LA TROBE UNIVERSITY

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Snippets from Infiltration: where Approximate Integral Analysis is Exact.

Hydrology of 1D Unsaturated Flow

in Darcy-Buckingham-Richards approach.

Nonlinear diffusion-convection equations

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}[D(\theta)\frac{\partial \theta}{\partial z}] - K'(\theta)\frac{\partial \theta}{\partial z}$$

$$D(\theta) = K(\theta)\Psi'(\theta)$$

θ = volumetric H_2O conc.

Ψ = capillary suction head.

Some strongly influential papers of the 70s:

J.-Y. Parlange, "Theory of water movement in soils, 8, One-dimensional infiltration with constant flux at the surface, Soil Sci 114, 1-4 (1972).
-Intro of approx analytical method based on mass conservation in integral form

W. Brutsaert, "Universal constants for scaling the exponential soil water diffusivity ?",
Water Resour Res 15, 481--483 (1979).
-Benefits of simplifying flow problems by sorptivity-based scaling

Realistic Integrable Model

$$D(\theta) = \frac{a}{(b-\theta)^2} \quad ; \quad K(\theta) = \frac{\lambda/2}{b-\theta} + \gamma(b-\theta) + \beta$$

Solution with constant-flux boundary cond:

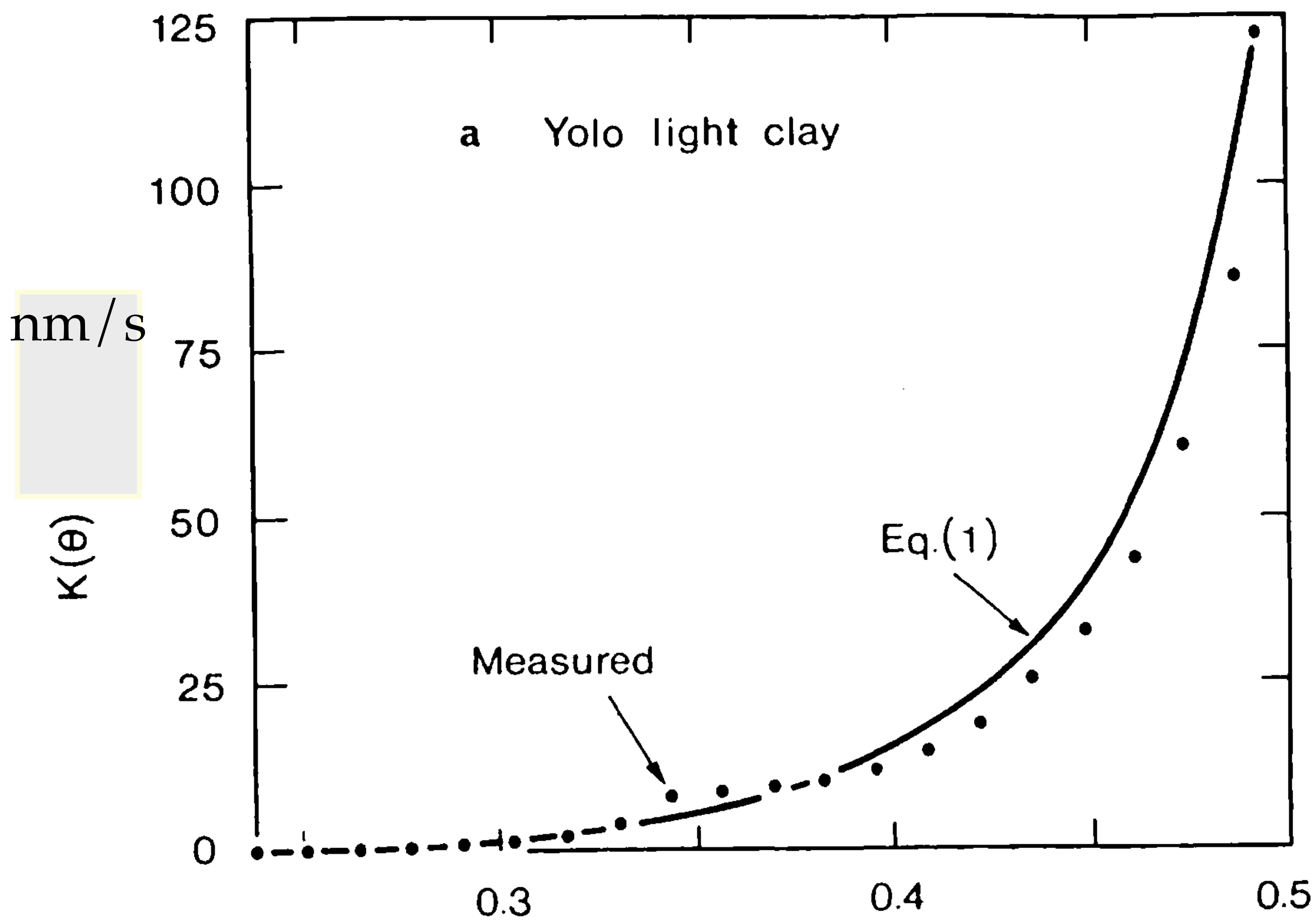
Sander, Parlange, Kuehnel, Hogarth, Lockington and O'Kane, J. Hydrol 97: 341-346 (1988).

Broadbridge and White, WRR 24(1), 145-154 (1988).

Exact ponding time: B&W, WRR 23(12), 2302-10 (1987).

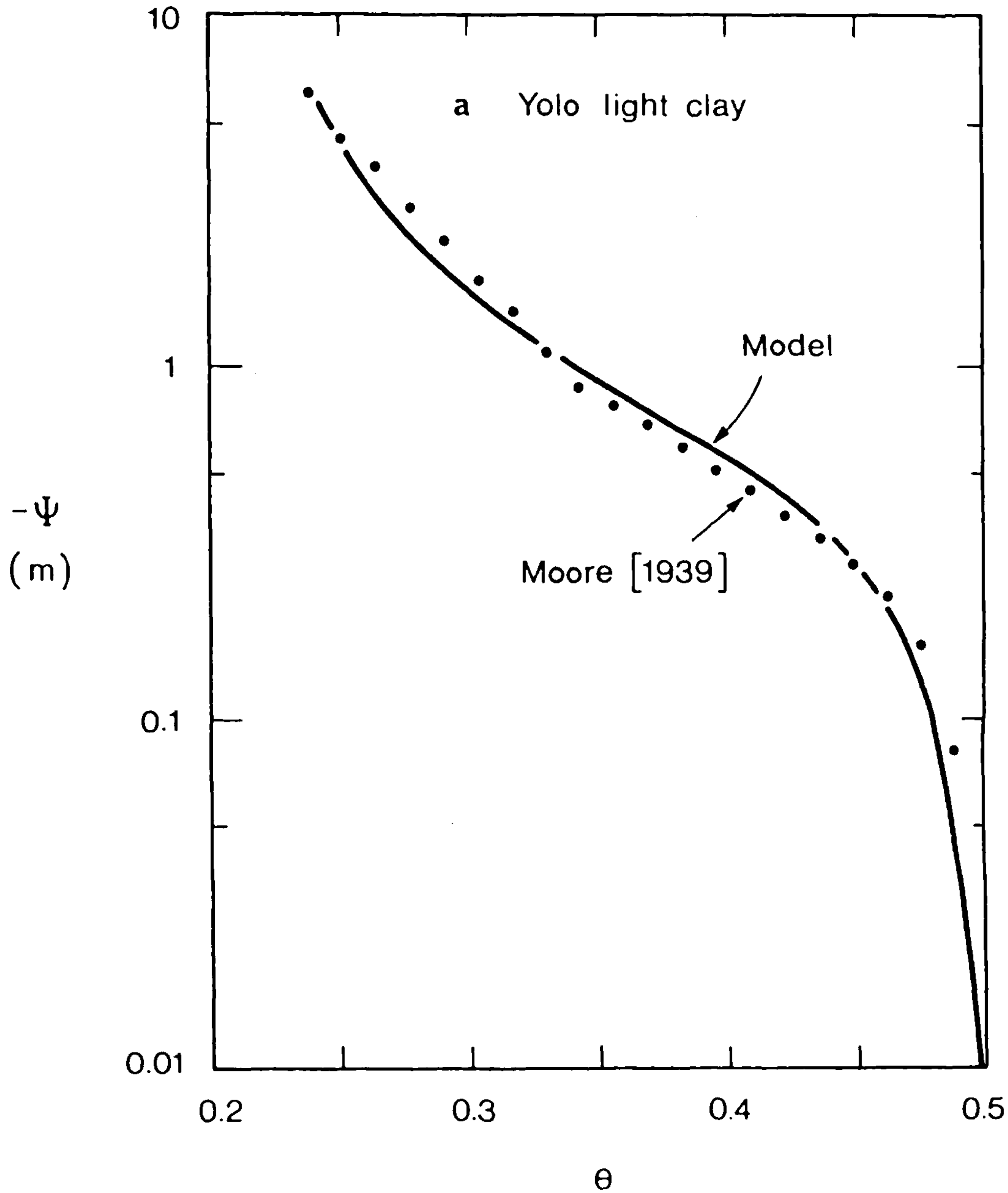
WHITE AND BROADBRIDGE: I

Water Resources Research 1988



Water Resources Research 1988

$$\Psi = \int_{\theta_s}^{\theta} \frac{D(\theta)}{K(\theta)} d\theta$$



Saturday, 21 April 2012

$$D(\theta) = \frac{a}{(b-\theta)^2} \quad ; \quad K(\theta) = \frac{\lambda/2}{b-\theta} + \gamma(b-\theta) + \beta$$

$$\ell_s = \frac{1}{K_s - K_n} \int_{\theta_n}^{\theta_s} D(\theta)\ d\theta \quad = \frac{1}{K_s - K_n} \int_{\Psi_n}^{0} K\ d\Psi$$

$$t_s = \frac{\ell_s(\theta_s - \theta_n)}{K_s - K_n}$$

$$\bar{D} = \frac{1}{\theta_s - \theta_n} \int_{\theta_n}^{\theta_s} D(\theta)\ d\theta \ = \ \frac{h(C)}{C(C-1)} \frac{S_0^2}{(\theta_s - \theta_n)^2}$$

$$1 < C < \infty \ ; \quad \frac{1}{2} < \frac{h(C)}{C(C-1)} < \frac{\pi}{4}$$

$1/\sqrt{h(C)}$ = sol'n of transcendent eq for location of melting front in classical Stefan prob.

Results will be shown for

$$
\begin{aligned}
D_*(\Theta) &= \frac{C(C-1)}{(C-\Theta)^2} \\
K_*(\Theta) &= \zeta\frac{C(C-1)}{C-\Theta} + (\zeta-1)(C-\Theta)
\end{aligned}
$$

$$\zeta = 1 + \frac{(\theta_s - \theta_n)\gamma}{K_s - K_n}.$$

$$\frac{\partial \Theta}{\partial t_*} = \frac{\partial}{\partial z_*}\left(D_*(\Theta)\frac{\partial \Theta}{\partial z_*} - K_*(\Theta)\right)$$

$$\Theta(z_*, 0) = 0 \qquad z_* \in (0, \infty)$$

$$\Theta(z_*, t_*) \ \to 0 \qquad \text{as } \mathrm{z}_* \to \infty$$

$+$ flux boundary cond

$$K_*(\Theta) - D_*(\Theta)\frac{\partial \Theta}{\partial z_*} = R_* \quad ; \quad z_* = 0$$

or concentration boundary cond

$$\Theta(0, t_*) \ = 1 \qquad t_* \in (0, \infty)$$

$\zeta = C$ implies model of Broadbridge & White 1988 with $K_*'(0) = 0$

$$\zeta = 1. \implies K'(\theta)/D(\theta) const. \implies K = e^{\alpha\Psi}$$

$$(C, \zeta) \to (1, 1) \implies D_*(\Theta) \to \delta(\Theta - 1),\ K_* \to H(\Theta - 1)$$

$$(C, \zeta) \to (\infty, \infty) \implies D_*(\Theta) \to 1,\ K_* \to \Theta^2$$

$$(C, \zeta) \to (\infty, 0) \implies D_*(\Theta) \to 1,\ \ K_*'(\Theta) \to 1$$

$$(C, \zeta) \to (1, 0) \implies D_*(\Theta) \to \delta(\Theta - 1),\ \ K_*'(\Theta) \to 1$$

leads to predictions of Green-Ampt model

“Quasi-analytic” integral approximation method: integrate by parts through Richards’ eq.,

$$-\frac{\partial}{\partial t}\int_{\theta_n}^{\theta} z(\bar{\theta},t)\,d\bar{\theta} = D\frac{\partial\theta}{\partial z} - K + K_n$$

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\theta_n}^{\theta_s} z^2(\theta,t)\,d\theta = \int_{\theta_n}^{\theta_s} D(\theta)\,d\theta - \int_{\theta_n}^{\theta_s}(K-K_n)\,\frac{\partial z}{\partial\theta}\,d\theta.$$

then make safe approximations within the integrand.

Time to ponding under constant irrigation rate:

$$\Theta(0, t_p^*) = 1$$

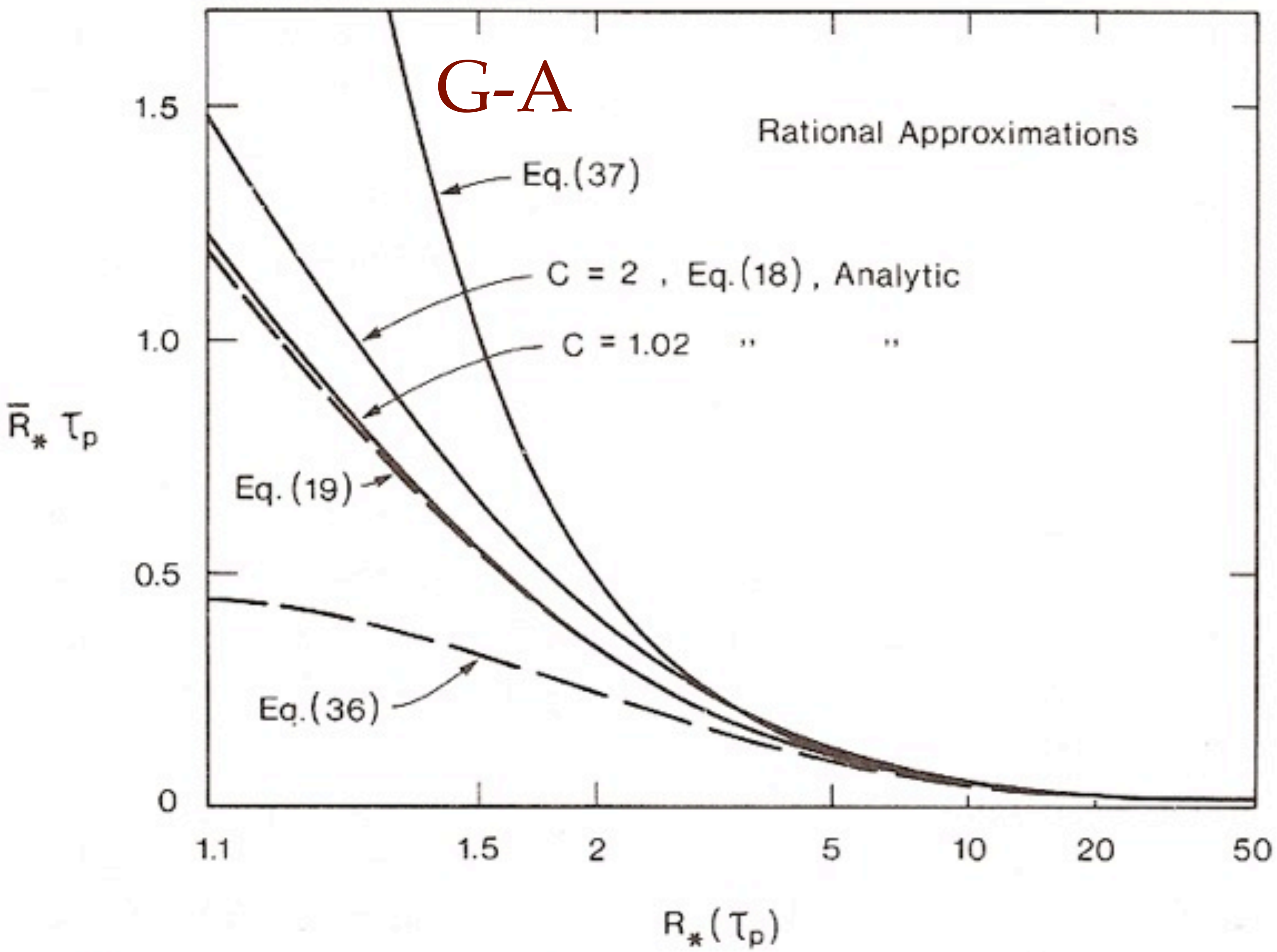
Approximate analysis Parlange and Smith 1976:

$$\bar{R}_* t_p^* = \ln\left(\frac{R_*(t_p^*)}{R_*(t_p^*) - 1}\right) \qquad ; R_* = \frac{R - K_n}{K_s - K_n}$$

Exactly solvable model BW 1987:

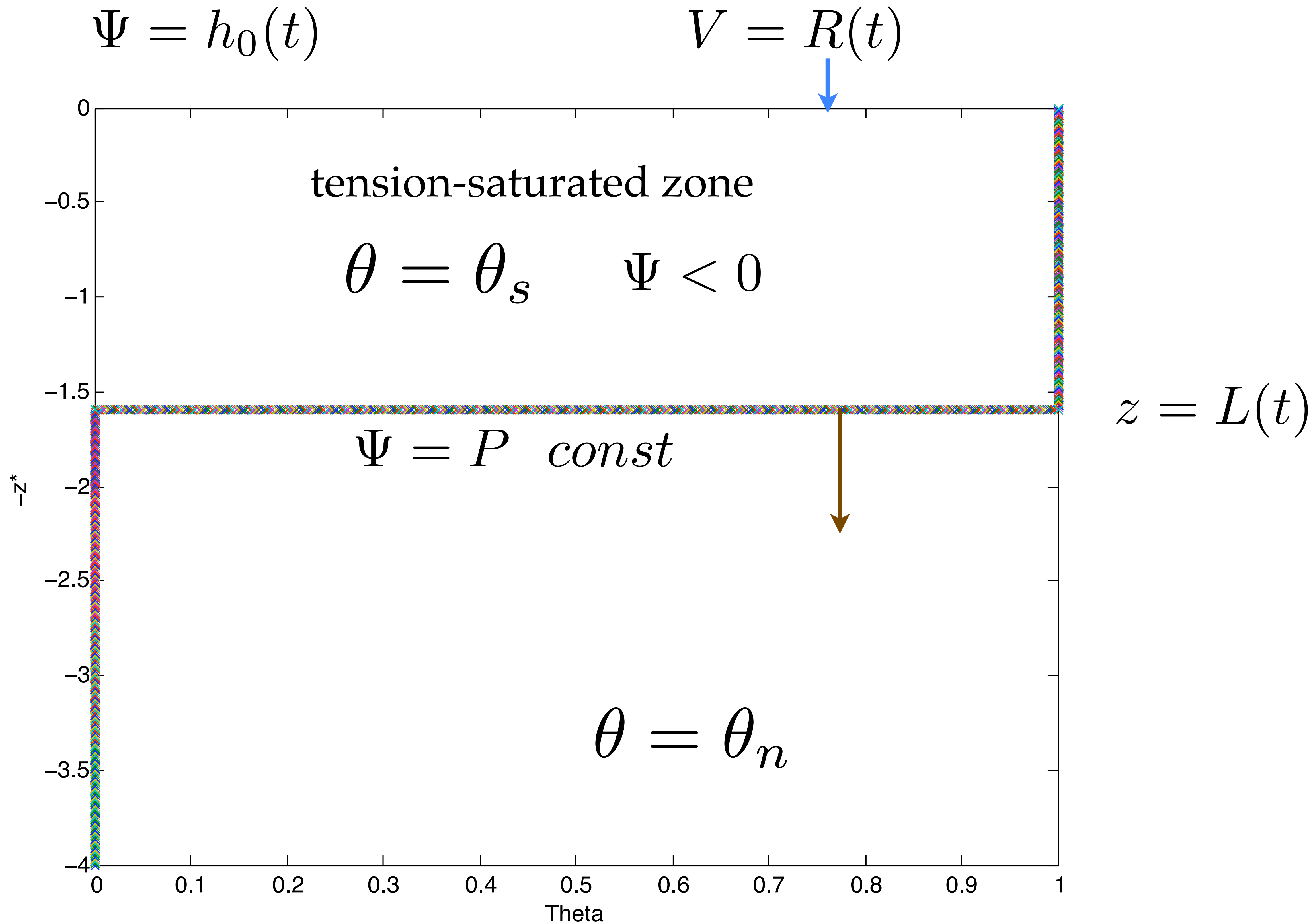
$$(C, \zeta) \to 1 \; ; \quad t_p^* \to \frac{1}{R_*} \ln\left(\frac{R_*}{R_* - 1}\right)$$

Broadbridge & White 1987 $C = \zeta$



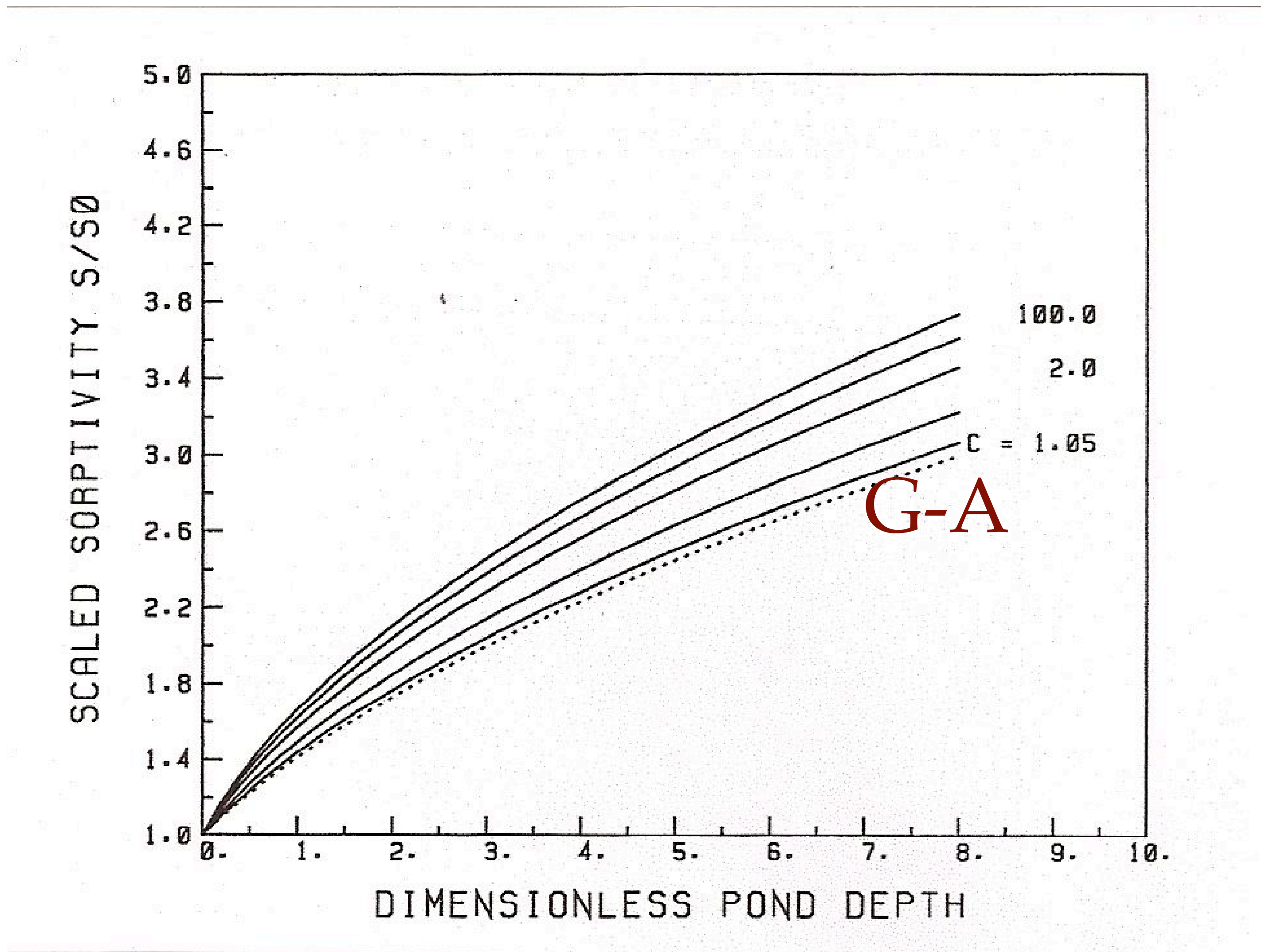
g=0 implies t_p=0.5 S_0^2/R^2 (Eq. 36 graphed above).

(Parlange & Smith 1976 Eq. 19 graphed above)



Green & Ampt 1911

Effect of pond depth on sorptivity: Broadbridge WRR 1990



Green-Ampt model:

$$S/S_0 = (1 + \Psi_0/|P|)^{1/2}$$

Parlange, Haverkamp & Touma 1985:

$$\mu = \frac{\int_{\theta_n}^{\theta_s} (\theta_s - \theta) D d\theta}{S_0^2 + 2K_s\psi_0(\theta_s - \theta_i)}$$

$$S/S_0 = (1 + [1 + \mu]\Psi_0/|P|)^{1/2} \quad \pm 1\%$$

For all values $C \in [1, \infty]$, error is less than 1%.

Exact infiltration coeff's: Triadis & Broadbridge 2011

$$i(t) = \int_0^\infty (\theta - \theta_n)dz \ + K_n t \quad = i^-(t) + K_n t$$

$$i^-(t) = S_0 t^{1/2} + S_1 t + \ldots S_j t^{(1+j)/2} + \ldots$$

Problem: Green-Ampt model predicts

$S_1/K_s = 2/3$

but field observations for soils show

$0.3 < S_1/K_s < 0.4$

e.g. Talsma 1969: $S_1/K_s = 1/2.8 = 0.357\ldots$

$$\frac{S_1(C,\zeta)}{K_s - K_n} = \frac{1}{2}$$

$$+(2\zeta - 1)\sqrt{C(C-1)}\ \frac{4\sqrt{(C-1)/C}\ \Psi(1,1/2,\gamma_0^2/4)}{4\Psi(1,1/2,\gamma_0^2/4) - \gamma_0^2\Psi(2,3/2,\gamma_0^2/4)}$$

Where Ψ is Kummer-U generalized hypergeometric function and $\gamma_0^2 = 1/h(C)$.

At least $S_1, \ldots, S_7$ agree exactly with

$$t_* = \frac{1}{\zeta - 1}[log\left(\frac{\zeta - 1 + e^{\zeta i_*}}{\zeta}\right) - i_*] \;\; ; \quad C \to 1$$

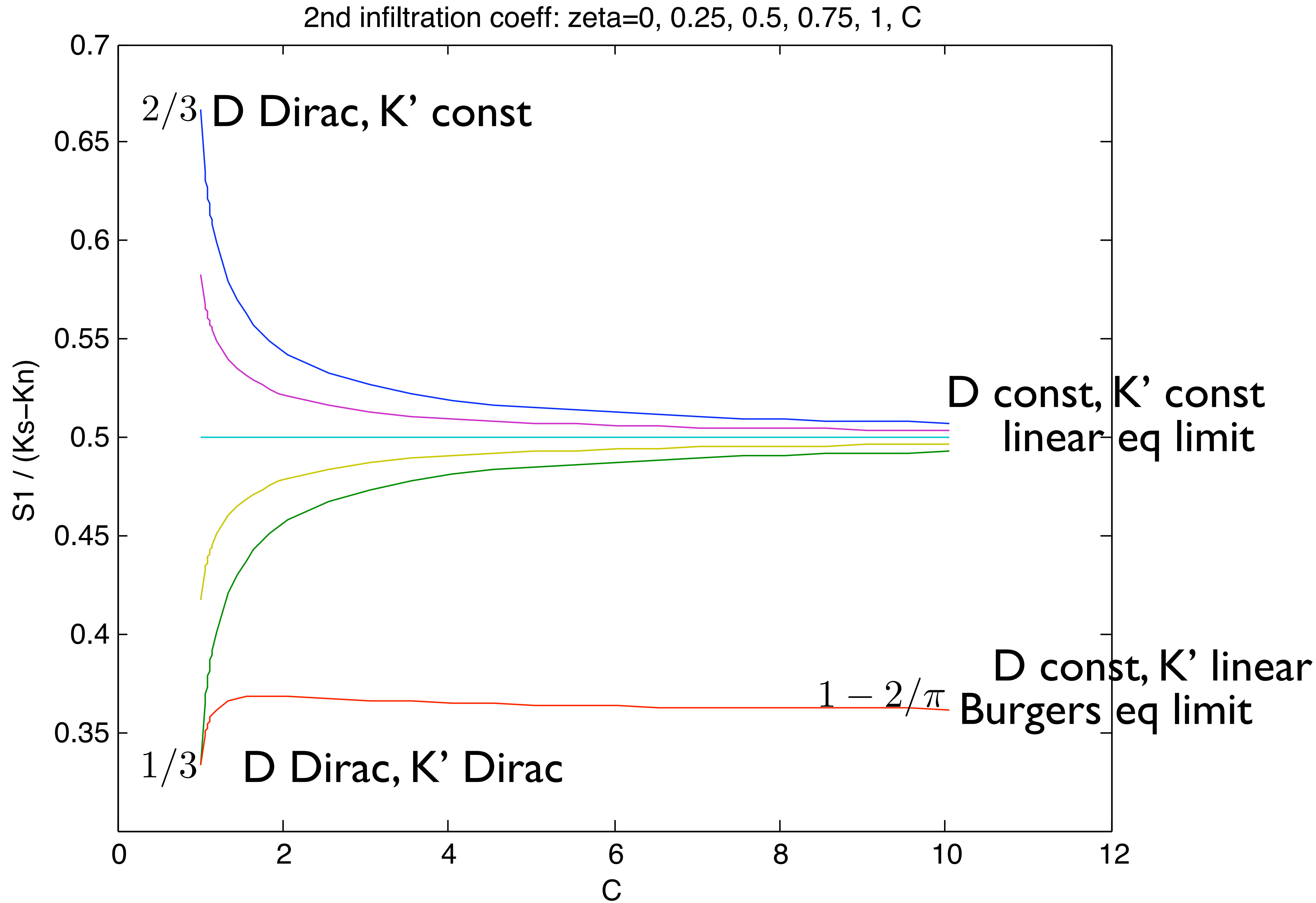
$$t_* = e^{-i_*} + i_* - 1 \;\; ; \quad (C, \zeta) \to 1$$

same as Parlange et al 1982 if their alpha=our zeta.

Talsma 1969: $S_1/K_s = 1/2.8 = 0.357\ldots$

exactly true when $(C, \zeta) = (1.05, 1.04)$.

Water content profile is close to step function but S_1 is quite different from G-A prediction.


2nd infiltration coeff: zeta=0, 0.25, 0.5, 0.75, 1, C
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.35
S1 / (Ks−Kn)
0
2
4
6
8
10
12
C
$2/3$ D Dirac, K' const
D const, K' const
linear eq limit
D const, K' linear
$1-2/\pi$ Burgers eq limit
$1/3$ D Dirac, K' Dirac

Main conclusions:

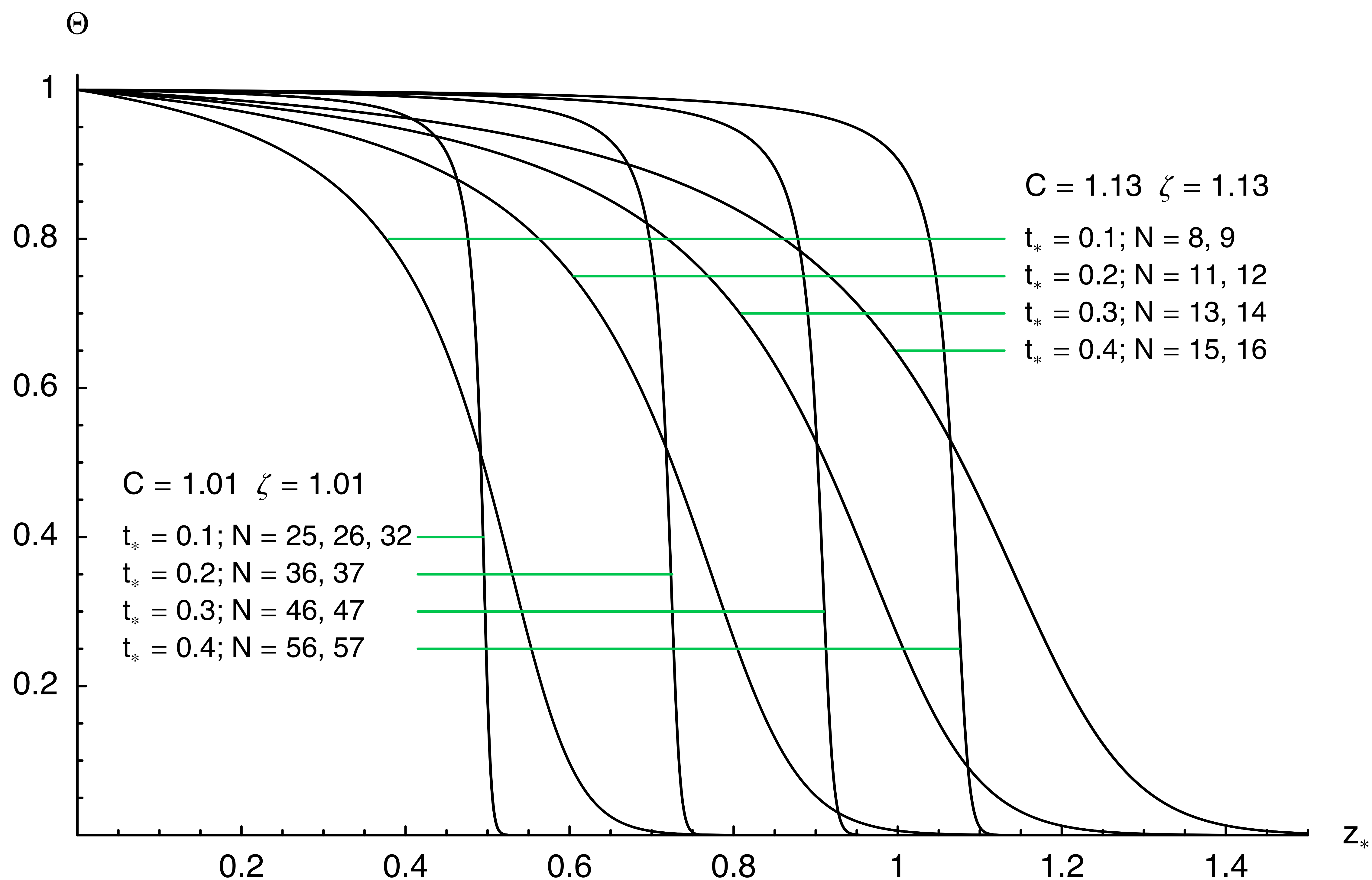
1. After judicious rescaling, approximate integral analysis leads to remarkably simple expressions for measurable quantities from infiltration experiments.

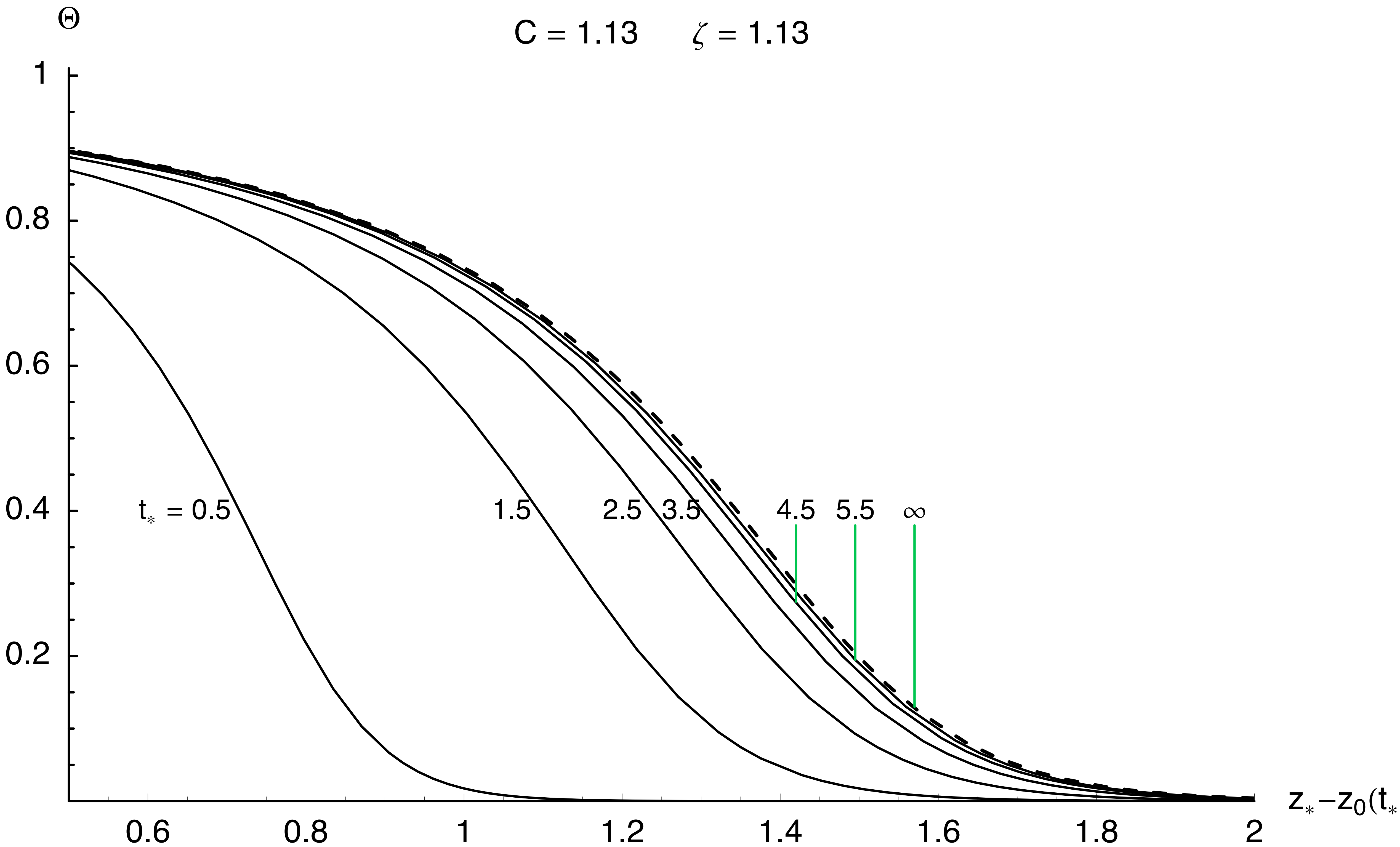
1a: time to ponding

1b: effect of pond depth on sorptivity

1c: second and higher infiltration coefficients

2. These expressions are satisfied exactly by some model Richards equations with reasonable nonlinear hydraulic functions.


Θ
1
0.8
0.6
0.4
0.2
C = 1.13 ζ = 1.13
t* = 0.1; N = 8, 9
t* = 0.2; N = 11, 12
t* = 0.3; N = 13, 14
t* = 0.4; N = 15, 16
C = 1.01 ζ = 1.01
t* = 0.1; N = 25, 26, 32
t* = 0.2; N = 36, 37
t* = 0.3; N = 46, 47
t* = 0.4; N = 56, 57
0.2
0.4
0.6
0.8
1
1.2
1.4
z*


Θ
C = 1.13 ζ = 1.13
1
0.8
0.6
0.4
0.2
t* = 0.5
1.5
2.5
3.5
4.5
5.5
∞
0.6
0.8
1
1.2
1.4
1.6
1.8
2
z*−z0(t*)