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SOLUTION CONCEPTS OF N-PERSON COOPERATIVE GAMES AS POINTS IN THE GAME SPACE

by

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TABLE OF CONTENTS

CHAPTER I

INTRODUCTION AND SUMMARY

In the theory of n-person cooperative games, the term "solution" is used to denote the outcome or expected outcome of a game played in a "rational" manner. The many interpretations of "rational" have lead to a variety of "solution" concepts for n-person cooperative games. The core of a game is perhaps the most widely studied solution concept. This paper characterizes those games which do have nonempty core. This characterization suggests a new way of looking at well known solution concepts such as the core and the nucleolus and also suggests definitions of new solution concepts.

In Section 2.1 we introduce the notion of the core of an m-dimensional vector with respect to a given $m \times n$ matrix B. In this case, an imputation is an n-dimensional probability vector. It is shown that the set of all m-dimensional vectors with nonempty core with respect to the matrix B is a closed unbounded convex polyhedron whose extreme points must be columns of B. In the case when all the columns are extreme points and no two columns are identical, there exists a one-to-one correspondence between the set of all imputations and the convex hull in R_m of the columns of B, and so the set of all imputations can be embedded in R_m in a natural way.

In Section 2.2 we use the results of Section 2.1 to determine the set of all 0-1 normalized n-person games with nonempty core and we describe this set by naming its extreme points. Using the above embedding we interpret the core and a modified nucleolus of an n-person game v in terms of v's "closeness" to the embedding.

In Chapter III, we use metric notions of "closeness" of a game to the embedded imputations to define new solution concepts for a game. Sections

3.1 and 3.2 explore the use of the Euclidean metric and Section 3.3 discusses the relationship between the use of ℓ_p-metrics and an absolute nucleolus which is defined in terms of the absolute values of excesses. If a game has nonempty core, the solution concept defined using the ℓ_1-metric is in fact the core of the game. This solution (called the 1-center) has the additional advantage that it exists for all games. In Section 3.4, we embed a subset of imputations into the game space in another way.

The results of Chapters II and III can be viewed in another way. Some types of games have natural solution concepts, e.g. a natural solution for a monotone simple game with veto players is an equal split among the veto players with the remainder of the players getting nothing. In Chapters II and III we use notions of "closeness" to approximate a given game by convex combinations of these natural games and then use the same convex combinations of the solutions for these natural games as a solution for the original game. The core and modified nucleolus, as well as the various centers defined in Chapter III, can be viewed in this way.

Chapters IV and V are miscellaneous in nature and explore lightly problems that to me appear to be of considerable difficulty. In Chapter IV we find the nonsimple extreme points of the set of all 4-person 0-1 normalized superadditive games. We use this result to demonstrate that the set of all 4-person 0-1 normalized totally balanced games has nonsimple extreme points. In Chapter V we examine games whose core contains its Shapley value. Such games form a closed convex polyhedron some of whose extreme points are nonsimple. The simple extreme points are determined.

CHAPTER II

IMPUTATIONS AS POINTS IN THE GAME SPACE

2.1 The B-core of a Vector

The notion of the core of an n-person game has been studied extensively in the literature of n-person game theory. However, not all n-person games have nonempty cores and it is important to determine which ones, in fact, do have nonempty cores. Bondareva [2], Shapley [16], and Charnes and Kortanek [3] have given the inequalities which determine the polyhedron consisting of all games with nonempty core. These inequalities, which introduce the combinatorial notion of balanced sets, become unwieldy when $n > 4$. One hopes for an alternate and easier way of characterizing this polyhedron. This can be done and the results can be related to Fulkerson's [7] notion of antiblocking polyhedra. This result, in turn, suggests using the game-theoretic notion of a core to interpret some of Fulkerson's results.

In this section we develop the notion of core for an m-dimensional real column vector with respect to a given $m \times n$ matrix. In the following section, these results are applied to 0-1 normalized n-person games. In this application, the set of imputations is embedded in a natural way in the space of n-person games.

Let B be an $m \times n$ matrix whose elements are real numbers and let v be any m-dimensional real column vector. An n-dimensional real column vector x with components x_i for $i = 1,2,\ldots,n$ is called an imputation if and only if $\sum_{i=1}^{n} x_i = 1$ and $x_i \geq 0$ for $i = 1,2,\ldots,n$.

If y and z are two r-dimensional real column vectors, then we write $y \geq z$ if and only if $y_i \geq z_i$ for $i = 1,2,\ldots,r$ where y_i and z_i are the i-th components of the vectors y and z, respectively.

Definition 1. The vector v is said to have nonempty B-core if there exists an imputation x such that

$$Bx \geq v \, . \tag{1}$$

If the vector v has nonempty B-core, then the set of all imputations x satisfying (1) is called the B-core of v.

Let C_B be the convex hull of the columns of B. Let R_m denote the set of all m-dimensional real column vectors and let R_m^+ denote the set of m-dimensional real column vectors all of whose components are nonnegative Also, if C and D are subsets of R_m, then $C + D$ $(C - D)$ is defined to be the set of all vectors $c + d$ $(c - d)$ where $c \in C$ and $d \in D$.

From (1) and the definition of an imputation, we have the following lemma.

Lemma 1. A vector v has nonempty B-core if and only if

$$v = w + u$$

where $w \in C_B$ and $u \in R_m$ and satisfying $u \leq 0$.

Thus, a vector v has nonempty B-core if and only if v lies "in or below" the set C_B, i.e. we have the following corollary.

Corollary 1. The set of all m-dimensional vectors having nonempty B-core is the closed convex polyhedron $C_B - R_m^+$.

Fulkerson [7] calls $(C_B - R_m^+) \cap R_m^+$ the antiblocking polyhedron of the polyhedron

$$B^T w \leq 1$$
(2)
$$w \geq 0 \ .$$

Fulkerson's Theorem 2.2 on page 10 of [7] is similar to our Theorem 1 for $(C_B - R_m^+) \cap R_m^+$ in the case when B is non-negative. The inequalities which determine $C_B - R_m^+$ are themselves determined by the extreme points of the polyhedron described by (2). However, we are primarily concerned with the extreme points of $C_B - R_m^+$. The extreme points of $(C_B - R_m^+) \cap R_m^+$ in the case of n-person games are determined in section 5.1.

Now, we wish to find the extreme points of $C_B - R_m^+$. We will show that each column of B which does not lie "in or below" the convex hull of the remaining columns of B is an extreme point of $C_B - R_m^+$. In fact, such columns are the only extreme points of $C_B - R_m^+$.

Lemma 2. If v is an extreme point of $C_B - R_m^+$, then v is a column of B.

Proof. Let $v \in C_B - R_m^+$ and suppose $v \notin C_B$. If the imputation x is in the B-core of v, then $B_k x > v_k$ for some $k \in \{1,2,\ldots,m\}$ where B_k denotes the k-th row of B. Choosing a number ε so that $0 < \varepsilon < B_k x - v_k$, we define vectors v^1 and v^2 by $v_j^1 = v_j = v_j^2$ if $j \neq k$ and

$$v_k^1 = v_k + \varepsilon, \quad v_k^2 = v_k - \varepsilon \ .$$

v^1 and v^2 are members of $C_B - R_m^+$ since the imputation x is a member of each of their respective B-cores. $v = \frac{1}{2} v^1 + \frac{1}{2} v^2$ and $v \neq v^1$ and so v cannot be an extreme point of $C_B - R_m^+$. Therefore, all the extreme points of $C_B - R_m^+$ are members of C_B. C_B is, in fact, the convex hull of the columns of B and so all the extreme points of $C_B - R_m^+$ must be columns of B. Δ*

For the remainder of this section, we assume that no two columns of B are identical. Therefore, if a column of B lies "in or below" the convex hull of the remaining columns of B, then clearly it cannot be an extreme point of $C_B - R_m^+$. The following lemma states that the converse also holds.

Lemma 3. If a column of B is not an extreme point of $C_B - R_m^+$, then it can be written $w + u$ where $u \leq 0$ and w is a member of the convex hull of the remaining columns of B.

Proof. Let b be a column of B which is not an extreme point of $C_B - R_m^+$. Therefore, $b = \frac{1}{2} v^1 + \frac{1}{2} v^2$ where $b \neq v^1$ and v^1 and v^2 are members of $C_B - R_m^+$. Since v^1 and v^2 are members of $C_B - R_m^+$, $v^1 = (1-\alpha)w^1 + \alpha b$ and $v^2 = (1-\beta)w^2 + \beta b$ where $0 \leq \alpha < 1$, $0 \leq \beta < 1$, w^1 and w^2 are members of $C_{B-b} - R_m^+$, and C_{B-b} is the convex hull of the $n-1$ columns of B which are not equal to b. Therefore,

$$b = \lambda w^1 + (1-\lambda)w^2$$

where $\lambda = \frac{1 - \alpha}{2 - \alpha - \beta}$. We have $1 > \lambda \geq 0$ and so $b \in C_{B-b} - R_m^+$. Δ

The results of Lemma 2 and Lemma 3 are summarized in the following theorem.

* In this and the remaining chapters of this paper, Δ indicates the end of a proof.

<u>Theorem 1</u>. The vector v is an extreme point of $C_B - R_m^+$ if and only if it is a column of B which cannot be written as $w + u$ where $u \leq 0$ and w is a member of the convex hull of the remaining columns of B.

Let B' be the submatrix of B whose columns are the extreme points of $C_B - R_m^+$. By repeated application of the proof of Lemma 3, we obtain the following corollary.

<u>Corollary 2</u>. $C_B - R_m^+ = C_{B'} - R_m^+$.

Corollary 3 discusses the B-cores of vectors in $C_{B'}$.

<u>Corollary 3</u>. If $v \in C_{B'}$, then the B-core of v consists of one and only one imputation. Furthermore, distinct members of $C_{B'}$ have disjoint B-cores.

<u>Proof</u>. Let $v \in C_{B'}$. $C_{B'}$ is the convex hull of the columns of B' and so $v = Bx$ where x is an imputation such that $x_i = 0$ if the i-th column of B, B_i, is not a column of B'. Therefore, the imputation x is in the B-core of v. Suppose the imputation $y \neq x$ is also in the B-core of v. $By \geq v$ and so $By \geq Bx$ or equivalently $B(y-x) \geq 0$. If $S = \{i: y_i - x_i \geq 0\}$, then S is not empty and there exists an $i \in \{1,2,\ldots,m\}$ such that $i \notin S$. Therefore,

$$\sum_{i\in S} B_i(y_i - x_i) \geq \sum_{i\notin S} B_i(x_i - y_i) \quad (3)$$

Let $\alpha = \sum_{i\in S}(y_i - x_i) = \sum_{i\notin S}(x_i - y_i) > 0$. If $i \notin S$, then $x_i > 0$ and B_i is an extreme point of $C_B - R_m^+$. By Lemma 3, if $i \notin S$, then

$$B_i > \frac{1}{\alpha}\sum_{k\in S} B_k(y_k - x_k) .$$

Therefore,

$$\sum_{i \notin S} B_i(x_i - y_i) > \sum_{i \in S} B_i(y_i - x_i) ,$$

and this contradicts (3). Therefore, the B-core of v consists of the single imputation x.

A similar argument proves that distinct members of $C_{B'}$ have disjoint B-cores. Δ

Corollary 4. If $v = B_k$ where B_k is an extreme point of $C_B - R_m^+$, then the B-core v consists of the single imputation x given by

$$x_i = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} . \end{cases}$$

The following corollary is an immediate consequence of Corollary 3 and states that the set of all imputations can, under certain circumstances, be imbedded in R_m in a natural way.

Corollary 5. If each column of B is an extreme point of $C_B - R_m^+$, then there exists a one-to-one correspondence between the set of all imputations and the set C_B.

Proof. If each column of B is an extreme point of $C_B - R_m^+$, then $B' = B$. By Corollary 3 the B-core of a vector v in C_B consists of one and only one imputation, call it x^v. By Corollary 3, the mapping f from C_B to the set of all imputations defined by $f(v) = x^v$ is one-to-one and onto. Δ

Corollary 6 is a restatement of Definition 1 using Lemma 1.

<u>Corollary 6.</u> A vector v has nonempty B-core if and only if $(\{v\} + R_m^+) \cap C_B$ is not empty, where $\{v\}$ denotes the subset of R_m containing the vector v only. Also, if v has nonempty B-core, then the B-core of v consists of all imputations x for which $Bx \in (\{v\} + R_m^+) \cap C_B$.

The important point about Corollary 6 is that the set C_B in R_m is used to determine whether or not a vector v has nonempty B-core. If the vector v has nonempty B-core, then C_B also determines the B-core of v.

2.2 Games with Nonempty Core and an Interpretation of Core in the Game Space

In this section, we want to apply the results of Section 2.1 to the classical theory of n-person cooperative games.

Let n be a fixed integer and let $N = \{1,2,\ldots,n\}$. Each member of N is called a <u>player</u> and each subset of N is called a <u>coalition</u>. An <u>n-person game</u> is a function from the nonempty subsets of N to the real numbers.*
An n-person game v is said to be in 0-1 <u>normal form</u> whenever

(1) $$v(N) = 1$$

(2) $$v(\{i\}) = 0 \quad \text{for} \quad i = 1,2,\ldots,n.$$

If w is an n-person game and if $w(N) - \sum_{i=1}^{n} w(\{i\}) > 0$, then w is said to be <u>strategically equivalent</u> to the 0-1 normal n-person game u if and only if

$$u(S) = \frac{w(S) - \sum_{i\in S} w(\{i\})}{w(N) - \sum_{i\in N} w(\{i\})} \quad \text{for all nonempty} \quad S \subseteq N.$$

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* It is common in the literature of game theory to also require that an n-person game satisfy the <u>superadditivity</u> requirement that $v(S) + v(T) \leq v(S \cup T)$ for all $S,T \subset N$ such that $S \cap T = \emptyset$. We will not impose this requirement here. The superadditivity requirement is considered only in Chapter IV.

In this and the following chapters, all games are assumed to be in 0-1 normal form unless specifically stated otherwise. Therefore, each 0-1 normal n-person game can be represented as a vector in R_{2^n-n-2}. R_{2^n-n-2} is called the <u>n-person game space</u>.

For a fixed n, an n-dimensional column vector x with components x_i for $i = 1,2,\ldots,n$ is called an <u>imputation</u> if $\sum_1^n x_i = 1$ and $x_i \geq v(\{i\}) = 0$ for $i = 1,2,\ldots,n$. An n-person game v is said to have <u>nonempty core</u> if there exists an imputation x such that

$$\text{(3)} \qquad \sum_{i\varepsilon S} x_i \geq v(S) \quad \text{for all} \quad S \subset N \quad \text{such that} \quad n > |S| > 1,$$

where $|S|$ is the number of players in coalition S. The set of all imputations x satisfying (3) for the game v is called the <u>core</u> of v.

If B is the $(2^n-n-2)Xn$ matrix of coefficients for the inequalities in (3), then the core of an n-person game v is, in fact, the B-core of v as defined in Section 2.1. Therefore, the results of Section 2.1 can be applied to the study of n-person games with nonempty core.

If for each player i we defined the n-person game v_i by

$$\text{(4)} \qquad v_i(S) = \begin{cases} 1 & \text{if } i \varepsilon S \text{ and } |S| > 1 \\ 0 & \text{otherwise} \end{cases}$$

then the games $v_1,v_2,\ldots,v_n$ are, in fact, the n columns of B. Also, the games $v_1,v_2,\ldots,v_n$ are monotone simple games each with one veto player.*

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* An n-person game v is <u>monotone</u> if and only if $T \subset S \subseteq N$ implies $v(T) \leq v(S)$. An n-person game v is <u>simple</u> if and only if $v(S)$ is 0 or 1 for all $S \subset N$. Player i is a <u>veto player</u> for the simple n-person game v if and only if $v(S) = 1$ implies $i \varepsilon S$.

Let C_n be the convex hull of the games $v_1, v_2, \ldots, v_n$. The following theorem is a direct application of Corollary 1 of Section 2.1.

Theorem 1. The set of all n-person games having nonempty core is the closed convex polyhedron $C_n - R^+_{2^n-n-2}$.

The next lemma is an application of Lemma 2 of Section 2.1.

Lemma 1. If v is an extreme point of $C_n - R^+_{2^n-n-2}$, then $v \in \{v_1, v_2, \ldots, v_n\}$.

Lemma 2. Each n-person game v_i defined in (4) is an extreme point of $C_n - R^+_{2^n-n-2}$.

Proof. If v_1 is not an extreme point of $C_n - R^+_{2^n-n-2}$, then from Lemma 3 of section 2.1 there exists non-negative real numbers $\lambda_2, \lambda_3, \ldots, \lambda_n$ such that $\sum_2^n \lambda_i = 1$ and satisfying

(5) $\quad v_1(S) \leq \lambda_1 v_2(S) + \ldots + \lambda_n v_n(S)$ for all $S \subset N$ such that $n > |S| > 1$.

For $n = 2$, (5) clearly cannot hold. If $n \geq 3$, then for $S = \{1,2\}$ and $S = \{1,3\}$, (5) gives the following inequalities

$$1 \leq \lambda_2$$

$$1 \leq \lambda_3$$

both of which clearly cannot hold. Therefore, v_1 must be an extreme point of $C_n - R^+_{2^n-n-2}$. Similarly, $v_2, v_3, \ldots, v_n$ are extreme points of $C_n - R^+_{2^n-n-2}$. Δ

Lemma 1 and Lemma 2 are summarized in the following theorem.

Theorem 2. The simple games $v_1, v_2, \ldots, v_n$ are the extreme points of $C_n - R^+_{2^n-n-2}$.

Applying Corollary 3 of section 2.1, we obtain the following corollary to Theorem 2.

Corollary 1. If $v \in C_n$, then the core of v consists of only one imputation. Furthermore, if $v' \in C_n$, $v' \neq v$, and x, x' are the imputations in their respective cores, then $x \neq x'$.

The next corollary follows from Corollary 5 of section 2.1 and states that the set of all imputations can be embedded in the game space R_{2^n-n-2} in a natural way.

Corollary 2. There exists a one-to-one correspondence between the set of imputations and the set C_n in R_{2^n-n-2}.

The next corollary is an application of Corollary 6 of section 2.1.

Corollary 3. An n-person game v has nonempty core if and only if $(\{v\} + R^+_{2^n-n-2}) \cap C_n$ is nonempty. Also, if v has nonempty core, then the core of v consists of all imputations x for which

$$x_1 v_1 + x_2 v_2 + \ldots + x_n v_n \in (\{v\} + R^+_{2^n-n-2}) \cap C_n .$$

The important point about Corollary 3 is that the set C_n in the game space R_{2^n-n-2} is used to determine whether or not a game has nonempty core. If v has nonempty core, C_n also determines the core of v.

2.3 A Modified Nucleolus and its Interpretation in the Game Space

Schmeidler [14] has introduced the nucleolus as a solution concept in n-person game theory. Schmeidler [14] has shown that the nucleolus consists of a unique imputation for each game. Also, the nucleolus is continuous [14] and piecewise linear [10], [4] as a function from games to imputations. The nucleolus is contained in the kernel of Davis and Maschler [6] which itself is contained in the bargaining set $M_1^{(i)}$ of Aumann and Maschler [1]. If a game has nonempty core, then it's core contains the nucleolus [14].

In this section we introduce a modified nucleolus which has the above properties of the nucleolus, except as regards the kernel, and can be interpreted in the game space R_{2^n-n-2} in terms of C_n. A modified kernel can be introduced which contains the modified nucleolus and itself is contained in the bargaining set $M_1^{(i)}$.

For each imputation x and each coalition $S \subset N$, we let

$$x(S) = \sum_{i\epsilon S} x_i$$

For a fixed n-person game v and a fixed imputation x, let $\Theta(x)$ be a vector in R_{2^n-n-2}, the components of which are the numbers $v(S) - x(S)$, arranged according to their magnitude, where S runs over the proper subsets of N such that $|S| > 1$, i.e. $i < j$ implies $\Theta_i(x) \geq \Theta_j(x)$. We say $\Theta(x)$ is lexicographically smaller than $\Theta(y)$, written $\Theta(x) <\cdot\ \Theta(y)$ if and only if the first non-zero component of $\Theta(y) - \Theta(x)$ is positive. The <u>modified nucleolus</u> for the n-person game v is the set $\bar{N}(v)$ given by

$$\bar{N}(v) = \{x \in A: \Theta(x) \underline{<}\cdot\ \Theta(y) \quad \text{for all} \quad y \in A\}\ .^*$$

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* A is the set of all n-dimensional imputations.

In Schmeidler's definition of the nucleolus (see page 1163 of [14]), the excesses $x(S) - v(S)$ for the one-person coalitions are also components of $\Theta(x)$ and hence in his solution concept, the one-person coalitions have two roles: first, they are included in the definition of an imputation by the requirement $x_i \geq v(\{i\})$ for $i \in N$; and second, the excesses $x_i - v(\{i\})$ are components of $\Theta(x)$. Given the first role, this second role seems perhaps to give the one-person coalitions "too much" influence in determining the final outcome of the game. One may argue that once each player i has been guaranteed at least $v(\{i\})$, the role of the one-person coalitions is ended and the final outcome must be determined by only the coalitions containing more than one player.

The following theorems on the modified nucleolus are proved exactly in the same manner as the corresponding theorems for the nucleolus.

Theorem 1. The modified nucleolus of a game consists of one and only one imputation (see Theorems 1 and 2 on page 1164 of [14]).

We now define a modified kernel. If v is an n-person game, then for each pair of distinct players i and j and each imputation x set

$$s_{ij}(x) = \text{maximum } \{v(S) - x(S): \ S \subsetneq N, \ i \in S, \ j \notin S, \ |S| > 1\}$$

An imputation x is said to be in the modified kernel $\bar{K}(v)$ of a game v if and only if

$$(s_{ij}(x) - s_{ji}(x))x_j \leq 0 \quad \text{for all pairs of distinct players } i \text{ and } j.$$

Theorem 2. For an n-person game v, $\bar{N}(v) \subset \bar{K}(v) \subset M_1^{(i)}$ (see Theorem 3 on page 1165 of [14] and Theorem 5.1 on page 233 of [6]).

Proof. To prove $\bar{N}(v) \subset \bar{K}(v)$ one can apply Schmeidler's proof on page 1167 of [14] of the corresponding result for the nucleolus. Similarly, to prove $\bar{K}(v) \subset M_1^{(i)}$ one can apply the proof on page 233 of [6] of the corresponding result of Davis and Maschler for the kernel. This result holds because in the definition of $M_1^{(i)}$ the one person coalitions cannot make objections (see [1]). Δ

Theorem 3. The modified nucleolus is a continuous piecewise linear function of the characteristic function (see Theorem 5 on page 1165 of [14] and Theorem 4 on page 64 of [11]).

Theorem 4. The modified nucleolus is contained in the core of any game with nonempty core (see Theorem 4 on page 1165 of [14]).

We now want to interpret the modified nucleolus in terms of C_n. For a given n-person game v and for a fixed game $u \varepsilon C_n$, let $\phi(u)$ be a vector in R_{2^n-n-2}, the components of which are the numbers $v(S) - u(S)$, arranged according to their magnitude, where S runs over the proper subsets of N such that $|S| > 1$, i.e. $i < j$ implies $\phi_i(x) \geq \phi_j(x)$. We say that the game u is lexicographically closest in C_n to v if and only if the first nonzero component of $\phi(w) - \phi(u)$ is positive for all $w \varepsilon C_n$.

Theorem 5. The game $u \varepsilon C_n$ is lexicographically closest to a given n-person game v if and only if

$$u = x_1 v_1 + x_2 v_2 + \ldots + x_n v_n$$

where $x \varepsilon \bar{N}(v)$.

Proof. For each imputation x, $\Theta(x) = \phi(x_1v_1 + x_2v_2 + \ldots + x_nv_n)$ since

$$x_1v_1(S) + x_2v_2(S) + \ldots + x_nv_n(S) = x(S) \quad \text{for all } S \subseteq N \text{ such that } |S| > 1.$$

Δ

CHAPTER III

CENTERS OF GAMES

3.1 The Two-center of a Game

In the previous chapter we interpreted the core and the modified nucleolus of an n-person game v in terms of v's "closeness" to the set C_n in R_{2^n-n-2}. This appears to be a reasonable thing to do because the set of all imputations was identified in a natural way with the set C_n. In this section we will define new solution concepts for a game v in terms of other notions of v's closeness to the set C_n.

The imputation associated with the game in C_n closest in Euclidean distance to a given game v seems a reasonable one point solution concept for the game v. This solution, called a two-center, is defined below, its properties are discussed, and an algorithm for its computation is developed.*

Let n be a fixed positive integer and let $N = \{1,2,\ldots,n\}$ be the set of players. Throughout this chapter we will use the following convenient notation:

$$Q = \{S \subset N: n > |S| > 1\}$$

$$Q_i = \{S \in Q: i \in S\} \quad \text{for } i = 1,2,\ldots,n .$$

Definition 1. Let v be an n-person game. An imputation is called the two-center of v if it is a solution to the following constrained minimization problem

* Charnes and Kortanek [4] and Keane [9] also define "nuclei" for n-person games using the Euclidean metric, but their definitions differ from the one presented here (in fact, their nuclei need not be imputations). My analysis proceeds in an entirely different direction than theirs.

$$\text{minimize } ||v - x_1v_1 - \ldots - x_nv_n||_2$$

subject to

(I)
$$x_1 + x_2 + \ldots + x_n = 1$$

$$x_i \geq 0 \quad \text{for} \quad i = 1,2,\ldots,n \ ,$$

where $||w||_2 = \sum_{S\varepsilon Q} [w(S)]^2$, i.e. $||\ \ ||_2^{1/2}$ is the Euclidean metric in R_{2^n-n-2}.

It follows from the definitions of the games $v_1, v_2, \ldots, v_n$ that the minimization problem (I) in Definition 1 can be rewritten:

$$\text{minimize } \sum_{S\varepsilon Q} [v(S) - x(S)]^2$$

subject to

(II)
$$x_1 + x_2 + \ldots + x_n = 1$$

$$x_i \geq 0 \quad \text{for} \quad i = 1,2,\ldots,n \ .$$

For a given game v, $v(S) - x(S)$ is called the excess of coalition S with respect to the imputation x.

Recall that C_n is a convex set in R_{2^n-n-2}. The following theorem is a direct application of a well-known theorem on minimal distance problems over convex sets in a Hilbert Space (see, for example, Theorem 1 on page 69 of [12]).

Theorem 1. If v is an n-person game, then there exists a unique game $u \in C_n$ such that

$$||v-u||_2 \leq ||v-w||_2 \quad \text{for all} \quad w \in C_n \ .$$

Furthermore, a necessary and sufficient condition that u be the unique minimization game is that

$$(1) \qquad \sum_{S \in Q} (v(S) - u(S))(w(S) - u(S)) \leq 0 \quad \text{for all} \quad w \in C_n \ .$$

Since C_n is the convex hull of the set $\{v_1, v_2, \ldots, v_n\}$, (1) holds for all $w \in C_n$ if and only if it holds for each v_i where $i = 1,2,\ldots,n$. Thus, (1) in the above theorem can be replaced by

$$(1') \quad \sum_{S \in Q_i} (v(S) - u(S)) \leq \sum_{S \in Q} u(S)(v_i(S) - u(S)) \quad \text{for} \quad i = 1,2,\ldots,n \ .$$

Corollary 2 of Section 2.2 states that each game in C_n is identified with a unique imputation. This identification together with Definition 1 imply that the two-center of a game v is uniquely determined by the game u of Theorem 1. This fact together with the simple equation

$$\sum_{S \in Q} x(S)(v(S) - x(S)) = \sum_{j=1}^{n} x_j \sum_{S \in Q_j} (v(S) - x(S)) \ ,$$

where x is an imputation, give the following theorem.

<u>Theorem 2</u>. Let v be an n-person game and for each imputation x let

$$w_i(x) = \sum_{S \in Q_i} (v(S) - x(S)) \quad \text{for} \quad i = 1,2,\ldots,n \ .$$

An imputation x is the two-center for the game v if and only if

$$w_i(x) \leq \sum_{j \in N} x_j w_j(x) \quad \text{for} \quad i = 1,2,\ldots,n \ .$$

The following corollaries follow immediately from Theorem 2.

<u>Corollary 1.</u> x is the two-center of game v if and only if $x_j > 0$ implies

$$w_j(x) = \text{maximum } \{w_i(x)\colon i = 1,2,\ldots,n\} \ .$$

<u>Corollary 2.</u> x is the two-center of game v if and only if $w_j(x) < w_i(x)$ for some players i and j implies $x_j = 0$.

The above corollaries hint that the two-center of a game has a "nucleolus-like" interpretation. Theorem 3 demonstrates that this is indeed so.

Given an n-person game v and an imputation x, let $\gamma(x)$ be a vector in R_n, the components of which are the numbers $w_i(x)$, defined in Theorem 2, arranged according to their magnitude, i.e. $i < j$ implies $\gamma_i(x) \geq \gamma_j(x)$. We say $\gamma(x)$ is lexicographically smaller than $\gamma(y)$, written $\gamma(x) <\cdot\ \gamma(y)$ if and only if the first non-zero component of $\gamma(y) - \gamma(x)$ is positive. The <u>total nucleolus</u> $\underline{N}(v)$ of the game v is the set of all imputations x for which

$$\gamma(x) \leq\cdot\ \gamma(y) \quad \text{for all imputations} \quad y \ .$$

<u>Lemma 1.</u> If v is an n-person game and x is an imputation, then

$$w_i(x) = p_i(v) - (2^{n-2}-1)(1+x_i) \quad \text{for} \quad i = 1,2,\ldots,n$$

where $p_i(v) = \sum_{S\varepsilon Q_i} v(S)$ for $i = 1,2,\ldots,n$.

Proof. For a given imputation x,

$$\begin{aligned} w_i(x) &= \sum_{S\varepsilon Q_i} (v(S) - x(S)) \\ &= p_i(v) - \sum_{S\varepsilon Q_i} x(S) \\ &= p_i(v) - (2^{n-1}-2)x_i - (2^{n-2}-1) \sum_{j\neq i} x_j \\ &= p_i(v) - (2^{n-1}-2)x_i - (2^{n-2}-1)(1-x_i) \\ &= p_i(v) - (2^{n-1}-1)(1+x_i) \quad \text{for } i = 1,2,\ldots,n . \end{aligned}$$

Δ

Theorem 3. The total nucleolus of a game consists of one and only one imputation and it is the two-center.

Proof. The existence of $\underline{N}(v)$ for a game v is verified by methods similar to those of Schmeidler (see proof of Theorems 1 and 2 on page 1166 of [14]).

Let x be an imputation in the total nucleolus of game v. Lemma 1 states that

$$(2) \qquad w_i(x) = p_i(v) - (2^{n-2}-1)(1+x_i) \quad \text{for } i = 1,2,\ldots,n .$$

If $x_i > 0$ and $x_j > 0$ for distinct players i and j, then (2) implies that $w_i(x) = w_j(x)$. If $x_j = 0$, then (2) also implies that $w_j(x) \leq w_i(x)$

for all i such that $x_i > 0$. Corollary 1 implies that x must be the two-center of game v. Therefore, $\underline{N}(v)$ must consist of one and only one imputation. Δ

Kohlberg uses the fact that the nucleolus of a game can be computed by means of a finite sequence of linear programs to show that the nucleolus is a continuous piecewise linear function from R_{2^n-n-2} to R_n (see Theorem 4 on page 64 of [11]). [Schmeidler [14] had previously shown the continuity of the nucleolus by other means.] The total nucleolus can also be computed by a sequence of linear programs (a simpler algorithm will be presented in the next section) and hence we have the following theorem.

Theorem 4. The two-center of a game is a continuous piecewise linear function from R_{2^n-n-2} to R_n.

The following corollary follows immediately from Theorem 2 and its corollaries.

Corollary 3. If V_S is the set of all n-person games whose two-centers have the i-th component positive when $i \in S$ and zero when $i \notin S$, then the two-center is a linear function over V_S.

The computational algorithm of the next section will determine the inequalities which characterize the regions of linearity for the two-center.

Two-centers for 0-1 normal games are defined by (II). If a game v is not 0-1 normalized, then the two-center for the game v is the unique solution to the following minimization problem

$$\text{minimize} \quad \sum_{\{S \subsetneq N:\ |S|>1\}} [v(S) - x(S)]^2$$

subject to

$$x_1 + x_2 + \ldots + x_n = v(N)$$

$$x_i \geq v(\{i\}) \quad \text{for} \quad i = 1,2,\ldots,n \ ;$$

where it is assumed that $v(N) - \sum_{i \in N} v(\{i\}) > 0$. This minimization problem differs from (II) only in the constraint region. All the above results hold for sets of n-person games for which $v(N)$ and $v(\{i\})$ for $i = 1,2,\ldots,n$ are fixed. In particular, Theorem 2 holds and the following properties of the two-center are simple consequences of Theorem 2.

Let x^v be the two-center of game v, not necessarily in 0-1 normal form.

Monotony. (i) x_i^v is a monotonically non-decreasing function of $v(S)$ for each S for which $i \in S$.

(ii) x_i^v is a monotomically non-increasing function of $v(S)$ for each S for which $i \notin S$.

Symmetry. A game v is symmetric in players i and j if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subset N$ such that $i \notin S$, $j \notin S$. If v is symmetric in players i and j, then $x_i^v = x_j^v$.

Equivalence. If, for n-person games v and w, there are constants $\{a_1, a_2, \ldots, a_n\}$ such that $w(S) = v(S) + \sum_{i \in S} a_i$ for all S, then $x_i^w = x_i^v + a_i$ for $i = 1,2,\ldots,n$.

Homogeneity. $x^{cv} = cx^v$ for every constant $c > 0$.

When the above two properties hold for a solution concept, the solution concept is said to satisfy the property of <u>strategic equivalence</u>.

<u>Inessential Game</u>. If v is additive, i.e. $v(S) = \sum_{i\varepsilon S} v(\{i\})$, for all $S \subseteq N$, then $x_i^v = v(\{i\})$ for $i = 1,2,\ldots,n$.

<u>Pure Bargaining Game</u>. If $v(S) = 0$ for all $S \subsetneq N$, then $x_i^v = \frac{1}{n} v(N)$ for $i = 1,2,\ldots,n$ if $v(N) \geq 0$.

Unfortunately, the following results hold:

(a) The two-center of a game is not necessarily in its core.

(b) A dummy player, i.e. a player i for which $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N$ such that $i \notin S$, for game v may have $x_i^v > 0$.

The following example illustrates both (a) and (b):

Let $N = \{1,2,3,4,5\}$ and let v be the simple game defined by

$$v(1234) = v(12345) = 1$$

$$v(S) = 0 \quad \text{for all other } S \subset N.$$

Player 5 is a dummy player for this game. The two-center of v is $(8/35,8/35,8/35,8/35,3/35)$ which is not in the core of v.

3.2 An Algorithm for Finding the Two-Center of a Game

The following algorithm is in fact a search for the Lagrange Multipliers of problem (II) to determine the two-center of a given game. The algorithm is computationally easy and has a certain intuitive appeal as a bargaining procedure among the players. Each player has an "index of power" and uses this power in the bargaining procedure to sequentially weed out those players whose power is too low to sustain their participation in the bargaining. At the conclusion of the bargaining procedure, the strong players are paid off in accordance with their index of power.

Let v be an n-person game. Recall that $p_i(v) = \sum_{S\varepsilon Q_i} v(S)$ for $i = 1,2,\ldots,n$. $p_i(v)$ is the "index of power" of player i. Suppose without loss of generality that

$$p_1(v) \leq p_2(v) \leq \ldots \leq p_n(v) . \tag{1}$$

Let $y^1 \varepsilon R_n$ and $\lambda_1 \varepsilon R_1$ be determined by the following $n + 1$ equations:

$$\sum_{S\varepsilon Q_i} v(S) - y^1(S) = \lambda_1 \quad \text{for } i = 1,2,\ldots,n$$

$$y_1^1 + y_2^1 + \ldots + y_n^1 = 1 \tag{2}$$

Hence,

$$y_i^1 = \frac{p_i(v) - \lambda_1}{2^{n-2} - 1} - 1 \quad \text{for } i = 1,2,\ldots,n$$

and by (2)

$$\lambda_1 = \frac{\sum_1^n p_i(v) - (2^{n-2}-1)}{n} - (2^{n-2}-1) .$$

Therefore, we have

$$y_i^1 = \frac{1}{n} + \frac{p_i(v) - \frac{1}{n}\sum_{j=1}^{n} p_j(v)}{2^{n-2} - 1} \quad \text{for } i = 1,2,\ldots,n .$$

If $y_1^1 \geq 0$, then Theorem 1 will demonstrate that y^1 is the two-center of game v.

If $y_1^1 < 0$, then define $y^2 \in R_n$ by

$$y_1^2 = 0$$

$$y_i^2 = \frac{1}{n-1} + \frac{p_i(v) - \frac{1}{n-1}\sum_{j=2}^{n} p_j(v)}{2^{n-2} - 1} \quad \text{for } j = 2,3,\ldots,n ,$$

i.e. player 1 is dropped from the bargaining and the remaining players reaccess the import of their own power in determining an outcome. If $y_2^2 \geq 0$, then Theorem 1 will demonstrate that y^2 is the two-center of game v.

In general, if $y_k^k < 0$, then define $y^{k+1} \in R_n$ by

$$y_i^{k+1} = 0 \quad \text{if } i = 1,2,\ldots,k$$

$$y_i^{k+1} = \frac{1}{n-k} + \frac{p_i(v) - \frac{1}{n-k}\sum_{j=k+1}^{n} p_j(v)}{2^{n-2} - 1} \quad \text{for } i = k+1,k+2,\ldots,n .$$

If $y_{k+1}^{k+1} \geq 0$, then by Theorem 1, y^{k+1} is the two center of v. Otherwise, define y^{k+2}. Clearly, this process must terminate after at most n steps. Theorem 1 will demonstrate that the imputation determined by this procedure is the two-center of v.

<u>Theorem 1</u>. If v is an n-person game satisfying (1) and x is the imputation defined by the above procedure, then x is the two-center for game v.

<u>Proof</u>. Suppose the above procedure defines $x = y^k$. In this case,

$$w_i(x) = p_i(v) - (2^{n-2}-1)\left(1 + \frac{1}{n-k+1} + \frac{p_i(v) - \frac{1}{n-k+1}\sum_{j=k}^{n} p_j(v)}{2^{n-2} - 1}\right)$$

$$= -(2^{n-2}-1)(1 + \frac{1}{n-k+1}) + \frac{1}{n-k+1}\sum_{j=k}^{n} p_j(v) \quad \text{for } i = k,k+1,\ldots,n .$$

Clearly, $w_i(x) = w_j(x)$ for $i = k,k+1,\ldots,n$. We must now show that $w_i(x) \leq w_k(x)$ for $i = 1,2,\ldots,k-1$. For $i < k$,

$$w_i(x) = p_i(v) - (2^{n-2}-1)$$

and by (1) we need only show that $w_{k-1}(x) \leq w_k(x)$, i.e. we must show

$$p_{k-1}(v) < \frac{\sum_{j=k}^{n} p_j(v) - (2^{n-2}-1)}{n-k+1} .$$

The fact that $y_{k-1}^{k-1} < 0$ implies that

$$p_{k-1}(v) < \frac{\sum_{j=k-1}^{n} p_j(v) - (2^{n-2}-1)}{n-k+2}$$

$$< \frac{\sum_{j=k}^{n} p_j(v) - (2^{n-2}-1)}{n-k+2} + \frac{\sum_{j=k-1}^{n} p_j(v) - (2^{n-2}-1)}{(n-k+2)^2}$$

$$\vdots$$

$$< \left(\left(\sum_{j=k}^{n} p_j(v)\right) - (2^{n-2}-1)\right) \sum_{m=1}^{\infty} \frac{1}{(n-k+2)^m}$$

$$= \frac{\sum_{j=k}^{n} p_j(v) - (2^{n-2}-1)}{n-k+1}$$

Therefore, the imputation x satisfies the conditions of Theorem 2 of Section 3.1 and so x is the two-center of game v. Δ

3.3 p-Centers of a Game and an Absolute Nucleolus

In the previous section the two-center of a game was determined by finding the game in C_n closest in Euclidean distance to the given game. Though C_n seems to be a reasonable set by which to define such solution concepts, it is not at all clear that the Euclidean metric (or any metric) is a reasonable motion of closeness for games. In this section we will define the p-center of a game using the ℓ_p-metric and show that as p approaches infinity, the p-center of a game converges to a "nucleolus" which differs from the modified nucleolus in that absolute values of excesses are used to define it rather than the excesses themselves.

Definition 1. Let v be an n-person game. An imputation is called a p-center of v if it is a solution to the following constrained minimization problem:

$$\text{minimize } ||v - x_1 v_1 - x_2 v_2 - \ldots - x_n v_n||_p$$

subject to

$$x_1 + x_2 + \ldots + x_n = 1$$

$$x_i \geq 0 \quad \text{for} \quad i = 1,2,\ldots,n$$

where $||w||_p = \sum_{S\varepsilon Q} [w(S)]^p$, i.e. $|| \ ||_p^{1/p}$ is the Minkowski p-metric in R_{2^n-n-2} associated with the real number $p \geq 1$.

From remarks in the previous section, the objective function of problem (I) can be written:

$$\sum_{S\varepsilon Q} |v(S) - x(S)|^p \ . \tag{1}$$

Since (1) defines a strictly convex function over R_{2^n-n-2} for $p > 1$ and since C_n is a convex set, the p-center of a game always exists and is a unique point (see for example page 263 of [13]). Clearly, the p-center is a continuous function of v from R_{2^n-n-2} to R_n and satisfies the condition of strategic equivalence.

For a fixed n-person game v and an imputation x, let $\psi(x)$ be a vector in R_{2^n-n-2}, the components of which are the numbers $|v(S) - x(S)|$, arranged according to their magnitude, where S runs over the proper subsets of N satisfying $|S| > 1$. $\psi(x)$ is lexicographically smaller than $\psi(y)$, written $\psi(x) < \psi(y)$, if and only if the first nonzero component of $\psi(x) - \psi(y)$ is positive. The <u>absolute nucleolus</u> $\hat{N}(v)$ of a game v is defined to be the set of all imputations x for which $\psi(x) \underline{<\cdot} \psi(y)$ for all imputations y.

By arguments analogous to Schmeidlers for the nucleolus, $\hat{N}(v)$ consists of one and only one imputation and $\hat{N}(v)$ is a continuous function of v from R_{2^n-n-2} to R_n (see the proofs of Theorem 2 on page 1166 and Theorem 5 on page 1167 of [14]).

<u>Lemma 1</u>. If v is an n-person game and x is an imputation not in $\hat{N}(v)$, then there exists a number $p_x > 1$ such that x is not a p-center for v for all $p \geq p_v$.

<u>Proof</u>. Suppose $y \in \hat{N}(v)$. Since $x \notin \hat{N}(v)$, the first nonzero component of the vector $\psi(x) - \psi(y)$ must be positive. Suppose it is the k-th component and let $d = \psi_k(x) - \psi_k(y)$. Then, there exists a $p_x > 1$ such that

$$|\psi_k(y)+d|^p > [2^n-n-1-k] \ |\psi_k(y)|^p \quad \text{for all} \quad p > p_x \ .$$

Thus,
$$\begin{aligned} ||v - \sum_{i=1}^{n} y_i v_i||_p &\leq [2^n-n-1-k] \ |\psi_k(y)|^p + \sum_{j=1}^{k-1} |\psi_k(y)|^p \\ &\leq |\psi_k(y)+d|^p + \sum_{j=1}^{k-1} |\psi_k(x)|^p \\ &\leq ||v - \sum_{i=1}^{n} x_i v_i||^p \quad \text{for all} \quad p \geq p_x \ . \end{aligned}$$

Hence, x cannot be the p-center of game v for all $p \geq p_x$. Δ

Lemma 2. If v is an n-person game, A is the set of all imputations, and D is a closed subset of $A - \hat{N}(v)$, then there exists a number $p(D) > 1$ such that the set D contains no p-centers of v for all $p \geq p(D)$.

Proof. By the previous lemma, for each point x in D there exists a number $p_x > 1$ such that x is not a p-center of v for all $p \geq p_x$. Since each component ψ_k of ψ is a continuous function over A, then there exists a neighborhood $M(x)$ of x in A such that no point of $M(x)$ is a p-center for v for all $p \geq p_x$ (p_x is determined as in the proof of Lemma 1.). Since $\{M(x): x \in D\}$ is an open cover of the compact set D, then D has a finite subcover $\{M(x_1),\ldots,M(x_m)\}$. Choose $p(D) =$ maximum $\{p(x_1),\ldots,p(x_m)\}$. Δ

Therefore, for each open set M containing $\hat{N}(v)$, there exists a number $p(A\text{-}M)$ such that all p-centers of v are contained in M for $p \geq p(A\text{-}M)$. Hence, we have the following theorem.

Theorem 1. If v is an n-person game, x^p is its p-center and $\hat{x}$ the unique imputation in its absolute nucleolus, then

$$\lim_{p \to \infty} x^p = \hat{x} .$$

The following theorem and corollary completes our discussion of p-centers.

Theorem 2. If for a given game v, x^p is the p-center of v, then x^p is a continuous function of p from $(1,\infty)$ to A.

Proof. For a fixed game v, let f be the continuous function of x and p defined by

$$f(x,p) = \sum_{\{S \subsetneq N:\ |S|>1\}} |v(S) - x(S)|^p .$$

Suppose the theorem is not true, i.e. there exists a $p > 1$, $\varepsilon > 0$, and a sequence $\{p(m): m = 1,2,\ldots\}$ such that for each $m = 1,2,\ldots$

$$(4) \qquad ||x^p - x^{p(m)}||_2 > \varepsilon \quad \text{and} \quad |p-p(m)| < \frac{1}{m} .$$

Since A is a compact set, the sequence $\{x^{p(m)}: m = 1,2,\ldots\}$ has a limit point y in A. From (4) we have $||x^p - y||_2 \geq \varepsilon$ and hence by the uniqueness of x^p we have $\delta > 0$ where $\delta = f(y,p) - f(x,p)$.

If $\{x^{p(m_k)}: k = 1,2,\ldots\}$ is the subsequence of $\{x^{p(m)}: m = 1,2,\ldots\}$ which converges to y then the sequence $\{(x^{p(m_k)}, p(m_k)): k = 1,2,\ldots\}$ converges to (y,p) and by the continuity of f there exists a $K \geq 1$ such that

$$|f(x^{p(m_k)}, p(m_k)) - f(y,p)| < \delta/4$$

and

$$|f(x^p, p(m_k)) - f(x^p,p)| < \delta/4$$

whenever $k > K$. Therefore,

$$f(x^{p(m_k)}, p(m_k)) - f(x^p, p(m_k)) > \delta/2$$

which contradicts the fact that $x^{p(m_k)}$ is the $p(m_k)$-center of v. Therefore, x^p is a continuous function of p. Δ

Combining Theorems 1 and 2, we have the following corollary:

Corollary 1. The p-centers of a given game v constitute a continuous path in A having an endpoint at the absolute nucleolus of the game v. This path may consist of one point only, e.g. when $v \in C_n$.

The following proposition is a slight variation of a result of Keane (see Proposition 4.2 on page 31 of [9]).

Proposition 1. If the game v has nonempty core, then the imputation x is a 1-center if and only if it is in the core of v.

It follows from Proposition 1 that a game may have more than one 1-center. The author conjectures that as p approaches 1 from the right, the p-centers of a game converge to a single 1-center, but he is unable to prove it. This completes our discussion of p-centers.

3.4 The Full-Center of a Game

In Chapter 2 it was shown that the set C_n was a rational subset in R_{2^n-n-2} on which to base a discussion of solution concepts. In all the above discussions, the solution for a game $v \in C_n$ was the imputation x satisfying

$$v = x_1 v_1 + x_2 v_2 + \ldots + x_n v_n \ .$$

The solution for the game v_i was the imputation e^i defined by

$$e^i_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \ . \end{cases}$$

Player i is the only veto player for game v_i but he is not a dictator as $v(\{i\}) = 0$ by our assumption of 0-1 normal form. Since player i needs the cooperation of at least one other player to insure that he is in a winning coalition, one can argue against e^i as a solution point for game v_i. Although there are other persuasive arguments for accepting e^i as <u>the</u> solution point of v_i, we see there is room for debate on this point.

The above paragraph raises the following question: Are there, in fact, games with solution points so natural as to be, in a sense, beyond dispute? In answer to this query, consider the following games: Let S be a coalition such that $S \subsetneq N$ and $|S| > 1$. Let v_S be the n-person game defined by

$$v_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise, i.e.} \end{cases}$$

v_S is the n-person monotone simple game for which coalition S is the set of veto players. In fact, the coalition S is a dictator in the sense that it wins by itself and only coalitions which contain S can win. A natural solution point for the game v_S is the imputation x^S defined by

$$x_i^S = \begin{cases} \frac{1}{|S|} & \text{if } i \in S \\ 0 & \text{otherwise ,} \end{cases}$$

i.e. the veto players divide 1 equally leaving nothing for the remaining players. It is difficult to imagine any dispute concerning the imputation x^S as a solution point for the game v_S.

The above analysis suggests utilizing the convex hull in R_{2^n-n-2} of the set $\{v_S: S \in Q\}$, denoted H_n, in the way C_n was in the previous sections for defining solution points. The computation difficulties are considerable in dealing with H_n, the convex hull of 2^n-n-2 games, as opposed to C_n, the convex hull of only n games. Because of these difficulties, we shall only briefly discuss the analog of the two-center.

For a given n-person game v, consider the following constrained minimization problem:

$$\text{minimize } ||v-w||_2$$

$$\text{subject to}$$

$$w \in H_n$$

where $||u||_2 = \sum_{S \in Q} [u(S)]^2$. The following theorem is another direct application of Theorem 1 on page 69 of [12].

<u>Theorem 1</u>. For a given n-person game v there exists a unique $v^* \in H_n$ such that

$$||v-v^*||_2 \leq ||v-w||_2 \quad \text{for all } w \in H_n .$$

Furthermore, a necessary and sufficient condition that v^* be the unique minimizing game is that

$$(1) \qquad \sum_{T \in Q} (v(T) - v^*(T))(w(T) - v^*(T)) \leq 0 \quad \text{for all } w \in H_n .$$

Since H_n is the convex hull of the set $\{v_S : S \in Q\}$, (1) in the above theorem can be replaced by

$$(1') \qquad \sum_{T\varepsilon Q} (v(T) - v^*(T))(v_S(T) - v^*(T)) \leq 0 \quad \text{for all} \quad S \varepsilon Q \ .$$

Therefore, v^* is the unique minimizing game if and only if

$$(1'') \quad \sum_{T\varepsilon Q} (v(T) - v^*(T)) \leq \sum_{T\varepsilon Q} v^*(T)(v(T) - v^*(T)) \quad \text{for all} \quad S \varepsilon Q \ .$$

Lemma 1. The set $\{v_S\colon S \varepsilon Q\}$ is linearly independent in R_{2^n-n-2}.

Proof. Suppose the lemma is not true, i.e. suppose for some $S \varepsilon Q$ we have

$$v_S = \sum_{\substack{T\varepsilon Q \\ T\neq S}} \lambda_T v_T$$

where the λ_T's are real numbers. Therefore,

$$(2) \qquad \sum_{\{T\varepsilon Q:\ T\subset S,\ |T|<|S|\}} \lambda_T = 1$$

$$(3) \qquad \sum_{\{T\varepsilon Q:\ T\subset U,\ |T|<|U|\}} \lambda_T = 0 \quad \text{for all} \quad U \subset S$$

where $U \neq S$. The relation (3) implies $\lambda_T = 0$ for all $T \subset S$ such that $|T| = 2$. This result and (3) imply that $\lambda_T = 0$ for all $T \subset S$ such that $|T| = 3$. Continuing in this way, we get $\lambda_T = 0$ for all $T \subset S$ such that $|S| > |T| > 1$. Therefore, (2) cannot hold and we have a contradiction. We conclude that the set $\{v_S\colon S \varepsilon Q\}$ is linearly independent in R_{2^n-n-2}. Δ

The set $\{v_S: S \varepsilon Q\}$ is linearly independent in R_{2^n-n-2} and so the game v^* is a member of H_n if and only if there exists a unique game $z^* \varepsilon R_{2^n-n-2}$ such that

$$\sum_{S\varepsilon Q} z^*(S) = 1 \ ,$$

$$z^*(S) \geq 0 \quad \text{for all} \quad S \subsetneq N, \ |S| > 1, \text{ and}$$

$$v^* = \sum_{S\varepsilon Q} z^*(S) v_S \ .$$

Clearly,

$$(4) \qquad v^*(T) = \sum_{\{U\varepsilon Q:\ U \subset S,\ |U|<|S|\}} z^*(U) \ .$$

If v is an n-person game and v^* and z^* are defined as above, then the imputation x^* defined by

$$(5) \qquad x^* = \sum_{S\varepsilon Q} z^*(S) x^S$$

is called the <u>full-center</u> of v.

An algorithm analogous to that for the two-center can be constructed by brute force but its complexity would be considerable. This complexity is due in part to the fact that z^* has 2^n-n-2 components and partly to relationship (4) which complicates the intermediate calculations in the search for z^*. As in the case of the two-center, the algorithm is a sequential search through a finite collection of systems of linear equations for the particular system which determines the Lagrange multipliers. This fact together with (5) implies that the full-center is a continuous piecewise linear function from R_{2^n-n-2} to R_n.

Below is a table for the full-center in the case when $n = 3$. Note that the games v_1, v_2, v_3 have full-centers $(\frac{1}{2},\frac{1}{4},\frac{1}{4})$, $(\frac{1}{4},\frac{1}{2},\frac{1}{4})$, $(\frac{1}{4},\frac{1}{4},\frac{1}{2})$, respectively. The table is based on the assumption that

$$v(12) \geq v(13) \geq v(23) \text{ .*}$$

full-center	inequalities for region of linearity in R_3
$x_1 = \frac{1}{3} + \frac{v(12) + v(13) - 2v(23)}{6}$ $x_2 = \frac{1}{3} + \frac{v(12) + v(23) - 2v(13)}{6}$ $x_3 = \frac{1}{3} + \frac{v(13) + v(23) - 2v(12)}{6}$	$v(23) \geq \frac{v(12) + v(13) + v(23) - 1}{3}$
$x_1 = \frac{1}{2}$ $x_2 = \frac{1}{4} + \frac{v(12) - v(13)}{4}$ $x_3 = \frac{1}{4} + \frac{v(13) - v(12)}{4}$	$v(23) \leq \frac{v(13) + v(12) - 1}{2}$ $v(13) \geq \frac{v(13) + v(12) - 1}{2}$
$x_1 = \frac{1}{2}$ $x_2 = \frac{1}{2}$	$v(12) - 1 \geq v(13)$

* ij denotes the coalition consisting of players i and j.

In the four-person case, if it is known that $z^*(S) > 0$ for $S \subsetneq N$, $|S| > 1$, then

$$x_i^* = \frac{f(i) - \frac{\sum_{j=1}^{4} f(j)}{4}}{3} + \frac{g(i) - \frac{\sum_{j=1}^{4} g(j)}{4}}{6} \quad \text{for } i = 1,2,3,4$$

where $f(i) = \sum_{\{S\varepsilon Q_i:\ |S|=3\}} v(S)$ and $g(i) = \sum_{\{S\varepsilon Q_i:\ |S|=2\}} v(S)$.

The complex problem of computing full-centers makes the enumeration of further cases difficult.

CHAPTER IV

4-PERSON SUPERADDITIVE GAMES

4.1 Extreme Points for 4-person Superadditive Games

A natural requirement for an n-person game v is that it be superadditive, i.e.

$$(1) \qquad v(S) + v(T) \leq v(S \cup T) \quad \text{for all disjoint coalitions } S \text{ and } T .$$

Since v is assumed to be 0-1 normalized, (1) implies that each component of v is non-negative. If K_n is the set of all n-person superadditive games, then $K_n \cap (C_n - R^+_{2^n-n-2})$ is the set of all n-person superadditive games with nonempty core. Condition (1) and its implication that each superadditive game is non-negative imply that K_n and $K_n \cap (C_n - R^+_{2^n-n-2})$ are closed bounded convex polyhedrons in R_{2^n-n-2}. Ideally, one would like to know the extreme points of these polyhedrons. This appears to be a very difficult problem. In this chapter, we determine the extreme points of K_4 and use this result to produce a counterexample to a conjecture concerning "totally balanced" games.

An n-person game v is constant-sum if and only if

$$(2) \qquad v(S) + v(N-S) = 1 \quad \text{for all coalitions } S \subseteq N .$$

If L_n is the set of all superadditive constant-sum n-person games, then from (1) and (2), L_n is a closed bounded convex polyhedron. With each n-person superadditive game v is identified an $n+1$-person constant-sum superadditive game v' defined by

$$(3) \qquad v'(S) = \begin{cases} v(S) & \text{if } n+1 \notin S \\ 1 - v(N-S) & \text{if } n+1 \in S . \end{cases}$$

Von Neumann and Morgenstern (see page 505 of edition 2 of [18]) call player $n+1$ in (3) the fictitious player. With player $n+1$ fixed as the fictitious player it is easy to see that (3) defines a one-to-one onto mapping from K_n to L_{n+1}. Theorem 1 will show that this mapping maps the extreme points of K_n onto the extreme points of L_{n+1}.

Theorem 1. The n-person game v is an extreme point of K_n if and only if v' is an extreme point of L_{n+1}.

Proof. If v is not an extreme point of K_n, then v' cannot be an extreme point of L_{n+1} since the components of v are also components of v'.

If v' is not an extreme point of L_{n+1}, then $v' = \frac{1}{2} w' + \frac{1}{2} u'$ where $w', u' \in L_{n+1}$ and $w' \neq v'$. By (2) w' must differ from v' in at least two components, one of which is the value of a coalition not containing player $n+1$. Therefore, $v = \frac{1}{2} w + \frac{1}{2} u$ where w and u are the super-additive "subgames" in K_n identified with w' and u', respectively. $w \neq v$ and so v cannot be an extreme point of K_n. Δ

Gurk [8] has determined the extreme points of L_5. We will use his results and Theorem 1 to obtain the extreme points of K_4. Gurk's results are summarized in Theorem 2.

Definition 1. (see Gurk [8].) An L-chain of coalitions for a 5-person constant-sum game v is an ordered collection $(T_1, \ldots, T_6)$ of subsets of $\{1,2,3,4,5\}$ of the form

$$(bd,ae,cd,ab,ce,bd)$$

where (a,b,c,d,e) is some permutation of $(1,2,3,4,5)$ and $v(T_i) = \frac{1}{2}$ for $i = 1,2,\ldots,6$. [bd denotes the subset of $\{1,2,3,4,5\}$ consisting of the elements b and d.]

Theorem 2 is a restatement of Gerk's Corollary 2 on pages 183-4 of [8].

__Theorem 2.__ If v is not a simple game, then v is an extreme point of L_5 if and only if $v(S) = 0,1,$ or $\frac{1}{2}$ for $S \subset \{1,2,3,4,5\}$ and v has at least one L chain.

We now define the notion of a chain for games in K_4. Theorem 3 characterizes the extreme points of K_4.

__Definition 2.__ A __K-chain__ of coalitions for a 4-person superadditive game is an ordered collection $(T_1,\ldots,T_5)$ of subsets of $\{1,2,3,4\}$ of the form

$$(ab,abc,bc,bcd,cd)$$

where (a,b,c,d) is some permutation of $(1,2,3,4)$ and $v(T_i) = \frac{1}{2}$ for $i = 1,2,\ldots,5$.

__Theorem 3.__ If v is not a simple game, then v is an extreme point of K_4 if and only if $v(S) = 0,1,$ or $\frac{1}{2}$ for $S \subset \{1,2,3,4\}$ and v has at least one K-chain.

__Proof.__ Suppose $v \in K_4$ such that $v(S) = 0,1,$ or $\frac{1}{2}$ for $S \subset \{1,2,3,4\}$ and suppose v has a K-chain. Suppose also, without loss of generality, that this K-chain is (12,123,23,234,34). Therefore, the game v' in L_5 has $v'(S) = 0, 1$, or $\frac{1}{2}$ for all $S \subset \{1,2,3,4,5\}$ and $v'(45) = v'(15) = \frac{1}{2}$. The game v' has the L-chain (12,45,23,15,34,12) and is therefore an extreme point of L_5. It follows from Theorem 1 that v is an extreme point of K_4.

Suppose v is a nonsimple* extreme point of K_4. It follows from Theorem 7 that v' is an extreme point of L_5. Therefore, by Theorem 2, the components of v' and hence the components of v must only have values 0,1, and $\frac{1}{2}$. Let (bd,ae,cd,ab,ce,bd) be the L-chain for game v'. Clearly, players b and d have symmetric roles in this L-chain; this is also true for players a,c,e. If b = 5, then game v has K-chain

$$(ae,ace,ce,cde,cd) \ .$$

If e = 5, then game v has the K-chain

$$(ab,abd,bd,bcd,cd) \ . \qquad \Delta$$

The following Table consists of the extreme points v of K_4 which are not simple games and which have the K-chain (12,123,23,234,34), i.e. $v(12) = v(123) = v(23) = v(234) = v(34) = \frac{1}{2}$.

---

* A game v is a nonsimple extreme point of K_4 if it is an extreme point of K_4 and if it is not a simple game.

Table 1

v(13)	v(14)	v(24)	v(124)	v(134)
0	0	0	1	1
½	0	0	1	1
0	½	0	1	1
0	0	½	1	1
0	0	0	½	1
0	0	0	1	½
½	½	0	1	1
½	0	½	1	1
½	0	0	½	1
½	0	0	1	½
0	½	½	1	1
0	½	0	½	1
0	½	0	1	½
0	0	½	½	1
0	0	½	1	½
0	0	0	½	½
½	½	½	1	1
½	½	0	½	1
½	½	0	1	½
½	0	½	½	1
½	0	½	1	½
½	0	0	½	½
0	½	½	½	1
0	½	½	1	½
0	½	0	½	½
0	0	½	½	½
½	½	½	½	1
½	½	½	1	½
½	½	0	½	½
½	0	½	½	½
0	½	½	½	½
½	½	½	½	½

4.2 Totally Balanced Games, a Counterexample

Let v be an n-person game. For each coalition $S \subset N$ where $|S| \geq 1$ there is an $|S|$-person game v^S defined by $v^S(T) = v(T)$ for all $T \subseteq S$. For each $S \subset N$ where $|S| \geq 1$, v^S is called a subgame of v. An n-person game is totally balanced [17] if and only if each subgame has nonempty core.

It is easily seen that the set H_n of all totally balanced n-person games is a closed bounded convex polyhedron contained in K_n. Therefore, if an extreme point of K_n is totally balanced, it is an extreme point of H_n. Attempts were made to prove that the only extreme points of H_n were the totally balanced simple games, i.e. the monotone simple games with veto players. This conjecture is false. Consider the game $v \in K_4$ given by

$$v(12) = v(123) = v(23) = v(234) = v(34) = v(124) = v(134) = \frac{1}{2}$$

$$v(1234) = 1$$

$$v(S) = 0 \quad \text{for all other } S \subset \{1,2,3,4\} .$$

It is easy to see that v is totally balanced. v contains the K-chain $(12,123,23,234,34)$ and therefore v is an extreme point of K_4. Therefore, v is also an extreme point of H_4.

CHAPTER V

THE SHAPLEY VALUE AND GAMES WITH NONEMPTY CORE

5.1 Extreme Points for Non-negative Games with Nonempty Core

In this chapter we attempt to characterize games (not necessarily superadditive) whose Shapley value is contained in its core.* To do this, we need the results of Theorem 1 of this section. In Theorem 1 we characterize the extreme points of $(C_n - R^+_{2^n-n-2}) \cap R^+_{2^n-n-2}$. This theorem is an unpublished result of L. S. Shapley. The proof used here was suggested to the author by Elon Kohlberg.

Lemma 1. If v is an extreme point of $(C_n - R^+_{2^n-n-2}) \cap R^+_{2^n-n-2}$ and if the imputation x is in the core of v, then $x(S) = v(S)$ for all $S \subseteq N$ for which $v(S) > 0$.

Proof. Let v be an extreme point of $(C_n - R^+_{2^n-n-2}) \cap R^+_{2^n-n-2}$. If x is an imputation in the core of v, then $x(S) \geq v(S)$ for all $S \subset N$. Suppose there exists a $T \subset N$ such that $x(T) > v(T) > 0$. Therefore, it is possible to choose a real number ε such that

$$\text{minimum } \{x(T) - v(T),\ v(T)\} > \varepsilon > 0 \ .$$

Define the n-person games u_1 and u_2 by

$$u_1(S) = \begin{cases} v(S) & \text{if } S \neq T \\ v(T) + \varepsilon & \text{if } S = T \end{cases}$$

---

* This problem was suggested by a remark of Charnes and Sorensen in [5].

$$u_2(S) = \begin{cases} v(S) & \text{if } S \neq T \\ v(T) - \varepsilon & \text{if } S = T \end{cases} .$$

The imputation x is also in the respective cores of u_1 and u_2 and so u_1 and u_2 are members of $(C_n - R^+_{2^n-n-2}) \cap R^+_{2^n-n-2}$. The fact that $v = \frac{1}{2} u_1 + \frac{1}{2} u_2$ where $u_1 \neq v$ contradicts the assumption that v is an extreme point of $(C_n - R^+_{2^n-n-2}) \cap R^+_{2^n-n-2}$. Therefore, $x(T) = v(T)$ for all $T \subset N$ for which $v(T) > 0$. Δ

<u>Lemma 2</u>. If v is an extreme point of $(C_n - R^+_{2^n-n-2}) \cap R^+_{2^n-n-2}$, then $v(S) = 0$ or 1 for all $S \subset N$.

<u>Proof</u>. Suppose v is an extreme point of $(C_n - R^+_{2^n-n-2}) \cap R^+_{2^n-n-2}$ and suppose there exists a coalition $T \subset N$ such that $0 < v(T) < 1$. Let x be an imputation in the core of v. It follows from Lemma 5 that $x(T) = v(T)$. Therefore, $x(N-T) = 1 - x(T) > 0$, and so there exists players $i \varepsilon T$ and $j \varepsilon N-T$ such that $x_i > 0$ and $x_j > 0$. Choose a real number d so that

$$\text{minimum } \{x_i, x_j, 1-x_i, 1-x_j\} > d > 0 ,$$

and define the imputations x^1 and x^2 by

$$x_k^1 = \begin{cases} x_k & \text{if } k \neq i,j \\ x_i + d & \text{if } k = i \\ x_j - d & \text{if } k = j \end{cases}$$

$$x_k^2 = \begin{cases} x_k & \text{if } k \neq i,j \\ x_i - d & \text{if } k = i \\ x_j + d & \text{if } k = j \end{cases} .$$

Define the games v^1 and v^2 by

$$v^k(S) = \begin{cases} x^k(S) & \text{if } v(S) > 0 \\ 0 & \text{if } v(S) = 0 \end{cases}$$

for $k = 1,2$. The imputation x^k is in the core of the game v^k for $k = 1,2$, and so v^1 and v^2 are members of $(C_n - R^+_{2^n-n-2}) \cap R^+_{2^n-n-2}$. However, $v = \frac{1}{2}v^1 + \frac{1}{2}v^2$ and $v^1(T) > v(T) > v^2(T)$. Therefore, v cannot be an extreme point of $(C_n - R^+_{2^n-n-2}) \cap R^+_{2^n-n-2}$, a contradiction. Therefore, $v(S)$ is 0 or 1 for all $S \subset N$. Δ

Theorem 1. Game v is an extreme point of $(C_n - R^+_{2^n-n-2}) \cap R^+_{2^n-n-2}$ if and only if it is a simple game with at least one veto player.

Proof. Let v be an extreme point of $(C_n - R^+_{2^n-n-2}) \cap R^+_{2^n-n-2}$. Lemma 6 implies that v is a simple game. Suppose v does not have a veto player, i.e. there exists disjoint subsets S_1 and S_2 of N such that $v(S_1) = v(S_2) = 1$. Clearly, no imputation x exists for which $x(S_1) \geq v(S_1)$ and $x(S_2) \geq v(S_2)$, and so $v \notin (C_n - R^+_{2^n-n-2})$. This contradicts the fact that v is an extreme point of the closed set $C_n - R_{2^n-n-2}$. Therefore, v must have at least one veto player.

Suppose v is a simple n-person game with at least one veto player. If player i is a veto player of the simple game v, then the imputation x

defined by

$$x_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

is in the core of v. Therefore, $v \in C_n - R^+_{2^n-n-2}$. Since no game w for which $w(S) > 1$ for some $S \subset N$ can have nonempty core, v must be an extreme point of $(C_n - R^+_{2^n-n-2}) \cap R^+_{2^n-n-2}$. Δ

5.2 Simple Games in which the Shapley Value is a Member of the Core

If v is an n-person game, then the Shapley value [16] of v is a vector $\phi(v) \in R_n$ defined by

$$\text{(1)} \qquad \phi(v)_i = \sum_{\{S \subseteq N:\ i \in S\}} \frac{(|S|-1)!(n-|S|)!}{n!} (v(S) - v(S - \{i\}))$$

for $i = 1,2,\ldots,n$. If the game v has nonempty core, it is not necessarily true that $\phi(v)$ is in the core of v. If Y_n is the set of all games with nonempty core containing the Shapley value, then Y_n is determined by the inequalities

$$\sum_{i \in S} \phi(v)_i \geq v(S) \quad \text{for all} \quad S \subseteq N .$$

Therefore, Y_n is a closed convex polyhedron.

An alternate description of Y_n would be the characterization of its extreme points. Since Y_n may be unbounded, we will restrict our attention to $Y_n \cap R^+_{2^n-n-2}$. Any extreme point of $(C_n - R^+_{2^n-n-2}) \cap R^+_{2^n-n-2}$ which is also a member of Y_n is clearly an extreme point of $Y_n \cap R^+_{2^n-n-2}$. Theorem 1 below describes those simple games with veto players which are contained in Y_n.

Theorem 1. If v is a simple game with at least one veto player, then the Shapley value of v is in the core of v if and only if v is a monotone game and contains more than one veto player.

Proof. Let v be a monotone simple game with more than one veto player. Let T be the set of veto players of the game v. If player $i \notin T$, then (1) implies $\phi(v)_i = 0$. $\sum_{j \in N} \phi(v)_j = 1$ and so $\sum_{j \in S} \phi(v)_j = 1$. The relation (1) also implies that if $i \in T$, then $\phi(v)_i \geq 0$. Therefore, $\phi(v)$ is an imputation in the core of v.

Let v be a nonmonotonic simple game with more than one veto player. Let T be the set of veto players of the game v. If $W = \{S \subset N\colon T \subset S,\ v(S) = 0\}$, then the game v can be written

$$v = v_T - \sum_{S \in W} e_S$$

where v_T is the monotonic simple game with veto players T and e_S is the n-person game (not in 0-1 normal form) given by

$$e_S(M) = \begin{cases} 1 & \text{if } M = S \\ 0 & \text{otherwise} . \end{cases}$$

The Shapley value $\phi(v_T)$ of the game v_T is given by

$$\phi(v_T)_i = \begin{cases} 1/|S| & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}$$

If $j \in S$, then $\phi(e_S)_j = \frac{(n-|S|)!(|S|-1)!}{n!} > 0$. The linearity of the Shapley value [16] implies that

$$\phi(v) = \phi(v_T) - \sum_{S \in W} \phi(e_S) \; .$$

Since W is nonempty and T is contained in each member of W,

$$\sum_{i \in T} \phi(v)_i < 1 \; .$$

Therefore, $\phi(v)$ cannot be in the core of v and so $v \notin Y_n$.

Let v be a monotone simple game with just one veto player, say player 1. The game v has a core consisting of the one imputation $(1,0,0,\ldots,0)$ in R_n. However, $\phi(v)_2 = \frac{1}{n(n-1)} > 0$. Therefore, $\phi(v)$ is not in the core of v and so $v \notin Y_n$. Δ

Unfortunately, there are extreme points of $Y_n \cap R^+_{2^n-n-2}$ which are not simple games. In the case $n = 3$, there are four such extreme points: $(\frac{4}{5},\frac{4}{5},0)$, $(\frac{4}{5},0,\frac{4}{5})$, $(0,\frac{4}{5},\frac{4}{5})$, $(\frac{2}{3},\frac{2}{3},\frac{2}{3})$.

CHAPTER VI

OPEN PROBLEMS

In Chapters IV and V, the open problems should be obvious even to the casual reader and will not be discussed here.

The embedding of imputations in the game space is introduced in Chapter II and utilized to define "centers" of games in Chapter III. This embedding provides a framework for interpreting new and old solution concepts for a game v in terms of v's relationship or "closeness", in some sense, to the embedded imputations. I think this idea can be explored further. In Chapter III we formulated the notion "closeness" as a metric concept and used it to define solution points for a game. However, the notion of "closeness" in the space of games does not have to be a metric one as the core and modified nucleolus of Chapter II illustrate. I believe it would be worthwhile to investigate new nonmetric notions of "closeness" in the game space. One might initially impose a few axioms on what is meant by a solution in order to avoid difficulties like those at the end of Section 3.1. Such a notion of "closeness" together with these axioms may provide very appealing solution concepts for n-person games.

The computational difficulties of Section 3.4 suggest that searches for other ways of embedding the imputations in the game space may not be very fruitfull.

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