On Global Convergence of A Trust Region and Affine Scaling Method for Nonlinearly Constrained Minimization

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Abstract. A nonlinearly constrained optimization problem can be solved by the exact penalty approach involving nondifferentiable functions $\sum_i |c_i(x)|$ and $\sum_i \max(0, c_i(x))$. In [11], a trust region affine scaling approach based on a 2-norm subproblem is proposed for solving a nonlinear l_1 problem. The (quadratic) approximation and the trust region subproblem are defined using affine scaling techniques. Explicit sufficient decrease conditions are proposed to obtain a limit point satisfying complementarity, dual feasibility, and second order optimality. In this paper, we present the global convergence properties of this new approach.

Key Words. nonlinearly constrained minimization, trust region, sufficient decrease conditions, affine scaling, exact penalty, nonlinear l_1 problem, global convergence

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1. Introduction. In [11], a trust region and affine scaling approach, based on a trust region subproblem with a 2-norm bound constraint, is proposed for solving a nonlinear l_1 problem:

$$\min_{x \in \Re^n} \Upsilon(x) \stackrel{\text{def}}{=} \|c(x)\|_1 + f(x), \tag{1.1}$$

where $f(x) : \Re^n \to \Re$ and $c(x) \stackrel{\text{def}}{=} [c_1(x); \cdots; c_m(x)] : \Re^n \to \Re^m$ are twice continuously differentiable functions.

Increasingly, trust region methods have become an attractive tool for unconstrained minimization [7]. Unfortunately, there is far less research on trust region methods for nonlinearly constrained minimization. Although trust region methods for nonlinear equality constrained problems have been suggested [2, 3, 15, 16], we are not aware of any trust region method, with explicit conditions on the steps for achieving optimality, for general nonlinearly constrained minimization including inequalities. We believe that the new trust region and affine scaling approach described in [11] offers, via exact nondifferentiable penalty functions, a trust region approach for a general nonlinearly constrained minimization.

The novelties of the approach proposed in [11] are the following. Firstly, affine scaling is employed to overcome the difficulty of nondifferentiability. Moreover, it is used to obtain a simple quadratic which provides, asymptotically, a second-order approximation to the nondifferentiable l_1 function. Secondly, a trust region subproblem with a 2-norm bound constraint, whose solution can asymptotically yield sufficient decrease, is employed. Thirdly, explicit conditions for complementarity, dual feasibility, and second-order optimality are proposed. The new approach is a further development of some of our previous research on solving simpler problems including linear l_1 [5], l_∞ [4], p-th norm minimization [10], and minimizing a nonlinear function with bound constraints [6].

Affine scaling, which has recently attracted a great deal of attention for linear programming problems (e.g.,[8, 1]), plays an important role in the proposed method. It is essential for deriving a second-order approximation to the nonlinear nondifferentiable l_1 function. It is indispensable in the trust region subproblem for handling nondifferentiability, generating sufficient decrease and maintaining second-order approximation to the nondifferentiable objective function. In addition, it facilitates satisfaction of the Kuhn-Tucker conditions.

The proposed method works in a surprisingly similar fashion to a trust region method for unconstrained minimization. At each iteration, an affine scaling (diagonal) matrix $\mathcal{M}_k$ is chosen to ensure a step can be generated for both complementarity and dual feasibility. A trust region subproblem with a 2-norm bound constraint is then approximately solved to determine a step based on second-order information. A second-order approximation to the change of the nonlinear l_1 objective function can be sufficiently decreased and the agreement between the approximation and the original objective function is measured. Finally, according to the agreement measurement, the trust region size Δ_k is adjusted in a simple fashion to ensure sufficient reduction of the nonlinear l_1 function at each iteration.

The main cost per iteration is an evaluation of the functions/gradients/Hessians of $f(x)$ and $c(x)$ and computing an approximate solution to a trust region subproblem with a 2-norm bound constraint and consistent linear equality constraints. Hence the techniques for solving a trust region subproblem, developed for unconstrained minimization, can be readily applied.

The main purpose of this paper is to analyze the global convergence properties of the trust region affine scaling method (TRASM) proposed in [11]. In particular, we prove that the three explicit conditions proposed in [11], (AC.1), (AC.2) and (AC.3) guarantee complementarity, dual feasibility and second-order optimality. The global convergence analysis is carried out in a parallel fashion to that of trust region methods for unconstrained minimization.

Notationally we use $(\cdot)$ to emphasize a dependence relation, e.g., $\lambda_k(D_k)$ depends on D_k. Our presentation follows many **Matlab** [12] notations. For example, a semicolon ; in $[c_1; c_2]$ is used to create a column vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ while , in $[c_1, c_2]$ is used to create a row vector $[c_1 c_2]$. Matrices can be generated by submatrices in the same fashion. In addition, given any $d \in \Re^m$, diag(d) denotes an m-by-m diagonal matrix with the vector s defining the diagonal entries in their natural order; $|d|$ denotes a vector of the same dimension with the ith component equal $|d_i|$. Moreover, for any nonsingular matrix $A \in \Re^{m\times m}$ and any $k > 0$, A^{-k} denotes the inverse of A^k, where A^k is the k-th power of A. The sign function is defined as below:

$$\text{sgn}(c_i) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } c_i > 0, \\ 1 & \text{if } c_i = 0, \\ -1 & \text{if } c_i < 0. \end{cases} \tag{1.2}$$

We also make the following smoothness and compactness assumptions throughout the presentation: Given an initial point $x_0 \in \Re^n$, we assume that the level set $\mathcal{L}$ of $\Upsilon(x)$ is compact, where $\mathcal{L} \stackrel{\text{def}}{=} \{x : x \in \Re^n \text{ and } \Upsilon(x) \le \Upsilon(x_0)\}$. The functions $f(x)$ and $c(x)$ are assumed to be at least continuously differentiable in $\mathcal{L}$.

2. Review of the Method TRASM. The main components of algorithm TRASM are motivated in great detail in [11]. In this section, we summarize the method.

Algorithm TRASM maintains an approximation $\Gamma_k(s)$ to the change of the original nonlinear l_1 function. This approximation is globally first-order and can be asymptotically second-order. The algorithm works in the usual trust region fashion: compute a step s_k, based on a trust region subproblem with a 2-norm bound constraint, which yields a sufficient reduction of the approximation $\Gamma_k(s)$. The approximation Γ_k and step s_k are chosen to achieve sufficient reduction for complementarity and dual feasibility. A correction step u_k can be computed if second-order optimality is desired. If $\Gamma_k(s_k)$ approximates well to the change of the nonlinear l_1 objective function, a step is taken and computation proceeds to the next iteration. Otherwise, the trust region size is reduced and the computation proceeds. This model algorithm is summarized in FIG. 1. Details of each step are described subsequently.

Affine Scaling Matrices D_k, $\mathcal{D}_k$ and $\mathcal{M}_k$. The diagonal matrix $\mathcal{M}_k$ is defined to be either D_k or $\mathcal{D}_k$ and D_k is a measurement of the distance to the nondifferentiable curves:

$$D_k \stackrel{\text{def}}{=} \text{diag}\{|c_k|^{\frac{1}{2}}\}. \tag{2.1}$$

The affine scaling matrix $\mathcal{D}_k$ is considered to ensure dual feasibility. It depends on the Lagrangian

TRASM (Trust Region and Affine Scaling Method):
Let $0 < \zeta < \eta < 1$ and $0 < \gamma_1 < 1 < \gamma_2$.
For $k = 0, 1, \cdots$

Step 1 Choose $\mathcal{M}_k$ between D_k and $\mathcal{D}_k$;
Step 2 Compute s_k and $\Gamma_k(s_k)$;
Step 3 If $s_k \in \mathcal{F}_k$ and $\Gamma_k(s_k) = \psi_k(s_k)$, compute u_k on the curve $\mathcal{P}_k$; Otherwise, $u_k = s_k$;
Step 4 Compute

$$\rho_k = \frac{\Upsilon(x_k + u_k) - \Upsilon(x_k)}{\Gamma_k(s_k)};$$

If $\rho_k > \zeta$ then set $x_{k+1} = x_k + u_k$. Otherwise set $x_{k+1} = x_k$;
Step 5 Update Δ_k as below:
1. If $\rho_k \le \zeta$ then set $\Delta_{k+1} \in (0, \gamma_1 \Delta_k]$.
2. If $\rho_k \in (\zeta, \eta)$ then set $\Delta_{k+1} \in [\gamma_1 \Delta_k, \Delta_k]$.
3. If $\rho_k \ge \eta$ then
   set $\Delta_{k+1} \in [\Delta_k, \gamma_2 \Delta_k]$.

FIG. 1. *A Trust Region and Affine Scaling Method for a Nonlinear l_1 Problem*

vector of multipliers $\lambda_k(D_k)$, which is a least squares solution to

$$\begin{bmatrix} J_k \\ -D_k \end{bmatrix} \lambda \overset{\text{LS}}{=} - \begin{bmatrix} \nabla f_k + J_k \text{sgn}(c_k) \\ 0 \end{bmatrix}, \tag{2.2}$$

where the Jacobian matrix $J_k \overset{\text{def}}{=} J(x_k)$ with $J(x) \overset{\text{def}}{=} [\nabla c_1(x), \cdots, \nabla c_m(x)] \in \Re^{n \times m}$.

As analyzed in [11], the Kuhn-Tucker conditions can be expressed as below:

$$\begin{cases} J(x)\lambda = -(\nabla f(x) + J(x)\text{sgn}(c)), \\ -\text{diag}(|c|^{\frac{1}{2}})\lambda = 0, \\ -2e \le \text{diag}(\text{sgn}(c))\lambda \le 0, \end{cases} \tag{2.3}$$

where $e \overset{\text{def}}{=} [1; \cdots; 1] \in \Re^m$. The first two equations in (2.3) are referred to as the *complementarity condition* and the inequality $-2e \le \text{diag}(\text{sgn}(c))\lambda \le 0$ is called *dual feasibility*. Furthermore, we say that the *strict complementarity condition* is satisfied at a point x if $|c| + \min\{|\lambda|, |\text{diag}(\text{sgn}(c))\lambda + 2e|\} > 0$.

Let ε be a small positive number (e.g., $\varepsilon = 10^{-3}$). Define the set $\mathcal{V}$ as the indices of the functions c_i which are approaching zero, relative to the corresponding multipliers with a tolerance ε, but the corresponding multipliers $\lambda_{ki}(D_k)$ predict that they should not:

$$\begin{aligned} \mathcal{V}_k \overset{\text{def}}{=} \{j \ : \quad &\textbf{either} \quad (\lambda_{kj}(D_k)\text{sgn}(c_{kj}) > 0 \textbf{ and } |c_{kj}| < \varepsilon|\lambda_{kj}(D_k)|) \\ &\textbf{or} \quad (\lambda_{kj}(D_k)\text{sgn}(c_{kj}) < -2 \textbf{ and } |c_{kj}| < \varepsilon|2 + \lambda_{kj}(D_k)\text{sgn}(c_{kj})|)\}. \end{aligned} \tag{2.4}$$

Identify a function $c_{k\imath}$, among $\mathcal{V}_k$, which is the "closest" to a nondifferentiable curve: $|c_{k\imath}| \stackrel{\text{def}}{=} \min(|c_{kj}| : j \in \mathcal{V}_k)$. Define the diagonal scaling matrix $\mathcal{D}_k$ as below:

$$\mathcal{D}_k \stackrel{\text{def}}{=} \begin{cases} D_k & \text{if } \mathcal{V}_k = \emptyset; \\ \mathcal{D}_{k ii} = D_{k ii}, \forall i \neq \imath \text{ and } \mathcal{D}_{k\imath\imath} = 1 & \text{otherwise.} \end{cases} \tag{2.5}$$

The definition of $\mathcal{D}_{k\imath\imath} = 1$ may seem arbitrary. However, this choice works well in our computations; we assume $\mathcal{D}_{k\imath\imath} = 1$ for simplicity.

Subsequently, we denote $\mathcal{M}_k$, which is either D_k or $\mathcal{D}_k$, as the scaling matrix under consideration. For notation simplicity, if a quantity depends on $\mathcal{M}_k$, this dependence is implicitly assumed. For example, we use P_k to denote the orthogonal projector onto the null space of $[J_k^T, -\mathcal{M}_k]$, while $P_k(\mathcal{D}_k)$ denotes the orthogonal projector onto the null space of $[J_k^T, -\mathcal{D}_k]$. Similarly, we use $[g_k; h_k]$ to denote the projection of the augmented gradient to the null space of $[J_k^T, -\mathcal{M}_k]$, i.e.,

$$\bar{g}_k \stackrel{\text{def}}{=} \begin{bmatrix} g_k \\ h_k \end{bmatrix} \stackrel{\text{def}}{=} -P_k \begin{bmatrix} \nabla f_k + J_k \text{sgn}(c_k) \\ 0 \end{bmatrix}. \tag{2.6}$$

The following assumption is made to ensure proper determination of g_k.

(AS.1) $\begin{bmatrix} J(x) \\ -D(x) \end{bmatrix}$ and $\begin{bmatrix} J(x) \\ -\mathcal{D}(x) \end{bmatrix}$ have full rank in $\mathcal{L}$.

Let $\phi_k(s)$ denote the following piecewise linear approximation to the nonlinear l_1 objective function:

$$\phi_k(s) \stackrel{\text{def}}{=} \nabla f_k^T s + \|c_k + J_k^T s\|_1 - \|c_k\|_1. \tag{2.7}$$

The affine scaling matrix $\mathcal{M}_k$ can be selected based on reduction of the first-order approximation $\phi_k(s)$ incurred by the projected gradients $g_k(D_k)$ and $g_k(\mathcal{D}_k)$. It is clear that $\mathcal{D}_k$ and D_k differ only by one diagonal element; this can be exploited computationally. An interested reader is referred to [11] for an example of determining $\mathcal{M}_k$.

Approximation $\Gamma_k(s)$. At each iteration, an approximation $\Gamma_k(s)$ is chosen to approximate the change of the nonlinear l_1 function $\Upsilon(x)$.

Let B_k denote an approximation to the Lagrangian Hessian of (1.1):

$$B_k \approx \nabla^2 f_k + \nabla^2 \|c_k\|_1 + \sum_{i=1}^{m} \lambda_{ki} \nabla^2 c_{ki},$$

where λ_k is a solution to the least squares problem:

$$\begin{bmatrix} J_k \\ -\mathcal{M}_k \end{bmatrix} \lambda \stackrel{\text{LS}}{=} - \begin{bmatrix} \nabla f_k + J_k \text{sgn}(c_k) \\ 0 \end{bmatrix}. \tag{2.8}$$

In [11], the approximation $\Gamma_k(s)$ is selected as either a piecewise linear $\phi_k(s)$ or a (extended) piecewise quadratic:

$$\Gamma_k(s) \stackrel{\text{def}}{=} \begin{cases} \text{either} & \phi_k(s), \\ \text{or} & \phi_k(s) + \frac{1}{2} s^T B_k s. \end{cases} \tag{2.9}$$

Let $\mathcal{F}_k$ denote the following region corresponding to a first-order sign restriction:

$$\mathcal{F}_k \stackrel{\text{def}}{=} \{s : \text{diag}(c_k)(c_k + J_k^T s) \geq 0\}. \tag{2.10}$$

When $\nabla^2 c_{ki}$ is available, let $\psi_k(s)$ denote the quadratic

$$\psi_k(s) \stackrel{\text{def}}{=} \nabla f_k^T s + \text{sgn}(c_k)^T J_k^T s + \frac{1}{2} s^T \nabla^2 f_k s + \frac{1}{2} s^T \sum_{i=1}^{m} \lambda_{ki} \nabla^2 c_{ki} s. \tag{2.11}$$

If is clear that, if $\Gamma_k(s) = \phi_k(s) + \frac{1}{2} s^T B_k s$, $B_k = \nabla^2 f_k + \nabla^2 \|c_k\|_1 + \sum_{i=1}^{m} \lambda_{ki} \nabla^2 c_{ki}$ and $s_k \in \mathcal{F}_k$, then $\Gamma_k(s)$ is the simple quadratic $\psi_k(s)$.

Let $[Z_k; \bar{Z}_k]$ be an orthonormal basis for the null space of $[J_k^T, -\mathcal{M}_k]$, where $Z_k \in \Re^{n \times n}$ and $\bar{Z}_k \in \Re^{m \times n}$. Assume further that

$$\begin{bmatrix} J_k \\ -\mathcal{M}_k \end{bmatrix} = \begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix} R_k, \quad \text{where } Y_k \in \Re^{n \times m}, \bar{Y}_k \in \Re^{m \times m} \text{ and } \begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix} \text{ is orthonormal}. \tag{2.12}$$

Let $\vartheta_k(s)$ denote the following addition to the first-order change $c_k + J_k^T s$ of $c(x)$ at x_k:

$$\vartheta_k(s) \stackrel{\text{def}}{=} - \begin{bmatrix} \frac{1}{2} s^T \nabla^2 c_{k1} s \\ \vdots \\ \frac{1}{2} s^T \nabla^2 c_{km} s \end{bmatrix}, \quad \text{for any } s \in \Re^n. \tag{2.13}$$

Assume that $s \in \Re^n, w \in \Re^m$ satisfy equality constraints: $D_k w - J_k^T s = 0$, i.e., $s = Z_k z$. Let $u_k(s)$ denote the parametric quadratic curve:

$$\mathcal{P}_k = \{u_k(s) : u_k(s) \stackrel{\text{def}}{=} s + Y_k R_k^{-T} \vartheta_k(s), \ s = Z_k z\}.$$

The following theorem, proved in [11], states that $\Gamma_k(s)$ is at least a first-order approximation to $\Upsilon(x_k + s) - \Upsilon(x_k)$ and the approximation can achieve second-order accuracy asymptotically.

THEOREM 2.1. [Theorem 3.5 in [11]] *Assume that the functions $f(x)$ and $c(x)$ are twice continuously differentiable on the compact level set $\mathcal{L}$ and the full rank assumption (AS.1) holds. Assume that $\{x_k\}$ converges to x^*, where $x_{k+1} = x_k + u_k(s_k)$, with $u_k(s_k) = s_k + Y_k R_k^{-T} \vartheta_k(s_k)$ and $J_k^T s_k - \mathcal{M}_k w_k = 0$. Then*

$$\Upsilon(x_k + u_k) - \Upsilon(x_k) = \Gamma_k(s_k) + o(\|s_k\|_2),$$

where Γ_k is defined by (2.9). Moreover, if, for sufficiently large k, $\mathcal{M}_k = D_k$, $B_k = \nabla^2 f_k + \nabla^2 \|c_k\|_1 + \sum_{i=1}^{m} \lambda_{ki} \nabla^2 c_{ki}$, $\Gamma_k(s) = \phi_k(s) + \frac{1}{2} s^T B_k s$ and $s_k \in \mathcal{F}_k$, then

$$\Upsilon(x_k + u_k) - \Upsilon(x_k) = \Gamma_k(s_k) + o(\|s_k\|_2^2).$$

The Trust Region Subproblem. The following trust region subproblem is derived in [11] to provide sufficient reductions for approximations:

$$
\begin{array}{lll}
 & & \min_{s,w} (\nabla f_k + J_k \mathrm{sgn}(c_k))^T s_k + \frac{1}{2} s^T B_k s + \frac{1}{2} w^T C_k w \\
(2.14) & \text{subject to} & J_k^T s - \mathcal{M}_k w = 0 \\
 & & \| \begin{bmatrix} s \\ w \end{bmatrix} \|_2 \le \Delta_k .
\end{array}
$$

Asymptotically, the solution $[p_k; q_k]$ of this subproblem is closely related to the Newton step of some nonlinear equations for fast local convergence [11].

Explicit Conditions for Optimality. Now we describe the acceptance conditions on the steps for producing sufficient decrease. First, we introduce some additional notations.

Let $\Psi_k(s,w)$ denote the following extended piecewise quadratic for the quadratic objective function in the trust region subproblem (2.14):

$$
(2.15) \qquad \begin{array}{l}
\Psi_k(s,w) \stackrel{\text{def}}{=} \phi_k(s) + \frac{1}{2} s^T B_k s + \frac{1}{2} w^T C_k w, \\
C_k \stackrel{\text{def}}{=} \mathrm{diag}(|\lambda_k|).
\end{array}
$$

Given $s_k \in \Re^n$ and $w_k \in \Re^m$ such that $\mathcal{M}_k w_k - J_k^T s_k = 0$, we can compute the stepsizes α_k and $\check{\alpha}_k$ which are the smallest and second smallest stepsizes along s_k to the boundary of the first-order sign restriction region $\mathcal{F}_k$:

$$
(2.16) \qquad \begin{array}{l}
\alpha_k \stackrel{\text{def}}{=} \min\{ -\frac{c_{k_i}}{\nabla {c_k}_i^T s_k} \; : \; -\frac{c_{k_i}}{\nabla {c_k}_i^T s_k} > 0, \; 1 \le i \le m \}, \\
\check{\alpha}_k \stackrel{\text{def}}{=} \min\{ -\frac{c_{k_i}}{\nabla {c_k}_i^T s_k} \; : \; -\frac{c_{k_i}}{\nabla {c_k}_i^T s_k} > \alpha_k, \; 1 \le i \le m \}.
\end{array}
$$

Let the superscript $\star$ denote the minimum value within the trust region while the superscript $*$ denote the minimum value within the trust region **and** the sign restriction region $\mathcal{F}_k$.

Specifically, denote these minimum values of $\Psi_k(s,w)$ as below:

$$
(2.17) \qquad \begin{array}{ll}
\Psi_k^\star[s_k, w_k] & \stackrel{\text{def}}{=} \Psi_k(\tau_k^\star s_k, \tau_k^\star w_k) \\
 & \stackrel{\text{def}}{=} \min_{\tau \ge 0} \{ \Psi_k(\tau s_k, \tau w_k) : \tau \le \check{\alpha}_k, \; \|\tau \begin{bmatrix} s_k \\ w_k \end{bmatrix} \|_2 \le \Delta_k \}, \\
\Psi_k^*[s_k, w_k] & \stackrel{\text{def}}{=} \Psi_k(\tau_k^* s_k, \tau_k^* w_k) \\
 & \stackrel{\text{def}}{=} \min_{\tau \ge 0} \{ \Psi_k(\tau s_k, \tau w_k) : x_k + \tau s_k \in \mathcal{F}_k, \|\tau \begin{bmatrix} s_k \\ w_k \end{bmatrix} \|_2 \le \Delta_k \}.
\end{array}
$$

The approximation Γ_k and s_k are chosen to satisfy the following sufficient decrease conditions:

Let $0 < \beta_{cs}, \beta_{df}, \beta_q, \beta_{2nd} < 1, \beta_s > 0$. Let $[g_k, h_k]$ denote the projected gradient (2.6) and $[p_k; q_k]$ be a global solution to (2.14). Assume that there exists $w_k \in \Re^m$ such that $J_k^T s_k - \mathcal{M}_k w_k = 0$ and $\|s_k\|_2 \le \beta_s \Delta_k$. (Recall that $p_k \stackrel{\text{def}}{=} p_k(\mathcal{M}_k)$ etc.)

(AC.1) $\Gamma_k(s_k) < \beta_{cs} \Psi_k^*[g_k(D_k), h_k(D_k)]$;

(AC.2) $\Gamma_k(s_k) < \beta_{df} \Psi_k^*[g_k(\mathcal{D}_k), h_k(\mathcal{D}_k)]$;

(AC.3) $\Gamma_k(s_k) < \beta_{2nd} \Psi_k^*[p_k, q_k]$. In addition, if $\Psi_k^*[p_k, q_k] \le \beta_q \Psi_k^*[g_k, h_k]$, then $B_k = \nabla^2 f_k + \nabla^2 \|c_k\|_1 + \sum_{i=1}^m \lambda_{ki} \nabla^2 c_{ki}$, $\Gamma_k(s_k) = \psi_k(s_k)$ and $s_k \in \mathcal{F}_k$.

There are different ways to construct algorithms in the model given. An example algorithm satisfying the conditions (AC.1), (AC.2) and (AC.3) is described in [11]. We subsequently prove that the conditions (AC.1), (AC.2) and (AC.3) lead to optimality. Specifically, (AC.1) is necessary for complementarity; (AC.2) for dual feasibility; (AC.3) for second-order optimality. Since the conditions (AC.1) and (AC.2) can be satisfied without the availability of the Hessians of $c(x)$, first-order optimality can be obtained without knowledge of the second-order information.

3. Global Convergence Properties. In this section, we study the global convergence properties of algorithm TRASM in FIG. 1.

The main results are in Theorems 3.6, 3.8 and 3.14. Assume that the level set $\mathcal{L}$ is compact, $c(x)$ and $f(x)$ are continuously differentiable and the full rank assumption (AS.1) holds. We subsequently prove that, if the reduction of the approximation $\Gamma_k(s_k)$ satisfies (AC.1), then every limit point satisfies the complementarity condition (see Theorem 3.6). If the reduction of the approximation $\Gamma_k(s_k)$ satisfies both (AC.1) and (AC.2) and strict complementarity holds, then every limit point satisfies the first-order necessary conditions (see Theorem 3.8). Assuming strict complementarity condition and twice continuous differentiability of $f(x)$ and $c(x)$, if the reduction of the approximation $\Gamma_k(s_k)$ satisfies both (AC.1), (AC.2) and (AC.3), then there exists at least one limit point at which the first and second-order necessary conditions are all satisfied (see Theorem 3.14).

From (2.6) and (2.8), it is easy to verify that

$$\begin{aligned} \bar{g}_k &= -(\begin{bmatrix} \nabla f_k + J_k \text{sgn}(c_k) \\ 0 \end{bmatrix} - \begin{bmatrix} J_k \\ -\mathcal{M}_k \end{bmatrix} \lambda_k \\ &= -\begin{bmatrix} \nabla f_k + J_k \text{sgn}(c_k) - J_k \lambda_k \\ \mathcal{M}_k \lambda_k \end{bmatrix}. \end{aligned} \tag{3.1}$$

The following Lemma 3.1 from [11] will be repeatedly used.

LEMMA 3.1. *Assume that the level set $\mathcal{L} = \{x : \Upsilon(x) \le \Upsilon(x_0)\}$ is compact. Let $\{x_k\}$ be any sequence in $\mathcal{L}$. Assume that the full rank assumption (1) holds. Then*

1. If $f(x)$ and $c_i(x)$ are continuously differentiable, then the multiplier function λ_k is bounded, i.e., *there exists χ_λ such that*

$$\|\lambda_k\|_2 \leq \chi_\lambda;$$

2. *If $f(x)$ and $c_i(x)$ are twice continuously differentiable and $J_k^T s - \mathcal{M}_k w = 0$, then*

$$\|u - s\|_2 = \| \begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix} R_k^{-T} \vartheta_k(s)\|_2 = O(\|s\|_2^2).$$

We first prove Theorem 3.6: the complementarity condition is satisfied at every limit point. Assume that the full rank assumption (AS.1) holds. The complementarity condition is satisfied at x^* if and only if the orthogonal projection of the argumented gradient $[\nabla f^* + J^*\text{sgn}(c^*); 0]$ onto the null space of $[J^{*T}, -D^*]$ is zero. Similarly, based on Lemma 3.3 in [11], under the additional strict complementarity condition, dual feasibility holds if and only if the orthogonal projection of the argumented gradient $[\nabla f^* + J^*\text{sgn}(c^*); 0]$ on to the null space of $[J^{*T}, -\mathcal{D}^*]$ is zero. Since the argumented gradient $[\nabla f(x) + J(x)\text{sgn}(c(x)); 0]$ is not a continuous function, we need to examine the asymptotic behavior of these projections.

LEMMA 3.2. *Assume that the full rank assumption (AS.1) holds and $\{x_k\}$ is any sequence in $\mathcal{L}$ which converges to x^*. Then*

1. *The complementarity condition is satisfied at x^* if and only if $\{\bar{g}_k(D_k)\}$ converges to zero.*
2. *Assume further that the Kuhn-Tucker condition with the strict complementarity is satisfied at x^*. Then, for sufficiently large k and any i such that $c_i^* = 0$,*

$$-2 < sgn(c_{ki})\lambda_{ki} < 0.$$

Moreover, $\mathcal{D}_k = D_k$ for sufficiently large k.

3. *Assume further that the strict complementarity holds at x^*. Then dual feasibility holds at x^* if and only if $\{\bar{g}_k(\mathcal{D}_k)\}$ converges to zero.*

Proof. We consider each result in order.

1. Let $\mathcal{A} = \{i : \ c_i^* = 0\}$ and a^* be any vector such that $a_j^* = 0$, if $j \notin \mathcal{A}$ and a_j^* is either 0 or 2 if $j \in \mathcal{A}$. Recall the definition (1.2) of sgn . It is clear that, if $\{\bar{g}_k(D_k)\}$ converges to zero, then

$$\begin{bmatrix} J^* \\ -D^* \end{bmatrix} \lambda^*(D^*) = -\begin{bmatrix} \nabla f^* + J^*(\text{sgn}(c^*) - a^*) \\ 0 \end{bmatrix}.$$

for some a^*. From (2.3) and (3.1), complementarity holds at x^*.

On the other hand, assume that the complementarity conditions are satisfied at x^*. Hence there exists $\lambda^*(D^*)$ such that

$$\begin{bmatrix} J^* \\ -D^* \end{bmatrix} \lambda^*(D^*) = -\begin{bmatrix} \nabla f^* + J^*\text{sgn}(c^*) \\ 0 \end{bmatrix}.$$

Then it is clear that

$$\begin{bmatrix} J^* \\ -D^* \end{bmatrix} (\lambda^*(D^*) + a^*) = -\begin{bmatrix} \nabla f^* + J^*(\text{sgn}(c^*) - a^*) \\ 0 \end{bmatrix}.$$

Since there are finite number of such a^* and $\text{sgn}(c_k)$ equals $\text{sgn}(c^*) - a^*$ for some a^*, we conclude that $\{\bar{g}_k(D_k)\}$ converges to zero.

2. Under the full rank assumption (AS.1) and the limit x^* being a Kuhn-Tucker point with the strict complementarity, there is a unique solution to

$$\begin{bmatrix} J^* \\ -D^* \end{bmatrix} y^* = -\nabla f^*, \quad \text{with } \; y_j^* = \text{sgn}(c_j^*), \text{ if } c_j^* \neq 0,$$

and $-1 < y_i^* < 1$, if $c_i^* = 0$.
From the first result proved, $\lim_{k\to\infty} \bar{g}_k(D_k) = 0$. This is equivalent to

$$\lim_{k\to\infty} \begin{bmatrix} J_k \\ -D_k \end{bmatrix} (\lambda_k(D_k) + \text{sgn}(c_k)) = -\nabla f^*.$$

Since $\begin{bmatrix} J^* \\ -D^* \end{bmatrix}$ has full rank, $\{\lambda_k(D_k) + \text{sgn}(c_k)\}$ converges to y^*. Thus, for sufficiently large k, for any i such that $c_i^* = 0$, $-1 < \lambda_{ki}(D_k) + \text{sgn}(c_{ki}) < 1$ or equivalently

$$-2 < \text{sgn}(c_{ki})\lambda_{ki}(D_k) < 0.$$

In addition, by definition (2.5) of $\mathcal{D}_k$, it is clear that $\mathcal{D}_k = D_k$ for sufficiently large k.

3. Using the above result, if x^* is a Kuhn-Tucker point with strict complementarity, then $\mathcal{D}_k = D_k$ for sufficiently large k. Hence, $\{\bar{g}_k(\mathcal{D}_k)\}$ converges to zero.
Next we consider the other direction. Assume that $\{\bar{g}_k(\mathcal{D}_k)\}$ converges to zero and strict complementarity holds at x^*. We show, by contradiction, that dual feasibility holds at x^*, i.e., $\mathcal{V}^* = \emptyset$ where $\mathcal{V}^*$ is defined by (2.4).
Assume that $\mathcal{V}^* \neq \emptyset$ and $\mathcal{D}_{ii}^* = 1$. From $\{\bar{g}_k(\mathcal{D}_k)\}$ converging to zero, it is clear that there exists some vector a^*, satisfying $a_j^* = 0$, if $c_j^* \neq 0$ and a_j^* equals either 0 or 2, such that

$$\begin{bmatrix} J^* \\ -\mathcal{D}^* \end{bmatrix} \lambda^*(\mathcal{D}^*) = -\begin{bmatrix} \nabla f^* + J^*(\text{sgn}(c^*) - a^*) \\ 0 \end{bmatrix}.$$

Hence $\lambda_i^*(\mathcal{D}^*) = 0$. But it is clear that

$$\begin{bmatrix} J^* \\ -D^* \end{bmatrix} \lambda^*(\mathcal{D}^*) = -\begin{bmatrix} \nabla f^* + J^*(\text{sgn}(c^*) - a^*) \\ 0 \end{bmatrix}.$$

If $a_i^* = 0$, this implies that the strict complementarity is violated. Hence $\mathcal{V}^* = \emptyset$.
Assume that $a_i^* = 2$. Since complementarity is satisfied at x^*, we have

$$\begin{bmatrix} J^* \\ -D^* \end{bmatrix} (\lambda^*(D^*) + a^*) = -\begin{bmatrix} \nabla f^* + J^*(\text{sgn}(c^*) - a^*) \\ 0 \end{bmatrix}. \tag{3.2}$$

But either $\lambda_i^*(D^*) < -2$ or $\lambda_i^*(D^*) > 0$. This means that $\lambda^*(D^*) + a_i^* \neq \lambda^*(\mathcal{D}^*) = 0$. This contradicts to the uniqueness of the solution to the equations (3.2) due to the full rank assumption. Therefore, $D^* = \mathcal{D}^*$. In other words, dual feasibility is satisfied at x^*.

This proof is completed. □

Lemma 3.2 indicates that, to establish satisfaction of the complementarity condition, we only need to prove that $\{\bar{g}_k(D_k)\}$ converges to zero. In addition, assuming that $\{\bar{g}_k(D_k)\}$ converges to zero, dual feasibility is satisfied at every limit point if $\{\bar{g}_k(\mathcal{D}_k)\}$ converges to zero.

Let $\bar{H}$ denote the Hessian of the quadratic in the trust region subproblem (2.14) and H_k denote the projected Hessian below:

$$\bar{H}_k \stackrel{\text{def}}{=} \begin{bmatrix} B_k & 0 \\ 0 & C_k \end{bmatrix} \quad \text{and} \quad H_k \stackrel{\text{def}}{=} \begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix}^T \bar{H}_k \begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix}, \tag{3.3}$$

where $[Y_k; \bar{Y}_k]$ is an orthonormal basis as defined in (2.12).

We subsequently assume that $\{B_k\}$ is uniformly bounded: there exists $\chi_B > 0$ such that $\|B_k\|_2 \leq \chi_B$. If $B_k = \nabla^2 f_k + \nabla^2 \|c_k\|_1 + \sum_{i=1}^m \lambda_{ki} \nabla^2 c_{ki}$, this condition is satisfied under the compactness assumption of $\mathcal{L}$, the full rank assumption (AS.1) and twice continuous differentiability of $c(x)$ and $f(x)$. If $\{B_k\}$ is uniformly bounded, the level set is compact and the full rank assumption (AS.1) holds, then there exists $\chi_H > 0$ such that

$$\|\bar{H}_k\| \leq \chi_H. \tag{3.4}$$

Now we prove a useful technical lemma. Recall that $\bar{g}_k = [g_k; h_k]$ is defined by (2.6).

LEMMA 3.3. *Assume that $\sigma_k \geq \chi_1 \|\bar{g}_k\|_2$ for some $\chi_1 > 0$ and b_k is any vector in $\Re^n$ with $b_k^T g_k \leq 0$. Moreover, $c(x)$ and $f(x)$ are continuously differentiable, $\{B_k\}$ is uniformly bounded, the level set is compact and the full rank assumption (AS.1) holds. Then, there exists $\chi_0 > 0$ such that,*

$$\min_{0 \leq \tau \leq \min(\Delta_k, \sigma_k)} \tau \frac{b_k^T g_k}{\|\bar{g}_k\|_2} + \frac{1}{2}\tau^2 \frac{\bar{g}_k^T \bar{H}_k \bar{g}_k}{\|\bar{g}_k\|_2^2} \leq \frac{1}{2} \min\left(\frac{\Delta_k}{\|\bar{g}_k\|_2}, \frac{\chi_0}{\|\bar{g}_k\|_2}, \chi_1\right) b_k^T g_k.$$

Proof. Let τ_k^* be the minimizer of $\omega(\tau)$ in $[0, \min(\Delta_k, \sigma_k)]$, where we consider

$$\min_{0 \leq \tau \leq \min(\Delta_k, \sigma_k)} \omega(\tau) \stackrel{\text{def}}{=} \tau \frac{b_k^T g_k}{\|\bar{g}_k\|_2} + \frac{1}{2}\tau^2 \mu_k, \qquad \text{where} \quad \mu_k \stackrel{\text{def}}{=} \frac{\bar{g}_k^T \bar{H}_k \bar{g}_k}{\|\bar{g}_k\|_2^2}.$$

If $\tau_k^* \in [0, \min(\Delta_k, \sigma_k))$, then $\mu_k \geq 0$ (thus $\bar{g}_k^T \bar{H}_k \bar{g}_k \geq 0$) and

$$\tau_k^* = -\frac{b_k^T g_k}{\|\bar{g}_k\|_2 \mu_k}.$$

Using $\mu_k \leq \|\bar{H}_k\| \leq \chi_H$ and $b_k^T g_k \leq 0$, we have

$$\omega(\tau_k^*) = \frac{1}{2} \frac{b_k^T g_k}{\|\bar{g}_k\|_2 \mu_k} \leq \frac{1}{2} \frac{b_k^T g_k}{\|\bar{g}_k\|_2 \|\bar{H}_k\|} \leq \frac{1}{2} \frac{b_k^T g_k}{\chi_H \|\bar{g}_k\|_2}.$$

Assume $\tau_k^* = \Delta_k$. Then $\Delta_k \leq -\frac{b_k^T g_k}{\|\bar{g}_k\|_2 \mu_k}$. Since $\mu_k \Delta_k \leq -\frac{b_k^T g_k}{\|\bar{g}_k\|_2}$ when $\mu_k > 0$, and $\omega(\Delta_k) \leq \Delta_k \frac{b_k^T g_k}{\|\bar{g}_k\|_2}$ otherwise, we have

$$\omega(\Delta_k) \leq \frac{1}{2} \Delta_k \frac{b_k^T g_k}{\|\bar{g}_k\|_2}.$$

Assume $\tau_k^* = \sigma_k$. Then $\sigma_k \leq -\frac{b_k^T g_k}{\|\bar{g}_k\|_2 \mu_k}$. Since $\mu_k \sigma_k \leq -\frac{b_k^T g_k}{\|\bar{g}_k\|_2}$ when $\mu_k > 0$, and $\omega(\sigma_k) \leq \sigma_k \frac{b_k^T g_k}{\|\bar{g}_k\|_2}$ otherwise, using the assumption $\sigma_k \geq \chi_1 \|\bar{g}_k\|_2$, we have

$$\omega(\sigma_k) \leq \frac{1}{2}\sigma_k \frac{b_k^T g_k}{\|\bar{g}_k\|_2} \leq \frac{1}{2}\chi_1 b_k^T g_k.$$

In conclusion, letting $\chi_0 = \frac{1}{\chi_H}$, we have

$$\min_{0 \leq \tau \leq \min(\Delta_k, \sigma_k)} \omega(\tau) \leq \frac{1}{2}\min(\frac{\Delta_k}{\|\bar{g}_k\|_2}, \frac{\chi_0}{\|\bar{g}_k\|_2}, \chi_1) b_k^T g_k.$$

This completes the proof. □

The following result expresses (AC.1) in a manageable form. It is similar to Lemma (4.8) in [13].

LEMMA 3.4. *Assume that the level set $\mathcal{L} = \{x : \Upsilon(x) \leq \Upsilon(x_0)\}$ is compact, $c(x)$ and $f(x)$ are continuously differentiable, $\{B_k\}$ is uniformly bounded and the full rank assumption (AS.1) holds. Then there exists $\chi_0, \chi_1 > 0$ such that*

$$\Psi_k^*[g_k(D_k), h_k(D_k)] \leq \frac{1}{2}\min(\frac{\Delta_k}{\|\bar{g}_k(D_k)\|_2}, \frac{\chi_0}{\|\bar{g}_k(D_k)\|_2}, \chi_1)(\nabla f_k + J_k sgn(c_k))^T g_k(D_k)$$

where Ψ_k^* is defined by (2.17).

Proof. In this proof, we assume the dependence on D_k and drop it from the notation, i.e., $g_k \stackrel{\text{def}}{=} g_k(D_k)$ etc.

We estimate $\Psi_k^*[g_k, h_k]$. Define $\omega(\tau) : \Re \rightarrow \Re$ as the following quadratic:

$$\omega(\tau) \stackrel{\text{def}}{=} \tau(\nabla f_k + J_k \text{sgn}(c_k)^T \frac{g_k}{\|\bar{g}_k\|_2} + \frac{1}{2}\tau^2 \frac{g_k^T B_k g_k}{\|\bar{g}_k\|_2^2} + \frac{1}{2}\tau^2 \frac{h_k^T C_k h_k}{\|\bar{g}_k\|_2^2}.$$

Equivalently, we have

$$\omega(\tau) = \tau \frac{(\nabla f_k + J_k \text{sgn}(c_k))^T g_k}{\|\bar{g}_k\|_2} + \frac{1}{2}\tau^2 \mu_k, \text{ where } \mu_k \stackrel{\text{def}}{=} \frac{\bar{g}_k^T \bar{H}_k \bar{g}_k}{\|\bar{g}_k\|_2^2}.$$

Let

$$\alpha_k = \min\{-\frac{c_{ki}}{\nabla c_{k_i}^T d_k} : -\frac{c_{ki}}{\nabla c_{k_i}^T d_k} > 0,\ 1 \leq i \leq m\} \quad \text{with } d_k = \frac{g_k}{\|\bar{g}_k\|_2}.$$

It is clear that $\omega(\tau) = \Psi_k(\tau g_k, \tau h_k)$ when $\tau \leq \min(\alpha_k, \Delta_k)$. Hence, by definition (2.17),

$$\Psi_k^*[g_k, h_k] = \min_{0 \leq \tau \leq \min(\Delta_k, \alpha_k)} \omega(\tau),$$

Since $\nabla c_{k_i}^T d_k = D_{k_{ii}} \frac{h_k}{\|\bar{g}_k\|_2}$, $h_k = D_k \lambda_k$ using (3.1), we have

$$\nabla c_{k_i}^T d_k = -D_{k_{ii}}^2 \frac{\lambda_{ki}}{\|\bar{g}_k\|_2}.$$

From Lemma 3.1, there exists $\chi_\lambda > 0$ such that $\|\lambda_k\|_2 \le \chi_\lambda$. We have

$$\alpha_k \ge \chi_1 \|\bar{g}_k\|_2, \quad \text{where } \chi_1 = \frac{1}{\chi_\lambda}. \tag{3.5}$$

Applying Lemma 3.3 with $b_k = \nabla f_k + J_k \mathrm{sgn}(c_k)$,

$$\min_{0 \le \tau \le \min(\Delta_k, \alpha_k)} \omega(\tau) \le \frac{1}{2} \min\left(\frac{\Delta_k}{\|\bar{g}_k\|_2}, \frac{\chi_0}{\|\bar{g}_k\|_2}, \chi_1\right)(\nabla f_k + J_k \mathrm{sgn}(c_k))^T g_k.$$

Hence

$$\Psi_k^*[g_k, h_k] \le \frac{1}{2} \min\left(\frac{\Delta_k}{\|\bar{g}_k\|_2}, \frac{\chi_0}{\|\bar{g}_k\|_2}, \chi_1\right)(\nabla f_k + J_k \mathrm{sgn}(c_k))^T g_k.$$

The proof is completed. □

From (3.1), we have

$$(\nabla f_k + J_k \mathrm{sgn}(c_k))^T g_k = -\|\bar{g}_k(D_k)\|_2^2.$$

If condition (AC.1) is satisfied, then

$$\Gamma_k(s_k) \le \beta_{cs} \Psi_k^*[g_k(D_k), h_k(D_k)] \le -\frac{\beta_{cs}}{2} \min(\Delta_k, \chi_0, \chi_1 \|\bar{g}_k(D_k)\|_2) \|\bar{g}_k(D_k)\|_2. \tag{3.6}$$

Assume that $x_{k+1} = x_k + u_k$ is a successful step. Lemma 3.4 implies that

$$\Upsilon(x_k) - \Upsilon(x_{k+1}) \ge -\zeta \Gamma_k(s_k) \ge \frac{\beta_{cs}\zeta}{2} \min(\Delta_k, \chi_0, \chi_1 \|\bar{g}_k(D_k)\|_2) \|\bar{g}_k(D_k)\|_2. \tag{3.7}$$

This inequality is important for the convergence proof.

Next, in Theorem 3.6, we prove that the complementarity condition is satisfied at every limit point of $\{x_k\}$. We first prove that there exists a limit point satisfying complementarity. The proof of the following lemma is a slight modification of Theorem (4.10) in [13].

LEMMA 3.5. *Assume that the level set $\mathcal{L} = \{x : \Upsilon(x) \le \Upsilon(x_0)\}$ is compact, $c(x)$ and $f(x)$ are continuously differentiable, $\{B_k\}$ is uniformly bounded and the full rank assumption (AS.1) holds. Assume further that $\Upsilon(x_k + u_k) - \Upsilon(x_k) = \Gamma_k(s_k) + o(\|s_k\|)$ and $\{s_k\}$ and $\{u_k\}$ are generated by TRASM in FIG. 1, if $\Gamma_k(s_k)$ satisfies (AC.1), then*

$$\liminf_{k \to \infty} \|\bar{g}_k(D_k)\|_2 = 0. \tag{3.8}$$

Proof. We show that $\{\|\bar{g}_k(D_k)\|_2\}$ is not asymptotically bounded away from zero by contradiction. Assume that there is an $\epsilon > 0$ such that $\|\bar{g}_k(D_k)\|_2 \ge \epsilon$ for all sufficiently large k. We first prove that

$$\sum_{k=1}^{\infty} \Delta_k < +\infty. \tag{3.9}$$

If there are a finite number of successful iterations then $\Delta_{k+1} \le \gamma_1 \Delta_k$ for all k sufficiently large and then (3.9) clearly holds. Assume that there is an infinite sequence $\{k_i\}$ of successful iterations. From (3.7), $\|\bar{f}_k\|_2 \ge \epsilon$ and $\{\Upsilon(x_k)\}$ is monotonically decreasing and bounded below, we have

$$\sum_{i=1}^{\infty} \Delta_{k_i} < +\infty.$$

Now the updating rules of TRASM in FIG. 1 imply that

$$\sum_{k=1}^{\infty} \Delta_k \le (1 + \frac{\gamma_2}{1-\gamma_1}) \sum_{i=1}^{\infty} \Delta_{k_i}.$$

Therefore, $\{\Delta_k\}$ converges to zero.

Next we prove that (3.9) implies that $\{|\rho_k - 1|\}$ converges to zero. First, using Lemma 3.1 and $\|s_k\| \le \beta_s \Delta_k$, we have

$$\|x_{k+1} - x_k\|_2 = \|u_k\|_2 \le \|s_k\|_2 + \|u_k - s_k\|_2 = O(\Delta_k) + O(\Delta_k^2).$$

Hence (3.9) shows that $\{x_k\}$ converges and $\{s_k\}$ and $\{u_k\}$ converge to zero.

By assumption that $\Gamma_k(x_k + u_k) - \Gamma_k(x_k) = \Gamma_k(s_k) + o(\|s_k\|_2)$, there exists a sequence of positive numbers $\{\epsilon_k\}$ converging to zero such that

$$|\Upsilon(x_k + u_k) - \Upsilon(x_k) - \Gamma_k(s_k)| = \epsilon_k \|s_k\|_2 \le \epsilon_k \beta_s \Delta_k.$$

From (3.7) and $\|\bar{g}_k(\mathcal{D}_k)\|_2 \ge \epsilon$, for sufficiently large k,

$$-\Gamma_k(s_k) \ge \frac{1}{2} \epsilon \beta_{cs} \Delta_k,$$

we readily obtain that $\{|\rho_k - 1|\}$ converges to zero. The updating rule of Δ_k implies that $\{\Delta_k\}$ cannot converge to zero. This contradicts (3.9) and establishes the result. □

The next theorem establishes that $\{\bar{g}_k(D_k)\}$ converges to zero. The proof of this theorem is similar to that of Theorem (4.14) in Moré [13].

THEOREM 3.6. *Assume that the level set* $\mathcal{L} = \{x : \Upsilon(x) \le \Upsilon(x_0)\}$ *is compact,* $c(x)$ *and* $f(x)$ *are continuously differentiable,* $\{B_k\}$ *is uniformly bounded and the full rank assumption (AS.1) holds. Assume further that* $\Upsilon(x_k + u_k) - \Upsilon(x_k) = \Gamma_k(s_k) + o(\|s_k\|)$ *and* $\{s_k\}$ *and* $\{u_k\}$ *are generated by TRASM in FIG. 1 and (AC.1) holds for* $\Gamma_k(s_k)$. *Then*

$$\lim_{k \to \infty} \|\bar{g}_k(D_k)\|_2 = 0. \tag{3.10}$$

Proof. Again we drop the dependence on D_k in our notation in this proof and prove the result by contradiction. Let ϵ_1 in $(0, 1)$ be given and assume that $\limsup_{k\to\infty} \|\bar{g}_k\|_2 \ge \epsilon_1$.

For any ϵ_2 in $(0, \epsilon_1)$, there exist two subsequences $\{l_i\}$ and $\{m_i\}$ with $\{l_i\}$ denoting the subsequence of all the indices with $\|\bar{g}_{l_i}\|_2 < \epsilon$ and

$$\|\bar{g}_k\|_2 \geq \epsilon_2, \quad m_i \leq k < l_i, \quad \|\bar{g}_{l_i}\|_2 < \epsilon_2. \tag{3.11}$$

Theorem 3.5 guarantees that $\liminf_{i\to\infty} \|\bar{g}_{l_i}\|_2 = 0$.

If the k-th iteration is successful, then according to (3.7),

$$\Upsilon(x_k) - \Upsilon(x_{k+1}) \geq \frac{1}{2}\beta_{cs}\zeta\epsilon_2 \min(\Delta_k, \chi_0, \chi_1\epsilon_2), \quad m_i \leq k < l_i. \tag{3.12}$$

Since $\Upsilon(x)$ is bounded below on $\mathcal{L}$ and $\{\Upsilon(x_k)\}$ monotonically converges, $\{\Upsilon(x_k) - \Upsilon(x_{k+1})\}$ converges to zero. Hence $\{\Delta_k\}$ converges to zero. From $\|x_{k+1} - x_k\|_2 \leq \|s_k\|_2 + \|u_k - s_k\|_2 = O(\Delta_k) + O(\Delta_k^2)$, it follows that, there exists $\epsilon_3 > 0$ such that, for sufficiently large i,

$$\Upsilon(x_k) - \Upsilon(x_{k+1}) \geq \epsilon_3\|x_{k+1} - x_k\|_2, \quad m_i \leq k < l_i. \tag{3.13}$$

Using (3.13) and the triangle inequality,

$$\Upsilon(x_{m_i}) - \Upsilon(x_k) \geq \epsilon_3\|x_k - x_{m_i}\|_2, \quad m_i \leq k \leq l_i.$$

Consider a subsequence of l_i (for notational simplicity we still denote it by l_i) such that $\{x_{l_i}\}$ converges to x^* and $\{\bar{g}_{l_i}\}$ converges to zero. Hence complementarity holds at x^* and $\{x_{m_i}\}$ converges to x^*. Since the complementarity condition is satisfied at x^*, from Lemma 3.2, $\{\bar{g}_{m_i}\}$ converges to zero. Hence, for i sufficiently large,

$$\|\bar{g}_{m_i}\|_2 < \epsilon_2.$$

This leads to

$$\epsilon_2 \leq \|\bar{g}_{m_i}\|_2 < \epsilon_2, \quad \text{for sufficiently large i.}$$

This is a contradiction and the proof is completed. □

Next, in Lemma 3.7 and Theorem 3.8, we prove that $\{g_k(\mathcal{D}_k)\}$ converges to zero.

LEMMA 3.7. *Assume that the conditions in Lemma 3.6 hold. Consider any subsequence $\{\bar{g}_{l_i}\}$ generated by TRASM. Moreover the strict complementarity condition is satisfied at every limit point and* $\liminf_{i\to\infty} \|\bar{g}_{l_i}(\mathcal{D}_{l_i})\|_2 > 0$. *Then, there exist $\chi_0, \chi_1, \chi_2 > 0$ such that, for sufficiently large i, if $\Gamma_{l_i}(s_{l_i})$ satisfies (AC.2) then*

$$-\Gamma_{l_i}(s_{l_i}) \geq \beta_{df}\Psi^\star_{l_i}[g_{l_i}(\mathcal{D}_{l_i}), h_{l_i}(\mathcal{D}_{l_i})] \geq \beta_{df}\chi_2 \min(\Delta_{l_i}, \chi_0, \chi_1\|\bar{g}_{l_i}(\mathcal{D}_{l_i})\|_2)\|\bar{g}_{l_i}(\mathcal{D}_{l_i})\|_2,$$

where Ψ^*_k is defined in (2.17).

Proof. For notational simplicity, we still denote the subsequence l_i by k and drop the dependence on $\mathcal{D}_k$ unless explicitly denoted, e.g., $g_k = g_k(\mathcal{D}_k)$ etc.

We estimate $\Psi^\star[g_k, h_k]$. From definition (2.17), it is clear that

$$\Psi_k^\star[g_k, h_k] = \min\{\Psi_k(\tau\frac{g_k}{\|\bar{g}_k\|_2}, \tau\frac{h_k}{\|\bar{g}_k\|_2}) : 0 \le \tau \le \min(\check{\alpha}_k, \Delta_k)\},$$

where α_k and $\check{\alpha}_k$ are as defined in (2.16) for $d_k = \frac{g_k}{\|\bar{g}_k\|_2}$:

$$\begin{aligned}\alpha_k &= \min\{-\tfrac{c_{k_i}}{\nabla c_{k_i}^T d_k} : -\tfrac{c_{k_i}}{\nabla c_{k_i}^T d_k} > 0,\ 1 \le i \le m\},\\ \check{\alpha}_k &= \min\{-\tfrac{c_{k_i}}{\nabla c_{k_i}^T d_k} : -\tfrac{c_{k_i}}{\nabla c_{k_i}^T d_k} > \alpha_k,\ 1 \le i \le m\}.\end{aligned}$$

Assume that at the k-th iteration,

$$\alpha_k = -\frac{c_{kj}}{\nabla c_{k_j}^T g_k}, \quad \text{for some } j. \tag{3.14}$$

From definition (2.17),

$$\Psi_k^\star[g_k, h_k] \le \min_{0\le\tau\le\min(\Delta_k,\alpha_k)} \tau(\nabla f_k + J_k\mathrm{sgn}(c_k))^T\frac{g_k}{\|\bar{g}_k\|_2} + \frac{1}{2}\tau^2\frac{\bar{g}_k^T\bar{H}_k\bar{g}_k}{\|\bar{g}_k\|_2^2}.$$

If $j \ne \imath$ or $D_k = \mathcal{D}_k$, then $\alpha_k \ge \chi_1\|\bar{g}_k\|_2$. Applying Lemma 3.3, there exist $\chi_0, \chi_1 > 0$, such that

$$\Psi_k^\star[g_k, h_k] \le \frac{1}{2}\min(\Delta_k, \chi_0, \chi_1\|\bar{g}_k\|_2)\|\bar{g}_k\|_2. \tag{3.15}$$

Now assume that $D_k \ne \mathcal{D}_k$ and $j = \imath$. We first prove the inequality below:

$$\liminf_{k\to\infty}\frac{(\nabla f_k + J_k\mathrm{sgn}(c_k) - 2J_{k\imath}\mathrm{sgn}(c_{k\imath}))^T g_k}{(\nabla f_k + J_k\mathrm{sgn}(c_k))^T g_k} > 0.$$

Since $\{\bar{g}_k(D_k)\}$ converges to zero, using (3.1), we have

$$\lim_{k\to\infty}\nabla f_k + J_k\mathrm{sgn}(c_k) + J_k\lambda_k(D_k) = 0 \quad\text{and}\quad \lim_{k\to\infty} D_k\lambda_k(D_k) = 0.$$

Hence there exists a sequence $\{\epsilon_k : \epsilon_k \in \Re^n\}$ converging to zero such that

$$\nabla f_k + J_k\mathrm{sgn}(c_k) = -J_k\lambda_k(D_k) + \epsilon_k.$$

In addition $\lim_{k\to\infty} D_k\lambda_k(D_k) = 0$. From $J_k^T g_k - \mathcal{D}_k h_k = 0$, $D_{kii} = \mathcal{D}_{kii}$, for any $i \ne \imath$, and $\lim_{k\to\infty} D_k\lambda_k(D_k) = 0$, we have

$$\begin{aligned}(\nabla f_k + J_k\mathrm{sgn}(c_k))^T g_k &= -g_k^T J_k\lambda_k(D_k) + O(\|\epsilon_k\|_2)\\ &= -h_{k_\imath}\lambda_{k_\imath}(D_k) + O(\hat{\epsilon}_k),\end{aligned} \tag{3.16}$$

where $\{\hat{\epsilon}_k\}$ converges to zero. Since

$$\liminf_{k\to\infty}(\nabla f_k + J_k\mathrm{sgn}(c_k))^T g_k = \liminf_{k\to\infty}(-\|\bar{g}_k\|_2^2) < 0,$$

$h_{k_\imath}\lambda_{k_\imath}(D_k) > 0$ for sufficiently large k.

From (3.14) and the assumption $j = \imath$,

$$\alpha_k = -\frac{c_{k\imath}\|\bar{g}_k\|_2}{(J_k^T g_k)_\imath} = -\frac{c_{k\imath}\|\bar{g}_k\|_2}{\mathcal{D}_{k\imath} h_{k\imath}}.$$

From definition (2.5) of $\mathcal{D}_k$ and $D_k \neq \mathcal{D}_k$, either $\lambda_{k\imath}(D_k)\mathrm{sgn}(c_{k\imath}) > 0$ or $\lambda_{k\imath}(D_k)\mathrm{sgn}(c_{k\imath}) < -2$. Since $\alpha_k \geq 0$, $\mathrm{sgn}(c_{k\imath})h_{k\imath} \leq 0$. But $h_{k\imath}\lambda_{k\imath}(D_k) > 0$ for sufficiently large k. This implies that $\lambda_{k\imath}(D_k)\mathrm{sgn}(c_{k\imath}) \leq 0$ for sufficiently large k. Hence we know $\lambda_{k\imath}(D_k)\mathrm{sgn}(c_{k\imath}) < -2$ for sufficiently large k.

From (3.16) and $J_{k\imath}^T g_{k\imath} = h_{k\imath}$, we have

$$\begin{aligned}(\nabla f_k + J_k \mathrm{sgn}(c_k) - 2J_{k\imath}\mathrm{sgn}(c_{k\imath}))^T g_k &= -h_{k\imath}(\lambda_{k\imath}(D_k) + 2\mathrm{sgn}(c_{k\imath})) + O(\hat{\epsilon}_k) \\ &= -h_{k\imath}\mathrm{sgn}(c_{k\imath})(\lambda_{k\imath}(D_k)\mathrm{sgn}(c_{k\imath}) + 2) + O(\hat{\epsilon}_k).\end{aligned} \tag{3.17}$$

From definition (2.5) of $\mathcal{D}$ and $\lim_{k\to\infty}\bar{g}_k = 0$,we conclude that $c_\imath^* = 0$. Using strict complementarity, there exists $\epsilon_d > 0$ such that, for sufficiently large k,

$$-\lambda_{k\imath}(D_k)\mathrm{sgn}(c_{k\imath}) - 2 > \epsilon_d. \tag{3.18}$$

Since $\mathrm{sgn}(c_{k\imath})\lambda_{k\imath}(D_{k\imath}) < -2$ and $\{\lambda_k(D_k)\}$ is bounded, it is clear that

$$\liminf_{k\to\infty} \frac{-\mathrm{sgn}(c_{k\imath})}{\lambda_{k\imath}(D_k)} > 0. \tag{3.19}$$

From $\liminf_{k\to\infty}(\nabla f_k + J_k\mathrm{sgn}(c_k))^T g_k < 0$ and (3.16),

$$\lim_{k\to\infty} \frac{O(\bar{\epsilon}_k)}{(\nabla f_k + J_k\mathrm{sgn}(c_k))^T g_k} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{-h_{k\imath}\lambda_{k\imath}(D_k)}{(\nabla f_k + J_k\mathrm{sgn}(c_k))^T g_k} = 1. \tag{3.20}$$

Hence, using (3.17), (3.18) and (3.20), we have

$$\begin{aligned}&\liminf_{k\to\infty} \frac{(\nabla f_k + J_k\mathrm{sgn}(c_k) - 2J_{k\imath}\mathrm{sgn}(c_{k\imath}))^T g_k}{(\nabla f_k + J_k\mathrm{sgn}(c_k))^T g_k} \\ \geq\ & \epsilon_d \liminf_{k\to\infty} \frac{-\mathrm{sgn}(c_{k\imath})}{\lambda_{k\imath}(D_k)} \frac{-h_{k\imath}\lambda_{k\imath}(D_k)}{(\nabla f_k + J_k\mathrm{sgn}(c_k))^T g_k} \\ >\ & 0.\end{aligned} \tag{3.21}$$

This means that, for sufficiently large k, $(\nabla f_k + J_k\mathrm{sgn}(c_k) - 2J_{k\imath}\mathrm{sgn}(c_{k\imath}))^T g_k < 0$. Hence the minimizer τ_k^* defining $\Psi_k^*[g_k, h_k]$ is greater than α_k. Therefore,

$$\begin{aligned}&\Psi_k^\star[g_k, h_k] \\ =\ & \min_{0\leq\tau\leq\min(\breve{\alpha}_k,\Delta_k)} \nabla f_k^T g_k + \|c_k + \tau J_k^T \frac{g_k}{\|\bar{g}_k\|_2}\|_1 - \|c_k\|_1 + \frac{1}{2}\frac{\tau^2}{\|\bar{g}_k\|_2^2}\bar{g}_k^T \bar{H}_k \bar{g}_k \\ =\ & \min_{0\leq\tau\leq\min(\breve{\alpha}_k,\Delta_k)} \tau(\nabla f_k + J_k\mathrm{sgn}(c_k) - 2J_{k\imath}\mathrm{sgn}(c_{k\imath}))^T \frac{g_k}{\|\bar{g}_k\|_2} - 2|c_{k\imath}| + \frac{1}{2}\frac{\tau^2}{\|\bar{g}_k\|_2^2}\bar{g}_k^T \bar{H}_k \bar{g}_k \\ \leq\ & \min_{0\leq\tau\leq\min(\breve{\alpha}_k,\Delta_k)} \tau(\nabla f_k + J_k\mathrm{sgn}(c_k) - 2J_{k\imath}\mathrm{sgn}(c_{k\imath}))^T \frac{g_k}{\|\bar{g}_k\|_2} + \frac{1}{2}\frac{\tau^2}{\|\bar{g}_k\|_2^2}\bar{g}_k^T \bar{H}_k \bar{g}_k.\end{aligned}$$

Since $\check{\alpha}_k \geq \frac{1}{\chi_\lambda}$, applying Lemma 3.3 with $b_k = \nabla f_k + J_k \text{sgn}(c_k) - 2J_{k_\imath}\text{sgn}(c_{k_\imath})$, we have

$$\begin{aligned} & \min_{0\leq\tau\leq\min(\check{\alpha}_k,\Delta_k)} \tau(\nabla f_k + J_k\text{sgn}(c_k) - 2J_{k_\imath}\text{sgn}(c_{k_\imath}))^T \frac{g_k}{\|\bar{g}_k\|_2} + \frac{1}{2}\frac{\tau^2}{\|\bar{g}_k\|_2^2}\bar{g}_k^T \bar{H}_k \bar{g}_k \\ \leq \quad & \min\{\frac{\Delta_k}{\|\bar{g}_k\|_2}, \frac{\chi_0}{\|\bar{g}_k\|_2}, \chi_1\}(\nabla f_k + J_k\text{sgn}(c_k) - 2J_{k_\imath}\text{sgn}(c_{k_\imath}))^T g_k. \end{aligned}$$

Applying (3.21), there exists $\chi_2 > 0$ such that $(\nabla f_k + J_k\text{sgn}(c_k) - 2J_{k_\imath}\text{sgn}(c_{k_\imath}))^T g_k \leq \chi_2(\nabla f_k + J_k\text{sgn}(c_k))^T g_k = -\chi_2\|\bar{g}_k\|_2^2$, the proof is concluded. □

THEOREM 3.8. *Assume that the level set $\mathcal{L} = \{x : \Upsilon(x) \leq \Upsilon(x_0)\}$ is compact, $c(x)$ and $f(x)$ are continuously differentiable, $\{B_k\}$ is uniformly bounded and the full rank assumption (AS.1) holds. Assume further that $\Upsilon(x_k + u_k) - \Upsilon(x_k) = \Gamma_k(s_k) + o(\|s_k\|)$, $\{s_k\}$ and $\{u_k\}$ are generated by TRASM in FIG.1, (AC.1) and (AC.2) hold for $\Gamma_k(s_k)$, and strict complementarity holds at every limit point. Then*

$$\lim_{k\to\infty} \|\bar{g}_k(\mathcal{D}_k)\|_2 = 0. \tag{3.22}$$

Proof. The proof is by contradiction and indeed very similar to Lemma 3.5. Once again, we drop the dependence notation $(\mathcal{D}_k)$ in this proof.

We observe that the difference between Lemma 3.7 and Lemma 3.6 is that the inequality below

$$\Upsilon(x_k) - \Upsilon(x_{k+1}) \geq -\zeta\Gamma_k(s_k) \geq \frac{1}{2}\zeta\beta_{df}\chi_2\|\bar{g}_k\|_2 \min\{\Delta_k, \chi_0, \chi_1\|\bar{g}_k\|_2,\}$$

is asymptotically true only under the assumption that $\liminf_{k\to\infty} \|\bar{g}_k\|_2 > 0$.

It is straightforward to verify that

$$\liminf_{k\to\infty} \|\bar{g}_k\|_2 = 0,$$

since, otherwise, Lemma 3.7 applies and the same arguments of Theorem 3.6 can be applied.

We now argue that $\{\bar{g}_k\}$ converges to zero.

Assume that $\limsup_{k\to\infty} \|\bar{g}_k\|_2 \geq \epsilon_1$ for some $\epsilon_1 > 0$.

For any ϵ_2 in $(0, \epsilon_1)$, there exist two subsequences $\{l_i\}$ and $\{m_i\}$ with $\{l_i\}$ denoting the subsequence of all the indices with $\|\bar{g}_{l_i}\|_2 < \epsilon$ and

$$\|\bar{g}_k\|_2 \geq \epsilon_2, \quad m_i \leq k < l_i, \quad \|\bar{g}_{l_i}\|_2 < \epsilon_2.$$

Since $\liminf_{k\to\infty} \|\bar{g}_k\| = 0$, we have $\liminf_{i\to\infty} \|\bar{g}_{l_i}\|_2 = 0$.

Consider the subsequence $\{\bar{g}_k :\ k = m_i, \cdots, l_i - 1\}$. Since

$$\liminf_{k\to\infty}\{\bar{g}_k :\ k = m_i, \cdots, l_i - 1\} \geq \epsilon_2,$$

Lemma 3.7 applies. Therefore, for sufficiently large i, we have an inequality similar to (3.12) in the proof of Lemma 3.6: If the k-th iteration is successful, then according to Lemma 3.7,

$$\Upsilon(x_k) - \Upsilon(x_{k+1}) \geq \beta_{cs}\zeta\chi_2\epsilon_2 \min\{\Delta_k, \chi_0, \chi_1\epsilon_2\}, \quad m_i \leq k < l_i.$$

The rest of the arguments are essentially the same as those after the inequality (3.12) in Lemma 3.6. □

We have so far established that, under the strict complementarity assumption and the conditions (AC.1) and (AC.2), both $\{\bar{g}_k(D_k)\}$ and $\{\bar{g}_k(\mathcal{D}_k)\}$ converge to zero. This means that, at every limit point of $\{x_k\}$ generated by TRASM in Fig. 1, the Kuhn-Tucker conditions are satisfied. Next we consider the second-order necessary conditions.

We recall the trust region subproblem (2.14). Assume that $B_k = \nabla^2 f_k + \nabla^2 \|c_k\|_1 + \sum_{i=1}^m \lambda_{ki} \nabla^2 c_{ki}$. Then the trust region subproblem (2.14) becomes:

$$
\begin{aligned}
& \min_{s,w} \psi_k(s) + \frac{1}{2} w^T C_k w \\
\text{subject to} \quad & J_k^T s - \mathcal{M}_k w = 0 \\
& \left\| \begin{bmatrix} s \\ w \end{bmatrix} \right\|_2 \le \Delta_k .
\end{aligned}
$$

Lemma 3.4 in [11] states that, under strict complementarity, the second-order sufficiency condition is equivalent to x^* satisfying the first-order necessary conditions and the projected Hessian being positive definite. Since the subspace restriction $J_k^T s - D_k w = 0$ changes at each iteration, we need to examine the asymptotic behavior of the projected Hessian. We subsequently denote $\mathrm{eig}_{\min}(H_k)$ as the smallest eigenvalue of the projected Hessian H_k defined by (3.3).

LEMMA 3.9. *Assume that the limit x^* of $\{x_k\}$ is a Kuhn-Tucker point and $B_k = \nabla^2 f_k + \nabla^2 \|c_k\|_1 + \sum_{i=1}^m \lambda_{ki} \nabla^2 c_{ki}$. If $\liminf_{k\to\infty} eig_{\min}(H_k) \ge 0$, then the second-order necessary condition is satisfied at x^*.*

Proof. We prove the result by contradiction. Assume that there exist s and w with $J^{*T} s - \mathcal{M}^* w = 0$ such that $\begin{bmatrix} s \\ w \end{bmatrix}^T \bar{H}^* \begin{bmatrix} s \\ w \end{bmatrix} < 0$. Let $\begin{bmatrix} s \\ w \end{bmatrix} = \begin{bmatrix} s_k \\ w_k \end{bmatrix} + \begin{bmatrix} \bar{s}_k \\ \bar{w}_k \end{bmatrix}$ where $J_k^T s_k - \mathcal{M}_k w_k = 0$ and $s_k^T \bar{s}_k + w_k \bar{w}_k = 0$. Since $J^{*T} s - \mathcal{M}^* w = 0$, $\{J_k^T s - \mathcal{M}_k w\}$ converges to zero. This implies that

$$\lim_{k\to\infty} J_k^T \bar{s}_k - D_k \bar{w}_k = 0.$$

From $s_k^T \bar{s}_k + w_k^T \bar{w}_k = 0$ and $J_k^T s_k - \mathcal{M}_k w_k = 0$, $\{\begin{bmatrix} \bar{s}_k \\ \bar{w}_k \end{bmatrix}\}$ converges to zero. It is clear that

$$
\begin{aligned}
\begin{bmatrix} s_k \\ w_k \end{bmatrix}^T \bar{H}_k \begin{bmatrix} s_k \\ w_k \end{bmatrix} &= \begin{bmatrix} s \\ w \end{bmatrix}^T \bar{H}_k \begin{bmatrix} s \\ w \end{bmatrix} + O(\| \begin{bmatrix} \bar{s}_k \\ \bar{w}_k \end{bmatrix} \|_2) \\
&= \begin{bmatrix} s \\ w \end{bmatrix}^T \bar{H}^* \begin{bmatrix} s \\ w \end{bmatrix} + O(\|\bar{H}_k - \bar{H}^*\|_2) + O(\| \begin{bmatrix} \bar{s}_k \\ \bar{w}_k \end{bmatrix} \|_2)
\end{aligned}
$$

and $\{\|\bar{H}_k - \bar{H}^*\|_2\}$ converges to zero. Hence $\lim_{k\to\infty} \begin{bmatrix} s_k \\ w_k \end{bmatrix}^T \bar{H}_k \begin{bmatrix} s_k \\ w_k \end{bmatrix} < 0$. This is a contradiction and thus the second-order necessary condition holds at x^*.

The proof is completed. □

Since the projected Hessian H^* is positive definite at x^* if the second-order sufficiency holds at x^* (Lemma 3.4 in [11]) and the continuity of $[J(x), D(x)]$, H_k is positive definite near x^*.

Now we consider the second-order optimality conditions. Recall that $[p_k; q_k]$ denotes a global solution of the trust region subproblem (2.14). Let

$$\begin{bmatrix} p_k \\ q_k \end{bmatrix} = \begin{bmatrix} Z_k \\ \bar{Z}_k \end{bmatrix} z_k, \tag{3.23}$$

where $\begin{bmatrix} Z_k \\ \bar{Z}_k \end{bmatrix}$ is an orthonormal basis for the null space of $[J_k^T, -\mathcal{M}_k]$. Then there exists $\mu_k \geq 0$ such that

$$H_k + \mu_k I_n = R_k^T R_k, \tag{3.24}$$

with $\mu_k(\Delta_k - \| \begin{bmatrix} p_k \\ q_k \end{bmatrix} \|_2) = 0$, R_k is an n-by-n an upper triangular matrix, $H_k + \mu_k I_n$ is positive semi-definite and a solution $[p_k; q_k]$ to (2.14) satisfies (e.g.,[9]):

$$(\bar{H}_k + \mu_k I_{m+n}) \begin{bmatrix} p_k \\ q_k \end{bmatrix} = - \begin{bmatrix} \nabla f_k + J_k \text{sgn}(c_k) \\ 0 \end{bmatrix} + \begin{bmatrix} J_k \\ -\mathcal{M}_k \end{bmatrix} y_{k+1} \tag{3.25}$$

for some $y_{k+1} \in \Re^m$. Then

$$[p_k^T, q_k^T](\bar{H}_k + \mu_k I_{m+n}) \begin{bmatrix} p_k \\ q_k \end{bmatrix} = -p_k^T(\nabla f_k + J_k \text{sgn}(c_k)), \tag{3.26}$$

$$-(\nabla f_k + J_k \text{sgn}(c_k))^T p_k = \|R_k z_k\|_2^2. \tag{3.27}$$

The above equations will be used repeatedly in the subsequent proofs.

LEMMA 3.10. *Assume that (AC.3) is satisfied. Let $[p_k; q_k]$ be a global solution to the trust region subproblem (2.14). Then*

$$-\Gamma_k(s_k) \geq \beta_{2nd} \Psi_k^*[p_k, q_k] \geq \frac{\beta_{2nd}}{2}[\min(1, \alpha_k^2)\mu_k \Delta_k^2 + \min(1, \alpha_k)\|R_k z_k\|_2^2],$$

where α_k the stepsize defined by (2.16) and $p_k = \begin{bmatrix} Z_k \\ \bar{Z}_k \end{bmatrix} z_k$ as in (3.23).

Proof. Recall definition (2.17): $\Psi_k^*[p_k, q_k] = \Psi_k(\tau_k^* p_k, \tau_k^* q_k)$ where

$$\tau_k^* = \text{augmin}\{\tau(\nabla f_k + J_k \text{sgn}(c_k))^T p_k + \frac{\tau^2}{2}[p_k; q_k]^T \bar{H}_k [p_k; q_k] : \tau \geq 0,\ x_k + \tau p_k \in \mathcal{F}_k \text{ and } \|\tau[p_k; q_k]\|_2 \leq \Delta_k\}.$$

It is easy to see that

$$\begin{aligned}
&= \tau(\nabla f_k + J_k \text{sgn}(c_k))^T p_k + \frac{1}{2}\tau^2 [p_k; q_k]^T \bar{H}_k [p_k; q_k] \\
&= \tau(\nabla f_k + J_k \text{sgn}(c_k))^T p_k - \frac{1}{2}\tau^2 (\nabla f_k + J_k \text{sgn}(c_k))^T p_k - \frac{1}{2}\tau^2 \mu_k \|[p_k; q_k]\|_2^2, \quad \text{(from (3.26))} \\
&= -\tau\|R_k z_k\|_2^2 + \frac{1}{2}\tau^2 \|R_k z_k\|_2^2 - \frac{1}{2}\tau^2 \mu_k \Delta_k^2, \qquad \text{(from (3.27))}.
\end{aligned}$$

From $\tau_k^* = \min(1, \alpha_k)$, $\tau_k^{*2} \le \tau_k^*$. Therefore,

$$\tau_k^*(\nabla f_k + J_k \mathrm{sgn}(c_k))^T p_k + \frac{1}{2}\tau_k^{*2}[p_k; q_k]^T \bar{H}_k [p_k; q_k] \le -\frac{1}{2}\min(1, \alpha_k)\|R_k z_k\|_2^2 - \frac{1}{2}\min(1, \alpha_k)^2 \mu_k \Delta_k^2.$$

Hence, from (AC.3), we have

$$\begin{aligned} -\Gamma_k(s_k) &\ge -\beta_{2nd}\Psi_k^*[p_k, q_k] \\ &\ge \frac{\beta_{2nd}}{2}[\min(1, \alpha_k^2)\mu_k \Delta_k^2 + \min(1, \alpha_k)\|R_k z_k\|_2^2]. \end{aligned}$$

The proof is completed. $\square$

The following lemma provides further estimation of the reductions in the quadratic model. We emphasize that the results hold for any *subsequence* generated by TRASM in FIG. 1 (consequently, it holds for the entire sequence as well).

LEMMA 3.11. *Assume that the conditions of Theorem 3.8 and (AC.3) hold. Furthermore, $\{x_{l_i}\}$ is any subsequence generated by TRASM in FIG. 1. Then, there exists $\chi_4 > 0$ such that, for i sufficiently large,*

$$-\Gamma_{l_i}(s_{l_i}) \ge -\beta_{2nd}\Psi_{l_i}^*[p_{l_i}, q_{l_i}] \ge \frac{\beta_{2nd}}{2}[\min(1, \chi_4)^2 \mu_{l_i}\Delta_{l_i}^2 + \min(1, \chi_4)\|R_{l_i} z_{l_i}\|_2^2],$$

and if the l_ith iteration is successful, then

$$\Upsilon(x_{l_i}) - \Upsilon(x_{l_i+1}) > \frac{\beta_{2nd}}{2}\zeta[\min(1, \chi_4)^2 \mu_{l_i}\Delta_{l_i}^2 + \min(1, \chi_4)\|R_{l_i} z_{l_i}\|_2^2].$$

Proof. For notational simplicity, we still use k to denote the subsequence index l_i.

Using Lemma 3.10,

$$-\Gamma_k(s_k) \ge \frac{\beta_{2nd}}{2}[\min(1, \alpha_k)^2 \mu_k \Delta_k^2 + \min(1, \alpha_k)\|R_k z_k\|_2^2], \tag{3.28}$$

where α_k is the stepsize along p_k as defined in (2.16):

$$\alpha_k = \min\{-\frac{c_{ki}}{\nabla c_{k_i}^T p_k} : -\frac{c_{ki}}{\nabla c_{k_i}^T p_k} > 0,\ 1 \le i \le m\}.$$

Following Theorems 3.6 and 3.8, $\{\bar{g}_k(D_k)\}$ and $\{\bar{g}_k(\mathcal{D}_k)\}$ converge to zero. Since strict complementarity holds at every limit point and $\{x_k\}$ is bounded, there exists $0 < \epsilon_d < 1$ such that, for sufficiently large k,

$$|c_k| + \min(|\lambda_k|, |\mathrm{sgn}(c_k)\lambda_k + 2|) > 2\epsilon_d e, \quad e = [1, \cdots, 1]^T \in \Re^n.$$

(Otherwise, there would be a degenerate limit point of $\{x_k\}$).

Assume that

$$\alpha_k = -\frac{c_{kj}}{\nabla c_{k_j}^T p_k} \quad \text{for some } j.$$

If $j \ne \imath$ or $D_k = \mathcal{D}_k$ for sufficiently large k, then it is clear that

$$\alpha_k \ge \frac{1}{\chi\lambda} \ge \frac{\epsilon_d}{\chi\lambda}.$$

Assume now $j = \imath$ and $D_k \neq \mathcal{D}_k$ for an infinite subsequence. For notational simplicity, we still use the subscript k to denote this subsequence.

Since $\lim_{k\to\infty} \bar{g}_k(D_k) = 0$, $\lim_{k\to\infty} D_k\lambda_k(D_k) = 0$. In addition, when $|c_{k_\imath}| < \epsilon_d$, we have

$$\min(|\lambda_{k_\imath}(D_k)|, |\text{sgn}(c_{k_\imath})\lambda_{k_\imath}(D_k) + 2|) > \epsilon_d.$$

Hence, for k sufficiently large, if $|c_{k_\imath}| < \epsilon_d$, $-2 < \text{sgn}(c_{k_\imath})\lambda_{k_\imath}(D_k) < 0$. (Otherwise, there would be a limit point at which the Kuhn-Tucker conditions are violated.) However, since $D_{k\imath\imath} \neq \mathcal{D}_{k\imath\imath}$, either $\text{sgn}(c_{k_\imath})\lambda_{k_\imath}(D_k) > 0$ or $\text{sgn}(c_{k_\imath})\lambda_{k_\imath}(D_k) < -2$. This implies that, for sufficiently large k, $|c_{k_\imath}| \geq \epsilon_d$. Hence

$$\alpha_k \geq \frac{\epsilon_d}{\chi_\lambda}.$$

Using (3.28)

$$-\Gamma_k(s_k) \geq \frac{\beta_{2nd}}{2}[\min(1, \frac{\epsilon_d}{\chi_\lambda})^2\mu_k\Delta_k^2 + \min(1, \frac{\epsilon_d}{\chi_\lambda})\|R_k z_k\|_2].$$

Let $\chi_4 = \frac{\epsilon_d}{\chi_\lambda}$, the results are established. $\square$

We subsequently refer the step $p_k^N \stackrel{\text{def}}{=} Z_k z_k^N$ as a Newton step for (2.14) where

$$H_k z_k^N = -Z_k^T(\nabla f_k + J_k \text{sgn}(c_k)), \tag{3.29}$$

whenever the projected Hessian H_k is positive definite.

Next, assuming (AC.3) is satisfied and $\{x_k\}$ converges to a strict complementarity point x^*, we prove below that asymptotically $s_k \in \mathcal{F}_k$ and $\Gamma_k(s_k) = \psi_k(s_k)$.

LEMMA 3.12. *Assume that the level set $\mathcal{L} = \{x : \Upsilon(x) \leq \Upsilon(x_0)\}$ is compact, $c(x)$ and $f(x)$ are continuously differentiable, $\{B_k\}$ is uniformly bounded and the full rank assumption (AS.1) holds. In addition, assume that $\{x_k\}$ converges to x^* satisfying Kuhn-Tucker conditions with strict complementarity. Assume further that $\{[p_k; q_k]\}$ converges to zero where $[p_k; q_k]$ solves (2.14). Then, for k sufficiently large,*

$$\Psi_k^*[p_k, q_k] < \beta_q \Psi_k^\star[g_k, h_k].$$

In addition, if (AC.3) is satisfied, then $\Gamma_k(s_k) = \psi_k(s_k)$ and $s_k \in \mathcal{F}_k$ for sufficiently large k.

Proof. Under the assumption that $\{x_k\}$ converges to x^* satisfying the Kuhn-Tucker conditions with strict complementarity, following Lemma 3.2, $\mathcal{M}_k = D_k$, for sufficiently large k.

From (3.25),

$$(\begin{bmatrix} B_k & 0 \\ 0 & C_k \end{bmatrix} + \mu_k I)\begin{bmatrix} p_k \\ q_k \end{bmatrix} = -\begin{bmatrix} \nabla f_k + J_k \text{sgn}(c_k) \\ 0 \end{bmatrix} + \begin{bmatrix} J_k \\ -D_k \end{bmatrix} y_{k+1}$$

for some $y_{k+1} \in \Re^m$ and $\mu_k \geq 0$.

Since $\{[p_k; q_k]\}$ converges to zero, this implies that

$$\lim_{k\to\infty} \begin{bmatrix} \nabla f_k + J_k \mathrm{sgn}(c_k) \\ 0 \end{bmatrix} + \begin{bmatrix} J_k \\ -D_k \end{bmatrix} y_{k+1} - \mu_k \begin{bmatrix} p_k \\ q_k \end{bmatrix} = 0. \tag{3.30}$$

Let α_k denote the stepsize (2.16). Next we prove that

$$\liminf_{k\to\infty} \alpha_k \geq 1.$$

Assume that the subsequence $\{\alpha_{l_i}\}$ converges to $\liminf_{k\to\infty} \alpha_k$ and the corresponding $\{\mu_{l_i}\}$ converges (possibly to $+\infty$). For notational simplicity, we still denote α_{l_i} by $\{\alpha_k\}$. There are two possibilities: either the corresponding $\lim_{k\to\infty} \mu_k = +\infty$ or $\lim_{k\to\infty} \mu_k < +\infty$.

First let us assume that $\lim_{k\to\infty} \mu_k < +\infty$. Under this assumption, from $\{[p_k; q_k]\}$ converges to zero and (3.30),

$$\lim_{k\to\infty} \begin{bmatrix} \nabla f_k + J_k \mathrm{sgn}(c_k) \\ 0 \end{bmatrix} + \begin{bmatrix} J_k \\ -D_k \end{bmatrix} y_{k+1} = 0.$$

But

$$\lim_{k\to\infty} \begin{bmatrix} \nabla f_k + J_k \mathrm{sgn}(c_k) \\ 0 \end{bmatrix} + \begin{bmatrix} J_k \\ -D_k \end{bmatrix} \lambda_k = 0.$$

Hence

$$\lim_{k\to\infty} \begin{bmatrix} J_k \\ -D_k \end{bmatrix} (y_{k+1} - \lambda_k) = 0.$$

From the full rank assumption (AS.1), this implies that

$$\lim_{k\to\infty} y_{k+1} - \lambda_k = 0.$$

Since $J_k^T p_k = D_k q_k$, the last m equations of (3.25) state that

$$J_k^T p_k = -D_k^2 (C_k + \mu_k I_m)^{-1} y_{k+1}. \tag{3.31}$$

From $\lim_{k\to\infty} y_{k+1} - \lambda_k = 0$ and (3.31), we have

$$\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} -\frac{c_{kj}}{J_{kj}^T p_k} = \lim_{k\to\infty} \frac{|\lambda_{kj}| + \mu_k}{|y_{k+1j}|} = 1 + \lim_{k\to\infty} \frac{\mu_k}{|y_{k+1j}|} \geq 1.$$

Now we assume that $\lim_{k\to\infty} \mu_k = +\infty$. From (3.31), we immediately have

$$\lim_{k\to\infty} \begin{bmatrix} J_k \\ -D_k \end{bmatrix} \frac{y_{k+1}}{\mu_k} = 0.$$

The full rank assumption (AS.1) implies that

$$\lim_{k\to\infty} \frac{y_{k+1}}{\mu_k} = 0 \quad \text{or equivalently} \quad \lim_{k\to\infty} \frac{\mu_k}{y_{k+1}} = \infty.$$

From

$$\alpha_k = \lim_{k\to\infty} \frac{|\lambda_{kj}| + \mu_k}{|y_{k+1j}|},$$

it is clear that $\lim_{k\to\infty} \alpha_k = +\infty$.

Hence we conclude that $\liminf_{k\to\infty} \alpha_k \geq 1$.

Next we prove that

$$(\nabla f_k + J_k \mathrm{sgn}(c_k))^T p_k + \frac{1}{2}[p_k; q_k]^T \bar{H}_k [p_k; q_k] \leq \Psi_k^\star[g_k, h_k]. \tag{3.32}$$

Recall definition (2.17) of $\Psi_k^\star$ and $\tau_k^\star$:

$$\Psi_k^\star[g_k, h_k] = \tau_k^\star \nabla f_k^T g_k + \|c_k + \tau_k^\star J_k^T g_k\|_1 - \|c_k\|_1 + \frac{1}{2}\tau_k^{\star 2} g_k^T B_k g_k + \frac{1}{2}\tau_k^{\star 2} h_k^T C_k h_k.$$

If $\tau_k^\star \leq \alpha_k$, since $\|\tau_k^\star[g_k; h_k]\| \leq \Delta_k$, we have

$$\Psi_k^\star[g_k, h_k] \geq (\nabla f_k + J_k \mathrm{sgn}(c_k))^T p_k + \frac{1}{2}[p_k; q_k]^T \bar{H}_k [p_k; q_k].$$

Assume $\alpha_k < \tau_k^\star \leq \check{\alpha}_k$ and $\alpha_k = -\frac{c_{kj}}{\nabla c_{kj}^T p_k}$. Then

$$\phi_k(\tau_k^\star g_k) = \tau_k^\star \nabla f_k^T g_k + \tau_k^\star (J_k \mathrm{sgn}(c_k))^T g_k - 2\mathrm{sgn}(c_{kj})(\tau_k^\star J_{kj}^T g_k + c_{kj}) \geq \tau_k^\star (\nabla f_k + J_k \mathrm{sgn}(c_k))^T g_k.$$

Hence, again, we have

$$\begin{aligned}
\Psi_k^\star[g_k, h_k] &\geq \tau_k^\star (\nabla f_k + J_k \mathrm{sgn}(c_k))^T g_k + \frac{1}{2}\tau_k^{\star 2}[g_k; h_k]^T \bar{H}_k [g_k; h_k] \\
&\geq (\nabla f_k + J_k \mathrm{sgn}(c_k))^T p_k + \frac{1}{2}[p_k; q_k]^T \bar{H}_k [p_k; q_k], \quad \text{since } \|\tau_k^\star[g_k; h_k]\| \leq \Delta_k,
\end{aligned}$$

i.e., (3.32) holds. From $\min(1, \alpha_k) \geq \min(1, \alpha_k)^2$, we have

$$\begin{aligned}
&\frac{\Psi_k^*[p_k, q_k]}{\Psi_k^\star[g_k, h_k]} \\
= \;& \frac{\min(1, \alpha_k) g_k^T p_k + \frac{1}{2}\min(1, \alpha_k)^2 [p_k; q_k]^T \bar{H}_k [p_k; q_k]}{(\nabla f_k + J_k \mathrm{sgn}(c_k))^T p_k + \frac{1}{2}[p_k; q_k]^T \bar{H}_k [p_k; q_k]} \times \frac{(\nabla f_k + J_k \mathrm{sgn}(c_k))^T p_k + \frac{1}{2}[p_k; q_k]^T \bar{H}_k [p_k; q_k]}{\Psi_k^\star[g_k, h_k]} \\
\geq \;& \min(\alpha_k, 1)^2 \times \frac{(\nabla f_k + J_k \mathrm{sgn}(c_k))^T p_k + \frac{1}{2}[p_k; q_k]^T \bar{H}_k [p_k; q_k]}{\Psi_k^\star[g_k, h_k]}
\end{aligned}$$

Using (3.32) and $\liminf_{k\to\infty} \alpha_k \geq 1$, we have for $0 < \beta_q < 1$, and k sufficiently large,

$$\Psi_k^*[p_k, q_k] < \beta_q \Psi_k^\star[g_k, h_k].$$

If, in addition, (AC.3) is satisfied, then it immediately follows that $\Gamma_k(s_k) = \psi_k(s_k)$ and $s_k \in \mathcal{F}_k$ for sufficiently large k. The proof is completed. $\square$

Before we state the second-order convergence result, we quote Lemma (4.10) in [14] below.

LEMMA 3.13. *Let x^* be an isolated limit point of a sequence $\{x_k\}$ in $\Re^n$. If $\{x_k\}$ does not converge then there is a subsequence x_{l_j} which converges to x^*, and an $\epsilon > 0$ such that*

$$\|x_{l_j+1} - x_{l_j}\| \geq \epsilon.$$

The next theorem indicates that the first-order and second-order necessary conditions can be satisfied.

THEOREM 3.14. *Assume that $c(x)$ and $f(x)$ are twice continuously differentiable on the compact level set $\mathcal{L}$. Assume further that $\Upsilon(x_k + u_k) - \Upsilon(x_k) = \Gamma_k(s_k) + o(\|s_k\|)$ and $\{s_k\}$ and $\{u_k\}$ are generated by TRASM in FIG. 1. then*

1. *If the condition (AC.1) is satisfied, the sequence $\{\bar{g}_k(D_k)\}$ converges to zero;*
2. *If the conditions (AC.1) and (AC.2) are satisfied and strict complementarity holds at every limit point, then*
   (a) *The sequence $\{\bar{g}_k(\mathcal{D}_k)\}$ converges to zero;*
   (b) *If, in addition, (AC.3) is satisfied, then*
      - *There is a limit point x^* at which the projected Hessian H^* is positive semi-definite;*
      - *If x^* is an isolated limit point, then the projected Hessian H^* is positive semi-definite;*
      - *If the projected Hessian H^* is nonsingular at some isolated limit point x^* of $\{x_k\}$, then the projected Hessian H^* is positive definite. Moreover, if $\Psi_k(s_k, w_k) < \beta_0 \Psi_k^*[p_k, q_k]$ for some $\beta_0 > 0$ and sufficiently large k, then $\{x_k\}$ converges to x^*, all iterations are eventually successful and $\{\Delta_k\}$ is bounded away from zero.*

Proof. We prove each result in order.

1. The sequences $\{\bar{g}_k(D_k)\}$ converges to zero: this has been proved in Theorems 3.6.
2. Now we assume, additionally, that (AC.2) is satisfied and strict complementarity holds at every limit point.
   (a) The sequence $\{\bar{g}_k(\mathcal{D}_k)\}$ converges to zero: this has been proved in Theorem 3.8.
   (b) Now assume further that (AC.3) is also satisfied.
      - Consider first the case when $\liminf_{k\to\infty} \mu_k = 0$, where μ_k is defined by (3.24). Let $\text{eig}_{\min}(H_k)$ denote the minimum eigenvalue of H_k. Since $\mu_k \geq \max(-\text{eig}_{\min}(H_k), 0)$, $\liminf_{k\to\infty} \max(-\text{eig}_{\min}(H_k), 0) = 0$. Hence $\liminf_{k\to\infty} \text{eig}_{\min}(H_k) \geq 0$. Using Lemma 3.9, there exists a limit point x^* at which H^* is positive semi-definite.
        Next we prove $\liminf_{k\to\infty} \mu_k = 0$ by contradiction. Assume that $\mu_k \geq \epsilon > 0$ for sufficiently large k. First we show that $\{\Delta_k\}$ converges to zero.
        Using Lemma 3.11, for sufficiently large k,

        $$-\Gamma_k(s_k) \geq \frac{\beta_{2nd}}{2} \min(1, \chi_4)^2 \mu_k \Delta_k^2. \tag{3.33}$$

        Moreover, for sufficiently large k and successful iterations

        $$\Upsilon(x_k) - \Upsilon(x_{k+1}) > \frac{\beta_{2nd}}{2} \zeta \min(1, \chi_4)^2 \mu_k \Delta_k^2. \tag{3.34}$$

The rest of arguments are similar to the proof of Theorem 3.5. If there are finite number of successful iterations, $\{\Delta_k\}$ converges to zero. Otherwise, let $\{l_i\}$ be the infinite sequence of successful iterations. The inequality (3.34) and the fact that $\{\Upsilon_k(x_k)\}$ converges imply that

$$\sum_{i=1}^{\infty} \Delta_{l_i}^2 \zeta \min(1, \chi_4)^2 \mu_{l_i} < \infty.$$

Since $\liminf_{k\to\infty} \mu_k > 0$, $\{\Delta_k\}$ converges to zero. Furthermore, $\{x_k\}$ converges from $\|x_{k+1} - x_k\|_2 = O(\Delta_k)$. Using $\|s_k\|_2 \le \beta_s \Delta_k$, $\|p_k\|_2 \le \Delta_k$ and $\|q_k\|_2 \le \Delta_k$, we conclude that both $\{s_k\}$, $\{p_k\}$ and $\{q_k\}$ converge to zero.

Using Lemma 3.12, for k sufficiently large $\Gamma_k(s_k) = \psi_k(s_k)$ and $s_k \in \mathcal{F}_k$. From Theorem 2.1, we have

$$|\Upsilon(x_k + u_k) - \Upsilon(x_k) - \psi_k(s_k)| = o(\|s_k\|^2).$$

This relation, $\|s_k\| \le \beta_s \Delta_k$, (3.33), $\liminf_{k\to\infty} \mu_k > 0$ and $\{s_k\}$ converging to zero imply that $\{|\rho_k - 1|\}$ converges to zero. In other words, the entire sequence $\{\rho_k\}$ converges to unity. According the trust region size updating rules, $\{\Delta_k\}$ cannot converge to zero, which is a contradiction.

In conclusion, $\liminf_{k\to\infty} \mu_k = 0$. Hence there exists a limit point with the projected Hessian H^* positive semi-definite.

- If $\{x_k\}$ converges to x^*, the result follows from above. If $\{x_k\}$ does not converge then Lemma 3.13 applies. Thus, if $\{x_{l_j}\}$ is the subsequence guaranteed by Lemma 3.13 then $\Delta_{l_j} \ge \frac{1}{\beta_s}\epsilon$, for some $\epsilon > 0$. From Lemma 3.11, $\{\mu_{l_j}\}$ converges to zero. Thus the projected Lagrangian Hessian H^* is positive semi-definite.

- Since x^* is an isolated limit point, the projected Lagrangian Hessian H^* is positive definite following (b).

  Consider any subsequence of $\{x_k\}$ which converges to the isolated limit x^* (for simplicity of notation, we still use the subscript k).

  From Lemma 3.12, for k sufficiently large,

$$\Psi_k^*[p_k, q_k] < \beta_q \Psi_k^*[g_k, h_k].$$

  But (AC.3) holds, hence $\Gamma_k = \psi_k(s_k)$, $s_k \in \mathcal{F}_k$ for k sufficiently large. Moreover, using Lemma 3.2, $\mathcal{M}_k = D_k$ for sufficiently large k.

  Since $D_k w_k - J_k^T s_k = 0$, there exists z_k such that

$$s_k = Z_k z_k, \quad w_k = \bar{Z}_k z_k.$$

  By assumption that $\Psi_k(s_k, w_k) \le \beta_0 \Psi_k^*[p_k, q_k]$, $\Psi_k(s_k, w_k) < 0$. From definition (2.7), $\psi_k(s_k) = (\nabla f_k + J_k \mathrm{sgn}(c_k))^T s_k + \frac{1}{2} s_k^T(\nabla^2 \|c_k\|_1 + \sum_{i=1}^m \lambda_{ki} \nabla^2 c_{ki}) s_k$. Hence, whenever the projected Lagrangian Hessian H_k is positive definite,

$$(\nabla f_k + J_k \mathrm{sgn}(c_k))^T s_k < -\frac{1}{2} z_k^T Z_k^T (\nabla^2 \|c_k\|_1 + \sum_{i=1}^m \lambda_{ki} \nabla^2 c_{ki}) Z_k z_k - \frac{1}{2} z_k^T \bar{Z}_k^T C_k \bar{Z}_k z_k < 0.$$

Therefore

$$\frac{1}{2}\|z_k\|_2 \leq \|H_k^{-1}\|_2 \|Z_k^T(\nabla f_k + J_k \mathrm{sgn}(c_k))\|_2, \tag{3.35}$$

whenever the projected Lagrangian Hessian H_k is positive definite. But $\{Z_k^T(\nabla f_k + J_k \mathrm{sgn}(c_k))\}$ converges to zero. We conclude that $\lim_{k\to\infty} z_k = 0$. Hence $\{s_k\}$ and $\{[p_k; q_k]\}$ converge to zero. Following Lemma 3.13, the entire sequence $\{x_k\}$ converges to x^*.

Next we prove that $\{\Delta_k\}$ is bounded away from zero. Assume that $\epsilon > 0$ is a lower bound on the eigenvalues of H_k. Let $p_k = Z_k z_k^p$.

Using Lemma 3.11, for sufficiently large k,

$$\Psi_k^*[p_k, q_k] \leq -\frac{1}{2}\min(1, \chi_4)\|R_k z_k^p\|_2^2.$$

From (3.24),

$$\|R_k z_k^p\|_2^2 \geq \epsilon \|z_k^p\|_2^2 + \mu_k \|z_k^p\|_2^2,$$

$\|z_k^p\|_2 \leq \Delta_k$ and $z_k^p = z_k^N$ if $\|z_k^p\|_2 < \Delta_k$, we have

$$|\Psi_k^*[p_k, q_k]| \geq \frac{1}{2}\epsilon \min(1, \chi_4) \min(\Delta_k^2, \|z_k^N\|_2^2), \quad \text{for sufficiently large } k. \tag{3.36}$$

Recall that, whenever H_k is positive definite,

$$\frac{1}{2}\|z_k\|_2 \leq \|H_k^{-1}\|_2 \|Z_k^T(\nabla f_k + J_k \mathrm{sgn}(c_k))\|_2$$

Let κ be an upper bound on the condition number of H_k. From $Z_k^T(\nabla f_k + J_k \mathrm{sgn}(c_k)) = -H_k z_k^N$ whenever Newton step exists, using (3.29) and (3.35),

$$\frac{1}{2}\|z_k\|_2 \leq \kappa \|z_k^N\|_2.$$

Hence, using (3.36), (AC.3), $\|z_k\|_2 \leq \beta_s \Delta_k$ and the above inequality, there exists $\bar{\epsilon} > 0$ such that

$$-\psi_k(s_k) \geq -\beta_{2nd}\Psi_k^*[p_k, q_k] \geq \bar{\epsilon}\|z_k\|_2^2.$$

Using Theorem 2.1,

$$|\Upsilon(x_k + u_k) - \Upsilon(x_k) - \psi_k(s_k)| = o(\|s_k\|^2) = o(\|z_k\|^2).$$

From $-\psi_k(s_k) \geq \bar{\epsilon}\|z_k\|_2^2$, $\rho_k > \eta$ for k sufficiently large.

We immediately conclude that $\{\Delta_k\}$ is bounded away from zero.

In conclusion, all the results hold. □

The local analysis of the method will be presented in a subsequent paper. Nonetheless, Theorem 3.14 indicates that Newton steps will be eventually successful and Maratos effect will not occur. We believe that the method is locally superlinearly convergent.

4. Conclusion. In this paper, we give a succinct description of the trust region affine scaling method (TRASM) proposed in [11] for solving a nonlinear l_1 problem (1.1) and present a global convergence analysis of this algorithm. Similar to analysis of trust region methods for unconstrained minimization, strong convergence results are obtained in an elegant fashion.

Sufficient decrease conditions (AC.1), (AC.2) and (AC.3) are proposed for optimality. The condition (AC.1) uses the projected gradient $g_k(D_k)$ as a benchmark and ensures complementarity. The condition (AC.2) uses the projected gradient $g_k(\mathcal{D}_k)$ and is necessary for dual feasibility. The condition (AC.3) is necessary for second-order optimality. In addition, Theorem 3.14 indicates that Newton steps will be eventually successful and Maratos effect will not occur.

Some preliminary computational results are presented in [11]. Our real objective is to develop a method which is capable of computing a solution for a nonlinearly constrained problem efficiently and reliably. Further investigation is needed to explore efficient computational implementation.

The local analysis of the method is also under investigation. We believe that the method is locally superlinearly convergent; This is certainly consistent with our computational observation.

In the large scale setting, it may be impractical to perform a correction step and one may choose to skip correction steps of TRASM in FIG. 1 entirely or partially. The resulting algorithm (without correction steps) retains the first order convergence result but the second-order convergence properties will not be guaranteed.

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