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ARRANGING POINTS IN \mathbb{R}^d :
A QUESTION OF BALANCE*

by

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Abstract

Several balancing problems of the following general form are considered. A family \mathcal{P} of configurations P is specified along with a function $\beta: \mathcal{P} \rightarrow \mathbb{R}_+ \cup \{\infty\}$ that indicates the "balance" of each $P \in \mathcal{P}$. A specified perturbation operation induces an equivalence relation, or neighborhood structure, on \mathcal{P} . The central question is whether every $P \in \mathcal{P}$ has a well-balanced neighbor. Typically \mathcal{P} consists of sets of points in a Euclidean space, possibly with some additional restrictions, and the perturbation operation is reflection of a point through the origin. The function β indicating balance might involve the disposition of P with respect to hyperplanes, or the location of the centroid of P . This work was motivated by an interpretation of the chromatic number problem in graph theory as such a balancing problem, by means of Minty's characterization of the chromatic number.

1. Introduction

This paper examines several different problems concerned with balancing arrangements of points. We begin with two examples. The first example, though simple, conveys the spirit of the collection of balancing problems examined here. The second example is more difficult; it is a main focus of this work.

Example 1. Suppose that n couples, $(f_1, m_1), \dots, (f_n, m_n)$ are seated at a circular table with $2n$ seats. An arc \mathcal{A} of the table is a contiguous subset of the $2n$ seats. The discrepancy of the seating arrangement on arc \mathcal{A} is the absolute value of the difference between the number of females (f_i 's) and males (m_i 's) seated on arc \mathcal{A} . Say that the seating arrangement is well-balanced along arc \mathcal{A} if the discrepancy on arc \mathcal{A} is small. The seating arrangement is well-balanced if it is well-balanced along every arc. There is no reason to expect that an arbitrary seating arrangement is well-balanced. Suppose, however, that we permit certain perturbations of the original arrangement in an effort to achieve balance. In particular, suppose for each $i = 1, \dots, n$, we permit f_i and m_i to swap seats. Can we then be certain that a well-balanced arrangement can be achieved? In Section 8 we show that for every choice of n and an initial arrangement, there is an arrangement that arises from such swaps with discrepancy at most two on every arc. This is best possible.

Example 2. Let P be a finite (multi-) set of points in \mathbb{R}^d . Given a nonzero vector $h \in \mathbb{R}^d$, say that P is well-balanced relative to the hyperplane $H = \{x \in \mathbb{R}^d: h^T x = \vec{0}\}$ if the ratio

$$(1.1) \quad |P \cap H^+| / |P \cap H^-|$$

is small; here H^+ and H^- denote the open halfspaces determined by H , $H^+ = \{x \in \mathbb{R}^d: h^T x > 0\}$ and $H^- = \{x \in \mathbb{R}^d: h^T x < 0\}$. The set P is well-balanced if it is well-balanced relative to every hyperplane H in \mathcal{H}_d , the set of hyperplanes through the origin in \mathbb{R}^d . There is no reason to expect an arbitrary P to be well-balanced. Suppose, however, that we permit certain perturbations of the original arrangement P in order to achieve balance. Specifically, for each $p \in P$ permit the replacement of p by $-p$, its reflection through the origin. Of course, reflection cannot help if there is some $H \in \mathcal{H}_d$ such that $|H \cap P| = |P| - 1$; call P degenerate if there exists such a hyperplane H . We prove in Section 3 that if $P \subseteq \mathbb{R}^d$ is full-dimensional (it spans \mathbb{R}^d) and nondegenerate, and $d \geq 2$, then there is a choice of reflections such that the ratio (1.1) is at most $2d-1$ for every hyperplane $H \in \mathcal{H}_d$. Furthermore, in Section 4 we exhibit choices of nondegenerate P in \mathbb{R}^d , for all $d \geq 1$, such that for every choice of reflections there is some $H \in \mathcal{H}_d$ with (1.1) at least $d + \lceil \sqrt{2d} \rceil - 2$.

Examples 1 and 2 are similar in spirit to several other examples of balancing problems analyzed in this paper. In each case there is a family \mathcal{P} of configurations, a map $\beta: \mathcal{P} \rightarrow \mathbb{R}_+ \cup \{\infty\}$ that associates with each configuration P an indicator $\beta(P)$ of its balance (small $\beta(P)$ indicates balance, large $\beta(P)$ indicates imbalance), and a neighborhood structure (equivalence relation) on \mathcal{P} determined by some permissible perturbation operations. For any $P \in \mathcal{P}$, its neighborhood (equivalence

class) $\mathcal{N}(P)$ is the subset of \mathcal{P} reachable by iterating permissible perturbations. In general our interest is in the value

$$\gamma(P) = \inf\{\beta(P') : P' \in \mathcal{N}(P)\}$$

associated with each $P \in \mathcal{P}$, and, especially in

$$\gamma = \sup\{\gamma(P) : P \in \mathcal{P}\}.$$

In Example 1 \mathcal{P} is the set of pairs (δ, ϵ) , where δ is a fixed-point-free involution on a set E of the form $E = \{1, 2, \dots, 2n\}$, and $\epsilon: E \rightarrow \{-1, +1\}$, with

$$(1.2) \quad \epsilon(j) + \epsilon(\delta(j)) = 0 \quad \text{for all } j \in E.$$

The indicator of balance is $\beta((\delta, \epsilon)) = \max_{1 \leq i < j \leq 2n} \{|\sum_{k=i}^j \epsilon(k)|\}$. The perturbation operation permits replacement of $(\delta, \epsilon) \in \mathcal{P}$ by (δ, ϵ') , with $\epsilon'(i) = -\epsilon(i)$, $\epsilon'(\delta(i)) = -\epsilon(\delta(i))$ for some $1 \leq i \leq n$, and $\epsilon'(j) = \epsilon(j)$ for all $j \neq i, \delta(i)$. So $\mathcal{N}((\delta, \epsilon))$ is the set of all $(\delta, \bar{\epsilon}) \in \mathcal{P}$ such that $\bar{\epsilon}$ satisfies (1.2) with respect to δ . Example 1 is discussed in Section 8, along with some other examples.

In Example 2 there is a family $\mathcal{P}(d)$ for each choice of the dimension d . $\mathcal{P}(d)$ is the set of all full dimensional, nondegenerate, finite subsets of \mathbb{R}^d . The balance indicator β is given by
$$\beta(P) = \sup_{H \in \mathcal{H}_d} \{|H^+ \cap P| / |H^- \cap P|\}. \quad \text{Since the perturbation operation is}$$

reflection of a point $p \in P \in \mathcal{P}(d)$ through the origin,

$\mathcal{N}(P) = \{\epsilon P \mid \epsilon: P \rightarrow \{-1, +1\}\}$. Here ϵP is shorthand for $\{\epsilon(p)p \mid p \in P\}$.

Example 2 is discussed in Sections 2-5, where we prove that for each

$d \geq 2$, $\gamma(d) = \sup_{P \in \mathcal{P}(d)} \inf_{P' \in \mathcal{N}(P)} \beta(P')$ satisfies

$$d + \lceil \sqrt{2d} \rceil - 2 \leq \gamma(d) \leq 2d-1.$$

This example is related to the densest hemisphere problem, shown by Johnson and Preparata [15] to be NP-complete.

Imre Barany pointed out to us a similarity between this result and a result of Larman [16] concerning the following problem of McMullen. For each positive integer d , what is the greatest positive integer $v(d)$ such that for any set S of $v(d)$ points in general position in \mathbb{R}^d there is a permissible projective transformation T such that $T(S)$ is the vertex set of a convex polytope? Larman conjectured that $v(d) = 2d+1$ for all $d \geq 1$, which would imply $\gamma_d = 2d-1$ for all $d \geq 2$. Our lower bound of $d + \lceil \sqrt{2d} \rceil - 2$ on γ_d (also discovered by Las Vergnas [17]) improves Larman's upper bound on $v(d)$; our upper bound, $\gamma_d \leq 2d - 1$, gives $v(d) \geq 2d + 1$, which coincides with Larman's lower bound on $v(d)$. The relationship of Example 2 with this and other problems is discussed in Sections 2-4.

Some variations on Example 2 arise by restricting further the choice of \mathcal{P} , and by defining β to be the supremum of (1.1) over a special subset of \mathcal{H}_d , the generated hyperplanes. The chromatic number problem

arises in this way (see Section 6). This reinterpretation of Minty's paper [20] motivated this work.

Other variations on Example 2 arise by: re-defining $\beta(P)$ to be the minimum distance over all $P' \in \mathcal{N}(P)$ from the centroid of P' to the origin (this problem can be regarded to be a multi-dimensional generalization of the number partition problem [11,SP12]); or by re-defining $\mathcal{P}(d)$ to be ordered subsets of \mathbb{R}^d , permitting interchanges as well as reflections, and taking $\beta(P)$ to be the maximum of the distances from the centroid of each initial sequence to the origin. These variations are discussed in Section 7. They are related to [1,3,4,21,24-28].

Section 8 concerns several examples that either come from graphs, or, as in Example 1, employ graph-theoretic tools in the determination of a well-balanced neighbor.

The balancing problems addressed here are related, in spirit, to work in "discrepancy theory" (see [3,4,18,23-28] and their references), and, as noted in the relevant passages, some of the results are variations on earlier work.

It is assumed that the reader knows some elementary graph theory. The notation is mostly standard. Given a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ denote by $S^+(x)$ the positive support of x , $S^+(x) = \{j: x_j > 0\}$, by $S^-(x)$ the negative support of x , $S^-(x) = \{j: x_j < 0\}$, and by $\underline{S}(x)$ the support of x , $\underline{S}(x) = S^+(x) \cup S^-(x)$. Given $S \subseteq \mathbb{R}^n$, $\text{conv}(S)$ denotes the convex hull of S .

2. Balancing points in \mathbb{R}^d with respect to hyperplanes

This section elaborates on Example 2 of the introduction. There are several equivalent formulations of this problem, and, at times, it will be

convenient to appeal to formulations other than that given in the introduction. So we begin with a discussion of those formulations. Let the dimension d be fixed.

Sets of points in \mathbb{R}^d

Let $\mathcal{P}_1(d)$ be the set of all finite $P \subseteq \mathbb{R}^d$ satisfying the condition

$$(2.1) \quad |P \cap H| \leq |P| - 2 \quad \text{for every choice of } \vec{0} \neq h \in \mathbb{R}^d \\ \text{and } H = \{x \in \mathbb{R}^d: h^T x = \vec{0}\}.$$

Note that (2.1) implies that P is full-dimensional and nondegenerate.

Let β_1 be given by

$$(2.2) \quad \beta_1(P) = \sup\{(|P \cap H^+|/|P \cap H^-|): h \in \mathbb{R}^d, h \neq \vec{0}\} \quad (\forall P \in \mathcal{P}_1(d)),$$

where $H^+ = \{x \in \mathbb{R}^d: h^T x > 0\}$ and $H^- = \{x \in \mathbb{R}^d: h^T x < 0\}$. We interpret

(2.2) to mean $\beta_1(P) = +\infty$ if for some h , $H^- \cap P = \emptyset$. Although h ranges over $\mathbb{R}^d \setminus \{\vec{0}\}$ in (2.2), it should be clear that the supremum is attained. Finally for each $P \in \mathcal{P}_1(d)$ we define

$$(2.3) \quad \gamma_1(P) = \inf\{\beta_1(\epsilon P) | \epsilon: P \rightarrow \{-1, +1\}\}, \quad (\forall P \in \mathcal{P}_1(d))$$

and

$$(2.4) \quad \gamma(d) = \sup\{\gamma_1(P) | P \in \mathcal{P}_1(d)\}.$$

It is clear that the infimum in (2.3) is attained. That the supremum in (2.4) is attained will be established in Section 3.

Note that in our concern with $\gamma_1(P)$ and $\gamma(d)$, we could restrict $\mathcal{P}_1(d)$ to finite (multi-) sets on the unit sphere S^{d-1} in \mathbb{R}^d satisfying (2.1). Any $p \in P$ at the origin does not contribute to the numerator or denominator of any of the ratios in (2.2). Furthermore, $\gamma_1(P)$ is invariant under nonzero scaling of each $p \in P$.

(2.5) Theorem. For each $d > 1$, $\gamma(P) < |P| - 1$ for all $P \in \mathcal{P}_1(d)$.

Proof. Let $P = (p_1, \dots, p_n) \in \mathcal{P}_1(d)$.

Think of P as a $d \times n$ matrix with no column of all zeros. For each nonzero $h \in \mathbb{R}^d$, the ratio $|H^+ \cap P|/|H^- \cap P|$ is just the ratio of the number of positive entries, $|S^+(y)|$, to the number of negative entries $|S^-(y)|$ in the vector $y = h^T P$ in the row space $\mathcal{R}(A)$ of A .

Furthermore, every nonzero vector y in $\mathcal{R}(A)$ arises as $y = h^T A$ for some hyperplane $H = \{x \in \mathbb{R}^d: h^T x = \vec{0}\}$, and $|H^+ \cap P|/|H^- \cap P| = |S^+(y)|/|S^-(y)|$. The nondegeneracy assumption guarantees that there is some $z \in \mathbb{R}^n$ such that $Az = \vec{0}$ and $z_j \neq 0$, $j = 1, \dots, n$. Choosing $\epsilon(p_j) = +1$ if $z_j > 0$ and $\epsilon(p_j) = -1$ if $z_j < 0$ produces ϵP , which has a dependence relation $(\epsilon P)\hat{z} = \vec{0}$, where $\hat{z}_j = \epsilon(p_j)z_j > 0$ for $j = 1, \dots, n$. Therefore the only nonnegative vector in $\mathcal{R}(\epsilon P)$ is the zero vector. Hence $\gamma_1(P) \leq |P| - 1$ for all $P \in \mathcal{P}_1(d)$. \square

The approach employed in the proof of Theorem 2.5 leads immediately to a reformulation of the original problem involving $\mathcal{P}_1(d)$ to a problem involving d -dimensional subspaces of \mathbb{R}^n .

d-dimensional subspaces of Euclidean spaces

Let $\mathcal{P}_2(d)$ be the set of d-dimensional vector subspaces \mathcal{U} of Euclidean spaces \mathbb{R}^n , over all finite n , satisfying the nondegeneracy condition

$$(2.6) \quad |\{j: x_j \neq 0\}| \geq 2 \quad \text{for all } x \in \mathcal{U}, \quad x \neq \vec{0}.$$

Let $\beta_2: \mathcal{P}_2(d) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ be given by

$$\beta_2(P) = \sup\{|S^+(x)|/|S^-(x)|: x \in \mathcal{U}, \quad x \neq \vec{0}\}.$$

For $\mathcal{U} \in \mathcal{P}_2$ denote by $n(\mathcal{U})$ the number of coordinates on which \mathcal{U} is defined and, given $\epsilon: \{1, \dots, n(\mathcal{U})\} \rightarrow \{-1, +1\}$, let $\epsilon\mathcal{U}$ denote $\{(\epsilon(1)x_1, \dots, \epsilon(n)x_n) | (x_1, \dots, x_n) \in \mathcal{U}\} \in \mathcal{P}_2(d)$. Finally let $\gamma_2(\mathcal{U}) = \inf\{\beta_2(\epsilon\mathcal{U}) | \epsilon: \{1, \dots, n(\mathcal{U})\} \rightarrow \{-1, +1\}\}.$

The next reformulation essentially takes the balancing problem in $\mathcal{P}_1(d)$ and projects it into \mathbb{R}^{d-1} .

Signed Subsets of \mathbb{R}^{d-1}

A signed subset Q of \mathbb{R}^d can be regarded to be a map $Q: \mathbb{R}^n \rightarrow \{-1, 0, +1\}$. We denote by Q^+ and Q^- , respectively, the subsets $Q^+ = \{x \in \mathbb{R}^n: Q(x) = +1\}$, $Q^- = \{x \in \mathbb{R}^n: Q(x) = -1\}$, and by \underline{Q} the underlying set $\underline{Q} = Q^+ \cup Q^-$. Such a signed subset Q of \mathbb{R}^n is sometimes regarded to be a two-colored subset of \mathbb{R}^n ; the elements of Q^+ are, say, red, and the elements of Q^- are, say, blue. Say that Q is finite if \underline{Q} is finite.

Let $\mathcal{P}_3(d-1)$, $d \geq 2$, be the set of all finite signed subsets Q of \mathbb{R}^{d-1} satisfying

$$(2.7) \quad |Q \cap K| \leq |Q| - 2 \quad \forall \quad h \in \mathbb{R}^{d-1}, h \neq \vec{0}, \quad r \in \mathbb{R} \quad \text{and} \\ K = \{x \in \mathbb{R}^{d-1} \mid h^T x = r\}.$$

Each nonzero $h \in \mathbb{R}^{d-1}$ and $r \in \mathbb{R}$, define a hyperplane K as in (2.7). Let $K^+ = \{x \in \mathbb{R}^{d-1} : h^T x > r\}$, $K^- = \{x \in \mathbb{R}^{d-1} : h^T x < r\}$, and let \mathcal{K}_d be the set of all signed sets K that arise in this way. For each $Q \in \mathcal{P}_3(d-1)$ define

$$\beta_3(Q) = \sup\{|(K^+ \cap Q^+) \cup (K^- \cap Q^-)| / |(K^- \cap Q^+) \cup (K^+ \cap Q^-)| : K \in \mathcal{K}_d\}.$$

Given $\epsilon: Q \rightarrow \{-1, +1\}$, let ϵQ denote the signed subset of \mathbb{R}^d having $\epsilon Q = Q$ and $\epsilon Q(x) = \epsilon(x) \cdot Q(x)$ for all $x \in Q$. For each $Q \in \mathcal{P}_3(d-1)$ define $\gamma_3(Q) = \inf\{\beta_3(\epsilon Q) \mid \epsilon: Q \rightarrow \{-1, +1\}\}$. Note that while $\beta_3(Q)$ depends on the partition (Q^+, Q^-) of Q , $\gamma_3(Q)$ depends only on Q . It will occasionally be convenient to write $\gamma_3(Q)$ for $Q \in \mathcal{P}_3(d-1)$.

There are other natural reformulations, e.g., one that arises by interchanging the roles of points and hyperplanes in $\mathcal{P}_3(d)$. However, the three above will be adequate for present purposes.

$$(2.9) \quad \textbf{Proposition.} \quad (a) \quad \gamma(d) = \sup\{\gamma_2(P) : P \in \mathcal{P}_2(d)\}, \quad \text{for all } d \geq 1; \\ (b) \quad \gamma(d) = \sup\{\gamma_3(P) : P \in \mathcal{P}_3(d-1)\}, \quad \text{for all } d \geq 2.$$

A proof of (2.9a) follows from showing that: (1) there is a natural map associating with each $P_1 \in \mathcal{P}_1(d)$ some $P_2 \in \mathcal{P}_2(d)$ having $\beta_2(P_2) = \beta_1(P_1)$; (2) there is another natural map associating with each $P_2 \in \mathcal{P}_2(d)$ some $P_1 \in \mathcal{P}_1(d)$ having $\beta_1(P_1) = \beta_2(P_2)$; and (3) neighborliness is preserved under the maps of (1) and (2). Part (b) is established similarly. This time the maps take $P_1 \in \mathcal{P}_1(d)$ to some $P_3 \in \mathcal{P}_3(d-1)$, and $P_3 \in \mathcal{P}_3(d-1)$ to some $P_1 \in \mathcal{P}_1(d)$.

(a). Map $P_1 = (p_1, \dots, p_n) \in \mathcal{P}_1(d)$ to the vector subspace of \mathbb{R}^n generated by the rows of the $d \times n$ matrix (p_1, \dots, p_n) . Given $P_2 \in \mathcal{P}_2(d)$, represent P_2 as the row space of a $d \times |P_2|$ matrix A ; map P_2 to P_1 whose points are the columns of A .

(b). Given $P_1 \in \mathcal{P}_1(d)$ the associated $P_3 \in \mathcal{P}_3(d)$ arises by first choosing a hyperplane W in \mathbb{R}^d that does not contain the origin and is not parallel to any line determined by a nonzero vector of P_1 . Let W_0 be the translation of W so that W_0 contains the origin. Now map P_1 into W via the projection through the origin. That is, for each nonzero vector p of P_1 , let q be the intersection point of W and the line determined by p . Let Q be the set of those images. Partition $Q = P_3$ into P_3^+ and P_3^- as follows: $q \in P_3^+$ if the corresponding $p \in P_1$ is on the same side of W_0 as W , and $q \in P_3^-$ otherwise. Similarly, given $P_3 \in \mathcal{P}_3(d)$ we can, essentially, invert the procedure above to "lift" P_3 to a $P_1 \in \mathcal{P}_1(d)$ with $\beta_1(P_1) = \beta_3(P_3)$. Regard the linear space spanned by P_3 in \mathbb{R}^{d-1} to be an affine subspace W in \mathbb{R}^d that does not contain the origin. Let \hat{P}_3 be the subset of \mathbb{R}^d corresponding in this way to $P_3 \subseteq \mathbb{R}^{d-1}$. Let W_0 be the linear subspace obtained by translating

W so that it hits the origin. Now for each $q \in \hat{P}_3$ there are two points of intersection of S^{d-1} with the line in \mathbb{R}^d generated by q --take p to be the intersection point in the slice of \mathbb{R}^d between W and W_0 if $q \in P_3^-$, otherwise take p to be the other intersection point. The set of points p arising in this way from $q \in \hat{P}_3$ constitutes P_1 .

3. An upper bound on $\gamma(d)$

This section begins with a proof of

Proposition 3.1. For all $d \geq 1$, $\gamma(d) \leq 2d$.

Later the upper bound on $\gamma(d)$ will be improved to $2d-1$ for all $d \geq 2$.

As noted earlier, there is no loss of generality in restricting $\mathcal{P}_1(d)$ to subsets of S^{d-1} . The argument that we will give to establish an upper bound of $2d$ on $\gamma(d)$ is geometric in spirit, and is, perhaps, more easily visualized if one considers $P \in P_1(d)$ to be on the unit sphere.

First some notation. Let $\epsilon: P \rightarrow \{-1, +1\}$. Recall that ϵP denotes the set $\{\epsilon(p)p: p \in P\}$. For $S \subseteq P$ let $\epsilon S = \{\epsilon(p)p: p \in S\}$; we may employ this notation even when ϵ is not determined off of S .

Proof of (3.1).

Each minimal linearly dependent set S in \mathbb{R}^{d-1} can be regarded to be a $(|S|-1)$ -dimensional simplex. We use the term simplex S to mean the convex hull of the set S . The reflection operation permits us to rearrange S to ϵS so that the unique (up to nonzero scalar multiples) linear dependence relation on ϵS is positive everywhere. Geometrically, this means that the rearranged simplex, ϵS , contains the origin in its

relative interior, implying that every hyperplane $H \in \mathcal{H}_d$ either contains ϵS , or has at least one vertex of ϵS in each of H^+ and H^- . If S satisfies (2.1), then H cannot contain it, and $\gamma_1(S) \leq |S|-1$.

Obviously, if $P \in \mathcal{P}_1(d)$ is a disjoint union of simplexes S_1, \dots, S_s , then we can independently reflect a subset of each S_i so that the newly arranged ϵS_i contain the origin in their relative interiors, giving $\gamma_1(P) \leq \beta_1(\epsilon P) \leq \max\{|S_i|-1: 1 \leq i \leq s\} \leq d$.

Let $P \in \mathcal{P}_1(d)$. Even if the set P is not a disjoint union of simplexes, (2.1) guarantees that P contains at least one simplex S_1 . Clearly we can partition P as $P = S_1 \cup \dots \cup S_s \cup I$, where $s \geq 1$, each S_i is a simplex, and I is linearly independent. In each block S_i , fix $\epsilon(p)$ over $p \in S_i$ so that ϵS_i contains the origin in its relative interior. Equivalently, find a nontrivial dependence relation $\sum_{p \in S_i} x(p)p = \vec{0}$ and for each $p \in S_i$ set $\epsilon(p) = +1$ if $x(p) > 0$ and $\epsilon(p) = -1$ if $x(p) < 0$. This fixes $\epsilon(p)$ for all $p \in S_1 \cup \dots \cup S_s$.

Let H be a hyperplane $\{x \in \mathbb{R}^d: h^T x = \vec{0}\}$. It follows from the discussion above that for $i = 1, \dots, s$ either (a) $T = S_1 \cup \dots \cup S_s \subseteq H$ or (b) $\epsilon T \cap H^- \neq \emptyset$. If (b) holds, $|\epsilon T \cap H^+|/|\epsilon T \cap H^-| \leq d$, which implies that for arbitrary choices of $\epsilon(p)$, $p \in I$, $|\epsilon P \cap H^+|/|\epsilon P \cap H^-| \leq 2d$. So the determination of appropriate $\epsilon(p)$, $p \in I$, to guarantee $\beta_1(\epsilon P) \leq 2d$ depends only on consideration of those hyperplanes H containing every S_i , $i = 1, \dots, s$.

It now suffices to show that there is a choice of ϵ on I such that

$$(3.2a) \quad H = \{x \in \mathbb{R}^d: h^T x = \vec{0}\} \supseteq T \text{ implies}$$

$$(3.2b) \quad H^- \cap \epsilon I \neq \emptyset.$$

Note that the hypothesis in (3.2a), $H \supseteq T$, implies $|H^+ \cap \epsilon P| \leq |I| \leq d$, and (3.2b) then implies $|H^+ \cap \epsilon P|/|H^- \cap \epsilon P| \leq d-1$. By the same argument employed in the proof of Theorem 2.5, (3.2) is achieved by fixing each $\epsilon(p)$, $p \in I$, according to the sign of $z(p)$ in some vector z such that

$$(3.3) \quad Pz = \vec{0}, \text{ and } z(p) \neq 0 \quad \forall p \in I,$$

so

$$(3.4) \quad \forall p \in I, \epsilon(p) = +1 \text{ if } z(p) > 0 \text{ and } \epsilon(p) = -1 \text{ if } z(p) < 0.$$

It is clear from (2.1) that a dependence relation z as in (3.3) can be determined easily. \square

An implication of choosing ϵ on I as in (3.3) and (3.4) is that for every $p \in I$ there is some $S_p \subseteq \epsilon P$ such that simplex S_p contains the origin in its relative interior and has $\epsilon(p)p$ as one of its extreme points. However, in determining ϵ on I it is not necessary to attempt to determine such simplexes S_p directly. For example, it suffices to fix ϵ on I iteratively by finding any simplex S'_q containing a $q \in I$ that has not yet had $\epsilon(q)$ fixed, determining ϵ' on S'_q such that $\epsilon' S'_q$ contains the origin, and then fixing $\epsilon(p) = \epsilon'(p)$ on those p with $\epsilon(p)$ not previously fixed. This iterative process continues until $\epsilon(p)$ has been fixed on all $p \in I$.

For $d = 1$ the upper bound of $2d$ is achieved by taking P to be any three points on S^0 . For each of $d \geq 2$ we will prove, in a moment, that $\gamma(d) \leq 2d-1$. This upper bound is achieved for $d = 2, 3, 4$. The proof of the improved bound exploits the following theorem of Paul Camion.

(3.5) Theorem [7]. Let A be a $d \times n$ real matrix of rank d . There exists a nonsingular $d \times d$ matrix D such that DA contains a $d \times d$ nonsingular diagonal matrix and every column of DA is either nonnegative or nonpositive.

Proof. Select some $b \in \mathbb{R}^d$ in linearly general position with respect to the columns of A . Consider the affine space $\mathcal{S} = \{x \in \mathbb{R}^n : Ax = b\}$. The hyperplanes H_1, \dots, H_n given by $H_j = \{x \in \mathbb{R}^d : x_j = 0\}$ cut \mathcal{S} into cells. Let \mathcal{B} be the set of subsets B of $\{1, \dots, n\}$ such that the columns of A indexed by B are linearly independent and $|B| = d$. With each $B \in \mathcal{B}$ associate the vertex $x(B)$ in the arrangement of hyperplanes in \mathcal{S} determined uniquely by the conditions that $x_j = 0$ for all $j \notin B$. Some of these vertices $x(B^*)$ are extreme points of the convex hull of $\{x(B) : B \in \mathcal{B}\}$; let $\mathcal{B}^* \subseteq \mathcal{B}$ be the associated set of Camion bases. Now for $B \in \mathcal{B}^*$ consider the $d \times (n+1)$ matrix $(\bar{A}, \bar{b}) = A_B^{-1}(A, b)$, where $A_B = (A_{B_1}, \dots, A_{B_d})$. Observe that since b is in general position, \bar{b} is nonzero in every row. The extremality property $B \in \mathcal{B}^*$ implies that for every $1 \leq j \leq n$ either \bar{a}_{ij} agrees in sign with \bar{b}_i for $i = 1, \dots, d$, or \bar{a}_{ij} disagrees in sign with \bar{b}_i for $i = 1, \dots, d$. Let D be the matrix obtained from A_B^{-1} by negating those rows i of A_B^{-1} such that $\bar{b}_i < 0$. Then DA has the desired property. \square

Let $c = (c_1, \dots, c_d)^T$ have each $c_j = \sum_{i=1}^d D_{ij}$, where D is the matrix described in the proof above. Suppose that A contains no column $A_j = \vec{0}$. Then the projective transformation $T: \mathbb{R} \rightarrow \mathbb{R}^d$ given by $T(x) = Dx/c^T x$ is permissible for $\{A_1, \dots, A_n\}$ and takes every column of A to a nonnegative vector. In particular, it takes the columns corresponding to

B to the unit vectors e_1, \dots, e_d in \mathbb{R}^d . If A contains a column $A_j = \vec{0}$, then the same effect as above can be achieved by the projective transformation $T'(x) = (1+\gamma)Dx/(c^T x + \gamma)$, for any sufficiently small positive scalar γ . Thus we get

(3.6) Proposition. If P is a finite set of vectors in \mathbb{R}^d , and $\text{rank}(P) = d$, then there is a projective transformation T permissible for P such that $T(p) \in \mathbb{R}_+^d$ for all $p \in P$, and for some $p_1, \dots, p_d \in P$, $T(p_i)$ is the i th unit vector, $i = 1, \dots, d$.

Camion's Theorem (3.5) will be useful in proving.

(3.7) Theorem. For each $d \geq 2$, $\gamma(d) \leq 2d-1$.

Proof. It is sufficient to show that for each $P \in \mathcal{P}_1(d)$

(3.8) there exists an $\epsilon: P \rightarrow \{-1, +1\}$ such that $\beta_1(\epsilon P) \leq 2d-1$.

Let $n = |P|$. If $n < 2d+1$, then (3.8) is immediate from Theorem 2.5.

So, either (i) $n = 2d+1$, or (ii) $n > 2d+1$.

(i) Suppose $n = 2d+1$. If P contains a minimal linearly dependent subset $T \subseteq P$, $|T| \leq d$, take $S_1 = T$ in the construction outlined in the proof of (3.1); it follows that ϵ , as determined by that construction, has $\beta_1(\epsilon P) \leq 2d-1$. Therefore we may assume that

(3.9) P contains no linearly dependent subset of cardinality less than $d+1$.

Camion's Theorem (3.5) implies the existence of an $\epsilon: P \rightarrow \{-1, +1\}$ and a nonsingular submatrix D of P (regarded to be a $d \times n$ matrix) such that $D(\epsilon P)$ can be put in the form

$$\begin{array}{cccccccccccc} + & 0 & 0 & & 0 & + & & + & - & - & & - \\ 0 & + & 0 & \dots & 0 & + & \dots & + & - & - & \dots & - \\ 0 & 0 & + & & 0 & + & & + & - & - & & - \\ \vdots & & & & & & & & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & + & + & \dots & + & - & - & \dots & - \end{array}$$

after permutation of its columns. Let \bar{A} denote the permuted form of $D(\epsilon P)$ schematically depicted above. By (3.9) each of the d "nonbasic" entries in columns $d+1$ to n of \bar{A} is nonzero. The signs of the ϵ 's in the nonbasic columns $\bar{A}_{d+1}, \dots, \bar{A}_n$ are chosen so that

(3.10a) at least half of the $(d+1)$ nonbasic columns are negative; and

(3.10b) there exist $x_{d+1}, \dots, x_n > 0$ such that $\sum_{j=d+1}^n \bar{A}_j x_j = \vec{0}$.

This can be achieved because (3.9) implies that the $d+1$ nonbasic columns of DP form a minimal linearly dependent set in \mathbb{R}^d .

It is enough to show that

$$(3.11a) \quad |H^- \cap \epsilon P| \geq 2, \quad \forall H \in \mathcal{H}_d,$$

which is equivalent to the condition that

$$(3.11b) \quad h^T \bar{A} \text{ has at least two strictly negative entries.}$$

If $h \neq \vec{0}$ is nonnegative, then (3.11b) certainly holds, because then

$h^T \bar{A}_j < 0$, for each of the negative columns of \bar{A} , of which there are at least $\lceil \frac{1}{2}(d+1) \rceil \geq 2$. So, suppose h has at least one negative entry, say h_i . This implies that $h^T \bar{A}_i < 0$. Moreover, (3.10b) implies that at least one of $h^T \bar{A}_j < 0$, $d+1 \leq j \leq n$. For suppose $h^T \bar{A}_j \geq 0$, $j = d+1, \dots, n$. By (3.9) $h^T \bar{A}_k > 0$ for some $j \leq k \leq n$. So $\sum_{j=d+1}^n h^T \bar{A}_j x_j > 0$, contradicting $\sum_{j=d+1}^n \bar{A}_j x_j = \vec{0}$.

(ii) Now suppose that $n > 2d+1$. Consider again the construction used to prove Proposition 3.1. If all S_i have $|S_i| \leq d$, or if $|I| \leq d-1$, then the resultant ϵ gives $\beta_1(\epsilon P) \leq 2d-1$. So, assume that some S_i , say S_1 , has $|S_1| = d+1$, and $|I| = d$. Now apply the approach (i) above to fix ϵ on $P' = S_1 \cup I$, $|P'| = 2d+1$, while fixing ϵ on $S_2 \cup \dots \cup S_s$ as before. Then, again, $\beta_1(\epsilon P) \leq 2d-1$. \square

The upper bound $2d-1$ is tight for $d = 2, 3, 4$. In $d = 2$ any subset $P \subseteq S^1$ of four points in general position has $\gamma_1(P) = 3$. In $d = 3$ the set P of six column vectors of the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

has $\gamma_1(P) = 5$. The following configuration Q in \mathbb{R}^3 is due to Larman [16]. Take Q to be the set of eight points

$$\begin{aligned} q_i &= (\cos(2\pi_i/3), \sin(2\pi_i/3), 0), \quad i = 1, 2, 3; \\ q_4 &= (0, 0, \delta); \\ q_i &= (\cos(\pi/6), \sin(\pi/6), 0) + q_{i-4}, \quad i = 5, 6, 7; \\ q_8 &= (0, 0, 1+\delta), \end{aligned}$$

where $0 < \delta < 0.1$ is a fixed scalar. Let $P \in \mathcal{P}_3(3)$ be any signed subset of \mathbb{R}^3 having $P = Q$. Then $\gamma_3(P) = 7$, implying $\gamma(4) = 7$.

Larman [16] reports that McMullen raised the following problem. Determine the least integer $\mu(d)$ such that for every set P of $\mu(d)$ points in linearly general position on S^{d-1} there exists $\epsilon: P \rightarrow \{-1, +1\}$ such that

$$|H^- \cap \epsilon P| \geq 2 \quad \forall \quad H \in \mathcal{H}_d.$$

It follows from the proof of Theorem 3.7 (in particular, the verification of (3.11b) in the case $n = 2d+1$), that

$$(3.12) \quad \mu(d) \leq 2d+1.$$

McMullen's interest in the problem above was motivated by its connection to another problem he raised. Determine the greatest integer $v(d)$ such that for every set P of $v(d)$ points in affinely general position in \mathbb{R}^d there is a permissible projective transformation T with $T(P)$ the set of vertices of a convex polytope. Larman [16] reports that McMullen showed that

$$\mu(k) = \min\{w: w \leq v(w-k-1)\}$$

and

$$v(d) = \max\{w: w \geq \mu(w-d-1)\}.$$

From (3.12) it follows that for all $d \geq 2$

$$(3.13) \quad v(d) \geq 2d+1,$$

which was also proved by Larman [16], who conjectured that $v(d) = 2d+1$.

Proposition 3.6 yields a direct proof of (3.13). Suppose \hat{P} is a set of $2d+1$ points in affinely general position in \mathbb{R}^d . Let $P = \{(\hat{p}^t, 1)^t : \hat{p} \in \hat{P}\}$, which is in linearly general position in \mathbb{R}^{d+1} . Let T be a projective transformation as in (3.6), with $T(p_i) = e_i$, $i = 1, \dots, d+1$, and $T(p_i) \in \mathbb{R}_+^d$ for $i = d+2, \dots, 2d+1$; $T(x) = Dx/c^T x$, for appropriate D and c . Note that the general position assumption implies that every component of every $T(p_i)$, $d+2 \leq i \leq 2d+1$, is positive. Consider the $(d+1) \times d$ matrix $R = (T(p_{d+2}), \dots, T(p_{2d+1}))$. There is a dependence relation on the rows of R that is nonzero everywhere, otherwise, for some $1 \leq i \leq d+1$, the $(d+1) \times (d+1)$ submatrix (e_i, R) of $T(P)$ has rank d , contradicting the general position assumption. Let $y \in \mathbb{R}^{d+1}$ have $R^t y = \vec{0}$, with $y_j \neq 0$, $j = 1, \dots, d+1$. Consider a projective transformation taking $x \rightarrow T'(x) = x/\hat{c}^T x$, where $\hat{c}_j \gg 0$ if $y_j > 0$, and $\hat{c}_j < 0$ with $|\hat{c}_j|$ small if $y_j < 0$, $j = 1, \dots, d+1$. The projective transformation $T'(T(\cdot))$ from \mathbb{R}^{d+1} to \mathbb{R}^{d+1} induces a projective transformation \bar{T} from \mathbb{R}^d to \mathbb{R}^d with the desired property: $\bar{T}(\hat{P})$ is the vertex set of a convex polytope. The transformation \bar{T} is of the form

$$\bar{T}(x) = \frac{\bar{A}x + \bar{b}}{\bar{c}^T x + \gamma}$$

where \bar{A} is the upper $d \times d$ submatrix of D , \bar{b} is the upper $d \times 1$

subvector of the $(d+1)$ st column of D , \bar{c} consists of the first d entries of $\hat{c}^T D$, and $\bar{\gamma}$ is equal to the $(d+1)$ st entry of $\hat{c}^T D$.

The balancing problem that we have been considering has a natural generalization to oriented matroids (see [5]). The proof of (3.1) remains valid in this setting. Camion's Theorem has not been proved for general oriented matroids, so the proof above of $\gamma(d) \leq 2d-1$ does not extend. However, Cordovil and Silva [10] have proved (3.13) for the natural generalization of McMullen's problem to oriented matroids. This can be used in place of Camion's theorem to dispose of case (i) in the proof of Theorem 3.7. The remainder of the proof extends with no difficulty.

It seems likely that for all $d \geq 1$ there is a $P \in \mathcal{P}_1(d)$ such that $\gamma(d) = \gamma(P) = |P| - 1$. For the moment we can only prove.

(3.14) Lemma. Suppose $P \in \mathcal{P}_1(d)$ has $P = S \cup P'$, $S \cap P' = \emptyset$, and S is a nonempty minimal set of linearly dependent vectors. If P' is nondegenerate, then

$$\text{either } \gamma_1(P) \leq d, \text{ or } \gamma_1(P) \leq \gamma_1(P').$$

Proof. Without loss of generality, suppose the origin is contained in the relative interior of the convex hull of S , and P' is optimally arranged, i.e., $\beta_1(P') = \gamma_1(P')$. Let $H \in \mathcal{H}_d$ and let

$$r = |H^+ \cap P| / |H^- \cap P|.$$

If H contains S , then

$$|H^+ \cap P| / |H^- \cap P| = |H^+ \cap P'| / |H^- \cap P'|.$$

If H does not contain S , then

$$H^+ \cap P = H^+ \cap (P' \cup S), \quad |H^+ \cap P| \leq |H^+ \cap P'| + (|S| - 1),$$

and

$$|H^- \cap P| \geq |H^- \cap P'| + 1.$$

In both cases,

$$r \leq \max(d, \gamma_1(P')).$$

So

$$\beta_1(P) \leq \max(d, \gamma_1(P')).$$

This completes the proof since $\gamma_1(P) \leq \beta_1(P)$. □

From the lemma above, the following is obvious.

(3.15) Corollary. For a real number $\alpha > d$, let $P \in \mathcal{P}_1(d)$ be of minimum cardinality such that $\alpha \leq \gamma_1(P)$. If $P = S \cup P'$, $S \cap P' = \emptyset$, and S is a minimal set of linearly dependent vectors, then P' is degenerate.

(3.16) Theorem. For all $d \geq 1$, $\gamma(d)$ is attained as $\gamma(P)$ for some $P \in \mathcal{P}_1(d)$, $|P| \leq d^2/4 + 2d + 1$.

To prove (3.16) we need some preliminaries.

It is easy to see that

$$(3.17) \quad \gamma(d) > d \quad \text{for all } d \geq 1.$$

For example, in $d \geq 2$ take $P \in \mathcal{P}_1(d)$ to be any $d+2$ points of S^{d-1} in linearly general position; $\gamma(P) = d+1$.

The approach in the proof of Proposition 3.1 suggests

(3.18) Lemma. Let α be a real number, $\alpha > d$. Suppose P is a finite nondegenerate configuration in \mathbb{R}^d such that $\gamma_1(P) \geq \alpha$ and $|P|$ is as small as possible. Then

$$|P| \leq d^2/4 + 2d + 1.$$

Proof. Consider any partition

$$P = S_1 \cup S_2 \cup \dots \cup S_m \cup T$$

of P such that every S_i is a minimal linearly dependent set and T is linearly independent. Suppose for some $1 \leq i < m$,

$$(3.19) \quad \text{rank}(S_1 \cup \dots \cup S_i) = \text{rank}(S_1 \cup \dots \cup S_i \cup S_{i+1}).$$

Let $P' = P \setminus S_{i+1}$. Then, by Corollary 3.15, P' must be degenerate. That is, there is a hyperplane $H \in \mathcal{H}_d$ such that either H contains P' or H misses only one point of P' . If H contains P' , then, because of (3.19), H must contain P also. If H misses one point, say q , of P' , then, since every S_i is a minimal linearly dependent set, q cannot be a member of any S_i . So, $q \in T$ and the set $S_1 \cup \dots \cup S_i$ is contained in H . Then, again by (3.19) the set $S_1 \cup \dots \cup S_i \cup S_{i+1}$ is contained in H , and hence H misses only one point q of P . In both cases, P also must be degenerate which contradicts the nondegeneracy assumption in the hypothesis. Therefore, for every $1 \leq i < m$,

$$\text{rank}(S_1 \cup \dots \cup S_i) < \text{rank}(S_1 \cup \dots \cup S_i \cup S_{i+1}).$$

By considering all i , we get

$$(3.20) \quad \text{rank}(S_1) < \text{rank}(S_1 \cup S_2) < \dots < \text{rank}(S_1 \cup \dots \cup S_m) \leq d.$$

Note that the inequality (3.20) does not depend on a particular order of the S_i 's. In particular, let $\sigma = |S_1| \geq |S_i|$ for all i . Then

$$\sigma - 1 = \text{rank}(S_1) < \text{rank}(S_1 \cup S_2) < \dots < \text{rank}(S_1 \cup \dots \cup S_m) \leq d,$$

and hence $m \leq d - \sigma + 2$. Since the decomposition is pairwise disjoint,

$$\begin{aligned} |P| \leq m\sigma + |T| &\leq (d - \sigma + 2)\sigma + |T| = -\{\sigma - (d+2)/2\}^2 + \{(d+2)/2\}^2 + |T| \\ &\leq d^2/4 + 2d + 1. \end{aligned} \quad \square$$

Proof of Theorem 3.16. Consider the set

$$\Gamma_1 = \{\gamma_1(P) : \gamma_1(P) > d, P \in \mathcal{P}_1(d)\}.$$

Then, by Lemma 3.15 above, Γ_1 is equal to

$$\Gamma_2 = \{\gamma_1(P) : \gamma_1(P) > d, P \in \mathcal{P}_1(d), |P| \leq d^2/4 + 2d + 1\}.$$

Hence, every number in Γ_1 can be represented as a quotient of integers with both numerator and denominator bounded by $d^2/4 + 2d + 1$. Therefore Γ_1 is finite and

$$\sup\{\gamma_1(P) \mid \gamma_1(P) \in \Gamma_1\} = \max\{\gamma_1(P) \mid \gamma_1(P) \in \Gamma_2\},$$

which equals $\gamma(d)$ by (3.17). \square

4. A lower bound on $\gamma(d)$ from a graphic case

We have seen that $\gamma(1) = 2$; $\gamma(d) = 2d-1$ for $d = 2, 3, 4$;
 $\gamma(d) \leq 2d-1$ for $d \geq 2$; and that $\gamma(d) \geq d+1$ for all $d \geq 1$. In this
 section we will consider an example that arises from directed graphs and
 yields a better lower bound, $\gamma(d) \geq d + \lceil \sqrt{2d} \rceil - 2$.

First we show that γ is monotone increasing.

(4.1) Lemma. Let $P = (p_1, \dots, p_n) \in \mathcal{P}_1(d)$. Extend P to
 $\hat{P} = (\hat{p}_1, \dots, \hat{p}_n, \hat{p}_{n+1}) \in \mathcal{P}_1(d+1)$, where $\hat{p}_i = (p_i, 0)$, $i = 1, \dots, n-1$,
 $\hat{p}_n = (p_n + q, \delta)$ and $\hat{p}_{n+1} = (p_n + q, -\delta)$, $q \in \mathbb{R}^{d+1}$, $p_n + q$ is not in any
 subspace of \mathbb{R}^d generated by $(d-1)$ points in P , $\|q\|$ is small, and δ
 is a small positive scalar. Then $\gamma_1(\hat{P}) > \gamma_1(P)$.

Proof. It is easy to see that \hat{P} is nondegenerate and full-dimensional in
 \mathbb{R}^{d+1} . The argument below shows that for each $\hat{\epsilon}: \hat{P} \rightarrow \{-1, +1\}$ there
 corresponds an $\epsilon: P \rightarrow \{-1, +1\}$ with $\beta_1(\hat{\epsilon}\hat{P}) > \beta_1(\epsilon P)$.

Let $\hat{\epsilon}: \hat{P} \rightarrow \{-1, +1\}$. Fix ϵ so that $\epsilon(p_i) = \hat{\epsilon}(\hat{p}_i)$, $i = 1, \dots, n-1$,
 and $\epsilon(p_n) = \hat{\epsilon}(\hat{p}_n)$. Let $H = \{x \in \mathbb{R}^d: h^T x = 0\}$ be a hyperplane in \mathbb{R}^d
 such that $\beta_1(\epsilon P) = |H^+ \cap \epsilon P| / |H^- \cap \epsilon P|$. Let H_1 and H_2 denote the
 hyperplanes in \mathbb{R}^{d+1} with normals $h_1 = (0, \dots, 0, 1)^t$ and $h_2 = (h, 0)^t$,
 respectively. Note that there exists a hyperplane H_3 in \mathbb{R}^{d+1} having
 $\hat{p}_n \in H_3^+$, $\hat{p}_{n+1} \in H_3$ and $\hat{p}_j \in H_3$ for all $1 \leq j \leq n-1$ such that $p_j \in H$.

We will consider several cases, depending on the sign of $\hat{\epsilon}(\hat{p}_{n+1})$ and the disposition of p_n with respect to H . We will show in each case that

there exists a hyperplane $\hat{H} = \{x \in \mathbb{R}^{d+1} : \hat{h}^t x = \vec{0}\}$ such that

$$|\hat{H}^+ \cap \hat{\epsilon P}| / |\hat{H}^- \cap \hat{\epsilon P}| > \beta_1(\epsilon P).$$

(a) Suppose that $\hat{\epsilon}(\hat{p}_{n+1}) = \hat{\epsilon}(\hat{p}_n)$.

(1) If $\epsilon(p_n)p_n \in H^+$, then take $\hat{H} = H_2$.

(2) If $\epsilon(p_n)p_n \in H^-$, let $\hat{h} = (h_2 + \lambda h_1)$, for $\lambda > 0$ and large.

(3) If $\epsilon(p_n)p_n \in H$, let $\hat{h} = h_2 + \lambda h_3$, for $\lambda > 0$ and small.

(b) Suppose that $\hat{\epsilon}(\hat{p}_{n+1}) = -\hat{\epsilon}(\hat{p}_n)$.

(1) If $\epsilon(p_n)p_n \in H^+$, take $\hat{h} = h_2 + \lambda h_1$, for $\lambda > 0$ and large.

(2) If $\epsilon(p_n)p_n \in H^-$, take $\hat{h} = h_2 + \lambda h_1$, for $\lambda > 0$ and large.

(3) If $\epsilon(p_n)p_n \in H$, take $\hat{h} = h_2 + \lambda h_3$, for $\lambda > 0$ and small. \square

Lemma 4.1 immediately implies

(4.2) Claim. For all $d \geq 1$, $\gamma(d+1) > \gamma(d)$.

Now we will show that for d of the form $(r-1)(r-2)/2$, $r \geq 3$, there exists $P \in \mathcal{P}_1(d)$ with $\gamma_1(P) \geq d + \sqrt{2d} - 1$. This uses an easy graph theoretic construction.

First recall that the circuit space $\mathcal{C}(G)$ of a directed graph $G = (V, E)$ is the null space $\{x \in \mathbb{R}^E : Ax = \vec{0}\}$ of its $(0, \pm 1)$ -vertex-edge incidence matrix, A , and if G is connected, $\dim(\mathcal{C}(G)) = |E| - |V| + 1$.

It is very well known that for any acyclic directed graph $G = (V, E)$ there is an ordering of the vertices, say $V = \{v_1, \dots, v_m\}$ such that every edge is of the form (v_i, v_j) , $i < j$. This is true, of course, also for acyclic directed graphs with multiple edges. Suppose that G is an

acyclic orientation of K_m , the complete graph on $m \geq 3$ vertices (possibly with some multiple edges). Ordering the vertices as above, it is easy to see that there is a vector $x \in \mathcal{E}(G)$ such that $x(1,m) < 0$ and $x(i,j) > 0$ for all $(i,j) \in E(G)$, $(i,j) \neq (1,m)$. On the other hand, if H is a strongly connected directed graph, then clearly $\mathcal{E}(H)$ contains a vector x that is positive on every edge of H . Combining these observations yields:

(4.3) Lemma. For $r \geq 3$ and for every orientation G of K_r , there is a vector $x \in \mathcal{E}(G)$ such that $|S^+(x)| \geq (r(r-1)/2) - 1$ and $|S^-(x)| \leq 1$.

Sketch of proof. If G has one strong component the result is immediate. If G has strong components G_1, \dots, G_m , apply the first observation above to the acyclic complete digraph \hat{G} (with multiple edges) formed from G by shrinking each $V(G_i)$ to a single point \hat{v}_i . The vector $\hat{x} \in \mathcal{E}(\hat{G})$ that arises in this way enables us to easily construct a vector $x \in \mathcal{E}(G)$ that is positive on all but one of the edges not having both ends in the same strong component. (Note that for each pair of vertices in $V(G_i)$ there is a directed path in G_i between them.) Now add to x an everywhere positive vector $x^i \in \mathcal{E}(G_i)$, $i = 1, \dots, m$. \square

(4.4) Corollary. Suppose A is a matrix such that the row vectors form a basis of $\mathcal{E}(K_r)$, $r \geq 3$. Let P be the set of column vectors of A . Then $P \in \mathcal{P}_1(d)$, $d = (r-1)(r-2)/2$, $|P| = r(r-1)/2$, and, for every $\epsilon: P \rightarrow \{-1, +1\}$, there is a hyperplane H through the origin such that

$$|H^+ \cap \epsilon P| = |P|, \text{ or } |H^+ \cap \epsilon P| = |P| - 1 \text{ and } |H^- \cap \epsilon P| = 1.$$

(4.5) Theorem. For every $d \geq 1$, there is a finite configuration $P \in \mathcal{P}_1(d)$ such that

$$|P| > d + (2d)^{1/2} - 1 \quad \text{and} \quad \gamma_1(P) = |P| - 1.$$

Hence, for all $d \geq 1$,

$$\gamma(d) \geq d + \lceil (2d)^{1/2} \rceil - 2.$$

Furthermore, if d is of the form $d = (r-1)(r-2)/2$, $r \geq 3$ an integer, then

$$|P| > d + (2d)^{1/2},$$

so $\gamma(d) \geq d + \lceil (2d)^{1/2} \rceil - 1$.

Proof. Suppose $d = (r-1)(r-2)/2$, $r \geq 3$ an integer. Let P be as in Corollary (4.4). Then

$$|P| = r(r-1)/2 = (r-1)(r-2)/2 + (r-1) > d + (2d)^{1/2},$$

and clearly $\gamma_1(P) = |P| - 1$.

For arbitrary $d \geq 1$, let r be the greatest integer such that

$$(r-1)(r-2)/2 \leq d,$$

and let

$$(r-1)(r-2)/2 + h = d < r(r-1)/2.$$

Then by applying Lemma 4.1 to P h times, we get $\tilde{P} \in \mathcal{P}_1(d)$ such that $|\tilde{P}| = h+r(r-1)/2 = d + r - 1 > d + \sqrt{2d} - 1$. Note that for $r \geq 3$, $(|\tilde{P}| - 2)/2 < |P| - 1$, and by Lemma 4.1, $\gamma_1(\tilde{P}) > \gamma_1(P)$. Hence $\gamma_1(\tilde{P}) = |\tilde{P}| - 1$. □

Lemma 4.3 provides an upper bound on $v(d)$, defined in Section 3. For any $d \geq 1$ let G_{d+2} be an orientation of K_{d+2} . Let $n = \frac{1}{2}(d+1)(d+2)$ and let $P^1 = (p_1^1, \dots, p_n^1)$ be a $(d+1) \times n$ matrix having $\mathcal{C}(G_{d+2})$ as its null space $n.s.(P^1) = \{x \in \mathbb{R}^n: P^1 x = \vec{0}\}$. By Lemma 4.3 this choice of $P = P^1$ has the property:

$$(4.6) \quad \forall \epsilon_1, \dots, \epsilon_n \in \{-1, +1\} \quad \exists x \in n.s.(eP) \quad \text{such that} \\ |S^+(x)| = n, \quad \text{or} \quad |S^+(x)| = n-1 \quad \text{and} \quad |S^-(x)| = 1.$$

Here eP denotes the matrix $(\epsilon_1 p_1^1, \dots, \epsilon_n p_n^1)$.

Let $P^2 = (p_1^2, \dots, p_n^2)$ be obtained from P^1 by a projective transformation T^1 of the form $T^1(x) = Ax/c^T x$, where

$$A = \left[\begin{array}{cccc} 1 & & \bigcirc & 0 \\ & \cdot & & \vdots \\ & & \cdot & \\ \bigcirc & & & 1 \\ \hline & & & 0 \\ & & c^T & \end{array} \right]$$

for any choice of c not in the union of the n d -dimensional linear subspaces of \mathbb{R}^{d+1} orthogonal to the p_i^2 's. Note that the $(d+1)$ st row of $P^2 = T^1(P^1)$ has all its entries equal to one. Since the numerator of T^1 is linear, property (4.6) still holds for P^2 .

Now let P^3 be obtained from P^2 by small perturbations while maintaining the $(d+1)$ st row of ones, so that in

$$P^3 = \begin{bmatrix} \bar{P}^3 \\ 1 \dots 1 \end{bmatrix}$$

the n points of $\bar{P}^3 \subseteq \mathbb{R}^d$ are in affinely general position. Note that under sufficiently small perturbations property (4.6) will be satisfied by P^3 .

Suppose there is a projective transformation \bar{T} in \mathbb{R}^d permissible for \bar{P}^3 such that $\bar{T}(\bar{P}^3)$ is the vertex set of a convex polytope. Let $\bar{T}(x) = (Ax+b)/(c^T x + \gamma)$. Then $T: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$ given by

$$T(x) = \begin{bmatrix} A & b \\ c^T & \gamma \end{bmatrix} x / (c^T x + \gamma) \text{ has } T(P^3) = \begin{bmatrix} \bar{T}(P^3) \\ 1 \dots 1 \end{bmatrix}. \text{ Hence } T(P^3) \text{ can}$$

have no vector in its null space with exactly one negative entry. However, P^3 has property (4.6), which is preserved under T , a contradiction.

Therefore $v(d) < \frac{1}{2} (d+1)(d+2)$. Las Vergnas [17] also derived this upper bound on $v(d)$ by, essentially, the same approach. This improves upon Larman's upper bound $v(d) \leq (d+1)^2$.

The cocircuit space $\mathcal{C}^*(G)$ of a directed graph $G = (V, E)$ is the row space of the $(0, \pm 1)$ -vertex-edge incidence matrix of G . In contrast to the results above concerning circuit spaces of orientations of complete graphs we have the following results on cocircuit spaces, which are proved in [9].

(4.8) Theorem. For $r > 2$ and G an orientation of K_r , $\gamma_2(\mathcal{C}^*(G)) \leq 3$ if r is odd and $\gamma_2(\mathcal{C}^*(G)) \leq 3 + r/2$ if r is even.

(4.9) Theorem. If the directed graph G is bridgeless, then

$$\gamma_2(\mathcal{C}^*(G)) \leq \frac{3}{2} |V| - 1.$$

5. Asymptotic behavior

Suppose $P \in \mathcal{P}_1(2)$ consists of a large number of distinct points on S^1 . Then it is easy to see that $\gamma(P)$ is close to 1. There is no loss of generality in assuming that P is in the upper semicircle, since we can achieve this with the choice of ϵ . Now reflect every second point as the semicircle is traversed from one end to the other. The resulting ϵ has

$$||H^+ \cap \epsilon P| - |H^- \cap \epsilon P|| \leq 2$$

and

$$|H \cap P| \leq 2$$

so

$$|H^+ \cap \epsilon P| / |H^- \cap \epsilon P| \approx 1$$

if P is large. In particular if $P_i = \{p_1, \dots, p_i\} \in \mathcal{P}_1(2)$,

$i = j \geq 3, j+1, j+2, \dots$, and the $|P_i|$ points of each P_i are distinct, then $\gamma_1(P_i) \rightarrow 1$. If we allowed repeated points in the P_i 's, we could get something like $P_3 = \{p_1, p_2, p_3\}$ with $\gamma_1(P_3) = 2$, and $p_4 \notin P_3$, $p_i = p_4$ for all $i \geq 4$, with $\gamma(P_i) = 2$ for all $i = 4, 5, \dots$, as in Figure 5.1.

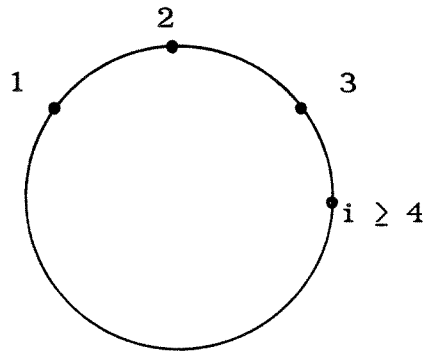


Figure 5.1

We can still get $\gamma(P_i) \rightarrow 1$ without requiring that the points are distinct, or even that they are in general position. It is enough to require that

$$(5.1) \quad \text{for every hyperplane } H \in \mathcal{H}_2, \quad |P \setminus H| \rightarrow \infty.$$

This condition may be sufficient for general d , but at this time we are only able to show that a condition that is intermediate in strength between (5.1) and general position suffices.

(5.2) Theorem. Let $\{P_i\}$ be a sequence of $P_i \in \mathcal{P}_1(d)$ such that $|P_i| \rightarrow \infty$. If there is a positive constant $\beta < 1$ such that for every $H \in \mathcal{H}_d$

$$|H \cap P_i| \leq \beta |P_i| \quad \text{for all } i, \quad \text{then } \gamma_1(P_i) \rightarrow 1.$$

The proof of (5.2) is probabilistic and follows an idea of Joel Spencer [29]. Like the demonstration above for $d = 2$, it first establishes an upper bound on $\inf_{\epsilon \in \mathcal{H}_d} \sup \{ ||H^+ \cap \epsilon P| - |H^- \cap \epsilon P| | \}$ in terms of $|P|$. Some preliminaries are required.

First recall Buck's result on the number of regions determined by n hyperplanes in \mathbb{R}^d .

(5.3) Lemma [6]. Suppose there are n hyperplanes in \mathbb{R}^d in general position. Let $R(d,n)$ be the number of regions and let $B(d,n)$ be the number of bounded regions determined by those n hyperplanes. Then

$$R(d,n) = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{d}.$$

and

$$B(d,n) = \binom{n-1}{d}.$$

Here we take $\binom{m}{i} = 0$ if $m < i$.

(5.4) Corollary. For the unit d -sphere S^d in \mathbb{R}^{d+1} , let $C(d,n)$ be the number of regions determined by n hyperplanes through the center of S^d in general position. Then

$$C(d,n) = R(d,n) + B(d,n),$$

where $R(d,n)$ and $B(d,n)$ are as in Lemma 5.3.

Proof. Consider a hyperplane H which does not meet with S^d and is not parallel to any of the given hyperplanes. Now project the sphere onto H through the center of S^d and count the number of regions to get the result. □

For a given finite subset P of S^d , define an equivalence relation on the set \mathcal{H}_d of all hyperplanes through the origin as follows:

$$(5.5) \quad \begin{aligned} H \text{ and } G \text{ are } \underline{\text{equivalent}} \text{ if and only if} \\ H^+ \cap P = G^+ \cap P \text{ and } H^- \cap P = G^- \cap P. \end{aligned}$$

(5.6) Corollary. For $d \geq 1$, the number of equivalence classes is no greater than $(ed+6)n^d$.

Proof. Let H be defined by $h^t x = 0$. A point $p \in P$ is in H^+ if and only if $h^t p > 0$. Hence if we define a hyperplane H_p for each $p \in P$ by $p^t x = 0$, then

$$p \in H^+ \text{ if and only if } h \in H_p^+.$$

Therefore the hyperplanes H_p , $p \in P$, determine the equivalence classes. In other words, if we consider H as a point h of the d -sphere, then the cell containing h among those cells determined by H_p 's, corresponds to the equivalence class of H . The cells are not necessarily full dimensional. Let $0 \leq r \leq d$. The r -dimensional cells arise by intersecting S^d with $d-r$ of the hyperplanes, to get an r -sphere, in which the remaining $n-d+r$ hyperplanes determine r -cells. The total number of r -cells is no greater than

$$\binom{n}{d-r} C(r, n-d+r),$$

because the number of cells is greatest when the hyperplanes are in general position. Therefore the total number of cells is bounded by

$$\begin{aligned} & \sum_{r=0}^d \binom{n}{d-r} \left\{ \binom{n-d+r-1}{r} + \sum_{i=0}^r \binom{n-d+r}{i} \right\} \\ &= \sum_{r=0}^d \frac{n!}{(d-r)!(n-d+r)r!(n-d-1)!} + \sum_{r=0}^d \sum_{i=0}^r \frac{n!}{(d-r)!i!(n-d+r-i)!} \\ &\leq \frac{n!}{d!(n-d-1)!} \sum_{r=0}^d \binom{d}{r} \frac{1}{n-d+r} + \sum_{r=0}^d \sum_{i=0}^r \frac{n!}{(d-r)!(n-d)!} \\ &\leq \binom{n}{d} (n-d) 2^d \frac{1}{n-d} + \frac{(d+1)n!}{(n-d)!} \sum_{r=0}^d \frac{1}{(d-r)!} \\ &< 2n^d + e(d+1)n^d \\ &< (ed + 6)n^d. \end{aligned}$$

□

From the corollary above, we can see that if V is a d -dimensional subspace of \mathbb{R}^n then the total number of signed supports of the vectors in V is bounded by $(ed+6)n^d$.

We need some probability. Let Y_1, Y_2, \dots, Y_m be

(5.7) independent random variables, each equal to $+1$
with probability $1/2$, or -1 with probability $1/2$.

Let

$$(5.8) \quad U_m = Y_1 + \dots + Y_m.$$

Then the following is well known. A proof is given in the appendix of [26].

(5.9) Lemma. For every positive real number α ,

$$\Pr(|U_m| > \alpha) \leq 2 \exp(-\alpha^2/2m).$$

Now we can prove

(5.10) Theorem. For every subset P of \mathbb{R}^d , $|P| = n$, there exists an ϵ such that

$$||H^+ \cap \epsilon P| - |H^- \cap \epsilon P|| \leq (2n \log(12n^{d-1}))^{1/2}$$

for all hyperplanes $H \in \mathcal{H}_d$.

Proof. Let $\epsilon(p_j)$, $j = 1, \dots, n$, be independent random variables as in (5.7). Let r be the number of equivalence classes as in (5.5), and let H_1, \dots, H_r be their representatives. For any H_i , $1 \leq i \leq r$, let $m = n - |H_i \cap P|$. Then the variables

$$|H_i^+ \cap \epsilon(p)p| - |H_i^- \cap \epsilon(p)p|, \quad p \in P \setminus H_i,$$

are also independent random variables as in (5.7). Hence

$$U^i = |H_i^+ \cap \epsilon P| - |H_i^- \cap \epsilon P|$$

is a random variable as in (5.8). So by Lemma (5.9)

$$\Pr(|U^i| > \alpha) \leq 2 \exp(-\alpha^2/2m) \leq 2 \exp(-\alpha^2/2n)$$

for all $\alpha > 0$. In particular, let $\alpha = (2n \log 2r)^{1/2}$, then we have

$$\Pr(|U^i| > \alpha) \leq r^{-1}$$

for all i . Note that this inequality is strict for those i with $|H_i \cap P| \neq 0$. Therefore,

$$\Pr(|U^i| > \alpha \text{ for some } i) < r \cdot r^{-1} = 1.$$

Hence

$$\Pr(|U^i| \leq \alpha \text{ for all } i) > 0.$$

Thus there exists some ϵ so that

$$||H_i^+ \cap \epsilon P| - |H_i^- \cap \epsilon P|| \leq \alpha$$

for all i . Note that $\alpha = (2n \log 2r)^{1/2}$. So

$$\begin{aligned} ||H_i^+ \cap \epsilon P| - |H_i^- \cap \epsilon P|| &\leq (2n \log 2r)^{1/2} \\ &\leq (2n \log(12n^{d-1}))^{1/2} \end{aligned}$$

by the inequality $r \leq 6dn^{d-1}$, which follows from Corollary (5.6). \square

Theorem 5.2 follows.

Spencer [29] gave a theorem of the same form as Theorem 5.10 based upon a looser estimate than Corollary 5.6 of the number of equivalence classes of hyperplanes.

Note that the hypothesis of (5.2) is satisfied if the points of each P_i are in general position. So if $P \in \mathcal{P}_1(d)$ is in general position and $|P| \gg 0$, then for some $\epsilon: P \rightarrow \{-1, +1\}$, $\beta_1(\epsilon P) \approx 1$. However, we know of no efficient deterministic algorithm for finding such an ϵ .

6. Two variations from graph theory

The balancing problem examined in the previous sections has a natural variation that arises by replacing $\beta_1(P)$, $P \in \mathbb{R}^d$, by $\beta'_1(P)$, defined like $\beta_1(P)$, except that the supremum is taken over generated hyperplanes only, i.e.

$$\beta'_1(P) = \sup\{|H^+ \cap P|/|H^- \cap P| : H \in \mathcal{H}(P)\},$$

where $\mathcal{H}(P)$ is the set of $(d-1)$ -dimensional subspaces of \mathbb{R}^d generated by a subset of P .

Example. Let $d = 3$ and $P = \{(1, -1, 0), (0, 1, -1), (0, 0, 1), (-1, 0, 0)\} \in \mathcal{P}_1(3)$. Then $\mathcal{H}(P)$ has six hyperplanes H , each with $|H^+ \cap P| = |H^- \cap P| = 1$; hence $\beta'_1(P) = 1$. However $\beta_1(P) = 3$; the hyperplane H with normal vector $h = (3, 2, 1)$ has $|H^+ \cap P| = 3$ and $|H^- \cap P| = 1$.

In a vector space \mathcal{U} a nonzero vector $y \in \mathcal{U}$ is elementary if there is no nonzero $z \in \mathcal{U}$ with $\underline{S}(z) \subsetneq \underline{S}(y)$. The choice of β'_2 on $\mathcal{P}_2(d)$

that corresponds to β'_1 on $\mathcal{P}_1(d)$ is $\beta'_2(\mathcal{U}) = \sup\{|S^+(y)|/|S^-(y)| : y \text{ an elementary vector of } \mathcal{U}\}$.

Other natural variations on the balancing problem arise by replacing $\mathcal{P}_1(d)$ (or, equivalently, $\mathcal{P}_2(d)$) by some $\mathcal{P}'_1(d) \subseteq \mathcal{P}_1(d)$ (or, $\mathcal{P}'_2(d) \subseteq \mathcal{P}_2(d)$) with special structure. For example, let $\mathcal{P}_2^G(d)$ be the set of $\mathcal{U}(G) \in \mathcal{P}_2(d)$ that arise as the circuit space of a directed graph. For a given directed graph $G = (V, E)$, with $E = \{e_1, \dots, e_m\}$, $P = \{p_1, \dots, p_m\}$ represents $\mathcal{U}(G)$ if the matrix (p_1, \dots, p_m) is of full row rank and its row space is $\mathcal{U}(G)$. Analogous to $\mathcal{P}_2^G(d)$ above is $\mathcal{P}_1^G(d)$, the set of all $P \subseteq \mathbb{R}^d$ such that P represents some $\mathcal{U}(G) \in \mathcal{P}_1(d)$. The nondegeneracy assumption (2.1) corresponds to G being loopless. Furthermore for any $\epsilon: P \rightarrow \{-1, +1\}$, ϵP represents the directed graph G' arising from G by reversing the orientation of those edges e_j of G for which $\epsilon(p_j) = -1$.

The more general balancing problem discussed in Sections 1-5 was motivated by the following theorem.

(6.1) Theorem. Suppose G is a directed graph with no loops and P represents G . Then

$$1 + \lceil \inf\{\beta'_2(\epsilon P) \mid \epsilon: P \rightarrow \{-1, +1\}\} \rceil$$

is the chromatic number of G .

This theorem is a reinterpretation of Minty's

(6.2) Theorem [20]. A loopless graph G is k -colorable if and only if for some orientation of G every circuit has at most $(k-1)$ times as many forward edges as reverse edges.

Theorem 6.1 follows from Theorem 6.2 and the fact that the elementary vectors in the circuit space $\mathcal{C}(G)$ of a directed graph G are just nonzero scalar multiples of signed incidence vectors of circuits of G .

Related to the graphic spaces $\mathcal{P}_2^G(\)$ are the cographic spaces $\mathcal{P}_2^C(\)$. Let $\mathcal{P}_2^C(d)$ be the set of $\mathcal{C}^*(G) \in \mathcal{P}_2(d)$ that arise as the cocircuit space of a directed graph. Define $\mathcal{P}_1^C(d)$ to be the set of sets P that represent some $\mathcal{C}^*(G) \in \mathcal{P}_1(d)$. The nondegeneracy assumption here corresponds to G being bridgeless, and, again, setting $\epsilon(p) = -1$ corresponds to reversing the orientation of the edge of G represented by p . For $P \in \mathcal{P}_2^C(d)$ and an integer $k \geq 1$,

$$(6.3) \quad \beta'_1(P) \leq k$$

corresponds to the condition that for every cocircuit of G the ratio of forward to reverse edges is at most k . By Hoffman's Circulation Theorem [13] and the integrality property of flows, (6.3) holds if and only if there is an integer flow $f: E \rightarrow \mathbb{Z}$ such that flow is conserved at every $v \in V$ (flow in = flow out), and $1 \leq x(e) \leq k$ for all $e \in E$. Therefore

$$(6.4) \quad 1 + \lceil \inf\{\beta'_1(\epsilon P) \mid \epsilon: P \rightarrow \{-1, +1\}\} \rceil$$

is just $\chi^*(G)$, the least integer j such that there exists an integer flow $f: E \rightarrow \mathbb{Z}$ that is conserved at every vertex, and $1 \leq |f(e)| \leq j-1$ on every edge $e \in E$. Tutte [30] studied χ^* and conjectured that $\chi^*(G) \leq 5$ for every bridgeless graph; Jaeger [14] proved $\chi^*(G) \leq 8$ and Seymour [22] improved this to $\chi^*(G) \leq 6$. The Petersen graph G_{10} has

$\chi^*(G_{10}) = 5$. Note that χ^* is, in a sense, dual to the chromatic number. If G is planar and G^* is dual to G , then $\chi(G) = \chi^*(G^*)$ and $\chi^*(G) = \chi(G^*)$.

7. Balancing with respect to centroids

Sections 1-6 concerned balancing problems in which the balance indicators were defined in terms of hyperplanes. This section examines some other indicators. Suppose P is a finite configuration of points of the unit ball B^d in \mathbb{R}^d . The distance of the centroid of P from the origin is a natural indicator of balance. In some problems where an ordering of the elements of the set P is important, the maximum of the distances from the origin to the centroids of all initial sequences is considered.

Suppose P is a finite subset of the unit ball B^d in \mathbb{R}^d . We want to find an $\epsilon: P \rightarrow \{-1, +1\}$ such that the centroid $(1/|P|)\sum \epsilon P$ of ϵP is close to the origin. Since the constant $1/|P|$ has no effect on measuring the relative balance of ϵP for different ϵ 's, the problem is equivalent to choosing ϵ to make the norm of the signed sum $\sum \epsilon P$ small.

Determining whether a specified value can be achieved is very hard, even for $d = 1$, because for $d = 1$ this problem is equivalent to the partition problem, which is known to be NP-complete. However, for any norm in \mathbb{R}^d there is an efficient algorithm that determines an ϵ and an order on P such that all the partial sums have norm no greater than d . For the Euclidean norm this can be improved so that the norm of the entire sum is at most \sqrt{d} , which is best possible.

In the following discussion, the norm will be Euclidean if it is not specified otherwise.

For a set $\{p_1, p_2, \dots, p_d\}$ of linearly independent vectors, the polytope

$$K = \{\sum t_i p_i : -1 \leq t_i \leq +1 \text{ for all } 1 \leq i \leq d\}$$

is called a d -dimensional parallelepiped.

(7.1) Lemma. If $\|p_i\| \leq 1$ for all $1 \leq i \leq d$, then, for any point u of the parallelepiped K , there is a vertex v of K within distance $d^{1/2}$ from u .

Proof. It is clear for $d = 1$. So suppose the lemma is true for all dimensions lower than d . If the given point u is on the boundary of K , then, by the induction hypothesis, we can find an appropriate vertex. If u is an interior point of K , then find a point u' on the boundary of K such that $\|u - u'\|$ is minimized. Then the vector $u - u'$ is orthogonal to the facet of K containing u' and $\|u - u'\| \leq 1$. This point u' can be considered to arise as follows: take a small sphere contained in the interior of K with center at u and expand it until it touches the boundary of K , then u' is the contact point. Point u' is on a facet of K and hence, by the induction hypothesis, there is a vertex v of the facet within distance $(d-1)^{1/2}$ from u' . By Pythagoras' Theorem, the vertex v is within distance $d^{1/2}$ from u . \square

Note that Lemma 7.1 is not true if the parallelepiped is degenerate, i.e., the vectors p_1, p_2, \dots, p_d are linearly dependent. For example, take, $d = 1$, $p_1 = p_2 = (1)$ and $u = 0$, then $K = [-2, +2]$. There is no vertex within distance $2^{1/2}$ from u . But if we consider the point $p_1 - p_2$ as a 'vertex' of K , then the assertion is true.

The following theorem was proved by Spencer [24] probabilistically.

We will give a constructive proof.

(7.2) Theorem [24]. If P is a finite subset of the unit ball B^d in \mathbb{R}^d , then there is a map $\epsilon: P \rightarrow \{-1, +1\}$ such that $\|\sum \epsilon(p)p\| \leq d^{1/2}$.

Proof. Consider polytope $Q = \{\sum_{p \in P} \epsilon(p)p \mid \epsilon: P \rightarrow \mathbb{R}, -1 \leq \epsilon(p) \leq 1 \ \forall p \in P\} \subseteq \mathbb{R}^d$. Clearly $\vec{0} \in Q$. Hence there must be an $\hat{\epsilon}: P \rightarrow \mathbb{R}$ such that

$$(7.3) \quad \begin{cases} \sum_{p \in P} \hat{\epsilon}(p)p = \vec{0}; \\ -1 \leq \hat{\epsilon}(p) \leq 1 \quad (\forall p \in P); \\ \{p \in P: |\hat{\epsilon}(p)| \neq 1\} \text{ is linearly independent.} \end{cases}$$

It is convenient to order the elements of P as $\{p_1, \dots, p_n\}$ so that we can think of P as a $d \times n$ matrix with $\{p_{n-d+1}, \dots, p_n\}$ linearly independent and $\hat{\epsilon}_j = \hat{\epsilon}(p_j)$ satisfying $\hat{\epsilon}_j = \pm 1$, $j = 1, \dots, n-d$. Let $u = -\sum_{j=1}^{n-d} \hat{\epsilon}_j p_j$. Since $\hat{\epsilon}_1 p_1 + \dots + \hat{\epsilon}_n p_n = \vec{0}$, $u = \sum_{j=n-d+1}^n \hat{\epsilon}_j p_j$, so u is in the parallelepiped $K = \{\sum_{i=n-d+1}^n \epsilon_i p_i: -1 \leq \epsilon_i \leq 1, n-d+1 \leq i \leq n\}$. By Lemma 7.1 K has a vertex v with $\|u-v\| \leq d^{1/2}$. Write v as $\sum_{i=n-d+1}^n \epsilon_i^* p_i$, where $\epsilon_i^* = \pm 1$, $n-d+1 \leq i \leq n$. Let $\epsilon_j = \hat{\epsilon}_j$, $1 \leq j \leq n-d$, and $\epsilon_j = \epsilon_j^*$, $n-d+1 \leq j \leq n$. Note that all $\epsilon_j = \pm 1$ and $\sum_{j=1}^n \epsilon_j p_j = v-u$, so $\|\sum_{j=1}^n \epsilon_j p_j\| \leq d^{1/2}$. \square

This proof is constructive in that it is easy to produce $\hat{\epsilon}$ as in (7.3), which determines all but d of the ϵ_j 's, and the construction

implicit in the proof of Lemma 7.1 fixes the rest. In fact one can prove a stronger result.

(7.4) Theorem. If P is a finite subset of B^d in \mathbb{R}^d , then there are an ϵ and an ordering on P such that

$$\|\sum \epsilon P\| \leq d^{1/2}, \quad \text{and} \quad \|\epsilon_1 p_1 + \epsilon_2 p_2 + \dots + \epsilon_k p_k\| \leq d$$

for all $k \leq |P|$.

First we will show how iterative construction of $\hat{\epsilon}$, in the proof of (7.2), leads to an ordering and an $\epsilon: P \rightarrow \{-1, +1\}$ such that

$$(7.5) \quad \|\epsilon p_1 + \dots + \epsilon_k p_k\| \leq d$$

for all $k \leq n-d$. Then we will address the cases $n-d < k \leq n$. At each stage of this iterative construction we will have an ordering $\{p_1, \dots, p_n\}$, an $\epsilon: P \rightarrow \{-1, +1\}$, and an integer $1 \leq k \leq n$, such that

$$(7.6a) \quad \epsilon_1 p_1 + \dots + \epsilon_n p_n = \vec{0};$$

$$(7.6b) \quad -1 \leq \epsilon_j \leq +1 \quad (j = 1, \dots, n);$$

$$(7.6c) \quad \epsilon_j = \pm 1 \quad (j = 1, \dots, k) \quad \text{and} \quad -1 < \epsilon_j < 1 \quad (j = k+1, \dots, n);$$

$$(7.6d) \quad F = \{p_j: 0 < |\epsilon_j| < 1\} \quad \text{is linearly independent.}$$

Note that k is just the number of entries of magnitude one in ϵ . At each iteration k is increased strictly, until termination, which occurs when

$$(7.7) \quad L = \{p_j: |\epsilon_j| < 1\} \text{ is linearly independent}$$

The ordering is fixed on p_1, \dots, p_k ; the only reordering during the subsequent computation is restricted to $\{p_j: n \geq j > k\}$. At each iteration the linearly independent set F is extended to a set E by adding elements of $L \setminus F$ one at a time, until E has rank $|E| - 1$. One then determines a nontrivial linear dependence relation $\sum_{j=1}^n y_j p_j = \vec{0}$ with $\underline{S}(y) \subseteq E$. The current iterate ϵ is updated to $\epsilon \leftarrow \epsilon + \lambda y$, with the scalar λ chosen so that (7.6b) is maintained and at least one $|\epsilon_j|$, $j > k$, increases to one; k is incremented by the number of new entries of magnitude one. The indices are then reordered so that (7.6c) remains satisfied. Clearly (7.6a) and (7.6d) are maintained. The procedure halts if (7.7) is satisfied, otherwise another iteration is performed. At termination, the order $\{p_1, \dots, p_n\}$ and $(\epsilon_1, \dots, \epsilon_n)$ satisfy (7.5) for all $1 \leq k \leq n-d$. By Lemma 7.1 we can fix $\epsilon_{n-d+1}, \dots, \epsilon_n$ such that

$$\|\epsilon_1 p_1 + \dots + \epsilon_k p_k\| \leq d+d^{1/2}$$

for all $n-d+1 \leq k \leq n$. The following lemma implies that there is a reordering of (p_{n-d+1}, \dots, p_n) and the associated $\epsilon_{n-d+1}, \dots, \epsilon_n$ such that (7.5) holds for all $1 \leq k \leq n$.

(7.8) Lemma. In \mathbb{R}^d , $d \geq 3$, suppose $1 \leq j \leq d$, and $\|\tilde{p}_i\| \leq 1$ for all i . If $q = p + \tilde{p}_1 + \dots + \tilde{p}_j$, $\|p\| \leq d$, and $\|q\| \leq d^{1/2}$, then there is an i , $1 \leq i \leq j$, such that $\|p + \tilde{p}_i\| \leq d$.

Proof of (7.8). If $\|p\| \leq d-1$ or $j \leq 2$, then it is obvious. So suppose $\|p\| > d-1$ and $j \geq 3$. Consider the projection of \mathbb{R}^d onto the line $\mathbb{R}p$. Let q', u_1, u_2, \dots, u_j be the projections of $q, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_j$. Then clearly we have

$$q' = p + u_1 + u_2 + \dots + u_j, \quad \text{and} \quad \|q'\| \leq d^{1/2} \leq d-1 < \|p\|,$$

and $\|u_i\| \leq 1$ for all i . Without loss of generality, suppose $p, q', u_1, u_2, \dots, u_j$, are all real numbers and $p > 0$. Let

$$u = \min\{u_i : 1 \leq i \leq j\}.$$

Without loss of generality again, suppose $u_1 = u$.

We will now prove that $\|p + \tilde{p}_1\| \leq d$. To prove this consider the plane determined by p and \tilde{p}_1 . Let α be the angle between $-p$ and \tilde{p}_1 . Note that $u \leq -(p - q')/j < 0$, so α is an acute angle. By the second cosine law,

$$\begin{aligned} \|p + \tilde{p}_1\|^2 &= p^2 + \|\tilde{p}_1\|^2 - 2p\|\tilde{p}_1\|\cos \alpha = p^2 + \|\tilde{p}_1\|^2 - 2p|u| \\ &\leq p^2 + \|\tilde{p}_1\|^2 - 2p(p - q')/j. \end{aligned}$$

Define a function f by

$$f(p, r, q') = p^2 + r^2 - 2p(p - q')/j, \quad d-1 \leq p \leq d, \quad 0 \leq r \leq 1, \quad -d^{1/2} \leq q' \leq d^{1/2},$$

and consider the partial derivatives of f . Then

$$f_{q'} = 2p/j \geq 0, \quad f_r = 2r \geq 0$$

and

$$f_p = 2p(1 - 2/j) + 2q'/j \geq 2d^{1/2}(1 - 2/j) - 2d^{1/2}/j = 2d^{1/2}(1 - 3/j) \geq 0.$$

So the maximum value of f is obtained when $p = d$, $r = 1$, and $q' = d^{1/2}$. Therefore,

$$\begin{aligned}\|p+p_1\|^2 &\leq d^2 + 1 - 2d(d-d^{1/2})/j \leq d^2 + 1 - 2d(d-d^{1/2})/d \\ &= d^2 + 1 - 2(d-d^{1/2}) \leq d^2.\end{aligned}\quad \square$$

Proof of (7.4). For $d \geq 3$, Theorem 7.4 is clear from the proof of Theorem 7.2 and Lemma 7.8. The cases when $d = 1$ and 2 , can be checked easily. \square

The upper bound $d^{1/2}$ for the total sum is tight. If P is the set of d standard unit vectors, then $\|\sum \epsilon P\| = d^{1/2}$ for every ϵ . But the bound d for the partial sums might be improved. The following partial result is due to P. Carvalho [8].

(7.9) Theorem. If $n \leq 2d$, then there exist an ϵ and an ordering on P such that

$$\|\sum \{\epsilon(p_i)p_i : 1 \leq i \leq k\}\| \leq d^{1/2}$$

for all $1 \leq k \leq n$.

Proof. We know that there is an ϵ such that $\|\sum \epsilon P\| \leq d^{1/2}$. Without loss of generality, suppose $\epsilon \equiv 1$. Then $\|p_1 + p_2 + \dots + p_n\| \leq d^{1/2}$ and

$$\begin{aligned}\sum_{j=1}^n \|\sum_{i \neq j} p_i\|^2 &= (n-1) \sum_{i=1}^n \|p_i\|^2 + 2(n-2) \sum_{i \neq j} p_i p_j \\ &= (n-2) \sum_{i=1}^n \|p_i\|^2 + \sum_{i=1}^n \|p_i\|^2 \leq (n-2)d + n.\end{aligned}$$

So at least one term must be less than or equal to $(1-2/n)d + 1 \leq d$ because $n \leq 2d$. Therefore, there is a vector in P , say p_n , such that the sum of the other $n-1$ vectors has norm at most $d^{1/2}$. By the same reason there is a vector, say p_{n-1} , among the $n-1$ remaining vectors such that the sum of the other $n-2$ vectors has norm at most $d^{1/2}$. And so on. By continuing this procedure $n-1$ times, we find a satisfactory order on P . \square

Until now we have considered the Euclidean norm only. For an arbitrary norm $\psi(\cdot)$ in \mathbb{R}^d , we can prove the following.

(7.10) Theorem. Let P be a finite set of points of \mathbb{R}^d such that $\psi(p) \leq 1$ for all $p \in P$. Then there exist an ordering and an ϵ on P such that

$$\psi(\epsilon(p_1)p_1 + \dots + \epsilon(p_k)p_k) \leq d$$

for all k , $1 \leq k \leq n = |P|$.

Proof. Use the procedure following the proof of Theorem 7.4. We started with $\epsilon = \vec{0}$ and gradually altered ϵ to get the desired result. In essence, for every k , $1 \leq k \leq n-d$, we had a vector ϵ^k satisfying:

$$|\epsilon_i^k| \begin{cases} = 1 & \text{for } i = 1, 2, \dots, k \\ \leq 1 & \text{for } i = k+1, \dots, k+d \\ = 0 & \text{for } i > k+d, \end{cases}$$

and

$$\epsilon_1^k p_1 + \dots + \epsilon_k^k p_k + \epsilon_{k+1}^k p_{k+1} + \dots + \epsilon_{k+d}^k p_{k+d} = \vec{0}.$$

Hence, if we set $\epsilon(p_i)$ to the terminal value of $\epsilon_i = \epsilon_i^{n-d}$, $1 \leq i \leq n-d$, then

$$\begin{aligned} & \mathfrak{v}(\epsilon(p_1)p_1 + \dots + \epsilon(p_k)p_k) \\ &= \mathfrak{v}(-\epsilon_{k+1}^k p_{k+1} - \dots - \epsilon_{k+d}^k p_{k+d}) \leq |\epsilon_{k+1}^k| + \dots + |\epsilon_{k+d}^k| \leq d \end{aligned}$$

for all $k \leq n-d$. The last d vectors may be ordered arbitrarily with ϵ defined by

$$\epsilon(p_i) = +1 \text{ if } \epsilon_i \geq 0, \text{ and } \epsilon(p_i) = -1 \text{ if } \epsilon_i < 0.$$

Without loss of generality, suppose $\epsilon_i \geq 0$ for all $i > n-d$. Then, from the equality

$$\epsilon_1 p_1 + \dots + \epsilon_{n-d} p_{n-d} + \epsilon_{n-d+1} p_{n-d+1} + \dots + \epsilon_n p_n = \vec{0},$$

we have

$$\begin{aligned} \mathfrak{v}(\epsilon(p_1)p_1 + \dots + \epsilon(p_k)p_k) &= \mathfrak{v}(\epsilon_1 p_1 + \dots + \epsilon_{n-d} p_{n-d} + \epsilon_{n-d+1} p_{n-d+1} + \\ &\quad \dots + \epsilon_n p_n + (1-\epsilon_{n-d+1})p_{n-d+1} + \dots + (1-\epsilon_k)p_k - \epsilon_{k+1}p_{k+1} - \dots - \epsilon_n p_n) \\ &= \mathfrak{v}((1-\epsilon_{n-d+1})p_{n-d+1} + \dots + (1-\epsilon_k)p_k - \epsilon_{k+1}p_{k+1} - \dots - \epsilon_n p_n) \\ &\leq (1-\epsilon_{n-d+1}) + \dots + (1-\epsilon_k) + \epsilon_{k+1} + \dots + \epsilon_n \leq d \end{aligned}$$

for all $k > n-d$. □

Theorem 3 of [2] implies a version of Theorem 7.10 with the looser bound $2d$, but without reordering. Note that the bound d is tight for the L_1 -norm, because the standard basis e_1, \dots, e_d of \mathbb{R}^d has the L_1 -norm of its sum equal to d for every ϵ .

Back to the Euclidean norm. When $d = 2$ and all $\|p_i\| = 1$, we can prove a stronger result.

(7.11) Lemma. Suppose P is a finite subset of the unit circle S^1 in \mathbb{R}^2 , P is of odd size and the points of P are located in the following way: if the points of $-P = \{-p: p \in P\}$ are inserted, then the points of P and the points of $-P$ alternate on the circle. Then $\|\Sigma P\| \leq 1$.

Proof. Suppose $|P| = 2m + 1$. By rotating and reflecting the coordinate system, if necessary, we can assume that the sum is on the x-axis, and, for an appropriate ordering of P as $p_1, p_2, \dots, p_{2m+1}$, the x-coordinates x_i of p_i satisfy

$$x_1 \leq -x_2 \leq x_3 \leq -x_4 \leq \dots \leq -x_{2m} \leq x_{2m+1}.$$

Then

$$\begin{aligned} -1 &\leq x_1 \leq x_1 + (x_2 + x_3) + \dots + (x_{2m} + x_{2m+1}) \\ &= (x_1 + x_2) + (x_3 + x_4) + \dots + (x_{2m-1} + x_{2m}) + x_{2m+1} \leq x_{2m+1} \leq 1. \end{aligned} \quad \square$$

(7.12) Theorem. If P is a finite subset of S^1 in \mathbb{R}^2 , then there exists an ordering and an ϵ on P such that

$$\|\Sigma\{\epsilon(p_i)p_i: 1 \leq i \leq k\}\| \leq 1 \quad \text{if } k \text{ is odd,}$$

and

$$\|\Sigma\{\epsilon(p_i)p_i: 1 \leq i \leq k\}\| \leq \sqrt{2} \quad \text{if } k \text{ is even,}$$

for all $1 \leq k \leq |P|$.

Proof. Suppose $n = |P|$ is odd. First define ϵ^1 on P so that all of the $\epsilon^1(p)p$'s are on an open semicircle, and order them along the

semicircle. Now define ϵ as: $\epsilon(p_i) = \epsilon^1(p_i)$ for odd i , and $-\epsilon^1(p_i)$ for even i . Note that for any k , $1 \leq k \leq n$, the set of points $\epsilon(p_1)p_1, \epsilon(p_2)p_2, \dots, \epsilon(p_k)p_k$, satisfies the hypothesis of Lemma (7.11). Hence

$$\|\Sigma\{\epsilon(p_i)p_i : 1 \leq i \leq k\}\| \leq 1$$

for all odd k , $1 \leq k \leq n$. Now, for each i , $1 \leq i < n/2$, interchange the pair p_{2i}, p_{2i+1} , if necessary, to get

$$\|\Sigma\{\epsilon(p_i)p_i : 1 \leq i \leq k\}\| \leq \sqrt{2}$$

for all even numbers k , $1 \leq k \leq n$. The bound 1 remains valid for odd k .

If n is even, then choose any one vector p from P . Determine an order and an ϵ for $P \setminus \{p\}$ as before. Then insert p as p_n with appropriate $\epsilon(p_n)$. □

8. Further variations

This section concerns several additional examples of balancing problems in which configurations are ordered. Each of the examples involves graphs, either in the definition of the configuration, or in the determination of an effective perturbation from the original configuration to one that is well balanced.

Let $G = (V, E)$ be a directed graph with $(0, \pm 1)$ -vertex-edge incidence matrix A . For each $i \in V$, let A_i be the i -th row of A . Then $P = \{A_i : i \in V\}$ is a finite subset of points in \mathbb{R}^E and $\Sigma P = 0$. Can we make all the partial sums 'small' by reordering the vertices?

(8.1) Theorem. The following problem is NP-complete. For a given directed graph G and a constant L , is there an ordering π of the vertices such that

$$\|A_{\pi(1)} + \dots + A_{\pi(k)}\|_1 \leq L$$

for all k , $1 \leq k \leq |V|$?

Proof. For any ordering $\pi(1), \pi(2), \dots, \pi(|V|)$ of the vertices and for $1 \leq k \leq |V|$, the partial sum

$$S_k = A_{\pi(1)} + \dots + A_{\pi(k)}$$

is equal to the $(0, \pm 1)$ incidence vector of the cutset D_k , consisting of edges having one end in $I_k = \{\pi(1), \pi(2), \dots, \pi(k)\}$, and one end in

$\bar{I}_k = V \setminus I_k$. Hence

$$\|S_k\|_1 = |D_k|.$$

Therefore the problem is equivalent to the Minimum Cut Linear Arrangement Problem which is known to be NP-complete ([GT44] of [11]). \square

Now let $P = \{A_e : e \in E\}$, where A_e is the column vector of A corresponding to an edge e . Then P is a finite subset of \mathbb{R}^V and, for each $i \in V$, the i -th entry of the sum is

$$\sum \{A_e : e \in E\} = \text{outd}(i) - \text{ind}(i).$$

Here $\text{outd}(i)$ and $\text{ind}(i)$ are the number of edges 'from' and 'to' the vertex i , respectively. Now if we change the directions of some edges, then we get a different sum. Note that for every choice of e the

terms in the sum corresponding to each vertex of odd degree in G will have magnitude at least one, and the terms corresponding to each vertex of even degree will be even. Hence the best we can expect for the L_1 -norm of the sum is that the sum is a $(0, \pm 1)$ -vector, i.e. its L_∞ norm is 0 or 1. In fact, this can be achieved because of the following characterization of totally unimodular matrices.

(8.2) Theorem [Ghouila-Houri, 12]. A $(0, \pm 1)$ -matrix A is totally unimodular if and only if, for every subset T of the column vectors of A , there are ± 1 multipliers for the columns in T such that the scaled sum of the columns in T is a $(0, \pm 1)$ -vector.

We can say more when A is the vertex-edge incidence matrix of a graph.

(8.3) Theorem. For any graph $G = (V, E)$, there exist an ordering of E and an orientation of G such that all the partial sums of $P = \{A_e : e \in E\}$ are $(0, \pm 1)$ -vectors.

The following procedure will give an appropriate orientation and an appropriate ordering of the edges simultaneously. The order of the edges will be recorded in the list list.

begin

$F := E$; $list := \phi$;

while F contains a circuit **do**

 find a circuit C in F ;

 make C directed and add the edges of C to list along C ;

$F := F \setminus C$;

end

```

while  $F \neq \phi$  do
    find a path  $Q$  from a vertex of degree 1 to another vertex
    of degree 1 in  $F$ ;
    make  $Q$  directed and add the edges of  $Q$  to list along  $Q$ ;
     $F := F \setminus Q$ ;
end
end

```

□

Now another problem. Consider the fundamental system $\mathcal{D} = \{D_e : e \in T\}$ of cocircuits with respect to a spanning tree T . Here we consider D_e to be the $(0, \pm 1)$ -incidence vector of the cocircuit determined by e and ENT and we think of \mathcal{D} as a matrix with rows D_e . Now the problem is to find (± 1) -multipliers and an ordering of the rows of \mathcal{D} such that all the partial sums have small norm. Since \mathcal{D} can be obtained from the rows of a totally unimodular matrix A , the vertex-edge incidence matrix, by a sequence of pivot operations, \mathcal{D} is totally unimodular. Hence there are multipliers such that the total sum is a $(0, \pm 1)$ -vector by Theorem 8.2.

(8.4) Theorem. For every graph $G = (V, E)$ and for every spanning forest T of G , there exist an orientation of G and an ordering of the edges of T such that all the partial sums of \mathcal{D} are $(0, \pm 1)$ -vectors.

Proof. Without loss of generality assume G is connected. Since the rows are indexed by the edges of the spanning tree T , all we have to do is to find an appropriate ordering and directions for the edges of T . Order the edges in such a way that, for each i , the edge set $\{e_1, e_2, \dots, e_i\}$ is

connected. Now determine the orientations of the edges of T so that, for each path P of T , the directions of the edges of P are alternating. Orientations on the edges of ENT are arbitrary. Then it is easy to check that the condition is satisfied. \square

Let $\mathcal{C} = \{C_e : e \in ENT\}$ be the fundamental system of circuits of G with respect to a spanning tree T ; for each $e \in ENT$, C_e is the $(0, \pm 1)$ -incidence vector of the circuit determined by e and T .

(8.5) Theorem [Lovász, 19]. There exist an orientation and an ordering of the edges of ENT such that each partial sum of \mathcal{C} is a $(0, \pm 1, \pm 2)$ -vector.

Proof [Lovász]. First note that if we add C_e 's 'along' a directed path P in ENT , then the tree part of the sum is the incidence vector of the path from the head of P to the tail of P . Hence if we add C_e 's along a directed cycle in ENT , then the tree part of the sum will be zero.

Suppose ENT is weakly Eulerian, i.e., every vertex has even degree in ENT . If ENT has components G_1, \dots, G_k , then by an Euler tour in ENT , we mean a sequence W_1, \dots, W_k , where each W_i is a Euler Tour in G_i . Order the C_e 's along an Euler tour and choose directions of the nontree edges so that the Euler tour becomes directed. Then the entries of the total sum corresponding to the tree edges will be zero, and all the partial sums will be $(0, \pm 1)$ -vectors. Therefore, the problem reduces to the case when ENT is not weakly Eulerian.

Suppose ENT is not weakly Eulerian. Choose a minimal set F of edges parallel to edges in T so that $(ENT) \cup F$ is weakly Eulerian. Then, for each $e \in T$, there is at most one $e' \in F$ parallel to e . Now determine an order and directions along an Euler tour made of $(ENT) \cup F$,

as above, and then delete the circuits corresponding to F . The resulting order and orientations satisfy the condition. To see this consider the sequence of the e -th entries for each edge e in T . If e has no duplicate in F , then the sequence on e is not changed. If e has a duplicate e' in F , then the sequence loses one $+1$ or -1 by deleting the circuit $C_{e'}$. Note that $C_{e'}$ consists of e and e' only. Hence the sequence of e -th entries of the partial sums will be changed by at most ± 1 . Therefore it must be a $(0, \pm 1, \pm 2)$ sequence. \square

It is not known if there is an example that attains the bound 2 in the theorem above. Can we find an order and directions such that all the partial sums are $(0, \pm 1)$ -vectors? A more general question would be:

(8.6) Problem. Is there a constant k such that, for every totally unimodular matrix, there is an ordering and an ϵ for the rows such that all the partial sums have L_∞ -norm $\leq k$?

By Theorem (8.2) we know that, for any given totally unimodular matrix A , there are (± 1) multipliers for the rows such that the row sum is a $(0, \pm 1)$ -vector. Hence if we multiply appropriate (± 1) -scalars on the columns too, and if we add all the entries of the matrix, then we would get either 0 or ± 1 . This is a special, and simple, case of an unsolved variation on a problem of L. Moser.

(8.7) Problem. Is there a constant k such that, for every $(0, \pm 1)$ -matrix, there exist (± 1) -multipliers for the rows and columns such that the sum of all entries of the scaled matrix is between 0 and k ?

Moser conjectured that for every $n \times n$ (± 1) -matrix there exist (± 1) -multipliers for the rows and columns such that the sum of the scaled

entries is 1 if n is odd and either 0 or 2 if n is even; this was proved probabilistically by Komlos and Sulyok and deterministically by Beck and Spencer (see [4]).

The final two examples do not arise explicitly from graphs. However, in each case the determination of an ϵ that yields a well-balanced neighbor is based on a simple graph-theoretic argument.

Consider again Example 1 from Section 1.

(8.8) Theorem. Suppose $P = \{f_1, m_1, f_2, m_2, \dots, f_n, m_n\}$ is a set of $2n$ distinct points on the circle S^1 . The order is arbitrary. Then there is a map $\epsilon: P \rightarrow \{-1, +1\}$ such that

$$(a) \quad \epsilon(f_i) + \epsilon(m_i) = 0$$

for all i , and

$$(b) \quad |\Sigma\{\epsilon(x): x \in P \cap \alpha\}| \leq 2$$

for every arc α of S^1 . Furthermore, there is an ϵ such that

$$(c) \quad |\Sigma\{\epsilon(x): x \in P \cap \alpha\}| \leq 1$$

for every arc α if and only if no pair (p_i, q_i) divides the rest of P into two odd sets.

Proof. Note that the bound 2 is tight, because if any pair (p_i, q_i) divides the rest of P into two odd sets, say S and T , then, for every ϵ ,

$$\text{either } |\epsilon(f_i) + \Sigma\{\epsilon(x): x \in S\}| \text{ or } |\epsilon(m_i) + \Sigma\{\epsilon(x): x \in S\}|$$

must be 2.

To prove the theorem, construct a 2-regular graph $G = (V, E)$ as follows. Let $V = P$. Let E be of the form $E = E_1 \cup E_2$, where E_1 is

a set of n disjoint pairs of vertices adjacent on S^1 , and E_2 is the set of n chords, $f_1m_1, f_2m_2, \dots, f_nm_n$. So $|E| = 2n$. Then clearly G is 2-regular, E is a vertex-disjoint union of circuits, and each circuit is of even length. Now define $\epsilon: V \rightarrow \{-1, +1\}$ by traversing the circuits so that vertices adjacent in G have opposite values. Then, since $\epsilon(u) + \epsilon(v) = 0$ for every edge $\{u, v\} \in E_1$, it is clear that this ϵ satisfies the condition (b) for every arc α of S^1 . See Figure 8.1.

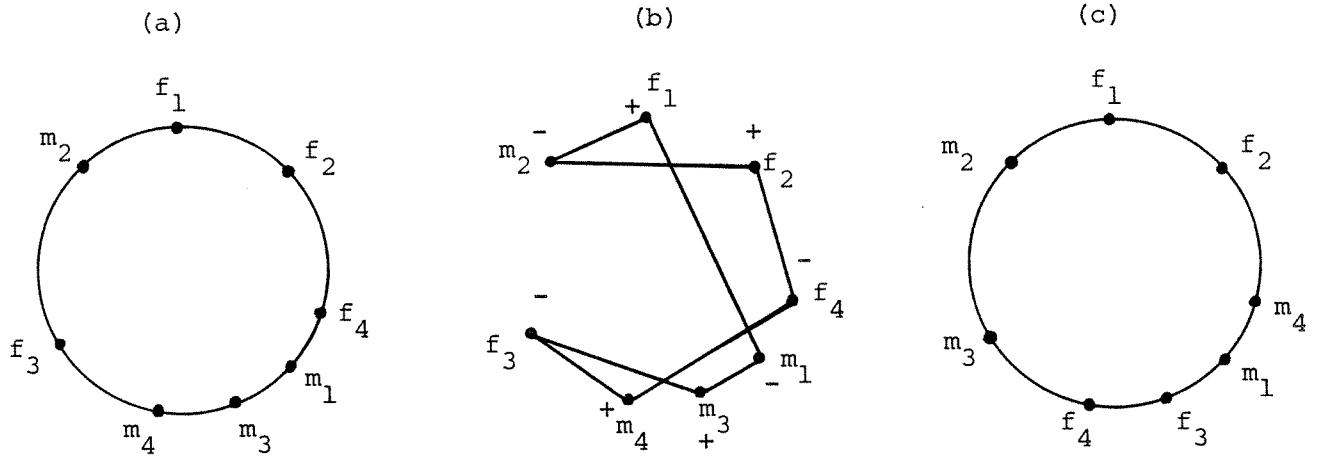


Figure 8.1 (a) An unbalanced arrangement;
 (b) the graph G ; (c) a balanced arrangement

In the language of Example 1 of Section 1, a balanced seating arrangement results from ϵ by having each couple (f_i, m_i) swap seats if and only if $\epsilon(f_i) = -1$.

For the second part, note that

$$|\Sigma\{\epsilon(x): x \in P \cap \alpha\}| \leq 1$$

for every arc α of S^1 if and only if the +1's and -1's alternate along the circle S^1 . Now if no pair (f_i, m_i) divides the rest of P into two

odd sets, then it is easy to see that the alternating assignment of ± 1 's along the circle satisfies (a) and hence (c). Conversely if a pair (f_i, m_i) divides the rest of P into two odd sets, then clearly an alternating assignment of ± 1 's along the circle violates (a). \square

Note that in the graph G above, traversing the circuits consistently, say clockwise, if possible, yields an optimal solution. If we can move clockwise on each edge of E_1 , then there is no pair (p, q) dividing the rest of P into two odd sets, and, conversely, if we cannot move clockwise on every edge of E_1 , then there is a pair (p, q) dividing the rest of P into two odd sets.

The requirement in the hypothesis of Theorem (8.8) that the $2n$ points be distinct is merely a convenience to facilitate the description of the graph G on the circle S^1 .

The following related problem was introduced to us by Lovász [19].

(8.9) Problem. For each positive integer d , let $\rho(d)$ be the least upper bound of

$$\min_{\epsilon: N \rightarrow \{-1, +1\}} \left\{ \max_{\substack{1 \leq i \leq d \\ 1 \leq k \leq n}} \left\{ \left| \sum_{j=1}^k \epsilon(\pi_i(j)) \right| \right\} \right\}$$

over all positive integers n , where $\pi_1, \pi_2, \dots, \pi_d$ are permutations of $N = \{1, 2, \dots, n\}$. Determine $\rho(d)$. In particular, is $\rho(d)$ finite for all d ?

The solution for Problem (8.9) is not known except for the case $d \leq 2$. Clearly $\rho(1) = 1$. Vesztergombi proved that $\rho(2) = 1$ [57]. The

following proof of $\rho(2) = 1$ is similar to the proof of Theorem 8.8. For given π_1 and π_2 , construct a 2-regular graph $G = (V, E)$ with $V = \{\pi_1(1), \dots, \pi_1(n), \pi_2(1), \dots, \pi_2(n)\}$. We treat $\pi_1(i)$'s and $\pi_2(i)$'s as distinct nodes. We assume that n is even. If n is odd, then add two more nodes $\pi_1(n+1)$ and $\pi_2(n+1)$. Let the edge set E be

$$E = \{\{\pi_1(1), \pi_1(2)\}, \{\pi_1(3), \pi_1(4)\}, \dots\} \\ \cup \{\{\pi_2(1), \pi_2(2)\}, \{\pi_2(3), \pi_2(4)\}, \dots\} \cup \{\{1, 1\}, \{2, 2\}, \dots, \{n, n\}\}.$$

Note that the graph G is 2-regular, and, therefore, is a vertex-disjoint union of circuits. Moreover, the number of edges of each circuit is an integer multiple of 4. Now define ϵ on V as follows. For each circuit C , traverse C and define ϵ on $V(C)$ in such a way that $\epsilon(i) = -\epsilon(j)$ if $\{i, j\} \in E$ and $i \neq j$ as members of N , and $\epsilon(i) = \epsilon(j)$ if $\{i, j\} \in E$ and $i = j$ as members of N . Hence the numbers ± 1 appear on the vertices of each circuit in sequences of two $+1$'s, two -1 's, two $+1$'s, and so on. Then clearly the map ϵ on N obtained from ϵ on V naturally satisfies $|\sum\{\epsilon(\pi_i(j)) : 1 \leq j \leq k\}| \leq 1$ for all $1 \leq k \leq n$ and for $i = 1$ and 2 . Therefore $\rho(2) = 1$. \square

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