BU-79-M

Analysis for a 4 x 4 balanced Lattice Square~~s~~ with an Additional Replicate~~s~~

W.T. Federer

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An arrangement of 16 treatments (0000, 0001, ···, 1111) in a 4 x 4 lattice square is given below:

Square I

0000	0010	0001	0011
1000	1010	1001	1011
0100	0110	0101	0111
1100	1110	1101	1111

Square II

0000	1000	0100	1100
1010	0010	1110	0110
0101	1101	0001	1001
1111	0111	1011	0011

Square III

0000	0101	1111	1010
1101	1000	0010	0111
1011	1110	0100	0001
0110	0011	1001	1100

Square IV

0000	0110	1101	1011
1110	1000	0011	0101
1001	1111	0100	0010
0111	0001	1010	1100

Square V

0000	1001	0111	1110
0001	1000	0110	1111
0011	1010	0100	1101
0010	1011	0101	1100

The system of confounding in the above 5 squares is:

	Pseudo-factors confounded	
Square	with rows	with columns
1	A, B, AB	C, D, CD
2	C, D, CD	AC, BD, ABCD
3	AC, BD, ABCD	AD, ABC, BCD
4	AD, ABC, BCD	BC, ABD, ACD
5	BC, ABD, ACD	A, B, AB

If a sixth replicate duplicating square 1 is used the analysis of variance is:

Source of variation	df	ss	ms
Total	6(16) = 96		
CT	1	CT	--
Replicate	5	Rep.	--
Treatments (ignoring rows and cols.)	15	T	--
Rows (elim. tr; ignoring col.)	18	R' + R_2	--
Cols. (elim. tr. and row)	$18 \begin{cases} 15 \\ 3 \end{cases}$	$\left.\begin{matrix} C_1 \\ C_2 \end{matrix}\right\} = C$	E_c
Error	39	E	E_e
Row (elim. tr. and col.)	$18 \begin{cases} 15 \\ 3 \end{cases}$	$\left.\begin{matrix} R_1 \\ R_2 \end{matrix}\right\}$ R	E_r
Col. (elim. tr; ignoring row)	18	C' + C_2	--

The sums of squares CT, Rep, and T are computed in the usual manner for a randomized complete block design. The remaining sums of squares are computed by a procedure similar to that described on pages 389-406 ~~in my book~~ in the book, "Experimental Design," by W. T. Federer.

The sum of squares for rows (eliminating treatment effects and ignoring columns) is:

$$R' = \frac{1}{8(1+1+1+1+4+4)} \sum_{u=0}^{1} \Big\{ [A_{.u} - 3(A_{1u}+A_{6u})]^2 + [B_{.u}-3(B_{1u}+B_{6u})]^2 + [AB_{.u}-3(AB_{1u}+AB_{6u})]^2 \Big\}$$

$$+ \frac{1}{8\binom{1+1+1+1}{+1+25}} \sum_{u=0}^{1} \Big\{ [C_{.u}-6C_{2u}]^2 + \ldots + [ACD_{.u}-6ACD_{5u}]^2 \Big\},$$

$$- \frac{3[X_{.....}-3X_{1....}-3X_{6....}]^2}{16(12)} - \frac{3}{16(30)} \Big\{ [X_{.....}-6X_{2....}]^2 + [X_{.....}-6X_{3....}]^2 + [X_{.....}-6X_{4....}]^2 + [X_{.....}-6X_{5....}]^2 \Big\},$$

where $A_{.u}$ = uth levels of effect A from all 6 arrangements and A_{vu} = uth level of effect A from the vth arrangement.

The sum of squares C_1 is computed as follows:

$$C_1 = \frac{1}{8(1+1+1+9)} \sum_{u=0}^{1} \Big\{ [A_{.u} - 4A_{5u} - A_{1u} - A_{6u}]^2$$

$$+ B + AB \text{ comparisons} \Big\} + \frac{1}{8(4+4+4+9+9)} \Sigma \Big\{ [2(C_{.u} - C_{2u}) - 5(C_{1u} + C_{6u})]^2$$

$$+ [2(D_{.u} - D_{2u}) - 5(D_{1u} + D_{6u})]^2 + [2(CD_{.u} - CD_{2u}) - 5(CD_{1u} + CD_{6u})]^2 \Big\}$$

$$+ \frac{1}{8(1+1+1+1+16)} \sum_{u=0}^{1} \Big\{ [AC_{.u} - 5AC_{2u} - AC_{3u}]^2$$

$$+ \text{comparisons for BD, ABCD, AD, ABC, BCD, BC, ABD, and ACD,} \Big\} .$$

X

$$- \frac{3}{16(12)} [X_{.....} - 4X_{5....} - X_{1...} - X_{6....}]^2$$

$$- \frac{3}{16(30)} [2X_{.....} - 2X_{2....} - 5(X_{1....} + X_{6....})]^2$$

$$- \frac{3}{16(20)} \Big\{ [X_{....} - 5X_{2....} - X_{3....}]^2 + [X_{.....} - 5X_{3....} - X_{4....}]^2$$

$$+ [X_{.....} - 5X_{4....} - X_{5....}]^2 \Big\} .$$

Likewise, the sum of squares for R_1 is:

$$R_1 = \frac{1}{8(4+4+4+9+9)} \sum_{u=0}^{1} \Big\{ [2A_{.u} - 2A_{5u} - 5(A_{1u} + A_{6u})]^2$$

$$+ B + AB \text{ comparisons} \Big\} + \frac{1}{8(1+1+1+9)} \sum_{u=0}^{1} \Big\{ [C_{.u} - C_{1u} - C_{6u} - 4C_{2u}]^2$$

$$+ C \text{ and } CD \text{ comparisons} \Big\}$$

$$+ \frac{1}{8(1+1+1+1+16)} \sum_{u=0}^{1} \Big\{ [AC_{.u} - AC_{2u} - 5AC_{3u}]^2$$

$$+ \text{comparisons for BD, ABCD, ..., ACD} \Big\}$$

$$- \Big\{ \frac{3}{16(30)} [2X_{.....} - 2X_{5....} - 5(X_{1....} + X_{6....})]^2$$

$$+ \frac{3}{16(12)} [X_{.....} - X_{1....} - X_{6....} - 4X_{2....}]^2$$

$$+ \frac{3}{16(20)} \Big\{ [X\ldots\ldots -X_{2}\ldots\ldots -5X_{3}\ldots\ldots]^2 + [X\ldots\ldots -X_{3}\ldots\ldots -5X_{4}\ldots\ldots]^2 + [X\ldots\ldots -X_{4}\ldots\ldots -5X_{5}\ldots\ldots]^2 \Big\} \Big\} .$$

The sum of squares C' is computed as:

$$C' = \frac{1}{8(1+1+1+1+4+4)} \sum_{u=0}^{1} \Big\{ [C_{\cdot u} - 3(C_{1u} + C_{6u})]^2$$

$$+ \text{ comparisons for D and CD } \Big\}$$

$$+ \frac{1}{8(1+1+1+1+1+25)} \sum_{u=0}^{1} \Big\{ [AC_{\cdot u} - 6AC_{2u}]^2 + [BD_{\cdot u} - 6BD_{2u}]^2$$

$$+ \ldots + [B_{\cdot u} - 6B_{5u}]^2 + [AB_{\cdot u} - 6AB_{5u}]^2 \Big\}$$

$$- \frac{3}{16(12)} [X\ldots\ldots -3(X_{1}\ldots\ldots + X_{6}\ldots\ldots)]^2 - \frac{3}{16(30)} \Big\{ [X\ldots\ldots -6X_{2}\ldots\ldots]^2$$

$$+ [X\ldots\ldots -6X_{3}\ldots\ldots]^2 + [X\ldots\ldots -6X_{4}\ldots\ldots]^2 + [X\ldots\ldots -6X_{5}\ldots\ldots]^2 \Big\}$$

C_2 = interaction sum of squares from the following table:

column number	Rep. 1	6
1		
2		
3		
4		

,

or from the following:

$$32 \; \frac{1}{\cancel{16}} \Big\{ (C_{11} - C_{61})^2 + (C_{10} - C_{60})^2 + (D_{11} - D_{61})^2 + (D_{10} - D_{60})^2 + (CD_{11} - CD_{61})^2 + (CD_{10} - CD_{60})^2 - \frac{3\,(X_{1}\ldots\ldots - X_{6}\ldots\ldots)^2}{2} \Big\} .$$

Likewise, R_2 is an interaction sum of squares for row totals in arrangements 1 and 6; it may be computed as follows:

$$32 \; \frac{1}{\cancel{16}} \Big\{ (A_{11} - A_{61})^2 + (A_{10} - A_{60})^2 + (B_{11} - B_{61})^2 + (B_{10} - B_{60})^2 + (AB_{11} - AB_{61})^2 + (AB_{10} - AB_{60})^2 - \frac{3}{2} (X_{1}\ldots\ldots - X_{6}\ldots)^2 \Big\} .$$

The sum of squares $C = C_1 + C_2$ and the sum of squares $R = R_1 + R_2$. The mean square $E_c = C/18$ and the mean square $E_r = R/18$.

~~The above coefficients should be verified. If you get stuck anywhere along the line be certain to ask me about it.~~

The weighted total levels of the effect are required next (see formula (1X-14)). For example,

$$(A_u)\text{wtd} = \frac{6\left[(A_{1u}+A_{6u})w_r+A_{5u}w_c+w(A_{2u}+A_{3u}+A_{4u})\right]}{k(2w_r+w_c+3w)}$$

$$(C_u)\text{wtd} = \frac{6\left[(C_{1u}+C_{6u})w_c+C_{2u}w_r+w(C_{3u}+C_{4u}+C_{5u})\right]}{k(2w_c+w_r+3w)}$$

$$(AC_u)\text{wtd} = \frac{6\left[\overset{AC}{C}_{3u}w_r+\overset{AC}{C}_{2u}w_c+w(\overset{AC}{C_{1u}+C_{4u}+C_{5u}+C_{6u}})\right]}{k(w_c+w_r+4w)}$$

$$w = 1/E_e$$

$$w_c = 1/\left(\frac{60}{49}E_c - \frac{11}{49}E_e\right) = 49/(60E_c - 11E_e)$$

$$w_r = 1/\left(\frac{60}{49}E_r - \frac{11}{49}E_e\right) = 49/(60E_r - 11E_e)$$

As a first step in obtaining the above analysis, I would obtain the levels of all pseudo-effect for each replicate. If you code your cards as per table VII-5 in my book, 15 columns will suffice for the coding operation. Then, with a sorter and tabulator you can obtain the desired totals.

With all levels of the 15 effects we can obtain the intrablock error sum of squares directly. Thus, the interactions of the level of the effects with the replicates in which they are unconfounded results in the intrablock error sum of squares. A, B, AB, C, D, and CD are unconfounded with rows or columns in three of the 6 replicates, yielding a total of 2+2+2+2+2+2=12 degrees of freedom. The remaining 9 effects are unconfounded in 4 of the 6 replicates yielding a total of 9(3)=27 degrees of freedom. The above 2 sets of sums of squares yield the error sum of squares with 12+27=39 degrees of freedom (see page 248 of ~~my book~~ Experimental Design for a similar example).

The average ~~eff~~ standard error of a difference between 2 adjusted means is

W. T. Federer
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$$s_{\bar{d}} = \sqrt{\frac{2}{15}\left\{\frac{3}{2w_r+w_c+3w}+\frac{3}{w_r+2w_c+3w}+\frac{9}{w_r+w_c+4w}\right\}}$$

and the efficiency is 100× 2(r. c. b. error)/ $6 s_{\bar{d}}^2$