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GENERALIZATIONS OF THE BASIC RENEWAL THEOREM FOR DEPENDENT VARIABLES

By

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Abstract

This paper is concerned with various sets of conditions on sequences of dependent real-valued random variables Z_n under which asymptotic statements about $N(t) \equiv \inf\{n \geq 1: Z_{n+1} \geq t\}$ and $EN(t)$ as $t \to \infty$ can be proved. The most satisfactory generalizations of the Basic Renewal Theorem require the assumption that $Z_n - Z_{n-1}$ is non-negative and $E\{Z_n - Z_{n-1} \mid Z_1, \ldots, Z_{n-1}\}$ is almost surely nonincreasing as a function of n. Two important classes of Markovian processes in Reliability - the proportional-time and proportional-hazard models - are introduced to illustrate and sharpen the general results.

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1. Introduction

One of the most severe limitations of standard Reliability Theory - as expounded, for example, by Barlow and Proschan (1981) - is its restriction to the study of independent failure-time random variables. Consider the case of Renewal Theory, which in the context of Reliability has led to the characterization of many classes of repair/replacement policies, and which appears to depend crucially on the assumption of independence for the times between successive failures. In practical life, it is clear that successive replacements of failed components in a complicated assembly (say, an aircraft) may have some cumulative effect tending to shorten future times between replacements. Additionally, one can imagine that shocks to the system from failures of single components can affect the lifetimes of the remaining components, or even that the age of important components can be reflected in the operating characteristics and therefore in the hazard of failure of the system.

The regression models of Cox (1972) in life table analysis gave a simple way for lifetimes to depend on (possibly time-dependent) covariate measurements. If we treat current lifetimes of system components as covariates, then these models imply an interesting and statistically identifiable dependence between component failure times. This idea has been used by Slud (1983) to study a class of multivariate dependent renewal processes in which a component's hazard of failure depends only on the current component lifetimes. Another approach, which we develop in the present paper, is to model the system's current failure distribution as depending on cumulative exposure variables (and possibly on covariates) observable up to the last previous failure time. Realistic reliability models in the context of successive failures after repair or replacement should typically take into account all three types of information: current component ages, cumulative exposures of the system, and environmental covariates.

Our main results concern generalization of the Basic Renewal Theorem to the family of point-processes $\{Z_n\}_{n=1}^{\infty}$ (of successive failure times) for which the non-negative waiting-time variables $Z_n - Z_{n-1}$ are <u>conditionally decreasing in expectation given the past</u>, in the sense that the sequence $E\{Z_n - Z_{n-1} \mid Z_1, \ldots, Z_{n-1}\}$ is almost surely non-increasing in n. Further assumptions and our general theorems are stated and proved in Section 2. Section 3 applies and sharpens these results in the important special <u>proportional-time</u> and <u>proportional-hazard</u> models for successive lifetimes $Z_n - Z_{n-1}$. Our concluding Section 4 contains miscellaneous remarks about our methods and results.

In the remainder of the Introduction, we discuss the reliability context of the processes characterized abstractly in Section 2 and of the special examples of Section 3. Consider a single device subject to failure and instantaneous repair (or partial replacement). Let T_i denote the time from the $(i-1)^{th}$ repair to the next (the i^{th}) failure, and $Z_n = \sum_{i=1}^{n} T_i$ the time from original installation to n^{th} failure. We imagine for simplicity that there is a nondecreasing sequence of cumulative-exposure variables W_n depending on $Z_1, \ldots, Z_{n-1}$ and possibly on the values $(\underline{C}(t),\ 0 \le t \le Z_{n-1})$ of some environmental process, which may for example describe system loading, such that the conditional law of T_n given $Z_1, \ldots, Z_{n-1}$ and $\{\underline{C}(t):\ t \le Z_{n-1}\}$ depends only on W_n. Apart from early system burn-in, which we ignore, we may reasonably suppose that the conditional expectation of T_n given W_n will always be less than or equal to that for T_{n-1} given W_{n-1}. An important comment is that "time" in reliability applications can be interpreted freely as operational time (or indeed, as time of operation during which a specified standard is met. Thus, the processes we describe can also model the degradation of performance over time.) For example, if the cumulative exposure W_n above were to take the simple form $W_n = \sum_{i=1}^{n} w(T_i)$ for a fixed function $w(\cdot)$, then $\overline{Z}_n \equiv W_{n+1} = \sum_{i=1}^{n} w(T_i)$ forms a Markovian sequence. In particular, we term a <u>proportional-time</u> [respectively <u>proportional-hazard</u>] process any increasing (Markovian) random sequence $\{Z_n\}_{n \geq 0}$ such that there exist positive deterministic functions $q_n(\cdot)$ [respectively, $Q_n(\cdot)$] for which $\{(Z_{n+1} - Z_n)/q_n(Z_n)\}_{n \geq 0}$ is a sequence of independent r.v.'s [respectively, for which $(\log P\{Z_{n+1} - Z_n > t \mid Z_1, \ldots, Z_n\})$ $/Q_n(Z_n)$ is for each t a constant not depending on n]. In other words, these are processes for which the successive lifetimes $Z_{n+1} - Z_n$ [respectively, their conditional cumulative hazards given $Z_1, \ldots, Z_n$] depend on $Z_1, \ldots, Z_n$ only through the multiplicative factor $q_n(Z_n)$ [respectively $Q_n(Z_n)$]. Our strongest asymptotic statements about the largest n for which $Z_n \leq t$ (for large t) will apply to special cases of these models.

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2. Counting-process asymptotics

Throughout the present section we consider sequences $\{Z_n\}_{n\geq 0}$ of integrable r.v.'s, with $Z_0 \equiv 0$ and $Z_n \geq Z_{n-1}$ a.s., which are adapted to an increasing family $\{F_n\}$ of σ-fields on a probability space (Ω, F, P). That is, Z_n is measurable with respect to F_n, which is regarded as containing all information about $\{Z_i\}_{i=1}^n$ and all relevant covariates observable up to time Z_n. Whenever $Z_n \to \infty$ a.s. as $n \to \infty$, we define for $t > 0$

$$N(t) \equiv \min\{n \geq 0: Z_{n+1} \geq t\}, \quad \tau(t) \equiv Z_{N(t)+1}.$$

We will show here how assumptions on the conditional mean and central moments of $Z_n - Z_{n-1}$ given F_{n-1} imply the asymptotic relationships

$$(2.1) \qquad \begin{aligned} &\tau(t) \sim t \sim \sum_{j=1}^{N(t)+1} E\{Z_j - Z_{j-1} \mid F_{j-1}\} \equiv W_{N(t)+1} \\ &E\tau(t) \sim t \sim EW_{N(t)+1}. \end{aligned} \qquad t \to \infty$$

By itself, (2.1) does not determine an asymptotic rate of growth for $N(t)$ or $EN(t)$, although in the suggestive form

$$(2.2) \qquad \begin{aligned} &t \sim \int_0^t g(s)dN(s) \sim E[\int_0^t g(s)dN(s)], \qquad t \to \infty \\ &g(s) \equiv E\{Z_k - Z_{k-1} \mid F_{k-1}\} \text{ whenever } N(s) = k-1,\ k \geq 1, \end{aligned}$$

it does indeed imply bounds on growth.

Our first set of conditions on $\{Z_n\}$ is

(A)

$$\sum_{i=1}^{n} E\{Z_i - Z_{i-1} \mid F_{i-1}\} \equiv W_n \to \infty \text{ a.s. and for some } 0 < \delta \leq 1,$$

$$\sum_{n=1}^{\infty} E\left[\frac{|Z_n - E\{Z_n \mid F_{n-1}\}|}{1+W_n}\right]^{1+\delta} < \infty.$$

Proposition 2.1. Assume (A). Then $Z_n \to \infty$ a.s. as $n \to \infty$, and

$$\tau(t) \sim \sum_{i=1}^{N(t)+1} E\{Z_i - Z_{i-1} \mid F_{i-1}\} \text{ a.s. as } t \to \infty,$$

where ~ means the limit of the ratio of two expressions is 1.

Proof. Under assumption (A), we apply Lemma 5.2' and Theorem A of Loève (1951, p. 286) to $f_n(x) \equiv x^{1+\delta}$ and $X_n \equiv (Z_n - E\{Z_n \mid F_{n-1}\})/(W_n+1)$ (with $\xi_n \equiv 0$ a.s.) to conclude (by the Kronecker Lemma, as in Loève's Theorem A)

$$\text{(2.3)} \qquad \sum_{n=n_0}^{N} (Z_n - E\{Z_n \mid F_{n-1}\})/W_N \overset{\text{a.s.}}{\to} 0 \text{ as } N \to \infty.$$

Therefore, as $N \to \infty$, $Z_N = \sum_{n=1}^{N}(Z_n - E\{Z_n \mid F_{n-1}\}) + \sum_{n=1}^{N} E\{Z_n - Z_{n-1} \mid F_{n-1}\}$ $\overset{\text{a.s.}}{\sim} \sum_{n=1}^{N} E\{Z_n - Z_{n-1} \mid F_{n-1}\}$, and $Z_N \to \infty$. For each $t > 0$, we now have $\tau(t) < \infty$ a.s., and $\tau(t) = Z_{N(t)+1}$ implies as $t \to \infty$, $N(t) \to \infty$ and $\tau(t) \sim W_{N(t)+1}$. □

Recall that $N(t)+1$ is a $\{F_n\}$ stopping-time (i.e. the event $[N(t)+1 > k]$ is F_k-measurable), and that $Z_n \geq Z_{n-1}$ a.s. For each $t > 0$ we calculate, whether the expectations are finite or not,

$$E\tau(t) = E \sum_{j=1}^{N(t)+1} (Z_j - Z_{j-1}) = E \sum_{j=1}^{\infty} I_{[N(t)+1 \geq j]}(Z_j - Z_{j-1})$$

$$= \sum_{j=1}^{\infty} E(I_{[N(t)+1 \geq j]} E\{Z_j - Z_{j-1} \mid F_{j-1}\})$$

$$= E \sum_{j=1}^{\infty} I_{[N(t)+1 \geq j]} E\{Z_j - Z_{j-1} \mid F_{j-1}\} = E \sum_{j=1}^{N(t)+1} E\{Z_j - Z_{j-1} \mid F_{j-1}\}.$$

Thus we have

<u>Lemma 2.2</u>. If (A) holds, then $E\tau(t) = EW_{N(t)+1}$.

<u>Remark</u>. In the special case of Chow and Robbins (1963) - who assumed $E\{Z_n - Z_{n-1} \mid F_{n-1}\} = \mu_n$ a.s. constant with $n^{-1}(\mu_1 + \ldots + \mu_n) \to \mu$, $0 < \mu < \infty$, but not $Z_n \geq Z_{n-1}$ - we have $\tau(t) \sim \mu N(t)$ and $E\tau(t) = \mu(1 + EN(t)) + E \sum_{j=1}^{N(t)+1} (\mu_j - \mu)$. To show that $E \sum_{j=1}^{N(t)+1} (\mu_j - \mu)$ is much smaller than $EN(t)$ for large t, observe that $E[| \sum_{j=1}^{N(t)+1} (\mu_j - \mu)| \cdot I_{[N(t)+1 \geq K]}]$ is for sufficiently large K at most $\varepsilon E(N(t)+1)$. Therefore, if $E\tau(t) < \infty$, then $EN(t) < \infty$ and $E\tau(t) \sim \mu EN(t)$ as $t \to \infty$. We generalize the theorem of Chow and Robbins, which states also that $E\tau(t) \sim \tau(t) \sim t$ as $t \to \infty$.

Theorem 2.3. Assume the a.s. increasing sequence $\{Z_n\}$ satisfies

(A) and in addition

(B) for each $\varepsilon > 0$ thre exists an integrable random variable H_ε such that for $n \geq 1$ (with the notation $x^+ \equiv \max(x,0)$), a.s.

$$(W_n - W_{n-1} - \varepsilon W_n)^+ \leq H_\varepsilon(1 + Z_{n-1}^+)$$

$$\sum_{j=1}^{n} (1+W_j)^{-\delta} E\{|Z_j - E\{Z_j | F_{j-1}\}|^{1+\delta} | F_{j-1}\} - \varepsilon W_n < H_\varepsilon$$

and as $n \to \infty$, $(W_n - W_{n-1})/\max(Z_{n-1}, W_n) \to 0$.

Then $E\tau(t) < \infty$, and as $t \to \infty$

$$\tau(t) \overset{a.s.}{\sim} t \sim E\tau(t).$$

Proof. Choose arbitrary $\varepsilon > 0$, and fix T,k. Observe that $N^* \equiv \min(k, N(T)+1)$ is a stopping time, and a.s.

$$T + I_{[N(T)+1 \leq k]}(Z_{N(T)+1} - Z_{N(T)}) \geq Z_{N^*}.$$

Then

$$T \geq EZ_{N^*} - \int_{[N(T)+1 \leq k]} (Z_{N(T)+1} - Z_{N(T)})dP$$

$$\geq E(W_{N^*} - \varepsilon W_{N(T)+1} I_{[N(T)+1 \leq k]}) -$$

$$E(W_{N(T)+1} - W_{N(T)} - \varepsilon W_{N(T)+1})^+ - E[I_{[N(T)+1 \leq k]} |Z_{N(T)+1} - Z_{N(T)} - W_{N(T)+1} + W_{N(T)}|]$$

$$\geq (1-\varepsilon)EW_{N^*} - E[(1+Z_{N(T)}^+)H_\varepsilon] - E^{\delta/(1+\delta)}[I_{[N(T)+1 \leq k]}(1+W_{N(T)+1})]$$

$$\cdot E^{1/(1+\delta)}[I_{[N(T)+1 \leq k]}(1+W_{N(T)+1})^{-\delta} |Z_{N(T)+1} - Z_{N(T)} - W_{N(T)+1} + W_{N(T)}|^{1+\delta}]$$

where we have used (B) and Hölder's inequality in the last step. But

$$E[I_{[N(T)+1\le k]}(1+W_{N(T)+1})^{-\delta}|Z_{N(T)+1}-Z_{N(T)}-W_{N(T)+1}+W_{N(T)}|^{1+\delta}]$$

$$\le E[\sum_{j=1}^{k} I_{[N(T)+1\ge j]}(1+W_j)^{-\delta}|Z_j-E\{Z_j|F_{j-1}\}|^{1+\delta}]$$

which by F_{j-1} measurability of W_j and $[N(T)+1\ge j]$, and then by (B), is

$$= E[\sum_{j=1}^{k} I_{[N(T)+1\ge j]}(1+W_j)^{-\delta}E\{|Z_j-E\{Z_j|F_{j-1}\}|^{1+\delta}|F_{j-1}\}]$$

$$\le E[H_\varepsilon+\varepsilon W_{N*}].$$

Since $Z^+_{N(T)} \le T$ by definition, so that $(W_{N(T)+1}-W_{N(T)}-\varepsilon W_{N(T)+1})^+ \ge (1+T)(1+Z^+_{N(T)})^{-1}(W_{N(T)+1}-W_{N(T)} - \varepsilon W_{N(T)+1})^+$, we have

$$T + (1+T)E[(1+Z^+_{N(T)})^{-1}(W_{N(T)+1}-W_{N(T)}-\varepsilon W_{N(T)+1})^+]$$

$$\ge (1-\varepsilon)EW_{N*} - (EW_{N*})^{\delta/(1+\delta)}(EH_\varepsilon+\varepsilon EW_{N*})^{1/(1+\delta)}.$$

Letting $k\to\infty$, we conclude $EW_{N(T)+1} < \infty$, and

$$T + (1+T)E[(1+Z^+_{N(T)})^{-1}(W_{N(T)+1}-W_{N(T)}-\varepsilon W_{N(T)+1})^+]$$

$$\ge (1-\varepsilon-\varepsilon^{1/(1+\delta)})EW_{N(T)+1} - EH_\varepsilon(EW_{N(T)+1})^{\delta/(1+\delta)}.$$

Since the integrand on the left-hand side converges (by (B)) dominatedly to 0, it follows by first taking $T\to\infty$ and then $\varepsilon\to 0$ that

$$\limsup_{T\to\infty} T^{-1} EW_{N(T)+1} \leq 1.$$

On the other hand, $\tau(T) \geq T$ implies $\liminf T^{-1}E\tau(T) \geq 1$. It follows now from Lemma 2.2 that $E\tau(T) \sim T$ as $T \to \infty$. Finally, the last part of (B) together with (2.3) implies as $T \to \infty$, a.s.

$$(Z_{N(T)+1} - Z_{N(T)})/\max(W_{N(T)+1}, Z_{N(T)}) \to 0.$$

Therefore $(\tau(T)-T)/\max(W_{N(T)+1}, T) \to 0$ as $T \to \infty$, and $T \sim W_{N(T)+1} \sim \tau(T)$. □

Remark. The proofs of Proposition 2.1 and Theorem 2.3 show, for $\{Z_n\}$ satisfying (A) and (B) but with the condition $Z_n \geq Z_{n-1}$ replaced by $W_n \geq W_{n-1}$ a.s., that $T \sim W_{N(T)+1} \sim \tau(T)$ a.s. as $T \to \infty$, that

$$\limsup T^{-1} EW_{N(T)+1} \leq 1, \quad \liminf T^{-1}E\tau(T) \geq 1$$

and

$$\limsup_{T\to\infty} E\left|Z_{N(T)+1} - Z_{N(T)}\right| / EW_{N(T)+1} \to 0.$$

Since for fixed T, k, and $N^* \equiv \min(k, N(T)+1)$, it is not hard to show that $E(\tau(T)-W_{N*}) = E((\tau(T)-Z_k)I_{[N(T)+1>k]}) \leq E\left|Z_{N(T)+1} - Z_{N(T)}\right|$, it follows that $T \sim E\tau(T) \sim EW_{N(T)+1}$. Thus our results generalize Theorem 1 of Chow and Robbins (1963), at least in the case where only finitely many of the constants $\mu_n \equiv E\{Z_n - Z_{n-1} | F_{n-1}\}$ are negative.

Theorem 2.4. Suppose $Z_n - Z_{n-1} \geq 0$ a.s., $Z_n \to \infty$ a.s. as $n \to \infty$, and $E\{Z_n - Z_{n-1} \mid F_{n-1}\}$ is a.s. non-increasing as a function of n. Assume also that for some fixed $0 \leq \delta' < \delta \leq 1$ and constant $K_1 < \infty$, and all $n \geq 1$,

$$(2.4) \qquad E\{|Z_n - E\{Z_n \mid F_{n-1}\}|^{1+\delta} \mid F_{n-1}\} \leq K_1 n^{\delta'} E^{1+\delta}\{Z_n - Z_{n-1} \mid F_{n-1}\}.$$

Then (A), (B), and (2.1) hold.

Proof. That (2.1) follows from (A) and (B) is the content of Proposition 2.1, Lemma 2.2, and Theorem 2.3. We show next that under the present hypotheses, the second part of (A) holds. In fact, since $E\{Z_n - Z_{n-1} \mid F_{n-1}\}$ decreases with n, and $W_n = \sum_{j=1}^{n} E\{Z_j - Z_{j-1} \mid F_{j-1}\}$,

$$(2.5) \qquad E\{Z_n - Z_{n-1} \mid F_{n-1}\}/W_n \leq n^{-1}.$$

But (2.4) and (2.5), together with the F_{n-1} measurability of W_n, yield

$$E[|Z_n - E\{Z_n \mid F_{n-1}\}|^{1+\delta}/W_n^{1+\delta}] = E[E\{|Z_n - E\{Z_n \mid F_{n-1}\}|^{1+\delta} \mid F_{n-1}\}/W_n^{1+\delta}\}$$

$$\leq K_1 n^{\delta'} E[E\{Z_n - Z_{n-1} \mid F_{n-1}\}/W_n]^{1+\delta} \leq K_1 n^{\delta'-1-\delta}$$

Theorem 2.4. Suppose $Z_n - Z_{n-1} \geq 0$ a.s., $Z_n \to \infty$ a.s. as $n \to \infty$, and $E\{Z_n - Z_{n-1} \mid F_{n-1}\}$ is a.s. non-increasing as a function of n. Assume also that for some fixed $0 \leq \delta' < \delta \leq 1$ and constant $K_1 < \infty$, and all $n \geq 1$,

$$(2.4) \qquad E\{|Z_n - E\{Z_n \mid F_{n-1}\}|^{1+\delta} \mid F_{n-1}\} \leq K_1 n^{\delta'} E^{1+\delta}\{Z_n - Z_{n-1} \mid F_{n-1}\}.$$

Then (A), (B), and (2.1) hold.

Proof. That (2.1) follows from (A) and (B) is the content of Proposition 2.1, Lemma 2.2, and Theorem 2.3. We show next that under the present hypotheses, the second part of (A) holds. In fact, since $E\{Z_n - Z_{n-1} \mid F_{n-1}\}$ decreases with n, and $W_n = \sum_{j=1}^{n} E\{Z_j - Z_{j-1} \mid F_{j-1}\}$,

$$(2.5) \qquad E\{Z_n - Z_{n-1} \mid F_{n-1}\}/W_n \leq n^{-1}.$$

But (2.4) and (2.5), together with the F_{n-1} measurability of W_n, yield

$$E[|Z_n - E\{Z_n \mid F_{n-1}\}|^{1+\delta}/W_n^{1+\delta}] = E[E\{|Z_n - E\{Z_n \mid F_{n-1}\}|^{1+\delta} \mid F_{n-1}\}/W_n^{1+\delta}\}$$

$$\leq K_1 n^{\delta'} E[E\{Z_n - Z_{n-1} \mid F_{n-1}\}/W_n]^{1+\delta} \leq K_1 n^{\delta'-1-\delta}$$

which is summable in n, proving the second part of (A). For (B), we remark first that $(W_n-W_{n-1}-\varepsilon W_n)^+/(1+Z_{n-1})$ is a.s. nonincreasing in n and integrable for $n=1$ and that $(W_n-W_{n-1})/W_n \to 0$ by (2.5). Next, by (2.4) and (2.5)

$$\sum_{j=1}^{n} (1+W_j)^{-\delta} E\{|Z_j - E\{Z_j|F_{j-1}\}|^{1+\delta}|F_{j-1}\} - \varepsilon W_n$$

$$\leq K_1 n^{\delta'-\delta} \sum_{j=1}^{n} E\{Z_j - Z_{j-1}|F_{j-1}\} - \varepsilon W_n$$

$$= W_n(K_1 n^{\delta'-\delta} - \varepsilon)$$

which is negative for all sufficiently large n. This proves (B).

It remains to show $W_n \to \infty$ a.s. Let $0 < \alpha < (\delta-\delta')/(1-\delta)$ be fixed arbitrarily, and define a sequence $\{Z_n'\}$ inductively by $Z_n' - Z_{n-1}' \equiv \min(n^\alpha, Z_n - Z_{n-1})$ for $n \geq 1$, and $Z_0' \equiv 0$. For $N \geq 1$ and K_2 a (large) positive constant, let

$$M_N \equiv \sum_{n=1}^{N} (I_{[Z_n'-Z_{n-1} \neq Z_n-Z_{n-1}]}$$

$$- P\{Z_n-Z_{n-1} > n^\alpha \mid F_{n-1}\}) I_{[\sum_{j=1}^{n} P\{Z_j-Z_{j-1} > j^\alpha \mid F_{j-1}\} \leq K_2]}.$$

which is summable in n, proving the second part of (A). For (B), we remark first that $(W_n-W_{n-1}-\varepsilon W_n)^+/(1+Z_{n-1})$ is a.s. nonincreasing in n and integrable for $n=1$ and that $(W_n-W_{n-1})/W_n \to 0$ by (2.5). Next, by (2.4) and (2.5)

$$\sum_{j=1}^{n} (1+W_j)^{-\delta} E\{|Z_j-E\{Z_j|F_{j-1}\}|^{1+\delta}|F_{j-1}\} - \varepsilon W_n$$

$$\leq K_1 n^{\delta'-\delta} \sum_{j=1}^{n} E\{Z_j-Z_{j-1}|F_{j-1}\} - \varepsilon W_n$$

$$= W_n(K_1 n^{\delta'-\delta}-\varepsilon)$$

which is negative for all sufficiently large n. This proves (B).

It remains to show $W_n \to \infty$ a.s. Let $0 < \alpha < (\delta-\delta')/(1-\delta)$ be fixed arbitrarily, and define a sequence $\{Z_n'\}$ inductively by $Z_n' - Z_{n-1}' \equiv \min(n^\alpha, Z_n - Z_{n-1})$ for $n \geq 1$, and $Z_0' \equiv 0$. For $N \geq 1$ and K_2 a (large) positive constant, let

$$M_N \equiv \sum_{n=1}^{N}(I_{[Z_n'-Z_{n-1} \neq Z_n-Z_{n-1}]}$$

$$- P\{Z_n-Z_{n-1} > n^\alpha \mid F_{n-1}\})I_{[\sum_{j=1}^{n} P\{Z_j-Z_{j-1} > j^\alpha \mid F_{j-1}\} \leq K_2]}.$$

Then $\{M_N\}$ is a square-integrable $\{F_N\}$-martingale with variance $\le K_2$. But $P\{Z_n - Z_{n-1} > n^\alpha \mid F_{n-1}\} \le P\{|Z_n - E\{Z_n \mid F_{n-1}\}| > n^\alpha/2 \mid F_{n-1}\}$ +

$I_{[E\{Z_n-Z_{n-1} \mid F_{n-1}\} > n^\alpha/2]} \le 2^{2+\delta} K_1 n^{\delta' - \alpha(1+\delta)} E^{1+\delta}\{Z_n - Z_{n-1} \mid F_{n-1}\}$,

and a.s. on the event $D_1 \equiv [\sum_{n=1}^{\infty} E\{Z_n - Z_{n-1} \mid F_{n-1}\} < \infty]$ the terms $E\{Z_n - Z_{n-1} \mid F_{n-1}\}$ must, because they are decreasing, for all sufficiently large n be $\le 2/n$. Therefore a.s. on D_1, $\sum_n P\{Z_n - Z_{n-1} > n^\alpha \mid F_{n-1}\} < \infty$, so that M_N converges a.s. as $N \to \infty$, no matter how large K_2 was, and

$\sum_{n=1}^{\infty} (I_{[Z'_n - Z'_{n-1} \neq Z_n - Z_{n-1}]} - P\{Z_n - Z_{n-1} > n^\alpha \mid F_{n-1}\}) < \infty$. It follows that $P\{D_1 \cap [Z_n - Z_{n-1} \neq Z'_n - Z'_{n-1} \text{ i.o.}]\} = 0$. Next, we check

$E\{(Z'_n - Z'_{n-1})^2 \mid F_{n-1}\} \le 2 \int_0^{n^\alpha} tP\{Z_n - Z_{n-1} \ge t \mid F_{n-1}\}dt \le$

$2n^{\alpha(1-\delta)} \int_0^{\infty} t^\delta P\{Z_n - Z_{n-1} \ge t \mid F_{n-1}\}dt \le 2(1+\delta)^{-1} n^{\alpha(1-\delta)} E\{(Z_n - Z_{n-1})^{1+\delta} \mid F_{n-1}\}$

$\le K_4 n^{\alpha(1-\delta)+\delta'} E^{1+\delta}\{Z_n - Z_{n-1} \mid F_{n-1}\}$, where $K_4 < \infty$ is a constant. As above, we know a.s. on the event D_1 that for all sufficiently large n,

$E\{Z_n - Z_{n-1} \mid F_{n-1}\} \le 2/n$, so that $E\{(Z'_n - Z'_{n-1})^2 \mid F_{n-1}\} \le 2^{1+\delta} K_4 n^{\delta' + \alpha(1-\delta) - 1 - \delta}$,

which by choice of α is summable in n. Therefore, for sufficiently large

Then $\{M_N\}$ is a square-integrable $\{F_N\}$-martingale with variance $\leq K_2$. But $P\{Z_n - Z_{n-1} > n^\alpha \mid F_{n-1}\} \leq P\{|Z_n - E\{Z_n \mid F_{n-1}\}| > n^\alpha/2 \mid F_{n-1}\} +$

$I_{[E\{Z_n - Z_{n-1} \mid F_{n-1}\} > n^\alpha/2]} \leq 2^{2+\delta} K_1 n^{\delta' - \alpha(1+\delta)} E^{1+\delta}\{Z_n - Z_{n-1} \mid F_{n-1}\}$,

and a.s. on the event $D_1 \equiv [\sum_{n=1}^{\infty} E\{Z_n - Z_{n-1} \mid F_{n-1}\} < \infty]$ the terms

$E\{Z_n - Z_{n-1} \mid F_{n-1}\}$ must, because they are decreasing, for all sufficiently large n be $\leq 2/n$. Therefore a.s. on D_1, $\sum_n P\{Z_n - Z_{n-1} > n^\alpha \mid F_{n-1}\} < \infty$, so that M_N converges a.s. as $N \to \infty$, no matter how large K_2 was, and

$\sum_{n=1}^{\infty} (I_{[Z_n' - Z_{n-1}' \neq Z_n - Z_{n-1}]} - P\{Z_n - Z_{n-1} > n^\alpha \mid F_{n-1}\}) < \infty$. It follows

that $P\{D_1 \cap [Z_n - Z_{n-1} \neq Z_n' - Z_{n-1}' \text{ i.o.}]\} = 0$. Next, we check

$E\{(Z_n' - Z_{n-1}')^2 \mid F_{n-1}\} \leq 2 \int_0^{n^\alpha} t P\{Z_n - Z_{n-1} \geq t \mid F_{n-1}\} dt \leq$

$2n^{\alpha(1-\delta)} \int_0^\infty t^\delta P\{Z_n - Z_{n-1} \geq t \mid F_{n-1}\} dt \leq 2(1+\delta)^{-1}\, n^{\alpha(1-\delta)}\, E\{(Z_n - Z_{n-1})^{1+\delta} \mid F_{n-1}\}$

$\leq K_4\, n^{\alpha(1-\delta)+\delta'}\, E^{1+\delta}\{Z_n - Z_{n-1} \mid F_{n-1}\}$, where $K_4 < \infty$ is a constant. As

above, we know a.s. on the event D_1 that for all sufficiently large n,

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which by choice of α is summable in n. Therefore, for sufficiently large constant K_5, on D_1 the a.s. convergent square-integrable martingale

$$\sum_{n=1}^{N} (Z_n' - E\{Z_n' \mid F_{n-1}\}) I_{[\sum_{i=1}^{n} E\{(Z_i'-Z_{i-1}')^2 \mid F_{i-1}\} \leq K_5]}$$

is equal to $\sum_{n=1}^{N} ((Z_n'-Z_{n-1}') - E\{Z_n'-Z_{n-1}' \mid F_{n-1}\})$. Since $Z_n'-Z_{n-1}' \leq Z_n-Z_{n-1}$ a.s., and since we have a.s. on D_1 for all sufficiently large n that $Z_n'-Z_{n-1}' = Z_n-Z_{n-1}$, we conclude

$$P\{D_1\} = P\{D_1 \cap [Z_n'-Z_{n-1}' \neq Z_n-Z_{n-1} \text{ i.o.}]^c \cap [\sum_{n=1}^{\infty} (Z_n'-Z_{n-1}') < \infty\}$$

$$\leq P\{D_1 \cap [\sup_n Z_n < \infty]\} = 0 \quad \text{by hypothesis.}$$

This proves the first part of (A). □

Remark. An examination of the proof of Theorem 2.4 reveals that the same result follows if the constant K_1 in (2.4) is replaced by a random variable A with expectation $\leq K_1$, for which also $E[A^{1/(\delta-\delta')}E\{Z_1|F_0\}] \leq K_1$.

We state next (without proof) a simple general condition guaranteeing that $Z_n \to \infty$ a.s.

Lemma 2.5. Assume for all $0 < T < \infty$ there exist sequences $\varepsilon_n = \varepsilon_n(T)$ and $\delta_n = \delta_n(T)$ with $0 < \delta_n, \varepsilon_n \leq 1$ such that $\sum_{n=1}^{\infty} \varepsilon_n \delta_n = \infty$ and a.s. on the event $[\max_{1\leq j\leq n} Z_j \leq T]$, $P\{Z_n-Z_{n-1} > \varepsilon_n \mid F_{n-1}\} \leq \delta_n$. Then a.s. $Z_n \to \infty$ as $n \to \infty$.

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$$\sum_{n=1}^{N}(Z'_n - E\{Z'_n \mid F_{n-1}\})I_{[\sum_{i=1}^{n} E\{(Z'_i-Z'_{i-1})^2 \mid F_{i-1}\} \leq K_5]}$$

is equal to $\sum_{n=1}^{N} ((Z'_n-Z'_{n-1}) - E\{Z'_n-Z'_{n-1} \mid F_{n-1}\})$. Since $Z'_n-Z'_{n-1} \leq Z_n-Z_{n-1}$ a.s., and since we have a.s. on D_1 for all sufficiently large n that $Z'_n-Z'_{n-1} = Z_n-Z_{n-1}$, we conclude

$$P\{D_1\} = P\{D_1 \cap [Z'_n-Z'_{n-1} \neq Z_n-Z_{n-1} \text{ i.o.}]^c \cap [\sum_{n=1}^{\infty} (Z'_n-Z'_{n-1}) < \infty\}$$

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It is natural to suppose that if $E\{Z_n - Z_{n-1} \mid F_{n-1}\}$ converges to 0 as $n \to \infty$ and if there is some control over the conditional variance of $Z_n - Z_{n-1}$ given F_{n-1}, then $\tau(t)-t$ might be proven small for large t. The following theorem gives a setting within which $Z_n - E\{Z_n \mid F_{n-1}\}$ is a.s. summable and uniformly integrable for $n \geq 1$, so that if in addition $E\{Z_n - Z_{n-1} \mid F_{n-1}\}$ were assumed to converge to 0 a.s. [respectively, in the mean], then $Z_n - Z_{n-1} \to 0$ as $n \to \infty$ and $\tau(t)-t \to 0$ as $t \to \infty$ a.s. [in the mean].

Theorem 2.6. Suppose that $Z_n - Z_{n-1} \geq 0$ a.s. and that there exists a Lebesgue-integrable decreasing but strictly positive function $\phi(\cdot)$ on $[0,\infty)$ such that for some constant $K < \infty$, a.s.

$$\text{(2.6)}\qquad \begin{aligned} &\text{(i)}\quad K\cdot E\{Z_n - Z_{n-1} \mid F_{n-1}\} \geq \phi(Z_{n-1}) \\ &\text{(ii)}\quad E\{(Z_n - E\{Z_n \mid F_{n-1}\})^2 \mid F_{n-1}\} \leq \phi^2(Z_{n-1}). \end{aligned}$$

Then as $n \to \infty$, $Z_n \to \infty$ and $\sum_{m=1}^{\infty} (Z_m - E\{Z_m \mid F_{m-1}\}) < \infty$ a.s.

Proof. We show first that $Z_n \to \infty$ a.s. Let $\tilde{M}_N \equiv (Z_N - W_N) I_{[Z_N \leq T]}$ where $T < \infty$ is an arbitrarily large positive constant. By (2.6)(i), a.s.

$$W_N I_{[Z_N \leq T]} = \sum_{j=1}^{N} E\{Z_j - Z_{j-1} \mid F_{j-1}\} I_{[Z_{j-1},\, Z_N \leq T]} \geq I_{[Z_N \leq T]} N\phi(T)/K.$$

By (2.6)(ii), we find for arbitrary $\gamma > 0$, $P\{|\tilde{M}_N| \geq \gamma N\} \leq E\tilde{M}_N^2/(\gamma N)^2 \leq$

$$(\gamma N)^{-2} E[\sum_{j=1}^{N} (Z_j - E\{Z_j \mid F_{j-1}\})]^2 \leq (\gamma N)^{-2} E \sum_{j=1}^{N} \phi^2(Z_{j-1}) \leq N^{-1}(\phi(0)/\gamma)^2.$$

By the Borel-Cantelli Lemma, a.s. for all sufficiently large $N = n^2$, $|M_N| < \gamma N$, so that $T \geq I_{[Z_N \leq T]} Z_N > W_N I_{[Z_N \leq T]} - \gamma N \geq N(\frac{\phi(T)}{K} I_{[Z_N \leq T]} - \gamma)$ implies $P(\sup_{n \geq 1} Z_n \leq T) = 0$. Since $T < \infty$ was arbitrary and Z_n increases with n, $Z_n \to \infty$ a.s. as $n \to \infty$.

Now fix any constant $c > 0$ and define sequence $\{s_j\}_{j=0}^{\infty}$ by: $s_0 = 0$, $s_{j+1} - s_j = c\phi(s_j)$. Then $s_j \uparrow \infty$ as $j \to \infty$; for if $s_j \uparrow s^*$ then $s_{j+1} \uparrow s^* + c \lim_{j \to \infty} \phi(s_j) > s^*$, which is a contradiction. The properties of $\phi(\cdot)$ now imply

$$\sum_{j=0}^{\infty} \phi(s_{j+1})(s_{j+1} - s_j) \leq \int_0^{\infty} \phi(x)dx < \infty.$$

Therefore, by definition of $s_{j+1} - s_j$, $\sum_{j=0}^{\infty} \phi(s_{j+1})\phi(s_j) < \infty$. It follows, using (2.6)(i),(ii), that

$$\sum_{n=N(s_j)+2}^{N(s_{j+1})+1} E\{(Z_n - E\{Z_n \mid F_{n-1})^2 \mid F_{n-1}\} \leq \sum_{n=N(s_j)+2}^{N(s_{j+1})+1} \phi^2(Z_{n-1})$$

$$\leq \phi(s_j)[\sum_{n=N(s_j)+2}^{N(s_{j+1})} \phi(Z_{n-1}) + \phi(s_{j+1})]$$

$$\leq \phi(s_j)[K \sum_{n=N(s_j)+2}^{N(s_{j+1})} E\{Z_n - Z_{n-1} \mid F_{n-1}\} + \phi(s_{j+1})]$$

$$\leq \phi(s_j)[K(s_{j+1} - s_j) + \phi(s_{j+1})].$$

Summing over j = 0,1,..., we have $\sum_{n=2}^{\infty} E\{Z_n - E\{Z_n \mid F_{n-1}\})^2 \mid F_{n-1}\} \leq$

$\sum_{j=0}^{\infty} \phi(s_j)[K(s_{j+1}-s_j) + \phi(s_{j+1})] < \infty$. Therefore the $\{F_n\}$-adapted martingale $\{Z_n-W_n\}_{n\geq 0}$ is square-integrable and converges a.s., implying

$\sum_{n=1}^{\infty} (Z_n - E\{Z_n \mid F_n\}) < \infty$ a.s. □

It is easy to check that when $E\{Z_n-Z_{n-1} \mid F_{n-1}\}$ is non-increasing in n, the hypotheses of Theorem 2.6 imply those of Theorem 2.4. Of course, the restrictive assumption (2.6) strongly suggests the important special class of proportional-time processes we consider in the next section.

3. Proportional-time and proportional-hazard models

In this section, we first discuss for some special processes the applicability of the previous theorems, and then sharpen the conclusions (2.1) to resemble more closely the asymptotics in Renewal Theory for N(t) and EN(t). Recall from the Introduction that an increasing random sequence $\{Z_n\}_{n\geq 0}$ follows a proportional-time model if there exists a sequence $\{V_n\}_{n\geq 1}$ of independent random variables and a sequence $\{q_n(\cdot)\}_{n\geq 0}$ of deterministic non-negative Borel functions such that $Z_n-Z_{n-1} = V_n q_{n-1}(Z_{n-1})$ a.s.; and that $\{Z_n\}$ follows a proportional-hazard model if there exists a sequence $\{Q_n(\cdot)\}_{n\geq 0}$ of positive Borel functions for which $P\{Z_n-Z_{n-1} > t \mid Z_1,\ldots,Z_{n-1}\} = \exp(-Q_{n-1}(Z_{n-1})\Lambda_0(t))$ a.s. where $\Lambda_0(\cdot)$ is a fixed cumulative hazard function. It is clear that $\{Z_n\}$ satisfying either of these models is nonstationary Markovian.

We say that such $\{Z_n\}$ is repsectively homogeneous proportional-time [respectively, proportional-hazard] if all $q_n(\cdot) \equiv q(\cdot)$ [all $Q_n(\cdot) \equiv Q(\cdot)$] and $\{V_n\}$ is i.i.d.

The following Proposition simply re-states Theorem 2.4 and Lemma 2.5 in the present context.

Proposition 3.1. Suppose that $\{Z_n\}$ is proportional-time with $(EV_{n+1})q_n(t) \leq (EV_n)q_{n-1}(s)$ for $n \geq 1$ and $s \leq t$, $\inf\{(EV_{n+1})q_n(t): \; n \geq 0, \; 0 \leq t \leq T\} > 0$ for each $T < \infty$, and for some $0 \leq \delta' < \delta \leq 1$ and $K_1 < \infty$,

$$EV_n^{1+\delta} \leq K_1 n^{\delta'} (EV_n)^{1+\delta}, \quad n \leq 1.$$

Then as $t \to \infty$,

$$t^{-1} \sum_{n=0}^{N(t)} (EV_{n+1})q_n(Z_n) \to 1 \text{ a.s. and in the mean.}$$

If $\{Z_n\}$ were instead homogeneous proportional-time with $q(\cdot)$ strictly positive non-increasing and with $EZ_1^{1+\delta} < \infty$, then the same conclusion holds.

In the proportional hazards model, conditional moments of $Z_n - Z_{n-1}$ are difficult to estimate, so that some auxiliary hypothesis is needed to imply a condition like (2.4).

Proposition 3.2. If $\{Z_n\}$ is proportional-hazard satisfying

(a) $Q_n(t) \geq Q_{n-1}(s)$ for $n \geq 1$, $s \leq t$,

(b) for some $\gamma > 0$, $C_1 < \infty$, $Q_n(t) \leq C_1 t^\gamma$ for all $n \geq 0$, $t \geq 0$, We say that such $\{Z_n\}$ is repsectively homogeneous proportional-time [respectively, proportional-hazard] if all $q_n(\cdot) \equiv q(\cdot)$ [all $Q_n(\cdot) \equiv Q(\cdot)$] and $\{V_n\}$ is i.i.d.

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$$EV_n^{1+\delta} \leq K_1 n^{\delta'} (EV_n)^{1+\delta}, \quad n \leq 1.$$

Then as $t \to \infty$,

$$t^{-1} \sum_{n=0}^{N(t)} (EV_{n+1})q_n(Z_n) \to 1 \text{ a.s. and in the mean.}$$

If $\{Z_n\}$ were instead homogeneous proportional-time with $q(\cdot)$ strictly positive non-increasing and with $EZ_1^{1+\delta} < \infty$, then the same conclusion holds.

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(a) $Q_n(t) \geq Q_{n-1}(s)$ for $n \geq 1$, $s \leq t$,

(b) for some $\gamma > 0$, $C_1 < \infty$, $Q_n(t) \leq C_1 t^{\gamma}$ for all $n \geq 0$, $t \geq 0$,

(c) for some fixed $0 \le \alpha \le \beta$ with $\gamma(\beta-\alpha) \le 1/4$, fixed $0 < \varepsilon_1 \le 1$ and positive finite constants C_2, C_3,

$$C_2 x^\beta \le \Lambda_0^{-1}(x) \le C_3 x^\alpha \quad \text{for } 0 < x \le \varepsilon_1,$$

and (d) $\int_0^\infty [\Lambda_0^{-1}(x/Q_0(0))]^2 \exp(-x)dx < \infty$,

then as $t \to \infty$

$$t^{-1} \sum_{n=0}^{N(t)} \int_0^\infty \exp[-Q_n(Z_n)\Lambda_0(t)]dt \sim 1 \text{ a.s. and in the mean.}$$

Proof. (a) implies for each $t > 0$, $P\{Z_n - Z_{n-1} > t \mid Z_1,\ldots,Z_{n-1}\}$ is decreasing in n, so that $E\{Z_n - Z_{n-1} \mid Z_1,\ldots,Z_{n-1}\}$ is decreasing. (b) implies $Z_n \to \infty$ a.s. by Lemma 2.5. (d) says $E(Z_1^2) < \infty$, and (c) implies (with $F_n \equiv \sigma(Z_1,\ldots,Z_n)$)

$$E\{(Z_n - E\{Z_n \mid F_{n-1}\})^2 \mid F_{n-1}\}/E^2\{Z_n - Z_{n-1} \mid F_{n-1}\}$$

$$\le C_4 Q_{n-1}(Z_{n-1})^{2(\beta-\alpha)} + C_5 Q_{n-1}(Z_{n-1})^{2\beta} \exp(-\varepsilon_1 Q_{n-1}(Z_{n-1}))$$

$$\le C_6 Z_{n-1}^{2\gamma(\beta-\alpha)} \quad \text{for finite constants } C_4, C_5, C_6.$$

Now $\{Z_n/n\}_{n\ge1}$ is an a.s. bounded and uniformly integrable sequence (by comparison with the ordinary Strong Law, since $P\{Z_n - Z_{n-1} > t \mid F_{n-1}\} \le P\{Z_1 > t\}$ a.s. and $EZ_1 < \infty$). Therefore Jensen's inequality implies, if we set $\delta' \equiv 2\gamma(\alpha-\beta) \le 1/2$, that $E(Z_n/n)^{\delta'}$ and $E(Z_n/n)^{\delta'/(1-\delta')}$ are ≤ 1. By Theorem 2.4 and the Remark following it (with $\delta \equiv 1$), our Proposition is proved.

In particular, if $\Lambda_0(x) \sim Cx^{\alpha}$ as $x \to 0$, for $\alpha \geq 0$, $C > 0$, then (c) holds and a weaker moment assumption than (d) would suffice for the result.

There is a further purpose to studying the homogeneous proportional-time model, namely that one can there improve (2.1) to provide asymptotic rates of growth for $N(t)$ and $EN(t)$.

<u>Theorem 3.3</u>. Suppose $\{Z_n\}$ is homogeneous proportional-time with positive non-increasing $q(\cdot)$, $EZ_1^{1+\delta} < \infty$ for some $0 < \delta \leq 1$, and for $0 < c \leq 1$

$$(3.1) \qquad \limsup_{x \to \infty} \sup_{cx \leq t \leq x} \max\left\{\left|\frac{q(cx)}{q(t)} - 1\right|, \left|\frac{\tilde{q}(cx)}{\tilde{q}(t)} - 1\right|\right\} \leq \psi(1-c)$$

where $\tilde{q}(t) \equiv \int_0^t (1/q(s))ds$ and $\psi(\cdot)$ is nondecreasing and right-continuous at 0 with $\psi(0) = 0$. Then a.s. as $t \to \infty$

$$(3.2) \qquad N(t) \sim EN(t) \sim \tilde{q}(t)/\mu$$

where $\mu = EV_1$.

<u>Proof</u>. Fix arbitrarily small $\varepsilon > 0$ and any $t_0 > 0$, and define $t_j \equiv (1+\varepsilon)^j t_0$ for $j \geq 1$. Let $g(s) \equiv q(Z_{N(s)})$ for $s \geq 0$, and 0 for $s < 0$, so that the random function $g(s)$ is non-increasing with $g(s) \geq q(s)$. Since Proposition 3.1 applies, we know a.s. as $j \to \infty$

$$t_{j+1} \sim \sum_{j=0}^{N(t_{j+1})} \mu q(Z_j) = \mu \int_0^{t_{j+1}} g(s)dN(s)$$

$$t_j \sim \mu \int_0^{t_j} g(s)dN(s).$$

Therefore $t_{j+1}-t_j = \varepsilon t_j \sim \mu \int_{t_j}^{t_{j+1}} g(s)dN(s)$, which implies $P\{N(t_{j+1}) = N(t_j) \text{ i.o.}\} = 0$. However, on the event $[N(t_j) \neq N(t_{j-1})]$ almost surely $q(t_{j-1})(N(t_{j+1}) - N(t_j)) \geq \int_{t_j}^{t_{j+1}} g(s)dN(s) \geq q(t_{j+1})(N(t_{j+1}) - N(t_j))$. Our assumptions imply $|q(t_{j-1})/q(t_{j+1}) - 1| \leq 2\psi(1-(1+\varepsilon)^{-2})$ for sufficiently large j. Thus a.s. for sufficiently large j, $N(t_{j+1}) - N(t_j)$ and $\mu^{-1} \int_{t_j}^{t_{j+1}} (1/q(s))ds$ differ at most by a factor $(1+2\psi(2\varepsilon))$, and the same is true asymptotically for $N(t_{j+1})$ and $\mu^{-1}\tilde{q}(t_{j+1})$. Since $N(\cdot)$ is a.s. nondecreasing, and by assumption $|\tilde{q}(t_{j+1})/\tilde{q}(t_j)-1| \leq 2\psi(\varepsilon)$ for sufficiently large j, we conclude $N(t) \sim \mu^{-1}\tilde{q}(t)$. Almost the same argument shows $EN(t) \sim \mu^{-1}\tilde{q}(t)$, since $E \int_{t_j}^{t_{j+1}} g(s)dN(s) \geq E[q(t_{j+1}) \int_{t_j}^{t_{j+1}} dN(s)] = q(t_{j+1})E(N(t_{j+1})-N(t_j))$, but we need a separate estimate to bound $E \int_{t_j}^{t_{j+1}} g(s)dN(s)$ above. Indeed, $E\int_{t_j}^{t_{j+1}} g(s)dN(s) = \sum_{k=1}^{j} E\Big[\int_{t_j}^{t_{j+1}} g(s)dN(s)I_{[\text{for some } n,\ t_{k-1} \leq Z_{n-1} \leq t_k,\ Z_n > t_j]}\Big] \leq q(t_{j-1})E(N(t_{j+1}) - N(t_j)) + \sum_{k=1}^{j-1} q(t_{k-1})E\Big[(N(t_{j+1}) - N(Z_n))$

$\cdot I_{[\text{for some } n,\ t_j < Z_n \leq t_{j+1} \text{ and } q(t_k)V_n > t_j-t_k]}\Big]$. The Markov property for $\{Z_n\}$ implies $N(t_{j+1}) - N(Z_n)$ and V_n are conditionally independent given Z_n. Moreover, a.s. on the event $[Z_n > t_j]$, $E\{N(t_{j+1})-N(Z_n)|Z_n\} \leq E(N(t_{j+1})-N(t_j))$. Therefore, $E\int_{t_j}^{t_{j+1}} g(s)dN(s) \leq q(t_{j-1})E(N(t_{j+1})-N(t_j)) + E(N(t_{j+1})-N(t_j))\cdot\sum_{k=1}^{j-1} q(t_{k-1})P(V_1 > (t_j-t_k)/q(t_k)) \leq$

$$E(N(t_{j+1})-N(t_j))[q(t_{j-1}) + \sum_{k=1}^{j-1} q(t_{k-1})^{2+\delta} EV_1^{1+\delta}(t_j-t_k)^{-1-\delta}]$$

$\leq$ $E(N(t_{j+1})-N(t_j))q(t_{j-1})(1+\varepsilon)$ for sufficiently large j. Proceeding as for N(t), we conclude a.s. $EN(t) \sim \mu^{-1}\tilde{q}(t)$ as $t \to \infty$. □

4. Discussion

The theme of this paper is that a renewal-type theorem for dependent waiting-times $\{Z_n - Z_{n-1}\}_{n\geq 1}$ divides naturally into two parts. The first is (2.1), which one can expect to prove for extremely general types of dependence as we have done in Theorems 2.3 and 2.4 and Lemma 2.5. The second, essentially (3.1), should hold only for very special types of processes $\{Z_n\}$. Theorem 3.3, which is our result of this kind, applies only to "homogeneous proportional-time" Markov processes for which $(Z_n - Z_{n-1})/q(Z_{n-1})$ is an i.i.d. sequence, where $q(\cdot)$ is non-increasing and satisfies a condition like regular variation.

With the aid of theorems of the type of 2.3, 2.4, or 3.3, one may hope to base repair/replacement policies on more complicated and realistic statistical models for the conditional distributions of life-times $Z_n - Z_{n-1}$ given the past than have previously been looked at. Such models are beginning to penetrate Reliability Theory. What this paper shows is that maintenance policies within such models can and should be compared on the basis of the rate of growth of conditional expected life given the past (i.e. of W_n defined in (A) of Section 2).

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