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AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NY 14853

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On the Continuous Time Capacitated Production/Inventory Problem with No Set Up Costs

by

A. Refik Güllü[1] and Peter L. Jackson

[1]Industrial Engineering Department, Middle East Technical University, 06531 Ankara Turkey. Tel. 11-4-210 1000 Ext. 2956

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ABSTRACT

In this paper, we address the problem of determining how much inventory to stock after a bottleneck operation to provide adequate service in meeting demand schedules that have a low to moderate degree of variability. Specifically, we consider a one-product, one-machine production/inventory problem in which the production occurs continuously. The production rate is constant, and the demand process is assumed to be time-homogeneous and additive. By limiting attention to the case of no set-up cost, we obtain simple, exact, and explicit formulas for the stationary optimal operating rules of this problem. Computational results are provided to reveal the behavior of the solutions.

1 Introduction

Consider a battery manufacturer whose daily production volume is on the order of 4500 batteries per day. The customers are automobile assembly plants and warehouses for general distribution. Daily demand for batteries is such that the battery manufacturing line is highly utilized (greater than 85%). The demand, after production smoothing, is relatively stable but it has some daily variation due to last-minute changes in assembly production schedules and the timing of warehouse replenishment orders. In the case of unanticipated variations in the demand, it is possible to run overtime to increase daily production capacity. Typically, however, it is cheaper to hold some inventory of batteries after the bottleneck stage of production to meet these variations and to avoid the costs of overtime. The line then operates on a produce-up-to-S type policy. That is, the line will shut down before achieving maximum production capacity if demand for the day has been met and the inventory has been restored to level S. Orders in excess of maximum daily production capacity are backordered to the next day. The purpose of this paper is to provide simple models that would yield a reasonable value of S to use in order to achieve acceptable customer service in a manufacturing line such as this. The option of running overtime is not explicitly considered. By applying stochastic models of storage processes, we are able to develop explicit formulas for key performance measures such as average inventory, average backorders, and demand fill rate. From these, we construct cost and fill rate models that can be optimized to determine the value of S.

In the actual battery manufacturing study alluded to above, there were multiple products being produced so that the actual policy recommended was to produce-up-to S_i, with a different S_i used for each product. A simple heuristic, not described here, was used to divide the single value of S among the different S_i. In Carr et al. [3], we address the multi-product version of this problem in more detail, but with more restrictive assumptions.

The particular model considered in this paper is a one-product, one-machine production/inventory problem in which the production occurs continuously, and the production rate is constant. Let the cumulative demand during the interval $(0, t]$ be denoted by $D(t)$. The demand process, $\{D(t);\ t \geq 0\}$, is assumed to be a time-homogenous, additive process (i.e. a process with stationary, independent increments). Such a process is an infinitely divisible process (see Prabhu [17], p.69). It is assumed that the production facility can produce at a rate r $(0 < r < \infty)$ units per unit time. Our main modelling assumption is that the fixed costs of the start-up and shut-down of the production facility are assumed to be negligible. We consider inventory holding and shortage costs as well as fill rate performance. The intent of this paper is to present simple, exact, and explicit formulas for the parameter S of an optimal produce-up-to S policy. These formulas would provide guidance as to the appropriate amount of capacity to store in the form of inventory in the face of stochastic demand and limited production capacity.

The demand process $D(t)$ is very general, in the sense that most of the demand processes considered in the literature of production/inventory theory are covered under the umbrella of the time homogenous, additive processes. In this paper we separately investigate the case where $D(t) \geq 0$ for all $t \geq 0$, and the case where $D(t)$ is unrestricted. In the former case, we show that the produce-up-to S policy minimizes the average expected holding and backorder costs per unit time. By using results from the study of storage processes we give a formula to compute the optimal value of S. However, the processes for which we give explicit formulas are single parameter demand processes. In the latter case, we further assume that $D(t)$ is Brownian Motion with drift . We compute the safety stock level required to achieve a stated fill rate policy. The advantage of the Brownian process is that it allows us to specify both an instantaneous mean and an instantaneous variance. For the Brownian process to make sense as a demand process we must limit consideration to relatively low instantaneous coefficient of variation.

There are a number of papers which attempt to analyze class of problems similar to the class that we consider here. Heyman [14] investigates optimal operating policies for M/G/1 queueing systems under the existence of server start-up and shut-down costs, and a cost per unit time spent in the system for each costumer. Sobel [22] considers a GI/G/1 queueing system operating under a very general cost structure. He shows that any pure stationary policy is equal to that of an (M, m) policy: If the queue length is less than or equal to m then do not provide service until it increases to M (where $M > m$), at which point service begins and continues until the queue length drops to m again. The equivalence mentioned above is shown to be in the sense that, there exists an epoch T after which the sequence of states generated by an arbitrary pure stationary policy will be the sequence that would have been generated by an (M, m) type of policy. An application of the (M, m) policy on a M/D/1 queueing sytem integrated with an inventory model is performed by Gavish and Graves [12]. Graves and Keilson [13] consider a one-product production/inventory problem where the demand arrivals are governed by a Poisson process and the demand sizes are exponentially distributed. Assming a two critical numbers type (S_1, S_2) policy and a constant rate continuous production system, they derive a closed form expression for the system cost. De Kok et. al [4] extend this model to accomodate two levels of production rates. Assuming a two-critical numbers (S_1, S_2) policy (depending on the stock level , one of the production rates is employed), they derive approximate solutions based on some service level objectives. Given the difference of the critical numbers $(S_2 - S_1)$, and the service levels (as the fraction of the demand or costumers that are lost), De Kok and Tijms [5] propose approximate solutions for the value of S_1.

Application of diffusion processes in the theory of production/inventory problems was initiated by Bather [1] where a positive set up cost for the production facility and infinite production rate (instantaneous replenishments) is assumed. Puterman [19] investigates solutions to the stationary (s, S) type policies under the assumption that the stock levels can be represented by a diffusion process. The parameters of the diffusion process are determined according to the position of the stock level (below or above s). Doshi [6] considers a similar setting where the stock level is modelled as a Brownian Motion(a special diffusion process), and there are two modes of control. A cost is incurred every time the mode of the control is changed. Furthermore, when the state of the process is x, a cost of cx^2 is incurred, $c > 0$.

Here, our intent is to limit the survey of the related literature to the continuous time, finite rate production models. For the treatment of the discrete time problems (with finite production capacity in each period) we refer the reader to Federgruen and Zipkin [8], Federgruen and Zipkin [9].

This paper differs from earlier work on continuous time models in that we limit attention to the special case in which there is no charge for changing the production rate. The fixed charge models are more difficult to analyze and few models have been found that are both exact and computationally tractable. The special case we consider does admit exact, explicit solutions. We believe that there are real situations, such as the battery manufacturing example, in which our formulas yield useful guidelines for operating practice.

The rest of this paper is organized as follows. In Section 2 we derive optimal policies for the continuous time production/inventory problem for the case $D(t) \geq 0$. The fill rate analysis of the case where $D(t)$ is approximated by a Brownian Motion is presented in Section 3. We present some computational findings in Section 4.

2 Application of the Storage Processes

The treatment of the production/inventory problems using tools developed in analysis of storage processes has been performed recently by Tayur [23]. In that paper, Tayur computes the optimal stationary policy of a capacitated, discrete time inventory problem (where the production capacity is finite, say C in each period) under the average expected cost per period criterion. In this section our goal is to point out a different class of problems that can be analyzed in the same spirit.

Consider the following dam (or water reservoir) model in the continuous time. Let $D(t)$ denote the total input (say, rainfall) into a dam during an interval $(0, t]$. $\{D(t); t \geq 0\}$ is assumed to be a process with stationary and independent increments (time homogenous additive). Furthermore, assume $D(t)$ is nonnegative for all $t > 0$. We assume a constant rate continuous release of water from the dam, and the units of time and of volume are chosen so that the release rate is unity except when the dam is empty. Let $Z(t)$ denote the water content of the dam at time t. Then, $Z(t)$ can be expressed as

$$Z(t) = Z(0) + D(t) - t + \int_0^t 1_{\{Z(\tau)=0\}} d\tau \tag{1}$$

where the integral in (1) gives the length of time the dam is empty during the interval $(0, t]$. We assume that the dam has infinite storage capacity. For an extensive treatment of dam models we refer the reader to Prabhu [Chapters 6,7] [16], Gani [10] and Prabhu [17].

Now, consider the produce-up-to S policy for the production/inventory problem. Let $I(t)$ denote the inventory position of the the item under control at time t. If $I(t)$ is less than a prespecified number S, then the production is continued until $I(t)$ reaches the level S. Whenever it does, the production is stopped until it drops below S again. A shortage cost p, and a holding cost h is charged linearly, per unit per unit time, depending on the position (below or above zero) of $I(t)$. We assume that production can be switched on and off without cost. The correspondance with the dam model described above can be seen as follows. Let the demand process of the inventory system be identical to the input process of the dam. Also, let the continuous production process be identical to the release process of the dam. Then, if the water level is $Z(t)$ at time t this corresponds to the inventory position $I(t) = S - Z(t)$. Assume that the inventory system has an inventory position of S at time 0, which corresponds to the case where the dam is empty at time 0. Note that If $Z(t) > S$, the inventory system is in the backorder state, which explains why we need a dam of infinite capacity. The following table, similar to Table 1 in Tayur [23], summarizes the above correspondence.

Dam Model	*Inventory Model*
Rainfall	Demand
Continuous Release	Continuous Production
Water Content	S−Inventory Position
Empty Dam	Inventory Position is S
Critical Level Is Crossed	Backorders

Stated simply, the process generated by the inventory system is what a person sitting in the dam upside down observes.

The limiting distribution of $Z(t)$ as $t \to \infty$ exists if and only if $0 < \rho < 1$ where ρ is the mean input per time. In this case it can be shown that (see the remarks by H.E. Daniels following Kendall [15], and see also p.248 of Prabhu [16]) for $z > 0$ the limiting distribution of $Z(t)$ is given by

$$F(z) = 1 - (1-\rho)\int_{w=0}^{\infty} dK(z+w, w), \tag{2}$$

where $K(x,t)$ is the input distribution function. Also it follows that

$$\int_{0+}^{\infty} dK(w,w) = \rho(1-\rho)^{-1},$$

so that $F(0) = 1 - \rho$.

Example 1: Gamma type input.

The input during an interval $(0,t]$ has the gamma distribution given by

$$\begin{aligned} k(x,t)dx &= P\{x < D(t) < x + dx\} \\ &= e^{-x/\rho}\frac{(x/\rho)^{t-1}dx}{\Gamma(t)\rho}, \end{aligned}$$

for $0 < x < \infty$, and $t > 0$. Then, for $z \geq 0$

$$F(z) = 1 - (1-\rho)\int_0^{\infty} \frac{(z+w)^{w-1}e^{-(z+w)/\rho}}{\rho^w \Gamma(w)} dw. \tag{3}$$

Note that this process is determined by the single parameter, ρ.

Example 2: Poisson type input.

The input during an interval $(0,t]$ has the distribution function

$$K(x,t) = \sum_{r=0}^{[x]} e^{-\rho t}\frac{(\rho t)^r}{r!},$$

where $[x]$ is the largest integer contained in x. For $z \geq 0$ we obtain

$$F(z) = 1 - (1-\rho)\sum_{r=[z]}^{\infty} \frac{\{\rho(r-z)\}^r}{r!} e^{-\rho(r-z)}. \tag{4}$$

This process is also determined by the single parameter, ρ.

Let $C(S)$ denote the average expected cost per unit time of the inventory system for a given value of S. Also let p and h denote the unit shortage and holding costs per time respectively. Then,

$$C(S) = h\int_0^S w dG(w) + p\int_{-\infty}^0 (-w) dG(w), \tag{5}$$

where $G(x)$ is the distribution function of $I := \lim_{t\to\infty} I(t)$. But (5) can be written as

$$C(S) = h\int_0^S (S-z)dF(z) + p\int_S^\infty (z-S)dF(z) \tag{6}$$

Hence, the optimum value of S can be found by using (6) and (2), which yields :

$$S^* = \inf\{s : F(s) \geq p/(h+p)\} \tag{7}$$

In some of the practical applications one needs to find the produce-up-to level (S_α) such that the (asymptotic) probability of being out of stock does not exceed, say $1-\alpha$ (see, for example, De Kok and Tijms [4], De Kok and Tijms [5]). Then

$$\begin{aligned}
1-\alpha &\geq \lim_{t\to\infty} Pr\{I(t) \leq 0\} \\
&= \lim_{t\to\infty} Pr\{S - Z(t) \leq 0\} \\
&= 1 - F(S) \\
&= (1-\rho)\int_{w=0}^{\infty} dK(S+w, w).
\end{aligned}$$

Thus, S_α is given by

$$S_\alpha = \inf\{S : (1-\rho)\int_{w=0}^{\infty} dK(S+w,w) \leq 1-\alpha\} \tag{8}$$

We conclude this section by showing the optimality of the order-up-to level policy for the continuous time production/inventory problem. Results (7), (8), and the following proposition are the principal contributions of this section.

Proposition 1 *The produce-up to-S policy defined by (7) is optimal for the continuous time production/inventory problem.*

Proof: Let the input into the dam during a time interval $(0,t]$ be denoted by $D(t)$. If $D(t)$ is a time-homogenous, nonnegative, infinitely divisible process with a continuous distribution, having finite mean and variance, then its Laplace transform is given by

$$E[e^{-\theta D(t)}] = e^{-t\xi(\theta)}$$

where

$$\xi(\theta) = \int_0^\infty (1 - e^{-\theta u})\lambda(u)du$$

so that $\lambda(u) \geq 0$, finite, $\lambda(u) \to \infty$ as $u \to 0$. Fix $\Delta = n^{-1}$ for a positive integer n. Consider the discrete time dam model in which

i. The length of each time interval is Δ

ii. The input into the dam, D_n, in the interval $(t\Delta, (t+1)\Delta]$ has probability distribution P_n

iii. Release from the dam occurs at the end of each interval except when the dam is empty. Furthermore, the release does not exceed Δ units.

By the properties of the process $\{D(t), t \geq 0\}$, we can always construct such a dam model. Moreover, as we have demonstrated earlier, this dam model corresponds to the discrete time, finite capacity inventory problem. Tayur [23] (also see Federgruen and Zipkin [8]) shows that the produce-up-to S policy is optimal for this problem. Define

$$W_n = Z_n + D_n$$

where Z_n is the stationary water content of the discretized dam at the beginning of the interval $(t\Delta, (t+1)\Delta]$ right after the release, with distribution function $F_n(w)$. Let

$$g(S, W_n) = \begin{cases} h(S - W_n) & \text{if } W_n \leq S \\ p(W_n - S) & \text{if } W_n > S \end{cases}$$

so that $g(S, W_n)$ maps the inventory positions(or the water level) at the end of a period to their respective cost values. Hence, by using the above, the long run average cost function of the constructed discrete time problem can be written for a given level of S as

$$C_n(S) = \int_0^\infty g(S, w)d(P_n * F_n)(w) \tag{9}$$

where * denotes the convolution operator.

It can be shown that (Prabhu [16], p.232) F_n has the corresponding Laplace transform

$$\psi_\Delta(\theta) = \frac{(1-\rho)(e^{\theta\Delta} - 1)}{e^{\theta\Delta} - \phi(\theta, \Delta)}$$

where $\phi(\theta, \Delta)$ is the Laplace transform of D_n, and is given by

$$\phi(\theta, \Delta) = e^{-\Delta\xi(\theta,\Delta)+o(\Delta)}.$$

It can also be shown that (Prabhu [16]) $\xi(\theta,\Delta) \to \xi(\theta)$ as $\Delta \to 0$. It follows by using Taylor's expansion that

$$\psi_\Delta(\theta) \to \frac{(1-\rho)\theta}{\theta - \xi(\theta)}$$

as $\Delta \to 0$. We note that this is the Laplace transform of the stationary water content, Z, in the continuous time process with the corresponding distribution function (2). Therefore, by the Continuity Theorem for Laplace Transforms (Durret [7], p.83) we have $Z_n \Longrightarrow Z$ where '$\Longrightarrow$' stands for weak convergence. Also note that as $n \to \infty$, $D_n \to 0$ with probability 1. Hence, $W_n = Z_n + D_n \Longrightarrow Z$ by Theorem 25.4. of Billingsley [2], sometimes referred as Slutsky's Lemma. Therefore, by using a corollary to the Continuous Mapping Theorem (Theorem 25.7. of Billingsley [2]) we can conclude that

$$C_n(S) = E[g(S,W_n)] \to E[g(S,Z)] = C(S)$$

As $n \to \infty$(as $\Delta \to 0$) the discrete time model approaches (by shrinking) the continuous time model. Therefore,

$$C(S) = \int_0^\infty g(S,w)dF(w)$$

is the true cost function of the continuous time model . Note that $C(S)$ is convex, whence the form of the optimal policy follows, by standard arguments.

□

3 Analysis of the Brownian Motion Demand Process

In this section we present simple control rules for the production/inventory problem introduced in Section 1 under the assumption that $D(t)$ follows a Brownian Motion with drift parameter μ, and instantaneous variance σ^2 (denoted by $B(\mu,\sigma^2)$. At first sight, modelling the demand process as a Brownian Motion may seem unrealistic. However, if we consider any time interval $(t, t+\delta]$, it is a property of the Brownian Motion that the increment $D(t+\delta)-D(t)$ has $N(\delta\mu, \delta\sigma^2)$ distribution. Therefore, if we have a discrete time problem with independent, identical distributed demands in each period, the total demand accumulated over a time interval should approximately have normal distribution by the Central Limit Theorem. Hence, it provides a reasonable approximation provided that the instantaneous coefficient of variation σ/μ is not large.

In this section the following operating rule will be investigated. If the inventory position of the product under control is less than a level S, then the production is continued at a finite, constant rate $r > \mu$ until the inventory position reaches the level S, where the production is stopped. It should be noted that, in the neighborhood of S this implies very rapid transitions between production and non-production. The intent of this section is to derive some performance measures based on a service level constraint so that we can solve for the value of S that satisfies a prespecified service level.

Let $I(t)$ denote the inventory position at time t. It easily follows that if $S < I(t)$ then $I(t)$ follows a $B(r-\mu,\sigma^2)$ process. Else, $I(t)$ follows a $B(-\mu,\sigma^2)$ process. Let $(S-\epsilon, S+\epsilon)$ be an ϵ neighborhood of S. Assume the production is stopped as soon as $I(t)$ enters in $(S-\epsilon, S+\epsilon)$ from below. We mark the times the process $I(t)$ hits $S+\epsilon$ from $S-\epsilon$ as α_i, and the times the process $I(t)$ hits $S-\epsilon$ from $S+\epsilon$ as β_i. Formally,

$$\alpha_i = \inf\{t > \beta_{i-1} : I(t) = S + \epsilon\},$$

and

$$\beta_i = \inf\{t > \alpha_i : I(t) = S - \epsilon\}.$$

The variable i indexes the production cycles. Since we are interested in stationary solutions, we can choose arbitrary initial conditions and take $I(0) = S - \epsilon$ and $\beta_0 = 0$. Let $\mathcal{C}_i$ be the collection of events between β_i, and β_{i-1}. By Markov property of the Brownian Motion $\{\mathcal{C}_i\}$ forms a regenerative process. Now, we introduce some more definitions. Let B_t be a Brownian Motion with drift $\tilde{\mu}$, and variance $\tilde{\sigma}^2$.

$$g_{a,b}(x) := E[\text{Time}\, B_t \,\text{spends below}\, a \;\text{ until it reaches}\, b > x | B_0 = x]$$

for $a < b < y$, and for some $y < \infty$. The following proposition is the key result of this section.

Proposition 2

$$g_{a,b}(x) = \begin{cases} \frac{\tilde{\sigma}^2}{2\tilde{\mu}^2}(1 - e^{-2\frac{\tilde{\mu}}{\tilde{\sigma}^2}(b-a)})P_{a,b}(x) & \textit{if } x \geq a \\ \frac{a-x}{\tilde{\mu}} + \frac{\tilde{\sigma}^2}{2\tilde{\mu}^2}(1 - e^{-2\frac{\tilde{\mu}}{\tilde{\sigma}^2}(b-a)}) & \textit{if } x < a \end{cases} \tag{10}$$

where

$$P_{a,b}(x) = \frac{e^{-2\frac{\tilde{\mu}}{\tilde{\sigma}^2}b} - e^{-2\frac{\tilde{\mu}}{\tilde{\sigma}^2}x}}{e^{-2\frac{\tilde{\mu}}{\tilde{\sigma}^2}b} - e^{-2\frac{\tilde{\mu}}{\tilde{\sigma}^2}a}}.$$

Proof: In order to derive the result, we follow the general theory of diffusion processes. For the diffusion process B_t, there exists a linear second order differential operator $\mathcal{L}$ of the form

$$\mathcal{L} = \tilde{\mu}\frac{d}{dx} + \frac{1}{2}\tilde{\sigma}^2\frac{d^2}{dx^2}$$

If we further parameterize $g_{a,b}(x)$ such that the starting time of the Brownian Motion is u, rather than 0, $g_{a,b}(x)$ satisfies the so called Kolmogorov's backward equation (see Gard [11], p.30)

$$\frac{dg}{du} + \mathcal{L}g = 0$$

which in our case yields

$$\tilde{\mu}\frac{dg_{a,b}(x)}{dx} + \frac{1}{2}\tilde{\sigma}^2\frac{d^2g_{a,b}(x)}{dx^2} = -1_{\{x<a\}} \tag{11}$$

where $1_{\{\}}$ is the indicator function. By using standard methods of differential equations (see Simmons [21], p.90) we can solve $g_{a,b}(x)$. One can also directly verify that (10) satisfies (11).

□

In light of the above development we can prove the final result of this section.

Proposition 3 *Assume* $r < \mu/(1-\alpha)$. *Then* $\lim_{t\to\infty} Pr\{I(t) \leq 0\} \leq 1-\alpha$ *if and only if* $S \geq \frac{-\sigma^2}{2(r-\mu)}\ln(\frac{r}{\mu}(1-\alpha))$

Proof: First note that by the regenerative property of the cycles $\{C_i\}$, we can use the Theory of Regenerative Processes, and by Proposition 5.9. of Ross [20] we conclude that

$$\lim_{t\to\infty} Pr\{I(t) \leq 0\} = \frac{g_{0,S+\epsilon}(S-\epsilon)}{E[\beta_1 - \beta_0]}$$

Next note that

$$\begin{aligned} E[\beta_1 - \beta_0] &= E[\beta_1] \\ &= E[\beta_1 - \alpha_1 + \alpha_1] \\ &= E[\beta_1 - \alpha_1] + E[\alpha_1] \\ &= \frac{2\epsilon}{r-\mu} + \frac{2\epsilon}{\mu} \\ &= \frac{2\epsilon r}{\mu(r-\mu)} \end{aligned}$$

Of course, $g_{0,S+\epsilon}(S-\epsilon)$ is obtained by choosing $a=0$, $b=S+\epsilon$, $x=S-\epsilon$, $\tilde{\mu}=\mu$, and $\tilde{\sigma}=\sigma$ in (10). Now, we let $\epsilon \to 0$ to obtain the result, which yields

$$\lim_{t\to\infty} Pr\{I(t) \leq 0\} = \frac{\mu}{r}e^{-2\frac{r-\mu}{\sigma^2}S} \tag{12}$$

Now, by using (12), the claim follows. The range of r is so chosen that $S > 0$, and hence the previous proposition applies (with $a = 0$, and $b = S + \epsilon$).

□

Let S_α denote the minimum safety stock level required to satisfy the fill rate:

$$S_\alpha = \frac{-\sigma^2}{2(r-\mu)} \ln(\frac{r}{\mu}(1-\alpha)). \tag{13}$$

It can quickly be checked that S_α has the following intuitively-satisfying characteristics:

$$\frac{dS_\alpha}{d\alpha} \geq 0,$$

$$\frac{dS_\alpha}{d\sigma} \geq 0,$$

$$\frac{dS_\alpha}{d\rho} \geq 0, \text{ and } \lim_{\rho \to 1} S_\alpha = \infty$$

where $\rho = \mu/r$. It should be noted that the safety stock level provided by equation (13) provides an analogous characterization to (8) for the Brownian Process demand case. Now, we note another form of (13), which exhibits interesting properties. Let $S_\alpha^\tau := S_\alpha/r$. Then we can write S_α^τ as

$$S_\alpha^\tau = \rho^2 \frac{-(\sigma/\mu)^2}{2(1-\rho)} \ln(\frac{1-\alpha}{\rho}), \tag{14}$$

which is a function of ρ, σ/μ, and α only. Hence, by considering only the utilization rate and the coefficient of variation of the process one can find the relative safety stock level to satisfy a specified fill rate. Then, the real safety stock levels can be obtained by multiplying S_α^τ by the production rate r. Computations on (14) are presented in Section 4.

4 Computational Results

In this section we present our implementation of the safety stock level characterizations given by equations (8), (13), and (14). Computations are performed on an IBM PS/2 PC. Two examples are considered for the implementation of equation (8): Gamma demand process, and Poisson demand process. For the case of Gamma demand process, in order to evaluate the integral in (3) we used Romberg Integration Algorithm (Press et al.[18]). For the Poisson demand case, the summation in (4) is evaluated until the tail of the distribution dies out.

Table 1: Comparison of the Safety Stock Levels

ρ	Gamma	Poisson	Brownian Process ($\sigma = \rho$)
	0.20	0.80	0.038
0.25	0.30	1.00	0.067
	0.70	1.70	0.134
	4.30	5.10	3.33
0.80	5.80	6.70	4.43
	9.30	10.40	7.01
	6.30	7.00	5.15
0.85	8.30	9.20	6.82
	13.20	14.30	10.70
	10.10	10.80	8.89
0.90	13.30	14.20	11.70
	20.80	21.90	18.22
	21.50	22.10	20.31
0.95	28.10	29.00	26.57
	43.50	44.40	41.10
	112.10	113.80	112.00
0.99	147.00	148.10	146.00
	226.10	228.00	225.00

The first set of computations we present is a comparison of safety stock levels for different demand processes. We perform our analysis for $\rho = (0.80, 0.85, 0.90, 0.99)$ where r is chosen to be equal to one for the Brownian demand process. Also, we set $\sigma = \rho$ for the Brownian demand case. Table (1) gives the minimum safety stock levels required to satisfy a fill rate of $\alpha = (0.90, 0.95, 0.99)$. Figure 1 plots the safety stock levels for $\alpha = 0.99$.

As expected, the safety stock level increases with ρ and α. More interesting observations can be summarized as follows:

i. The Poisson Process yields the highest safety stock level at a fixed utilization and fill rate. This is partly expected becouse the Poisson Process has the highest instantenous coefficient of variation among the three processes (note that $\sigma = \rho$ for the Brownian demand process in this set of computations).

ii. The safety stock levels become less sensitive to the demand process used as the utilization rate approaches to one. Note that for low ρ (such as 0.25), the ratio of safety stock levels for different demand processes may be as high as 5, whereas it is very close to 1 for high ρ (such as 0.99).

Table 2: Relative Safety Stock Levels for the Brownian Process Demand

ρ	$\sigma/\mu = 0.1$	$\sigma/\mu = 0.3$	$\sigma/\mu = 0.5$	$\sigma/\mu = 0.8$
	0.051	0.464	1.289	3.299
0.85	0.068	0.614	1.706	4.367
	0.107	0.963	1.675	6.848
	0.090	0.800	2.225	5.695
0.90	0.117	1.053	2.927	7.492
	0.182	1.640	4.556	11.663
	0.020	1.828	5.079	13.003
0.95	0.266	2.391	6.643	17.007
	0.411	3.700	10.275	26.303
	1.123	10.111	28.086	71.901
0.99	1.463	13.170	36.578	93.640
	2.250	20.267	56.296	144.118

iii. As the utilization rate increase the safety stock levels increases exponentially. This effect can be observed for $\alpha = 0.99$ in Figure 1.

The next set of computations involves the implementation of the Brownian motion model (equation (14)) for $\rho = (0.85, 0.90, 0.95, 0.99)$, and $\sigma = (0.5, 3.0, 10.0)$ at $\alpha = (0.90, 0.95, 0.99)$. Table (2) summarizes the results. Using Table(2) we can easily determine the actual safety stock levels. For example, at $\rho = 0.95$, and $\sigma/\mu = 0.3$, if $r = 10$, then $S_{0.95} = 3.7 * 10 = 37$. Figure 2 shows the effect of the coefficient of variation on the relative safety stock level at 99% fill rate and different utilization rates.

Finally, it is interesting to note a different characterization of S_α, for (14)

$$S_\alpha = k_\alpha\, \sigma \tag{15}$$

where

$$k_\alpha = -\frac{\sigma/\mu}{2^{\frac{1-\rho}{\rho}}} \ln\left(\frac{1-\alpha}{\rho}\right)$$

is a unitless quantity. For example, for the battery manufacturer example, the standard deviation of the daily demand is on the order of 250 units, then by using (15),

$$\begin{aligned} k_\alpha &= -\frac{250/4500}{2^{\frac{1-0.95}{0.95}}} \ln\left(\frac{1-0.95}{0.95}\right) \\ &= 1.555, \end{aligned}$$

and $S_\alpha = 388.75$.

Figure 1. Comparison of Safety Stock Levels at 0.99 Fill Rate

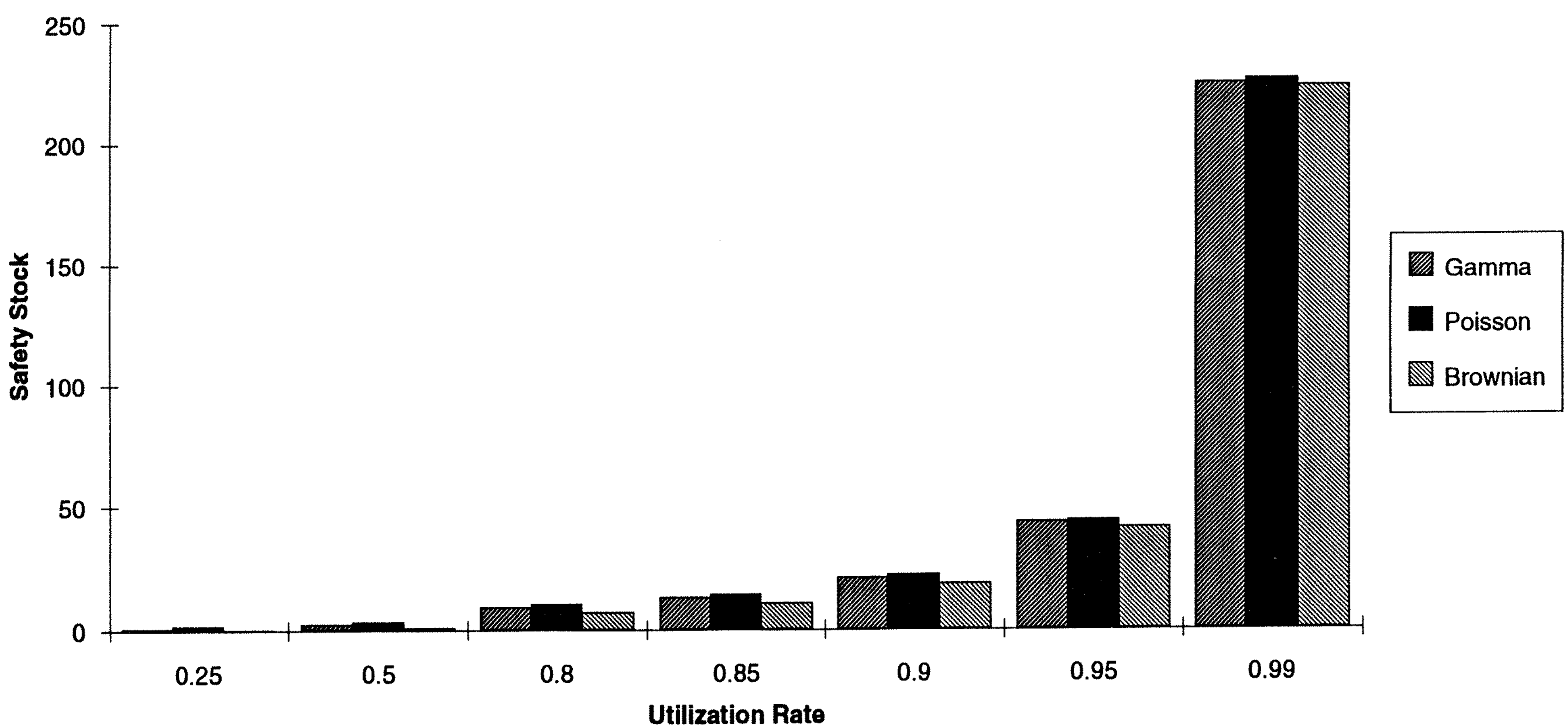
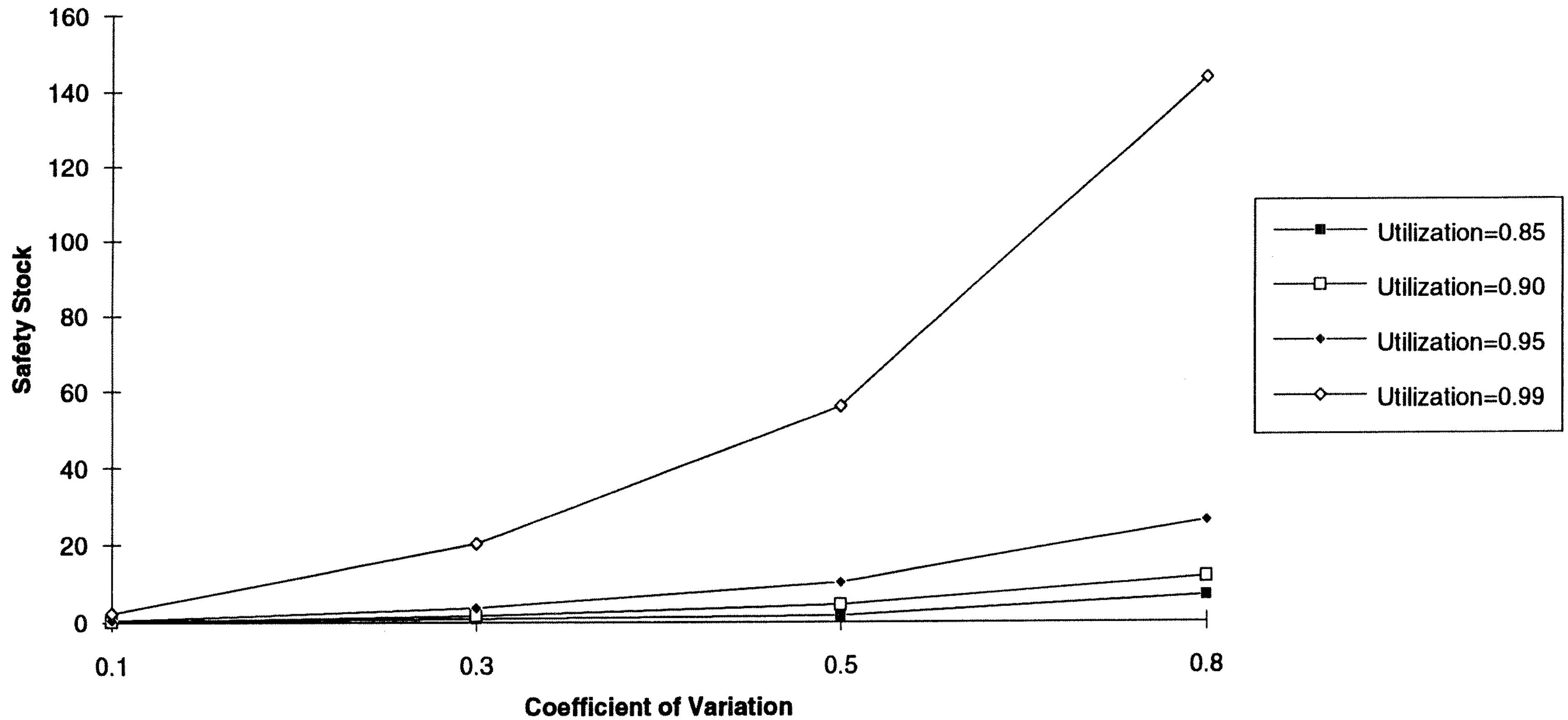


Figure 2. Safety Stock Levels For Brownian Demand Process at 0.99 Fill Rate



5 Conclusion

In this paper we considered a single-item, capacitated, continuous time production/inventory problem with no setup costs. Under a very general demand structure, we proved that the produce-up-to level type of an operating policy is optimal for the continuous time problem. We presented explicit formulas (3), (4) and (7) to compute this level when the demand is Gamma or Poisson distributed. For the case that the demand follows a Brownian Process, we derived a formula (14) for the produce-up-to level to satisfy a pre-specified fill rate. We also provide various computations in this case, including comparisons with the other demand distributions

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