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# QUANTIFYING CLOSENESS OF DISTRIBUTIONS OF SUMS AND MAXIMA WHEN TAILS ARE FAT

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# QUANTIFYING CLOSENESS OF DISTRIBUTIONS OF SUMS AND MAXIMA WHEN TAILS ARE FAT

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## ABSTRACT

Let $X_1, X_2, \ldots, X_n$ be $n$ independent, identically distributed, non–negative random variables and put $S_n = \Sigma_{i=1}^n X_i$ and $M_n = \vee_{i=1}^n X_i$. Let $\rho(X,Y)$ denote the uniform distance between the distributions of random variables $X$ and $Y$; i.e. $\rho(X,Y) = \sup_{x \in \mathbb{R}} |P(X \le x) - P(Y \le x)|$. We consider $\rho(S_n, M_n)$ when $P(X_1 > x)$ is slowly varying and we provide bounds for the asymptotic behaviour of this quantity as $n \to \infty$, thereby establishing a uniform rate of convergence result in Darling's law for distributions with slowly varying tails.

Keywords and phrases: slow variation, partial sums, partial maxima.

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## 1. Introduction

Suppose that $X_1, X_2, \ldots$ is a sequence of non–negative, independent, identically distributed (i.i.d.) random variables with common distribution function (d.f.) $F$, and denote $\bar{F} = 1 - F$. Put $S_n = \Sigma_{i=1}^n X_i$ and $M_n = \vee_{i=1}^n X_i$, $n = 1,2,3,\ldots$ .

$\bar{F}$ is said to be regularly varying at infinity with index $-\alpha$ $(\alpha \geq 0)$ iff

(1.1) $$\lim_{x \to \infty} \frac{\bar{F}(xt)}{\bar{F}(x)} = t^{-\alpha}, \quad \text{for every } t > 0.$$

If $\alpha = 0$ in (1.1), $\bar{F}$ is called slowly varying. In the sequel, we will denote (1.1) as $\bar{F} \in \mathscr{R}_{-\alpha}$.

If $\bar{F} \in \mathscr{R}_{-\alpha}$ with $\alpha \neq 0$, it is well known that there exist linear normalizations such that $S_n$ and $M_n$ converge weakly to (different) non–degenerate limit laws. Moreover, the concept of regular variation is widely accepted to be the natural way of characterizing domains of attraction in these limit relations, see e.g. Doeblin [6], Feller [7], de Haan [4], Bingham et al [2], Resnick [12].

If $\bar{F}$ is slowly varying $(\alpha = 0)$, $EX_1^\rho = \infty$ for every $\rho > 0$ and Lévy [9] pointed out that for such distributions, every linear normalization of $S_n$ (or $M_n$) leads to a degenerate limit law. Hence one is forced to consider nonlinear normalizing functions and in this setup, Darling [3] showed that if $\bar{F} \in \mathscr{R}_0$,

(1.2) $$n\bar{F}(S_n) \Longrightarrow E$$

where $\Longrightarrow$ denotes weak convergence and $E$ is an exponential random variable with parameter 1. Also

$$n\bar{F}(M_n) \Longrightarrow E$$

so that by uniform convergence,

$$(1.3)\qquad \rho(S_n,M_n) := \sup_{x\geq 0} |P[n\bar{F}(S_n) \leq x] - P[n\bar{F}(M_n) \leq x]|$$

$$= \sup_{x\geq 0} |P(S_n \leq x) - P(M_n \leq x)| \to 0 \text{ as } n \to \infty.$$

Another interpretation of this result is given in Resnick [11, section 5] where it is shown that

$$a_n^{-1}(M_n,S_n) \Rightarrow (\xi,\xi)$$

where $n\bar{F}(a_n) = 1$ $(n = 1,2,\ldots)$ and $\xi$ is such that $P(\xi = 0) = e^{-1} = 1 - P(\xi = \infty)$. Thus $\bar{F} \in \mathscr{R}_0$ implies that $\rho(S_n,M_n) \to 0$ as $n \to \infty$ and in this paper we are interested in the rate of convergence to zero of $\rho(S_n,M_n)$. In order to obtain a precise rate, it is natural to specify the manner in which $\bar{F}$ is slowly varying. This is done in the next section where we discuss $\Pi$–varying tails. Section 3 contains the results on the rate of decay of $\rho(S_n,M_n)$ under various conditions on $\bar{F}$.

## 2. Preliminaries

From Karamata's Theorem ([2], [4], [7], [12]) it follows that $\bar{F} \in \mathscr{R}_0$ iff

$$\frac{1}{x}\int_0^x u\,dF(u) = o(\bar{F}(x)) \quad (x \to \infty).$$

We can specify the way in which $\bar{F}$ is slowly varying by being more precise about the o–term in this relation. Therefore, suppose that

$$x^{-1}\int_0^x u\,dF(u) = V(1/\bar{F}(x)), \tag{2.1}$$

where $V$ is a non–negative measurable function such that $xV(x) \to 0$. More precise conditions on $V$ will be given later.

In section 3 we show that (2.1) is a natural condition for obtaining a rate of convergence to zero of $\rho(S_n, M_n)$. Here our first concern is to interpret the condition in (2.1) by translating it into an equivalent form containing only $\bar{F}$. In order to state the result, we introduce some necessary definitions and notations: A non–negative measurable function $U$ is $\Pi$–varying $(U \in \Pi)$ iff there exists a function $b \in \mathscr{R}_0$ such that

$$\lim_{x\to\infty} \frac{U(tx) - U(x)}{b(x)} = \log t. \tag{2.2}$$

(Cf. [2], [4], [12].) $b$ is usually called an auxiliary function (a.f.) of $U$ and it is shown in [4] that $U \in \Pi$ iff $x^{-1}\int_0^x s\,dU(s) \in \mathscr{R}_0$ in which case we may take $b(x) = x^{-1}\int_0^x s\,dU(s)$. If $U$ is monotone, non–decreasing and right continuous, the inverse of $U$ is defined as $U^{\leftarrow}(x) = \inf\{y\colon U(y) \ge x\}$ and it is well known in this case that $U \in \Pi$ with a.f. $b$ iff $U^{\leftarrow}$ is $\Gamma$–varying with a.f. $f(x) = b(U^{\leftarrow}(x))$; i.e.

$$\lim_{x\to\infty} \frac{U^{\leftarrow}(x + tf(x))}{U^{\leftarrow}(x)} = e^t \quad \text{for every } t \in \mathbb{R}. \tag{2.3}$$

(Cf. [2], [4], [12].) One can show (cf. [2], [4], [8], [12]) that if $f$ is the a.f. of a function in the class $\Gamma$, then $f$ is self–neglecting $(f \in SN)$; i.e.

$$\lim_{x\to\infty} \frac{f(x + uf(x))}{f(x)} = 1,$$

locally uniformly in $u \in \mathbb{R}$. Furthermore, if $f$ is any SN function we have $\exp\{\int_1^x (1/f(u))du\} \in \Gamma$. In order to prove the main result of this section we need some special relations between $\Pi$ and $\Gamma$ which are gathered in the following lemma.

<u>Lemma 2.1</u>. Suppose $U, H$ are non–decreasing on $(0, \infty)$.

A. (i) If $U \in \Gamma$ with a.f. $f(t) \in \mathscr{R}_1 \cap SN$ then $\log U \in \Pi$ with a.f. $a(t) = t/f(t)$.

(ii) If $H \in \Pi$ with a.f. $H(t)L(t)/\log t$ where $t/L(e^t) \in SN$, then $H(e^t) \in \Gamma$ with a.f. $t/L(e^t)$.

B. (i) If $U \in \Gamma$ with a.f. $f \in \mathscr{R}_{1-\alpha}$, $\alpha > 0$ then $\log U(x) \sim \alpha^{-1}x/f(x) \in \mathscr{R}_\alpha$.

(ii) If $H \in \Pi$ with a.f. $H(t)/\alpha \log t$ for some $\alpha > 0$ then $H(e^x) \in \mathscr{R}_{1/\alpha}$.

C. (i) If $U(x) \to \infty$ and $U \in \Gamma$ with a.f. $f$ where $t^2/f(t) \in \Gamma$ with a.f. $h$, then $\log U \in \Gamma$ with a.f. $h$.

(ii) If $H \in \Pi$ with a.f. $H(t)L(t)/\log t$ where $L(t) \to 0$ and $L(e^t) \in \mathscr{R}_0$ then $H(e^x) \in \Pi$ with a.f. $H(e^t)L(e^t)$.

<u>Proof</u>. A. (i) If $U \in \Gamma$, we have the Balkema–de Haan representation (cf. [12], for example)

$$U(x) = c(x)\exp\left\{\int_1^x (1/f_1(u))du\right\}$$

where $c(x) \to c > 0$ and $f_1 \sim f$, so that $f_1 \in \mathscr{R}_1 \cap SN$. Hence

$$\log U(x) = \log c(x) + \int_1^x (1/f_1(u))du. \tag{2.4}$$

Now $\int_1^x (1/f_1(u))du \in \Pi$ with a.f. $t/f_1(t) \to \infty$ because it is the integral of a $-1$–varying function. Since $\log c(x) \to \log c$, it follows from (2.4) that $\log U \in \Pi$.

(ii) Since we can always represent the a.f. of H as $x^{-1}\int_0^x u\,dH(u) = H(x) - x^{-1}\int_0^x H(u)du$ we have for some function $b(x)$, $b(x) \to 1$, that

$$H(x) - x^{-1}\int_0^x H(u)du = b(x)H(x)L(x)/\log x$$

whence

$$\frac{H(x)}{\int_0^x H(u)du} = \left(x\left(1 - \frac{b(x)L(x)}{\log x}\right)\right)^{-1}$$

and integrating from 1 to x produces

$$\int_0^x H(u)du = c\exp\left\{\int_1^x \left(s\left(1 - \frac{b(s)L(s)}{\log s}\right)\right)^{-1} ds\right\}.$$

Since

$$\int_0^x H(u)du = xH(x)\left[1 - \frac{b(x)L(x)}{\log x}\right]$$

we get

$$\begin{aligned} H(x) &= cx^{-1}\left[1 - \frac{b(x)L(x)}{\log x}\right]^{-1}\exp\left\{\int_1^x \left(s\left(1 - \frac{b(s)L(s)}{\log s}\right)\right)^{-1} ds\right\} \\ &= c\left[1 - \frac{b(x)L(x)}{\log x}\right]^{-1}\exp\left\{\int_1^x \left(\frac{\log s}{b(s)L(s)} - 1\right)^{-1}\frac{ds}{s}\right\} \end{aligned}$$

and thus

$$H(e^x) = c(1 - x^{-1}b(e^x)L(e^x))^{-1}\exp\left\{\int_0^x \left(\frac{y}{b(e^y)L(e^y)} - 1\right)^{-1} dy\right\}. \tag{2.5}$$

Set $f^*(x) = x(b(e^x)L(e^x))^{-1}$ and we get

$$H(e^x) = c((f^*(x) - 1)/f^*(x))\exp\left\{\int_0^x 1/(f^*(s) - 1))ds\right\}.$$

Now observe that since the auxiliary function of $H$ is $H(x)L(x)/\log x$ we have $H(x)/(H(x)L(x)(\log x)^{-1}) = \log x/L(x) \to \infty$ (cf. [4], [12]) and thus $f^*(x) \to \infty$ whence $(f^*(x) - 1)/f^*(x) \to 1$ and $f^*(x) - 1 \sim f^* \in SN$. Thus $H(e^x) \in \Gamma$.

B. (i) From (2.4) and Karamata's Theorem

$$\log U(x) \sim \alpha^{-1}x/f_1(x) \sim \alpha^{-1}x/f(x).$$

(ii) From (2.5) we have with $L(x) \equiv 1/\alpha$

$$H(e^x) \sim c\exp\left\{\int_0^x \frac{y}{(\alpha y/b(e^y)) - 1}\frac{dy}{y}\right\}$$

and since $y/((\alpha y/b(e^y)) - 1) \to \alpha^{-1}$, the result follows from Karamata's representation of a regularly varying function ([2], [4], [12]).

C. (i) From (2.4) and the assumption $U(x) \to \infty$ we have $\log U(x) \sim \int_1^x (1/f_1(u))du$ where $1/f_1(u) = \gamma(u)/u^2$ and $\gamma \in \Gamma$ with a.f. $h$. Now $\gamma \in \Gamma$ with a.f. $h$ implies $\gamma(u)/u^2 \in \Gamma$ with a.f. $h$ and this in turn implies $\int_1^x \gamma(u)/u^2du \in \Gamma$ with a.f. $h$ (cf. [4], p. 45.).

(ii) From (2.5) it follows that

$$H(e^x) \sim c\exp\left\{\int_0^x b^*(s)L(e^s)/s\,ds\right\}$$

where $b^*(s) := b(e^s)(1-s^{-1}b(e^s)L(e^s))^{-1} \to 1$ and since $L(x) \to 0$ we get from the Karamata representation that $H(e^x) \in \mathscr{R}_0$. Because $H \in \Pi$ we may write ([1],[2])

$$H(x) = d(x) + \int_1^x a_1(s)/s\, ds$$

where $d = o(a_1)$ and $a_1(t) \sim H(t)L(t)/\log t$. Thus

$$H(e^x) = d(e^x) + \int_0^x a_1(e^y)dy$$

where

$$a_1(e^y) \sim H(e^y)L(e^y)/y \in \mathcal{R}_{-1}$$

and

$$\lim_{x\to\infty} d(e^x)/H(e^x)L(e^x) = \lim_{x\to\infty} \frac{d(x)a_1(x)}{a_1(x)H(x)L(x)} = \lim_{x\to\infty} \frac{d(x)}{a_1(x)\log x} = 0.$$

Now $\int_0^x a_1(e^y)dy$, being the integral of a $-1$–varying function, is in $\Pi$ with a.f. $H(e^t)L(e^t)$ and the same is true of $H(e^x)$. □

We are now ready to formulate our theorem which interprets (2.1).

<u>**Theorem 2.1**</u>. Define $g = 1/(1-F)$ and consider the following relations:

(i) For some non–negative, measurable function V satisfying $\lim_{x\to\infty} xV(x) = 0$

(2.1) $$x^{-1} \int_0^x u dF(u) = V(g(x)).$$

(ii) For some function $L(x) \geq 0$, $g \in \Pi$ with a.f. $g(x)L(x)/\log x$, or equivalently,

(2.6) $$\frac{\bar{F}(tx)}{\bar{F}(x)} - 1 \sim (-\log t)(L(x)/\log x), \quad x \to \infty.$$

Then we have

A. (i) holds and $V \in \mathcal{R}_{-1}$ iff (ii) holds and $x/L(e^x) \in SN$.

B. (i) holds and $V \in \mathcal{R}_{-1-\alpha}$ $(\alpha > 0)$ iff (ii) holds and $\lim_{x\to\infty} L(x) = \alpha^{-1}$.

C. (i) holds and $1/V \in \Gamma$ iff (ii) holds, $L(x) \to 0$, and $L(e^x) \in \mathcal{R}_0$.

If one of the equivalences in A, B, or C holds, there is a function $b(x) \to 1$ and $\bar{F}$ is of the form $(c > 0)$

$$\text{(iii)} \qquad \bar{F}(x) = c\left(1 + \frac{b(x)L(x)}{\log x}\right)^{-1} \exp\left\{-\int_1^x \left(\frac{b(u)L(u)}{\log u + b(u)L(u)}\right)\frac{du}{u}\right\}$$

and furthermore L and V determine each other asymptotically through the relation

$$L(x) \sim g(x)V(g(x))\log x.$$

<u>Proof</u>. The proofs of A, B and C heavily rely on the corresponding statements in Lemma 2.1 A, B and C. Suppose (2.1) holds for some function $V(x)$ satisfying $xV(x) \to 0$. Since from (2.1)

$$x\left(g^2(x)\int_0^x u\,dF(u)\right)^{-1} = \left(g^2(x)V(g(x))\right)^{-1}$$

we get upon integrating with respect to $dg(x)$ that for $T \geq 1$

$$\int_1^T \frac{x\,dF(x)}{\int_0^x u\,dF(u)} = \int_1^T \left(g^2(x)V(g(x))\right)^{-1} dg(x)$$

$$= \int_{g(1)}^{g(T)} \left(y^2V(y)\right)^{-1} dy$$

and since the left side is

$$\log\left(\int_0^T x\,dF(x)\Big/\int_0^1 x\,dF(x)\right)$$

we obtain for some $c > 0$ the representation

$$\int_0^T x dF(x) = c \exp\left\{ \int_1^{g(T)} (y^2 V(y))^{-1} dy \right\}.$$

So using (2.1)

$$x = (c/V(g(x)))\exp\left\{ \int_1^{g(x)} (y^2 V(y))^{-1} dy \right\}. \tag{2.7}$$

Thus if we set

$$H(x) = (c/V(x))\exp\left\{ \int_1^{x} (y^2 V(y))^{-1} dy \right\} \tag{2.8}$$

then

$$x = H \circ g(x)$$

and $g$ is the inverse of $H$.

To prove (A), suppose that both (2.1) holds and $V \in \mathscr{R}_{-1}$. Since $V \in \mathscr{R}_{-1}$ and $xV(x) \to 0$ it follows that $f(x) := x^2 V(x) \in SN$ since $f(x)/x = xV(x) \to 0$ and thus as $t \to \infty$

$$\frac{f(t + xf(t))}{f(t)} = \frac{(t + xf(t))^2}{t^2} \frac{V(t + xf(t))}{V(t)} \to 1.$$

Hence $H \in \Gamma$ with a.f. $f(x)$ whence $g \in \Pi$ with a.f. $f \circ g(x) = g^2(x)V(g(x))$. This proves (ii) and it remains to set $L(x) = g(x)V(g(x))\log x$ and show

$$x/L(e^x) \sim \frac{1}{g(e^x)V(g(e^x))} \in SN.$$

However since $H \in \Gamma$ with a.f. $f \in SN \cap \mathscr{R}_1$ it follows from Lemma 2.1.A.(i) that $\log H \in \Pi$ with a.f. $a(t) = t/f(t) = 1/tV(t)$ and therefore $(\log H)^{\leftarrow} \in \Gamma$ with a.f.

$a((\log H)^{\leftarrow}(t)) = 1/((\log H)^{\leftarrow}(t))V((\log H)^{\leftarrow}(t)) \in SN$ and the desired result follows since $(\log H)^{\leftarrow}(x) = g(e^x)$.

Suppose now that (ii) holds and $x/L(e^x) \in SN$. We show (i) holds with $V \in \mathscr{R}_{-1}$. We assume $g \in \Pi$ with a.f. $g(t)L(t)/\log t$ which implies $F \in \Pi$ with a.f. $\bar{F}(t)L(t)/\log t$ whence

$$\bar{F}(t)L(t)/\log t \sim t^{-1} \int_0^t u dF(u).$$

From Lemma 2.1.A.(ii) we have $g(e^x) \in \Gamma$ with a.f. $x/L(e^x)$ whence by inversion $\log g^{\leftarrow}(y) \in \Pi$ with a.f. $\log g^{\leftarrow}(y)/L(g^{\leftarrow}(y)) \in \mathscr{R}_0$ and thus we conclude

$$V(t) := \frac{L(g^{\leftarrow}(t))}{t \ \log g^{\leftarrow}(t)} \in \mathscr{R}_{-1}.$$

So we have

$$V(g(t)) \sim \bar{F}(t)L(t)/\log t \sim t^{-1} \int_0^t u dF(u)$$

as desired.

The derivation of (iii) is carried out as in Lemma 2.1.A.(ii).

B. Given (2.1) with $V \in \mathscr{R}_{-1-\alpha}$, we have $f(t) := t^2V(t) \in \mathscr{R}_{1-\alpha} \subset SN$ and hence from (2.8) we have $H(x) \in \Gamma$ with a.f. $f(t) = t^2V(t)$ so $H^{\leftarrow}(x) = g(x) \in \Pi$ with a.f. $g^2(t)V(g(t))$. From Lemma 2.1.B.(i) we have $\log H(x) \sim \alpha^{-1}x/f(x) \in \mathscr{R}_\alpha$ so

$$\log H(g(x)) \sim \log x \sim (\alpha g(x)V(g(x)))^{-1}$$

and so the a.f. of $g$ is

$$g^2(t)V(g(t)) \sim g(t)(\alpha \log t)^{-1}$$

as desired.

Conversely assume $g \in \Pi$ with a.f. $g(t)/\alpha \log t$. Then $F \in \Pi$ with a.f. $\bar{F}(t)/\alpha \log t$ and so

$$t^{-1} \int_0^t u dF(u) \sim \bar{F}(t)/\alpha \log t.$$

From Lemma 1.B.ii we have $g(e^x) \in \mathcal{R}_{1/\alpha}$ whence $\log g^{\leftarrow}(y) \in \mathcal{R}_{\alpha}$. So $V(t) := (\alpha t \log g^{\leftarrow}(t))^{-1} \in \mathcal{R}_{-1-\alpha}$ and

$$V(g(t)) \sim \bar{F}(t)/\alpha \log t \sim t^{-1} \int_0^t u dF(u)$$

as desired.

C. Given (2.1) and $1/V \in \Gamma$ with a.f. $h$ we have that $(1/V)^{\leftarrow} \in \Pi$ with a.f. $h \circ (1/V)^{\leftarrow} \in \mathcal{R}_0$. We use this to check that $y^2 V(y) \in SN$. Note $\lim_{t\to\infty} t^2 V(t)/h(t) = 0$ since this limit equals

$$\lim_{y\to\infty} ((1/V)^{\leftarrow}(y))^2 y^{-1}/h((1/V)^{\leftarrow}(y))$$

which is the limit of a function in $\mathcal{R}_{-1}$. Therefore

$$\lim_{t\to\infty} \frac{(t + xt^2V(t))^2 V(t + xt^2V(t))}{t^2 V(t)}$$

$$= \lim_{t\to\infty} (1 + xtV(t))^2 V(t + xh(t)(t^2V(t)/h(t)))/V(t)$$

$$= \exp\{-\lim_{t\to\infty} xt^2V(t)/h(t)\} = 1$$

which says that $y^2V(y) \in SN$. Furthermore $t^2V(t)/h(t) \to 0$ implies $V(t)/h(t) \to 0$ and the above argument can be repeated to show $V \in SN$. Thus $H$ in (2.8) is in $\Gamma$ with a.f. $y^2V(y)$ whence from Lemma 2.1.C.(i) $\log H \in \Gamma$ with a.f. $h$ and inverting we conclude $g \in \Pi$ (one desired conclusion) with a.f. $g^2V(g)$ and $g(e^y) \in \Pi$ with a.f. $h(g(e^y)) \in \mathscr{R}_0$.

It remains to show that the a.f. of $g$

$$g^2(x)V(g(x)) \sim g(x)L(x)/\log x$$

where $L(e^x) \in \mathscr{R}_0$; i.e. we show

$$xg(e^x)V(g(e^x)) \in \mathscr{R}_0.$$

However $1/V \in \Gamma$ with a.f. $h$ implies $(x^2V(x))^{-1} \in \Gamma$ with a.f. $h$ so that ([4], p. 45)

$$h(x) \sim x^2V(x) \int_1^x 1/(y^2V(y))dy$$

and from (2.8)

$$h(x) \sim x^2V(x)\log g^{\leftarrow}(x)$$

so that since $h(g(e^x)) \in \mathscr{R}_0$ we get

$$h(g(e^x)) \sim g^2(e^x)V(g(e^x))x \in \mathscr{R}_0$$

and since $g(e^x) \in \Pi \subset \mathscr{R}_0$ we also get

$$xg(e^x)V(g(e^x)) \in \mathscr{R}_0.$$

Furthermore since $h(t)/t \to 0$ as a consequence of $h$ being an auxiliary function, we have

$$L(g(e^x)) \sim h(g(e^x))/g(e^x) \to 0$$

whence $L(x) \to 0$.

Conversely, suppose $g \in \Pi$ with a.f. $g(x)L(x)/\log x$ where $L(x) \to 0$, $L(e^x) \in \mathscr{R}_0$. As in A and B we have

$$\bar{F}(x)L(x)/\log x \sim x^{-1} \int_0^x u dF(u)$$

so it remains to check that

$$V(x) := L(g^{\leftarrow}(x))/(x \log g^{\leftarrow}(x))$$

satisfies $1/V \in \Gamma$. However from Lemma 2.1.C.(ii) $g(e^t) \in \Pi$ with a.f. $g(e^t)L(e^t)$ whence $\log g^{\leftarrow} \in \Gamma$ with a.f. $tL(g^{\leftarrow}(t)) =: h(t)$. This implies

$$\log g^{\leftarrow}(x)/(xL(g^{\leftarrow}(x))) \in \Gamma \text{ with a.f. } h$$

and further that

$$x^2 \log g^{\leftarrow}(x)/(xL(g^{\leftarrow}(x))) = x \log g^{\leftarrow}(x)/L(g^{\leftarrow}(x)) = 1/V \in \Gamma$$

with a.f. $h$ as desired. □

Theorem 2.1 informs us that condition (2.1) means $F$ is $\Pi$–varying with a special form for the auxiliary function. In the next section we will show that (2.1) is a condition which is natural for obtaining a rate of convergence for $\rho(S_n, M_n)$.

## 3. Rates of convergence

Darling [3] showed that if $\bar{F} \in \mathscr{R}_0$,

$$E \frac{S_n}{M_n} \to 1 \text{ as } n \to \infty.$$

Defining $\epsilon_n^2 := E\left[\frac{S_n}{M_n}\right] - 1$, we thus have that $\epsilon_n \to 0$ as $n \to \infty$. The first simple step expresses $\rho(S_n, M_n)$ in terms of $\epsilon_n$.

<u>Lemma 3.1</u>. Let $\bar{F} \in \mathscr{R}_0$. Then

$$\rho(S_n,M_n) \le \epsilon_n + \sup_{x\ge 0} (F^n(x) - F^n(x(1+\epsilon_n)^{-1}). \tag{3.1}$$

<u>Proof</u>. We have for any $x \ge 0$,

$$\begin{aligned} P(M_n > x) \le P(S_n > x) &= P(S_n > x,\ M_n^{-1}\cdot S_n > 1+\epsilon_n) \\ &\quad + P(S_n > x,\ M_n^{-1}\cdot S_n \le 1+\epsilon_n) \\ &\le P(M_n^{-1}\cdot S_n - 1 > \epsilon_n) + P(M_n(1+\epsilon_n) > x). \end{aligned}$$

Since $M_n^{-1}\cdot S_n - 1 \ge 0$, we can apply Markov's inequality giving that

$$P(M_n^{-1}\cdot S_n - 1 > \epsilon_n) \le \frac{1}{\epsilon_n} E(M_n^{-1}\cdot S_n - 1) = \epsilon_n.$$

Using this upper bound, we get that

$$P(M_n > x) \le P(S_n > x) \le \epsilon_n + P(M_n > x(1+\epsilon_n)^{-1})$$

whence

$$0 \le P(S_n > x) - P(M_n > x) \le \epsilon_n + F^n(x) - F^n(x(1+\epsilon_n)^{-1}).$$

Taking suprema over $x$ gives the result. □

It is clear from Lemma 3.1 that in order to bound $\rho(S_n,M_n)$ we need to examine the two terms in the right hand side of (3.1). We first show that the conditions on $F$ assumed in the previous section allow us to establish the precise asymptotic behaviour of $\epsilon_n$ as $n \to \infty$. This is done in the next lemma.

Lemma 3.2. Suppose (2.1) is satisfied.

(i) If $V \in \mathscr{R}_{-1-\alpha}$, $0 \le \alpha$, then $\epsilon_n^2 \sim \Gamma(\alpha+2)\cdot nV(n)$ $(n \to \infty)$.

(ii) Set $\Psi(x) = x^{-1}(-\log V)^{\leftarrow}(x^{-1})$. If $-\log V \in \mathscr{R}_\beta$, $\beta > 0$ then $-\log \epsilon_n \sim \frac{1}{2}(1+\beta^{-1})\beta^{1/(1+\beta)}/\Psi^{\leftarrow}(n)$ $(n \to \infty)$ and $\epsilon_n = \exp\{-W(n)\}$ where $W \in \mathscr{R}_{\beta/(1+\beta)}$.

Proof. We have from Darling [3] or from Maller and Resnick [10, Lemma 1.1] that

$$\epsilon_n^2 = n(n-1)\int_0^\infty F^{n-2}(y)\,(y^{-1}\int_0^y u\,dF(u))\,dF(y),$$

and using (2.1) this becomes

$$\epsilon_n^2 = n(n-1)\int_0^\infty F^{n-2}(y)V\left(\frac{1}{\bar{F}(y)}\right)dF(y).$$

Define $V_1$ by $(0 < s < 1)$

$$V\left(\frac{1}{1-s}\right) = V_1\left(\frac{1}{-\log s}\right)$$

and set $q(x) = -\log F(x)$, $x \ge 0$. Then

$$\begin{aligned}\epsilon_{n+1}^2 &= (n+1)n\int_0^\infty e^{-(n-1)q(y)}\,V_1\left(\frac{1}{q(y)}\right)de^{-q(y)}\\ &= (n+1)n\int_0^\infty e^{-ns}\,V_1\left(\frac{1}{s}\right)ds\end{aligned}$$

and it seems irresistable to get the asymptotic behavior of $\epsilon_n$ from well known Abel–Tauber theorems for Laplace transforms; see [2]. If $V \in \mathscr{R}_{-1-\alpha}$, $\alpha \ge 0$, it follows that $V(x) \sim V_1(x)$ $(x \to \infty)$, so that via standard methods [2],

$$\epsilon_{n+1}^{2} \sim nV(n)\cdot\Gamma(\alpha+2) \quad (n \to \infty).$$

This proves (i).

As for (ii), we use an Abel–Tauber theorem for Kohlbecker transforms [2, Theorem 4.12.11.9iii)] which immediately implies the result. □

Remarks. 1. It would be worthwhile to establish a general Abel–Tauber theorem for Laplace transforms of functions in the class $\Gamma$. Since this is not known, we concentrated in Lemma 3.2(ii) on the special case that $-\log V \in \mathscr{R}_\beta$, $\beta > 0$, which covers most cases.
2. We can get the converse assertions in Lemma 3.2(i) (or (ii)) by imposing a Tauberian condition on $V$ (or $-\log V$), see Bingham et al. [2].

It is clear from Lemmas 3.1 and 3.2 that we can estimate $\rho(S_n, M_n)$ if we bound the second term in the right hand side of (3.1).

Lemma 3.3. If (2.1) holds and $xV(x) \to 0$ and either

$$V \in \mathscr{R}_{-1-\alpha}, \quad \alpha \geq 0$$

or

$$1/V \in \Gamma \quad \text{and} \quad -\log V \in \mathscr{R}_\beta, \quad \beta > 0$$

then

$$\sup_{x\geq 0} |F^n(x) - F^n(x(1+\epsilon_n)^{-1})| = o(\epsilon_n).$$

Proof. Clearly for every $0 \leq z \leq y$,

$$F^n(y) - F^n(z) = \int_z^y nF^{n-1}(t)dF(t) \tag{3.2}$$

$$\leq nF^{n-1}(y)(F(y) - F(z)).$$

From Theorem 2.1 we have $F \in \Pi$ with a.f. $V(g)$ and so given $\delta > 0$ there exists $x_0 = x_0(\delta)$ such that if $x \geq x_0$ we have

$$|F(x) - F(x(1+\epsilon_n)^{-1})| \leq (1+\delta)\log(1+\epsilon_n)V(g(x(1+\epsilon_n)^{-1}))$$

where we have used the fact that convergence in the definition of $\Pi$–variation is locally uniform. Combining this with (3.2) gives

$$F^n(x) - F^n(x(1+\epsilon_n)^{-1}) \leq nF^{n-1}(x)(F(x) - F(x(1+\epsilon_n)^{-1}))$$

$$\leq (1+\delta)nF^{n-1}(x) \log(1+\epsilon_n)V(g(x(1+\epsilon_n)^{-1})), \quad x > x_0(\delta).$$

Therefore,

$$\sup_{x\geq 0} |F^n(x) - F^n(x(1+\epsilon_n)^{-1})| \tag{3.3}$$

$$\leq nF^{n-1}(x_0) + (1+\delta)n \log(1+\epsilon_n) \cdot \sup_{x\geq x_0} F^{n-1}(x)\, V(g(x(1+\epsilon_n)^{-1})).$$

Since $x_0$ is a fixed number and $F(x_0) < 1$, it follows from Lemma 3.2 that $nF^{n-1}(x_0) = o(\epsilon_n)$ $(n \to \infty)$.

We now consider the second term in the right hand side of (3.3). To prove that this is $o(\epsilon_n)$ obviously requires us to show that

$$\sup_{x\geq x_0} nF^{n-1}(x)V(g(x(1+\epsilon_n)^{-1})) \to 0 \quad (n \to \infty).$$

Let $(x_n)_{n=1}^{\infty}$ be a sequence such that $x_n \to x_\infty$.

If $x_\infty < \infty$, clearly

$$nF^{n-1}(x_n)V(g(x_n(1+\epsilon_n)^{-1})) \sim nF^{n-1}(x_\infty)V(g(x_\infty)) \to 0 \quad (n \to \infty).$$

If $x_\infty = \infty$, we use $F = 1 - g^{-1}$ and

$$nF^{n-1}(x_n)V(g(x_n(1+\epsilon_n)^{-1})) \sim \frac{n}{g(x_n)} e^{-n/g(x_n)} g(x_n)V(g(x_n)) \quad (n \to \infty).$$

which tends to zero since $xe^{-x}$ is bounded on $[0,\infty)$ and $xV(x) \to 0$ $(x \to \infty)$. This proves the lemma.

Combining Theorem 2.1 and Lemmas 3.1–3.3, we have proved the following theorem which gives a rate of convergence for $\rho(S_n,M_n)$.

<u>Theorem 3.1</u>. Suppose that $x^{-1}\int_0^x udF(u) = V(1/(1-F(x)))$ where $xV(x) \to 0$.

(i) If $V \in \mathscr{R}_{-1-\alpha}$, $0 \le \alpha$, then

$$\limsup_{n\to\infty} \rho(S_n,M_n)/(nV(n))^{1/2} \le (\Gamma(\alpha+2))^{1/2}$$

(ii) Suppose $1/V \in \Gamma$ and $-\log V \in \mathscr{R}_\beta$, $\beta > 0$. Set $\Psi(x) = x^{-1}(-\log V)^{\leftarrow}(x^{-1})$ and $W(x) = (1+o(1))\,\frac{1}{2}\,(1+\beta^{-1})\beta^{1/(1+\beta)}/\Psi^{\leftarrow}(x)$ where $o(1) \to 0$ as $x \to \infty$ so that $W(x) \in \mathscr{R}_{\beta/(1+\beta)}$. Then

$$\limsup_{n\to\infty} \rho(S_n,M_n)\exp\{W(n)\} \le 1.$$

<u>Remarks</u>. 1. The o–term in Theorem 3.1(ii) stems from the fact that we only have an asymptotic expression for $-\log \epsilon_n$ in Lemma 3.2(ii). If we want to specify this term we need more information on $V$ which enables us to use an Abel–Tauber theorem with remainder for Kohlbecker transform in Lemma 3.2(ii).

2. We assumed in Theorem 2.1 that $V$ is regularly varying or that $1/V$ is $\Gamma$–varying. Clearly this can be generalized to $O(o)$–versions (see [2]), leading to $O(o)$–expressions for the behaviour of $\epsilon_n$ as $n \to \infty$. This then gives $O(o)$–type of results in Theorem 3.1.

We now give some examples.

1) Suppose $\bar{F}(x) = (\log x)^{-\gamma}$, $x \geq e$, $\gamma > 0$. Then

$$F'(x) = \gamma(\log x)^{-\gamma-1}x^{-1} \in \mathscr{R}_{-1}$$

so that $F \in \Pi$ with a.f. $a(t) = \gamma(\log t)^{-\gamma-1}$. Since $g(x) = (\log x)^{\gamma}$ we have

$$V(x) = a(g^{\leftarrow}(x)) = \gamma/x^{1+\gamma^{-1}} \in \mathscr{R}_{-1-\gamma^{-1}}$$

and therefore from Theorem 3.1

$$\limsup_{n\to\infty} \rho(S_n,M_n)n^{1/2\gamma} \leq (\gamma\Gamma(2+\gamma^{-1}))^{1/2}.$$

If $\gamma = 1$

$$\limsup_{n\to\infty} \sqrt{n}\, \rho(S_n,M_n) \leq \sqrt{2}\,.$$

2) If $\bar{F}(x) = \exp\{-(\log x)^{\gamma}\}$, $x \geq 1$, $0 < \gamma < 1$, then

$$V(x) = \frac{\gamma}{x}(\log x)^{\frac{\gamma-1}{\gamma}}$$

so that

$$\limsup_{n\to\infty} \rho(S_n,M_n)(\log n)^{\frac{1-\gamma}{2\gamma}} \leq \gamma^{1/2}.$$

3) If $\bar{F}(x) = (\log\log x)^{-\gamma}$, $x \geq e^{e}$, $\gamma > 0$, then

$$V(x) = \gamma\cdot x^{-\frac{1+\gamma}{\gamma}}e^{-x^{1/\gamma}}$$

so that

$$\limsup_{n\to\infty} \rho(S_n,M_n)\exp\{\tfrac{1}{2}(1+o(1))(1+\gamma)\gamma^{-\gamma/(1+\gamma)}n^{1/(1+\gamma)}\} \le 1.$$

The pattern of the previous three examples, suggests that the more longtailed the underlying distribution, the faster the rate at which $\rho(S_n,M_n)$ tends to zero. The next theorem supports this view.

<u>Theorem 3.2</u>. Let $F_1$, $F_2$ be two distributions satisfying (2.1) and denote the corresponding V–functions appearing in (2.1) by $V_1$ and $V_2$. If $g_i := 1/(1-F_i) \in \Pi$, $i = 1,2$ then

(i) $\lim_{x\to\infty} \frac{V_1(x)}{V_2(x)} = d$ $(0 < d < \infty)$ iff $F_1(x) = F_2(U_d(x))$ with $U_d$ a monotone function in $\mathscr{R}_d$ $(0 < d < \infty)$.

(ii) $\lim_{x\to\infty} \frac{V_1(x)}{V_2(x)} = 0$ iff $\lim_{x\to\infty} \frac{V_1(x)}{V_2(x)}$ exists and $F_2(x) \ge F_1(x^M)$ for all $M \ge 1$ and $x > x_0(M)$.

<u>Proof</u>. Set $g_i := \frac{1}{1-F_i}$, $i = 1,2$. As $g_i$ is $\Pi$–varying, $g_i^{\leftarrow}$ belongs to $\Gamma$, and we denote the auxiliary function of $g_i^{\leftarrow}$ by $h_i$. Solving (2.1) in terms of $V_i$ yields

$$V_i(x) = \frac{1}{g_i^{\leftarrow}(x)} \int_1^x g_i^{\leftarrow}(z)\,\frac{dz}{z^2}\,.$$

From the remark in [4, p. 45] we get that $x^2 V_i(x) \sim h_i(x)$ $(x \to \infty)$. Therefore

$$\frac{V_1(x)}{V_2(x)} \sim \frac{h_1(x)}{h_2(x)} \quad (x \to \infty).$$

(i) First suppose $V_1(x)/V_2(x) \to d \in (0,\infty)$. Since $g_1^{\leftarrow} \in \Gamma$ with a.f. $h_i$ we note that $g_1^{\leftarrow} \in \Gamma$ with a.f. $dh_2$ and therefore $(g_1^{\leftarrow}(x))^d \in \Gamma$ with a.f. $h_2$. Thus both $(g_1^{\leftarrow}(x))^d$ and $g_2^{\leftarrow}(x)$ have the same a.f. and from [5, Theorem 2.1] there exists monotone $U \in \mathscr{R}_1$ such that

$$g_2^{\leftarrow}(x) = U((g_1^{\leftarrow}(x))^d).$$

Set $U_d(x) = U(x^d) \in \mathscr{R}_d$ and we get

$$g_2^{\leftarrow}(x) = U_d(g_1^{\leftarrow}(x))$$

or

$$g_1(x) = g_2(U_d(x))$$

as desired.

Conversely, $F_1(x) = F_2(U_d(x))$ iff $g_1(x) = g_2(U_d(x))$ iff $g_2^{\leftarrow}(x) = U_d(g_1^{\leftarrow}(x))$ $= U((g_1^{\leftarrow}(x))^d)$ where $U(x) = U_d(x^{1/d}) \in \mathscr{R}_1$. So $g_2^{\leftarrow}(x)$ and $(g_1^{\leftarrow}(x))^d$ are $\Gamma$–functions with the same a.f. and the result follows.

(ii) If $\lim_{x\to\infty} \frac{V_1(x)}{V_2(x)} = 0$, we can use [5, Theorem 2.1] to show that for every $\epsilon$, $0 < \epsilon < 1$, there exist monotone functions $U_1^{(1)}$, $U_1 \in \mathscr{R}_1$ ($U$ possibly depending on $\epsilon$) such that

$$g_1^{\leftarrow}(x) = U_1^{(1)}\left(\exp\left\{\int_1^x \frac{du}{h_1(u)}\right\}\right) \geq U_1\left(\exp\left\{\frac{1}{\epsilon}\int_1^x \frac{du}{h_2(u)}\right\}\right)$$

$$= U_{1/\epsilon}(g_2^{\leftarrow}(x))$$

with $U_{1/\epsilon}$ a monotone function in $\mathscr{R}_{1/\epsilon}$. Hence $\lim_{x\to\infty} \frac{g_1^{\leftarrow}(x)}{(g_2^{\leftarrow}(x))^M} = \infty$ for any $1 \le M < 1/\epsilon$. This implies that $g_1^{\leftarrow}(x) \ge (g_2^{\leftarrow}(x))^M$ for all $x > x_0'(M)$ for some $x_0'(M)$ whence there exists $x_0(M)$ such that $g_2(x) \ge g_1(x^M)$ for all $x \ge x_0(M)$.

We now prove the converse. Since $F_2(x) \ge F_1(x^M)$ for all $M \ge 1$ and $x > x_0(M)$, we have that

$$\lim_{x\to\infty} \frac{g_1^{\leftarrow}(x)}{(g_2^{\leftarrow}(x))^M} = \infty \text{ for any } M \ge 1. \tag{3.4}$$

Now denote $\lim_{x\to\infty} \frac{V_1(x)}{V_2(x)} = d$. Using the same reasoning as before, it is not hard to show that $d > 0$ leads to a contradiction with (3.4). Hence $d = 0$. □

Clearly Theorem 3.2 implies that if $V_1(x) \sim dV_2(x)$, $0 \le d < 1$, i.e. $V_1$ is asymptotically smaller than $V_2$, then $F_2(x) \ge F_1(x)$ for $x > x_0$, which means that $F_1$ has a fatter tail than $F_2$.

Refinements of Theorem 3.2 are possible. For instance, if we assume that Theorem 2.1.B holds, i.e., $V_i \in \mathscr{R}_{-1-\alpha}$, $\alpha > 0$, $i = 1,2$, then in Theorem 3.2 it is easy to see that the assumption $\lim_{x\to\infty} V_1(x)/V_2(x)$ exists may be dropped from the right side of (ii).

An obvious problem which still remains is to assess how accurate are the bounds in Theorem 3.1. We have not successfully calculated $\rho(S_n,M_n)$ for any specific example so this issue is unresolved. Two simulation studies (see figure 1, 2) show that the asymptotic bound does not perform particularly well for values of $n$ up to 200. The simulations were for $\overline{F}(x) = (\log\log x)^{-1/2}$, $x > e^e$ and $\overline{F}(x) = \exp\{-(\log x)^{1/2}\}$, $x > 1$. The simulations are somewhat inconclusive as it may be that the asymptotic bound performs better for much larger values of $n$.

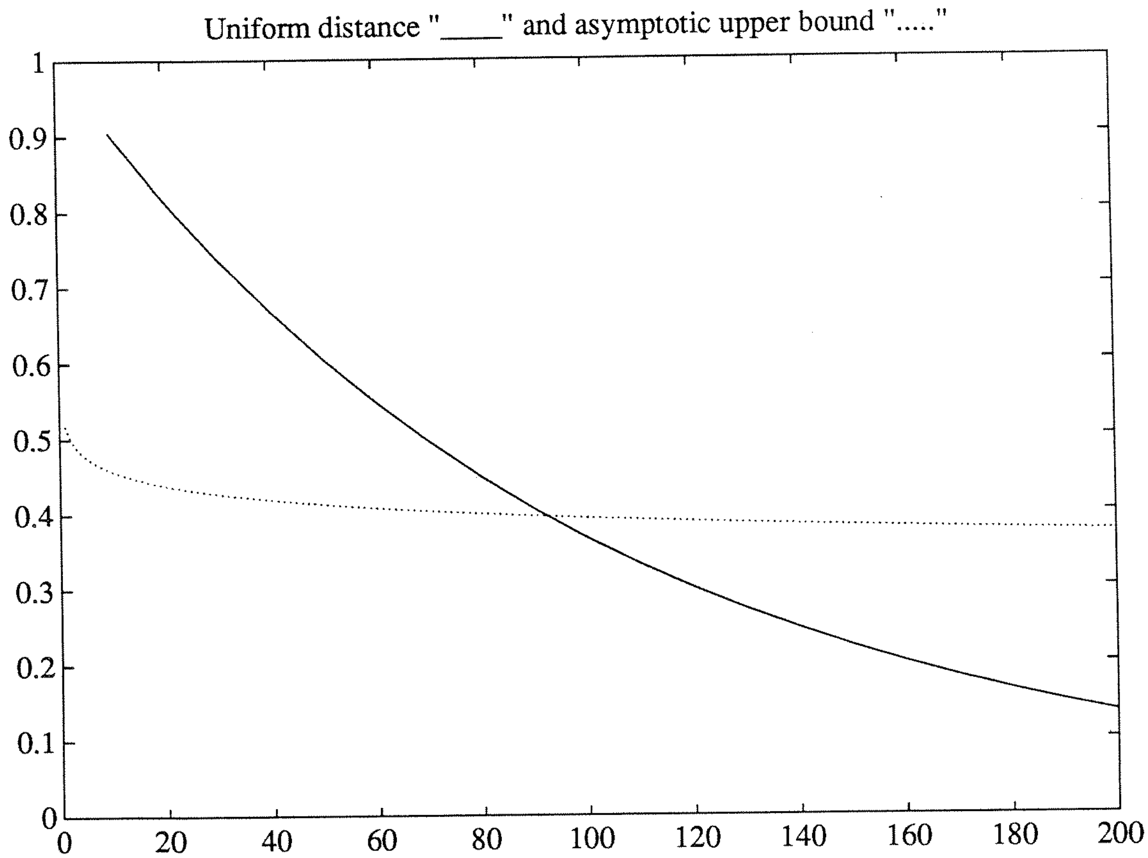


Figure 1. $\overline{F}(x) = (\log \log x)^{-1/2}$

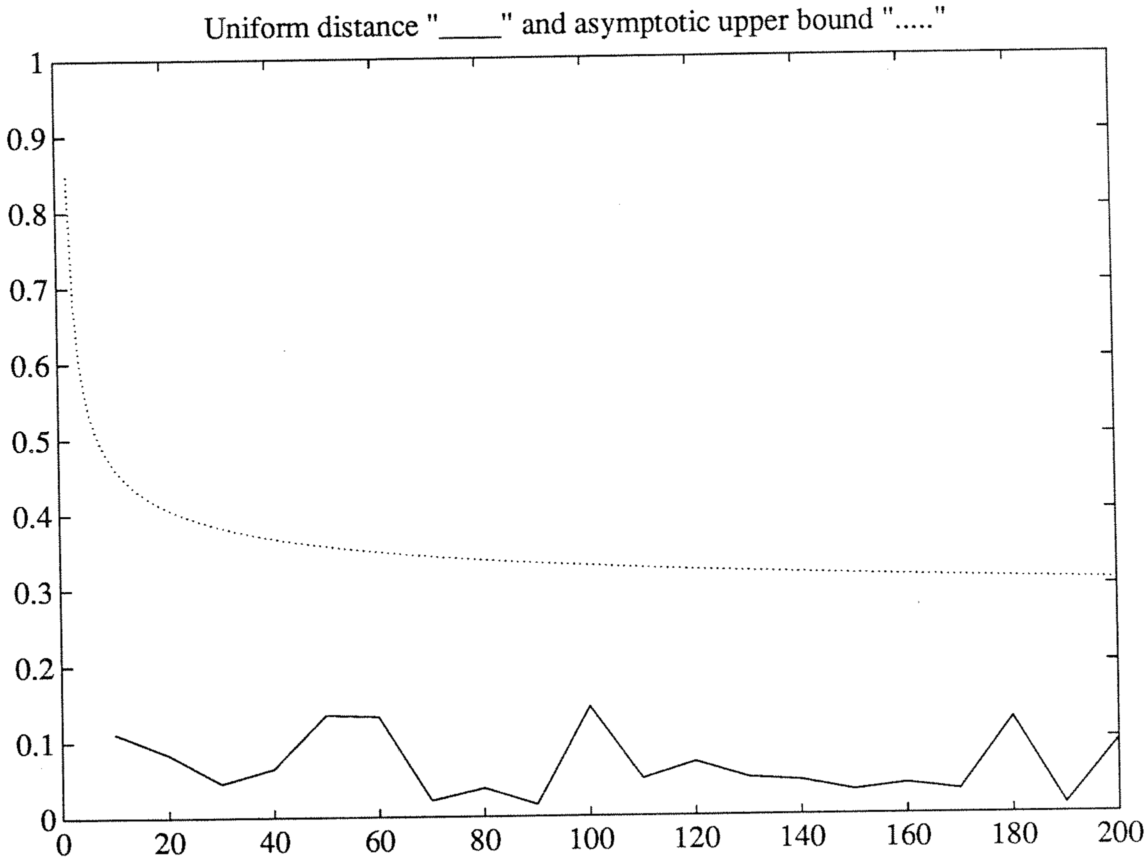


Figure 2. $\overline{F}(x) = \exp\{-(\log x)^{1/2}\}$.

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