Economics Letters 15 (1984) 219–221
North-Holland

UTILITY MEASUREMENT
A Direct Proof of Lange's Conjecture *

Kaushik BASU

Delhi School of Economics, Delhi 110007, India

Received 18 January 1984

A simple proof, making use of no prior results, is given of Oskar Lange's conjecture regarding the axiomization of cardinal utility.

1. Oskar Lange's attempt to axiomatize cardinal utility has once been the source of a prolonged and unresolved controversy [see Lange (1933), Samuelson (1938), Zeuthen (1936), Majumdar (1958)]. It can, however, be shown that Lange's conjecture is valid. I have tried to prove this elsewhere by establishing a *general* proposition by using some earlier theorems on meta-orderings and then showing that Lange's conjecture is a corollary [Basu (1982)]. [1]

What is interesting is that if we want to prove only Lange's conjecture *and no more*, it is possible to give a surprisingly simple proof. The proof hinges, in essence, on the construction of a two-way induction and it does not require any prior results. This highlights that what led to the controversy was not the need for more mathematics but greater conceptual clarity, which is, in itself, a useful lesson.

2. Let χ be the set of alternatives and R the set of real numbers. A *utility function* is a mapping from χ to R, and a *transformation* is a

* This paper was written while the author was a visitor at the University of Warwick. I am grateful to Ranen Das for a step in the proof.

[1] At a higher level of abstraction, axiomatizations of both cardinal utility and co-cardinal interpersonal comparability can be derived from a general property of positive affine transformations [Basu (1983)].

mapping from R to R. An individual is characterised by a reference utility function, u, and a set of permitted transformations, Ω. The idea is that if $f \in \Omega$, then the composite function, fu, is a utility function of the person (u, Ω).[2] An individual, (u, Ω), has *cardinal* utility iff $\forall f \in \Omega$, $\exists a$, $b \in R$, with $b > 0$, such that $\forall r \in R$, $f(r) = a + br$.

Lange's conjecture was that if a person can compare his changes in utility or first-difference in utility then his utility must be cardinal.

Definition. An individual (u, Ω) *can compare first-difference in utility*

$$\text{iff } \forall f \in \Omega, \quad \forall r_1, r_2, r_3, r_4 \in R,$$
$$r_1 - r_2 \geqslant r_3 - r_4 \leftrightarrow f(r_1) - f(r_2) > f(r_3) - f(r_4).$$

Remark. It is easy to check that if a person (u, Ω) has cardinal utility then he can compare first-differences in utility. The reverse proposition is Lange's conjecture and the source of controversy. In the proof that follows I use Q and I to denote the sets of rational numbers and integers.

Theorem. If an individual can compare first-differences in utility, then his utility is cardinal.

Proof. Suppose (u, Ω) can compare first-differences in utility. Let $f \in \Omega$. The theorem is established by proving that $\exists a$, $b \in R$, with $b > 0$, such that $\forall r \in R$, $f(r) = a + br$. The proof is broken up into three parts.

(i) It will be proved here that $f(r) = a + br$, $\forall r \in Q$. Let $p, q \in I (q \neq 0)$. Since $p/q - 1/q = (p-1)/q - 0$, by the definition,

$$f(p/q) - f(1/q) = f((p-1)/q) - f(0). \tag{1}$$

Given (1), it is easy to check that the following two equations are equivalent, i.e., (2) $\leftrightarrow$ (3):

$$f((p-1)/q) = (p-1)[f(1/q) - f(0)] + f(0), \tag{2}$$

$$f(p/q) = p[f(1/q) - f(0)] + f(0). \tag{3}$$

[2] It is assumed that Ω contains the identity mapping. This ensures that the reference utility function is a utility function of the person.

Since for $p = 0$, (3) is valid and for $p = 1$, (2) is valid, hence by induction on p [recall (2) $\leftrightarrow$ (3)] we see that (3) is valid $\forall p \in I$. Now,

$$f(1) = f(q/q) = q[f(1/q) - f(0)] + f(0), \quad \text{by (3). Hence,}$$

$$f(1/q) = (1/q)[f(1) - f(0)] + f(0). \tag{4}$$

Let $r = p/q$ be a rational number, where $p, q \in I$.

$$\begin{aligned} f(r) &= p[f(1/q) - f(0)] + f(0), \quad \text{by (3)} \\ &= r[f(1) - f(0)] + f(0), \quad \text{by substituting (4).} \end{aligned}$$

Let $a \equiv f(0)$ and $b \equiv f(1) - f(0)$. Since $1 - 0 > 0 - 0$, using the definition we get $f(1) - f(0) > 0$. Thus $b > 0$.

(ii) It will now be proved that f is a continuous function. Let $r_0 \in R$ and δ be any positive scalar. Since $\forall r_i \in Q$, $f(r_i) = a + br_i$, we can find $r_1, r_2 \in Q$ such that $|f(r_1) - f(r_2)| < \delta$. Pick such a pair, r_1 and r_2, and denote $|r_1 - r_2|$ by α. Let $r_j \in R$ be such that $|r_0 - r_j| < \alpha$. Then $|r_0 - r_j| < |r_1 - r_2|$. By the definition, $|f(r_0) - f(r_j)| < |f(r_1) - f(r_2)|$. Hence $|f(r_0) - f(r_j)| < \delta$.

(iii) Now we extend the result from Q to R. Let $t \in R$ and let $\{r_n\}$ be a sequence in Q such that $\lim_{n \to \infty} r_n = t$. Since f is continuous, $\lim_{n \to \infty} f(r_n) = f(t)$. Since $r_n \in Q$, $\forall n$, $\forall n$, $f(r_n) = a + br_n$. Therefore $f(t) = \lim_{n \to \infty}[a + br_n] = a + bt$. Q.E.D.

References

Basu, K., 1982, Determinateness of the utility function: Revisiting a controversy of the thirties, Review of Economic Studies 49.

Basu, K., 1983, Cardinal utility, utilitarianism and a class of invariance axioms in welfare analysis, Journal of Mathematical Economics 12, no. 3.

Lange, O., 1933, The determinateness of the utility function, Review of Economic Studies 1.

Majumdar, T., 1958, The measurement of utility (Macmillan, London).

Samuelson, P.A., 1938, The numerical representation of ordered classifications and the concept of utility, Review of Economic Studies 6.

Zeuthen, F., 1936, On the determinateness of the utility function, Review of Economic Studies 4.