# DIFFERENTIATING THE TRACE OF CERTAIN MATRIX PRODUCTS

S. R. Searle

Biometrics Unit, Cornell University, Ithaca, N. Y.

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Abstract

Results $\partial[\mathrm{tr}(\underset{\sim}{X}\underset{\sim}{A})]/\partial\underset{\sim}{X} = \underset{\sim}{A}'$ , $\partial[\mathrm{tr}(\underset{\sim}{X}'\underset{\sim}{B})]/\partial\underset{\sim}{X} = \underset{\sim}{B}$ and $\partial[\mathrm{tr}(\underset{\sim}{A}\underset{\sim}{X}'\underset{\sim}{B}\underset{\sim}{X}\underset{\sim}{C})]/\partial\underset{\sim}{X} = \underset{\sim}{B}\underset{\sim}{X}\underset{\sim}{C}\underset{\sim}{A} + \underset{\sim}{B}'\underset{\sim}{X}\underset{\sim}{A}'\underset{\sim}{C}'$ are derived.

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Numerous results on matrix differentiation useful in statistics are given in Neudecker [1969] and the papers that he cites. However, the following simple results are additionally useful.

Define $\underset{\sim}{X} = \{x_{ij}\}$ and $\underset{\sim}{A} = \{a_{ij}\}$ so that $\underset{\sim}{A}\underset{\sim}{X}$ is square. Then

$$\mathrm{tr}(\underset{\sim}{X}\underset{\sim}{A}) = \Sigma\Sigma x_{ij}a_{ji}$$

and so

$$\frac{\partial}{\partial x_{ij}}\mathrm{tr}(\underset{\sim}{X}\underset{\sim}{A}) = a_{ji} \ . \tag{1}$$

Now define

$$\frac{\partial}{\partial \underset{\sim}{X}}\mathrm{tr}(\underset{\sim}{X}\underset{\sim}{A}) \overset{\mathrm{def}}{=} \left\{\frac{\partial}{\partial x_{ij}}\mathrm{tr}(\underset{\sim}{X}\underset{\sim}{A})\right\} \tag{2}$$

i.e., $\partial[\mathrm{tr}(\underset{\sim}{X}\underset{\sim}{A})]/\partial\underset{\sim}{X}$ is defined as each element of $\underset{\sim}{X}$ replaced by the differential of $\mathrm{tr}(\underset{\sim}{X}\underset{\sim}{A})$ with respect to that element. Then by (1)

$$\frac{\partial}{\partial \mathbf{X}}\mathrm{tr}(\mathbf{XA}) = \{a_{ji}\} = \mathbf{A}' \ . \qquad (3)$$

Similarly, for $\mathbf{X}'\mathbf{B}$ being square,

$$\mathrm{tr}(\mathbf{X}'\mathbf{B}) = \mathrm{tr}(\mathbf{BX}') = \mathrm{tr}(\mathbf{XB}')$$

and so by (3)

$$\frac{\partial}{\partial \mathbf{X}}\mathrm{tr}(\mathbf{X}'\mathbf{B}) = \mathbf{B} \ . \qquad (4)$$

Finally, for $\mathbf{AX}'\mathbf{BXC}$ being a square matrix,

$$\mathrm{tr}(\mathbf{AX}'\mathbf{BXC}) = \mathrm{tr}(\mathbf{XCAX}'\mathbf{B}) = \mathrm{tr}(\mathbf{X}'\mathbf{BXCA})$$

so that application of both (3) and (4) gives

$$\frac{\partial}{\partial \mathbf{X}}\mathrm{tr}(\mathbf{AX}'\mathbf{BXC}) = (\mathbf{CAX}'\mathbf{B})' + \mathbf{BXCA} = \mathbf{B}'\mathbf{XA}'\mathbf{C}' + \mathbf{BXCA} \ . \qquad (5)$$

Numerous special cases of (5) can be imagined, as well as obvious extensions to products involving $\mathbf{X}$ and $\mathbf{X}'$ more than once.

In all cases of (3), (4) and (5) it is easily seen from conformability considerations that the dimensionality implied by (2) is always maintained.

## Reference


Neudecker, H. [1969]. Some theorems on matrix differentiation with special reference to Kronecker matrix products. J. of Am. Stat. Ass., 64, 953-963.