig. 256.

The various forms of the cylindric chain repeat themselves with the conic one, but with certain differences in their relations. The principal of these relates to the relative lengths of the links, which would vary if they were measured upon spheric surfaces of different radii,—if they were taken, that is, at different distances from the point of intersection. The ratio, however, between the length of a link and its radius remains constant for all values of the latter, and these ratios are simply the values in circular measure of the angles 1 *M* 2, 2 *M* 3, 3 *M* 4, and 4 *M* 1, subtended by the links. Instead of the link lengths therefore we must consider the relative magnitudes of these angles, which we can also indicate by the letters *a*, *b*, *c*, and *d*.

The series of alterations in these lengths which we supposed in the former case, and which we carried on until all the links became infinite, are here represented by corresponding angular changes. The infinitely long link corresponds to an angle of 90°. For the case where two links are infinite but have a finite difference (§ 73) we have now one subtending a right, and the other an obtuse, angle. As however we must always imagine the axis of the links prolonged through and beyond the centre of the sphere, the obtuse angle between two axes gives on the other side also an acute angle between them,—so that no real difference exists between acute and obtuse-angled links. A similar simplification affects the centroids and axoids. The infinitely distant points of the centroids in the chain (C''_4), of which we had illustrations in § 8, are here represented by the points in which the common normal to the fixed axes cuts the sphere. The axoids here are consequently cones (circular or non-circular) upon some closed base.

Keeping these points in view we may now proceed to examine the mechanisms formed from the conic quadric crank-chain, which we shall do as far as possible in the same order as before.

A. Conic quadric crank-chain $(C_4^{\angle})$ Fig. 257. All links subtend less angles than 90.° We obtain from it, as from (C''_4), eight mechanisms for its eight principal special cases or positions; to these we can give the same names as before, only prefixing the word conic in each case. Their formulæ, also, are analogous to the former ones, the form-symbol for oblique replacing that for parallel. I do not know of any applications of these mechanisms, but it is quite possible they may exist, disguised under dissimilar constructive forms.

The parallel and anti-parallel cranks repeat themselves in the conic chain along with the others. The arrangements necessary for passing the dead-points are not, however, those examined before. If we join two conic parallel crank-chains in a way corresponding to Fig. 206, we obtain a mechanism by which it might appear at first sight that a uniform rotation could be transmitted between shafts whose axes are neither coincident nor parallel, a problem for which a solution has often been attempted. The formula of such a train would be $2(C_2^{\angle} \parallel C_2^{\angle})^{d}s$ In reality, however, this combination is an impossible one. For the chain $(C_2^{\angle} \parallel C_2^{\angle})$ has only four positions—the four cardinal ones—in which its opposite links lie parallel to each other; in all other positions the opposite angles of what was the parallelogram are unequal, and the rotation of the cranks is therefore not uniform. While therefore the chain $(C_2^{\angle} \parallel C_2^{\angle})$ has its own special interest, it will be seen that it is not entirely analogous with $(C_2'' \parallel C_2'')$.

B. Conic slider-crank chain $(C_2^{\angle} C_2^{\perp})$, Fig. 258. The links d and c are right-angled, that is, the angle between the axes 1 and 4

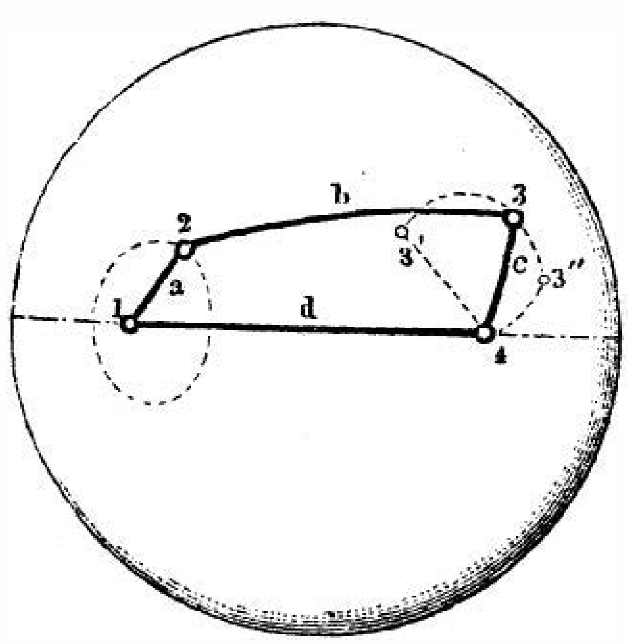


FIG. 257.

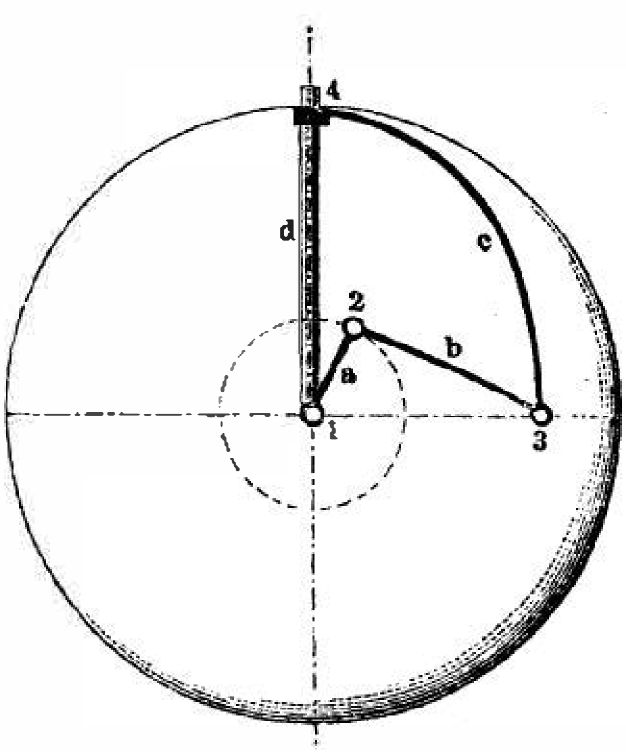


FIG. 258.

and between 4 and 3 = 90°. The comprehension of this chain, which may present at first difficulties to some of my readers, may perhaps be made more easy by the help of Fig. 259. Here the principle of pin-expansion is applied to the mechanism. For the arm M 3 (which the figure shows as the projection of a quadrant like c, Fig. 258), turning about an axis at M (corresponding to the

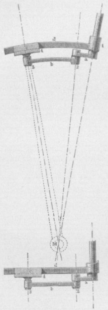


FIG. 259.

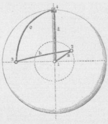


FIG. 260.

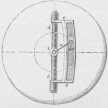


FIG. 261.

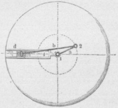


FIG. 262.

Fig. 259.—Conic slider-crank chain, $(C_2^{\llcorner}C_2^{\perp})$ compared with cylindric chain $(C_3'' P^{\perp})$.

Fig. 260–2. — Normal conic double slider-crank chain $(C_3^{\perp} C^{\llcorner})$.

rod 4, Fig. 258) perpendicular to the plane of the paper,—we substitute the small section 4 of a cylinder, sliding upon a corresponding section d of another cylinder; c is now the block, d the frame, a the crank and b the coupler as before. Below the conic chain a similar cylindric chain is shown; the juxtaposition of the two makes it very easy to realise that the latter is simply the conic chain with the point M removed to an infinite distance. The conic slider-crank chain, like its cylindric counterpart, gives us six mechanisms, four principal forms and two secondary ones. We shall give them the same names as before with the prefix conic to each. There appears to be very little, if any, use made of them in practice.

C. Conic (normal) double slider chain $(C^{\perp}_{3} C^{\angle})$, Fig. 260. Here the links b, c and d are right-angled, and a only acute-angled. This chain corresponds to the one bearing the same name in the cylindric series, and by applying the method of pin-expansion, it can be brought into a very similar form, as in Fig. 261. The mechanism of Fig. 262 is essentially identical with that of the one before it. The slide d is nothing more than a portion of a cross section of the cylinder which in Fig. 261 appears as a round bar, marked with the same letter. The link b subtends an angle of 90°, and is thus identical with the sector b in Fig. 261. This chain, like the cylindric double slider, gives us three mechanisms, which will be called

15.* The conic (turning) double slider, or conic swinging cross-block } $(C^{\perp}_{3} C^{\angle})^{d=b}$.
16. The conic (turning) cross-block ... $(C^{\perp}_{3} C^{\angle})^{a}$.
17. The conic swinging double-slider ... $(C^{\perp}_{3} C^{\angle})^{c}$.

Considerable use is made of these mechanisms in practice. One well-known application of No. 16 is to be found in the mechanism known as the universal or Hooke's joint.† Writing out the formula $(C^{\perp}_{3} C^{\angle})^{a}$ in full we have

$$\overbrace{C^{+} \ldots \perp \ldots \underline{C^{\pm}}}^{b}\;\overbrace{C^{-} \ldots \perp \ldots C^{\mp}}^{c}\;\overbrace{C^{+} \ldots \perp \ldots \underline{C^{\pm}}}^{d}\;\underline{\overbrace{C^{-} \ldots \angle \ldots C^{\mp}}^{a}}$$

and this formula, corresponding to the chain in Fig. 260, we have already found (§ 58) to be that of the universal joint Fig. 263.

* The chains $(C^{\angle}_{4})$ and $C^{\angle}_{2} C^{\perp}_{2})$ together, as we have seen, give 14 mechanisms.

† In Germany also as Cardano's coupling.

As either b or d may be the driving link its special formula runs $(C_{\overline{3}}^{\perp}C^{\llcorner})_{\frac{a}{b}}$ or $(C_{\overline{3}}^{\perp}C^{\llcorner})_{\frac{a}{d}}$. It may again be noticed that the links b, c and d are completely identical, as indeed becomes visible in Fig.

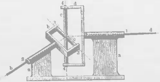


Fig. 263.

260, although in the universal joint they commonly appear so extremely different. We shall shortly have to examine some other very important applications of this chain.

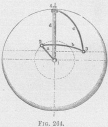


Fig. 264.

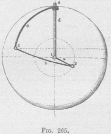


Fig. 265.

D. Crossed conic slider-crank chain $(C_{\overline{3}}^{\llcorner}C^{\perp})$. The crossing of the cylindric slider-chain expresses itself here in the altered length of the links, of which one only, in Fig. 264 the link d, remains right-angled. We obtain as before four mechanisms (Nos. 18 to 21), of which very few applications occur.

E. Skew double slider-crank chain $(C^{\angle}C^{\perp})_2$, Fig. 265, *a* and *c* are acute, *b* and *d* right-angled. This chain corresponds both to the cylindric skew double slider chain and to the cylindric crossed slide chains *F* and *G* (p. 322). It gives us three mechanisms (22 to 24), of which very occasionally we find an application existing.

In all, therefore, this conic crank chain gives us 24 mechanisms, dividing themselves into five different classes. The majority of these have been hitherto unknown; whether they are "practical" or "unpractical" is not a question which concerns us here. Our unerring analysis will allow us further on to obtain very important results from them. Summing up the results of the last ten sections we find that the number of mechanisms formed or essentially derived from the quadric crank chain has been 54, and that they have occurred in 12 distinct classes.

§ 76.

Reduction of a Kinematic Chain.

If we wish to obtain the motion of any particular link in a complete mechanism, without requiring at the same time to use the motions of any other of its links, it is often possible to remove one of these, its place being supplied by a suitable pairing between the two links which it connected. The number of links in the chain can thus be diminished without affecting the particular motion which is required, and it is evident that this may often be very advantageous. We shall examine some examples of it.

Suppose that it be wished to obtain a reciprocating sliding motion by means of the turning slider-crank $(C_3''P^{\perp})^d$ Fig. 266, and that none of the other motions in the chain be required, then the coupler *b* may be removed if we pair the crank *a* to the block *c* direct. This can be done, for example, by attaching to *a* a pin of suitable diameter, and connecting with the block an envelope (§ 3, Fig. 4) for it,—which will in this case take the form of a curved slot touched by the pin upon both sides, as in Fig. 267. The pin and its envelope form together a higher pair of elements. The simplest arrangement will be obtained by using the former crank pin, which will pair with a slot described from the centre of the coupler with a radius equal to its length,—for the slots which would be required as envelopes of or pins of further from the centre of have forms much more troublesome to deal with, such as the path

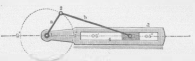


Fig. 206.

(shown dotted) of the point 2'. If the link b be removed and the chain afterwards closed in this simple way, its complete formula, (placed on d), will run—

$$\overbrace{C^{+}\ldots\|\ldots C^{+}}^{a},\ \overbrace{A^{\pm}_{=}\ldots\perp\ldots P^{+}_{=}}^{c}\ \overbrace{\underline{P^{-}\ldots\perp\ldots C^{-}_{=}}}^{d}$$

We must find means for distinguishing this chain from the former $(C''_3 P^{\perp})^4$:—

$$C^{+}\ldots\|\ldots C^{\pm}_{=}\ C^{-}\ldots\|\ldots C^{-}_{=}\ C^{+}\ldots\perp\ldots \underline{P^{\pm}_{=}\ P^{-}\ldots\perp\ldots C^{-}_{=}}$$

As it gives us no new motions, but only fewer motions than before, we shall not make a new class for it, but shall treat it as a derived form of the four-linked chain, obtained from it by

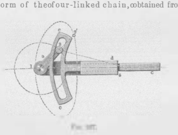


Fig. 207.

the removal of one link, in this case b. We shall call this removal of a link from a complete chain and its replacement by higher pairing a reduction of the chain by that link, and shall indicate it in the contracted formula for the new chain by writing the latter:—

$$(C_3'' P^{\perp}) - b.$$

The chain so reduced has three links;—it can therefore be placed in three ways only, so that only three mechanisms can be formed from it. These are: $(C_3'' P^{\perp})^d - b$; the turning slider-crank, $(C_3'' P^{\perp})^a - b$ the turning block, and $(C_3'' P^{\perp})^c - b$ the swinging slider-crank. All three mechanisms occur in practice.

Instead of removing b any other of the links may be taken away, provided only that the particular motion required can be obtained without it. The two following figures show two methods

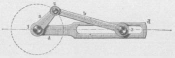


Fig. 268.

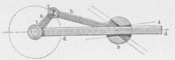


Fig. 269.

of reducing the chain $(C_3'' P^{\perp})$ by omitting the block c, they represent therefore $(C_3'' P^{\perp}) - c$. In the first of them there is added to the coupler, conaxially with the former cylinder pair 3, a solid cylinder, and this is paired with its envelope in the slider, which is a straight slot, or negative prism. In Fig. 269, on the other hand, the slider is made a positive prism, and paired with its envelope in the end of the coupler. The latter takes the form of an X-shaped recess, which lies closely on the slider only at the points of greatest pressure,—(*i.e.*, when the axes of a and b are at right angles),—but in all other positions has considerable freedom, and works therefore under force-closure. The pair-closure could be complete only if the prism d were infinitely thin. When the arrangement of Fig. 269 occurs in practice it is most frequently with well-rounded corners in the recess in b. The manner in which the true form of these round corners can be obtained is not unimportant;—it may be looked at as follows. If we have assumed in the first place some considerable breadth for the slider d, and found the required shape of the envelope in the coupler, we can then draw equidistants to the profile thus obtained (§ 35), and may choose such of these as give us for the prism a narrower profile than before. The corresponding equidistants for the recess in b will then give us the rounded corners required, as is shown in the figure above. The pairing is still, however, incomplete (force-closed) for there is some freedom left between b and d in all positions but those of greatest pressure.

FIG. 270.

The form in which Leonardo's elliptic chuck is most commonly constructed furnishes us with a remarkable example of a reduced crank chain. This form, to which we have already alluded in § 72, is shown in Fig. 271. The reduction here consists in the omission of the coupler b from the turning cross-block,—the form of the mechanism is that, therefore, which would be obtained from Fig. 267 if the radius of the slot were made infinite. Its contracted general and special formulæ are $(C_2^z P_2^{\perp})^a - b$ and $(C_2^z P_2^{\perp})^{\frac{a}{d}} - b$ respectively,—for it is driven by the slider d, turning about the pin 1 (compare the mechanism in § 72, where b is the driving link). a_1 is the end of the headstock, to which the piece a_2 is secured by means of adjusting screws. These two form together the fixed link of the mechanism, the crank a. This carries the bearing for the spindle 1, to which the slide d, carrying the open prism 4, is attached;—d is therefore of the form $C^+ \ldots \perp \ldots P^-$. The cross-block c, $(P^+ \ldots \perp \ldots P^-)$ consists here of the full prism paired with d at 4, and of the two pieces 3 3, forming the sides of an open prism, which are shown in the lowest of the accompanying figures. These pieces envelop thegreatlytexpanded pin 2 of the fixed link a, and formt with it—the link b which connects them inthe completetchain having

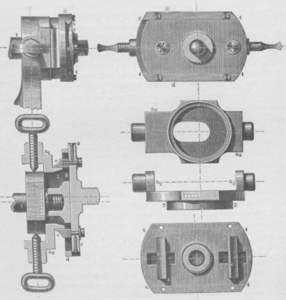


Fig.

beenremoved—a higher pair. The expansion of an element and the reductiou ofta chain are here, therefore, used together. By the adjusting screws andtscale the relativet positions of a_2 and

a_1,—that is the length of the link a_3—can be readily adjusted at pleasure,—the piece a_2 being made very open inside to allow of this. On the face of the cross-block there is a screw to which the chuck can be attached in the usual manner. This apparatus gives us an excellent illustration of the extraordinary way in which the constructive details of a mechanism may hide its real nature. It can easily be seen how this elliptic chuck, although so often used, has been hitherto so little understood. The way in which the very disadvantageous reduction of the chain has perpetuated itself is really extraordinary. The higher pair 2.3 wears rapidly on account of its deficiency in surface, and the motion of the mechanism becomes therefore inexact. The

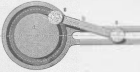


FIG. 272.

arrangement shown in Fig. 247,* in which the chain is used unreduced, and to which our theoretical investigations directly led us, has not this disadvantage. It is also in other ways more convenient than the common plan, especially in its simple arrangement of the prism pairs 3 and 4 upon the back of a disc.

We have seen that it is possible to reduce a four-linked chain to one of three links:—it must in the same way be possible to reduce a three-linked to a two-linked chain under suitable circumstances. This can be done as well with a chain previously reduced as with one containing three links in its complete form, as an illustration will show. Fig. 272 shows the chain represented in Fig. 268 reduced by another link, namely the crank a. For

* Fig. 247 represents the form [illegible] Reuleaux has chosen for the model of this mechanism in the kinematic collection of the Königl. Gewerbe Akademie in Berlin. A machine for cutting elliptical grooves (for man-holes etc.), in which the mechan[illegible] was used unreduced, was [illegible]ted at the V[illegible] Ex[illegible] by a Chemnitz firm. See *Record of the V. Univ. Ex.* (Maw and Dredge), Pl. 195.

this purpose a full cylinder on the end 2 of the coupler is paired with its envelope on the frame, an open annular cylinder having 1 for its centre. Such a chain is written $(C_3'' P^{\perp})^c - a - c$.

It is two-linked:—in other words it has been reduced to a pair of elements, the two pieces *b* and *d*. It can be placed therefore in two ways only. Placed upon *d* it gives us the turning slider-crank $(C_3'' P^{\perp})^d - a - c$, in a form which might be used in cases where the only motion required is that of the coupler. Placed upon *b* we have the swinging block, and can utilise the motion of the slider only.

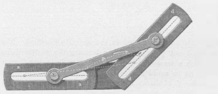


Fig. 273.

Fig. 273, an arrangement which came before us at the very outset of our work, is another example of a reduced chain. It can easily be seen that it is really a portion of a twice reduced skew double-slider chain such as Fig. 253 (Cf. No. 24 page 326). The piece marked *aa dd* is the skew cross-block, and *b c* the crank. Its formula is therefore

$$(C_2'' P_{\frac{\angle}{2}}) - b - d.$$

Any point *p* in the crank moves, as we have frequently seen, in an ellipse. The pairing rendered necessary by the omission of the links is here, as in the other cases, higher pairing, the pair of elements obtained by ultimate reduction from a chain is a higher pair. The reduction can be carried no further, for a machinal arrangement cannot be made with less than two bodies.

Practical use is made of this process of reduction in other chains than those which we have been considering. I may just give one example of this. The spur chain (Fig. 274) can be reduced to the higher pair of elements shown in Fig. 275 by omittingtthe frame.t The complete chain has the formula $(C_z^+C_z'')$ in its reduced form it will therefore be written $(C_z^+C_z'')$—*c*.

Itmust not be supposed that the reduced chains haveactually beent derived in thist way from the complete ones. We haveou the contraryt (Chapter VI.) seentcause to supposet that the process

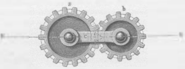


Fig. [illegible]

of development has moved i n the opposite direction. This however need not prevent us from trea ting the matter deductively when such a treatment greatly facilitates its comprehension, and enables us to avoidt endlesstrepetitions. If we had to consider each reduced chain as a separate form of kinematic linkage we should ob taiu, through the very various formst which the higher

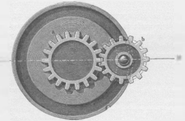


Fig. [illegible]

pairingtcauttake, an enormous number of additionaltcombinations, without at the same time having added a single new motion to those alreadyt existingt in thet completet chains. Thet admittance therefore, of the principle of chain-reduction intoourt workhelps us very greatly in simplfying its arrangement. In thet ease of compound mechanisms it can often be very advantageously employed, and the exact analysis of these combinations forms a very interesting and instructive problem in applied Kinematics.[49]

§ 77.

Augmentation of Kinematic Chains.

The augmentation of a chain stands logically as the contra-positive of its reduction. A chain which has been already reduced can obviously be restored to its original completeness by such a process. But there is no reason why the augmentation should stop here ;—the pairing between any two pieces may be replaced by chaining, *i.e.* by linkage, if the link introduced between them be so arranged as not to alter their relative motions. If the chain already possess the largest number of links which it can have as a simple closed chain, any augmentation must be so arranged as to make it a compound closed chain (§ 3, p. 49). In general therefore the process leads us to such chains, which do not belong to the part of our subject at present under consideration. We may merely mention a few illustrations of it. The (so-called) parallel motions are augmentations of this kind: the parallel motions of Watt or Evans, for instance, replace a prism pair by a kinematic chain having only turning pairs, which therefore essentially is a crank mechanism. A common train of wheel-work again, which is used really as a substitute for a pair of wheels of inconveniently large diametral ratio, may be considered an augmentation in the same way. It will be seen from these examples alone that a very extended use is made in practice of this method of chain augmentation. We shall content ourselves here with having thus stated the general nature of the principle, and shall not go further into the matter. Its further consideration forms indeed a part rather of applied than of theoretical Kinematics.